④ Consider random vector $X \sim N(\mu, \Sigma = \sigma^2 I_n)$.
Derive the distribution of CX (for C being a non-singular matrix of constants) either through MGFs or through density transformation of X.

$$f_X(x) = (2\pi)^{-n/2} |\Sigma|^{-1/2} \exp\left\{-\frac{1}{2}(X-\mu)' \Sigma^{-1} (X-\mu)\right\}.$$

$$X \sim N(\mu, \Sigma = \sigma^2 I_n)$$

$$g(X) = Y = CX$$

$$g^{-1}(Y) = X = C^{-1}Y$$

$$\frac{\partial g^{-1}(Y)}{\partial Y} = \begin{bmatrix} \frac{\partial g^{-1}(Y_1)}{\partial Y_1} & \cdots & \frac{\partial g^{-1}(Y_n)}{\partial Y_1} \\ \vdots & & \vdots \\ \frac{\partial g^{-1}(Y_1)}{\partial Y_n} & \cdots & \frac{\partial g^{-1}(Y_n)}{\partial Y_n} \end{bmatrix}$$

$$C^{-1}Y = \begin{bmatrix} c^{11} & \cdots & c^{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} c^{1i} Y_i \\ \vdots \\ \sum_{i=1}^{n} c^{ni} Y_i \end{bmatrix}$$

⊛ In the Jacobian:

$$\frac{\partial g^{-1}(Y)}{\partial Y} = \begin{bmatrix} \frac{\partial \sum_{i=1}^{n} c^{1i} Y_i}{\partial Y_1} & \cdots & \frac{\partial \sum_{i=1}^{n} c^{1i} Y_i}{\partial Y_n} \\ \vdots & & \vdots \\ \frac{\partial \sum_{i=1}^{n} c^{ni} Y}{\partial Y_n} & \cdots & \frac{\partial \sum_{i=1}^{n} c^{ni} Y_i}{\partial Y_n} \end{bmatrix} = \begin{bmatrix} c^{11} & \cdots & c^{1n} \\ \vdots & & \vdots \\ c^{n1} & \cdots & c^{nn} \end{bmatrix} = C^{-1}$$

$$f_Y(y) = f_X(g^{-1}y)\,|C^{-1}|$$

$$= |C^{-1}|\,(2\pi)^{-n/2}\,|\sigma^2 I_n|^{-1/2} \exp\left\{-\frac{1}{2}(C^{-1}y-\mu)'(\sigma^2 I)^{-1}(C^{-1}y-\mu)\right\}$$

$$= (2\pi)^{-n/2}\,|\sigma^2 C'C|^{-1/2} \exp\left\{-\frac{1}{2}\left(\frac{CC^{-1}y-C\mu}{C}\right)'(\sigma^2 I)^{-1}\left(\frac{CC^{-1}y-\mu}{C}\right)\right\}$$

$$= (2\pi)^{-n/2}\,|\sigma^2 C'C|^{-1/2} \exp\left\{-\frac{1}{2}(\sigma^2 C'C)^{-1}(y-C\mu)'(y-C\mu)\right\}$$

therefore $CX \sim N(C\mu, \sigma^2 C'C)$