Commonality Analysis of a Library of Simplex Method Solvers (LoSMS) [Put the name of your library in the title —SS][done —HZ]

Hanane Zlitni

October 4, 2018

1 Revision History

Date	Version	Notes
December 17, 2018	2.0	Final Draft (includes making the document consistent with the rest of the deliverables)
December 16, 2018	1.2	Applied Dr. Smith's Comments
October 13, 2018	1.1	Applied the comments obtained from Jennifer Garner's review
October 4, 2018	1.0	First Draft

2 Reference Material

This section records information for easy reference.

2.1 Table of Units

This section is not applicable for LoSMS.

2.2 Table of Symbols

The table that follows summarizes the symbols used in this document.

symbol	unit	description
Z	-	Optimal solution of the objective function
Z'	-	The negation of the objective function
R	-	The set of real numbers
N	-	The set of natural numbers

2.3 Abbreviations and Acronyms

symbol	description
A	Assumption
C	Calculation
DD	Data Definition
GD	General Definition
GS	Goal Statement
IM	Instance Model
LC	Likely Change
PS	Physical System Description
R	Requirement
SRS	Software Requirements Specification
CA	Commonality Analysis
LoSMS	Library of Simplex Method Solvers
T	Theoretical Model
$s.\ t.$	Subject to
2D	2 Dimentional

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3 Introduction

Linear programming is a method to achieve the best possible outcome in a mathematical model which is represented by linear relationships. It can also be called "Linear Optimization".

The simplex method is a linear programming algorithm that is considered one of the most popular algorithms which has significant influence in the fields of science and engineering (Dongarra and Sullivan (2000)).

[In your introduction it would be nice to say what linear programming actually is. —SS][done. —HZ]

The algorithm can be used in a variety of fields and its goal is to make the most of the available resources to achieve the optimal solution. For example, the simplex method is used in the sand casting process to optimize the sand casting parameters to produce the best results (Nadar (2016)). Moreover, the simplex method was used in chemistry to maximize the yield of a chemical reaction (Rozycki (1993)).

Since the simplex method has various applications in different fields, a software that facilitates solving objective functions using the simplex method for different purposes can be useful.

This commonality analysis (CA) provides detailed documentation of a general-purpose program family, called LoSMS (Library of Simplex Method Solvers), that solves linear programming problems using the simplex method. The reason for choosing this specific algorithm is because of its high efficiency, its numerous applications and its influence in various fields including science and engineering. The CA template is based on Smith (2006).

3.1 Purpose of Document

The purpose of this document is to formally describe the requirements for the development of the LoSMS tool which simplifies obtaining the optimal solution of linear programs. Having a thorough and comprehensive documentation of this tool would be useful for future use of LoSMS, possible enhancements and maintenance.

3.2 Scope of the Family

The scope of LoSMS is limited to solving linear equations using the simplex method. The tool supports both maximization and minimization linear programs.

3.3 Characteristics of Intended Reader

The intended reader of this document must have basic knowledge of linear programming which is typically given to year 3 undergraduate students. The intended reader must also have basic knowledge of linear algebra and calculus. No technical background is required.

3.4 Organization of Document

The document begins by providing a general description of the system, which includes potential system contexts, user characteristics and system constraints, in Section 4. Then, Section 5 describes the commonalities in the LoSMS program family by giving a background overview of the tool, terminology definitions, data definitions that are used to build instance models, goal statements and theoretical models. Next, Section 6 details the variabilities in the tool and consists of instance models and assumptions. This is followed by the functional and nonfunctional requirements of the LoSMS tool in Section 7 and likely changes in variabilities in Section 8. Finally, Section 9 visualises the way different sections of this document can be traced to one another.

4 General System Description

This section identifies the interfaces between the system and its environment, describes the potential user characteristics and lists the potential system constraints.

4.1 Potential System Contexts

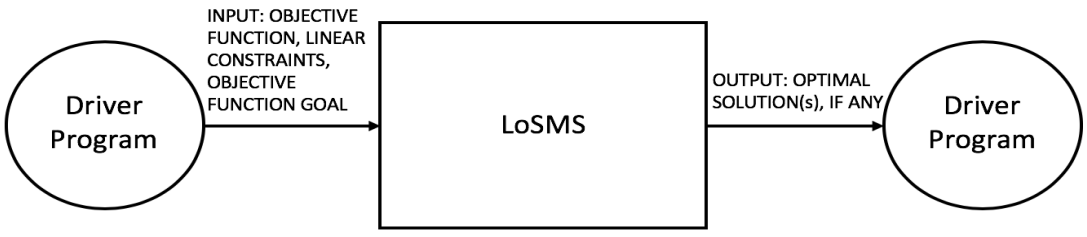

Figure 1: System Context

[Nice to see the drive program is interacting with your library, and not a user. —SS][thank you. —HZ]

Figure 1 describes the system context of LoSMS. The user inputs the correct data and LoSMS displays the output, if any.

- Driver Program Responsibilities: [Rather than user responsibilities, you should probably discuss the calling programs responsibilities —SS][done. —HZ]
  - Input the objective function, linear constraints and the objective function goal.
  - Handle the case where an input is completely missing.

  - Handle the case where an input is entered in a wrong format.

- LoSMS Responsibilities:
  - Handle the case where an input is empty.
  - Find and display the optimal value of the objective function along with the points where that value occurs (the values of the decision variables), if any.

[You say optimal solution, but it is unclear whether you mean the value of the optimal value of the objective function, or the optimal value and the corresponding values of the decision variables. —SS][I elaborated in the explanation. —HZ]

4.2 Potential User Characteristics

The end user of LoSMS should have basic knowledge of linear programming which is typically given in an operations research course to year 3 undergraduate students in Mathematics and in some Engineering programs like Software Engineering. [A little more information would be good here. Undergraduate students in what programs? What course subject would typically cover linear programming - operations research? systems? —SS][done. —HZ]

4.3 Potential System Constraints

LoSMS does not have any system constraints.

5 Commonalities

This section begins by providing a general idea about the LoSMS tool, followed by terminology and data definitions, goal statements and theoretical models.

5.1 Background Overview

LoSMS is a program family that facilitates obtaining the optimal solution of linear programs given the objective function and linear constraints. The tool can be beneficial for users coming from various fields, including physics and chemistry.

The following figure visualizes the simplex method in a 2D graph:

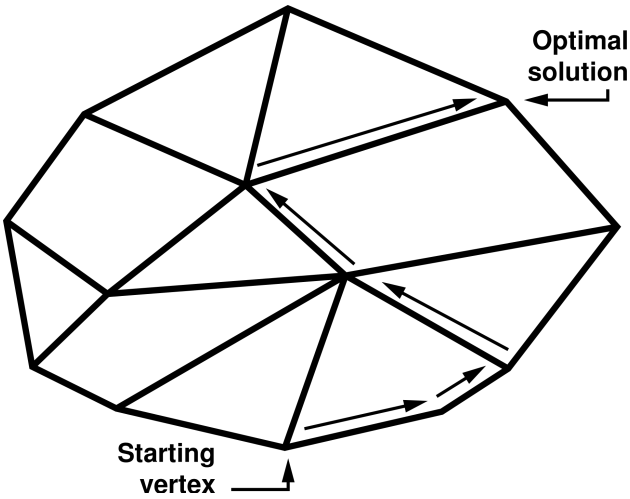

Figure 2: Simplex Method 2D Graph

[The background section would be a great place to show the "classic" 2D picture of the simplex method. —SS][done —HZ]

5.2 Terminology and Definitions

This subsection provides a list of terms that are used in the subsequent sections and their meaning, with the purpose of reducing ambiguity and making it easier to correctly understand the requirements:

- Linear program: An optimization problem where the goal is to minimize or maximize the objective function.
- Objective function: The function to be minimized or maximized.
- Objective function goal: The choice of minimizing or maximizing the objective function.
- Linear constraints: Linear constraints are some of the main inputs needed to solve a linear programming problem. They can be equalities or inequalities. There are two types of linear constraints: *main constraints* (e.g. $2x_1 + 3x_2 \leq 10$) and *non-negativity constraints* (e.g. $x_1, x_2 \geq 0$).
- Decision variables: They are variables that represent the parameters that the user wishes to optimize, and they are written as: x_1, x_2, ..., x_k, where $k \in N$.
- Entering variable: An entering variable is the entry in the simplex tableau that becomes the pivot instead of a previous entry.
- Departing variable: A departing variable is the orevious entry in the simplex tableau that was the pivot and got replaced by the entering variable.

- Feasible solution: The point that satisfies all constraints and sign restrictions.
- Feasible region: The set of all feasible points.
- Optimal solution/The optimum: A feasible solution with the maximum value in maximization objective functions or the minimum value in minimization objective functions.

5.3 Data Definitions

This section collects and defines all the data needed to build the instance models. The dimension of each quantity is also given.

Number	DD1
Label	**The Negation of a Minimization Linear Program**
Symbol	Z'
Equation	$Z' = -Z$
Description	To convert the linear program goal from minimization to maximization, negate the minimization function by multiplying it by -1.
Sources	-
Ref. By	IM2

[It is nice to see that IM2 actually does reference DD1. Good! —SS][thank you (: —HZ]

Number	DD2
Label	**The Simplex Tableau**
Symbol	sTableau
Equation	sTableau $= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \dots & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & \dots & b_2 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \dots & b_n \end{bmatrix}$, where: $a, b \in R$ and $n =$ the row number
Description	The simplex tableau is the augmented matrix form that LoSMS creates from the linear program's equations.
Sources	Murty (1997)
Ref. By	IM1

[I feel like there should be a symbol for the tableau and a source. —SS][done —HZ]

Number	DD3
Label	**The Slack Variable**
Symbol	S_n
Equation	$a_1x_1 + a_2x_2 \leq b_n$ becomes $a_1x_1 + a_2x_2 + S_n = b_n$
Description	The slack variable is a variable that represents zero or a positive real number. It is added to the simplex tableau to convert it to the canonical form (see T1). This can also be called an "artificial variable".
Sources	Stacho (2014)
Ref. By	IM1

5.4 Goal Statements

Given the objective function, linear constraints and the objective function goal, the goal statement of LoSMS is:

GS1: Use the simplex method to find and output the optimal value of the objective function and the values of the decision variables satisfying all linear constraints and sign restrictions.

[Rather than display the optimal solution, say "output" the optimal solution. This is a more abstract way to say it. —SS][done —HZ] [As mentioned previously, you should be clear on what you mean by the optimal solution. —SS][done —HZ]

5.5 Theoretical Models

This section focuses on the general equations and laws that LoSMS is based on.

Number	T1
Label	**The Canonical Form of a Linear Program**
Equation	sTableau $= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \dots & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & \dots & b_2 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \dots & b_n \\ c_{n1} & c_{n2} & c_{n3} & c_{n4} & \dots & 0 \end{bmatrix}$, where: $a, b, c \in R$; $n =$ the row number ; the last row = the coefficients of the objective function ; the rest of the rows = the coeffiecients of the linear constraints [a and b should have the subscripts. Without the subscripts, they would not have the type real, but multidimensional sequences of real. —SS][I don't really understand which subscripts should be added. —HZ]
Description	A linear program is in its canonical form when it satisfies the following conditions: • The objective function is a maximization function • All constraints are equalities • All decision variables are greater than or equal to zero. To convert a simplex tableau to the canonical form, slack variables (see DD3) must be added to the tableau.
Source	Stacho (2014)
Ref. By	IM1, R4

[The names of functions, like max, should not be in italic font. You can use the following

LaTeXcode to change fonts inside an equation: max Z = ... —SS][done. —HZ] [Isn't Z usually in lower case in this formula? —SS][I saw that both upper and lower cases are used online and in text books. I used the upper case because that's what I'm used to when I studied operations research. —HZ]

[Do you want to consider using matrix notation for these equations? I guess it depends on how many times you need the equation. If you are using it frequently, the more succinct matrix notation would help. —SS][You're right, the matrix notation is much better. I will change all occurences of the equations to the matrix form. —HZ]

[Isn't this where you want to mention slack variables? —SS][done. —HZ]

[Note: Instead of having two theoretical models for the standard and canonical forms, I merged them into one under the name of canonical form because I noticed that they aren't differentiated online and in text books. So I thought that there's no need to do that here. —HZ]

Number	T2
Label	**Pivoting in an Augmented Matrix**
Equation	$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & b_1 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & b_n \end{bmatrix}$, where: $a \in R$ Let R_i and R_j be any two rows in the matrix, $i, j \in N$: Pivoting includes one or more of the following operations: • Row switching: $R_i \leftrightarrow R_j$ • Row addition: $R_i + R_j$ • Multiplying a row by a non-zero constant: kR_i ; $k \neq 0$
Description	Pivoting is the process of performing what's similar to Gaussian elimination with row operations to clear the entries of the pivot column (set them to 0). The element that performs the pivoting operation is called the "pivot". It is determined by finding the minimum ratio between the entries in the pivot column and the ones in the same row in the constants column (the last column). This is done to "guarantee that the new basis obtained after the pivot step will also be a feasible basis".
Source	Murty (1997)
Ref. By	IM1

[The information on T3 doesn't actually tell me what pivoting is. You have just defined elementary row operations, not why you would do them. —SS][That's true. I added an explanation for the actually pivoting process. Thank you. —HZ]

6 Variabilities

This section describes the variabilities in the LoSMS tool and details the instance models, assumptions and variabilities in the calculation and the output.

6.1 Instance Models

This section transforms the documented problem into one which is expressed in mathematical terms.

Number	IM1
Label	**The Simplex Method for Solving Linear Programs: Maximization Functions**
Input	1. Max objective function in the following form: $max\ Z = c_1x_1 + c_2x_2 + ... + c_kx_k$ 2. Linear constraint(s) in the following form: $a_{11}x_1 + a_{12}x_2 + ... + a_{nm}x_k \leq b_n$ $x_1, ..., x_k \geq 0$, where: $c, a, b \in R$; $k, n, m \in N$
Output	The optimal solution Z and the points x_i where Z occurs (the values of the decision variables) [I'm surprised you just want Z. Don't you also want the x_i values that give you Z? —SS][Yes. I added it. —HZ]
Description	The purpose of this instance model is to provide details about solving linear programs that intend to maximize a parameter. The steps to solve a maximization problem are: 1. Convert linear program to its canonical form using slack variables. 2. Set up the simplex method tableau. 3. Perform pivoting. 4. Set up the new simplex tableau. 5. Repeat steps 2-4 until there are no negative numbers in the bottom row of the tableau. 6. The optimal solution Z is found in the basic feasible solution derived from the final tableau.
Sources	Stacho (2014)
Ref. By	IM2, Section 6.3, R5

Number	IM2
Label	**The Simplex Method for Solving Linear Programs: Minimization Functions**
Input	1. Min objective function in the following form: $min\ Z = c_1x_1 + c_2x_2 + ... + c_kx_k$ 2. Linear constraint(s) in the following form: $a_{11}x_1 + a_{12}x_2 + ... + a_{nm}x_k \leq b_n$, $x_1, ..., x_k \geq 0$, where: $c, a, b \in R$; $k, n, m \in N$
Output	The optimal solution Z and the points x_i where Z occurs (the values of the decision variables)
Description	The purpose of this instance model is to provide details about solving linear programs that intend to minimize a parameter. The steps to solve a minimization problem are: 1. Convert the minimization linear program to a maximization linear program by finding Z' defined in DD1. 2. Solve IM1.
Sources	Stacho (2014)
Ref. By	Section 6.3, R5

[You have missed the notion of binding time. You could have family members where the number of equations are fixed at design time. This information could be hard-coded into a family member. I don't think you need to add this now, but you could clarify that the size variabilities are all left to run-time. —SS][clarified in 6.3. —HZ]

[You could have mentioned the "covering" variation (see the Wikipedia page for Linear Programming. —SS]

Derivation of the Simplex Method

The origin of the simplex method is detailed in Dantzig (1987).

6.2 Assumptions

A1: The objective function $Z \in R$.

A2: If the linear constraints are inequalities, they are of type less than or equal to. This is to ensure that there are no negative variables.

A3: It is assumed that all linear programs have the non-negativity constraint (i.e. the decision variables are greater than 0), even if the non-negativity constraint is not present in a linear program.

A4: The default linear program goal is maximization.

6.3 Calculation

LoSMS has some variabilities in its calculations. For instance, IM1 is solved when the linear program is a maximization problem, while IM2 is solved when the linear program is a minimization problem.

Moreover, the size of the linear program (i.e. the number of linear constraints and decision variables) varies and it is left to be determined at run-time.

[It would be better if you phrased these as variabilities. You could also have mentioned the variabilities I mentioned above. —SS][done. —HZ]

6.4 Output

Similar to its calculations, LoSMS has variabilities in its output, as well. The library currently outputs the optimal solution Z and the values of the decision variables x_i. However, with a small change, the library can just output Z or x_i, depending on what's desired. Furthermore, the format of the output varies. The library can output the solution to the screen and write it to a file. It can also be expanded in the future to accomodate more output format options.

[You have more output variabilities than this. You have the decision on whether the output includes the x values, or not. You could output to the screen, or to a file, or to memory. The format of the output could be a variability. —SS][I made changes to this section. Thank you for the clarification. —HZ]

7 Requirements

This section provides the functional requirements, the business tasks that the software is expected to complete, and the nonfunctional requirements, the qualities that the software is expected to exhibit.

7.1 Functional Requirements

R1: The LoSMS tool shall read the objective function, linear constraints and the objective function goal from the user.

R2: The LoSMS tool shall verify that all inputs are valid and satisfy A1 and A2.

R3: If there are invalid inputs, the LoSMS tool shall display a corresponding message to the user.

R4: The LoSMS tool shall convert the linear program to its canonical form (T1).

R5: The LoSMS tool shall find the optimal solution of the linear program and the points where it occurs by solving IM1 or IM2, depending on the objective function goal.

R6: The LoSMS tool shall display the optimal solution and the points where it occurs to the user, if any.

R7: If the given linear program does not have an optimal solution, the LoSMS tool shall display a corresponding message to the user.

7.2 Nonfunctional Requirements

Usability

NFR1: It shall take at most 10 minutes for the user to learn to use the LoSMS tool.

Portability

NFR2: The LoSMS tool shall be operable in Windows, Mac and Linux operating systems.

Accuracy

NFR3: The relative error of the output of the LoSMS tool shall be less than 0.2.

Correctness

NFR4: The difference between the output of LoSMS and MatLab for the same linear program shall be less than 10%.

Performance

NFR5: It shall take no more than 3 seconds for LoSMS to calculate the solution.

Stability

NFR6: The LoSMS tool shall handle a heavy load of inputs and not crash.

8 Likely Changes

LC1: The support for greater than or equal to inequalities in the linear constraints.

LC2: The support for additional linear programming algorithms like the criss-cross and interior point algorithms. [You could name some other algorithms, like the criss-cross algorithm, or interior point methods. —SS][done —HZ]

9 Traceability Matrices and Graphs

The purpose of a traceability matrix is to visualise the way different components of this CA are dependent on one another. Every time a component in a row is changed, the component in the corresponding column marked with an "X" may have to be changed as well. Table 1 shows the traceability between the data definitions, theoretical models, instance models, assumptions and the calculation.

	DD1	DD2	DD3	T1	T2	IM1	IM2	A1	A2
DD1							X		
DD2						X			
DD3						X			
T1						X			
T2						X			
IM1							X		
IM2									
A1									
A2						X			

Table 1: Traceability Matrix Showing the Dependencies between Components of this CA

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10 Appendix

This section provides additional content related to this commonality analysis.

10.1 Symbolic Parameters

There are no symbolic parameters used in this document.