③ $Y = Z\beta + \varepsilon \qquad E\{\varepsilon\} = 0, \quad Var\{\varepsilon\} = \sigma^2 I_n$.

ⓐ Let $\hat{Y}$ be the vector of predicted response variables based on least squares estimation of the model.

Let $\hat{\varepsilon}$ be the estimator error vector. Show that $Cov\{\hat{Y}, \hat{\varepsilon}\} = 0$

Let H be the hat matrix

⊛ $\hat{Y} = HY$.

⊛ $\hat{\varepsilon} = (I - H)Y$.

$$
\begin{aligned}
Cov(\hat{\varepsilon}, \hat{Y}) &= Cov[(I-H)Y, HY] \\
&= (I-H)(\sigma^2 I)H \\
&= \sigma^2 (H - H^2) \\
&= \sigma^2 (H - H) = 0
\end{aligned}
$$

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3 (b) Let $\Sigma = \sigma^2$

$$L(\beta, \sigma^2) = (2\pi\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}(y - Z\beta)'(y - Z\beta)\right\}$$

$$l(\beta, \sigma^2) = \log(\beta, \sigma^2) = -\frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}(y - Z\beta)'(y - Z\beta) - \frac{n}{2}\log(2\pi)$$

$$\frac{\partial l}{\partial \beta} = -\frac{1}{2\sigma^2}(-2X'y + 2X'X\beta)$$

$$\frac{\partial l}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}(y - Z\beta)'(y - Z\beta)$$

Setting $\frac{\partial l}{\partial \beta} = 0$, we get the least square estimate of β.

This gives $X'X\hat{\beta} = X'y \Rightarrow \hat{\beta} = (X'X)^{-1}X'Y$

Setting $\frac{\partial l}{\partial \sigma^2} = 0$, we get $\hat{\sigma}^2 = \frac{(y - Z\hat{\beta})'(y - Z\hat{\beta})}{n}$

$$l(\hat{\beta}, \hat{\sigma}^2) - l(\hat{\beta}, \sigma^2) = -\frac{n}{2}\left[\log\left(\frac{\hat{\sigma}^2}{\sigma^2}\right) + 1 - \frac{\hat{\sigma}^2}{\sigma^2}\right] \geq 0$$

$$L(\beta, \sigma^2) \leq L(\hat{\beta}, \hat{\sigma}^2)$$

Thus $\hat{\beta}$ and $\hat{\sigma}^2$ are the MLE of β and σ^2