(1) The canonical form of the exponential family:

$$f_Y(y, \theta, \phi) = \exp\left\{ \frac{y\theta - b(\theta)}{a(\phi)} + c(y, \phi) \right\}$$

(a) $Y \sim \text{Poisson}(\lambda)$

$$f_Y(y) = e^{-\lambda} \frac{\lambda^y}{y!}$$

$$= e^{-\lambda} e^{\log\left(\frac{\lambda^y}{y!}\right)}$$

$$= \exp\left\{ y \log(\lambda) - \lambda - \log(y!) \right\}.$$

* Canonical parameter: $\theta = \log(\lambda)$
* Dispersion parameter: $\phi = 1$
* $a(\phi) = 1$

$b(\theta) = \lambda = e^{\theta}$.

$c(y, \phi) = -\log(y!)$

* Variance function: $V(\mu) = b''(\theta) = \lambda$ where $\mu = \lambda$.

(b) $Y \sim \text{Exponential}(\lambda)$

$$f_Y(y) = \lambda e^{-\lambda y}$$

$$= e^{\log \lambda} e^{-\lambda y}.$$

$$= \exp\{-\lambda y + \log \lambda\}.$$

* Canonical parameter: $\theta = \lambda$
* Dispersion parameter: $\phi = 1$
* $a(\phi) = 1$

$$b(\theta) = -\log \lambda = -\log(-\theta)$$

$$c(y, \phi) = 0.$$

* Variance function:

$$b'(\theta) = -\frac{1}{\theta} \Rightarrow \theta(\mu) = -\frac{1}{\mu}$$

$$b''(\theta) = \frac{1}{\theta^2} = \mu^2$$

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③ $Y_i \sim$ ind Bernoulli(p_i).

$$f_{Y_i}(y_i) = p_i^{y_i}(1-p_i)^{1-y_i}$$

ⓐ Estimate p_i with y_i.

$$\prod_{i=1}^{n} f_{Y_i}(y_i) = \prod_{i=1}^{n}\left[(y_i^{y_i})(1-y_i)^{1-y_i}\right] = L_{max, reduced}$$

ⓑ
$$\prod_{i=1}^{n}(\hat{p}_i^{y_i})(1-\hat{p}_i)^{1-y_i} = L_{max, full}$$

ⓒ
$$2\log\frac{L_{max, reduced}}{L_{max\ full}} = 2\log\frac{\prod_{i=1}^{n}\left[(y_i^{y_i})(1-y_i)^{1-y_i}\right]}{\prod_{i=1}^{n}\left[(\hat{p}_i^{y_i})(1-\hat{p}_i)^{1-y_i}\right]}$$

$$= 2\log\prod_{i=1}^{n}\left[(y_i^{y_i})(1-y_i)^{1-y_i}\right] - 2\log\prod_{i=1}^{n}\left[(\hat{p}_i^{y_i})(1-\hat{p}_i)^{1-y_i}\right]$$

$$= 2\sum_{i=1}^{n}\left(y_i\log y_i + (1-y_i)\log(1-y_i)\right) - 2\sum_{i=1}^{n}\left(y_i\log(\hat{p}_i) + (1-y_i)\log(1-\hat{p}_i)\right)$$

$$= D_0 - D_F$$

So 2 log of ratio = difference in deviance ($D_{null} - D_{full}$)

(4) $\rho_{ij} = \dfrac{\sigma_{ij}}{\sqrt{\sigma_{ii}}\sqrt{\sigma_{jj}}} \Rightarrow \hat{\rho}_{ij} = \dfrac{s_{ij}}{\sqrt{s_j}\sqrt{s_j}}$

We have

$$x^* = \left[\frac{x_1 - \bar{x}_1}{\sqrt{S_{x_1}}}, \dots, \frac{x_q - \bar{x}_q}{\sqrt{S_{x_q}}} \right]$$

$$S = \left(\left(\frac{1}{n} x_i' x_j - x_i \bar{x}_j \right)\right)_{i,j}$$

$$S_{x^*} = \left(\left(\frac{1}{n} x_i^{*\prime} x_j^* - \bar{x}_i^* \cdot \bar{x}_j^* \right)\right)_{i,j}$$

$$= \frac{1}{n} x_i^* x_j^*$$

$$= \frac{1}{n} \begin{bmatrix} \frac{n s_{11}}{s_{11}} & \frac{n s_{12}}{\sqrt{s_{11}}\sqrt{s_{12}}} & \cdots & \frac{n s_{1q}}{\sqrt{s_{11}}\sqrt{s_{qq}}} \\ \vdots & \ddots & & \vdots \\ \frac{n s_{1q}}{\sqrt{s_{11}}\sqrt{s_{qq}}} & \frac{n s_{2q}}{\sqrt{s_{22}}\sqrt{s_{qq}}} & \cdots & \frac{n s_{qq}}{s_{qq}} \end{bmatrix}$$

$$= R$$

⑤ Suppose $X_{q\times 1} \sim N(\mu, \Sigma)$.

We know that $Y_i = (X-\mu)' e_i \sim N(0, \lambda_i)$

Derive the distribution of $Y_i^* = X^* e_i$ when

X is standardized to form $X^* = (X-\mu)\Sigma^{-1/2}$

$$Y_i^* = X^* e_i$$
$$= (X-\mu)\,\Sigma^{-1/2}\, e_i \qquad i = 1, \cdots, q$$

$$E\{(x-\mu)\Sigma^{-1/2} e_i\} = E\{(x-\mu)\}\underbrace{\Sigma^{-1/2} e_i}_{\text{constant}}$$
$$= (E\{x\} - \mu)\ \text{constant}$$
$$= (\mu - \mu)\ \text{constant} = 0$$

Let $(x-\mu)\Sigma^{-1/2} = Z$

$$Var\{Y_i^*\} = e_i'\ Var(z)\, e_i$$
$$= e_i'\ I\ e_i = e_i' e_i = 1$$

Since x^* is standardly normally distributed.

$\Rightarrow Y \sim N(0, 1)$