Senior High School

Basic Calculus

Learner's Material

This learning resource was collaboratively developed and reviewed by educators from public and private schools, colleges, and/or universities. We encourage teachers and other education stakeholders to email their feedback, comments and recommendations to the Department of Education at action@deped.gov.ph.

We value your feedback and recommendations.

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Learner's Material
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Preface

These learning modules are intended as the primary learning material for the Basic Calculus course in senior high school. It contains the main definitions, theorems, operations, formulas and techniques for the course. The material includes numerous worked-out examples to help you understand the different principles and gain proficiency in the various problem-solving skills and techniques.

Calculus is one of the most important inventions in mathematics. Developed in the second half of the 17th century by Isaac Newton and Gottfried Leibniz, it is now a fundamental area of mathematics and is a powerful tool used extensively not only by mathematicians but also by engineers, scientists, economists and others in a wide variety of applications. Calculus provides the language and concepts that allow us to model natural phenomena.

An introductory course in calculus, such as the Grade 11 Basic Calculus course, is now a standard course in the senior high school curriculum in almost all countries all over the world. It is important that you learn the fundamental concepts and skills now. In particular, you will use the knowledge from this course in learning physics, which will be taught to you in Grade 12 using a calculus-based approach. There are still many things to learn about calculus, and you will encounter them in college.

The best way to master the concepts is to study very well, and not just read, these modules. By studying, we mean that you should take your pen and paper, and work carefully through the examples, and solve the exercises given in learning modules until you are comfortable with the ideas and techniques. This is the best way to learn mathematics.

The Basic Calculus course will require many concepts and skills that you have already learned in previous math courses, such as equations, functions, polynomials and their graphs. However, there will be some new ideas that you will encounter for the first time. Some of these ideas may appear abstract and complicated, but we expect all students to appreciate and learn how to use them. Because senior high school is a transition to college, mastering this course will prepare you for a higher level of academic rigor and precision. We are confident that you can do it.

Contents

Chapter 1

Limits and Continuity

LESSON 1: The Limit of a Function: Theorems and Examples

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Illustrate the limit of a function using a table of values and the graph of the function;
2. Distinguish between $\lim_{x \to c} f(x)$ and $f(c)$;
3. Illustrate the limit theorems; and
4. Apply the limit theorems in evaluating the limit of algebraic functions (polynomial, rational, and radical).

TOPIC 1.1: The Limit of a Function

LIMITS

Consider a function f of a single variable x. Consider a constant c which the variable x will approach (c may or may not be in the domain of f). The limit, to be denoted by L, is the unique real value that $f(x)$ will approach as x approaches c. In symbols, we write this process as

$$\lim_{x \to c} f(x) = L.$$

This is read, **"The limit of $f(x)$ as x approaches c is L."**

LOOKING AT A TABLE OF VALUES

To illustrate, let us consider

$$\lim_{x \to 2} (1 + 3x).$$

Here, $f(x) = 1+3x$ and the constant c, which x will approach, is 2. To evaluate the given limit, we will make use of a table to help us keep track of the effect that the approach of x toward 2 will have on $f(x)$. Of course, on the number line, x may approach 2 in two ways: through values on its left and through values on its right. We first consider approaching 2 from its left or through values less than 2. Remember that the values to be chosen should be close to 2.

x	$f(x)$
1	4
1.4	5.2
1.7	6.1
1.9	6.7
1.95	6.85
1.997	6.991
1.9999	6.9997
1.9999999	6.9999997

Now we consider approaching 2 from its right or through values greater than but close to 2.

x	$f(x)$
3	10
2.5	8.5
2.2	7.6
2.1	7.3
2.03	7.09
2.009	7.027
2.0005	7.0015
2.0000001	7.0000003

Observe that as the values of x get closer and closer to 2, the values of $f(x)$ get closer and closer to 7. This behavior can be shown no matter what set of values, or what direction, is taken in approaching 2. In symbols,

$$\lim_{x \to 2}(1 + 3x) = 7.$$

EXAMPLE 1: Investigate

$$\lim_{x \to -1}(x^2 + 1)$$

by constructing tables of values. Here, $c = -1$ and $f(x) = x^2 + 1$.

We start again by approaching -1 from the left.

x	$f(x)$
-1.5	3.25
-1.2	2.44
-1.01	2.0201
-1.0001	2.00020001

Now approach -1 from the right.

x	$f(x)$
-0.5	1.25
-0.8	1.64
-0.99	1.9801
-0.9999	1.99980001

The tables show that as x approaches -1, $f(x)$ approaches 2. In symbols,

$$\lim_{x \to -1}(x^2 + 1) = 2.$$

EXAMPLE 2: Investigate $\lim_{x \to 0} |x|$ through a table of values.

Approaching 0 from the left and from the right, we get the following tables:

x	$\lvert x \rvert$
-0.3	0.3
-0.01	0.01
-0.00009	0.00009
-0.00000001	0.00000001

x	$\lvert x \rvert$
0.3	0.3
0.01	0.01
0.00009	0.00009
0.00000001	0.00000001

Hence,

$$\lim_{x \to 0} |x| = 0.$$

EXAMPLE 3: Investigate

$$\lim_{x \to 1} \frac{x^2 - 5x + 4}{x - 1}$$

by constructing tables of values. Here, $c = 1$ and $f(x) = \dfrac{x^2 - 5x + 4}{x - 1}$.
Take note that 1 is not in the domain of f, but this is not a problem. In evaluating a limit, remember that we only need to go very close to 1; we will not go to 1 itself.

We now approach 1 from the left.

x	$f(x)$
1.5	-2.5
1.17	-2.83
1.003	-2.997
1.0001	-2.9999

Approach 1 from the right.

x	$f(x)$
0.5	-3.5
0.88	-3.12
0.996	-3.004
0.9999	-3.0001

The tables show that as x approaches 1, $f(x)$ approaches -3. In symbols,

$$\lim_{x \to 1} \frac{x^2 - 5x + 4}{x - 1} = -3.$$

EXAMPLE 4: Investigate through a table of values

$$\lim_{x \to 4} f(x),$$

if

$$f(x) = \begin{cases} x + 1 & \text{if } x < 4 \\ (x-4)^2 + 3 & \text{if } x \geq 4. \end{cases}$$

This looks a bit different, but the logic and procedure are exactly the same. We still approach the constant 4 from the left and from the right, but note that we should evaluate the appropriate corresponding functional expression. In this case, when x approaches 4 from the left, the values taken should be substituted in $f(x) = x + 1$. Indeed, this is the part of the function which accepts values less than 4. So,

x	$f(x)$
3.7	4.7
3.85	4.85
3.995	4.995
3.99999	4.99999

On the other hand, when x approaches 4 from the right, the values taken should be substituted in $f(x) = (x - 4)^2 + 3$. So,

x	$f(x)$
4.3	3.09
4.1	3.01
4.001	3.000001
4.00001	3.0000000001

Observe that the values that $f(x)$ approaches are not equal, namely, $f(x)$ approaches 5 from the left while it approaches 3 from the right. In such a case, we say that the limit of the given function *does not exist* **(DNE)**. In symbols,

$$\lim_{x\to 4} f(x) \ \text{DNE}.$$

Remark 1: We need to emphasize an important fact. We do not say that $\lim\limits_{x\to 4} f(x)$ "equals DNE", nor do we write "$\lim\limits_{x\to 4} f(x) = \text{DNE}$", because "DNE" is not a value. In the previous example, "DNE" indicated that the function moves in different directions as its variable approaches c from the left and from the right. In other cases, the limit fails to exist because it is undefined, such as for $\lim\limits_{x\to 0} \frac{1}{x}$ which leads to division of 1 by zero.

Remark 2: Have you noticed a pattern in the way we have been investigating a limit? We have been specifying whether x will approach a value c from the left, through values less than c, or from the right, through values greater than c. This direction may be specified in the limit notation, $\lim\limits_{x\to c} f(x)$ by adding certain symbols.

- If x approaches c from the left, or through values less than c, then we write $\lim\limits_{x\to c^-} f(x)$.
- If x approaches c from the right, or through values greater than c, then we write $\lim\limits_{x\to c^+} f(x)$.

Furthermore, we say

$$\lim_{x\to c} f(x) = L$$

if and only if

$$\lim_{x\to c^-} f(x) = L \ \text{ and } \ \lim_{x\to c^+} f(x) = L.$$

In other words, for a limit L to exist, the limits from the left and from the right must both exist and be equal to L. Therefore,

$$\lim_{x\to c} f(x) \ \text{DNE} \ \text{ whenever } \ \lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x).$$

These limits, $\lim\limits_{x\to c^-} f(x)$ and $\lim\limits_{x\to c^+} f(x)$, are also referred to as ***one-sided limits,*** since you only consider values on one side of c.

Thus, we may say:

- in our very first illustration that $\lim\limits_{x\to 2}(1+3x) = 7$ because $\lim\limits_{x\to 2^-}(1+3x) = 7$ and $\lim\limits_{x\to 2^+}(1+3x) = 7$.

- in Example 1, $\lim_{x\to -1}(x^2+1) = 2$ since $\lim_{x\to -1^-}(x^2+1) = 2$ and $\lim_{x\to -1^+}(x^2+1) = 2$.
- in Example 2, $\lim_{x\to 0}|x| = 0$ because $\lim_{x\to 0^-}|x| = 0$ and $\lim_{x\to 0^+}|x| = 0$.
- in Example 3, $\lim_{x\to 1}\dfrac{x^2-5x+4}{x-1} = -3$ because $\lim_{x\to 1^-}\dfrac{x^2-5x+4}{x-1} = -3$ and $\lim_{x\to 1^+}\dfrac{x^2-5x+4}{x-1} = -3$.
- in Example 4, $\lim_{x\to 4} f(x)$ DNE because $\lim_{x\to 4^-} f(x) \neq \lim_{x\to 4^+} f(x)$.

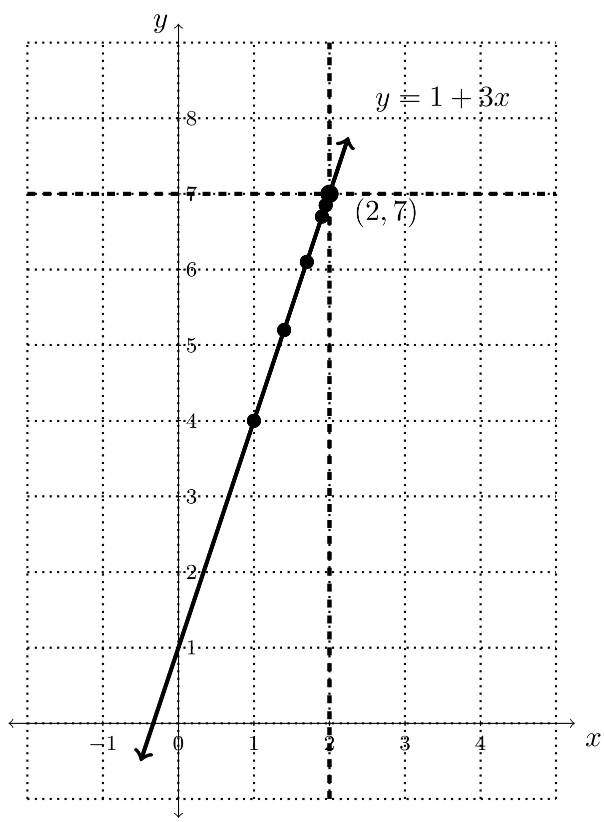
LOOKING AT THE GRAPH OF $y = f(x)$

If one knows the graph of $f(x)$, it will be easier to determine its limits as x approaches given values of c.

Consider again $f(x) = 1 + 3x$. Its graph is the straight line with slope 3 and intercepts $(0, 1)$ and $(-1/3, 0)$. Look at the graph in the vicinity of $x = 2$.

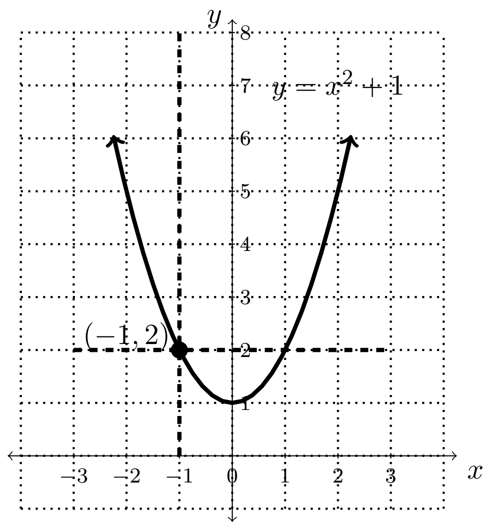
You can easily see the points (from the table of values in page 4) $(1, 4)$, $(1.4, 5.2)$, $(1.7, 6.1)$, and so on, approaching the level where $y = 7$. The same can be seen from the right (from the table of values in page 4). Hence, the graph clearly confirms that

$$\lim_{x\to 2}(1 + 3x) = 7.$$



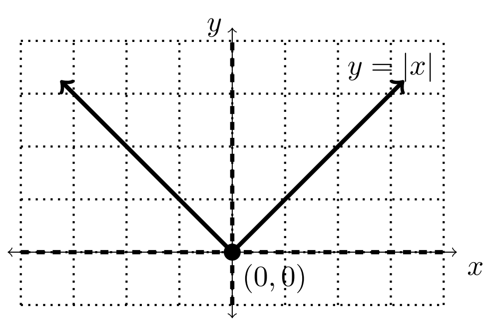
Let us look at the examples again, one by one.

Recall Example 1 where $f(x) = x^2 + 1$. Its graph is given by



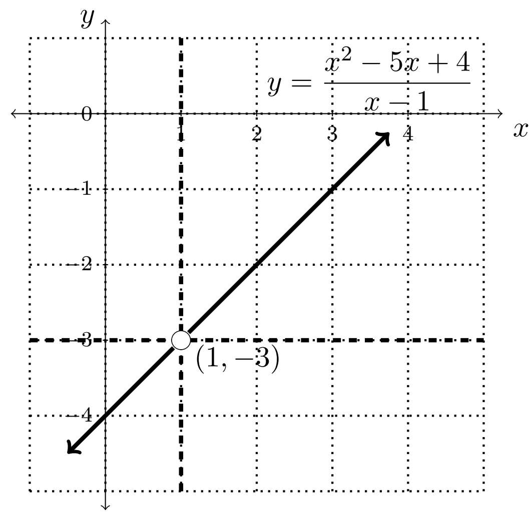
It can be seen from the graph that as values of x approach -1, the values of $f(x)$ approach 2.

Recall Example 2 where $f(x) = |x|$.



It is clear that $\lim_{x \to 0} |x| = 0$, that is, the two sides of the graph both move downward to the origin $(0, 0)$ as x approaches 0.

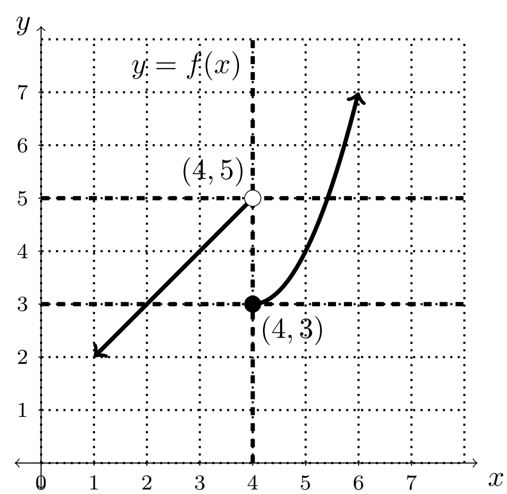
Recall Example 3 where $f(x) = \dfrac{x^2 - 5x + 4}{x - 1}$.



Take note that $f(x) = \dfrac{x^2 - 5x + 4}{x - 1} = \dfrac{(x-4)(x-1)}{x-1} = x - 4$, provided $x \neq 1$. Hence, the graph of $f(x)$ is also the graph of $y = x - 1$, excluding the point where $x = 1$.

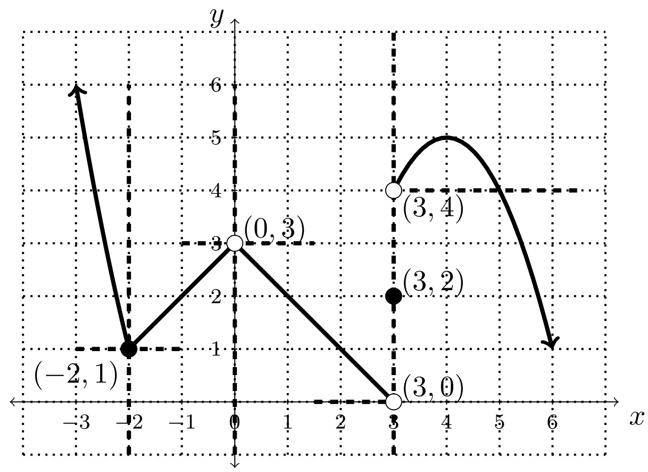
Recall Example 4 where

$$f(x) = \begin{cases} x + 1 & \text{if } x < 4 \\ (x-4)^2 + 3 & \text{if } x \geq 4. \end{cases}$$



Again, we can see from the graph that $f(x)$ has no limit as x approaches 4. The two separate parts of the function move toward different y-levels ($y = 5$ from the left, $y = 3$ from the right) in the vicinity of $c = 4$.

So, in general, if we have the graph of a function, such as below, determining limits can be done much faster and easier by inspection.



For instance, it can be seen from the graph of $y = f(x)$ that:

a. $\lim\limits_{x \to -2} f(x) = 1$.

b. $\lim\limits_{x \to 0} f(x) = 3$. Here, it does not matter that $f(0)$ does not exist (that is, it is undefined, or $x = 0$ is not in the domain of f). Always remember that what matters is the behavior of the function close to $c = 0$ and not precisely at $c = 0$. In fact, even if $f(0)$ were defined and equal to any other constant (not equal to 3), like 100 or -5000, this would still have no bearing on the limit. In cases like this, $\lim\limits_{x \to 0} f(x) = 3$ prevails regardless of the value of $f(0)$, if any.

c. $\lim\limits_{x \to 3} f(x)$ DNE. As can be seen in the figure, the two parts of the graph near $c = 3$ do not move toward a common y-level as x approaches $c = 3$.

Solved Examples

LOOKING AT TABLES OF VALUES

EXAMPLE 1: Determine $\lim\limits_{x \to 1} \left(x^3 - 1\right)$.

Solution. Approaching $x = 1$ from the left,

x	$f(x)$
0.9	-0.271
0.99	-0.029701
0.999	-0.002997001
0.9999	-0.00029997
0.99999	-0.0000299997

Now, taking values from the right of $x = 1$,

x	$f(x)$
1.1	0.331
1.01	0.030301
1.001	0.003003001
1.0001	0.00030003
1.00001	0.0000300003

Thus, $\lim\limits_{x \to 1} \left(x^3 - 1\right) = 0$.

EXAMPLE 2: Determine $\lim\limits_{x \to 0} |x + 2|$.

Solution. Taking values from the left of 0,

x	$f(x)$
-0.1	1.9
-0.05	1.95
-0.01	1.99
-0.005	1.995
-0.001	1.999

Approaching 0 from the right,

x	$f(x)$
0.1	2.1
0.05	2.05
0.01	2.01
0.005	2.005
0.001	2.001

Thus, $\lim\limits_{x \to 0} |x+2| = 2$.

EXAMPLE 3: Given

$$f(x) = \begin{cases} 2x - 3 & , \quad x \leq 4, \\ x^2 - x + 1 & , \quad x > 4. \end{cases}$$

Evaluate $\lim\limits_{x \to 4} f(x)$.

Solution. Approaching $x = 4$ from the left,

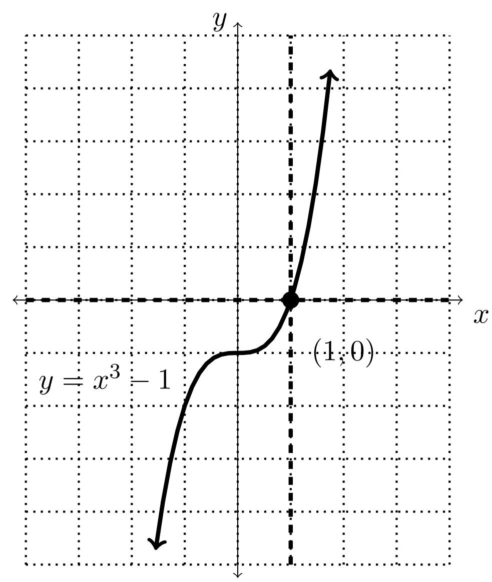
x	$f(x)$
3.9	4.8
3.95	4.9
3.99	4.98
3.995	4.99
3.999	4.998

Taking values from the right of 4,

x	$f(x)$
4.1	13.71
4.05	13.3525
4.01	13.0701
4.005	13.035025
4.001	13.007001

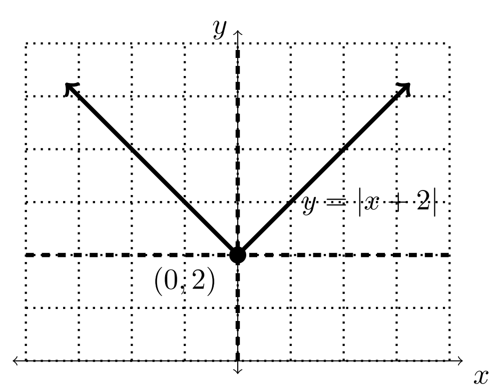
Hence, $\lim\limits_{x \to 4} f(x)$ DNE.

EXAMPLE 4: Recall Example 1, where $f(x) = x^3 - 1$ and whose graph is as follows:



Solution. It is clear from the graph that $\lim_{x \to 1} \left(x^3 - 1\right) = 0$.

EXAMPLE 5: Recall Example 2, where $f(x) = |x + 1|$. Its graph is as follows:

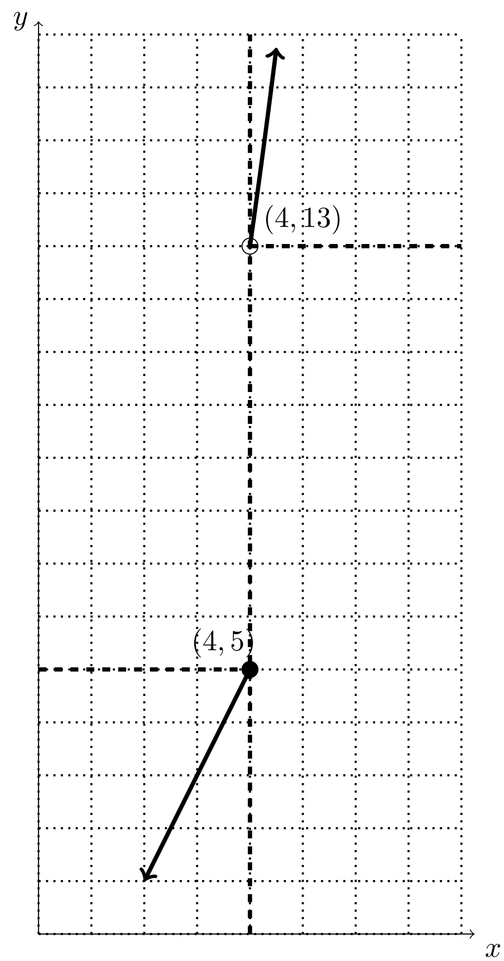


Solution. It is clear from the graph above that $\lim_{x \to 0} |x + 2| = 2$.

EXAMPLE 6: Recall Example 3, where

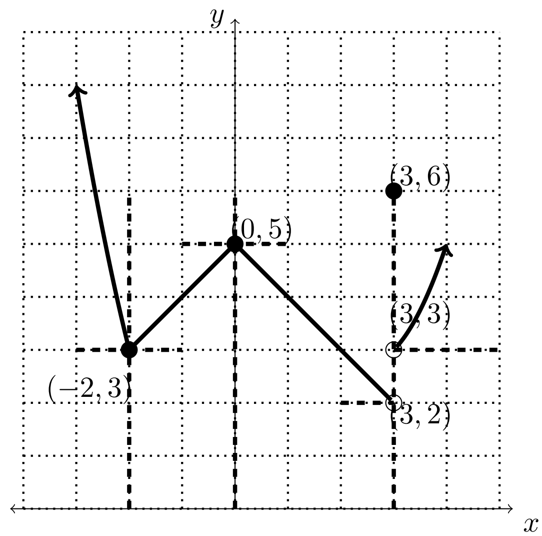
$$f(x) = \begin{cases} 2x - 3 & , \quad x \leq 4, \\ x^2 - x + 1 & , \quad x > 4, \end{cases}$$

and whose graph is as follows:



Solution. From the graph above, $\lim_{x \to 4^-} f(x) = 5$ and $\lim_{x \to 4^+} f(x) = 13$. Hence, $\lim_{x \to 4} f(x)$ DNE.

EXAMPLE 7: Now, consider the following graph of function:



Solution. From the graph,

1. $\lim_{x \to -2} f(x) = 3$,
2. $\lim_{x \to 0} f(x) = 5$,
3. $\lim_{x \to 3} f(x)$ DNE.

Supplementary Problems

1. Using tables of values, determine the limits of the following.

   (a) $\lim_{x \to 0} (x)$

   (b) $\lim_{x \to 1})2x)$

   (c) $\lim_{x \to 2} (-x + 1)$

   (d) $\lim_{x \to -1} (1 - x)$

   (e) $\lim_{x \to 0} (2x - 1)$

   (f) $\lim_{x \to 3} (x - 3)$

   (g) $\lim_{x \to 2} (3x^2 - 2)$

   (h) $\lim_{x \to 0} (5 - x - x^2)$

   (i) $\lim_{x \to 1} (x^3 + 1)$

   (j) $\lim_{x \to 0} (1 - x^2 - x^3)$

   (k) $\lim_{x \to 1} \sqrt{x^3 - 1}$

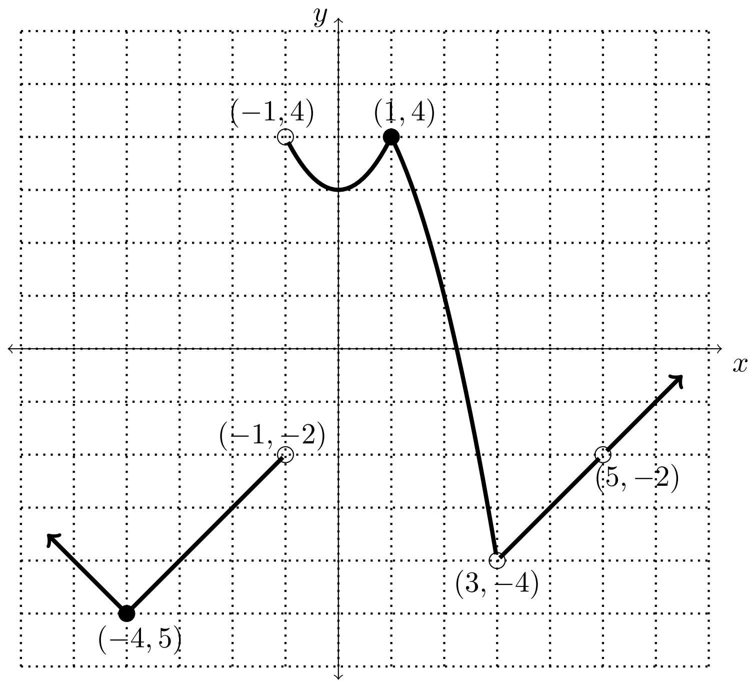
   (l) $\lim_{x \to 0} \sqrt{x^2 - x^3}$

   (m) $\lim_{x \to 1} \sqrt{x^4 - x^2 + 1}$

   (n) $\lim_{x \to 0} \dfrac{x}{1 - x}$

   (o) $\lim_{x \to 1} \dfrac{1 - x^2}{x + 1}$

2. The graph of $f(x)$ is given by



Determine the following limits.

(a) $\lim_{x \to -4} f(x)$

(b) $\lim_{x \to -1} f(x)$

(c) $\lim_{x \to 1} f(x)$

(d) $\lim_{x \to 3} f(x)$

(e) $\lim_{x \to 5} f(x)$

TOPIC 1.2: The Limit of a Function at c versus the Value of the Function at c

We will mostly recall our discussions and examples in Lesson 1.

Let us again consider

$$\lim_{x\to 2}(1+3x).$$

Recall that its tables of values are:

x	$f(x)$
1	4
1.4	5.2
1.7	6.1
1.9	6.7
1.95	6.85
1.997	6.991
1.9999	6.9997
1.9999999	6.9999997

x	$f(x)$
3	10
2.5	8.5
2.2	7.6
2.1	7.3
2.03	7.09
2.009	7.027
2.0005	7.0015
2.0000001	7.0000003

and we had concluded that $\lim_{x\to 2}(1+3x)=7$.

In comparison, $f(2)=7$. So, in this example, $\lim_{x\to 2}f(x)$ and $f(2)$ are equal. Notice that the same holds for the next examples discussed:

$\lim_{x\to c}f(x)$	$f(c)$
$\lim_{x\to -1}(x^2+1)=2$	$f(-1)=2$
$\lim_{x\to 0}\lvert x\rvert=0$	$f(0)=0$

This, however, is not always the case. Let us consider the function

$$f(x)=\begin{cases}|x| & \text{if } x\neq 0\\ 2 & \text{if } x=0.\end{cases}$$

In contrast to the second example above, the entries are now unequal:

$\lim_{x\to c}f(x)$	$f(c)$
$\lim_{x\to 0}\lvert x\rvert=0$	$f(0)=2$

Does this in any way affect the existence of the limit? Not at all. This example shows that $\lim_{x \to c} f(x)$ and $f(c)$ may be distinct.

Furthermore, consider the third example in Lesson 1 where

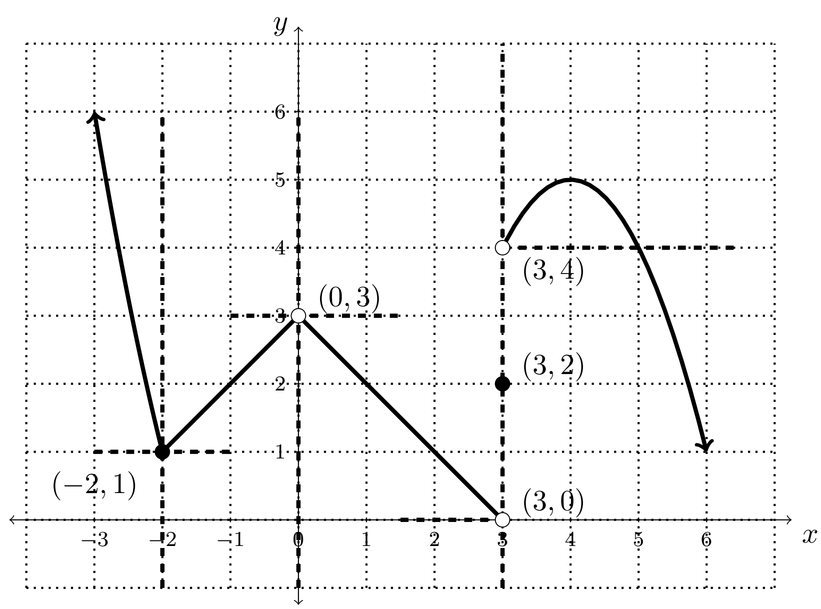
$$f(x) = \begin{cases} x+1 & \text{if } x < 4 \\ (x-4)^2+3 & \text{if } x \geq 4. \end{cases}$$

We have:

$\lim_{x \to c} f(x)$	$f(c)$
$\lim_{x \to 4} f(x)$ DNE	$f(4) = 3$

Once again we see that $\lim_{x \to c} f(x)$ and $f(c)$ are not the same.

A review of the graph given in Lesson 1 (redrawn below) will emphasize this fact.



We restate the conclusions, adding the respective values of $f(c)$:

1. $\lim_{x \to -2} f(x) = 1$ and $f(-2) = 1$.
2. $\lim_{x \to 0} f(x) = 3$ and $f(0)$ does not exist (or is undefined).

3. $\lim_{x \to 3} f(x)$ DNE and $f(3)$ also does not exist (or is undefined).

Solved Examples

EXAMPLE 1: Consider now $f(x) = \left(x^3 - 1\right)$. Compare $\lim_{x \to 1} f(x)$ and $f(1)$.

Solution. In this example, $f(1)$ and $\lim_{x \to 1} f(x)$ are equal.

$\lim_{x \to c} f(x)$	$f(c)$
$\lim_{x \to 1} (x^3 - 1) = 0$	$f(1) = 0$

EXAMPLE 2: Cinsider $f(x) = \lvert x + 2 \rvert$ and compare $\lim_{x \to 0} f(x)$ and $f(0)$.

Solution. Here, $f(0)$ and $\lim_{x \to 0} f(x)$ are equal.

$\lim_{x \to c} f(x)$	$f(c)$
$\lim_{x \to 0} \lvert x + 2 \rvert = 2$	$f(0) = 2$

EXAMPLE 3: Given the function

$$f(x) = \begin{cases} 2x - 3 & , \quad x \leq 4, \\ x^2 - x + 1 & , \quad x > 4. \end{cases}$$

Compare $\lim_{x \to 4} f(x)$ and $f(4)$.

Solution. Here, we the entries are different.

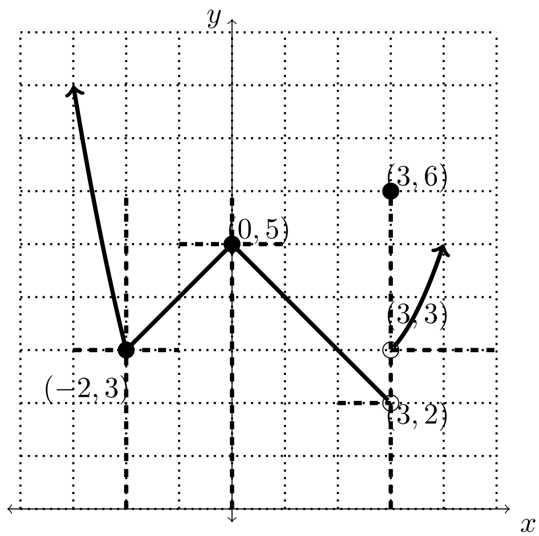
$\lim_{x \to c} f(x)$	$f(c)$
$\lim_{x \to 4} (f(x))\textbf{DNE}$	$f(4) = 5$

EXAMPLE 4: Given the following graph of $f(x)$, determine if $\lim_{x\to c} f(x) = f(c)$ at

i. $c = -2$,

ii. $c = 0$ and

iii. $c = 3$.



Solution. It is clear from the graph that

i. $\lim_{x\to -2} f(x) = 3 = f(-2)$,

ii. $\lim_{x\to 0} f(x) = 5 = f(0)$ and

iii. $f(3) = 6$ but $\lim_{x\to 3} f(x)$ DNE.

Supplementary Problems

Determine if $\lim_{x\to c} f(x) = f(c)$.

1. $f(x) = x + 2;\ \ c = -1$
2. $f(x) = x - 2;\ \ c = 0$
3. $f(x) = x^2 + 2;\ \ c = 1$
4. $f(x) = x^2 - 1;\ \ c = -1$
5. $f(x) = x^3 - x;\ \ c = 0$
6. $f(x) = x^3 - 3x;\ \ c = 1$
7. $f(x) = x^2 - 4;\ \ c = 2$
8. $f(x) = x^4 - 1;\ \ c = 1$

9. $f(x) = \dfrac{x^3 - x}{x}$; $c = 0$

10. $f(x) = \dfrac{x^3 - 3x}{x}$; $c = 1$

11. $f(x) = \dfrac{x^2 - 4}{x - 2}$; $c = 2$

12. $f(x) = \dfrac{x^4 - 1}{x - 1}$; $c = 1$

13. $f(x) = \dfrac{\sqrt{x^4} - 1}{x - 1}$; $c = 1$

14. $f(x) = \dfrac{\sqrt{x} - x^2}{x^2 - x}$; $c = 0$

15. $f(x) = \dfrac{\sqrt{x^3} - 1}{x - 1}$; $c = 1$

16. (at $c = 2$)

$$f(x) = \begin{cases} x - 1 & \text{if } x < 2, \\ (x-2)^2 + 1 & \text{if } x \geq 2; \end{cases}$$

17. (at $c = -1$)

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x < -1, \\ (x-1)^2 - 4 & \text{if } x \geq -1; \end{cases}$$

18. (at $c = 1$)

$$f(x) = \begin{cases} x^3 - 1 & \text{if } x < 1, \\ x^2 + 4 & \text{if } x \geq 1; \end{cases}$$

19. (at $c = 2$)

$$f(x) = \begin{cases} |x| & \text{if } x < 2, \\ x - 1 & \text{if } x \geq 2; \end{cases}$$

20. (at $c = 0$)

$$f(x) = \begin{cases} x^2 + x & \text{if } x < 0, \\ x^2 - 1 & \text{if } x \geq 0; \end{cases}$$

TOPIC 1.3: Illustration of Limit Theorems

In the following statements, c is a constant, and f and g are functions which may or may not have c in their domains.

1. The limit of a constant is itself. If k is any constant, then

$$\lim_{x \to c} k = k.$$

For example,

(a) $\lim_{x \to c} 2 = 2$

(b) $\lim_{x \to c} -3.14 = -3.14$

(c) $\lim_{x \to c} 789 = 789$

2. The limit of x as x approaches c is equal to c. This may be thought of as the substitution law because x is simply substituted by c.

$$\lim_{x \to c} x = c.$$

For example,

(a) $\lim_{x \to 9} x = 9$

(b) $\lim_{x \to 0.005} x = 0.005$

(c) $\lim_{x \to -10} x = -10$

For the remaining theorems, we will assume that the limits of f and g both exist as x approaches c and that they are L and M, respectively. In other words,

$$\lim_{x \to c} f(x) = L, \quad \text{and} \quad \lim_{x \to c} g(x) = M.$$

3. <u>The Constant Multiple Theorem</u>: This says that the limit of a multiple of a function is simply that multiple of the limit of the function.

$$\lim_{x \to c} k \cdot f(x) = k \cdot \lim_{x \to c} f(x) = k \cdot L.$$

For example, if $\lim_{x \to c} f(x) = 4$, then

(a) $\lim_{x \to c} 8 \cdot f(x) = 8 \cdot \lim_{x \to c} f(x) = 8 \cdot 4 = 32.$

(b) $\lim_{x \to c} -11 \cdot f(x) = -11 \cdot \lim_{x \to c} f(x) = -11 \cdot 4 = -44.$

(c) $\lim_{x \to c} \frac{3}{2} \cdot f(x) = \frac{3}{2} \cdot \lim_{x \to c} f(x) = \frac{3}{2} \cdot 4 = 6.$

4. The Addition Theorem: This says that the limit of a sum of functions is the sum of the limits of the individual functions. Subtraction is also included in this law, that is, the limit of a difference of functions is the difference of their limits.

$$\lim_{x \to c} (\ f(x) + g(x)\ ) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = L + M.$$

$$\lim_{x \to c} (\ f(x) - g(x)\ ) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x) = L - M.$$

For example, if $\lim_{x \to c} f(x) = 4$ and $\lim_{x \to c} g(x) = -5$, then

(a) $\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = 4 + (-5) = -1.$

(b) $\lim_{x \to c} (f(x) - g(x)) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x) = 4 - (-5) = 9.$

5. The Multiplication Theorem: This is similar to the Addition Theorem, with multiplication replacing addition as the operation involved. Thus, the limit of a product of functions is equal to the product of their limits.

$$\lim_{x \to c} (f(x) \cdot g(x)) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x) = L \cdot M.$$

Again, let $\lim_{x \to c} f(x) = 4$ and $\lim_{x \to c} g(x) = -5$. Then

$$\lim_{x \to c} f(x) \cdot g(x) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x) = 4 \cdot (-5) = -20.$$

Remark 1: The Addition and Multiplication Theorems may be applied to sums, differences, and products of more than two functions.

Remark 2: The Constant Multiple Theorem is a special case of the Multiplication Theorem. Indeed, in the Multiplication Theorem, if the first function $f(x)$ is replaced by a constant k, the result is the Constant Multiple Theorem.

6. The Division Theorem: This says that the limit of a quotient of functions is equal to the quotient of the limits of the individual functions, provided the denominator limit is not equal to 0.

$$\begin{aligned}\lim_{x\to c}\frac{f(x)}{g(x)} &= \frac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)}\\ &= \frac{L}{M}, \quad \text{provided } M \neq 0.\end{aligned}$$

For example,

(a) If $\lim\limits_{x\to c} f(x) = 4$ and $\lim\limits_{x\to c} g(x) = -5$,

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)} = \frac{4}{-5} = -\frac{4}{5}.$$

(b) If $\lim\limits_{x\to c} f(x) = 0$ and $\lim\limits_{x\to c} g(x) = -5$,

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{0}{-5} = 0.$$

(c) If $\lim\limits_{x\to c} f(x) = 4$ and $\lim\limits_{x\to c} g(x) = 0$, it is not possible to evaluate $\lim\limits_{x\to c}\frac{f(x)}{g(x)}$, or we may say that the limit DNE.

7. The Power Theorem: This theorem states that the limit of an integer power p of a function is just that power of the limit of the function. If $\lim\limits_{x\to c} f(x) = L$, then

$$\lim_{x\to c}(f(x))^p = (\lim_{x\to c} f(x))^p = L^p.$$

For example,

(a) If $\lim\limits_{x\to c} f(x) = 4$, then

$$\lim_{x\to c}(f(x))^3 = (\lim_{x\to c} f(x))^3 = 4^3 = 64.$$

(b) If $\lim\limits_{x\to c} f(x) = 4$, then

$$\lim_{x\to c}(f(x))^{-2} = (\lim_{x\to c} f(x))^{-2} = 4^{-2} = \frac{1}{4^2} = \frac{1}{16}.$$

8. The Radical/Root Theorem: This theorem states that if n is a positive integer, the limit of the nth root of a function is just the nth root of the limit of the function, provided the nth root of the limit is a real number. Thus, it is important to keep in mind that if n is even, the limit of the function must be positive. If $\lim_{x \to c} f(x) = L$, then

$$\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)} = \sqrt[n]{L}.$$

For example,

(a) If $\lim_{x \to c} f(x) = 4$, then

$$\lim_{x \to c} \sqrt{f(x)} = \sqrt{\lim_{x \to c} f(x)} = \sqrt{4} = 2.$$

(b) If $\lim_{x \to c} f(x) = -4$, then it is not possible to evaluate $\lim_{x \to c} \sqrt{f(x)}$ because then,

$$\sqrt{\lim_{x \to c} f(x)} = \sqrt{-4},$$

and this is not a real number.

Solved Examples

EXAMPLE 1: Evaluate the following limits.

1. $\lim_{x \to -1} 0$ (Ans. 0)
2. $\lim_{x \to 32910} -1$ (Ans. -1)
3. $\lim_{x \to 0.00001} 1$ (Ans. 1)

EXAMPLE 2: Evaluate the given limits.

1. $\lim_{x \to 1} x$ (Ans. 1)
2. $\lim_{x \to -0.01} x$ (Ans. -0.01)
3. $\lim_{x \to 300} x$ (Ans. 300)

EXAMPLE 3: Solve the following completely.

1. Given $\lim_{x \to 3} f(x) = -1$, evaluate $\lim_{x \to 3} (5 \cdot f(x))$.

2. Given $\lim\limits_{x \to 0} g(x) = 2$, determine $\lim\limits_{x \to 0}(-2 \cdot g(x))$.

Solution. Using limit theorems, we have

1. $\lim\limits_{x \to 3}(5 \cdot f(x)) = 5 \cdot \lim\limits_{x \to 3} f(x) = 5 \cdot -1 = -5$,
2. $\lim\limits_{x \to 0}(-2 \cdot g(x)) = -2 \cdot \lim\limits_{x \to 0} g(x) = -2 \cdot 2 = -4$.

EXAMPLE 4: Use limit theorems to evaluate the following limits.

1. Determine $\lim\limits_{x \to 1}(f(x) + g(x))$ if $\lim\limits_{x \to 1} f(x) = 2$ and $\lim\limits_{x \to 1} g(x) = -1$.
2. Evaluate $\lim\limits_{x \to -1}(f(x) - g(x))$ given that $\lim\limits_{x \to -1} f(x) = 0$ and $\lim\limits_{x \to -1} g(x) = -1$.

Solution. We use limit theorems to get

1. $\lim\limits_{x \to 1}(f(x) + g(x)) = \lim\limits_{x \to 1} f(x) + \lim\limits_{x \to 1} g(x) = 2 + (-1) = 1$,
2. $\lim\limits_{x \to -1}(f(x) - g(x)) = \lim\limits_{x \to -1} f(x) - \lim\limits_{x \to -1} g(x) = 0 - (-1) = 1$.

EXAMPLE 5: Evaluate the following limits.

1. Given $\lim\limits_{x \to 2} f(x) = 3$ and $\lim\limits_{x \to 2} g(x) = -1$, determine $\lim\limits_{x \to 2} f(x) \cdot g(x)$.
2. If $\lim\limits_{x \to -3} f(x) = 9$ and $\lim\limits_{x \to -3} g(x) = -3$, evaluate $\lim\limits_{x \to -3} \dfrac{f(x)}{g(x)}$.

Solution. Using the theorems above, we have

1. $\lim\limits_{x \to 2} f(x) \cdot g(x) = \lim\limits_{x \to 2} f(x) \cdot \lim\limits_{x \to 2} g(x) = 3 \cdot (-1) = -3$,
2. $\lim\limits_{x \to -3} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to -3} f(x)}{\lim\limits_{x \to -3} g(x)} = \dfrac{9}{-3} = -3$.

EXAMPLE 6: Solve the following completely.

1. Evaluate $\lim_{x\to 1}(f(x))^3$ if $\lim_{x\to 1} f(x) = -1$.
2. Evaluate $\lim_{x\to 0}(g(x))^{-2}$ if $\lim_{x\to 0} g(x) = 2$.

Solution. Using the limit theorems, we obtain

1. $\lim_{x\to 1}(f(x))^3 = (\lim_{x\to 1} f(x))^3 = (-1)^3 = -1,$
2. $\lim_{x\to 0}(g(x))^{-2} = (\lim_{x\to 0} g(x))^{-2} = 2^{-2} = \dfrac{1}{4}.$

EXAMPLE 7: Evaluate the following limits.

1. Determine $\lim_{x\to -1} \sqrt{f(x)}$ given that $\lim_{x\to -1} f(x) = 1$.
2. Evaluate $\lim_{x\to -2} \sqrt{g(x)}$ if $\lim_{x\to -2} g(x) = 9$.

Solution. We use limit theorems to get

1. $\lim_{x\to -1} \sqrt{f(x)} = \sqrt{\lim_{x\to -1} f(x)} = \sqrt{1} = 1,$
2. $\lim_{x\to -2} \sqrt{g(x)} = \sqrt{\lim_{x\to -2} g(x)} = \sqrt{9} = 3.$

Supplementary Problems

1. Given $\lim_{x\to 1} f(x) = 3$ and $\lim_{x\to 1} g(x) = -1$, evaluate the following limits.

   (a) $\lim_{x\to 1} 2 \cdot f(x)$

   (b) $\lim_{x\to 1} -3 \cdot g(x)$

   (c) $\lim_{x\to 1}(f(x) + g(x))$

   (d) $\lim_{x\to 1}(f(x) - g(x))$

   (e) $\lim_{x\to 1}(f(x) \cdot g(x))$

   (f) $\lim_{x\to 1} \dfrac{f(x)}{g(x)}$

2. Given $\lim_{x \to -1} f(x) = 2$ and $\lim_{x \to -1} g(x) = -2$, evaluate the following limits.

(a) $\lim_{x \to -1} \frac{1}{2} \cdot f(x)$

(b) $\lim_{x \to -1} (f(x) + g(x))$

(c) $\lim_{x \to -1} (f(x) - g(x))$

(d) $\lim_{x \to -1} (f(x) \cdot g(x))$

(e) $\lim_{x \to -1} \frac{f(x)}{g(x)}$

(f) $\lim_{x \to -1} (f(x))^2$

(g) $\lim_{x \to -1} (g(x))^{-3}$

3. Given $\lim_{x \to 0} f(x) = 0$ and $\lim_{x \to 0} g(x) = 1$, evaluate the following limits.

(a) $\lim_{x \to 0} 139401 \cdot f(x)$

(b) $\lim_{x \to 0} (f(x) + g(x))$

(c) $\lim_{x \to 0} (f(x) - g(x))$

(d) $\lim_{x \to 0} (f(x) \cdot g(x))$

(e) $\lim_{x \to 0} \frac{f(x)}{g(x)}$

(f) $\lim_{x \to 0} \sqrt{f(x)}$

(g) $\lim_{x \to 0} \sqrt{g(x)}$

TOPIC 1.4: Limits of Polynomial, Rational, and Radical Functions

In the previous lesson, we presented and illustrated the limit theorems. We start by recalling these limit theorems.

Theorem 1. *Let c, k, L and M be real numbers, and let $f(x)$ and $g(x)$ be functions defined on some open interval containing c, except possibly at c.*

1. *If $\lim_{x \to c} f(x)$ exists, then it is unique. That is, if $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} f(x) = M$, then $L = M$.*

2. $\lim_{x \to c} c = c$.

3. $\lim_{x \to c} x = c$

4. *Suppose $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$.*

   (a) *(Constant Multiple)* $\lim_{x \to c} [k \cdot g(x)] = k \cdot M$.

   (b) *(Addition)* $\lim_{x \to c} [f(x) \pm g(x)] = L \pm M$.

   (c) *(Multiplication)* $\lim_{x \to c} [f(x)g(x)] = LM$.

   (d) *(Division)* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, *provided* $M \neq 0$.

   (e) *(Power)* $\lim_{x \to c} [f(x)]^p = L^p$ *for p, a positive integer.*

   (f) *(Root/Radical)* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$ *for positive integers n, and provided that $L > 0$ when n is even.*

In this lesson, we will show how these limit theorems are used in evaluating algebraic functions. Particularly, we will illustrate how to use them to evaluate the limits of polynomial, rational and radical functions.

LIMITS OF ALGEBRAIC FUNCTIONS

We start with evaluating the limits of polynomial functions.

EXAMPLE 1: Determine $\lim_{x\to 1}(2x+1)$.

Solution. From the theorems above,

$$\begin{aligned}
\lim_{x\to 1}(2x+1) &= \lim_{x\to 1} 2x + \lim_{x\to 1} 1 && \text{(Addition)}\\
&= \left(2\lim_{x\to 1} x\right) + 1 && \text{(Constant Multiple)}\\
&= 2(1)+1 && \left(\lim_{x\to c} x = c\right)\\
&= 2+1\\
&= 3.
\end{aligned}$$

EXAMPLE 2: Determine $\lim_{x\to -1}(2x^3-4x^2+1)$.

Solution. From the theorems above,

$$\begin{aligned}
\lim_{x\to -1}(2x^3-4x^2+1) &= \lim_{x\to -1} 2x^3 - \lim_{x\to -1} 4x^2 + \lim_{x\to -1} 1 && \text{(Addition)}\\
&= 2\lim_{x\to -1} x^3 - 4\lim_{x\to -1} x^2 + 1 && \text{(Constant Multiple)}\\
&= 2(-1)^3 - 4(-1)^2 + 1 && \text{(Power)}\\
&= -2-4+1\\
&= -5.
\end{aligned}$$

EXAMPLE 3: Evaluate $\lim_{x\to 0}(3x^4-2x-1)$.

Solution. From the theorems above,

$$\begin{aligned}
\lim_{x\to 0}(3x^4-2x-1) &= \lim_{x\to 0} 3x^4 - \lim_{x\to 0} 2x - \lim_{x\to 0} 1 && \text{(Addition)}\\
&= 3\lim_{x\to 0} x^4 - 2\lim_{x\to 0} x^2 - 1 && \text{(Constant Multiple)}\\
&= 3(0)^4 - 2(0) - 1 && \text{(Power)}\\
&= 0-0-1\\
&= -1.
\end{aligned}$$

We will now apply the limit theorems in evaluating rational functions. In evaluating the limits of such functions, recall from Theorem 1 the Division Rule, and all the rules stated in Theorem 1 which have been useful in evaluating limits of polynomial functions, such as the Addition and Product Rules.

EXAMPLE 4: Evaluate $\lim\limits_{x\to 1} \dfrac{1}{x}$.

Solution. First, note that $\lim\limits_{x\to 1} x = 1$. Since the limit of the denominator is nonzero, we can apply the Division Rule. Thus,

$$\begin{aligned} \lim_{x\to 1} \frac{1}{x} &= \frac{\lim\limits_{x\to 1} 1}{\lim\limits_{x\to 1} x} && \text{(Division)} \\ &= \frac{1}{1} \\ &= 1. \end{aligned}$$

EXAMPLE 5: Evaluate $\lim\limits_{x\to 2} \dfrac{x}{x+1}$.

Solution. We start by checking the limit of the polynomial function in the denominator.

$$\lim_{x\to 2}(x+1) = \lim_{x\to 2} x + \lim_{x\to 2} 1 = 2 + 1 = 3.$$

Since the limit of the denominator is not zero, it follows that

$$\lim_{x\to 2} \frac{x}{x+1} = \frac{\lim\limits_{x\to 2} x}{\lim\limits_{x\to 2}(x+1)} = \frac{2}{3} \qquad \text{(Division)}$$

EXAMPLE 6: Evaluate $\lim\limits_{x\to 1} \dfrac{(x-3)(x^2-2)}{x^2+1}$.

Solution. First, note that

$$\lim_{x\to 1}(x^2+1) = \lim_{x\to 1} x^2 + \lim_{x\to 1} 1 = 1 + 1 = 2 \neq 0.$$

Thus, using the theorem,

$$\begin{aligned} \lim_{x\to 1} \frac{(x-3)(x^2-2)}{x^2+1} &= \frac{\lim\limits_{x\to 1}(x-3)(x^2-2)}{\lim\limits_{x\to 1}(x^2+1)} && \text{(Division)} \\ &= \frac{\lim\limits_{x\to 1}(x-3) \cdot \lim\limits_{x\to 1}(x^2-2)}{2} && \text{(Multipication)} \\ &= \frac{\left(\lim\limits_{x\to 1} x - \lim\limits_{x\to 1} 3\right)\left(\lim\limits_{x\to 1} x^2 - \lim\limits_{x\to 1} 2\right)}{2} && \text{(Addition)} \\ &= \frac{(1-3)(1^2-2)}{2} \\ &= 1. \end{aligned}$$

Theorem 2. *Let f be a polynomial of the form*

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0.$$

If c is a real number, then

$$\lim_{x \to c} f(x) = f(c).$$

EXAMPLE 7: Evaluate $\lim\limits_{x \to -1} (2x^3 - 4x^2 + 1)$.

Solution. Note first that our function

$$f(x) = 2x^3 - 4x^2 + 1,$$

is a polynomial. Computing for the value of f at $x = -1$, we get

$$f(-1) = 2(-1)^3 - 4(-1)^2 + 1 = 2(-1) - 4(1) + 1 = -5.$$

Therefore, from Theorem 2,

$$\lim_{x \to -1} (2x^3 - 4x^2 + 1) = f(-1) = -5.$$

Note that we get the same answer when we use limit theorems.

Theorem 3. *Let h be a rational function of the form $h(x) = \dfrac{f(x)}{g(x)}$ where f and g are polynomial functions. If c is a real number and $g(c) \neq 0$, then*

$$\lim_{x \to c} h(x) = \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}.$$

EXAMPLE 8: Evaluate $\lim\limits_{x \to 1} \dfrac{1 - 5x}{1 + 3x^2 + 4x^4}$.

Solution. Since the denominator is not zero when evaluated at $x = 1$, we may apply Theorem 3:

$$\lim_{x \to 1} \frac{1 - 5x}{1 + 3x^2 + 4x^4} = \frac{1 - 5(1)}{1 + 3(1)^2 + 4(1)^4} = \frac{-4}{8} = -\frac{1}{2}.$$

We will now evaluate limits of radical functions using limit theorems.

EXAMPLE 9: Evaluate $\lim_{x\to 1} \sqrt{x}$.

Solution. Note that $\lim_{x\to 1} x = 1 > 0$. Therefore, by the Radical/Root Rule,

$$\lim_{x\to 1} \sqrt{x} = \sqrt{\lim_{x\to 1} x} = \sqrt{1} = 1.$$

EXAMPLE 10: Evaluate $\lim_{x\to 0} \sqrt{x+4}$.

Solution. Note that $\lim_{x\to 0}(x+4) = 4 > 0$. Hence, by the Radical/Root Rule,

$$\lim_{x\to 0} \sqrt{x+4} = \sqrt{\lim_{x\to 0}(x+4)} = \sqrt{4} = 2.$$

EXAMPLE 11: Evaluate $\lim_{x\to -2} \sqrt[3]{x^2+3x-6}$.

Solution. Since the index of the radical sign is odd, we do not have to worry that the limit of the radicand is negative. Therefore, the Radical/Root Rule implies that

$$\lim_{x\to -2} \sqrt[3]{x^2+3x-6} = \sqrt[3]{\lim_{x\to -2}(x^2+3x-6)} = \sqrt[3]{4-6-6} = \sqrt[3]{-8} = -2.$$

EXAMPLE 12: Evaluate $\lim_{x\to 2} \dfrac{\sqrt{2x+5}}{1-3x}$.

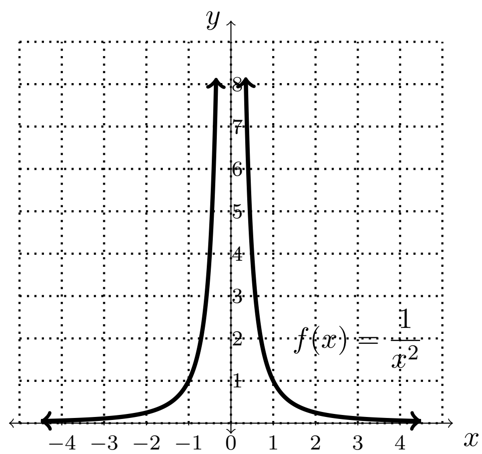
Solution. First, note that $\lim_{x\to 2}(1-3x) = -5 \neq 0$. Moreover, $\lim_{x\to 2}(2x+5) = 9 > 0$. Thus, using the Division and Radical Rules of Theorem 1, we obtain

$$\lim_{x\to 2} \frac{\sqrt{2x+5}}{1-3x} = \frac{\lim_{x\to 2} \sqrt{2x+5}}{\lim_{x\to 2} 1-3x} = \frac{\sqrt{\lim_{x\to 2}(2x+5)}}{-5} = \frac{\sqrt{9}}{-5} = -\frac{3}{5}.$$

INTUITIVE NOTIONS OF INFINITE LIMITS

We investigate the limit at a point c of a rational function of the form $\dfrac{f(x)}{g(x)}$ where f and g are polynomial functions with $f(c) \neq 0$ and $g(c) = 0$. Note that Theorem 3 does not cover this because it assumes that the denominator is nonzero at c.

Now, consider the function $f(x) = \dfrac{1}{x^2}$. Note that the function is not defined at $x = 0$ but we can check the behavior of the function as x approaches 0 intuitively. We first consider approaching 0 from the left.



x	$f(x)$
-0.9	1.2345679
-0.5	4
-0.1	100
-0.01	$10,000$
-0.001	$1,000,000$
-0.0001	$100,000,000$

Observe that as x approaches 0 from the left, the value of the function increases without bound. When this happens, we say that the limit of $f(x)$ as x approaches 0 from the left is *positive infinity*, that is,

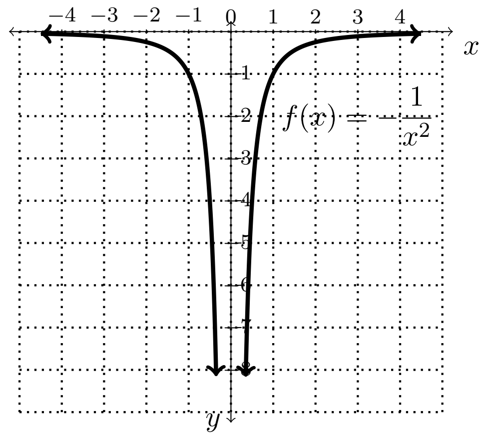
$$\lim_{x \to 0^-} f(x) = +\infty.$$

x	$f(x)$
0.9	1.2345679
0.5	4
0.1	100
0.01	$10,000$
0.001	$1,000,000$
0.0001	$100,000,000$

Again, as x approaches 0 from the right, the value of the function increases without bound, so, $\lim_{x\to 0^+} f(x) = +\infty$.

Since $\lim_{x\to 0^-} f(x) = +\infty$ and $\lim_{x\to 0^+} f(x) = +\infty$, we may conclude that $\lim_{x\to 0} f(x) = +\infty$.

Now, consider the function $f(x) = -\dfrac{1}{x^2}$. Note that the function is not defined at $x = 0$ but we can still check the behavior of the function as x approaches 0 intuitively. We first consider approaching 0 from the left.



x	$f(x)$
-0.9	-1.2345679
-0.5	-4
-0.1	-100
-0.01	$-10,000$
-0.001	$-1,000,000$
-0.0001	$-100,000,000$

This time, as x approaches 0 from the left, the value of the function decreases without bound. So, we say that the limit of $f(x)$ as x approaches 0 from the left is negative infinity, that is,

$$\lim_{x\to 0^-} f(x) = -\infty.$$

x	$f(x)$
0.9	-1.2345679
0.5	-4
0.1	-100
0.01	$-10,000$
0.001	$-1,000,000$
0.0001	$-100,000,000$

As x approaches 0 from the right, the value of the function also decreases without bound, that is, $\lim_{x\to 0^+} f(x) = -\infty$.

Since $\lim\limits_{x \to 0^-} f(x) = -\infty$ and $\lim\limits_{x \to 0^+} f(x) = -\infty$, we are able to conclude that $\lim\limits_{x \to 0} f(x) = -\infty$.

We now state the intuitive definition of *infinite limits* of functions:

The limit of $f(x)$ as x approaches c is positive infinity, denoted by,

$$\lim_{x \to c} f(x) = +\infty$$

if the value of $f(x)$ increases without bound whenever the values of x get closer and closer to c. The limit of $f(x)$ as x approaches c is negative infinity, denoted by,

$$\lim_{x \to c} f(x) = -\infty$$

if the value of $f(x)$ decreases without bound whenever the values of x get closer and closer to c.

Let us consider $f(x) = \dfrac{1}{x}$. The graph on the right suggests that

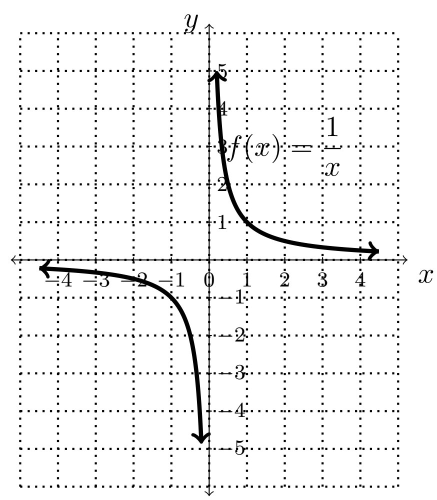
$$\lim_{x \to 0^-} f(x) = -\infty$$

while

$$\lim_{x \to 0^+} f(x) = +\infty.$$

Because the one-sided limits are not the same, we say that

$$\lim_{x \to 0} f(x) \text{ DNE}.$$



Remark 1: Remember that ∞ is NOT a number. It holds no specific value. So, $\lim\limits_{x \to c} f(x) = +\infty$ or $\lim\limits_{x \to c} f(x) = -\infty$ describes the behavior of the function near $x = c$, but it does not exist as a real number.

Remark 2: Whenever $\lim\limits_{x \to c^+} f(x) = \pm\infty$ or $\lim\limits_{x \to c^-} f(x) = \pm\infty$, we normally see the dashed vertical line $x = c$. This is to indicate that the graph of $y = f(x)$ is *asymptotic* to $x = c$, meaning, the graphs of $y = f(x)$ and $x = c$ are very close to each other as x-values approach c. In this case, we call $x = c$ a ***vertical asymptote*** of the graph of $y = f(x)$.

Solved Examples

Evaluate the following limits.

EXAMPLE 1: $\lim_{x \to 2} (3x - 5)$

Solution. $\lim_{x \to 2} (3x - 5) = \lim_{x \to 2} (3x) - \lim_{x \to 2} 5 = \lim_{x \to 2} (3x) - 5 = 3(\lim_{x \to 2} x) - 5 = 3(2) - 5 = 1.$

EXAMPLE 2: $\lim_{x \to -1} (2x^4 - 4x^3 + x - 2)$

Solution. $\lim_{x \to -1} (2x^4 - 4x^3 + x - 2) = 2(-1)^4 - 4(-1)^3 + (-1) - 2 = 3.$

EXAMPLE 3: $\lim_{x \to 2} \dfrac{2}{x}$

Solution. Note that $\lim_{x \to 2} x = 2 \neq 0$. Therefore,

$$\lim_{x \to 2} \frac{2}{x} = \frac{\lim_{x \to 2} 2}{\lim_{x \to 2} x} = \frac{2}{2} = 1.$$

EXAMPLE 4: $\lim_{x \to \frac{1}{2}} \dfrac{x - 1}{2x}$

Solution. Note that $\lim_{x \to \frac{1}{2}} 2x = 2(\dfrac{1}{2}) = 1 \neq 0$. Thus, we have

$$\lim_{x \to \frac{1}{2}} \frac{x - 1}{2x} = \frac{\lim_{x \to \frac{1}{2}} (x - 1)}{\lim_{x \to \frac{1}{2}} 2x} = \frac{\frac{1}{2} - 1}{1} = -\frac{1}{2}.$$

EXAMPLE 5: $\lim_{x \to -3} \dfrac{(2x + 1)(x^2 + 3)}{x + 4}$

Solution. We note that $\lim_{x \to -3} (x+4) = -3+4 = 1 \neq 0$. Hence, we obtain

$$
\begin{aligned}
\lim_{x \to -3} \frac{(2x+1)(x^2+3)}{x+4} &= \frac{\lim_{x \to -3} (2x+1)(x^2+3)}{\lim_{x \to -3} x+4} \\
&= \frac{\lim_{x \to -3} (2x+1) \cdot \lim_{x \to -3} (x^2+3)}{\lim_{x \to -3} (x+4)} \\
&= \frac{(2(-3)+1) \cdot ((-3)^2+3)}{1} \\
&= (-7)(12) = -84.
\end{aligned}
$$

EXAMPLE 6: $\lim_{x \to 0} (5x^3 - 3x^2 + 1)$

Solution. We remark that $x = 0$ is in the domain of the polynomial function $f(x) = 5x^3 - 3x^2 + 1$. Therefore,

$$\lim_{x \to 0} (5x^3 - 3x^2 + 1) = f(0) = 5(0)^3 - 3(0)^2 + 1 = 1.$$

EXAMPLE 7: $\lim_{x \to -2} \frac{x^2 + x - 2}{x+1}$

Solution. Note that $x = -2$ is in the domain of $g(x) = \frac{x^2 + x - 2}{x+1}$. Thus,

$$\lim_{x \to -2} \frac{x^2 + x - 2}{x+1} = g(-2) = \frac{(-2)^2 + (-2) - 2}{-2+1} = \frac{0}{-1} = 0.$$

EXAMPLE 8: $\lim_{x \to 1} \left(\frac{2 - 3x^2}{x^3 - 3x + 1} \right)^3$

Solution. Note that $x = 1$ is in the domain of the polynomial function

$$h(x) = \frac{2 - 3x^2}{x^3 - 3x + 1}$$

and

$$h(1) = \frac{2 - 3(1)^2}{(1)^3 - 3(1) + 1} = \frac{-1}{-1} = 1.$$

Thus, we have

$$\lim_{x\to 1}\left(\frac{2-3x^2}{x^3-3x+1}\right)^2 = \left(\lim_{x\to 1}\left(\frac{2-3x^2}{x^3-3x+1}\right)\right)^2 = (h(1))^2 = 1.$$

EXAMPLE 9: $\lim\limits_{x\to -1}\sqrt{x+1}$

Solution. Note that $\lim\limits_{x\to -1}(x+1) = 0$. Hence, we get

$$\lim_{x\to -1}\sqrt{x+1} = \sqrt{\lim_{x\to -1}(x+1)} = \sqrt{0} = 0.$$

EXAMPLE 10: $\lim\limits_{x\to 3}\sqrt{x^2+2x+1}$

Solution. Note that $\lim\limits_{x\to 3}(x^2+2x+1) = (3)^2+2(3)+1 = 16 > 0$. Thus,

$$\lim_{x\to 3}\sqrt{x^2+2x+1} = \sqrt{\lim_{x\to 3} x^2+2x+1} = \sqrt{16} = 4.$$

EXAMPLE 11: $\lim\limits_{x\to 0}\dfrac{1}{x^4}$

Solution. Approaching 0 from the left,

x	$f(x)$
-0.9	1.524157903
-0.5	16
-0.1	$10,000$
-0.01	$100,000,000$
-0.001	$1,000,000,000,000$
-0.0001	$10,000,000,000,000,000$

Approaching 0 from the right,

x	$f(x)$
0.9	1.524157903
0.5	16
0.1	$10,000$
0.01	$100,000,000$
0.001	$1,000,000,000,000$
0.0001	$10,000,000,000,000,000$

Hence, $\lim_{x \to 0} \frac{1}{x^4} = +\infty$.

EXAMPLE 12: $\lim_{x \to 0} \frac{1}{x^3}$

Solution. Approaching 0 from the left,

x	$f(x)$
-0.9	-1.371742112
-0.5	-8
-0.1	-1000
-0.01	$-1,000,000$
-0.001	$-1,000,000,000$
-0.0001	$-1,000,000,000,000$

Approaching 0 from the right,

x	$f(x)$
0.9	1.371742112
0.5	8
0.1	$1,000$
0.01	$1,000,000$
0.001	$1,000,000,000$
0.0001	$1,000,000,000,000$

Hence, $\lim_{x \to 0} \frac{1}{x^3}$ DNE.

Supplementary Problems

1. Evaluate the following limits using the limit theorems.

(a) $\lim_{x \to 1} (x + 1)$

(b) $\lim_{x \to 2} (x - 2)$

(c) $\lim_{x \to -1} (2x + 3)$

(d) $\lim_{x \to 0} (1 - 3x)$

(e) $\lim_{x \to -2} (x^2 + 3x - 5)$

(f) $\lim_{x \to -1} (2x - 1 - 5x^2)$

(g) $\lim_{x \to 2} (x^3 + x^2 - 1)$

(h) $\lim_{x \to 1} (x^4 - x^2 - x + 1)$

(i) $\lim_{x \to 0} \dfrac{x}{x + 1}$

(j) $\lim_{x \to 3} \dfrac{x - 1}{2x - 4}$

(k) $\lim_{x \to -3} -\dfrac{x^2 - 1}{x + 1}$

(l) $\lim_{x \to 2} \dfrac{1 + x^2}{x + 3}$

(m) $\lim_{x \to 5} \sqrt{x - 1}$

(n) $\lim_{x \to 1} (\sqrt{x^2} - 1)$

2. Determine the following limits using tables of values.

(a) $\lim_{x \to -2} \dfrac{1}{x + 2}$

(b) $\lim_{x \to 1} \dfrac{1}{x^2 - 1}$

(c) $\lim_{x \to 1} \dfrac{1}{x - 1}$

(d) $\lim_{x \to -1} \dfrac{1}{x + 1}$

(e) $\lim_{x \to 0} \dfrac{1}{x^2 - x}$

(f) $\lim_{x \to 0} \dfrac{1}{x^3 - x}$

LESSON 2: Limits of Some Transcendental Functions and Some Indeterminate Forms

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Compute the limits of exponential, logarithmic, and trigonometric functions using tables of values and graphs of the functions;
2. Evaluate the limits of expressions involving $\dfrac{\sin t}{t}$, $\dfrac{1-\cos t}{t}$, and $\dfrac{e^t - 1}{t}$ using tables of values; and
3. Evaluate the limits of expressions resulting in the indeterminate form $\text{``}\dfrac{0}{0}\text{''}$.

TOPIC 2.1: Limits of Exponential, Logarithmic, and Trigonometric Functions

Real-world situations can be expressed in terms of functional relationships. These functional relationships are called mathematical models. In applications of calculus, it is quite important that one can generate these mathematical models. They sometimes use functions that you encountered in precalculus, like the exponential, logarithmic, and trigonometric functions. Hence, we start this lesson by recalling these functions and their corresponding graphs.

1. If $b > 0$, $b \neq 1$, the *exponential function with base* b is defined by

$$f(x) = b^x,\ x \in \mathbb{R}.$$

2. Let $b > 0, b \neq 1$. If $b^y = x$ then y is called the *logarithm of* x *to the base* b, denoted $y = \log_b x$.

EVALUATING LIMITS OF EXPONENTIAL FUNCTIONS

First, we consider the natural exponential function $f(x) = e^x$, where e is called the *Euler number*, and has value 2.718281....

EXAMPLE 1: Evaluate the $\lim\limits_{x \to 0} e^x$.

Solution. We will construct the table of values for $f(x) = e^x$. We start by approaching the number 0 from the left or through the values less than but close to 0.

x	$f(x)$
-1	0.36787944117
-0.5	0.60653065971
-0.1	0.90483741803
-0.01	0.99004983374
-0.001	0.99900049983
-0.0001	0.999900049983
-0.00001	0.99999000005

Intuitively, from the table above, $\lim\limits_{x \to 0^-} e^x = 1$. Now we consider approaching 0 from its right or through values greater than but close to 0.

x	$f(x)$
1	2.71828182846
0.5	1.6487212707
0.1	1.10517091808
0.01	1.01005016708
0.001	1.00100050017
0.0001	1.000100005
0.00001	1.00001000005

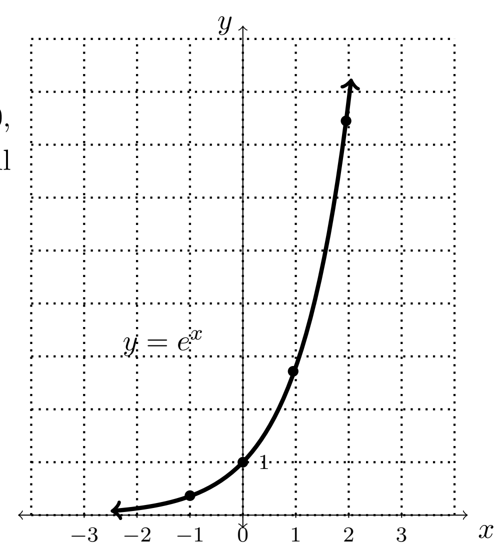
From the table, as the values of x get closer and closer to 0, the values of $f(x)$ get closer and closer to 1. So, $\lim_{x\to 0^+} e^x = 1$. Combining the two one-sided limits allows us to conclude that

$$\lim_{x\to 0} e^x = 1.$$

We can use the graph of $f(x) = e^x$ to determine its limit as x approaches 0. The figure below is the graph of $f(x) = e^x$.

Looking at Figure 1.1, as the values of x approach 0, either from the right or the left, the values of $f(x)$ will get closer and closer to 1. We also have the following:

1. $\lim_{x\to 1} e^x = e = 2.718...$
2. $\lim_{x\to 2} e^x = e^2 = 7.389...$
3. $\lim_{x\to -1} e^x = e^{-1} = 0.367...$



EVALUATING LIMITS OF LOGARITHMIC FUNCTIONS

Now, consider the natural logarithmic function $f(x) = \ln x$. Recall that $\ln x = \log_e x$. Moreover, it is the inverse of the natural exponential function $y = e^x$.

EXAMPLE 2: Evaluate $\lim_{x\to 1} \ln x$.

Solution. We will construct the table of values for $f(x) = \ln x$. We first approach the number 1 from the left or through values less than but close to 1.

x	$f(x)$
0.1	-2.30258509299
0.5	-0.69314718056
0.9	-0.10536051565
0.99	-0.01005033585
0.999	-0.00100050033
0.9999	-0.000100005
0.99999	-0.00001000005

Intuitively, $\lim_{x\to 1^-} \ln x = 0$. Now we consider approaching 1 from its right or through values greater than but close to 1.

x	$f(x)$
2	0.69314718056
1.5	0.4054651081
1.1	0.0953101798
1.01	0.00995033085
1.001	0.00099950033
1.0001	0.000099995
1.00001	0.00000999995

Intuitively, $\lim_{x\to 1^+} \ln x = 0$. As the values of x get closer and closer to 1, the values of $f(x)$ get closer and closer to 0. In symbols,

$$\lim_{x\to 1} \ln x = 0.$$

We now consider the *common logarithmic function* $f(x) = \log_{10} x$. Recall that $f(x) = \log_{10} x = \log x$.

EXAMPLE 3: Evaluate $\lim_{x \to 1} \log x$.

Solution. We will construct the table of values for $f(x) = \log x$. We first approach the number 1 from the left or through the values less than but close to 1.

x	$f(x)$
0.1	-1
0.5	-0.30102999566
0.9	-0.04575749056
0.99	-0.0043648054
0.999	-0.00043451177
0.9999	-0.00004343161
0.99999	-0.00000434296

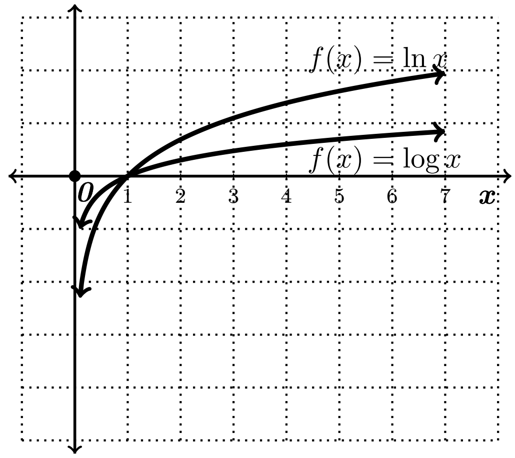
Now we consider approaching 1 from its right or through values greater than but close to 1.

x	$f(x)$
2	0.30102999566
1.5	0.17609125905
1.1	0.04139268515
1.01	0.00432137378
1.001	0.00043407747
1.0001	0.00004342727
1.00001	0.00000434292

As the values of x get closer and closer to 1, the values of $f(x)$ get closer and closer to 0. In symbols,

$$\lim_{x \to 1} \log x = 0.$$

Consider now the graphs of both the natural and common logarithmic functions. We can use the following graphs to determine their limits as x approaches 1.



The figure helps verify our observations that $\lim\limits_{x\to 1} \ln x = 0$ and $\lim\limits_{x\to 1} \log x = 0$. Also, based on the figure, we have

1. $\lim\limits_{x\to e} \ln x = 1$
2. $\lim\limits_{x\to 10} \log x = 1$
3. $\lim\limits_{x\to 3} \ln x = \ln 3 = 1.09...$
4. $\lim\limits_{x\to 3} \log x = \log 3 = 0.47...$
5. $\lim\limits_{x\to 0^+} \ln x = -\infty$
6. $\lim\limits_{x\to 0^+} \log x = -\infty$

TRIGONOMETRIC FUNCTIONS

EXAMPLE 4: Evaluate $\lim\limits_{x\to 0} \sin x$.

Solution. We will construct the table of values for $f(x) = \sin x$. We first approach 0 from the left or through the values less than but close to 0.

x	$f(x)$
-1	-0.8414709848
-0.5	-0.4794255386
-0.1	-0.09983341664
-0.01	-0.00999983333
-0.001	-0.00099999983
-0.0001	-0.00009999999
-0.00001	-0.00000999999

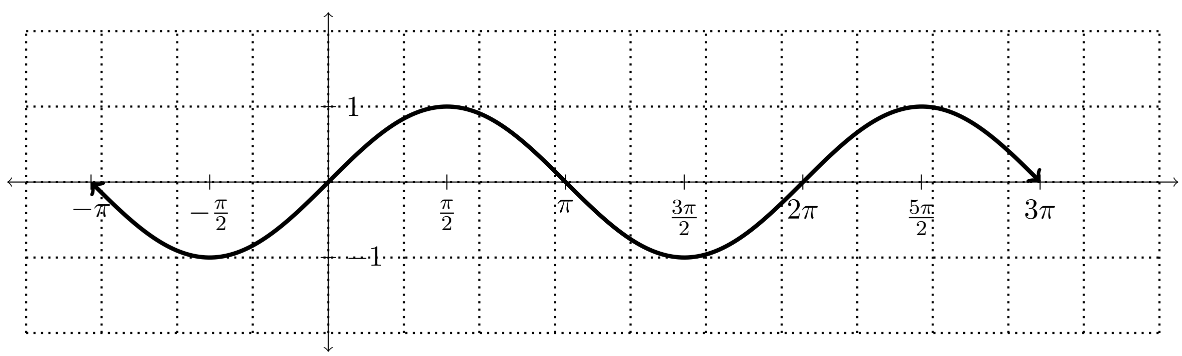
Now we consider approaching 0 from its right or through values greater than but close to 0.

x	$f(x)$
1	0.8414709848
0.5	0.4794255386
0.1	0.09983341664
0.01	0.00999983333
0.001	0.00099999983
0.0001	0.00009999999
0.00001	0.00000999999

As the values of x get closer and closer to 1, the values of $f(x)$ get closer and closer to 0. In symbols,

$$\lim_{x\to 0} \sin x = 0.$$

We can also find $\lim_{x\to 0} \sin x$ by using the graph of the sine function. Consider the graph of $f(x) = \sin x$.



The graph validates our observation in Example 4 that $\lim_{x\to 0} \sin x = 0$. Also, using the graph, we have the following:

1. $\lim_{x\to \frac{\pi}{2}} \sin x = 1.$
2. $\lim_{x\to \pi} \sin x = 0.$
3. $\lim_{x\to -\frac{\pi}{2}} \sin x = -1.$
4. $\lim_{x\to -\pi} \sin x = 0.$

Solved Examples

EXAMPLE 1: Evaluate $\lim_{x\to 0} e^{x+1}$.

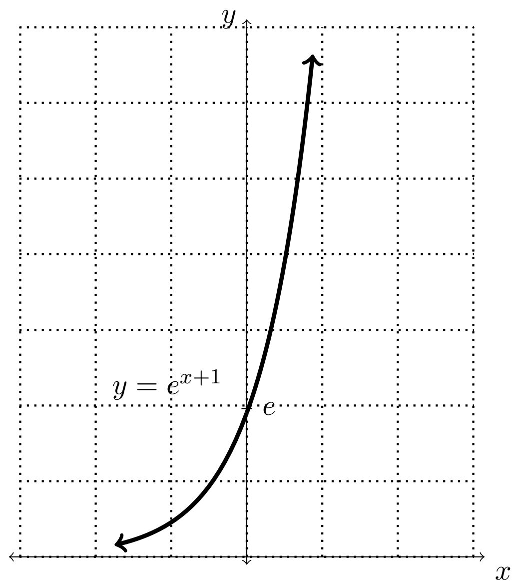
Solution. The following table shows the values of $f(x) = e^{x+1}$ at values from the left of $x = 0$.

x	$f(x)$
-1	1
-0.5	1.648721271
-0.1	2.459603111
-0.01	2.691234472
-0.001	2.715564905
-0.0001	2.718010014
-0.00001	2.718254646

Approaching $x = 0$ from the right,

x	$f(x)$
1	7.389056099
0.5	4.48168907
0.1	3.004166024
0.01	2.745601015
0.001	2.72100147
0.0001	2.71855367
0.00001	2.718309011

Thus, $\lim_{x\to 0} e^{x+1} = 2.71....$



The figure tells us that:

1. $\lim_{x \to 1} e^{x+1} = 7.38905...$
2. $\lim_{x \to 2} e^{x+1} = 20.08553...$
3. $\lim_{x \to -1} e^{x+1} = 1$
4. $\lim_{x \to -2} e^{x+1} = 0.36787...$

EXAMPLE 2:

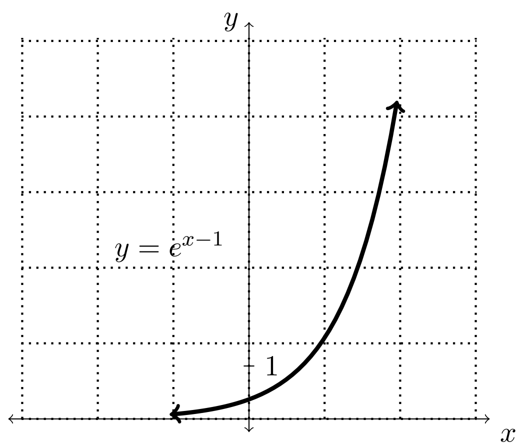
Evaluate $\lim_{x \to 1} e^{x-1}$.

Solution. Approaching $x = 1$ from the left,

x	$f(x)$
0	0.36787944117
0.5	0.60653065971
0.1	0.90483741803
0.01	0.99004983374
0.001	0.99900049983
0.0001	0.99900049983
0.00001	0.99999000005

Taking values close to $x = 1$ from the right,

x	$f(x)$
2	2.71828182846
1.5	1.6487212707
1.1	1.10517091808
1.01	1.01005016708
1.001	1.00100050017
1.0001	1.000100005
1.00001	1.00001000005



From the figure above, we have

1. $\lim_{x \to 0} e^{x-1} = 0.36787...$
2. $\lim_{x \to 2} e^{x-1} = 2.71828...$
3. $\lim_{x \to -1} e^{x-1} = 0.13533...$
4. $\lim_{x \to -2} e^{x-1} = 0.04978...$

EXAMPLE 3:

Evaluate $\lim_{x \to 1} 2^x$.

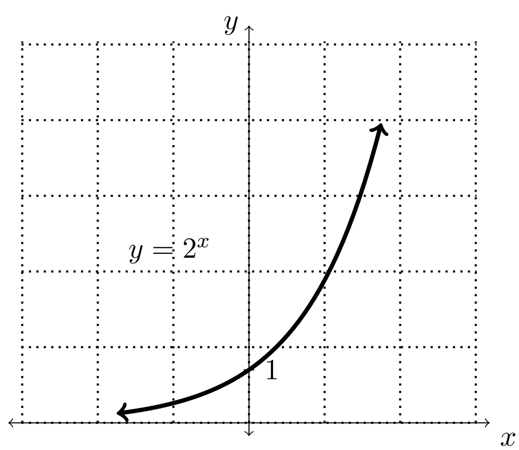
Solution. Approaching $x = 1$ from the left,

x	$f(x)$
0	1
0.5	1.414213562
0.9	1.866065983
0.99	1.986184991
0.999	1.998614186
0.9999	1.999861375
0.99999	1.999986137

Taking values close to $x = 1$ from the right,

x	$f(x)$
2	4
1.5	2.828427125
1.1	2.143546925
1.01	2.0139111
1.001	2.001386775
1.0001	2.000138634
1.00001	2.000013863

Thus, $\lim_{x \to 1} 2^x = 2$.



Using the figure above,

1. $\lim_{x \to 0} 2^x = 1$
2. $\lim_{x \to 2} 2^x = 4$
3. $\lim_{x \to -1} 2^x = 0.5$
4. $\lim_{x \to -2} 2^x = 0.25$

EXAMPLE 4: Evaluate $\lim\limits_{x \to 1} \ln(x+1)$.

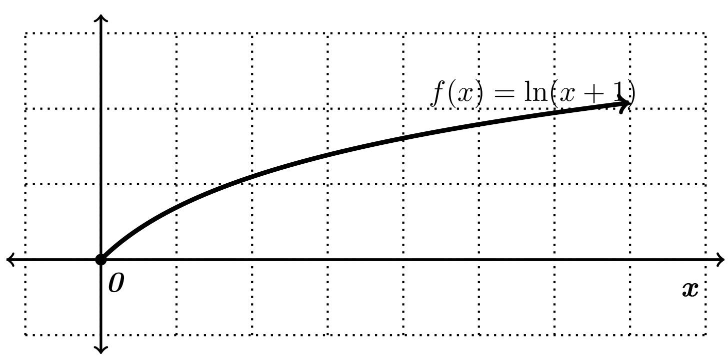
Solution. Approaching values from the left of $x = 1$,

x	$f(x)$
0	0
0.5	0.405465108
0.9	0.641853886
0.99	0.688134638
0.999	0.692647055
0.9999	0.693097179
0.99999	0.69314218

Approaching $x = 1$ from the right,

x	$f(x)$
2	1.098612289
1.5	0.916290731
1.1	0.741937344
1.01	0.698134722
1.001	0.693647055
1.0001	0.693197179
1.00001	0.69315218

Therefore, $\lim\limits_{x \to 1} \ln(x+1) = 0.69....$



Based on the figure,

1. $\lim_{x \to 0} \ln(x+1) = 0$
2. $\lim_{x \to 2} \ln(x+1) = 1.09861...$
3. $\lim_{x \to 3} \ln(x+1) = 1.38629...$
4. $\lim_{x \to 4} \ln(x+1) = 1.60943...$

EXAMPLE 5: Evaluate $\lim_{x \to 0} \ln(x^2+1)$.

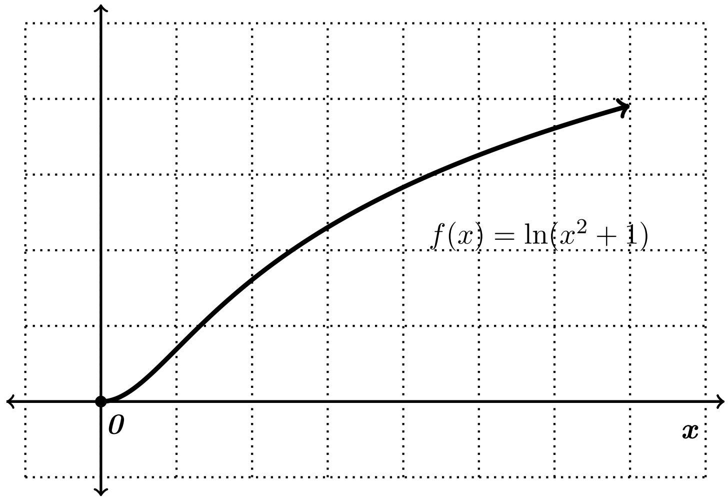
Solution. Approaching values from the left of $x = 0$,

x	$f(x)$
-1	0.69314718
-0.5	0.223143551
-0.1	0.00995033
-0.01	0.000099995
-0.001	0.000000999
-0.0001	0.000000009
-0.00001	0.0000000009

Approaching $x = 0$ from the right,

x	$f(x)$
1	0.69314718
0.5	0.223143551
0.1	0.00995033
0.01	0.000099995
0.001	0.000000999
0.0001	0.000000009
0.00001	0.0000000009

Therefore, $\lim_{x \to 0} \ln(x^2+1) = 0$.



The figure above gives us:

1. $\lim_{x\to 1} \ln(x^2+1) = 0.69314...$
2. $\lim_{x\to 2} \ln(x^2+1) = 1.60943...$
3. $\lim_{x\to -1} \ln(x^2+1) = 0.69314...$
4. $\lim_{x\to -2} \ln(x^2+1) = 1.60943...$

EXAMPLE 6: Evaluate $\lim_{x\to 1} \log(x+1)$.

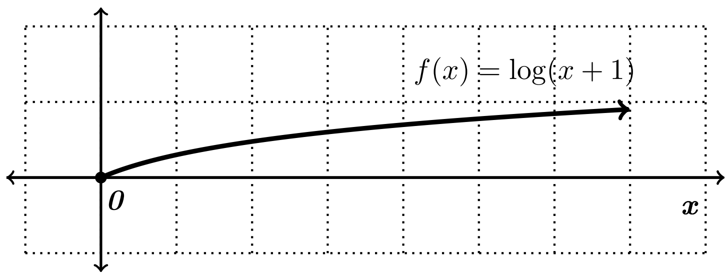
Solution. Approaching values from the left of $x = 1$,

x	$f(x)$
0	0
0.5	0.176091259
0.9	0.278753601
0.99	0.298853076
0.999	0.300812794
0.9999	0.30100828
0.99999	0.301027824

Approaching $x = 1$ from the right,

x	$f(x)$
2	0.477121254
1.5	0.397940008
1.1	0.322219294
1.01	0.303196057
1.001	0.301247088
1.0001	0.301051709
1.00001	0.301032167

Hence, $\lim_{x \to 1} \log(x+1) = 0.301....$



From the figure above,

1. $\lim_{x \to 2} \log(x+1) = 0.47712...$
2. $\lim_{x \to 3} \log(x+1) = 0.60205...$
3. $\lim_{x \to 4} \log(x+1) = 0.69897...$
4. $\lim_{x \to 5} \log(x+1) = 0.77815...$

EXAMPLE 7:

Evaluate $\lim_{x \to -1} \log(x^2)$.

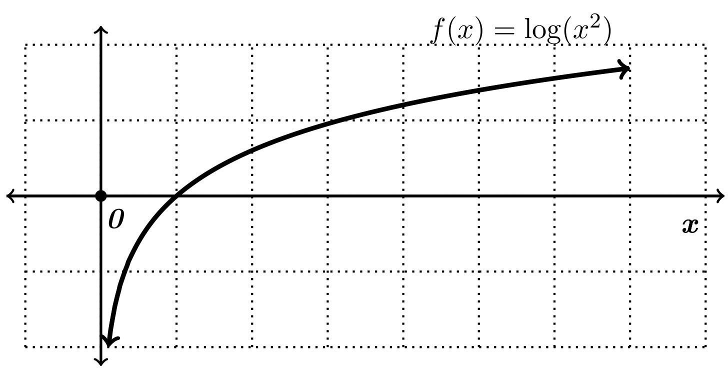
Solution. Approaching values from the left of $x = -1$,

x	$f(x)$
-2	0.603059991
-1.5	0.352182518
-1.1	0.08278537
-1.01	0.008642747
-1.001	0.00868154
-1.0001	0.000086854
-1.00001	0.000008685

Approaching $x = -1$ from the right,

x	$f(x)$
-0.1	-2
-0.5	-0.602059991
-0.9	-0.091514981
-0.99	-0.00872961
-0.999	-0.000869023
-0.9999	-0.000086863
-0.99999	-0.000008685

Hence, $\lim_{x \to -1} \log(x^2) = 0$.



Using the figure, we have

1. $\lim_{x \to 2} \log(x^2) = 0.60205...$
2. $\lim_{x \to 3} \log(x^2) = 0.95424...$
3. $\lim_{x \to 4} \log(x^2) = 1.20411...$
4. $\lim_{x \to 5} \log(x^2) = 1.39794...$

EXAMPLE 8: Evaluate $\lim_{x\to 0} \cos x$.

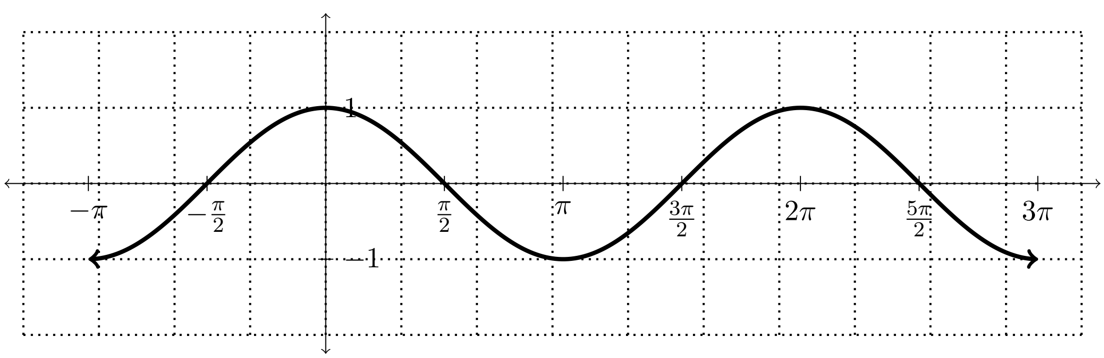
Solution. Approaching values from the left of $x = 0$,

x	$f(x)$
-1	0.999847695
-0.5	0.999961923
-0.1	0.999998476
-0.01	0.999999984
-0.001	0.999999999
-0.0001	0.9999999999
-0.00001	0.99999999999

Approaching $x = 0$ from the right,

x	$f(x)$
1	0.999847695
0.5	0.999961923
0.1	0.999998476
0.01	0.999999984
0.001	0.999999999
0.0001	0.9999999999
0.00001	0.99999999999

Hence, $\lim_{x\to 0} \cos x = 1$.



Based on the figure,

1. $\lim_{x\to\frac{\pi}{2}} \cos x = 0$
2. $\lim_{x\to\frac{\pi}{4}} \cos x = 0.70710...$
3. $\lim_{x\to\frac{\pi}{3}} \cos x = 0.5$
4. $\lim_{x\to\pi} \cos x = -1$

EXAMPLE 9: Evaluate $\lim_{x\to 0} \cos\left(x + \frac{\pi}{2}\right)$.

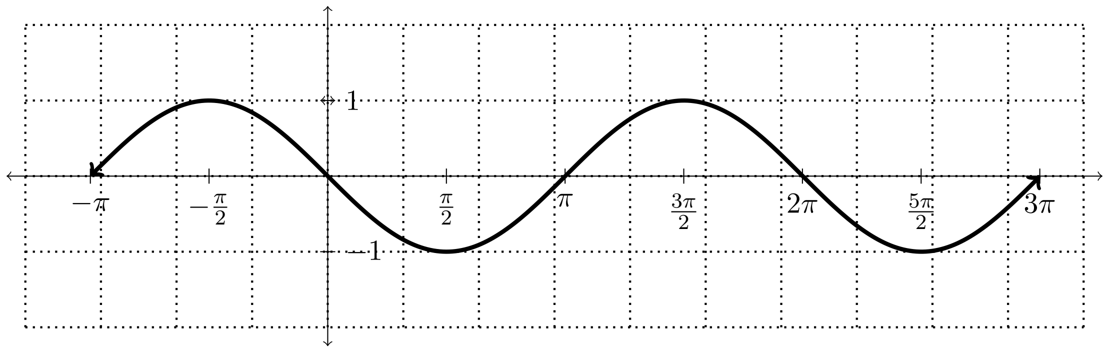
Solution. Approaching values from the left of $x = 0$,

x	$f(x)$
-1	0.841470984
-0.5	0.479425538
-0.1	0.099833416
-0.01	0.009999833
-0.001	0.000999999
-0.0001	0.000099999
-0.00001	0.000009999

Approaching $x = 0$ from the right,

x	$f(x)$
1	-0.841470984
0.5	-0.479425538
0.1	-0.099833416
0.01	-0.009999833
0.001	-0.000999999
0.0001	-0.000099999
0.00001	-0.000009999

Hence, $\lim_{x\to 0} \cos\left(x + \frac{\pi}{2}\right) = 0$.



Using the graph above,

1. $\displaystyle\lim_{x\to\frac{\pi}{2}} \cos\left(x+\frac{\pi}{2}\right) = -1$
2. $\displaystyle\lim_{x\to\frac{\pi}{4}} \cos\left(x+\frac{\pi}{2}\right) = -0.70710...$
3. $\displaystyle\lim_{x\to\frac{\pi}{3}} \cos\left(x+\frac{\pi}{2}\right) = -0.86602...$
4. $\displaystyle\lim_{x\to\pi} \cos\left(x+\frac{\pi}{2}\right) = 0.$

Supplementary Problems

Evaluate the following limits.

1. $\displaystyle\lim_{x\to-1} e^x$
2. $\displaystyle\lim_{x\to 2} e^x$
3. $\displaystyle\lim_{x\to 1} e^{x-2}$
4. $\displaystyle\lim_{x\to 3} e^{x-1}$
5. $\displaystyle\lim_{x\to 0} 2^x$
6. $\displaystyle\lim_{x\to 0} 3^x$
7. $\displaystyle\lim_{x\to-1} 3^{x+1}$
8. $\displaystyle\lim_{x\to 0} 2^{x-1}$
9. $\displaystyle\lim_{x\to 1} \ln(x^2+x-1)$
10. $\displaystyle\lim_{x\to 2} \ln(x^2-2)$
11. $\displaystyle\lim_{x\to 0} \ln(x^2-x+1)$
12. $\displaystyle\lim_{x\to-1} \ln(x+2)$
13. $\displaystyle\lim_{x\to\frac{3\pi}{2}} \sin x$
14. $\displaystyle\lim_{x\to\frac{\pi}{4}} \sin x$
15. $\displaystyle\lim_{x\to\frac{\pi}{4}} \sin\left(x+\frac{\pi}{2}\right)$
16. $\displaystyle\lim_{x\to\frac{\pi}{2}} \sin\left(\frac{\pi}{4}-x\right)$
17. $\displaystyle\lim_{x\to\frac{\pi}{4}} \cos x$
18. $\displaystyle\lim_{x\to\frac{\pi}{2}} \cos x$
19. $\displaystyle\lim_{x\to\frac{3\pi}{2}} \cos\left(x+\frac{\pi}{2}\right)$
20. $\displaystyle\lim_{x\to\frac{3\pi}{4}} \cos\left(\frac{\pi}{2}-x\right)$

TOPIC 2.2: Some Special Limits

We will determine the limits of three special functions; namely, $f(t) = \dfrac{\sin t}{t}$, $g(t) = \dfrac{1 - \cos t}{t}$, and $h(t) = \dfrac{e^t - 1}{t}$. These functions will be vital to the computation of the derivatives of the sine, cosine, and natural exponential functions in Chapter 2.

THREE SPECIAL FUNCTIONS

We start by evaluating the function $f(t) = \dfrac{\sin t}{t}$.

EXAMPLE 1: Evaluate $\displaystyle\lim_{t \to 0} \frac{\sin t}{t}$.

Solution. We will construct the table of values for $f(t) = \dfrac{\sin t}{t}$. We first approach the number 0 from the left or through values less than but close to 0.

t	$f(t)$
-1	0.84147099848
-0.5	0.9588510772
-0.1	0.9983341665
-0.01	0.9999833334
-0.001	0.9999998333
-0.0001	0.99999999983

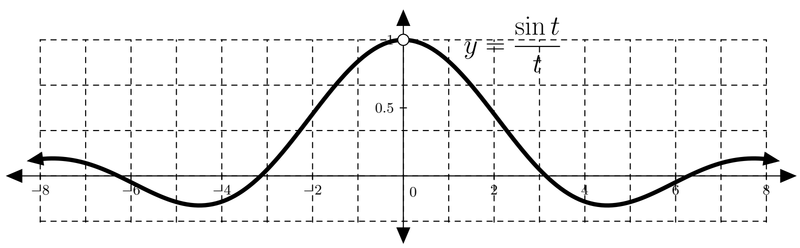
Now we consider approaching 0 from the right or through values greater than but close to 0.

t	$f(t)$
1	0.8414709848
0.5	0.9588510772
0.1	0.9983341665
0.01	0.9999833334
0.001	0.9999998333
0.0001	0.9999999983

Since $\lim_{t\to 0^-} \frac{\sin t}{t}$ and $\lim_{t\to 0^+} \frac{\sin t}{t}$ are both equal to 1, we conclude that

$$\lim_{t\to 0} \frac{\sin t}{t} = 1.$$

The graph of $f(t) = \frac{\sin t}{t}$ below confirms that the y-values approach 1 as t approaches 0.



Now, consider the function $g(t) = \frac{1-\cos t}{t}$.

EXAMPLE 2: Evaluate $\lim_{t\to 0} \frac{1-\cos t}{t}$.

Solution. We will construct the table of values for $g(t) = \frac{1-\cos t}{t}$. We first approach the number 1 from the left or through the values less than but close to 0.

t	$g(t)$
-1	-0.4596976941
-0.5	-0.2448348762
-0.1	-0.04995834722
-0.01	-0.0049999583
-0.001	-0.0004999999
-0.0001	-0.000005

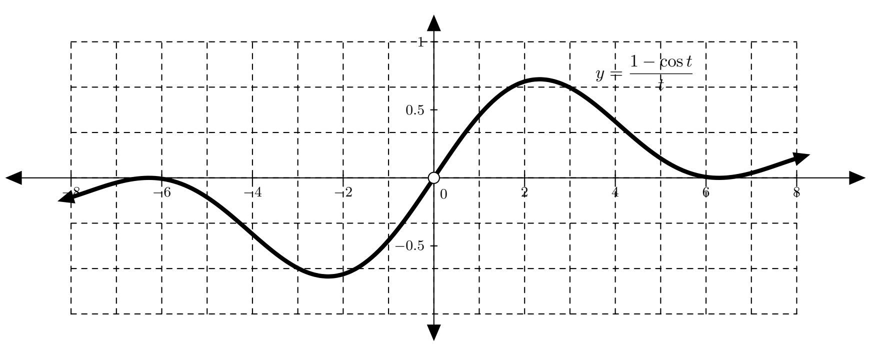
Now we consider approaching 0 from the right or through values greater than but close to 0.

t	$g(t)$
1	0.4596976941
0.5	0.2448348762
0.1	0.04995834722
0.01	0.0049999583
0.001	0.0004999999
0.0001	0.000005

Since $\lim\limits_{t \to 0^-} \dfrac{1-\cos t}{t} = 0$ and $\lim\limits_{t \to 0^+} \dfrac{1-\cos t}{t} = 0$, we conclude that

$$\lim_{t \to 0} \frac{1-\cos t}{t} = 0.$$

Below is the graph of $g(t) = \dfrac{1-\cos t}{t}$. We see that the y-values approach 0 as t tends to 0.



We now consider the special function $h(t) = \dfrac{e^t - 1}{t}$.

EXAMPLE 3: Evaluate $\lim\limits_{t \to 0} \dfrac{e^t - 1}{t}$.

Solution. We will construct the table of values for $h(t) = \dfrac{e^t - 1}{t}$. We first approach the number 0 from the left or through the values less than but close to 0.

t	$h(t)$
-1	0.6321205588
-0.5	0.7869386806
-0.1	0.9516258196
-0.01	0.9950166251
-0.001	0.9995001666
-0.0001	0.9999500016

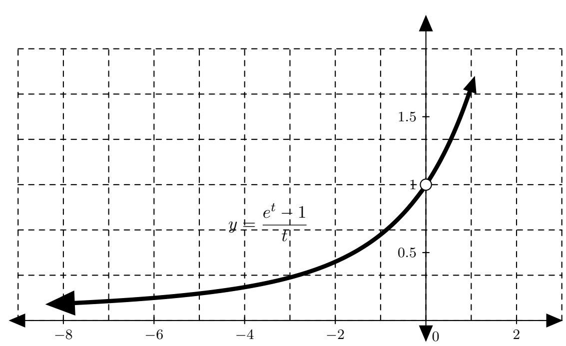
Now we consider approaching 0 from the right or through values greater than but close to 0.

t	$h(t)$
1	1.718281828
0.5	1.297442541
0.1	1.051709181
0.01	1.005016708
0.001	1.000500167
0.0001	1.000050002

Since $\lim\limits_{x\to 0^-} \dfrac{e^t - 1}{t} = 1$ and $\lim\limits_{x\to 0^+} \dfrac{e^t - 1}{t} = 1$, we conclude that

$$\lim_{x\to 0} \frac{e^t - 1}{t} = 1.$$

The graph of $h(t) = \dfrac{e^t - 1}{t}$ below confirms that $\lim\limits_{t\to 0} h(t) = 1$.



INDETERMINATE FORM $\dfrac{\text{“}0\text{”}}{0}$

There are functions whose limits cannot be determined immediately using the Limit Theorems we have so far. In these cases, the functions must be manipulated so that the limit, if it exists, can be calculated. We call such limit expressions *indeterminate forms*.

In this lesson, we will define a particular indeterminate form, $\dfrac{\text{“}0\text{”}}{0}$, and discuss how to evaluate a limit which will initially result in this form.

INDETERMINATE FORM OF TYPE $\dfrac{\text{“}0\text{”}}{0}$

If $\lim\limits_{x\to c} f(x) = 0$ and $\lim\limits_{x\to c} g(x) = 0$, then $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)}$ is called an **indeterminate form of type** $\dfrac{\text{“}0\text{”}}{0}$.

Remark 1: A limit that is indeterminate of type $\dfrac{\text{“}0\text{”}}{0}$ may exist. To find the actual value, one should find an expression equivalent to the original. This is commonly done by factoring or by rationalizing. Hopefully, the expression that will emerge after factoring or rationalizing will have a computable limit.

EXAMPLE 4: Evaluate $\lim\limits_{x\to -1} \dfrac{x^2+2x+1}{x+1}$.

Solution. The limit of both the numerator and the denominator as x approaches -1 is 0. Thus, this limit as currently written is an indeterminate form of type $\dfrac{0}{0}$. However, observe that $(x+1)$ is a factor common to the numerator and the denominator, and

$$\frac{x^2+2x+1}{x+1} = \frac{(x+1)^2}{x+1} = x+1, \text{ when } x \neq -1.$$

Therefore,

$$\lim_{x\to -1} \frac{x^2+2x+1}{x+1} = \lim_{x\to -1} (x+1) = 0.$$

EXAMPLE 5: Evaluate $\lim\limits_{x\to 1} \dfrac{x^2-1}{\sqrt{x}-1}$.

Solution. Since $\lim\limits_{x\to 1} x^2 - 1 = 0$ and $\lim\limits_{x\to 1} \sqrt{x} - 1 = 0$, then $\lim\limits_{x\to 1} \dfrac{x^2-1}{\sqrt{x}-1}$ is an indeterminate form of type $\dfrac{\text{“}0\text{”}}{0}$. To find the limit, observe that if $x \neq 1$, then

$$\frac{x^2-1}{\sqrt{x}-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1} = \frac{(x-1)(x+1)(\sqrt{x}+1)}{x-1} = (x+1)(\sqrt{x}+1).$$

So, we have

$$\lim_{x\to 1} \frac{x^2-1}{\sqrt{x}+1} = \lim_{x\to 1} (x+1)(\sqrt{x}+1) = 4.$$

Solved Examples

EXAMPLE 1: Evaluate $\lim\limits_{t\to 1} \dfrac{\sin(1-t)}{1-t}$.

Solution. Approaching values from the left of 1,

t	$f(t)$
0	0.841470984
0.5	0.958851077
0.9	0.998334166
0.99	0.999983333
0.999	0.999999833
0.9999	0.9999999998

Approaching $x = 1$ from the right,

t	$f(t)$
2	0.841470984
1.5	0.958851077
1.1	0.998334166
1.01	0.999983333
1.001	0.999999833
1.0001	0.999999998

Thus, $\lim\limits_{t\to 1} \dfrac{\sin(1-t)}{1-t} = 1$.

EXAMPLE 2: Evaluate $\lim\limits_{t \to 1} \dfrac{\sin(1-t^2)}{1-t^2}$.

Solution. Approaching values from the left of 1,

t	$f(t)$
0	0.841470984
0.5	0.90885168
0.9	0.993994184
0.99	0.999933999
0.999	0.000000334
0.9999	0.999999993

Approaching $x = 1$ from the right,

t	$f(t)$
2	0.047040002
1.5	0.759187695
1.1	0.992666189
1.01	0.999932666
1.001	0.999999332
1.0001	0.999999993

Thus, $\lim\limits_{t \to 1} \dfrac{\sin(1-t^2)}{1-t^2} = 1$.

EXAMPLE 3: Evaluate $\lim\limits_{t \to 1} \dfrac{1-\cos(t+1)}{t+1}$.

Solution. Approaching values from the left of 1,

t	$f(t)$
0	-0.459697694
0.5	-0.244834876
0.9	-0.049958347
0.99	-0.004999958
0.999	-0.000499999
0.9999	-0.000049999

Approaching $x = 1$ from the right,

t	$f(t)$
2	0.459697694
1.5	0.244834876
1.1	0.049958347
1.01	0.004999958
1.001	0.000499999
1.0001	0.000049999

Thus, $\lim_{t \to 1} \dfrac{1 - \cos(t+1)}{t+1} = 0$

EXAMPLE 4: Evaluate $\lim_{t \to -1} \dfrac{1 - \cos(t^2 - 1)}{t^2 - 1}$.

Solution. Approaching values from the left of $x = -1$,

t	$f(t)$
-2	0.663330832
-1.5	0.54774211
-1.1	0.104614691
-1.01	0.010049661
-1.001	0.001000499
-1.0001	0.000100004

Approaching $x = -1$ from the right,

t	$f(t)$
0	-0.45697694
-0.5	-0.357748174
-0.9	-0.094714552
-0.99	-0.009949671
-0.999	-0.000999499
-0.9999	-0.000099994

Thus, $\lim_{t \to -1} \dfrac{1 - \cos(t^2 - 1)}{t^2 - 1} = 0.$

EXAMPLE 5: Evaluate $\lim\limits_{t\to 1} \dfrac{e^{t-1}-1}{t-1}$.

Solution. Approaching values from the left of 1,

t	$f(t)$
0	0.632120558
0.5	0.78693868
0.9	0.951625819
0.99	0.995016625
0.999	0.999500166
0.9999	0.999950002

Approaching $x = 1$ from the right,

t	$f(t)$
2	1.718281828
1.5	1.297442541
1.1	1.051709181
1.01	1.005016708
1.001	1.000500167
1.0001	1.000050001

Thus, $\lim\limits_{t\to 1} \dfrac{e^{t-1}-1}{t-1} = 1$.

EXAMPLE 6: Evaluate $\lim\limits_{t\to 1} \dfrac{e^{t^2-1}-1}{t^2-1}$.

Solution. Approaching values from the left of 1,

t	$f(t)$
0	0.632120558
0.5	0.703511263
0.9	0.9107414
0.99	0.990115674
0.999	0.999001165
0.9999	0.9990001

Approaching $x = 1$ from the right,

t	$f(t)$
2	6.361845641
1.5	1.992274366
1.1	1.112752666
1.01	1.010117675
1.001	1.1.001001168
1.0001	1.000100011

Thus, $\lim_{t \to 1} \dfrac{e^{t^2-1} - 1}{t^2 - 1} = 1$.

EXAMPLE 7: Evaluate $\lim_{x \to 0} \dfrac{x}{x^2 - x}$.

Solution. $\lim_{x \to 0} \dfrac{x}{x^2 - x} \left(\dfrac{0}{0}\right) = \lim_{x \to 0} \dfrac{x}{x(x-1)} = \lim_{x \to 0} \dfrac{1}{x-1} = \dfrac{1}{-1} = -1.$

EXAMPLE 8: Evaluate $\lim_{x \to 1} \dfrac{\sqrt{x^2 + x - 1} - 1}{x - 1}$.

Solution.

$$\begin{aligned}
\lim_{x \to 1} \frac{\sqrt{x^2 + x - 1} - 1}{x - 1} \left(\frac{0}{0}\right) &= \lim_{x \to 1} \frac{\sqrt{x^2 + x - 1} - 1}{x - 1} \cdot \frac{\sqrt{x^2 + x - 1} + 1}{\sqrt{x^2 + x - 1} + 1} \\
&= \lim_{x \to 1} \frac{x^2 + x - 1 - 1}{(x-1)(\sqrt{x^2 + x - 1} + 1)} \\
&= \lim_{x \to 1} \frac{x^2 + x - 2}{(x-1)(\sqrt{x^2 + x - 1} + 1)} \\
&= \lim_{x \to 1} \frac{(x-1)(x+2)}{(x-1)(\sqrt{x^2 + x - 1} + 1)} \\
&= \lim_{x \to 1} \frac{x+2}{\sqrt{x^2 + x - 1} + 1} = \frac{3}{2}.
\end{aligned}$$

Supplementary Problems

Evaluate the following limits.

1. $\lim_{t\to 3} \frac{\sin(t-3)}{t-3}$
2. $\lim_{t\to -2} \frac{\sin(t+2)}{t+2}$
3. $\lim_{t\to 0} \frac{\sin(t^2+t)}{t^2+t}$
4. $\lim_{t\to 1} \frac{\sin(t^3+2t-3)}{t^3+2t-3}$
5. $\lim_{t\to -1} \frac{\sin(t^3-t)}{t^3-t}$
6. $\lim_{t\to 2} \frac{1-\cos(t-2)}{t-2}$
7. $\lim_{t\to -2} \frac{1-\cos(t+2)}{t+2}$
8. $\lim_{t\to 0} \frac{1-\cos(t^2+t)}{t^2+t}$
9. $\lim_{t\to 3} \frac{1-\cos(t^2-9)}{t^2-9}$
10. $\lim_{t\to 2} \frac{1-\cos(t^2-4)}{t^2-4}$
11. $\lim_{t\to 3} \frac{e^{t-3}-1}{t-3}$
12. $\lim_{t\to 1} \frac{e^{t-1}-1}{t-1}$
13. $\lim_{t\to 0} \frac{e^{t^2}-1}{t^2}$
14. $\lim_{t\to -1} \frac{e^{t^2-1}-1}{t^2-1}$
15. $\lim_{x\to 0} \frac{x}{x}$
16. $\lim_{x\to -1} \frac{x^2-1}{x+1}$
17. $\lim_{x\to 1} \frac{x^2-1}{x-1}$
18. $\lim_{x\to 0} \frac{x^2-x}{x}$
19. $\lim_{x\to 3} \frac{\sqrt{x}-3}{x^2-9}$
20. $\lim_{x\to 1} \frac{\sqrt{x}-1}{x^2-1}$

LESSON 3: Continuity of Functions

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Illustrate continuity of a function at a point;
2. Determine whether a function is continuous at a point or not;
3. Illustrate continuity of a function on an interval; and
4. Determine whether a function is continuous on an interval or not.

LESSON OUTLINE:

1. Continuity at a point
2. Determining whether a function is continuous or not at a point
3. Continuity on an interval
4. Determining whether a function is continuous or not on an interval

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TOPIC 3.1: Continuity at a Point

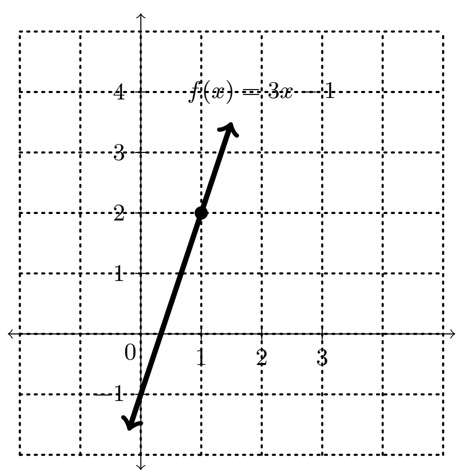
As we have observed in our discussion of limits in Topics 1.1 and 1.2, there are functions whose limits are not equal to the function value at $x = c$, meaning, $\lim_{x\to c} f(x) \neq f(c)$.

This leads us to the study of continuity of functions. In this section, we will be focusing on the continuity of a function at a specific point.

LIMITS AND CONTINUITY AT A POINT

What does "continuity at a point" mean? Intuitively, this means that in drawing the graph of a function, the point in question will be traversed. We start by graphically illustrating what it means to be continuity at a point.

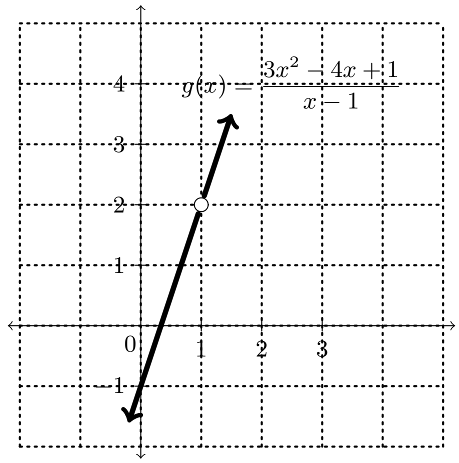
EXAMPLE 1: Consider the graph below.



Is the function continuous at $x = 1$?

Solution. To check if the function is continuous at $x = 1$, use the given graph. Note that one is able to trace the graph from the left side of the number $x = 1$ going to the right side of $x = 1$, without lifting one's pen. This is the case here. Hence, we can say that the function is continuous at $x = 1$.

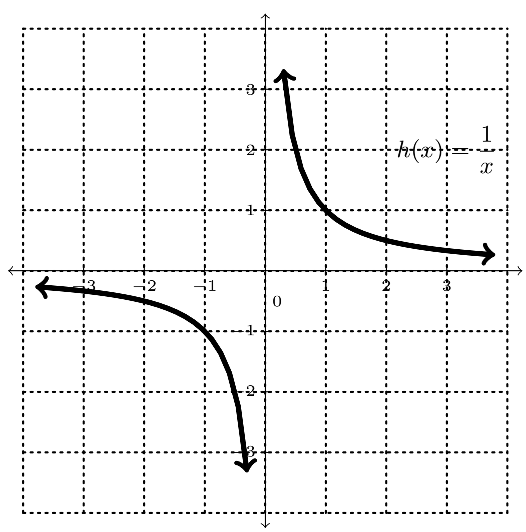
EXAMPLE 2: Consider the graph of the function $g(x)$ below.



Is the function continuous at $x = 1$?

Solution. We follow the process in the previous example. Tracing the graph from the left of $x = 1$ going to right of $x = 1$, one finds that s/he must lift her/his pen briefly upon reaching $x = 1$, creating a hole in the graph. Thus, the function is discontinuous at $x = 1$.

EXAMPLE 3: Consider the graph of the function $h(x) = \dfrac{1}{x}$.



Is the function continuous at $x = 0$?

Solution. If we trace the graph from the left of $x = 0$ going to right of $x = 0$, we have to lift our pen since at the left of $x = 0$, the function values will go downward indefinitely, while at the right of $x = 0$, the function values will go to upward indefinitely. In other words,

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty \text{ and } \lim_{x \to 0^+} \frac{1}{x} = \infty$$

Thus, the function is discontinuous at $x = 0$.

EXAMPLE 4: Consider again the graph of the function $h(x) = \frac{1}{x}$. Is the function continuous at $x = 2$?

Solution. If we trace the graph of the function $h(x) = \frac{1}{x}$ from the left of $x = 2$ to the right of $x = 2$, you will not lift your pen. Therefore, the function h is continuous at $x = 2$.

Suppose we are not given the graph of a function but just the function itself. How do we determine if the function is continuous at a given number? In this case, we have to check three conditions.

THREE CONDITIONS OF CONTINUITY

A function $f(x)$ is said to be **continuous** at $x = c$ if the following three conditions are satisfied:

(i) $f(c)$ exists;

(ii) $\lim_{x \to c} f(x)$ exists; and

(iii) $f(c) = \lim_{x \to c} f(x)$.

If at least one of these conditions is not met, f is said to be **discontinuous** at $x = c$.

EXAMPLE 5: Determine if $f(x) = x^3 + x^2 - 2$ is continuous or not at $x = 1$.

Solution. We have to check the three conditions for continuity of a function.

1. If $x = 1$, then $f(1) = 0$.
2. $\lim_{x \to 1} f(x) = \lim_{x \to 1}(x^3 + x^2 - 2) = 1^3 + 1^2 - 2 = 0$.
3. $f(1) = 0 = \lim_{x \to 1} f(x)$.

Therefore, f is continuous at $x = 1$.

EXAMPLE 6: Determine if $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ is continuous or not at $x = 0$.

Solution. We have to check the three conditions for continuity of a function.

1. If $x = 0$, then $f(0) = 1$.
2. $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \dfrac{x^2 - x - 2}{x - 2} = \lim\limits_{x \to 0} \dfrac{(x-2)(x+1)}{x - 2} = \lim\limits_{x \to 0}(x + 1) = 1.$
3. $f(0) = 1 = \lim\limits_{x \to 0} f(x)$.

Therefore, f is continuous at $x = 0$.

EXAMPLE 7: Determine if $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ is continuous or not at $x = 2$.

Solution. Note that f is not defined at $x = 2$ since 2 is not in the domain of f. Hence, the first condition in the definition of a continuous function is not satisfied. Therefore, f is discontinuous at $x = 2$.

EXAMPLE 8: Determine if

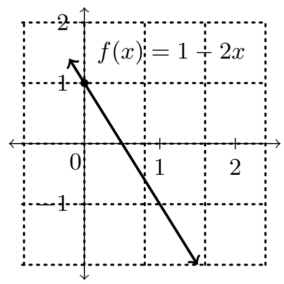
$$f(x) = \begin{cases} x + 1 & \text{if } x < 4, \\ (x - 4)^2 + 3 & \text{if } x \geq 4 \end{cases}$$

is continuous or not at $x = 4$. (This example was given in Topic 1.1.)

Solution. Note that f is defined at $x = 4$ since $f(4) = 3$. However, $\lim\limits_{x \to 4^-} f(x) = 5$ while $\lim\limits_{x \to 4^+} f(x) = 3$. Therefore $\lim\limits_{x \to 4^-} f(x)$ DNE, and f is discontinuous at $x = 4$.

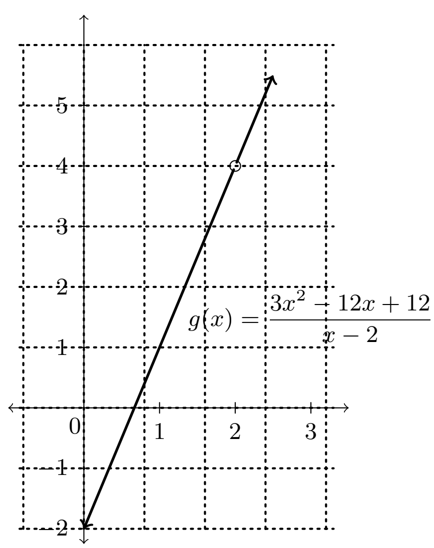
Solved Examples

EXAMPLE 1: Consider the graph below.



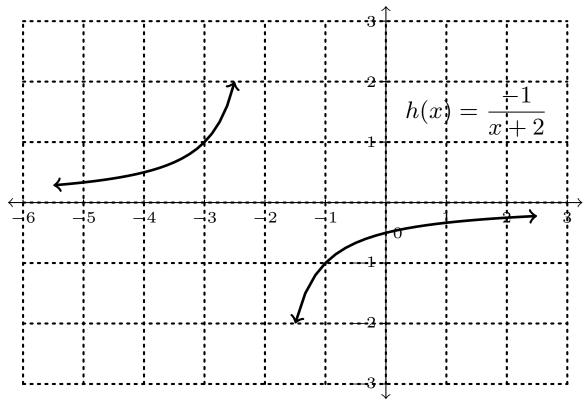
Solution. The function $f(x)$ is continuous at $x = 0$ because we can trace the graph from the left of 1 and from the right of 1 without lifting the pen.

EXAMPLE 2: Consider the graph of a function below.



Solution. If we trace the graph of $g(x)$, we need to lift the pen at the hole at $(2, 4)$. Thus, $g(x)$ is not continuous at $x = 2$.

EXAMPLE 3: Consider the graph of the function $h(x) = \dfrac{-1}{x+2}$.



Solution. The function values from the left of -2 go to ∞ while from the left, they approach $-\infty$. Therefore, $h(x)$ is discontinuous at $x = -2$.

EXAMPLE 4: Consider again the graph of $h(x) = \dfrac{-1}{x+2}$.

Solution. The function $h(x)$ is continuous at $x = 0$ since we need not lift the pen as we trace the graph from the left and right of 0.

EXAMPLE 5: Determine if $f(x) = x^4 - x^2 + 1$ is continuous at $x = -1$.

Solution. Note that

1. $f(-1) = 1$
2. $\lim\limits_{x \to -1} f(x) = \lim\limits_{x \to -1} (x^4 - x^2 + 1) = (-1)^4 - (-1)^2 + 1 = 1$
3. $f(-1) = 1 = \lim\limits_{x \to -1} f(x)$

Thus, $f(x)$ is continuous at $x = -1$.

EXAMPLE 6: Determine if $g(x) = \dfrac{x^2 - 9}{x + 3}$ is continuous at $x = -3$.

Solution. Note that $g(-3)$ is undefined and hence, $g(x)$ is discontinuous at $x = -3$.

EXAMPLE 7: Determine if $h(x) = \dfrac{\sqrt{x} - 1}{x - 1}$ is continuous at $x = 1$.

Solution. Note that $h(1)$ is undefined and therefore, h is discontinuous at $x = 1$.

EXAMPLE 8: Determine if $j(x) = \dfrac{\sqrt{x-1}}{x+1}$ is continuous at $x = 1$.

Solution. Note that

1. $j(1) = 0$;
2. $\lim\limits_{x\to 1} j(x) = \lim\limits_{x\to 1} \dfrac{\sqrt{x-1}}{x+1} = \dfrac{\sqrt{1-1}}{1+1} = \dfrac{0}{1} = 0$;
3. $j(1) = 0 = \lim\limits_{x\to 1} j(x)$.

   Hence, j is continuous at $x = 1$.

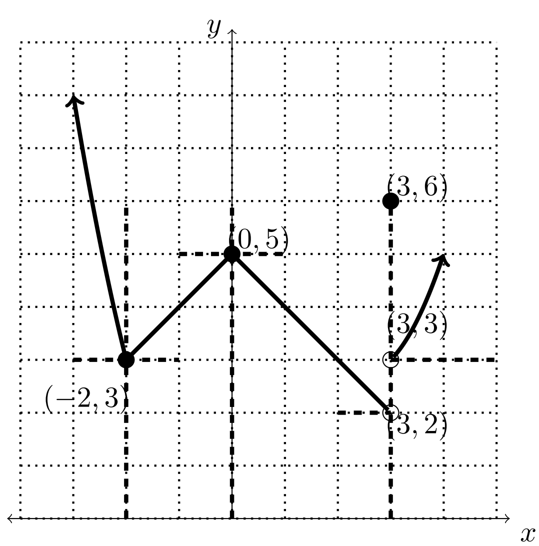
EXAMPLE 9: Determine if

$$f(x) = \begin{cases} 2x - 3 & , \quad x \leq 4, \\ x^2 - x + 1 & , \quad x > 4, \end{cases}$$

is continuous at $x = 4$.

Solution. Note that $f(4) = 5$ but $\lim\limits_{x\to 4^-} f(x) = 5 \neq 13 = \lim\limits_{x\to 4^+} f(x)$, that is, $\lim\limits_{x\to 4} f(x)$ DNE. Thus, f is discontinuous at $x = 4$.

EXAMPLE 10: The graph of a function k is given below.



From the graph, we have

1. $k(-2) = 3 = \lim_{x \to -2} k(x)$;
2. $k(0) = 5 = \lim_{x \to 0} k(x)$;
3. $k(3) = 6$ but $\lim_{x \to 3} k(x)$ DNE.

Thus, $k(x)$ is continuous at $x = -2$ and at $x = 0$, but is discontinuous at $x = 3$.

Supplementary Problems

1. Determine if the following functions are continuous at $x = c$.

   (a) $f(x) = x + 2$; $c = -1$

   (b) $f(x) = x - 2$; $c = 0$

   (c) $f(x) = x^2 + 2$; $c = 1$

   (d) $f(x) = x^2 - 1$; $c = -1$

   (e) $f(x) = x^3 - x$; $c = 0$

   (f) $f(x) = x^3 - 3x$; $c = 1$

   (g) $f(x) = x^2 - 4$; $c = 2$

   (h) $f(x) = x^4 - 1$; $c = 1$

   (i) $f(x) = \dfrac{x^3 - x}{x}$; $c = 0$

   (j) $f(x) = \dfrac{x^3 - 3x}{x}$; $c = 1$

   (k) $f(x) = \dfrac{x^2 - 4}{x - 2}$; $c = 2$

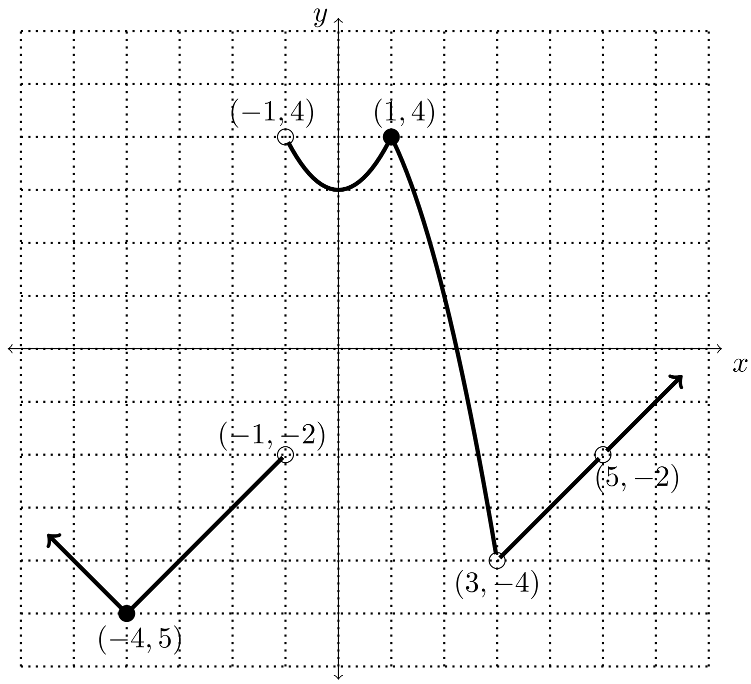
   (l) $f(x) = \dfrac{x^4 - 1}{x - 1}$; $c = 1$

   (m) $f(x) = \dfrac{\sqrt{x^4} - 1}{x - 1}$; $c = 1$

   (n) $f(x) = \dfrac{\sqrt{x} - x^2}{x^2 - x}$; $c = 0$

   (o) $f(x) = \dfrac{\sqrt{x^3} - 1}{x - 1}$; $c = 1$

2. The graph of $f(x)$ is given by



Determine is f is continuous at $x = c$.

(a) $x = -4$

(b) $x = -1$

(c) $x = 1$

(d) $x = 3$

(e) $x = 5$

TOPIC 3.2: Continuity on an Interval

A function can be continuous on an interval. This simply means that it is continuous at every point on the interval. Equivalently, if we are able to draw the entire graph of the function on an interval without lifting our tracing pen, or without being interrupted by a hole in the middle of the graph, then we can conclude that the the function is continuous on that interval.

We begin our discussion with two concepts which are important in determining whether a function is continuous at the endpoints of closed intervals.

ONE-SIDED CONTINUITY

1. A function f is said to be **continuous from the left at** $x = c$ if
$$f(c) = \lim_{x \to c^-} f(x).$$
2. A function f is said to be **continuous from the right at** $x = c$ if
$$f(c) = \lim_{x \to c^+} f(x).$$

Here are known facts on continuities of functions on intervals:

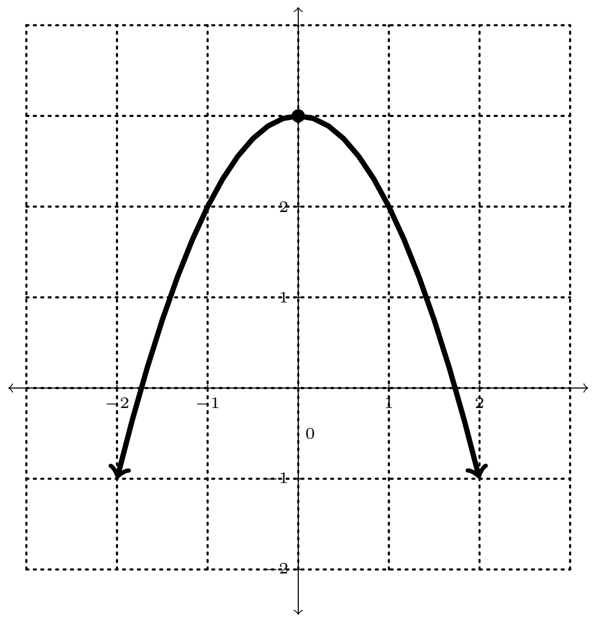
CONTINUITY OF POLYNOMIAL, ABSOLUTE VALUE, RATIONAL AND SQUARE ROOT FUNCTIONS

1. Polynomial functions are continuous everywhere.
2. The absolute value function $f(x) = |x|$ is continuous everywhere.
3. Rational functions are continuous on their respective domains.
4. The square root function $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

LIMITS AND CONTINUITY ON AN INTERVAL

We first look at graphs of functions to illustrate continuity on an interval.

EXAMPLE 1: Consider the graph of the function f given below.



Using the given graph, determine if the function f is continuous on the following intervals:

1. $(-1, 1)$
2. $(-\infty, 0)$
3. $(0, +\infty)$

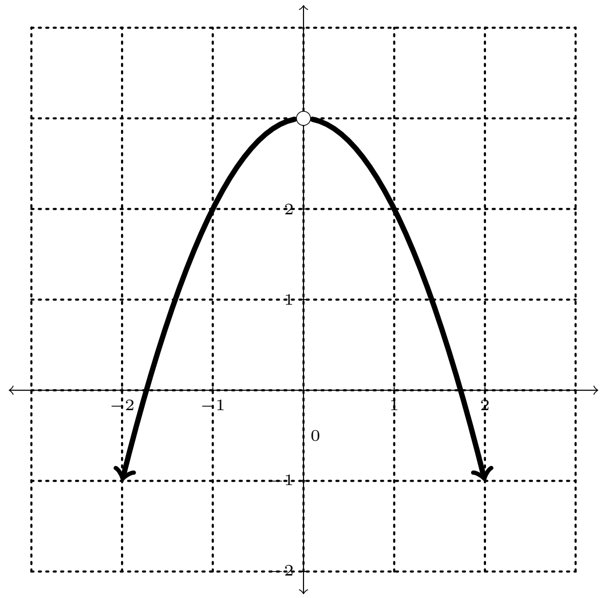
Solution. Remember that when we say "trace from the right side of $x = c$", we are tracing not from $x = c$ on the x-axis, but from the point $(c, f(c))$ *along the graph.*

1. We can trace the graph from the right side of $x = -1$ to the left side of $x = 1$ without lifting the pen we are using. Hence, we can say that the function f is continuous on the interval $(-1, 1)$.

2. If we trace the graph from any negatively large number up to the left side of 0, we will not lift our pen and so, f is continuous on $(-\infty, 0)$.

3. For the interval $(0, +\infty)$, we trace the graph from the right side of 0 to any large number, and find that we will not lift our pen. Thus, the function f is continuous on $(0, +\infty)$.

EXAMPLE 2: Consider the graph of the function h below.



Determine using the given graph if the function f is continuous on the following intervals:

a. $(-1, 1)$

b. $[0.5, 2]$

Solution. Because we are already given the graph of h, we characterize the continuity of h by the possibility of tracing the graph without lifting the pen.

1. If we trace the graph of the function h from the right side of $x = -1$ to the left side of $x = 1$, we will be interrupted by a hole when we reach $x = 0$. We are forced to lift our pen just before we reach $x = 0$ to indicate that h is not defined at $x = 0$ and continue tracing again starting from the right of $x = 0$. Therefore, we are not able to trace the graph of h on $(-1, 1)$ without lifting our pen. Thus, the function h is not continuous on $(-1, 1)$.

2. For the interval $[0.5, 2]$, if we trace the graph from $x = 0.5$ to $x = 2$, we do not have to lift the pen at all. Thus, the function h is continuous on $[0.5, 2]$.

Now, if a function is given without its corresponding graph, we must find other means to determine if the function is continuous or not on an interval. Here are definitions that will help us:

A function f is said to be **continuous**

1. everywhere if f is continuous at every real number. In this case, we also say f is continuous on $\mathbb{R}$.
2. on (a, b) if f is continuous at every point x in (a, b).
3. on $[a, b)$ if f is continuous on (a, b) and from the right at a.
4. on $(a, b]$ if f is continuous on (a, b) and from the left at b.
5. on $[a, b]$ if f is continuous on $(a, b]$ and on $[a, b)$.
6. on (a, ∞) if f is continuous at all $x > a$.
7. on $[a, \infty)$ if f is continuous on (a, ∞) and from the right at a.
8. on $(-\infty, b)$ if f is continuous at all $x < b$.
9. on $(-\infty, b]$ if f is continuous on $(-\infty, b)$ and from the left at b.

EXAMPLE 3: Determine the largest interval over which the function $f(x) = \sqrt{x+2}$ is continuous.

Solution. Observe that the function $f(x) = \sqrt{x+2}$ has function values only if $x + 2 \geq 0$, that is, if $x \in [-2, +\infty)$. For all $c \in (-2, +\infty)$,

$$f(c) = \sqrt{c+2} = \lim_{x \to c} \sqrt{x+2}.$$

Moreover, f is continuous from the right at -2 because

$$f(-2) = 0 = \lim_{x \to -2^+} \sqrt{x+2}.$$

Therefore, for all $x \in [-2, +\infty)$, the function $f(x) = \sqrt{x+2}$ is continuous.

EXAMPLE 4: Determine the largest interval over which $h(x) = \dfrac{x}{x^2 - 1}$ is continuous.

Solution. Observe that the given rational function $h(x) = \dfrac{x}{x^2 - 1}$ is not defined at $x = 1$ and $x = -1$. Hence, the domain of h is the set $\mathbb{R} \setminus \{-1, 1\}$. As mentioned at the start of this topic, a rational function is continuous on its domain. Hence, h is continuous over $\mathbb{R} \setminus \{-1, 1\}$.

EXAMPLE 5: Consider the function $g(x) = \begin{cases} x & \text{if } x \leq 0, \\ 3 & \text{if } 0 < x \leq 1, \\ 3 - x^2 & \text{if } 1 < x \leq 4, \\ x - 3 & \text{if } x > 4. \end{cases}$

Is g continuous on $(0, 1]$? on $(4, \infty)$?

Solution. Since g is a piecewise function, we just look at the 'piece' of the function corresponding to the interval specified.

1. On the interval $(0, 1]$, $g(x)$ takes the constant value 3. Also, for all $c \in (0, 1]$,

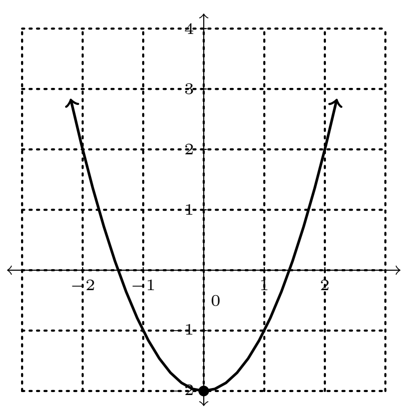
$$\lim_{x \to c} g(x) = 3 = g(c).$$

   Thus, g is continuous on $(0, 1]$.

2. For all $x > 4$, the corresponding 'piece' of g is $g(x) = x - 3$, a polynomial function. Recall that a polynomial function is continuous everywhere in $\mathbb{R}$. Hence, $f(x) = x - 3$ is surely continuous for all $x \in (4, +\infty)$.

Solved Examples

EXAMPLE 1: Consider graph of the function f below.



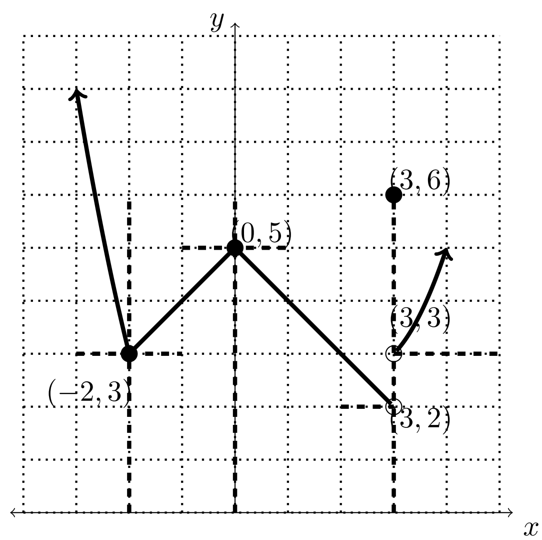
Determine using the given graph if the function f is continuous on the following intervals.

1. $(-2, 2)$.
2. $(-\infty, 0)$.
3. $(0, +\infty)$.

Solution. Note that tracing the graph of f from the left of $x = -2$ to the right of $x = 2$, we need not lift our pen. Thus, f is continuous on $(-2, 2)$. Moreover, f is continuous on $(-\infty, 0)$ and on $(0, +\infty)$.

EXAMPLE 2:

The graph of a function h is given below.



Solution. From the given graph,

1. h is continuous on $(-\infty, 2]$.
2. h is continuous on $(-\infty, 0)$.
3. h is continuous on $(-3, 0)$.
4. h is continuous on $[-2, 2]$.
5. h is discontinuous on $(2, +\infty)$.
6. h is discontinuous on $[2, +\infty)$.
7. h is discontinuous on $(0, 4)$.
8. h is discontinuous on $[-4, 4]$.

EXAMPLE 3: Determine the largest interval in which $f(x) = x^4 - x^3 + 1$ is continuous.

Solution. Since f is a polynomial function, then it is continuous everywhere. Hence, f is continuous on $\mathbb{R}$.

EXAMPLE 4: Determine the largest interval where $g(x) = \dfrac{x^2 - 4}{x - 2}$ is continuous.

Solution. Note that $g(x)$ is undefined at $x = 2$. Thus, if $x \neq 2$, $g(x) = x + 2$, which is a polynomial function. Therefore, $g(x) = \dfrac{x^2 - 4}{x - 2}$ is continuous for all $x \in \mathbb{R} \setminus \{2\}$.

EXAMPLE 5: Determine the largest interval where $h(x) = \sqrt{x^2 + 1}$ is continuous.

Solution. Since $x^2 + 1 \geq 0$, for any $x \in \mathbb{R}$, h is defined on $\mathbb{R}$. Moreover, for any $c \in \mathbb{R}$,

$$h(c) = \sqrt{c^2 + 1} = \lim_{x \to c} h(x).$$

Hence, h is continuous on $\mathbb{R}$

EXAMPLE 6: Determine the largest interval where $j(x) = \sqrt{x^2 - 1}$ is continuous.

Solution. Note that j is defined only whenever $x^2 - 1 \geq 0$, that is, $x \leq -1$ or $x \geq 1$. Also, for all $c \in (-\infty, -1) \cup (1, +\infty)$,

$$j(c) = \sqrt{c^2 - 1} = \lim_{x \to c} j(x).$$

Therefore, j is continuous on $(-\infty, -1) \cup (1, +\infty)$.

EXAMPLE 7: Consider the function given by

$$f(x) = \begin{cases} x - 1 & \text{if } x < 4, \\ \sqrt{x} + 1 & \text{if } x \geq 4. \end{cases}$$

Solution. We have the following:

1. For $x < 4$, $f(x) = x - 1$ is a polynomial function and therefore, continuous everywhere. Thus, f is continuous on $(-\infty, 4)$.

2. For $x > 4$, $f(x) = \sqrt{x} + 1$ is always defined. Moreover, $f(a) = \sqrt{a} + 1 = \lim_{x \to a} f(x)$, for any $a > 4$. Thus, $f(x)$ is continuous for all $x > 4$.

EXAMPLE 8: Consider the function given by

$$g(x) = \begin{cases} \dfrac{1}{x+1} & \text{if } x < 0, \\ x^2 - x + 1 & \text{if } 0 \leq x \geq 2, \\ 5 & \text{if } x > 2. \end{cases}$$

Solution. We have the following:

1. Note that for $x < 0$, $g(x) = \dfrac{1}{x-1}$ is undefined at $x = -1$. Thus, g is discontinuous on $(-\infty, 0)$.
2. For $x \in (0, 2)$, $g(x)$ is a polynomial function, which is continuous on $\mathbb{R}$. Therefore, g is continuous on $(0, 2)$.
3. Since $g(x) = 5$ is a constant function for any $x > 2$, $g(x)$ is continuous for all $x \in (2, +\infty)$.

Supplementary Problems

1. Determine if the following functions are continuous on the interval I.

   (a) $f(x) = x - 1;\;\; I = [-2, 2]$

   (b) $f(x) = x^2 - 5;\;\; I = [-5, 5]$

   (c) $f(x) = x^3 - x + 1;\;\; I = (2, +\infty)$

   (d) $f(x) = \dfrac{x^3 - x}{x};\;\; I = (-1, 1)$

   (e) $f(x) = \dfrac{x^3 - 3x}{x};\;\; I = (-2, 2)$

   (f) $f(x) = \dfrac{x^2 - 4}{x - 2};\;\; I = (2, 3)$

   (g) $f(x) = \dfrac{x^4 - 1}{x - 1};\;\; I = (1, 5)$

   (h) $f(x) = \dfrac{\sqrt{x^4 - 1}}{x - 1};\;\; I = (-2, 2)$

   (i) $f(x) = \dfrac{\sqrt{x} - x^2}{x^2 - x};\;\; I = (0, 1)$

   (j) $f(x) = \dfrac{\sqrt{x^3 - 1}}{x - 1};\;\; I = (2, +\infty)$

   (k) $f(x) = \dfrac{\sqrt{x^2 - 1}}{x - 3};\;\; I = [-4, 4]$

2. Consider

$$f(x) = \begin{cases} x^2 - 2x + 1 & \text{if } x < -1, \\ \dfrac{x}{x+1} & \text{if } -1 \leq x \leq 0, \\ \sqrt{x - 1} & \text{if } 0 < x < 1, \\ 0 & \text{if } x \geq 1. \end{cases}$$

Dertermine if the following functions are continuous on the interval I.

(a) $I = (-\infty, -1)$

(b) $I = [-1, 0]$

(c) $I = (0, 1)$

(d) $I = (1, +\infty)$

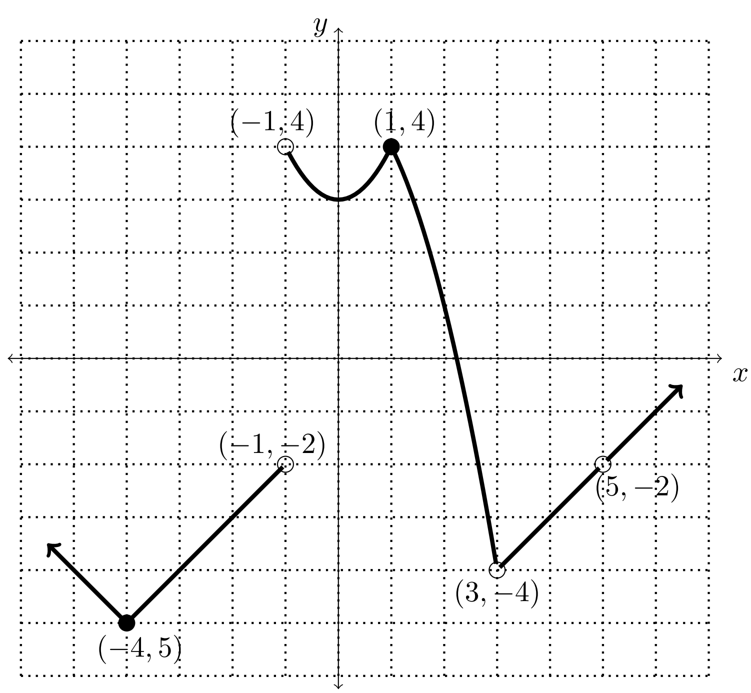
3. Determine the largest intervals in which the following functions are continuous.

(a) $f(x) = 2 - 5x - x^2$

(b) $f(x) = \dfrac{x^3 - 1}{x - 1}$

(c) $f(x) = \sqrt{x^2 - 4}$

4. The graph of $f(x)$ is given by



Determine if f is continuous on the interval I.

(a) $I = (-\infty, -2)$

(b) $I = [-2, 0)$

(c) $I = [-1, 1]$

(d) $I = [0, 2]$

(e) $I = [2, +\infty)$

LESSON 4: More on Continuity

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Illustrate different types of discontinuity (hole/removable, jump/essential, asymptotic/infinite);
2. Illustrate the Intermediate Value and Extreme Value Theorems; and
3. Solve problems involving the continuity of a function.

TOPIC 4.1: Different Types of Discontinuities

In Topics 1.2, it was emphasized that the value of $\lim_{x \to c} f(x)$ may be distinct from the value of the function itself at $x = c$. Recall that a limit may be evaluated at values which are not in the domain of $f(x)$.

In Topics 3.7 - 3.8, we learned that when $\lim_{x \to c} f(x)$ and $f(c)$ are equal, $f(x)$ is said to be continuous at c. Otherwise, it is said to be discontinuous at c. We will revisit the instances when $\lim_{x \to c} f(x)$ and $f(c)$ have unequal or different values. These instances of inequality and, therefore, discontinuity are very interesting to study. This section focuses on these instances.

Consider the functions $g(x)$, $h(x)$ and $j(x)$ where

$$g(x) = \begin{cases} \dfrac{3x^2 - 4x + 1}{x - 1} & \text{if } x \neq 1, \\ 1 & \text{if } x = 1. \end{cases}$$

$$h(x) = \begin{cases} x + 1 & \text{if } x < 4, \\ (x - 4)^2 + 3 & \text{if } x \geq 4. \end{cases}$$

and

$$j(x) = \frac{1}{x}, \quad x \neq 0.$$

We examine these for continuity at the respective values 1, 4, and 0.

1. $\lim_{x \to 1} g(x) = 2$ but $g(1) = 1$.

2. $\lim_{x \to 4} h(x)$ DNE but $h(4) = 3$.

3. $\lim_{x \to 0} j(x)$ DNE and $f(0)$ DNE.

All of the functions are discontinuous at the given values. A closer study shows that they actually exhibit different types of discontinuity.

REMOVABLE DISCONTINUITY

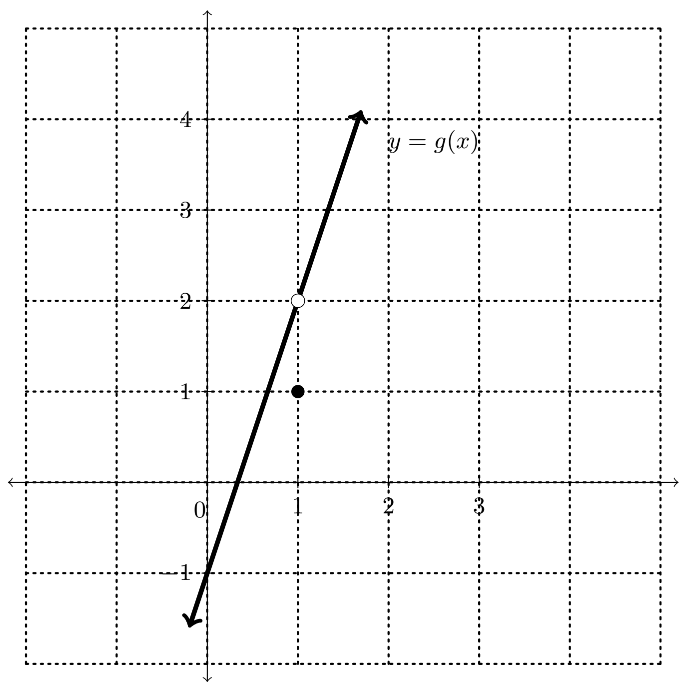
A function $f(x)$ is said to have a ***removable discontinuity*** at $x = c$ if

1. $\lim\limits_{x \to c} f(x)$ exists; and
2. either $f(c)$ does not exist or $f(c) \neq \lim\limits_{x \to c} f(x)$.

It is said to be *removable* because the discontinuity may be removed by *redefining* $f(c)$ so that it will equal $\lim\limits_{x \to c} f(x)$. In other words, if $\lim\limits_{x \to c} f(x) = L$, a removable discontinuity is remedied by the redefinition:

$$\text{Let } f(c) = L.$$

Recall $g(x)$ above and how it is discontinuous at 1. In this case, $g(1)$ exists. Its graph is as follows:

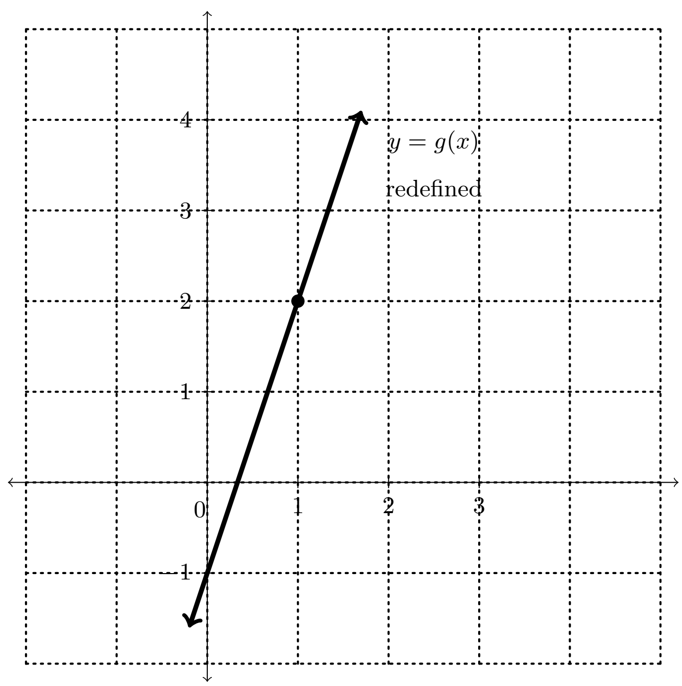


The discontinuity of g at the point $x = 1$ is manifested by the hole in the graph of $y = g(x)$ at the point $(1, 2)$. This is due to the fact that $g(1)$ is equal to 1 and not 2, while $\lim\limits_{x \to 1} g(x) = 2$. We now demonstrate how this kind of a discontinuiy may be removed:

$$\text{Let } g(1) = 2.$$

This is called a *redefinition* of g at $x = 1$. The redefinition results in a "transfer" of the point $(1, 1)$ to the hole at $(1, 2)$. In effect, the hole is filled and the discontinuity is removed! This is why the discontinuity is called a removable one. This is also why, sometimes, it is called a *hole* discontinuity.

We go back to the graph of $g(x)$ and see how redefining $f(1)$ to be 2 removes the discontinuity:



and revises the function to its continuous counterpart,

$$G(x) = \begin{cases} g(x) & \text{if } x \neq 1, \\ 2 & \text{if } x = 1. \end{cases}$$

ESSENTIAL DISCONTINUITY

A function $f(x)$ is said to have an **essential discontinuity** at $x = c$ if $\lim_{x \to c} f(x)$ DNE.

Case 1. If for a function $f(x)$, $\lim_{x \to c} f(x)$ DNE because the limits from the left and right of $x = c$ both exist but are not equal, that is,

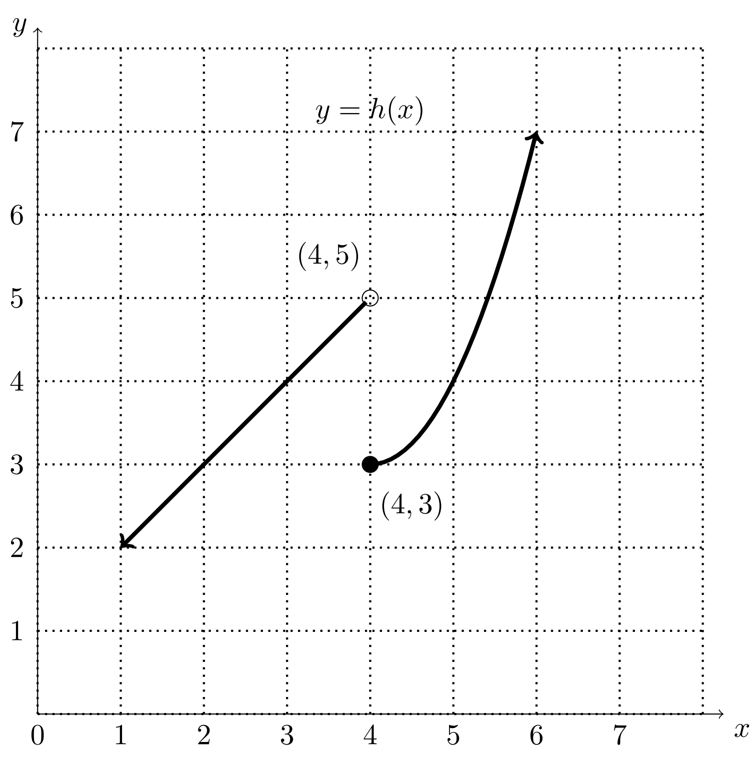
$$\lim_{x \to c^-} f(x) = L \text{ and } \lim_{x \to c^+} f(x) = M, \text{ where } L \neq M,$$

then f is said to have a *jump* essential discontinuity at $x = c$.

Recall the function $h(x)$ where

$$h(x) = \begin{cases} x + 1 & \text{if } x < 4, \\ (x-4)^2 + 3 & \text{if } x \geq 4. \end{cases}$$

Its graph is as follows:



From Lesson 2, we know that $\lim_{x \to 4} h(x)$ DNE because

$$\lim_{x \to 4^-} h(x) = 5 \text{ and } \lim_{x \to 4^+} h(x) = 3.$$

The graph confirms that the discontinuity of $h(x)$ at $x = 4$ is certainly not removable. See, the discontinuity is not just a matter of having one point missing from the graph and putting it in; if ever, it is a matter of having a part of the graph entirely out of place. If we force to remove this kind of discontinuity, we need to connect the two parts by a vertical line from $(4, 5)$ to $(4, 3)$. However, the resulting graph will fail the Vertical Line Test and will not be a graph of a function anymore. Hence, this case has no remedy. From the graph, it is clear why this essential discontinuity is also called a *jump* discontinuity.

Case 2. If a function $f(x)$ is such that $\lim_{x\to c} f(x)$ DNE because either

(i) $\lim_{x\to c^-} f(x) = +\infty$, or

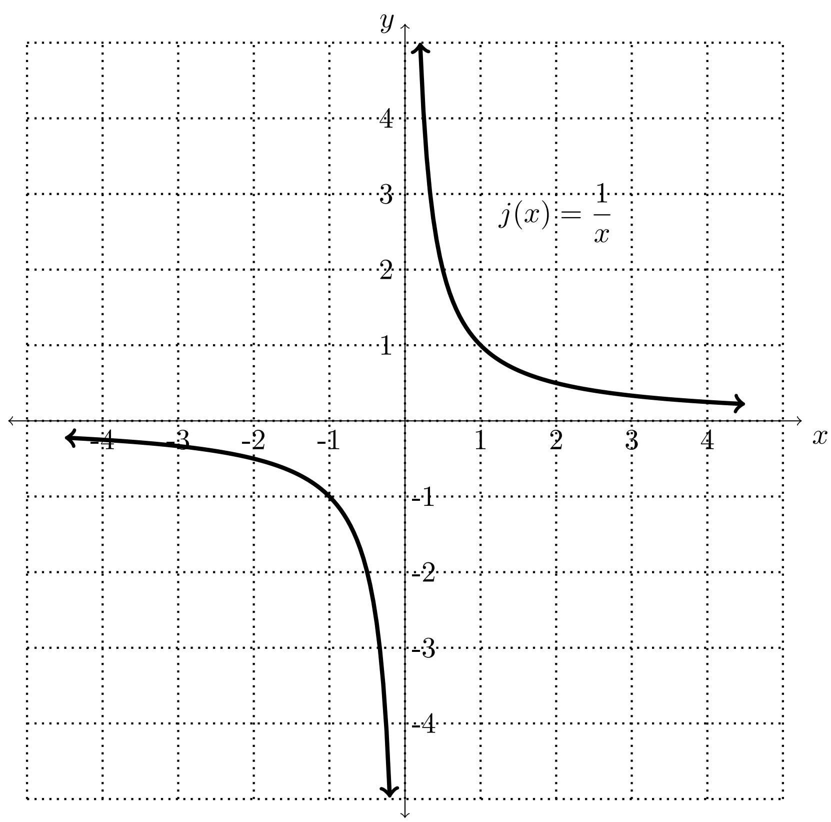
(ii) $\lim_{x\to c^-} f(x) = -\infty$, or

(iii) $\lim_{x\to c^+} f(x) = +\infty$, or

(iv) $\lim_{x\to c^+} f(x) = -\infty$,

then $f(x)$ is said to have an *infinite* discontinuity at $x = c$.

Recall $j(x) = \dfrac{1}{x}$, $x \neq 0$, as mentioned earlier. Its graph is as follows:

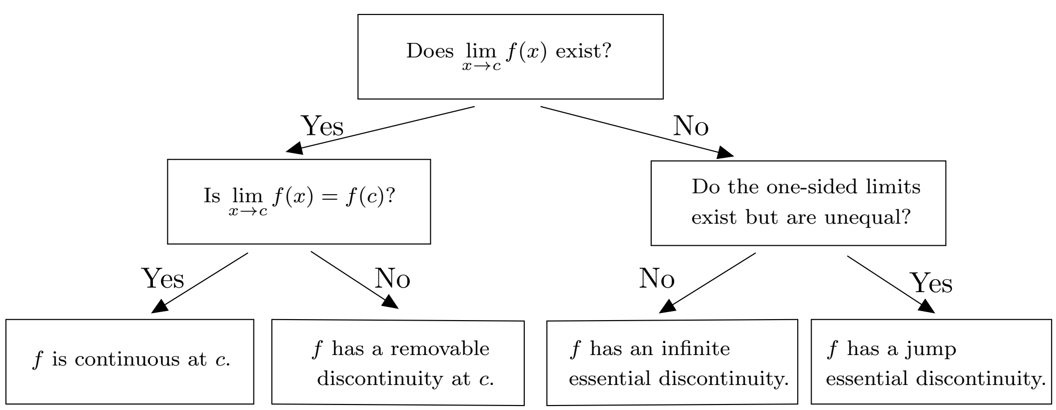


We have seen from Topics 2.5 and 2.6 that

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty \quad \text{and} \quad \lim_{x\to 0^+} \frac{1}{x} = +\infty.$$

Because the limits are infinite, the limits from both the left and the right of $x = 0$ do not exist, and the discontinuity cannot be removed. Also, the absence of a left-hand (or right-hand) limit from which to "jump" to the other part of the graph means the discontinuity is permanent. As the graph indicates, the two ends of the function that approach $x = 0$ continuously move away from each other: one end goes upward without bound, the other end goes downward without bound. This translates to an asymptotic behavior as x-values approach 0; in fact, we say that $x = 0$ is a vertical asymptote of $f(x)$. Thus, this discontinuity is called an *infinite* essential discontinuity.

<u>FLOWCHART</u>. Here is a flowchart which can help evaluate whether a function is continuous or not at a point c. Before using this, make sure that the function is defined on an open interval containing c, except possibly at c.

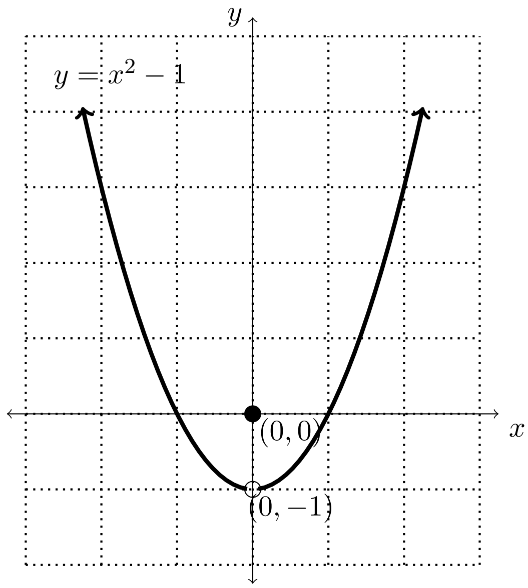


Solved Examples

EXAMPLE 1: Consider $g(x) = \begin{cases} x^2 - 1 & , \quad x \neq 0, \\ 0 & , \quad x = 0. \end{cases}$

Determine if g is continuous at $x = 0$.

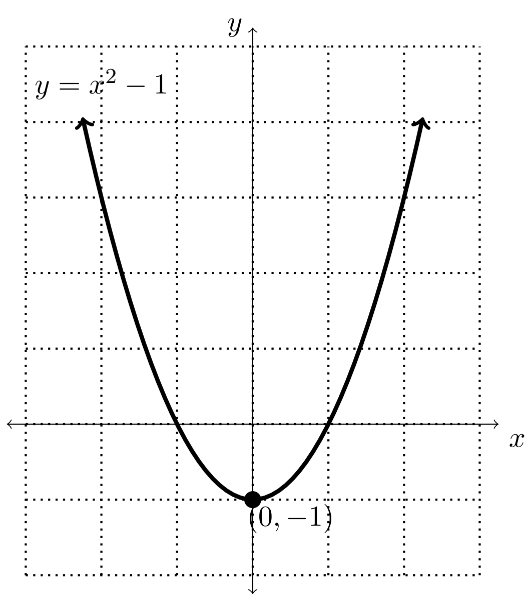
Solution. The graph of g is given by:



1. $g(0) = 0$ exists;
2. $\lim_{x \to 0} g(x) = -1$;
3. $g(0) = 0 \neq -1 = \lim_{x \to 0} g(x)$.

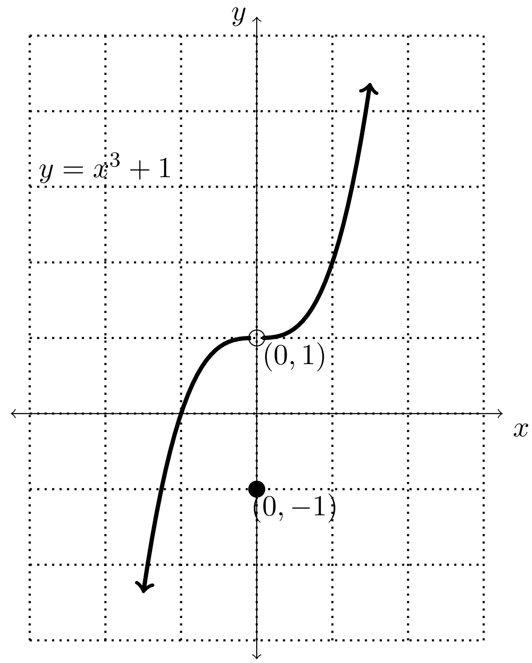
Thus, $g(x)$ has a removable discontinuity at $x = 0$.

To remove the discontinuity, redefine $g(0) = -1$. We now have the new graph of $g(x)$.



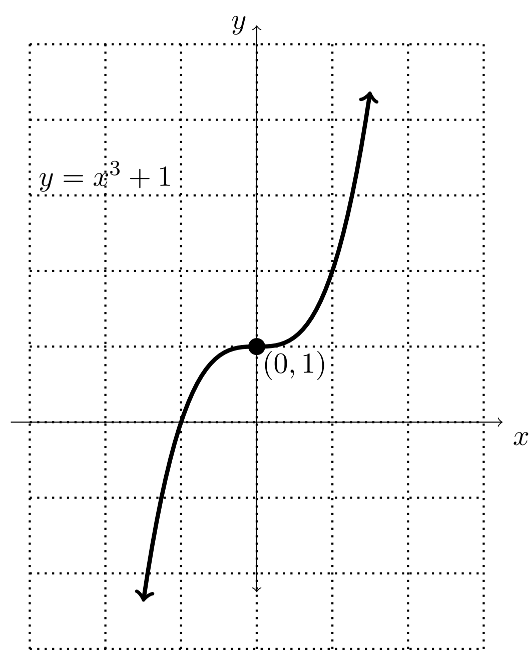
Therefore, $g(0) = -1 = \lim_{x \to 0} g(x)$ and hence, $g(x)$ is now continuous at $x = 0$.

EXAMPLE 2: Given the function $h(x) = \begin{cases} x^3+1 & , \quad x \neq 0, \\ -1 & , \quad x = 0. \end{cases}$ Determine if h is continuous at $x=0$. The graph of h is given by:



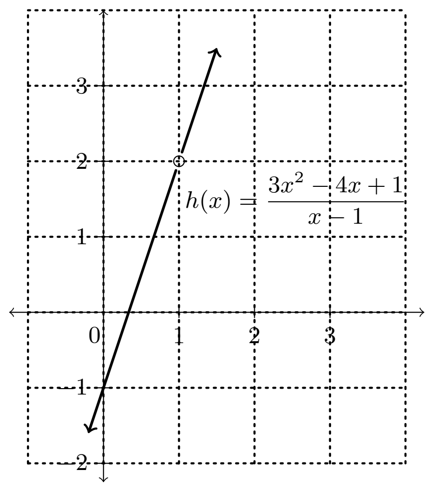
Solution. Note that $h(0) = -1$. Now, evaluating $\lim_{x \to 0} h(x) = 1$. Since, $h(0) = -1 \neq 1 = \lim_{x \to 0} h(x)$, therefore, $h(x)$ has a removable discontinuity at $x = 0$.

To remove the discontinuity, redefine $h(0) = 1$. We now have the new graph of $h(x)$.



Now, $h(0) = 1 = \lim_{x \to 0} h(x)$ and therefore, $h(x)$ is now continuous at $x = 0$.

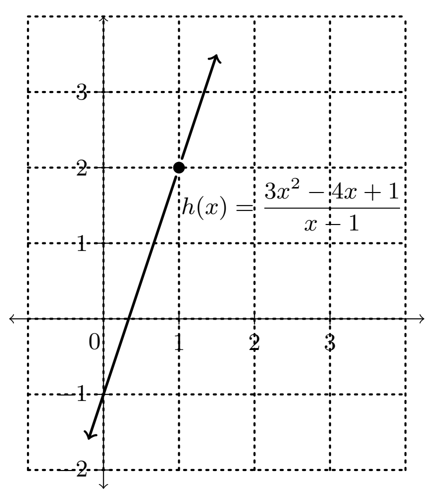
EXAMPLE 3: Determine if $h(x) = \dfrac{3x^2 - 4x + 1}{x - 1}$ is continuous at $x = 1$.



Solution. Note that $h(1)$ is undefined but

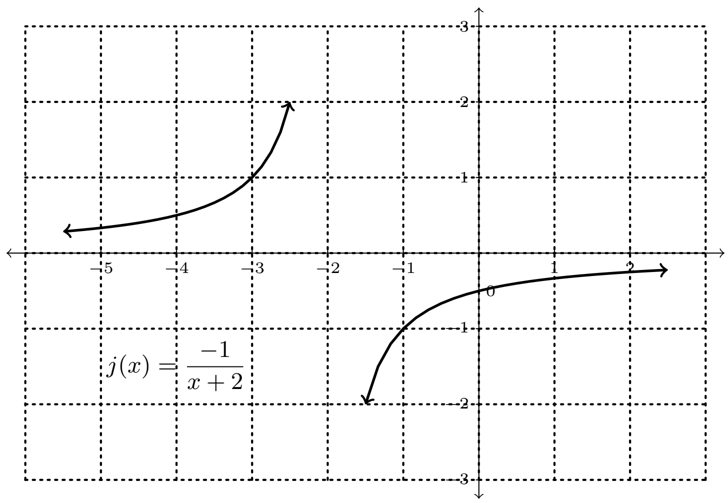
$$\lim_{x \to 1} h(x) = \lim_{x \to 1} \frac{3x^2 - 4x + 1}{x - 1} = \lim_{x \to 1} \frac{(3x - 1)(x - 1)}{x - 1} = \lim_{x \to 1}(3x - 1) = 2.$$

Thus, $h(x)$ has a removable discontinuity at $x = 1$. We remove this discontinuity by redefining $h(1) = 2$ and therefore have the following graph.



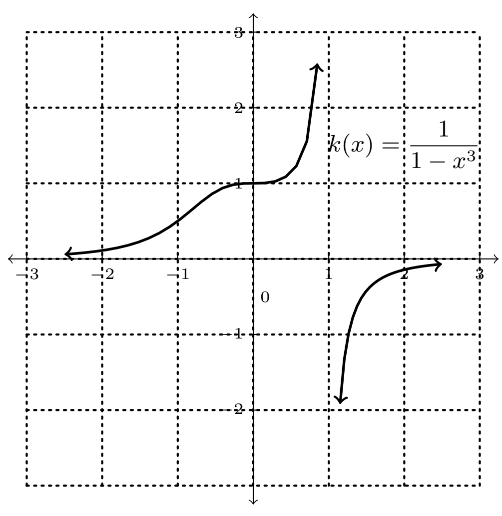
So, $h(1) = 2 = \lim_{x \to 1} h(x)$ and hence, $h(x)$ is now continuous at $x = 1$.

EXAMPLE 4: Determine if $j(x) = \dfrac{-1}{x+2}$ is continuous at $x = -2$.



Solution. The function values from the left of -2 go to ∞ while from the left, they approach $-\infty$. Therefore, $h(x)$ has an infinite essential discontinuity at $x = -2$.

EXAMPLE 5: Determine if $k(x) = \dfrac{-1}{1-x^3}$ is continuous at $x = 1$.

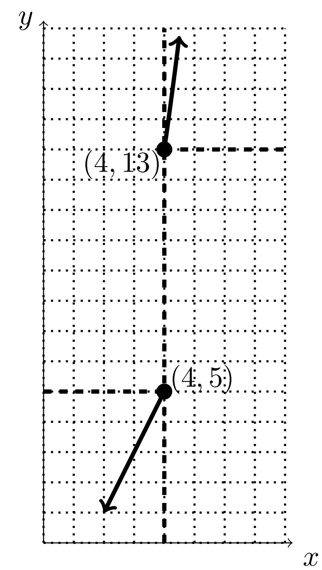


Solution. The function values from the left of 1 go to ∞ while from the left, they approach $-\infty$. Therefore, $h(x)$ has an infinite essential discontinuity at $x = 1$.

EXAMPLE 6: Consider now

$$f(x) = \begin{cases} 2x - 3 & , \quad x \leq 4, \\ x^2 - x + 1 & , \quad x > 4. \end{cases}$$

Determine if f is continuous at $x = 4$.

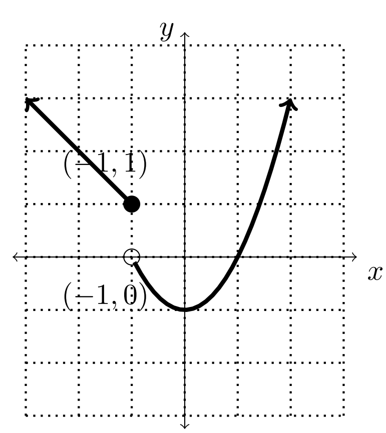


Solution. From the graph above, $\lim_{x \to 4^-} f(x) = 5$ and $\lim_{x \to 4^+} f(x) = 13$. Hence, $\lim_{x \to 4} f(x)$ DNE, i.e., $f(x)$ has a jump essential discontinuity at $x = 4$.

EXAMPLE 7: Given the function

$$g(x) = \begin{cases} |x| & , \quad x \leq -1, \\ x^2 - 1 & , \quad x > -1, \end{cases}$$

determine if g is continuous at $x = -1$.



Solution. From the graph above, $\lim_{x \to -1^-} g(x) = 1$ and $\lim_{x \to -1^+} g(x) = 0$. Hence, $\lim_{x \to -1} g(x)$ DNE, i.e., $g(x)$ has a jump essential discontinuity at $x = -1$.

Supplementary Problems

Determine if the following functions are continuous at the point $x = c$. If not, classify the discontinuity as to removable, jump essential or inifinite essential.

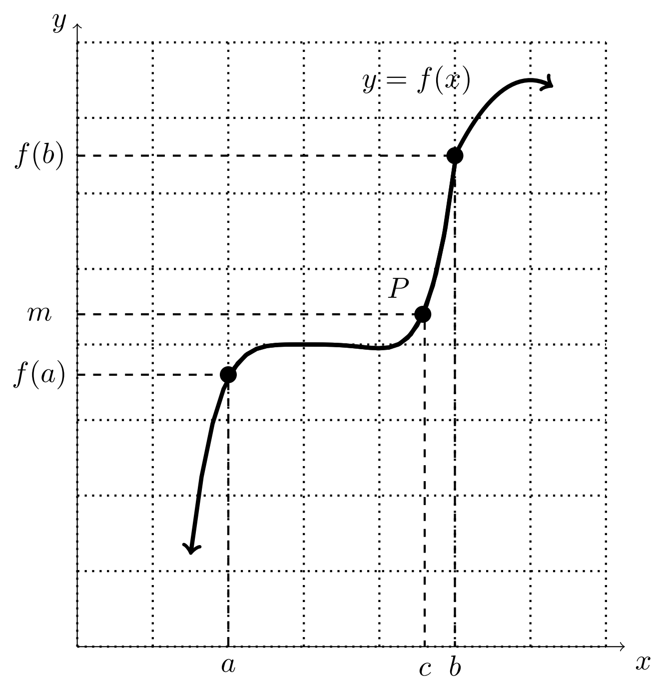
1. $f(x) = \dfrac{1}{x-3}$; $x = 3$
2. $f(x) = \dfrac{1}{x+5}$; $x = -5$
3. $f(x) = \dfrac{x^2-1}{x-1}$; $x = 1$
4. $f(x) = \dfrac{x^2-1}{x+1}$; $x = -1$
5. $f(x) = \dfrac{x-2}{x-1}$; $x = 1$
6. $f(x) = \dfrac{x-1}{x+1}$; $x = -1$
7. $f(x) = \dfrac{x^3-1}{x-1}$; $x = 1$
8. $f(x) = \dfrac{x^3+1}{x+1}$; $x = -1$
9. $f(x) = \dfrac{x^2+2}{x-1}$; $x = 1$
10. $f(x) = \dfrac{x^2-3}{x+1}$; $x = -1$
11. $f(x) = \dfrac{x^3-8}{x-2}$; $x = 2$
12. $f(x) = \dfrac{x^3+8}{x+2}$; $x = -2$
13. $f(x) = \dfrac{2x^3-16}{x-2}$; $x = 2$
14. $f(x) = \dfrac{x^3+27}{x+3}$; $x = -3$
15. $f(x) = \begin{cases} 3x^2 - 1 & , \quad x \leq 0, \\ x - 2 & , \quad x > 0, \end{cases}$
16. $f(x) = \begin{cases} x - 1 & , \quad x \leq 1, \\ x^2 + 1 & , \quad x > 1, \end{cases}$
17. $f(x) = \begin{cases} |x - 1| & , \quad x \leq \frac{1}{2}, \\ -(x - 1) & , \quad x > \frac{1}{2}, \end{cases}$
18. $f(x) = \begin{cases} x^3 - 8 & , \quad x < -2, \\ x & , \quad x \geq -2, \end{cases}$
19. $f(x) = \begin{cases} 3x^2 - 1 & , \quad x < 0, \\ x - 2 & , \quad x \geq 0, \end{cases}$
20. $f(x) = \begin{cases} \left|x^2 - 3\right| & , \quad x < 0, \\ x - 3 & , \quad x \geq 0, \end{cases}$

TOPIC 4.2: The Intermediate Value and the Extreme Value Theorems

The Intermediate Value Theorem

The first theorem we will illustrate says that a function $f(x)$ which is found to be continuous over a closed interval $[a, b]$ will take any value between $f(a)$ and $f(b)$.

Theorem 4 (Intermediate Value Theorem (IVT)). *If a function $f(x)$ is continuous over a closed interval $[a, b]$, then for every value m between $f(a)$ and $f(b)$, there is a value $c \in [a, b]$ such that $f(c) = m$.*



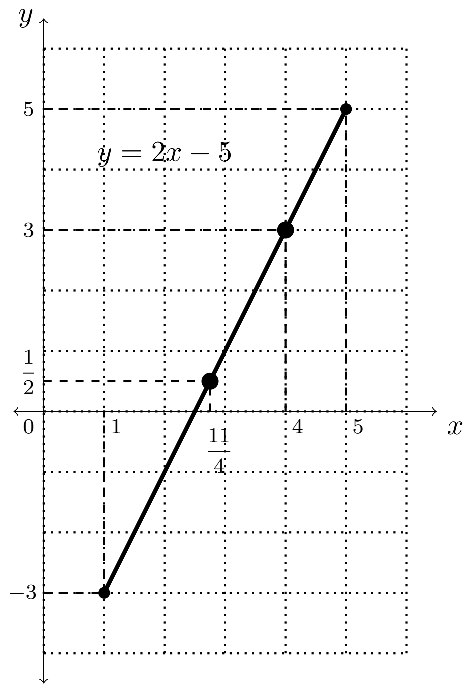
Look at the graph as we consider values of m between $f(a)$ and $f(b)$. Imagine moving the dotted line for m up and down between the dotted lines for $f(a)$ and $f(b)$. Correspondingly, the dot P will move along the thickened curve between the two points, $(a, f(a))$ and $(b, f(b))$.

We make the following observations:

- As the dark dot moves, so will the vertical dotted line over $x = c$ move.
- In particular, the said line moves between the vertical dotted lines over $x = a$ and $x = b$.
- More in particular, for any value that we assign m in between $f(a)$ and $f(b)$, the consequent position of the dark dot assigns a corresponding value of c between a and b. This illustrates what the IVT says.

EXAMPLE 1: Consider the function $f(x) = 2x - 5$.

Since it is a linear function, we know it is continuous everywhere. Therefore, we can be sure that it will be continuous over any closed interval of our choice.



Take the interval $[1, 5]$. The IVT says that for any m intermediate to, or in between, $f(1)$ and $f(5)$, we can find a value intermediate to, or in between, 1 and 5.

Start with the fact that $f(1) = -3$ and $f(5) = 5$. Then, choose an $m \in [-3, 5]$, to exhibit a corresponding $c \in [1, 5]$ such that $f(c) = m$.

Choose $m = \dfrac{1}{2}$. By IVT, there is a $c \in [1, 5]$ such that $f(c) = \dfrac{1}{2}$. Therefore,

$$\frac{1}{2} = f(c) = 2c - 5 \quad \Longrightarrow \quad 2c = \frac{11}{2} \quad \Longrightarrow \quad c = \frac{11}{4}.$$

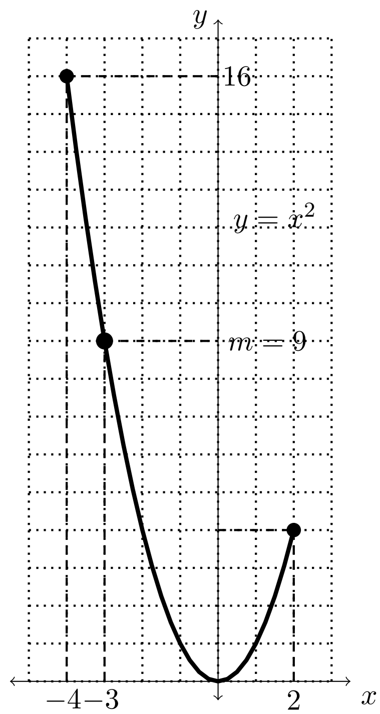
Indeed, $\dfrac{11}{4} \in (1, 5)$.

We can try another m-value in $(-3, 5)$. Choose $m = 3$. By IVT, there is a $c \in [1, 5]$ such that $f(c) = 3$. Therefore,

$$3 = f(c) = 2c - 5 \quad \Longrightarrow \quad 2c = 8 \quad \Longrightarrow \quad c = 4.$$

Again, the answer, 4, is in $[1, 5]$. The claim of IVT is clearly seen in the graph of $y = 2x - 5$.

EXAMPLE 2: Consider the simplest quadratic function $f(x) = x^2$.



Being a polynomial function, it is continuous everywhere. Thus, it is also continuous over any closed interval we may specify.

We choose the interval $[-4, 2]$. For any m in between $f(-4) = 16$ and $f(2) = 4$, there is a value c inside the interval $[-4, 2]$ such that $f(c) = m$.

Suppose we choose $m = 9 \in [4, 16]$. By IVT, there exists a number $c \in [-4, 2]$ such that $f(c) = 9$. Hence,

$$9 = f(c) = c^2 \quad \Longrightarrow \quad c = \pm 3.$$

However, we only choose $c = -3$ because the other solution $c = 3$ is not in the specified interval $[-4, 2]$.

Note: In the previous example, if the interval that was specified was [0,4], then the final answer would instead be $c = +3$.

Remark 1: The value of $c \in [a, b]$ in the conclusion of the Intermediate Value Theorem is not necessarily unique.

EXAMPLE 3: Consider the polynomial function

$$f(x) = x^3 - 4x^2 + x + 7$$

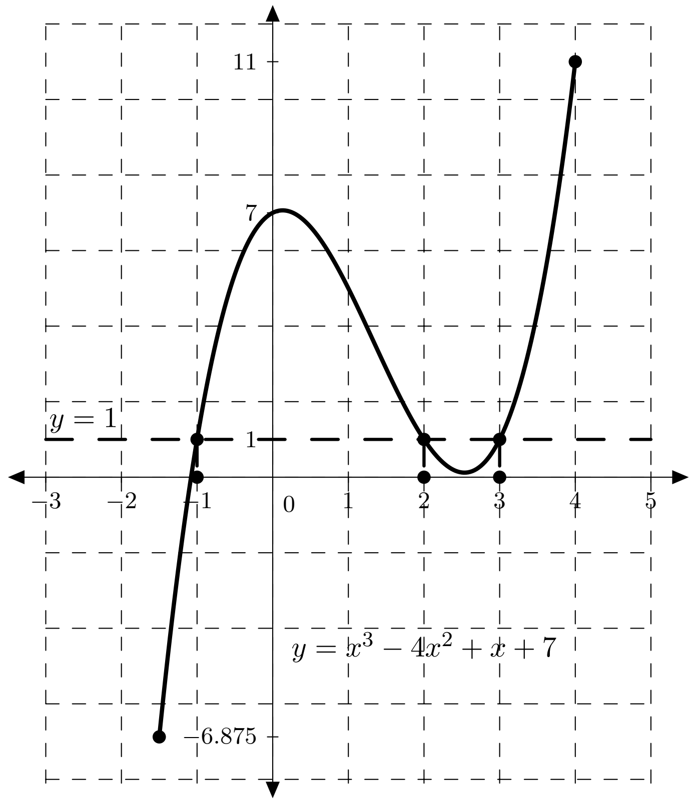
over the interval $[-1.5, 4]$ Note that

$$f(-1.5) = -6.875 \quad \text{and} \quad f(4) = 11.$$

We choose $m = 1$. By IVT, there exists $c \in [-1.5, 4]$ such that $f(c) = 1$. Thus,

$$\begin{aligned} & f(c) = c^3 - 4c^2 + c + 7 = 1 \\ \implies \quad & c^3 - 4c^2 + c + 6 = 0 \\ \implies \quad & (c+1)(c-2)(c-3) = 0 \\ \implies \quad & c = -1 \text{ or } c = 2 \text{ or } c = 3. \end{aligned}$$

We see that there are three values of $c \in [-1.5, 4]$ which satisfy the conclusion of the Intermediate Value Theorem.



The Extreme Value Theorem

The second theorem we will illustrate says that a function $f(x)$ which is found to be continuous over a closed interval $[a, b]$ is guaranteed to have extreme values in that interval.

An extreme value of f, or *extremum*, is either a minimum or a maximum value of the function.

- A *minimum* value of f occurs at some $x = c$ if $f(c) \leq f(x)$ for all $x \neq c$ in the interval.
- A *maximum* value of f occurs at some $x = c$ if $f(c) \geq f(x)$ for all $x \neq c$ in the interval.

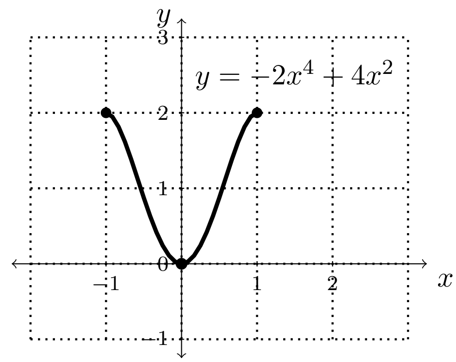
Theorem 5 (Extreme Value Theorem (EVT))**.** *If a function $f(x)$ is continuous over a closed interval $[a, b]$, then $f(x)$ is guaranteed to reach a maximum and a minimum on $[a, b]$.*

Note: In this section, we limit our illustration of extrema to graphical examples. More detailed and computational examples will follow once derivatives have been discussed.

EXAMPLE 4: Consider the function $f(x) = -2x^4 + 4x^2$ over $[-1, 1]$.

From the graph, it is clear that on the interval, f has

- The maximum value of 2, occurring at $x = \pm 1$; and
- The minimum value of 0, occurring at $x = 0$.



Remark 2: Similar to the IVT, the value $c \in [a, b]$ at which a minimum or a maximum occurs is not necessarily unique.

Here are more examples exhibiting the guaranteed existence of extrema of functions continuous over a closed interval.

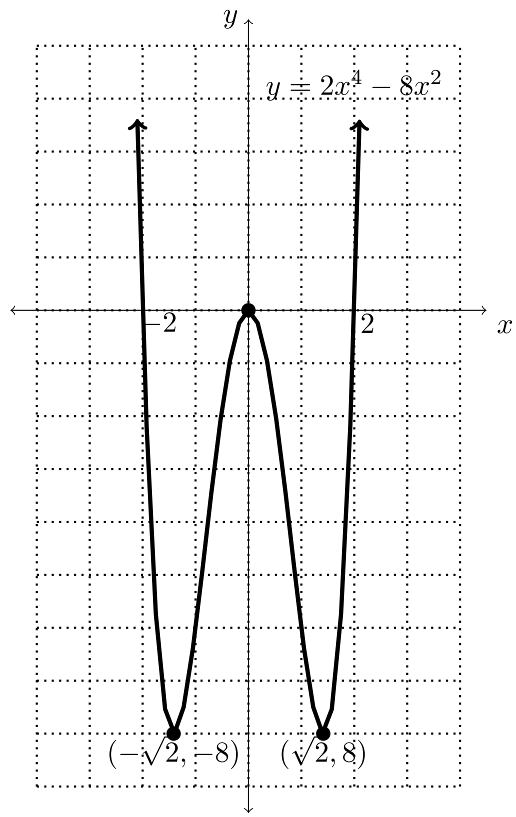
EXAMPLE 5: Consider Example 1. Observe that $f(x) = 2x - 5$ on $[1, 5]$ exhibits the extrema at the endpoints:

- The minimum occurs at $x = 1$, giving the minimum value $f(1) = -3$; and
- The maximum occurs at $x = 5$, giving the maximum value $f(5) = 5$.

EXAMPLE 6: Consider Example 2. $f(x) = x^2$ on $[-4, 2]$ exhibits an extremum at one endpoint and another at a point inside the interval (or, an *interior* point):

- The minimum occurs at $x = 0$, giving the minimum value $f(0) = 0$; and
- The maximum occurs at $x = -4$, giving the maximum value $f(-4) = 16$.

EXAMPLE 7: Consider $f(x) = 2x^4 - 8x^2$.



- On the interval $[-2, -\sqrt{2}]$, the extrema occur at the endpoints.
  - Endpoint $x = -2$ yields the maximum value $f(-2) = 0$.
  - Endpoint $x = -\sqrt{2}$ yields the minimum value $f(-\sqrt{2}) = -8$.
- On the interval $[-2, -1]$, one extremum occurs at an endpoint, another at an interior point.
  - Endpoint $x = -2$ yields the maximum value $f(-2) = 0$.
  - Interior point $x = -\sqrt{2}$ yields the minimum value $f(-\sqrt{2}) = -8$.
- On the interval $[-1.5, 1]$, the extrema occur at interior points.
  - Interior point $x = -\sqrt{2}$ yields the minimum value $f(-\sqrt{2}) = -8$.
  - Interior point $x = 0$ yields the maximum value $f(0) = 0$.
- On the interval $[-2, 2]$, the extrema occur at both the endpoints and several interior points.
  - Endpoints $x = \pm 2$ and interior point $x = 0$ yield the maximum value 0.
  - Interior points $x = \pm\sqrt{2}$ yield the minimum value -8.

Remark 3: Keep in mind that the IVT and the EVT are existence theorems ("*there is a value c ...*"), and their statements do not give a method for finding the values stated in their respective conclusions. It may be difficult or impossible to find these values algebraically especially if the function is complicated.

Solved Examples

EXAMPLE 1: Consider $f(x) = 3x - 1$ on $[-1, 1]$.

Solution. Since f is a polynomial function, it is continuous everywhere. Note that $f(-1) = -4$ and $f(1) = 2$. IVT on $[-1, 1] \implies$ for any $m \in (-4, 2)$, there is a $c \in [-1, 1]$ such that $f(c) = m$.

- Take $m = -1 \in (-4, 2)$. Note $c = 0 \in (-1, 1)$ and $f(0) = -1 = m$.
- Choose $m = 1 \in (-4, 2)$. Note $c = \dfrac{2}{3} \in (-1, 1)$ and $f\left(\dfrac{2}{3}\right) = 1 = m$.
- Take $m = 0 \in (-4, 2)$. Note $c = \dfrac{1}{3} \in (-1, 1)$ and $f\left(\dfrac{1}{3}\right) = 0 = m$.

EXAMPLE 2: Consider $g(x) = 2x - 3$ on $[0, 3]$.

Solution. g, polynomial $\implies$ g, continuous everywhere. Note $g(0) = -3$ and $g(3) = 3$ Thus, IVT on $[0, 3] \implies$ for all $m \in (-3, 3)$, there is a $c \in (0, 3)$, with $g(c) = m$.

- Take $m = 0 \in (-3, 3)$. Then $c = \dfrac{3}{2} \in (0, 3)$ and $g\left(\dfrac{3}{2}\right) = 0 = m$.
- Take $m = -1 \in (-3, 3)$. Then $c = 1 \in (0, 3)$ and $g(1) = -1 = m$.
- Take $m = 1 \in (-3, 3)$. Then $c = 2 \in (0, 3)$ and $g(2) = 1 = m$.

EXAMPLE 3: Consider $h(x) = x^2 - 1$ on $[-1, 1]$.

Solution. h, polynomial $\implies$ h, continuous everywhere. Note $h(-1) = 0 = h(1)$. Thus, IVT cannot be applied to h on $[-1, 1]$.

EXAMPLE 4: Consider again $h(x) = x^2 - 1$, but on $[-1, 2]$.

Solution. h, polynomial $\implies$ h, continuous everywhere. Note $h(-1) = 0$ and $h(2) = 3$ Thus, IVT on $[-1, 2] \implies$ for all $m \in (0, 3)$, there is a $c \in (-1, 2)$, with $h(c) = m$.

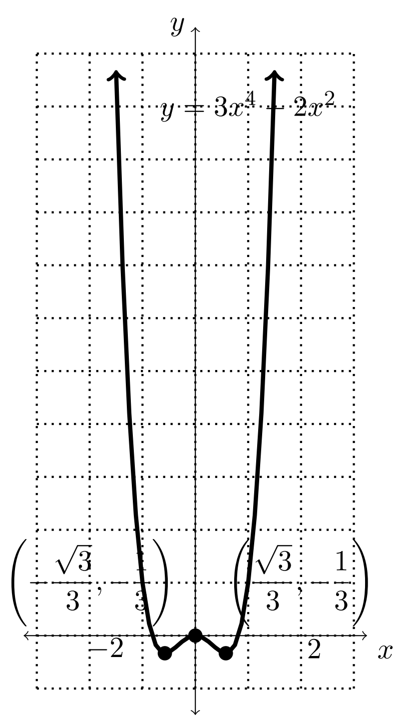
- Take $m = 1 \in (0, 3)$. Then $c = 0 \in (-1, 2)$ and $h(0) = -1 = m$.
- Take $m = 0 \in (0, 3)$. Then $c = 1 \in (-1, 2)$ and $h(1) = 0 = m$.
- Take $m = 2 \in (0, 3)$. Then $c = \sqrt{3} \in (-1, 2)$ and $h(\sqrt{3}) = 2 = m$.

EXAMPLE 5: Consider $j(x) = x^4 - 2$ on $[-2, 1]$.

Solution. j, polynomial $\implies$ j, continuous everywhere. Note $j(-2) = 14$ and $j(1) = -1$ Thus, IVT on $[-2, 1]$ $\implies$ for all $m \in (-1, 14)$, there is a $c \in (-2, 1)$, with $j(c) = m$.

- Take $m = 1 \in (-1, 14)$. Then $c = \sqrt[4]{3} \in (-2, 1)$ and $j(\sqrt[4]{3}) = 1 = m$.
- Take $m = 0 \in (-1, 14)$. Then $c = \sqrt[4]{2} \in (-2, 1)$ and $j(c) = 0 = m$.

EXAMPLE 6: Consider $f(x) = 3x^4 - 2x^2$. Its graph is the following.



Solution.

- On the interval $[-2, -\sqrt{3}]$
  - f attains its maximum at $x = -2$ $(f(-2) = 40)$
  - f attains its minimum at $x = -\dfrac{\sqrt{3}}{3}$ $\left(f\left(-\dfrac{\sqrt{3}}{3}\right) = -\dfrac{1}{3}\right)$
- On the interval $[-\sqrt{3}, 0]$
  - f attains its minimum at $x = \dfrac{\sqrt{3}}{3}$ $\left(f\left(-\dfrac{\sqrt{3}}{3}\right) = -\dfrac{1}{3}\right)$
  - f attains its maximum at $x = 0$ $(f(0) = 0)$
- On the interval $[0, \sqrt{3}]$
  - f attains its minimum at $x = \dfrac{\sqrt{3}}{3}$ $\left(f\left(\dfrac{\sqrt{3}}{3}\right) = -\dfrac{1}{3}\right)$
  - f attains its maximum at $x = 0$ $(f(0) = 0)$
- On the interval $[\sqrt{3}, 2]$
  - f attains its minimum at $x = \dfrac{\sqrt{3}}{3}$ $\left(f\left(\dfrac{\sqrt{3}}{3}\right) = -\dfrac{1}{3}\right)$
  - f attains its maximum at $x = 2$ $(f(40) = 0)$

Supplementary Problems

1. Find at least one value of $c \in I = [a, b]$ such that the IVT holds for the function f for $m \in (f(a), f(b))$.

   (a) $f(x) = x$; $I = [1, 2]$

   (b) $f(x) = x - 2$; $I = [-1, 1]$

   (c) $f(x) = -x^2$; $I = [0, 1]$

   (d) $f(x) = -x^2 - 1$; $I = [-1, 2]$

   (e) $f(x) = x^2 - x$; $I = [0, 2]$

   (f) $f(x) = -x^2 + x$; $I = [-1, 0]$

   (g) $f(x) = x^3 - 2$; $I = [-1, 0]$

   (h) $f(x) = x^4 + x$; $I = [-1, 1]$

2. Determine if the IVT is applicable to the following functions on the interval I.

(a) $f(x) = x$; $I = [1, 2]$

(b) $f(x) = x^2 + 1$; $I = [-1, 1]$

(c) $f(x) = 3 - x^3$; $I = [0, 1]$

(d) $f(x) = \dfrac{x+1}{x-1}$; $I = [0, 2]$

(e) $f(x) = \sqrt{x^2 - 1}$; $I = [-1, 0]$

(f) $f(x) = x^2 - 1$; $I = [-1, 1]$

(g) $f(x) = \sqrt{x^2 + x + 1}$; $I = [0, 2]$

(h) $f(x) = \dfrac{3x+1}{x+2}$; $I = [-1, 1]$

3. Give the maximum and minimum points of the functions on the interval I.

(a) $f(x) = x - 2$; $I = [-1, 1]$

(b) $f(x) = x^2$; $I = [-2, 2]$

(c) $f(x) = 1 - x^3$; $I = [0, 1]$

(d) $f(x) = \sin x$; $I = [-\pi, \pi]$

TOPIC 4.3: Problems Involving Continuity

For every problem that will be presented, we will provide a solution that makes use of continuity and takes advantage of its consequences, such as the Intermediate Value Theorem (IVT).

APPROXIMATING ROOTS (Method of Bisection)

Finding the roots of polynomials is easy if they are special products and thus easy to factor. Sometimes, with a little added effort, roots can be found through synthetic division. However, for most polynomials, roots, can at best, just be approximated.

Since polynomials are continuous everywhere, the IVT is applicable and very useful in approximating roots which are otherwise difficult to find. In what follows, we will always choose a closed interval $[a, b]$ such that $f(a)$ and $f(b)$ differ in sign, meaning, $f(a) > 0$ and $f(b) < 0$, or $f(a) < 0$ and $f(b) > 0$.

In invoking the IVT, we take $m = 0$. This is clearly an intermediate value of $f(a)$ and $f(b)$ since $f(a)$ and $f(b)$ differ in sign. The conclusion of the IVT now guarantees the existence of $c \in [a, b]$ such that $f(c) = 0$. This is tantamount to looking for the roots of polynomial $f(x)$.

EXAMPLE 1: Consider $f(x) = x^3 - x + 1$. Its roots cannot be found using factoring and synthetic division. We apply the IVT.

- Choose any initial pair of numbers, say -3 and 3.
- Evaluate f at these values.

$$f(-3) = -23 < 0 \text{ and } f(3) = 25 > 0.$$

  Since $f(-3)$ and $f(3)$ differ in sign, a root must lie between -3 and 3.

- To approach the root, we trim the interval.
  - Try $[0, 3]$. However, $f(0) = 1 > 0$ like $f(3)$ so no conclusion can be made about a root existing in $[0, 3]$.
  - Try $[-3, 0]$. In this case, $f(0)$ and $f(-3)$ differ in sign so we improve the search space for the root from $[-3, 3]$ to $[-3, 0]$.

- We trim further.
  - $f(-1) = 1 > 0$ so the root is in $[-3, -1]$.
  - $f(-2) = -5 < 0$ so the root is in $[-2, -1]$.
  - $f\left(-\dfrac{3}{2}\right) = -\dfrac{7}{8} < 0$ so the root is in $\left[-\dfrac{3}{2}, -1\right]$.
  - $f\left(-\dfrac{5}{4}\right) = \dfrac{19}{64} > 0$ so the root is in $\left[-\dfrac{3}{2}, -\dfrac{5}{4}\right]$.
- Further trimming and application of the IVT will yield the approximate root which is $x = -\dfrac{53}{40} = -1.325$. This gives $f(x) \approx -0.0012$.

The just-concluded procedure gave one root, a negative one. There are two more possible real roots.

FINDING INTERVALS FOR ROOTS

When finding an exact root of a polynomial, or even an approximate root, proves too tedious, some problem-solvers are content with finding a small interval containing that root

EXAMPLE 2: Consider again $f(x) = x^3 - x + 1$. If we just need an interval of length 1, we can already stop at $[-2, -1]$. If we need an interval of length $1/2$, we can already stop at $\left[-\dfrac{3}{2}, -1\right]$. If we want an interval of length $1/4$, we stop at $\left[-\dfrac{3}{2}, -\dfrac{5}{4}\right]$.

EXAMPLE 3: Consider $f(x) = x^3 - x^2 + 4$. Find three distinct intervals of length 1, or less, containing a root of $f(x)$.

When approximating, we may choose as sharp an estimate as we want. The same goes for an interval. While some problem-solvers will make do with an interval of length 1, some may want a finer interval, say, of length $1/4$. We should not forget that this type of search is possible because we are dealing with polynomials, and the continuity of polynomials everywhere allows us repeated use of the IVT.

SOME CONSEQUENCES OF THE IVT

Some interesting applications arise out of the logic used in the IVT.

EXAMPLE 4: We already know from our first lessons on polynomials that the degree of a polynomial is an indicator of the number of roots it has. Furthermore, did you know that a polynomial of odd degree has at least one real root?

Recall that a polynomial takes the form,

$$f(x) = a_0 x^n + a_1 x^{n-1} + ... + a_{n-1}x + a_n$$

where $a_0, a_1, ..., a_n$ are real numbers and n is an odd integer.

Take for example $a_0 = 1$. So,

$$f(x) = x^n + a_1 x^{n-1} + ... + a_{n-1}x + a_n.$$

Imagine x taking bigger and bigger values, like ten thousand or a million. For such values, the first term will far outweigh the total of all the other terms. See, if x is positive, for big n the value of $f(x)$ will be positive. If x is negative, for big n the value of $f(x)$ will be negative.

We now invoke the IVT. Remember, n is odd.

- Let a be a large-enough negative number. Then, $f(a) < 0$.
- Let b be a large-enough positive number. Then, $f(b) > 0$.

By the IVT, there is a number $c \in (a, b)$ such that $f(c) = 0$. In other words, $f(x)$ does have a real root!

Solved Examples

EXAMPLE 1: Find an approximation of a solution of $f(x) = x^3 + x - 1$.

Solution. Note that f is a polynomial function, continuous everywhere, so we can apply IVT for any $x \in \mathbb{R}$.

- Choose $[0, 1]$.
- $f(0) = -1 < 0$ and $f(1) = 1 > 0$. By IVT, f has a root between 0 and 1.
- Trim the interval:
  - $f\left(\frac{1}{2}\right) = -\frac{3}{8} < 0 \implies$ root is in $\left[\frac{1}{2}, 1\right]$
  - $f\left(\frac{3}{4}\right) = \frac{11}{64} > 0 \implies$ root is in $\left[\frac{1}{2}, \frac{3}{4}\right]$
  - $f\left(\frac{5}{16}\right) \approx -0.13 < 0 \implies$ root is in $\left[\frac{5}{16}, \frac{3}{4}\right]$
  - $f\left(\frac{11}{16}\right) \approx 0.01 > 0 \implies$ root is in $\left[\frac{5}{16}, \frac{11}{16}\right]$
- Further trimming gives us an approximate root $x \approx 0.683$ which gives $f(x) \approx 0.002$

EXAMPLE 2: Find an approximation of a root of $g(x) = x^3 - 2x + 1$.

Solution. Note that g is a polynomial function, continuous everywhere, so we can apply IVT for any $x \in \mathbb{R}$.

- Choose $[-2, -1]$.
- $g(-2) = -3 < 0$ and $g(-1) = 2 > 0$. By IVT, f has a root between -2 and -1.
- Trim the interval:
  - $g\left(-\frac{7}{4}\right) < 0 \implies$ root is in $\left[-\frac{7}{4}, -1\right]$
  - $g\left(-\frac{5}{4}\right) > 0 \implies$ root is in $\left[-\frac{7}{4}, -\frac{5}{4}\right]$
  - $g\left(-\frac{3}{2}\right) < 0 \implies$ root is in $\left[-\frac{3}{2}, -\frac{5}{4}\right]$
- Further trimming gives us an approximate root $x \approx -1.615$ which gives $f(x) \approx 0.01...$

EXAMPLE 3: Find three distinct intervals which contain the roots of $h(x) = x^3 - 3x + 1$.

Solution. Since h is a polynomial function, then h is continuous on $\mathbb{R}$. Note that

- $h(-2) = -1 < 0$ and $h(0) = 1 > 0 \implies h$ has a root in [-1,0];
- $h(0) = 1 > 0$ and $h(1) = -1 < 0 \implies h$ has a root in [0,1]; and
- $h(1) = -1 < 0$ and $h(2) = 3 > 0 \implies h$ has a root in [1,2].

EXAMPLE 4: Show that $j(x) = \cos(x + 1) - \sin(x - 1)$ has a root between 0 and $\dfrac{3\pi}{4}$.

Solution. Since j is continuous everywhere, we can apply IVT on $\left[0, \dfrac{3\pi}{4}\right]$. Note that $j(0) = 1.382 > 0$ and $j\left(\dfrac{3\pi}{4}\right) = -1.954 < 0$. Thus, by IVT, j has a solution in $\left[0, \dfrac{3\pi}{4}\right]$.

EXAMPLE 5: Show that $j(x) = \cos(x + 1) - \sin(x - 1)$ has a negative root greater than $\dfrac{-5\pi}{4}$.

Solution. Consider $I = \left[-\dfrac{5\pi}{4}, 0\right]$. Since j is continuous on $\mathbb{R}$, we can apply IVT on I. Now, $j\left(-\dfrac{5\pi}{4}\right) = -1.954 < 0$ and $j(0) = 1.382 > 0$ gives our desired conclusion.

EXAMPLE 6: Use IVT to show that $k(x) = x - \dfrac{1}{x}$ has a positive root less than 2.

Solution. Note that $k(0)$ is undefined. Thus, k is discontinuous at $x = 0$. Consider $[0.5, 2]$ so, $k(0.5) = -1 < 0$ and $k(2) = 0.5 > 0$. Hence, by IVT, k has a solution on [0.5,2].

Supplementary Problems

1. Approximate the roots of the following functions.

   (a) $f(x) = x^3 + x - 1$

   (b) $f(x) = x^3 + x - 3$

   (c) $f(x) = x^3 - 2x - 4$

   (d) $f(x) = x^3 - x + 2$

   (e) $f(x) = x^3 + 3x + 1$

   (f) $f(x) = x^3 - x - 2$

2. Find an interval containing a solution of the following.

   (a) $f(x) = x^3 - 3x + 5$

   (b) $f(x) = 2x^3 - x + 1$

   (c) $f(x) = x^3 - 2x - 2$

   (d) $f(x) = x^3 + x + 1$

   (e) $f(x) = x^4 - x^3 - 1$

   (f) $f(x) = 1 - x - x^4$

   (g) $f(x) = \dfrac{\sin(x+1)}{x+1}$

   (h) $f(x) = \dfrac{\cos x}{x+1}$

3. Use IVT to show the following.

   (a) $f(x) = x^2 - 4$ has a zero on [-3,-1]

   (b) $f(x) = x^2 - 81$ has a zero on [8,10]

   (c) $f(x) = \cos(x-1) - x$ has a root between 0 and 2

   (d) $f(x) = \dfrac{\sin x}{x-1}$ has a root between -4 and -2

   (e) $f(x) = \sin x - 2\cos x$ has a positive root less than 3

   (f) $f(x) = x + \cos x$ has a negative root greater than -2

CHAPTER 1 REVIEW

I. Evaluate the following limits.

1. $\lim\limits_{x \to 0} x^4 - 3x^2 + 1$
2. $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x - 1}$
3. $\lim\limits_{x \to -1} \dfrac{\sqrt{x^2 - 1}}{x + 1}$

II. Given

$$f(x) = \begin{cases} -\dfrac{|x+6|}{x}, & x \in (-\infty, -5], \\ \dfrac{x^2 + 8x + 16}{x^2 + 4x}, & x \in (-5, +\infty). \end{cases}$$

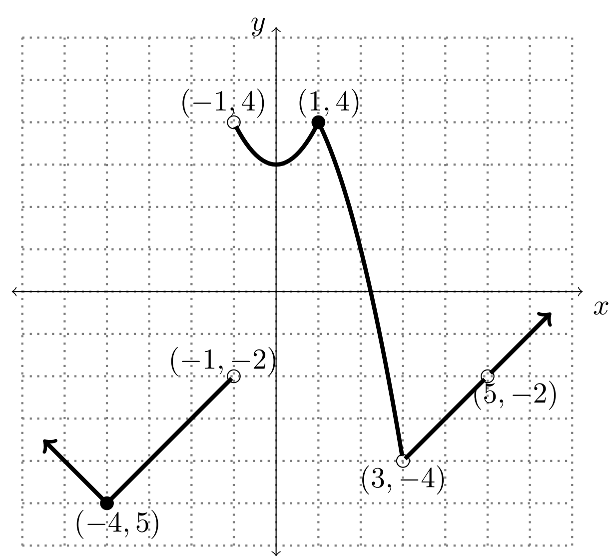
Discuss continuity of f at $x = -5$ and $x = -4$. Classify each discontinuity, if any.

III. Do as indicated.

1. Use the Intermediate Value Theorem to show that the function $f(x) = x^2 - x - 5$ has a zero between 2 and 3.
2. Given that $\lim\limits_{x \to 1} \sin(\pi x) = 0$ and $\lim\limits_{x \to 1} e^{x-1} = 1$ Determine $\lim\limits_{x \to 1} \dfrac{\sin(\pi x)}{e^{x-1}}$.
3. Identify the points where the function $g(x) = x^4 - 1$ will attain its maximum and minimum values on $[-1, 1]$.

IV. From the graph of the function h given below, evaluate the following limits.

1. $\lim\limits_{x \to -4} h(x)$
2. $\lim\limits_{x \to -1} h(x)$
3. $\lim\limits_{x \to 1} h(x)$
4. $\lim\limits_{x \to 3} h(x)$
5. $\lim\limits_{x \to 5} h(x)$



Chapter 2

Derivatives

LESSON 5: The Derivative as the Slope of the Tangent Line

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Illustrate the tangent line to the graph of a function at a given point;
2. Apply the definition of the derivative of a function at a given number; and
3. Relate the derivative of a function to the slope of the tangent line.

TOPIC 5.1: The Tangent Line to the Graph of a Function at a Point

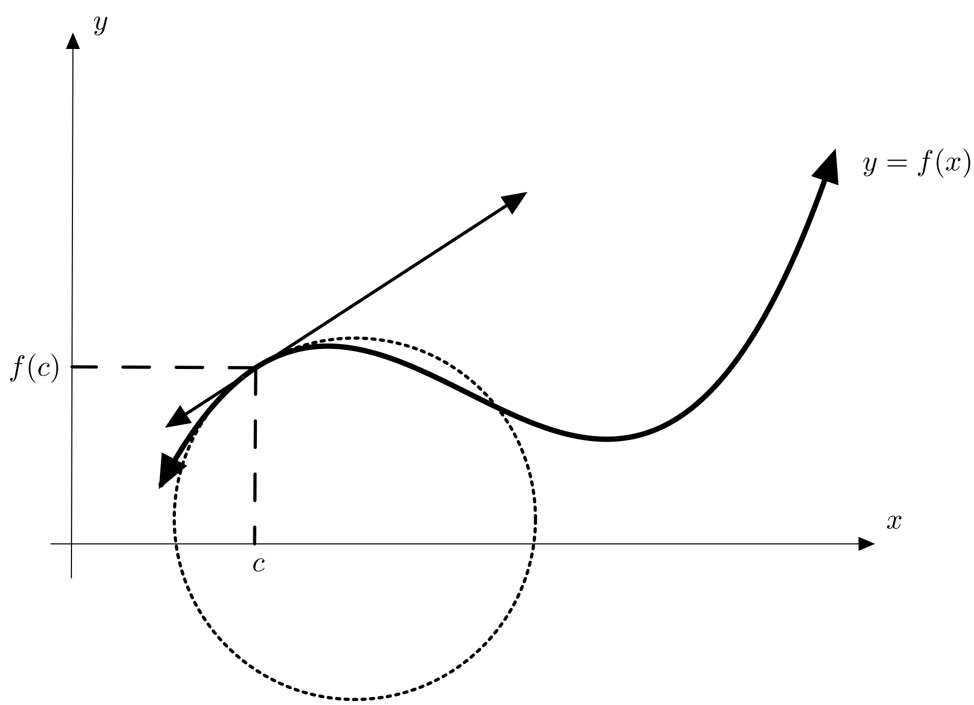
TANGENT LINES TO CIRCLES

- Recall from geometry that a **tangent line to a circle** centered at O is a line intersecting the circle at exactly one point. It is found by constructing the line, through a point A on the circle, that is perpendicular to the segment (radius) $\overline{OA}$.

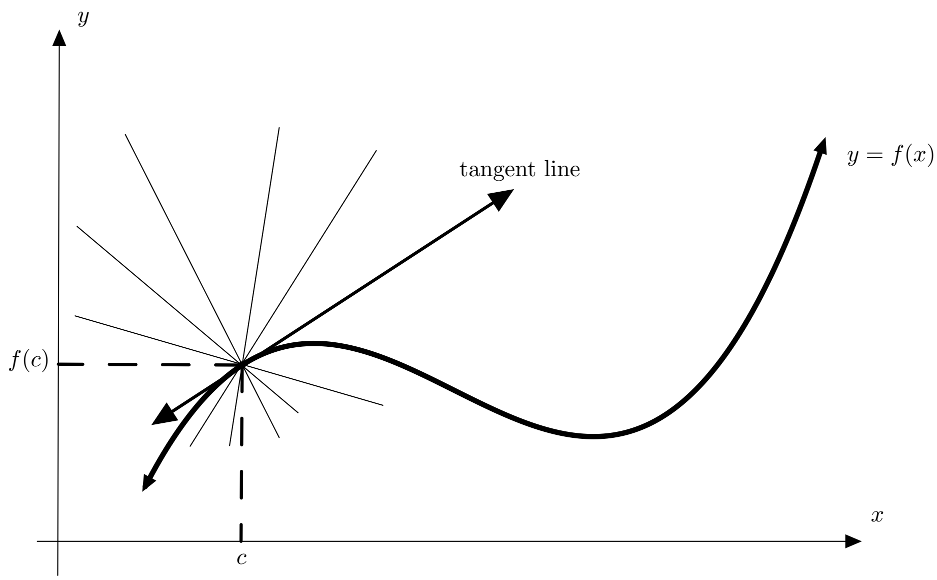
- A **secant line to a circle** is a line intersecting the circle at two points.

HOW TO DRAW TANGENT LINES TO CURVES AT A POINT

The definition of a tangent line is not very easy to explain without involving limits. One can imagine that locally, the curve looks like an arc of a circle. Hence, we can draw the tangent line to the curve as we would to a circle.

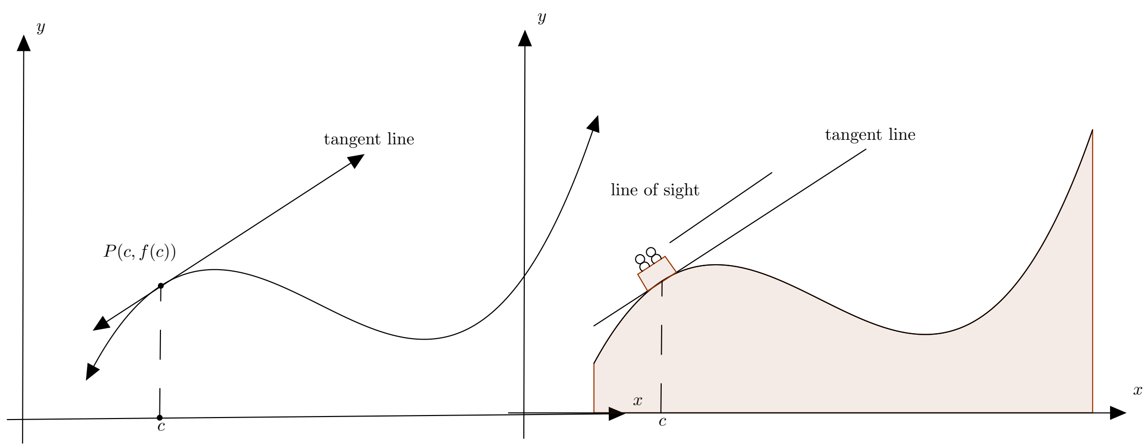


One more way to see this is to choose the line through a point that locally looks most like the curve. Among all the lines through a point $(c, f(c))$, the one which best approximates the curve $y = f(x)$ near the point $(c, f(c))$ is the **tangent line** to the curve at that point.

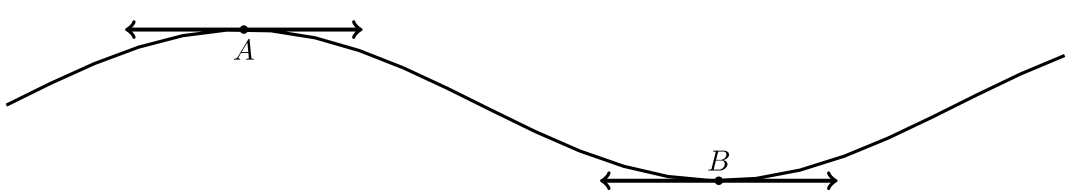


Among all lines passing through $(c, f(c))$, the tangent line is the closest to the curve locally.

Another way of qualitatively understanding the tangent line is to visualize the curve as a roller coaster. The tangent line to the curve at a point is parallel to the line of sight of the passengers looking straight ahead and sitting erect in one of the wagons of the roller coaster.

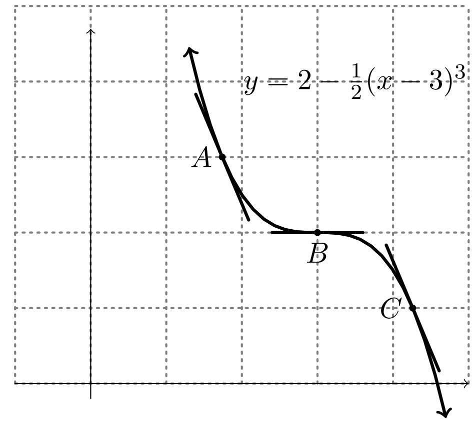


EXAMPLE 1: What do you think are the tangent lines at the "peaks" and "troughs" of a smooth curve?



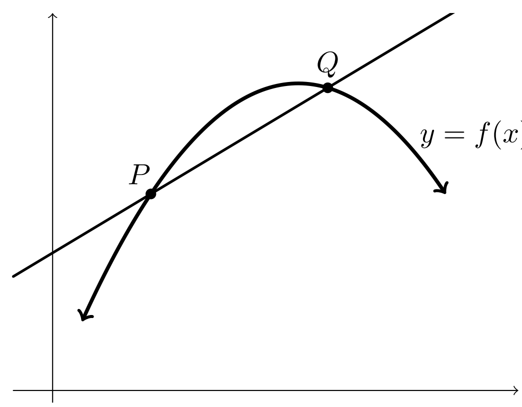
Whenever the graph is smooth (meaning, there are no sharp corners), the tangent lines at the "peaks" and "troughs" are always **horizontal**.

EXAMPLE 2: The following is the graph of $y = 2 - \frac{1}{2}(x-3)^3$. Drawn are the tangent lines at each of the given points A, B, and C.



THE TANGENT LINE DEFINED MORE FORMALLY

The precise definition of a tangent line relies on the notion of a *secant line.* Let C be the graph of a continuous function $y = f(x)$ and let P be a point on C. A secant line to $y = f(x)$ through P is any line connecting P and another point Q on C. In the figure on the right, the line $\overleftrightarrow{PQ}$ is a secant line of $y = f(x)$ through P.

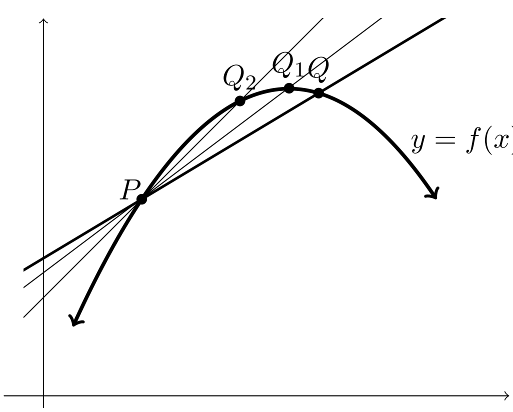


We now construct the tangent line to $y = f(x)$ at P.

Choose a point Q on the right side of P, and connect the two points to construct the secant line $\overleftrightarrow{PQ}$.

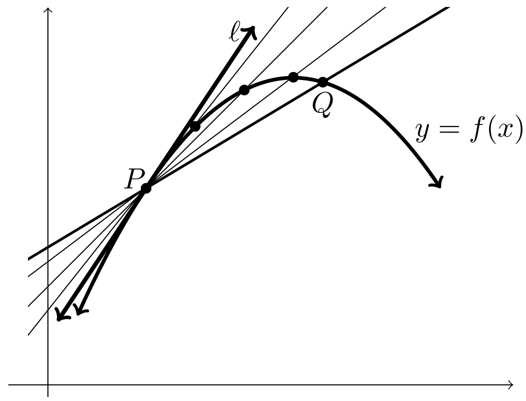
Choose another point Q_1 in between P and Q. Connect the two points P and Q_1 to construct the secant line $\overleftrightarrow{PQ_1}$.

Choose another point Q_2 in between P and Q_1. Construct the secant line $\overleftrightarrow{PQ_2}$.



Consider also the case when Q is to the left of P and perform the same process. Intuitively, we can define the tangent line through P to be the limiting position of the secant lines $\overleftrightarrow{PQ}$ as the point Q (whether to the left or right of P) approaches P.

If the sequence of secant lines to the graph of $y = f(x)$ through P approaches one limiting position (in consideration of points Q to the left and from the right of P), then we define this line to be the tangent line to $y = f(x)$ at P.



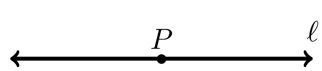
We summarize below the definitions of the secant line through a point, and the tangent line at a point of the graph of $y = f(x)$.

Definition

Let C be the graph of a continuous function $y = f(x)$ and let P be a point on C.

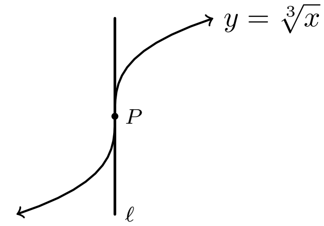
1. A **secant line** to $y = f(x)$ through P is any line connecting P and another point Q on C.
2. The **tangent line** to $y = f(x)$ at P is the limiting position of all secant lines $\overleftrightarrow{PQ}$ as $Q \to P$.

EXAMPLE 3: The tangent line to another line at any point is the line itself. (This debunks the idea that a tangent line touches the graph at only one point!) Indeed, let ℓ be a line and let P be on ℓ. Observe that no matter what point Q on ℓ we take, the secant line $\overleftrightarrow{PQ}$ is ℓ itself. Hence, the limiting position of a line ℓ is ℓ itself.



The tangent line to ℓ at any point P is itself!

EXAMPLE 4: Our definition of the tangent line allows for a vertical tangent line. We have seen this on the unit circle at points $(1, 0)$ and $(-1, 0)$. A vertical tangent line may also exist even for continuous functions. Draw the curve $y = \sqrt[3]{x}$ and mark the point $P(0, 0)$. Allow the class to determine the tangent line to the graph at P using the formal definition. Consider the two cases: when Q is to the right of P and when Q is to the left of P.

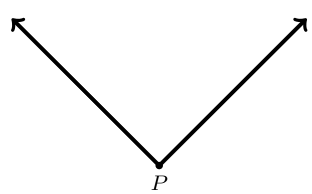


CURVES THAT DO NOT HAVE TANGENT LINES

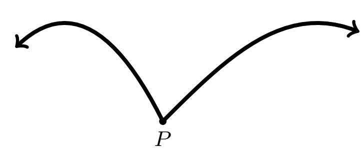
It is possible that the tangent line to a graph of a function at a point $P(x_0, f(x_0))$ does not exist. There are only two cases when this happens:

1. *The case when the function is not continuous at* x_0: It is clear from the definition of the tangent line that the function must be continuous.
2. *The case when the function has a sharp corner/cusp at* P: This case produces different limiting positions of the secant lines $\overline{PQ}$ depending on whether Q is to the left or to the right of P.

Remark 1: The word "sharp corner" is more commonly used for joints where only lines are involved. For example, the absolute value function $y = |x|$ has a sharp corner at the origin. In contrast, the term "cusp" is often used when at least one graph involved represents a nonlinear function. See the graphs below.



corner at P



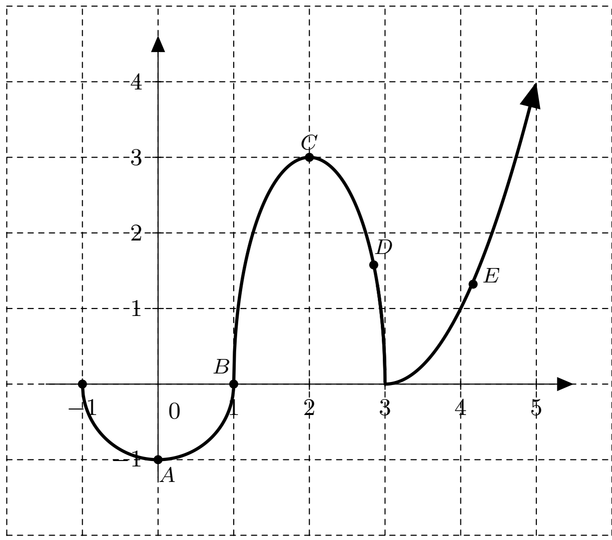
cusp at P

In the above examples, each has a sharp corner/cusp at P. Choosing Q to be points to the left of P produces a different limiting position than from choosing Q to the right of P. Since the two limiting positions do not coincide, then the tangent line at P does not exist. (This is the same thing that happens when the limit from the left of c differs from the limit from the right of c, where we then conclude that the limit does not exist.)

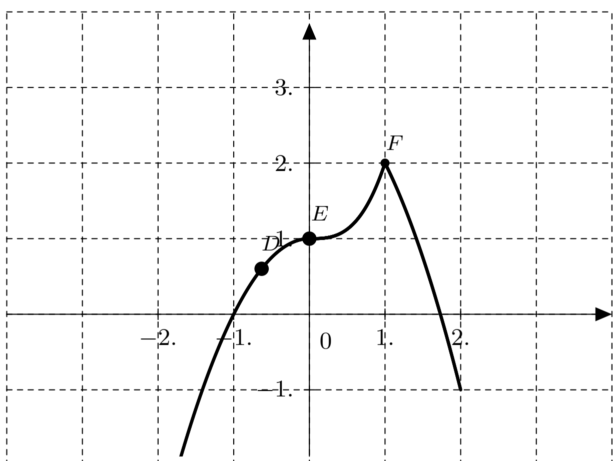
Supplementary Problems

Construct tangent lines at the labeled points.

1.



2.



TOPIC 5.2: The Equation of the Tangent Line

> **Recall: Slope of a Line**
> A line ℓ passing through distinct points (x_0, y_0) and (x, y) has slope
> $$m_\ell = \frac{y - y_0}{x - x_0}.$$

EXAMPLE 1: Given $A(1, -3)$, $B(3, -2)$, and $C(-1, 0)$, what are the slopes of the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$?

Solution. The slope of $\overleftrightarrow{AB}$ is

$$m_{\overleftrightarrow{AB}} = \frac{-2 - (-3)}{3 - 1} = \frac{1}{2}.$$

The slope of $\overleftrightarrow{AC}$ is

$$m_{\overleftrightarrow{AC}} = \frac{0 - (-3)}{-1 - 1} = -\frac{3}{2}.$$

The slope of $\overleftrightarrow{BC}$ is

$$m_{\overleftrightarrow{BC}} = \frac{0 - (-2)}{-1 - 3} = -\frac{2}{4} = -\frac{1}{2}.$$

> **Recall: Point-Slope Form**
> The line passing through (x_0, y_0) with slope m has the equation
> $$y - y_0 = m(x - x_0).$$

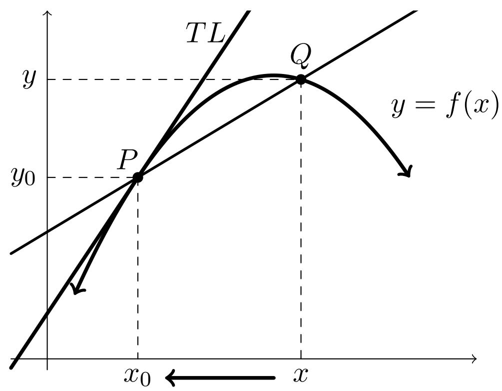
EXAMPLE 2: From Example 1 above, since $m_{\overleftrightarrow{AB}} = \frac{1}{2}$, then using $A(1, -3)$ as our point, then the point-slope form of the equation of $\overleftrightarrow{AB}$ is

$$y - (-3) = \frac{1}{2}(x - 1) \quad \text{or} \quad y + 3 = \frac{1}{2}(x - 1).$$

THE EQUATION OF THE TANGENT LINE

Given a function $y = f(x)$, how do we find the equation of the tangent line at a point $P(x_0, y_0)$?

Consider the graph of a function $y = f(x)$ whose graph is given below. Let $P(x_0, y_0)$ be a point on the graph of $y = f(x)$. Our objective is to find the equation of the tangent line (TL) to the graph at the point $P(x_0, y_0)$.



- Find any point $Q(x, y)$ on the curve.
- Get the slope of this secant line $\overleftrightarrow{PQ}$.

$$m_{\overleftrightarrow{PQ}} = \frac{y - y_0}{x - x_0}.$$

- Observe that **letting Q approach P is equivalent to letting x approach x_0.**

  We use the formal definition of the tangent line.

- Since the tangent line is the limiting position of the secant lines as Q approaches P, it follows that the slope of the tangent line (TL) at the point P is the limit of the slopes of the secant lines $\overleftrightarrow{PQ}$ as x approaches x_0. In symbols,

$$m_{TL} = \lim_{x \to x_0} \frac{y - y_0}{x - x_0} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

- Finally, since the tangent line passes through $P(x_0, y_0)$, then its equation is given by

$$y - y_0 = m_{TL}(x - x_0).$$

Equation of the Tangent Line

To find the equation of the tangent line to the graph of $y = f(x)$ at the point $P(x_0, y_0)$, follow this 2-step process:

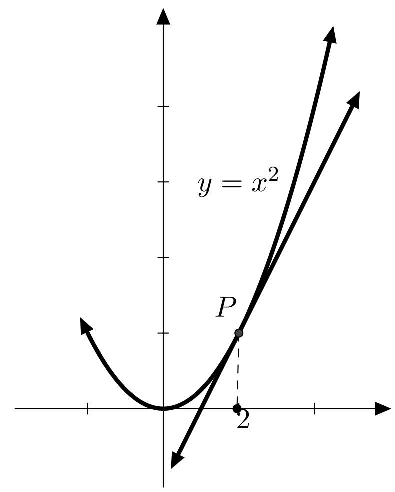
- Get the slope of the tangent line by computing

$$m = \lim_{x \to x_0} \frac{y - y_0}{x - x_0} \quad \text{or} \quad m = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

- Substitute this value of m and the coordinates of the known point $P(x_0, y_0)$ into the point-slope form to get

$$y - y_0 = m(x - x_0).$$

EXAMPLE 3: Find the equation of the tangent line to $y = x^2$ at $x = 2$.



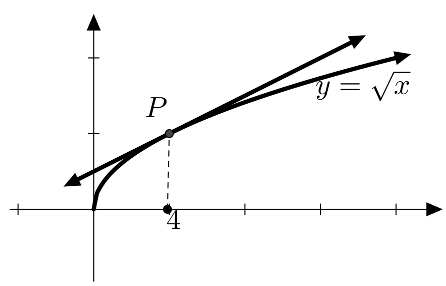
Solution. To get the equation of the line, we need the point $P(x_0, y_0)$ and the slope m. We are only given $x_0 = 2$. However, the y-coordinate of x_0 is easy to find by substituting $x_0 = 2$ into $y = x^2$. This gives us $y_0 = 4$. Hence, P has the coordinates $(2, 4)$. Now, we look for the slope:

$$\lim_{x \to x_0} \frac{y - y_0}{x - x_0} = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4.$$

Finally, the equation of the tangent line with slope $m = 4$ and passing through $P(2, 4)$ is

$$y - 4 = 4(x - 2) \quad \text{or} \quad y = 4x - 4.$$

EXAMPLE 4: Find the slope-intercept form of the tangent line to $f(x) = \sqrt{x}$ at $x = 4$.



Solution. Again, we find the y-value corresponding to $x_0 = 4$: $y_0 = f(x_0) = \sqrt{x_0} = \sqrt{4} = 2$. Hence, P has coordinates $(4, 2)$. Now, we look for the slope of the tangent line. Notice that we have to rationalize the numerator to evaluate the limit.

$$\begin{aligned} m = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} &= \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} \cdot \frac{\sqrt{x} + 2}{\sqrt{x} + 2} \\ &= \lim_{x \to 4} \frac{x - 4}{(x - 4)(\sqrt{x} + 2)} \\ &= \lim_{x \to 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{4}. \end{aligned}$$

Finally, with point $P(4, 2)$ and slope $m = \frac{1}{4}$, the equation of the tangent line is

$$y - 2 = \frac{1}{4}(x - 4) \quad \text{or} \quad y = \frac{x}{4} + 1.$$

The next example shows that our process of finding the tangent line works even for horizontal lines.

EXAMPLE 5: Show that the tangent line to $y = 3x^2 - 12x + 1$ at the point $(2, -11)$ is horizontal.

Solution. Computing for the slope, we get:

$$\begin{aligned} m = \lim_{x\to x_0} \frac{y-y_0}{x-x_0} &= \lim_{x\to 2} \frac{(3x^2-12x+1)-(-11)}{x-2} \\ &= \lim_{x\to 2} \frac{3(x^2-4x+4)}{x-2} \\ &= \lim_{x\to 2} (3(x-2)) = 0. \end{aligned}$$

Recall that a horizontal line has zero slope. Since the slope of the tangent line is 0, it must be horizontal. Its equation is

$$y - (-11) = 0(x-2) \quad \text{or} \quad y = -11.$$

EXAMPLE 6: Verify that the tangent line to the line $y = 2x + 3$ at $(1, 5)$ is the line itself.

Solution. We first compute for the slope of the tangent line. Note that $x_0 = 1$ and $y_0 = 5$.

$$m = \lim_{x\to x_0} \frac{y-y_0}{x-x_0} = \lim_{x\to 1} \frac{(2x+3)-5}{x-1} = \lim_{x\to 1} \frac{2x-2}{x-1} = 2.$$

Therefore, substituting this into the point-slope form with $P(1, 5)$ and $m = 2$, we get

$$y - 5 = 2(x-1) \quad \text{or,} \quad y = 2x + 3.$$

This is the same equation as that of the given line.

Solved Examples

EXAMPLE 1: Given $A\,(2, -1)$, $B\,(0, 5)$, and $C\,(-3, 1)$, what are the slopes of the segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$?

Solution. The slope of $\overline{AB}$ is

$$m_{\overline{AB}} = \frac{5-(-1)}{0-2} = \frac{6}{-2} = -3.$$

The slope of $\overline{BC}$ is

$$m_{\overline{BC}} = \frac{1-5}{-3-0} = \frac{-4}{-3} = \frac{4}{3}.$$

The slope of $\overline{AC}$ is

$$m_{\overline{AC}} = \frac{1-(-1)}{-3-2} = \frac{2}{-5} = -\frac{2}{5}.$$

EXAMPLE 2: Find the equation of the line containing segment $\overline{BC}$.

Solution. The slope of this line was already solved in the previous example, i.e., $m_{\overline{BC}} = \frac{4}{3}$. Since the line passes through point $B\,(0,5)$, applying the point-slope form, we get

$$y - 5 = \frac{4}{3}\,(x-0) \quad \text{or} \quad y = \frac{4}{3}x + 5.$$

One also gets the same equation if the point $C(-3,1)$ is used instead.

EXAMPLE 3: Show that the equation of the tangent line to the line $y = 1-2x$ at $(1,-1)$ is itself.

Solution. We first compute for the slope of the tangent line at the given point. Here, we have $x_0 = 1$ and $y_0 = -1$.

$$m = \lim_{x\to x_0} \frac{y-y_0}{x-x_0} = \lim_{x\to 1} \frac{(1-2x)-(-1)}{x-1} = \lim_{x\to 1} \frac{2-2x}{x-1} = \lim_{x\to 1} \frac{-2\,(x-1)}{x-1} = -2.$$

Thus, substituting this in the point-slope form with $P\,(1,-1)$ and $m = 2$, we get

$$y-(-1) = -2\,(x-1) \quad \text{or} \quad y = -2x+1 = 1-2x.$$

EXAMPLE 4: Find the equation of the tangent line to the curve $y = 2x^2 - 1$ at $x = 2$.

Solution. We first need to get the coordinates of the point of tangency $P(x_0, y_0)$, and the slope at $P(x_0, y_0)$. In this problem, $x_0 = 2$. Therefore, $y_0 = 2\,(2)^2 - 1 = 7$. Thus, P has coordinates

$(2,7)$. We now compute for the slope:

$$\begin{aligned} m = \lim_{x \to x_0} \frac{y - y_0}{x - x_0} &= \lim_{x \to 2} \frac{\left(2x^2 - 1\right) - 7}{x - 2} \\ &= \lim_{x \to 2} \frac{2x^2 - 8}{x - 2} \\ &= \lim_{x \to 2} \frac{2\left(x^2 - 4\right)}{x - 2} \\ &= \lim_{x \to 2} \frac{2\left(x + 2\right)\left(x - 2\right)}{x - 2} \\ &= \lim_{x \to 2} 2\left(x + 2\right) \\ &= 8. \end{aligned}$$

The equation of the tangent line with slope $m = 8$ passing through $P\left(2, 7\right)$ is

$$y - 7 = 8\left(x - 2\right) \quad \text{or} \quad y = 8x - 9.$$

EXAMPLE 5: Find the equation of the tangent line to the curve $f\left(x\right) = \sqrt{3x - 1}$ at $x = \dfrac{5}{3}$.

Solution. The y-coordinate of the point on the graph when $x_0 = \dfrac{5}{3}$ is $y_0 = f\left(x_0\right) = \sqrt{3\left(\dfrac{5}{3}\right) - 1} = \sqrt{5 - 1} = \sqrt{4} = 2$. We now find the slope of the tangent line at $\left(\dfrac{5}{3}, 2\right)$:

$$\begin{aligned} m = \lim_{x \to x_0} \frac{f\left(x\right) - f\left(x_0\right)}{x - x_0} &= \lim_{x \to \frac{5}{3}} \frac{\sqrt{3x - 1} - 2}{x - \frac{5}{3}} \cdot \frac{\sqrt{3x - 1} + 2}{\sqrt{3x - 1} + 2} \\ &= \lim_{x \to \frac{5}{3}} \frac{3x - 1 - 4}{\left(x - \frac{5}{3}\right)\left(\sqrt{3x - 1} + 2\right)} \\ &= \lim_{x \to \frac{5}{3}} \frac{3\left(x - \frac{5}{3}\right)}{\left(x - \frac{5}{3}\right)\left(\sqrt{3x - 1} + 2\right)} \\ &= \frac{3}{\sqrt{3\left(\frac{5}{3}\right) - 1} + 2} \\ &= \frac{3}{4}. \end{aligned}$$

The equation of the tangent line with slope $m = \dfrac{3}{4}$ passing through $\left(\dfrac{5}{3}, 2\right)$ is

$$y - 2 = \frac{3}{4}\left(x - \frac{5}{3}\right).$$

EXAMPLE 6: Show that the tangent line to the curve $y = 2x^2 - 12x + 19$ at the point $(3, 1)$ is horizontal.

Solution. We need to show that the slope of the tangent line is 0.

$$\begin{aligned} m = \lim_{x \to x_0} \frac{y - y_0}{x - x_0} &= \lim_{x \to 3} \frac{\left(2x^2 - 12x + 19\right) - 1}{x - 3} \\ &= \lim_{x \to 3} \frac{2x^2 - 12x + 18}{x - 3} \\ &= \lim_{x \to 3} \frac{2\left(x - 3\right)^2}{x - 3} \\ &= \lim_{x \to 3} 2\left(x - 3\right) \\ &= 0. \end{aligned}$$

Since the slope is 0, the tangent line at $(3, 1)$ is horizontal.

EXAMPLE 7: Show that the tangent line to the curve $f\left(x\right) = \sqrt{x - 1}$ at $x = 1$ is vertical. What is the equation of this line?

Solution. Observe that

$$\begin{aligned} m &= \lim_{x \to x_0} \frac{f\left(x\right) - f\left(x_0\right)}{x - x_0} \\ &= \lim_{x \to 1} \frac{\sqrt{x - 1} - \sqrt{1 - 1}}{x - 1} \\ &= \lim_{x \to 1} \frac{\sqrt{x - 1}}{x - 1} \cdot \frac{\sqrt{x - 1}}{\sqrt{x - 1}} \\ &= \lim_{x \to 1} \frac{x - 1}{\left(x - 1\right)\sqrt{x - 1}} \\ &= \lim_{x \to 1} \frac{1}{\sqrt{x - 1}} \end{aligned}$$

This limit assumes the form $\dfrac{1}{0}$ and is thus undefined. In this case, the tangent line is vertical with equation $x = 1$.

Supplementary Problems

Find the standard (slope-intercept form) equation of the tangent line to the following:

1. $f(x) = x^2 - 3x + 1$ at $x = 1$
2. $f(x) = \frac{5}{2}x - \frac{2}{3}$ at $x = -1$
3. $f(x) = x^3 + 8$ at $x = -2$
4. $f(x) = 2x^2 - 3x + 4$ at $x = 5$
5. $f(x) = 1 - 2x - x^2$ at $x = -1$
6. $f(x) = x^3 + 3x^2 + 3x + 1$ at $x = -1$
7. $f(x) = x^3 - x^2 + 1$ at $x = 1$
8. $f(x) = x^4 - 3x^3 + 5x^2 - 2x + 1$ at $x = -1$
9. $f(x) = x^5 - x^4 + x^3 - x^2 + x + 1$ at $x = 1$
10. $f(x) = 6x^2 - 5x + 1$ at $x = \frac{1}{2}$
11. $f(x) = \sqrt{x+2}$ at $x = 4$
12. $f(x) = \sqrt{2-3x}$ at $x = -2$
13. $f(x) = \sqrt{1+2x}$ at $x = 3$
14. $f(x) = \sqrt{5-9x^2}$ at $x = -\frac{1}{3}$
15. $f(x) = \sqrt{x^2+2x}$ at $x = 0$
16. $f(x) = \sqrt{3x^2+5}$ at $x = 1$
17. $f(x) = \sqrt[3]{x^3+1}$ at $x = 1$
18. $f(x) = \sqrt[3]{1-x^3}$ at $x = -1$
19. $f(x) = \sqrt[3]{x^3+3x^2+3x+1}$ at $x = 3$
20. $f(x) = \sqrt{x+1} - (x+1)$ at $x = 0$

TOPIC 5.3: The Definition of the Derivative

Definition of the Derivative

Let f be a function defined on an open interval $I \subseteq \mathbb{R}$, and let $x_0 \in I$. The derivative of f at x_0 is defined to be

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

if this limit exists. That is, the derivative of f at x_0 is the slope of the tangent line at $(x_0, f(x_0))$, if it exists.

Notations: If $y = f(x)$, the derivative of f is commonly denoted by

$$f'(x),\ D_x[f(x)],\ \frac{d}{dx}[f(x)],\ \frac{d}{dx}[y],\ \frac{dy}{dx}.$$

Remark 1: Note that the limit definition of the derivative is inherently *indeterminate*! Hence, the usual techniques for evaluating limits which are indeterminate of type $\frac{0}{0}$ are applied, e.g., **factoring**, **rationalization**, or **using one of the following established limits**:

(i) $\displaystyle\lim_{x \to 0} \frac{\sin x}{x} = 1$ (ii) $\displaystyle\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$ (iii) $\displaystyle\lim_{x \to 0} \frac{e^x - 1}{x} = 1.$

EXAMPLE 1: Compute $f'(1)$ for each of the following functions:

1. $f(x) = 3x - 1$
2. $f(x) = 2x^2 + 4$
3. $f(x) = \dfrac{2x}{x+1}$
4. $f(x) = \sqrt{x+8}$

Solution. Here, x_0 is fixed to be equal to 1. Using the limit definition of the derivative,

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}.$$

Remember that what we are computing, $f'(1)$, is just the slope of the tangent line to $y = f(x)$ at $x = 1$.

1. Note that $f(1) = 2$, so by factoring

$$\begin{aligned} f'(1) &= \lim_{x \to 1} \frac{(3x-1)-2}{x-1} \\ &= \lim_{x \to 1} \frac{3(x-1)}{x-1} \\ &= \lim_{x \to 1} 3 \\ &= 3. \end{aligned}$$

2. Here, $f(1) = 6$ so again, by factoring,

$$\begin{aligned} f'(1) &= \lim_{x \to 1} \frac{(2x^2+4)-6}{x-1} \\ &= \lim_{x \to 1} \frac{2(x+1)(x-1)}{x-1} \\ &= \lim_{x \to 1} 2(x+1) \\ &= 4. \end{aligned}$$

3. We see that $f(1) = 1$. So, from the definition,

$$f'(1) = \lim_{x \to 1} \frac{\frac{2x}{x+1} - 1}{x-1}.$$

We multiply both the numerator and the denominator by $x + 1$ to simplify the complex fraction:

$$\begin{aligned} f'(1) &= \lim_{x \to 1} \frac{\frac{2x}{x+1} - 1}{x-1} \cdot \frac{x+1}{x+1} \\ &= \lim_{x \to 1} \frac{2x-(x+1)}{(x-1)(x+1)} \\ &= \lim_{x \to 1} \frac{x-1}{(x-1)(x+1)} \\ &= \lim_{x \to 1} \frac{1}{x+1} = \frac{1}{2}. \end{aligned}$$

4. Note that $f(1) = 3$. Therefore, by rationalizing the numerator (meaning, multiplying by $\sqrt{x+8} + 3$),

$$\begin{aligned} f'(1) &= \lim_{x \to 1} \frac{\sqrt{x+8}-3}{x-1} \cdot \frac{\sqrt{x+8}+3}{\sqrt{x+8}+3} \\ &= \lim_{x \to 1} \frac{(x+8)-9}{(x-1)(\sqrt{x+8}+3)} \\ &= \lim_{x \to 1} \frac{1}{\sqrt{x+8}+3} \\ &= \frac{1}{6}. \end{aligned}$$

Alternative Definition of the Derivative

Let f be a function defined on an open interval $I \subseteq \mathbb{R}$, and let $x \in I$. The derivative of f at x is defined to be

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}, \tag{2.1}$$

if this limit exists.

EXAMPLE 2: Let $f(x) = \sin x$, $g(x) = \cos x$, and $s(x) = e^x$. Find $f'(2\pi)$, $g'(\pi)$, and $s'(3)$.

Solution. We use the alternative definition of the derivative.

1. Here, we substitute $x_0 = 2\pi$.

$$\begin{aligned} f'(2\pi) &= \lim_{h \to 0} \frac{f(2\pi + h) - f(2\pi)}{h} \\ &= \lim_{h \to 0} \frac{\sin(2\pi + h) - 0}{h}. \end{aligned}$$

Using the sum identity of the sine function: $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, and noting that $\sin(2\pi) = 0$ and $\cos(2\pi) = 1$, we get

$$\begin{aligned} f'(2\pi) &= \lim_{h \to 0} \frac{\sin(2\pi)\cos h + \cos(2\pi)\sin h}{h} \\ &= \lim_{h \to 0} \frac{\sin h}{h} \\ &= 1. \qquad \text{(Why?)} \end{aligned}$$

2.

$$\begin{aligned} g'(\pi) &= \lim_{h \to 0} \frac{g(\pi + h) - g(\pi)}{h} \\ &= \lim_{h \to 0} \frac{\cos(\pi + h) - (-1)}{h}. \end{aligned}$$

Using the sum identity of the cosine function: $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, and noting that $\cos\pi = -1$ and $\sin\pi = 0$, we get

$$\begin{aligned} g'(\pi) &= \lim_{h \to 0} \frac{\cos\pi\cos h - \sin\pi\sin h + 1}{h} \\ &= \lim_{h \to 0} \frac{-\cos h + 1}{h} \\ &= 0. \qquad \text{(Why?)} \end{aligned}$$

3.

$$
\begin{aligned}
s'(3) &= \lim_{h\to 0} \frac{s(3+h)-s(3)}{h} \\
&= \lim_{h\to 0} \frac{e^{3+h}-e^3}{h}.
\end{aligned}
$$

Using the exponent laws, $e^{3+h} = e^3 e^h$. Moreover, since e^3 is just a constant, we can factor it out of the limit operator. So,

$$
\begin{aligned}
s'(3) &= \lim_{h\to 0} \frac{e^3 e^h - e^3}{h} \\
&= e^3 \lim_{h\to 0} \frac{e^h - 1}{h} \\
&= e^3. \qquad\qquad \text{(Why?)}
\end{aligned}
$$

INSTANTANEOUS VELOCITY OF A PARTICLE IN RECTILINEAR MOTION

The derivative of a function is also interpreted as the instantaneous rate of change. We discuss here a particular quantity which is important in physics – the instantaneous velocity of a moving particle.

Suppose that an object or a particle starts from a fixed point A and moves along a straight line towards a point B. Suppose also that its position along line $\overleftrightarrow{AB}$ at time t is s. Then the motion of the particle is completely described by the ***position function***

$$s = s(t), \quad t \geq 0$$

and since the particle moves along a line, it is said to be in ***rectilinear motion***.

EXAMPLE 3: Suppose that a particle moves along a line with position function $s(t) = 2t^2 + 3t + 1$ where s is in meters and t is in seconds.

a. What is its initial position?

b. Where is it located after $t = 2$ seconds?

c. At what time is the particle at position $s = 6$?

Solution.

a. The initial position corresponds to the particle's location when $t = 0$. Thus,

$$s(0) = 2(0)^2 + 3(0) + 1 = 1.$$

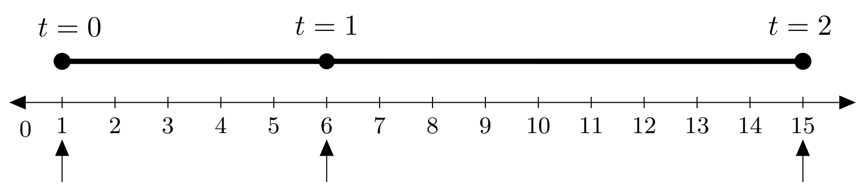
This means that the particle can be found 1 meter to the right of the origin.

b. After 2 seconds, it can now be found at position $s(2) = 2(2)^2 + 3(2) + 1 = 15$ meters.

c. We equate $s(t) = 2t^2 + 3t + 1 = 6$. So,

$$2t^2 + 3t - 5 = 0 \iff (2t+5)(t-1) = 0 \iff t = -\frac{5}{2} \text{ or } t = 1.$$

Since time cannot be negative, we choose $t = 1$ second.



Now, we ask: What is the particle's velocity at the instant when time $t = 1$?

Recall that the formula for the ***average velocity*** of a particle is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time elapsed}} = \frac{s(t_{\text{final}}) - s(t_{\text{initial}})}{t_{\text{final}} - t_{\text{initial}}}.$$

This poses a problem because at the instant when $t = 1$, there is no elapsed time since no $t_{\text{final}} is given.$ We remedy this by computing the velocity at short time intervals with an endpoint at $t = 1$.

For example, on the time interval $[1, 2]$, the velocity of the particle is

$$v = \frac{s(2) - s(1)}{2 - 1} = \frac{15 - 6}{2 - 1} = 9 \text{ m/s}.$$

We compute the particle's velocity on shorter intervals:

Time Interval	Average Velocity
$[1, 1.5]$	8
$[1, 1.1]$	7.2
$[1, 1.01]$	7.02
$[1, 1.001]$	7.002

Time Interval	Average Velocity
$[0.5, 1]$	6
$[0.9, 1]$	6.8
$[0.99, 1]$	6.98
$[0.999, 1]$	6.998

We see from the tables above that the velocities of the particle on short intervals ending or starting at $t = 1$ approach 7 m/s as the lengths of the time intervals approach 0. This limit

$$\lim_{t \to 1} \frac{s(t) - s(1)}{t - 1}$$

is what we refer to as the ***instantaneous velocity*** of the particle at $t = 1$. However, the limit expression above is precisely the definition of the derivative of s at $t = 1$, and the instantaneous velocity is actually the slope of the tangent line at the point $t = t_0$ if the function in consideration is the position function. We make the connection below.

Instantaneous velocity

Let $s(t)$ denote the position of a particle that moves along a straight line at each time $t \geq 0$. The instantaneous velocity of the particle at time $t = t_0$ is

$$s'(t_0) = \lim_{t \to t_0} \frac{s(t) - s(t_0)}{t - t_0},$$

if this limit exists.

EXAMPLE 4: A ball is shot straight up from a building. Its height (in meters) from the ground at any time t (in seconds) is given by $s(t) = 40 + 35t - 5t^2$. Find

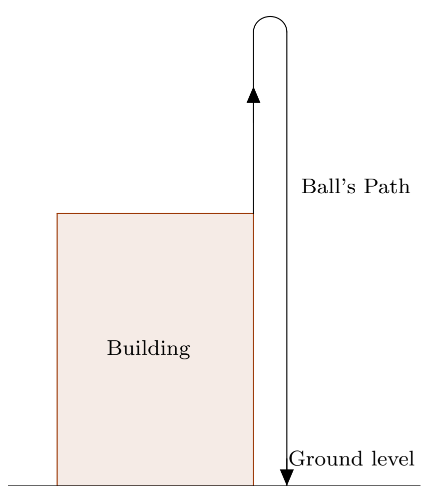
a. the height of the building.

b. the time when the ball hits the ground.

c. the average velocity on the interval $[1, 2]$.

d. the instantaneous velocity at $t = 1$ and 2.

e. the instantaneous velocity at any time t_0.



Solution.

a. The height of the building is the initial position of the ball. So, the building is $s(0) = 40$ meters tall.

b. The ball is on the ground when the height s of the ball from the ground is zero. Thus we solve the time t when $s(t) = 0$:

$$30 + 40t - 5t^2 = 0 \iff 5(8-t)(1+t) = 0 \iff t = 8 \text{ or } t = -1.$$

Since time is positive, we choose $t = 8$ seconds.

c. The average velocity of the ball on $[1, 2]$ is $\dfrac{s(2) - s(1)}{2-1} = \dfrac{90-70}{2-1} = 20$ m/s.

d. Then instantaneous velocity at time $t = 1$ is

$$\lim_{t\to 1} \frac{s(t)-s(1)}{t-1} = \lim_{t\to 1} \frac{(40+35t-5t^2)-70}{t-1} = \lim_{t\to 1} \frac{-5(t-6)(t-1)}{t-1} = 25 \text{ m/s}.$$

At time $t = 2$,

$$\lim_{t\to 2} \frac{s(t)-s(2)}{t-2} = \lim_{t\to 2} \frac{(40+35t-5t^2)-90}{t-2} = \lim_{t\to 2} \frac{-5(t-5)(t-2)}{t-2} = 15 \text{ m/s}.$$

e. The instantaneous velocity at any time t_0 is

$$\begin{aligned}
\lim_{t\to t_0} \frac{s(t)-s(t_0)}{t-t_0} &= \lim_{t\to 1} \frac{(40+35t-5t^2)-(40+35t_0-5t_0^2)}{t-t_0} \\
&= \lim_{t\to 1} \frac{5(t-t_0)(7-(t+t_0))}{t-t_0} \\
&= (35-10t_0) \text{ m/s}.
\end{aligned}$$

Solved Examples

EXAMPLE 1: Compute $f'(2)$ for the following functions:

1. $f(x) = x^2 + 1$
2. $f(x) = x^3 - 1$
3. $f(x) = \dfrac{x}{2x+1}$
4. $f(x) = \sqrt{x+8}$

Solution. In this example, x_0 is fixed and is equal to 2. Using the definition of the derivative,

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}.$$

1. Note that $f(2) = 2^2 + 1 = 5$. We get,

$$\begin{aligned} f'(2) &= \lim_{x \to 2} \frac{\left(x^2 + 1\right) - 5}{x - 2} \\ &= \lim_{x \to 2} \frac{x^2 - 4}{x - 2} \\ &= \lim_{x \to 2} (x + 2) \\ &= 4. \end{aligned}$$

2. Here, $f(2) = 2^3 - 1 = 7$. Thus,

$$\begin{aligned} f'(2) &= \lim_{x \to 2} \frac{\left(x^3 - 1\right) - 7}{x - 2} \\ &= \lim_{x \to 2} \frac{x^3 - 8}{x - 2} \\ &= \lim_{x \to 2} \left(x^2 + 2x + 4\right) \\ &= 12. \end{aligned}$$

3. We see that $f(2) = \dfrac{2}{2(2) + 1} = \dfrac{2}{5}$. So,

$$\begin{aligned} f'(2) &= \lim_{x \to 2} \frac{\left(\dfrac{x}{2x + 1}\right) - \dfrac{2}{5}}{x - 2} \\ &= \lim_{x \to 2} \frac{\left(\dfrac{5x - 4x - 2}{5(2x + 1)}\right)}{x - 2} \\ &= \lim_{x \to 2} \frac{(x - 2)}{5(2x + 1)(x - 2)} \\ &= \frac{1}{5(2(2) + 1)} \\ &= \frac{1}{25}. \end{aligned}$$

4. Note that $f(2) = \sqrt{2+8} = \sqrt{10}$. Therefore,

$$\begin{aligned} f'(2) &= \lim_{x \to 2} \frac{\sqrt{x+8} - \sqrt{10}}{x-2} \cdot \frac{\sqrt{x+8} + \sqrt{10}}{\sqrt{x+8} + \sqrt{10}} \\ &= \lim_{x \to 2} \frac{x+8-10}{(x-2)\left(\sqrt{x+8} + \sqrt{10}\right)} \\ &= \lim_{x \to 2} \frac{(x-2)}{(x-2)\left(\sqrt{x+8} + \sqrt{10}\right)} \\ &= \frac{1}{\sqrt{10} + \sqrt{10}} \\ &= \frac{1}{2\sqrt{10}}. \end{aligned}$$

EXAMPLE 2: Let $f(x) = \sin 2x$, $g(x) = \cos(x+\pi)$, and $h(x) = e^{x-1}$. Find $f'(\pi)$, $g'(2\pi)$, and $h'(1)$.

Solution. Using the alternative definition of the derivative, we obtain

$$\begin{aligned} f'(\pi) &= \lim_{h \to 0} \frac{f(\pi+h) - f(\pi)}{h} \\ &= \lim_{h \to 0} \frac{\sin(2(\pi+h)) - 0}{h} \\ &= \lim_{h \to 0} \frac{\sin(2\pi+2h)}{h}. \end{aligned}$$

We apply the sum identity for the sine function: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ and noting that $\sin(2\pi) = 0$ and $\cos(2\pi) = 1$, we obtain

$$\begin{aligned} f'(\pi) &= \lim_{h \to 0} \frac{\sin(2\pi)\cos(2h) + \cos(2\pi)\sin(2h)}{h} \\ &= \lim_{h \to 0} \frac{\sin(2h)}{h} \cdot \frac{2}{2} \\ &= \lim_{h \to 0} \frac{\sin(2h)}{2h} \cdot 2 \\ &= 2 \cdot \lim_{h \to 0} \frac{\sin(2h)}{2h} \\ &= 2 \cdot 1 \\ &= 2. \end{aligned}$$

Now, note that $g(2\pi) = \cos(2\pi+\pi) = -1$. Thus,

$$g'(2\pi) = \lim_{x \to 2\pi} \frac{\cos(x+\pi) - (-1)}{x - 2\pi}.$$

We use the identity, $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$. And so,

$$\begin{aligned} g'(2\pi) &= \lim_{x\to 2\pi} \frac{\cos x\cos\pi - \sin x\sin\pi + 1}{x-2\pi} \\ &= \lim_{x\to 2\pi} \frac{1-\cos x}{x-2\pi}. \end{aligned}$$

At this point, we make a substitution, $t = x - 2\pi$. Thus, $x = t + 2\pi$, and as $x \to 2\pi$, $t \to 0$. Thus,

$$\begin{aligned} g'(2\pi) &= \lim_{t\to 0} \frac{1-\cos(t+2\pi)}{t} \\ &= \lim_{t\to 0} \frac{1-(\cos t\cos 2\pi - \sin t\sin 2\pi)}{t} \\ &= \lim_{t\to 0} \frac{1-\cos t}{t} \\ &= 0. \end{aligned}$$

Lastly, $h(1) = e^{1-1} = 1$. Calculating the derivative,

$$h'(1) = \lim_{x\to 1} \frac{e^{x-1}-1}{x-1}.$$

Applying the substitution $t = x - 1$, and noting that as $x \to 1$, $t \to 0$, we obtain,

$$\begin{aligned} h'(1) &= \lim_{t\to 0} \frac{e^t-1}{t} \\ &= 1. \end{aligned}$$

EXAMPLE 3: Suppose the cost y (in pesos) of manufacturing x kilos of a certain product is given by

$$y = C(x) = x^4 - 2x^3 + x^2 + 1.$$

Compute for the average rate of change of the cost y of producing x products over $[1,2]$, $[1,5]$, and $[0,5]$.

Solution. To solve this, we need the values of $C(1)$, $C(2)$, $C(5)$, and $C(0)$. Computing, we get $C(1) = 1$, $C(2) = 5$, $C(5) = 401$, and $C(0) = 1$. Thus,

$$\begin{aligned} \frac{C(2)-C(1)}{2-1} &= \frac{5-1}{1} &&= \text{Php}\,4/\text{kg} \\ \frac{C(5)-C(1)}{5-1} &= \frac{401-1}{5-1} &&= \text{Php}\,100/\text{kg} \\ \frac{C(5)-C(0)}{5-0} &= \frac{401-1}{5-0} &&= \text{Php}\,80/\text{kg} \end{aligned}$$

EXAMPLE 4: Solve for the instantaneous rate of change of the cost function in Ex. 3 at $x = 1$.

Solution. The instantaneous rate of change at $x = 1$ is precisely the derivative of the cost function at $x = 1$. Since $C(1) = 1$, we get

$$\begin{aligned} C'(1) &= \lim_{x\to 1} \frac{C(x) - C(1)}{x-1} \\ &= \lim_{x\to 1} \frac{(x^4 - 2x^3 + x^2 + 1) - 1}{x-1} \\ &= \lim_{x\to 1} \frac{x^4 - 2x^3 + x^2}{x-1} \\ &= \lim_{x\to 1} \frac{x^2\left(x^2 - 2x + 1\right)}{x-1} \\ &= \lim_{x\to 1} \left(x^2(x-1)\right) \\ &= 0. \end{aligned}$$

Supplementary Problems

1. For each of the following functions, find the indicated derivative.

   (a) $f(x) = 2x^2 + x - 1; \quad f'(1)$

   (b) $f(x) = 2 - x^3; \quad f'(-1)$

   (c) $f(x) = x^5 - x^4 + x^3 - x^2 + x + 1; \quad f'(0)$

   (d) $f(x) = \sqrt{3x-5}; \quad f'(2)$

   (e) $f(x) = 1 - \sqrt{x^2 - 3x + 4}; \quad f'(3)$

   (f) $f(x) = \dfrac{x}{2x-1}; \quad f'(-3)$

   (g) $f(x) = \dfrac{x^2+1}{x-4}; \quad f'(0)$

   (h) $f(x) = \dfrac{\sqrt{3-2x} - x^2 + 1}{x-2}; \quad f'(-5)$

   (i) $f(x) = 2\sin\left(\dfrac{\pi x}{4}\right); \quad f'(2)$

   (j) $f(x) = xe^{-x}; \quad f'(1)$

   (k) $f(x) = \sqrt[3]{x^3+1}; \quad f'(2)$

   (l) $f(x) = \cot\left(2x + \dfrac{\pi}{2}\right); \quad f'\left(\dfrac{\pi}{2}\right)$

2. Suppose that the total revenue, $R(x)$ (in pesos), from the sale of x products is given by

$$R(x) = 2x^2 - 3x + 5.$$

Find the following:

(a) The average rate of change of the revenue in the interval:

i. $[10, 50]$

ii. $[10, 100]$

iii. $[0, 100]$

(b) Find the instantaneous rate of change of the revenue at:

i. $x = 10$

ii. $x = 50$

iii. $x = 100$

(c) Find the instantaneous rate of change of the slope of the tangent line at each point of the graph of $f(x) = x^3 - 2x^2 + x + 1$, where the slope of the tangent line is zero.

LESSON 6: Rules of Differentiation

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Determine the relationship between differentiability and continuity;
2. Derive the differentiation rules; and
3. Apply the differentiation rules in computing the derivatives of algebraic, exponential, and trigonometric functions.

TOPIC 6.1: Differentiability Implies Continuity

The difference between continuity and differentiability is a critical issue. Most, but not all, of the functions we encounter in calculus will be differentiable over their entire domain. Before we can confidently apply the rules regarding derivatives, we need to be able to recognize the exceptions to the rule.

Recall the following definitions:

Definition 1 (Continuity at a Number)**.** *A function f is* ***continuous at a number*** *c if all of the following conditions are satisfied:*

(*i*) *$f(c)$ is defined;*

(*ii*) *$\lim_{x \to c} f(x)$ exists; and*

(*iii*) *$\lim_{x \to c} f(x) = f(c)$.*

If at least one of the these conditions is not satisfied, the function is said to be ***discontinuous*** *at c.*

Definition 2 (Continuity on $\mathbb{R}$)**.** *A function f is said to be* ***continuous everywhere*** *if f is continuous at every real number.*

Definition 3. *A function f is* ***differentiable at the number*** *c if*

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists.

We now present several examples of determining whether a function is continuous or differentiable at a number.

EXAMPLE 1:

1. The piecewise function defined by

$$f(x) = \begin{cases} \dfrac{x^2+2x-3}{x-1} & \text{if } x \neq 1, \\ 4 & \text{if } x = 1, \end{cases}$$

is continuous at $c = 1$. This is because $f(1) = 4$,

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{(x-1)(x+3)}{x-1} = 4,$$

and $f(1) = \lim\limits_{x\to 1} f(x)$.

2. The function defined by

$$f(x) = \begin{cases} -x^2 & \text{if } x < 2, \\ 3-x & \text{if } x \geq 2. \end{cases}$$

is not continuous at $c = 2$ since $\lim\limits_{x\to 2^-} f(x) = -4 \neq 1 = \lim\limits_{x\to 2^+} f(x)$, that is, the $\lim\limits_{x\to 2} f(x)$ does not exist.

3. Consider the function $f(x) = \sqrt[3]{x}$. By definition, its derivative is

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h} \cdot \frac{\sqrt[3]{(x+h)^2}+\sqrt[3]{(x+h)(x)}+\sqrt[3]{x^2}}{\sqrt[3]{(x+h)^2}+\sqrt[3]{(x+h)(x)}+\sqrt[3]{x^2}} \\ &= \lim_{h\to 0} \frac{(x+h)-x}{h(\sqrt[3]{(x+h)^2}+\sqrt[3]{(x+h)(x)}+\sqrt[3]{x^2})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt[3]{(x+h)^2}+\sqrt[3]{(x+h)(x)}+\sqrt[3]{x^2}} \\ &= \frac{1}{3\sqrt[3]{x^2}}. \end{aligned}$$

Since $f'(1) = \dfrac{1}{3\sqrt[3]{1^2}} = \dfrac{1}{3}$, then f is differentiable at $x = 1$. On the other hand, $f'(0)$ does not exist. Hence f is not differentiable at $x = 0$.

4. The function defined by

$$f(x) = \begin{cases} 5x & \text{if} \quad x < 1 \\ 2x+3 & \text{if} \quad x \geq 1 \end{cases}$$

is continuous but not differentiable at $x = 1$. Indeed, $f(1) = 2(1) + 3 = 5$. Now,

- If $x < 1$, then $f(x) = 5x$ and so $\lim\limits_{x\to 1^-} 5x = 5$.
- If $x > 1$, then $f(x) = 2x + 3$ and so $\lim\limits_{x\to 1^+} (2x+3) = 5$.

Since the one-sided limits exist and are equal to each other, the limit exists and equals 5. So,

$$\lim_{x\to 1} f(x) = 5 = f(1).$$

This shows that f is continuous at $x = 1$. On the other hand, computing for the derivative,

- For $x < 1$, $f(x) = 5x$ and $\displaystyle\lim_{h\to 0^-} \frac{5(x+h) - (5x)}{h} = 5$.
- For $x > 1$, $f(x) = 2x + 3$ and $\displaystyle\lim_{h\to 0^+} \frac{(2(x+h)+3) - (2x+3)}{h} = 2$.

Since the one-sided limits at $x = 1$ do not coincide, the limit at $x = 1$ does not exist. Since this limit is the definition of the derivative at $x = 1$, we conclude that the derivative does not exist. Therefore, f is not differentiable at $x = 1$.

5. Another classic example of a function that is continuous at a point but not differentiable at that point is the absolute value function $f(x) = |x|$ at $x = 0$. Clearly, $f(0) = 0 = \lim_{x\to 0} |x|$. However, if we look at the limit definition of the derivative,

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{|0+h| - |0|}{h} = \lim_{h\to 0} \frac{|h|}{h}.$$

Note that the absolute value function is defined differently to the left and right of 0 so we need to compute one-sided limits. Note that if h approaches 0 from the left, then it approaches 0 through negative values. Since $h < 0 \implies |h| = -h$, it follows that

$$\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} -1 = -1.$$

Similarly, if h approaches 0 from the right, then h approaches 0 through positive values. Since $h > 0 \implies |h| = h$, we obtain

$$\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1.$$

Hence, the derivative does not exist at $x = 0$ since the one-sided limits do not coincide.

The previous two examples prove that continuity does not necessarily imply differentiability. That is, there are functions which are continuous at a point, but not differentiable at that point. The next theorem however says that the converse is always TRUE.

Theorem 6. *If a function f is differentiable at a, then f is continuous at a.*

Remark 1:

1. If f is continuous at $x = a$, it does not mean that f is differentiable at $x = a$.
2. If f is not continuous at $x = a$, then f is not differentiable at $x = a$.
3. If f is not differentiable at $x = a$, it does not mean that f is not continuous at $x = a$.
4. A function f is not differentiable at $x = a$ if one of the following is true:
   (a) f is not continuous at $x = a$.
   (b) the graph of f has a *vertical* tangent line at $x = a$.
   (c) the graph of f has a corner or cusp at $x = a$.

Solved Examples

EXAMPLE 1: The function

$$f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$$

is continuous at $x = 3$ since $f(3) = 6$ and $\lim\limits_{x\to 3} f(x) = 6$.

EXAMPLE 2: The function

$$f(x) = \begin{cases} x^3 & \text{if } x \geq 2 \\ x + 1 & \text{if } x < 2 \end{cases}$$

is not continuous at $x = 2$ since $\lim\limits_{x\to 2^+} f(x) = 8$ while $\lim\limits_{x\to 2^-} f(x) = 3$. Since f is not continuous at $x = 2$, it cannot be differentiable at $x = 2$.

EXAMPLE 3: The derivative of the function $f(x) = \dfrac{1}{x}$ is $f'(x) = -\dfrac{1}{x^2}$. Observe that f is differentiable at $x = 1$ since $f'(1) = -1$. However, f is not differentiable at $x = 0$ since $f'(0)$ is undefined.

EXAMPLE 4: Consider the function

$$f(x) = \begin{cases} 2x & \text{if } x \geq 0, \\ 5x & \text{if } x < 0. \end{cases}$$

This function is continuous at $x = 0$ since $f(0) = \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^-} f(x) = 0$. However, this function is not differentiable at $x = 0$. To see this, note that

$$f'(x) = \begin{cases} 2 & \text{if } x > 0, \\ 5 & \text{if } x < 0, \end{cases}$$

by taking the derivatives of $f(x) = 2x$ and $f(x) = 5x$, respectively. Note also the change in the inequalities involved in the intervals present. Since $\lim_{x \to 0^+} f'(x) = 2 \neq 5 = \lim_{x \to 0^-} f'(x)$, then the limit does not exist. The derivative, being defined as this limit, would also not exist at $x = 0$. Graphically, a corner is found at $x = 0$.

Note also that failure to be differentiable at a point does not imply discontinuity at that point. This is shown in this example, whereas f is not differentiable at $x = 0$ however it is continuous at $x = 0$.

Supplementary Problems

1. Suppose f is a function such that $f'(1)$ is undefined. Which of the following statements is ALWAYS true?

   (a) f must be continuous at $x = 1$.

   (b) f is NOT continuous at $x = 1$.

   (c) There is not enough information to determine whether or not f is continuous at $x = 1$.

2. Which of the following statements is/are ALWAYS true?

   (a) A function that is continuous at $x = a$ must be differentiable at $x = a$.

   (b) A function that is NOT continuous at $x = a$ must NOT be differentiable at $x = a$.

   (c) A function that is NOT differentiable at $x = a$ must NOT be continuous at $x = a$.

   (d) A function that is differentiable at $x = a$ must NOT be continuous at $x = a$.

3. Suppose that f is a function whose domain is the set of all real numbers, $\mathbb{R}$, which is NOT continuous at $x = -1$. Which of the following statements is/are FALSE?

   (a) f is NOT differentiable at $x = -1$.

   (b) Either $f(-1) \neq \lim_{x \to -1^+} f(x)$ or $f(-1) \neq \lim_{x \to -1^-} f(x)$.

   (c) $f(-1)$ is defined.

4. Consider the function defined by

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 5 \\ 6x - 4 & \text{if } x \geq 5. \end{cases}$$

   (a) Is f continuous at $x = 5$?

   (b) Is f differentiable at $x = 5$?

5. Determine the values for which f is continuous.

(a) $f(x) = \dfrac{x+5}{x-5}$

(b) $f(x) = \dfrac{\sqrt{x-1}}{x-2}$

(c) $f(x) = \dfrac{2x-\sqrt{3-x}}{x^2-5x+6}$

(d) $f(x) = \sqrt{\dfrac{3-x}{x+2}}$

(e) $f(x) = \sqrt{x^2-7x+12}$

6. Let

$$f(x) = \begin{cases} \dfrac{2x-1}{3x+2} & , x > 1, \\ \sqrt{x+5} & , 0 \leq x \leq 1, \\ x & , x < 0. \end{cases}$$

At which points is f NOT continuous?

7. Suppose a and b are real numbers. Consider,

$$f(x) = \begin{cases} ax+b, & \text{if } x \geq 2, \\ x^2-1, & \text{if } x < 2. \end{cases}$$

(a) Is it possible to find a and b such that f is differentiable at $x = 2$? If so, what are these numbers?

(b) If we only require continuity at $x = 2$, what are the possible values for a and b such that this is satisfied?

8. Is the function defined by $g(x) = x^3 - \tan\left(\dfrac{x}{4}\right) + 1$ continuous at $x = 2\pi$?

9. Consider the function

$$f(x) = \left|x^2 + x - 12\right|.$$

(a) For which values is this function continuous?

(b) For which values is this function differentiable?

TOPIC 6.2: The Differentiation Rules and Examples Involving Algebraic, Exponential, and Trigonometric Functions

We first recall the definition of the derivative of a function.

The derivative of the function f the function f' whose value at a number x in the domain of f is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{2.2}$$

if the limit exists.

For example, let us compute the derivative of the first function of the seatwork above: $f(x) = 3x^2 + 4$. Let us first compute the numerator of the quotient in (2.2):

$$\begin{aligned}
f(x+h) - f(x) &= (3(x+h)^2 + 4) - (3x^2 + 4) \\
&= (3x^2 + 6xh + 3h^2 + 4) - (3x^2 + 4) \\
&= 6xh + 3h^2 \\
&= h(6x + 3h).
\end{aligned}$$

Therefore,

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{3(x+h)^2 + 4 - (3x^2 + 4)}{h} \\
&= \lim_{h \to 0} \frac{h(6x + 3h)}{h} \\
&= \lim_{h \to 0} (6x + 3h) \\
&= 6x.
\end{aligned}$$

We see that computing the derivative using the definition of even a simple polynomial is a lengthy process. What follows next are rules that will enable us to find derivatives easily. We call them **DIFFERENTIATION RULES**.

DIFFERENTIATING CONSTANT FUNCTIONS

The graph of a constant function is a horizontal line and a horizontal line has zero slope. The derivative measures the slope of the tangent, and so the derivative is zero.

RULE 1: <u>The Constant Rule</u>

If $f(x) = c$ where c is a constant, then $f'(x) = 0$. The derivative of a constant is equal to zero.

EXAMPLE 1:

1. If $f(x) = 10$, then $f'(x) = 0$.
2. If $h(x) = -\sqrt{3}$, then $h'(x) = 0$.
3. If $g(x) = 5\pi$, then $g'(x) = 0$.

DIFFERENTIATING POWER FUNCTIONS

RULE 2: <u>The Power Rule</u>

If $f(x) = x^n$ where $n \in \mathbb{N}$, then $f'(x) = nx^{n-1}$.

EXAMPLE 2:

1. If $f(x) = x^3$, then $f'(x) = 3x^{3-1} = 3x^2$.
2. Find $g'(x)$ where $g(x) = \dfrac{1}{x^2}$.

   Solution. In some cases, the laws of exponents must be used to rewrite an expression before applying the power rule. Thus, we first write $g(x) = \dfrac{1}{x^2} = x^{-2}$ before we apply the Power Rule. We have:
   $$g'(x) = (-2)x^{-2-1} = -2x^{-3} \quad \text{or} \quad \frac{-2}{x^3}.$$

3. If $h(x) = \sqrt{x}$, then we can write $h(x) = x^{\frac{1}{2}}$. So we have,
$$\begin{aligned} h'(x) &= \frac{1}{2}x^{\frac{1}{2}-1} \\ &= \frac{1}{2}x^{-\frac{1}{2}} \qquad \text{or} \qquad \frac{1}{2\sqrt{x}} \end{aligned}$$

DIFFERENTIATING A CONSTANT TIMES A FUNCTION

RULE 3: The Constant Multiple Rule

If $f(x) = k\, h(x)$ where k is a constant, then $f'(x) = k\, h'(x)$.

EXAMPLE 3:

Find the derivatives of the following functions.

1. $f(x) = 5x^{\frac{3}{4}}$
2. $g(x) = \frac{1}{3}\sqrt[3]{x}$
3. $h(x) = -\sqrt{3}\,x$

Solution. We use Rule 3 in conjunction with Rule 2.

1. $f'(x) = 5 \cdot \dfrac{3}{4}x^{\frac{3}{4}-1} = \dfrac{15}{4}x^{-\frac{1}{4}}$.
2. $g(x) = \dfrac{1}{3}x^{\frac{1}{3}} \implies g'(x) = \dfrac{1}{3} \cdot \dfrac{1}{3}x^{\frac{1}{3}-1} = \dfrac{1}{9}x^{-\frac{2}{3}}$.
3. $h'(x) = -\sqrt{3}x^{1-1} = -\sqrt{3}$.

DIFFERENTIATING SUMS AND DIFFERENCES OF FUNCTIONS

RULE 4: The Sum Rule

If $f(x) = g(x) + h(x)$ where g and h are differentiable functions, then $f'(x) = g'(x) + h'(x)$.

EXAMPLE 4:

1. Differentiate the following:

   (i) $f(x) + g(x)$ (ii) $g(x) + h(x)$ (iii) $f(x) + h(x)$

2. Use Rules 3 and 4 to differentiate the following: (Hint: $f(x) - g(x) = f(x) + (-1)g(x)$.)

   (i) $f(x) - g(x)$ (ii) $g(x) - h(x)$ (iii) $f(x) - h(x)$

Solution.

1. Copying the derivatives in the solution of Example (3), and substituting them into the formula of the Sum Rule, we obtain

   (i) $\frac{15}{4}x^{-\frac{1}{4}} + \frac{1}{9}x^{-\frac{2}{3}}$. (ii) $\frac{1}{9}x^{-\frac{2}{3}} + (-\sqrt{3})$. (iii) $\frac{15}{4}x^{-\frac{1}{4}} + (-\sqrt{3})$.

2. Using Rules 3 and 4, we deduce that the derivative of $f(x) - g(x)$ is equal to the difference of their derivatives: $f'(x) - g'(x)$. Therefore we obtain

   (i) $\frac{15}{4}x^{-\frac{1}{4}} - \frac{1}{9}x^{-\frac{2}{3}}$. (ii) $\frac{1}{9}x^{-\frac{2}{3}} - (-\sqrt{3})$. (iii) $\frac{15}{4}x^{-\frac{1}{4}} - (-\sqrt{3})$.

RULE 5: The Product Rule

If f and g are differentiable functions, then

$$D_x[f(x)\, g(x)] = f(x)\, g'(x) + g(x)\, f'(x).$$

Rule 5 states that the derivative of the product of two differentiable functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.

EXAMPLE 5:

1. Find $f'(x)$ if $f(x) = (3x^2 - 4)(x^2 - 3x)$

   Solution.

$$\begin{aligned} f'(x) &= (3x^2 - 4)D_x(x^2 - 3x) + (x^2 - 3x)D_x(3x^2 - 4) \\ &= (3x^2 - 4)(2x - 3) + (x^2 - 3x)(6x) \\ &= 6x^3 - 9x^2 - 8x + 12 + 6x^3 - 18x^2 \\ &= 12x^3 - 27x^2 - 8x + 12. \end{aligned}$$

Remark 1: In the above example, we could have also multiplied the two factors and get

$$f(x) = 3x^4 - 9x^3 - 4x^2 + 12x.$$

Then, by the Rules 2,3 and 4, the derivative of f is

$$f'(x) = 12x^3 - 27x^2 - 8x + 12$$

which is consistent with the one derived from using the product rule.

2. Find $f'(x)$ if $f(x) = \sqrt{x}\,(6x^3 + 2x - 4)$.

Solution. Using product rule,

$$\begin{aligned} f'(x) &= x^{1/2}\, D_x(6x^3 + 2x - 4) + D_x(x^{1/2})\,(6x^3 + 2x - 4) \\ &= x^{1/2}\,(18x^2 + 2) + \frac{1}{2}x^{-1/2}\,(6x^3 + 2x - 4) \\ &= 18x^{5/2} + 2x^{1/2} + 3x^{5/2} + x^{1/2} - 2x^{-1/2} \\ &= 21x^{5/2} + 3x^{1/2} - 2x^{-1/2}. \end{aligned}$$

DIFFERENTIATING QUOTIENTS OF TWO FUNCTIONS

RULE 6: The Quotient Rule

Let $f(x)$ and $g(x)$ be two differentiable functions with $g(x) \neq 0$. Then

$$D_x\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$

The rule above states that the derivative of the quotient of two functions is the fraction having as its ***denominator*** the *square of the original denominator*, and as its ***numerator*** the *denominator times the derivative of the numerator minus the numerator times the derivative of the denominator*.

EXAMPLE 6:

1. Let $h(x) = \dfrac{3x+5}{x^2+4}$. Compute $h'(x)$.

Solution. If $h(x) = \dfrac{3x+5}{x^2+4}$, then $f(x) = 3x+5$ and $g(x) = x^2+4$ and therefore $f'(x) = 3$ and $g'(x) = 2x$. Thus,

$$\begin{aligned} h(x) &= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \\ &= \frac{(x^2+4)(3) - (3x+5)(2x)}{(x^2+4)^2} \\ &= \frac{3x^2+12-6x^2-10x}{(x^2+4)^2} \\ &= \frac{12-10x-3x^2}{(x^2+4)^2}. \end{aligned}$$

2. Find $g'(x)$ if $g(x) = \dfrac{2x^4+7x^2-4}{3x^5+x^4-x+1}$.

Solution.

$$\begin{aligned} g'(x) &= \frac{(3x^5+x^4-x+1)\,D_x(2x^4+7x^2-4) - (2x^4+7x^2-4)\,D_x(3x^5+x^4-x+1)}{(3x^5+x^4-x+1)^2} \\ &= \frac{(3x^5+x^4-x+1)(8x^3+14x) - (2x^4+7x^2-4)(15x^4+4x^3-1)}{(3x^5+x^4-x+1)^2}. \end{aligned}$$

DIFFERENTIATING TRIGONOMETRIC FUNCTIONS

RULE 7: Derivatives of trigonometric functions

1. $D_x(\sin x) = \cos x$
2. $D_x(\cos x) = -\sin x$
3. $D_x(\tan x) = \sec^2 x$
4. $D_x(\cot x) = -\csc^2 x$
5. $D_x(\sec x) = \sec x \tan x$
6. $D_x(\csc x) = -\csc x \cot x$

EXAMPLE 7: Differentiate the following functions:

1. $f(x) = \sec x + 3\csc x$
2. $g(x) = x^2 \sin x - 3x\cos x + 5\sin x$

Solution. Applying the formulas above, we get

1. If $f(x) = \sec x + 3\csc x$, then

$$f'(x) = \sec x \tan x + 3(-\csc x \cot x) = \sec x \tan x - 3\csc x \cot x.$$

2. If $g(x) = x^2 \sin x - 3x \cos x + 5 \sin x$, then

$$\begin{aligned} g'(x) &= [(x^2)(\cos x) + (\sin x)(2x)] - 3[(x)(-\sin x) + (\cos x)(1)] + 5 \cos x \\ &= x^2 \cos x + 2x \sin x + 3x \sin x - 3 \cos x + 5 \cos x \\ &= x^2 \cos x + 5x \sin x + 2 \cos x. \end{aligned}$$

DIFFERENTIATING AN EXPONENTIAL FUNCTION

RULE 8: Derivative of an exponential function

$$\text{If } f(x) = e^x, \text{ then } f'(x) = e^x.$$

EXAMPLE 8:

1. Find $f'(x)$ if $f(x) = 3e^x$.

   Solution. Applying Rules 3 and 7, we have

$$f'(x) = 3D_x[e^x] = 3e^x.$$

2. Find $g'(x)$ if $g(x) = -4x^2e^x + 5xe^x - 10e^x$.

   Solution. Applying Rule 5 to the first two terms and Rule 3 to the third term, we have

$$\begin{aligned} g'(x) &= [(-4x^2)(e^x) + (e^x)(-8x)] + [(5x)(e^x) + (e^x)(5) - 10 \cdot e^x] \\ &= -4x^2e^x - 3xe^x - 5e^x. \end{aligned}$$

3. Find $h'(x)$ if $h(x) = e^x \sin x - 3e^x \cos x$.

   Solution. We apply the Product Rule to each term.

$$\begin{aligned} h'(x) &= [(e^x)(\cos x) + (\sin x)(e^x)] - 3 \cdot [(e^x)(-\sin x) + (\cos x)(e^x)] \\ &= e^x \cos x + e^x \sin x + 3e^x \sin x - 3e^x \cos x \\ &= e^x(4 \sin x - 2 \cos x). \end{aligned}$$

4. Find $\dfrac{dy}{dx}$ where $y = \dfrac{17}{e^x x^e + 2x - 3\sqrt{x}}$.

Solution. Using Quotient Rule (also Product Rule when differentiating $x^e e^x$), we obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{(e^x x^e + 2x - 3\sqrt{x})(0) - (17)([(e^x)(ex^{e-1}) + (x^e)(e^x)] + 2 - \frac{3}{2\sqrt{x}})}{(e^x x^e + 2x - 3\sqrt{x})^2} \\ &= \frac{-17e^{x+1}x^{e-1} - 17e^x x^e - 34 + \frac{51}{2\sqrt{x}}}{(e^x x^e + 2x - 3\sqrt{x})^2}.\end{aligned}$$

Solved Examples

EXAMPLE 1:

1. If $f(x) = 25$, then $f'(x) = 0$.
2. If $g(x) = -\dfrac{\sqrt{7}}{\sqrt[3]{2}}$, then $g'(x) = 0$.
3. If $h(x) = e^{1.2345}$, then $h'(x) = 0$.
4. If $r(x) = c$, where $c \in \mathbb{R}$, then $r'(x) = 0$.

EXAMPLE 2:

1. If $f(x) = x^3$, then $f'(x) = 3x^{3-1} = 3x^2$.
2. If $f(x) = \sqrt[5]{x^3}$, then we can write f as $f(x) = x^{\frac{3}{5}}$. Thus, $f'(x) = \frac{3}{5}x^{\frac{3}{5}-1} = \frac{3}{5}x^{-\frac{2}{5}}$.
3. If $f(x) = -\dfrac{1}{x^4}$, then f can be written as $f(x) = -x^{-4}$. Thus, $f'(x) = -4x^{-4-1} = -4x^{-5}$.

EXAMPLE 3:

1. If $f(x) = -4x^{\frac{4}{3}}$, then $f'(x) = -4\left(\frac{4}{3}\right)x^{\frac{4}{3}-1} = -\frac{16}{3}x^{\frac{1}{3}}$.
2. If $g(x) = 2\sqrt[4]{x^3}$, then g can be written as $g(x) = 2x^{\frac{3}{4}}$. Thus, $g'(x) = 2\left(\frac{3}{4}\right)x^{\frac{3}{4}-1} = \frac{3}{2}x^{-\frac{1}{4}}$.
3. If $h(x) = -\sqrt{7}x^2$, then $h'(x) = -\sqrt{7}\left(2x^{2-1}\right) = -2\sqrt{7}x$.

EXAMPLE 4:

1. If $f(x) = 3x^2 - 6\sqrt{x} + 1$, then $f'(x) = 3\left(2x^{1-1}\right) - 6\left(\dfrac{1}{2\sqrt{x}}\right) + 0 = 6x - \dfrac{3}{\sqrt{x}}$.
2. If $f(x) = x^{-2} - x$, then $f'(x) = -2x^{-3} - 1$.

EXAMPLE 5:

1. If $f(x) = 3x \sin x$. Then $f'(x) = D_x(3x)\sin x + (3x)\, D_x(\sin x) = 3\sin x + 3x\cos x$.
2. If $f(x) = \left(4x^3 - 2x^2 + 3x + 1\right)\left(x^{-2} - \sqrt{x}\right)$, then

$$\begin{aligned} f'(x) &= \left(x^{-2} - \sqrt{x}\right) D_x\left(4x^3 - 2x^2 + 3x + 1\right) + \left(4x^3 - 2x^2 + 3x + 1\right) D_x\left(x^{-2} - \sqrt{x}\right) \\ &= \left(x^{-2} - \sqrt{x}\right)\left(12x^2 - 4x + 3\right) + \left(4x^3 - 2x^2 + 3x + 1\right)\left(-2x^{-3} - \frac{1}{2\sqrt{x}}\right). \end{aligned}$$

EXAMPLE 6: Find $f'(x)$ if $f(x) = \dfrac{x^3 - 1}{2x^2 + 3x}$.

$$\begin{aligned} f'(x) &= \frac{\left(2x^2 + 3x\right) D_x\left(x^3 - 1\right) - \left(x^3 - 1\right) D_x\left(2x^2 + 3x\right)}{\left(2x^2 + 3x\right)^2} \\ &= \frac{\left(2x^2 + 3x\right)\left(3x^2\right) - \left(x^3 - 1\right)(4x + 3)}{\left(2x^2 + 3x\right)^2} \\ &= \frac{6x^4 + 9x^3 - \left(4x^4 + 3x^3 - 4x - 3\right)}{\left(2x^2 + 3x\right)^2} \\ &= \frac{2x^4 + 6x^3 + 4x + 3}{\left(2x^2 + 3x\right)^2}. \end{aligned}$$

EXAMPLE 7: Find $g'(x)$ if $g(x) = 2x^3 e^x - 4xe^x + e^x$.

Solution. Note that g can be written as $g(x) = e^x\left(2x^3 - 4x + 1\right)$. Thus, by applying the Product Rule, we get

$$\begin{aligned} g'(x) &= e^x\left(2x^3 - 4x + 1\right) + e^x\left(6x^2 - 4\right) \\ &= e^x\left(2x^3 - 4x + 1 + 6x^2 - 4\right) \\ &= e^x\left(2x^3 + 6x^2 - 4x - 3\right) \\ &= 2x^3 e^x + 6x^2 e^x - 4xe^x - 3e^x. \end{aligned}$$

EXAMPLE 8: Find $f'(x)$ if $f(x) = \dfrac{xe^x}{x^2 + 2e^x}$.

Solution. We apply the Quotient Rule to obtain

$$\begin{aligned} f'(x) &= \frac{\left(x^2 + 2e^x\right)\left(e^x + xe^x\right) - xe^x\left(2x + 2e^x\right)}{\left(x^2 + 2e^x\right)^2} \\ &= \frac{x^2e^x + x^3e^x + 2e^{2x} + 2xe^{2x} - 2x^2e^x - 2xe^{2x}}{\left(x^2 + 2e^x\right)^2} \\ &= \frac{x^3e^x - x^2e^x + 2e^{2x}}{\left(x^2 + 2e^x\right)^2}. \end{aligned}$$

EXAMPLE 9: Differentiate the following trigonometric functions:

1. $f(x) = \cot x + 2\tan x \sec x$

$$\begin{aligned} f'(x) &= -\csc^2 x + 2\left(\left(\sec^2 x\right)\sec x + \tan x\left(\sec x \tan x\right)\right) \\ &= -\csc^2 x + 2\sec^3 x + 2\sec x \tan^2 x. \end{aligned}$$

2. $g(x) = 3x^3 \cos x - 2x \sin x + 5\csc x$

$$\begin{aligned} g'(x) &= \left(9x^2 \cos x + 3x^3\left(-\sin x\right)\right) - 2\left(\sin x + x\cos x\right) - 5\csc x \cot x \\ &= 9x^2 \cos x - 3x^3 \sin x - 2\sin x - 2x\cos x - 5\csc x \cot x. \end{aligned}$$

$\square$

Supplementary Problems

1. Find the derivative of the following functions:

   (a) $f(x) = \left(2x^2 - 3x\right)\left(-4x^{-2} + 5\right)$

   (b) $f(x) = 4x^4 - 5x^{-\frac{3}{2}} + 6x^2 - 3$

   (c) $g(x) = 5x^{-3} - \sqrt{x^{-2}} + x^2$

   (d) $h(x) = 3x^2 - 4x^{\frac{1}{2}} + \dfrac{3}{x}$

   (e) $f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$

   (f) $f(x) = x^6 + x^5 + x^4 - x^3 - x^2 - x - 1$

   (g) $h(t) = \dfrac{t + 2}{t^2 - 3t}$

   (h) $g(u) = \sqrt{u^3} - \cos u$

   (i) $f(x) = \dfrac{\sin x}{\sec x + x^2}$

   (j) $g(x) = \dfrac{x}{x\tan x} - \dfrac{2}{x\sqrt{x}}$

2. Find $\frac{dy}{dx}$ and simplify if possible.

(a) $y = \sqrt{x^3} - \frac{1}{\sqrt{x^3}}$

(b) $y = x^4 + \pi - e^{\pi}x$

(c) $y = \frac{e^x}{x}$

(d) $y = \sin x \cos x \tan x$

(e) $y = x^2 - e^x \sin x$

3. Find the equation of the tangent lines to the following curves at the given points.

(a) $f(x) = 3x^2 - 2x + 1, \quad x = 2$

(b) $g(x) = \sin x + \cos x, \quad x = \frac{\pi}{4}$

(c) $h(x) = x^2 e^x + \sec x, \quad \text{at } (0, 1)$

(d) Find all points on the graph of $y = (2x+1)^2$ at which the tangent line is parallel to the line with equation $y + 3x - 5 = 0$.

(e) Find all points on the graph of $y = \frac{x^3}{3} - x^2 - 2x$ such that the tangent line is perpendicular to the line with equation $y + x = 2$.

LESSON 7: Optimization

LEARNING OUTCOME: At the end of the lesson, the learner shall be able to solve optimization problems.

TOPIC 7.1: Optimization using Calculus

REVIEW OF MATHEMATICAL MODELING

Before we start with problem solving, we recall key concepts in mathematical modeling. Functions are used to describe physical phenomena. For example:

- The number of people y in a certain area that is infected by an epidemic after some time t;
- The concentration c of a drug in a person's bloodstream t hours after it was taken;
- The mice population y as the snake population x changes, etc.

We model physical phenomena to help us predict what will happen in the future. We do this by finding or constructing a function that exhibits the behavior that has already been observed. In the first example above, we want to find the function $y(t)$. For example, if $y(t) = 1000 \cdot 2^{-t}$, then we know that initially, there are $y(0) = 1000$ affected patients. After one hour, there are $y(1) = 1000 \cdot 2^{-1} = 500$ affected patients.

Observe that the independent variable here is time t and that the quantity y depends on t. Since y is dependent on t, it now becomes possible to optimize the value of y by controlling at which time t you will measure y.

We now look at some examples.

EXAMPLE 1: For each of the following, determine the dependent quantity $Q(x)$ and the independent quantity x on which it depends. Then, find the function $Q(x)$ that accurately models the given scenario.

1. The product P of a given number x and the number which is one unit bigger.
   Answer: $P(x) = x(x + 1) = x^2 + x$
2. The volume V of a sphere of a given radius r Answer: $V(r) = 4/3\,\pi r^3$
3. The volume V of a right circular cone with radius 3cm and a given height h
   Answer: $V(h) = 3\pi h$

4. The length ℓ of a rectangle with width 2cm and a given area A

   Answer: $\ell(A) = A/2$

5. The total manufacturing cost C of producing x pencils if there is an overhead cost of ₱100 and producing one pencil costs ₱2

   Answer: $C(x) = 100 + 2x$

6. The volume of the resulting open-top box when identical squares with side x are cut off from the four corners of a 3 meter by 5 meter aluminum sheet and the sides are then turned up Answer: $V(x) = x(3 - 2x)(5 - 2x)$.

CRITICAL POINTS AND POINTS WHERE EXTREMA OCCUR

Definition

Let f be a function that is continuous on an open interval I containing x_0.

- We say that x_0 is a **critical point** of f if $f'(x_0) = 0$ or $f'(x_0)$ does not exist (that is, f has a corner or a cusp at $(x_0, f(x_0))$).
- We say that the **maximum** occurs at x_0 if the value $f(x_0)$ is the largest among all other functional values on I, that is,

$$f(x_0) \geq f(x) \quad \text{for all } x \in I.$$

- We say that the **minimum** of f occurs at x_0 if the value $f(x_0)$ is the smallest among all the other functional values on I, that is,

$$f(x_0) \leq f(x) \quad \text{for all } x \in I.$$

- We say that an **extremum** of f occurs at x_0 if either the maximum or the minimum occurs at x_0.

EXAMPLE 2: Find all critical points of the given function f.

1. $f(x) = 3x^2 - 3x + 4$
2. $f(x) = x^3 - 9x^2 + 15x - 20$
3. $f(x) = x^3 - x^2 - x - 10$
4. $f(x) = x - 3x^{1/3}$
5. $f(x) = x^{5/4} + 10x^{1/4}$

Solution. We differentiate f and find all values of x such that $f'(x)$ becomes zero or undefined.

1. Note that f is differentiable everywhere, so critical points will only occur when f' is zero. Differentiating, we get $f'(x) = 6x - 3$. Therefore, $f'(x) = 0 \Leftrightarrow 6x - 3 = 0 \Leftrightarrow x = 1/2$. So $x = 1/2$ is a critical point.

2. $f'(x) = 3x^2 - 18x + 15 = 3(x^2 - 6x + 5) = 3(x - 5)(x - 1)$. Hence the critical points are 1 and 5.

3. $f'(x) = 3x^2 - 2x - 1 = (3x + 1)(x - 1)$. So, the critical points are $-1/3$ and 1.

4. $f'(x) = 1 - x^{-2/3} = \dfrac{x^{2/3} - 1}{x^{2/3}}$. Observe that f' is zero when the numerator is zero, or when $x = 1$. Moreover, f' is undefined when the denominator is zero, i.e., when $x = 0$. So, the critical points are 0 and 1.

5. $f'(x) = \dfrac{5}{4}x^{1/4} + \dfrac{10}{4}x^{-3/4} = \dfrac{5(x + 2)}{4x^{3/4}}$. Note that the domain of f is $[0, \infty)$; therefore -2 cannot be a critical point. The only critical point is 0.

FERMAT'S THEOREM AND THE EXTREME VALUE THEOREM

Theorem 7 (Fermat's Theorem)**.** *Let f be continuous on an open interval I containing x_0. If f has an extremum at x_0, then x_0 must be a critical point of f.*

Theorem 8 (Extreme Value Theorem)**.** *Let f be a function which is continuous on a closed and bounded interval $[a, b]$. Then the extreme values (maximum and minimum) of f always exist, and they occur either at the endpoints or at the critical points of f.*

EXAMPLE 3: Find the extrema of the given functions on the interval $[-1, 1]$. (These functions are the same as in the previous example.)

1. $f(x) = 3x^2 - 3x + 4$

2. $f(x) = x^3 - 9x^2 + 15x - 20$

3. $f(x) = x^3 - x^2 - x - 10$

4. $f(x) = x - 3x^{1/3}$

Solution. We had already solved all critical points of f in the previous exercise. We will only consider those critical points on the interval $[-1, 1]$. By the Extreme Value Theorem, we also have to consider the endpoints. So, what remains to be done is the following:

- Get the functional values of all the critical points inside $[-1, 1]$;
- Get the functional values at the endpoints; and
- Compare the values. The highest one is the maximum value while the lowest one is the minimum value.

1. There is only one critical point, $x = 1/2$, and the endpoints are $x = \pm 1$. We present the functional values in a table.

x	-1	$1/2$	1
$f(x)$	10	$13/4$	4

   Clearly, the maximum of f occurs at $x = -1$ and has value 10. The minimum of f occurs at $x = 1/2$ and has value $13/4$.

2. The critical points of f are 1 and 5, but since we limited our domain to $[-1, 1]$, we are only interested in $x = 1$. Below is the table of functional values for this critical point, as well as those at the endpoints.

x	-1	1
$f(x)$	-45	-13

   Therefore, the maximum value -13 occurs at $x = 1$ while the minimum value -45 occurs at $x = -1$.

3. We consider the functional values at $-1/3, -1$ and 1:

x	-1	$-1/3$	1
$f(x)$	-11	$-25/27$	-11

   Thus, the maximum point is $(-1/3, -25/27)$ while the minimum points are $(1, -11)$ and $(-1, -11)$.

4. $f(0) = 0$, $f(1) = -2$ and $f(-1) = 2$. So, the maximum point is $(-1, 2)$ while the minimum point is $(1, -2)$.

OPTIMIZATION: APPLICATION OF EXTREMA TO WORD PROBLEMS

Many real-life situations require us to find a value that best suits our needs. If we are given several options for the value of a variable x, how do we choose the "best value?" Such a problem is classified as an ***optimization problem***. We now apply our previous discussion to finding extremum values of a function to solve some optimization problems.

Suggested Steps in Solving Optimization Problems

1. If possible, draw a diagram of the problem.
2. Assign variables to all unknown quantities involved.
3. Specify the objective function. This function must be continuous.
   (a) Identify the quantity, say q, to be maximized or minimized.
   (b) Formulate an equation involving q and other quantities. Express q in terms of a single variable, say x. If necessary, use the information given and relationships between quantities to eliminate some variables.
   (c) The objective function is

$$\begin{aligned} &\textbf{maximize} \quad q = f(x) \\ \text{or}\ \ &\textbf{minimize} \quad q = f(x). \end{aligned}$$

4. Determine the domain or constraints of q from the physical restrictions of the problem. The domain must be a closed and bounded interval.
5. Use appropriate theorems involving extrema to solve the problem. Make sure to give the exact answer (with appropriate units) to the question.

EXAMPLE 4: Find the number in the interval $[-2, 2]$ so that the difference of the number from its square is maximized.

Solution. Let x be the desired number. We want to maximize

$$f(x) = x^2 - x$$

where $x \in [-2, 2]$. Note that f is continuous on $[-2, 2]$ and thus, we can apply the Extreme Value Theorem.

We first find the critical numbers of f in the interval $(-2, 2)$. We have

$$f'(x) = 2x - 1,$$

which means that we only have one critical number in $(-2, 2)$: $x = \dfrac{1}{2}$.

Then we compare the function value at the critical number and the endpoints. We see that

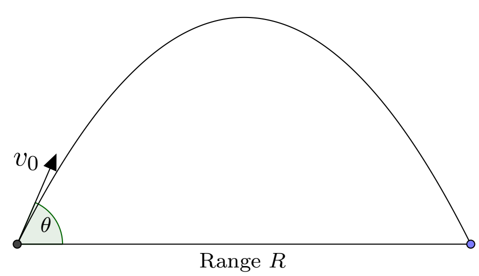
$$f(-2) = 6, \quad f(2) = 2, \quad f\left(\frac{1}{2}\right) = -\frac{1}{4}.$$

From this, we conclude that f attains a maximum on $[-2, 2]$ at the left endpoint $x = -2$. Hence, the number we are looking for is -2.

EXAMPLE 5: The range R (distance of launch site to point of impact) of a projectile that is launched at an angle $\theta \in [0°, 90°]$ from the horizontal, and with a fixed initial speed of v_0, is given by

$$R(\theta) = \frac{v_0^2}{g} \sin 2\theta,$$

where g is the acceleration due to gravity. Show that this range is maximized when $\theta = 45°$.



Solution. Let $R(\theta)$ denote the range of the projectile that is launched at an angle θ, measured from the horizontal. We need to maximize

$$R(\theta) = \frac{v_0^2}{g} \sin 2\theta$$

where $\theta \in [0, \pi/2]$. Note that R is continuous on $[0, \pi/2]$ and therefore the Extreme Value Theorem is applicable.

We now differentiate R to find the critical numbers on $[0, \pi/2]$:

$$R'(\theta) = \frac{v_0^2}{g} 2 \cos 2\theta = 0 \iff 2\theta = \pi/2.$$

Hence $\theta = \pi/4 = 45^\circ$ is a critical number.

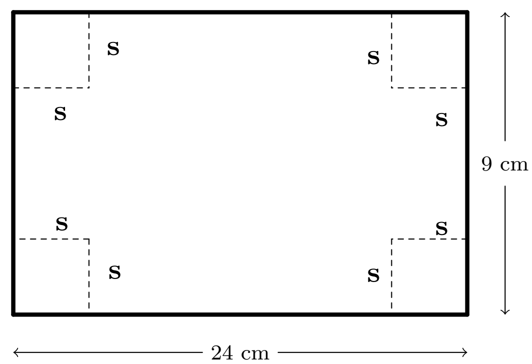
Finally, we compare the functional values:

$$f(0) = 0, \quad f(\pi/4) = \frac{v_0^2}{g}, \quad f(\pi/2) = 0.$$

Thus, we conclude that f attains its maximum at $\theta = \pi/4$, with value v_0^2/g.

EXAMPLE 6:

An open-typed rectangular box is to be made from a piece of cardboard 24 cm long and 9 cm wide by cutting out identical squares from the four corners and turning up the sides. Find the volume of the largest rectangular box that can be formed.



Solution. Let s be the length of the side of the squares to be cut out, and imagine the "flaps" being turned up to form the box. The length, width and height of the box would then be $24 - 2s$, $9 - 2s$, and s, respectively. Therefore, the volume of the box is

$$V(s) = (24 - 2s)(9 - 2s)s = 2(108s - 33s^2 + 2s^3).$$

We wish to maximize $V(s)$ but note that s should be nonnegative and should not be more than half the width of the cardboard. That is, $s \in [0, 4.5]$. (The case $s = 0$ or $s = 4.5$ does not produce any box because one of the dimensions would become zero; but to make the interval closed and bounded, we can think of those cases as *degenerate* boxes with zero volume). Since V is just a polynomial, it is continuous on the closed and bounded interval $[0, 4.5]$. Thus, the Extreme Value Theorem applies. Now

$$V'(s) = 216 - 132s + 12s^2 = 4(54 - 33s + 3s^2) = 4(3s - 6)(s - 9)$$

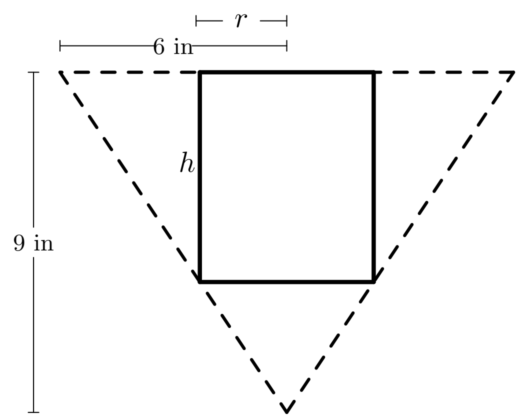
and hence the only critical number of V in $(0, 4.5)$ is 2 ($s = 9$ is outside the interval).

We now compare the functional values at the endpoints and at the critical points:

s	0	2	4.5
$V(s)$	0	200	0

Therefore, from the table, we see that V attains its maximum at $s = 2$, and the maximum volume is equal to $V(2) = 200$ cm^3.

EXAMPLE 7: Determine the dimensions of the right circular cylinder of greatest volume that can be inscribed in a right circular cone of radius 6 cm and height 9 cm.



Solution. Let h and r respectively denote the height and radius of the cylinder. The volume of the cylinder is $\pi r^2 h$.

Looking at the central cross-section of the cylinder and the cone, we can see similar triangles, and so

$$\frac{6}{6-r} = \frac{9}{h}. \tag{2.3}$$

We can now write our objective function as

$$V(r) = 9\pi r^2 - \frac{3}{2}\pi r^3 = 3\pi r^2 \left(3 - \frac{r}{2}\right),$$

and it is to be maximized. Clearly, $r \in [0, 6]$. Since V is continuous on $[0, 6]$, the Extreme Value Theorem is applicable.

Now,

$$V'(r) = 18\pi r - \frac{9}{2}\pi r^2 = 9\pi r \left(2 - \frac{r}{2}\right)$$

and hence, our only critical number on $(0, 6)$ is 4. We now compare the functional values at the endpoints and at the critical points:

r	0	4	6
$V(r)$	0	48π	0

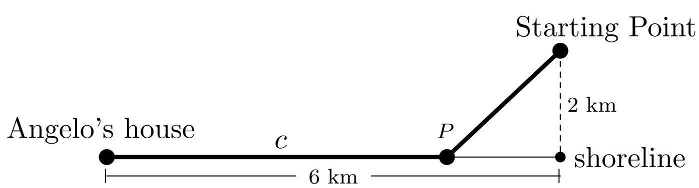
We see that the volume is maximized when $r = 4$, with value $V(4) = 48\pi$. To find the dimensions, we solve for h from (2.3).

If $r = 4$,

$$\frac{6}{6-4} = \frac{9}{h} \quad \Longrightarrow \quad h = 3.$$

Therefore, the largest circular cylinder that can be inscribed in the given cone has dimensions $r = 4$ cm and height $h = 3$ cm.

EXAMPLE 8: Angelo, who is in a rowboat 2 kilometers from a straight shoreline, wishes to go back to his house which is on the shoreline and 6 kilometers from the point on the shoreline nearest Angelo. If he can row at 6 kph and run at 10 kph, how should he proceed in order to get to his house in the least amount of time?



Solution. Let c be the distance between the house and the point P on the shore from which Angelo will start to run. Using the Pythagorean Theorem, we see that the distance he will travel by boat is $\sqrt{4 + (6-c)^2}$.

Note that speed$=\frac{\text{distance}}{\text{time}}$. Thus, he will sail for $\frac{\sqrt{4+(6-c)^2}}{6}$ hours and run for $\frac{c}{10}$ hours. We wish to minimize

$$T(c) = \frac{\sqrt{4+(6-c)^2}}{6} + \frac{c}{10}.$$

We can assume that $c \in [0, 6]$. We now differentiate T. Observe that our previously discussed rules of differentiation are not applicable to this function because we have not yet discussed how to differentiate compositions of functions. We use the definition of the derivative instead.

$$
\begin{aligned}
T'(c) &= \lim_{x\to c} \frac{T(x)-T(c)}{x-c} \\
&= \lim_{x\to c} \frac{\left(\dfrac{\sqrt{4+(6-x)^2}}{6}+\dfrac{x}{10}\right)-\left(\dfrac{\sqrt{4+(6-c)^2}}{6}+\dfrac{c}{10}\right)}{x-c} \\
&= \frac{1}{10}\lim_{x\to c}\frac{x-c}{x-c}+\frac{1}{6}\lim_{x\to c}\left(\frac{\sqrt{4+(6-x)^2}-\sqrt{4+(6-c)^2}}{x-c}\right).
\end{aligned}
$$

We now rationalize the expression in the first limit by multiplying the numerator and denominator by $\sqrt{4+(6-x)^2}+\sqrt{4+(6-c)^2}$. This yields:

$$
\begin{aligned}
T'(c) &= \frac{1}{10}+\frac{1}{6}\lim_{x\to c}\frac{(4+(6-x)^2)-(4+(6-c)^2)}{(x-c)(\sqrt{4+(6-x)^2}+\sqrt{4+(6-c)^2}} \\
&= \frac{1}{10}+\frac{1}{6}\lim_{x\to c}\frac{-12(x-c)+(x-c)(x+c)}{(x-c)(\sqrt{4+(6-x)^2}+\sqrt{4+(6-c)^2})} \\
&= \frac{1}{10}+\frac{1}{6}\lim_{x\to c}\frac{-12+x+c}{\sqrt{4+(6-x)^2}+\sqrt{4+(6-c)^2}} \\
&= \frac{1}{10}+\frac{1}{6}\cdot\frac{-12+2c}{2\sqrt{4+(6-c)^2}}.
\end{aligned}
$$

Solving for the critical numbers of f on $(0,6)$, we solve

$$
T'(c)=\frac{1}{10}+\frac{1}{6}\cdot\frac{-12+2c}{2\sqrt{4+(6-c)^2}}=\frac{3\sqrt{4+(6-c)^2}-30+5c}{30\sqrt{4+(6-c)^2}}=0,
$$

and we get $c=\dfrac{9}{2}$.

Comparing function values at the endpoints and the critical number,

$$
T(0)=\frac{\sqrt{40}}{6},\qquad T\left(\frac{9}{2}\right)=\frac{13}{15},\qquad T(6)=\frac{14}{15},
$$

we see that the minimum of T is attained at $c=\frac{9}{2}$. Thus, Angelo must row up to the point P on the shore $\frac{9}{2}$ kilometers from his house and $\frac{3}{2}$ kilometers from the point on the shore nearest him. Then he must run straight to his house.

Solved Examples

EXAMPLE 1: For each of the following, provide a model that depicts the first quantity being expressed as a function of the other quantities.

1. The quotient Q of a number x and the sum of its absolute value and 1.

2. The product P of a number x and the number equal to 2 less than its square.

3. The volume V of a cone of radius r and height $2r$.

4. The volume V of a rectangular prism with length l, height $2l$, and width $3l$.

5. The length l of a square of area A.

6. The area A of a 4 m by 6 m cardboard sheet when identical squares with side x are cut off from the four corners of the sheet.

Solution.

1. $Q\left(x\right) = \dfrac{x}{|x| + 1}$

2. $P\left(x\right) = x\left(x^2 - 2\right) = x^3 - 2x$

3. $V\left(r\right) = \dfrac{1}{3}\pi r^2 \left(2r\right) = \dfrac{2\pi}{3} r^3$

4. $V\left(l\right) = l \cdot 2l \cdot 3l = 6l^3$

5. $l\left(A\right) = \sqrt{A}$

6. $A\left(x\right) = 24 - 4x^2$

EXAMPLE 2: Find all the critical points of the given functions.

1. $f\left(x\right) = 4x^2 - 3x = 5$

2. $f\left(x\right) = \dfrac{x^3}{3} + x^2 + x + 5$

3. $f\left(x\right) = \sin x$

4. $f\left(x\right) = e^x - x$

5. $f\left(x\right) = \ln x$

Solution.

1. To get the critical points of f, we differentiate f and look for the zeroes of f'. Indeed, $f'(x) = 8x - 3$. Thus, the critical points are the solutions of $8x - 3 = 0$. Hence, the critical point is $x = \frac{3}{8}$.

2. $f'(x) = x^2 + 2x + 1 = (x+1)^2$. Hence, the critical point is $x = -1$.

3. $f'(x) = \cos x$. This is zero when $x = \frac{k\pi}{2}$, where k is an odd integer.

4. $f'(x) = e^x - 1$. Thus, we want to look for x such that $e^x = 1$. The critical point is $x = 0$.

5. $f'(x) = \frac{1}{x}$. This function has no zeroes. Also, even though f' is undefined at the origin, $x = 0$ is not classified as a criticial point since it is not in the domain of f.

EXAMPLE 3: Find the extrema of the given functions in the interval $[0, 10]$.

1. $f(x) = x^3 - 12x + 3$

2. $f(x) = 4x^2 + 5x - 1$

3. $f(x) = 5x^3 + 3x^2 + 4x + 5$

4. $f(x) = e^x$

Solution.

1. $f'(x) = 3x^2 - 12$. Equating this to zero, we have that $3x^2 = 12$, or that $x^2 = 4$. Thus, the critical points are $x = \pm 2$. We now evaluate $f(0)$, $f(10)$, and $f(2)$ and compare these values. Note that we did not evaluate $f(-2)$ since $-2 \notin [0, 10]$. $f(0) = 3$, $f(10) = 1000 - 120 + 3 = 883$ and $f(2) = 8 - 24 + 3 = -13$. Thus, the minimum is $(2, -13)$ and the maximum is $(10, 883)$.

2. $f'(x) = 8x+5$. Thus, the critical point is $x = -\frac{5}{8}$. We evaluate the following: $f(0) = -1$, $f(10) = 400 + 50 - 1 = 449$, and $f\left(-\frac{5}{8}\right) = 4\left(-\frac{5}{8}\right)^2 + 5\left(-\frac{5}{8}\right) - 1 = -\frac{41}{16}$. So, the minimum is $\left(-\frac{5}{8}, -\frac{41}{16}\right)$ and the maximum is $(10, 449)$.

3. $f'(x) = 15x^2 + 6x + 4$. Applying the quadratic formula, we obtain

$$\begin{aligned} x &= \frac{-6 \pm \sqrt{36 - 4(15)(4)}}{30} \\ &= \frac{-6 \pm \sqrt{-204}}{30} \\ &= \frac{-6 \pm 2i\sqrt{51}}{30}. \end{aligned}$$

Since these are complex numbers with nonzero imaginary parts, f has no critical points. We now evaluate $f(0) = 5$ and $f(10) = 5345$. Thus, the minimum is $(0, 5)$ and the maximum is $(10, 5345)$.

4. $f'(x) = e^x$. This function has no zeroes and so f has no critical points. However, $f'(x)$ exists for any $x \in \mathbb{R}$. Comparing $f(0) = 1$ and $f(10) = e^{10}$, we see that $f(10) = e^{10} > 1 = f(0)$. Therefore, the minimum is $(0, 1)$ and the maximum is $\left(10, e^{10}\right)$.

EXAMPLE 4: Find two numbers whose product is 25 and whose sum is a maximum.

Solution. We want to maximize the sum of two numbers whose product is 25. In order to write this as a function of a single variable, we have to write one of the numbers in terms of the other. Since their product is 25, we can denote as x one of the numbers, $x \neq 0$, and $\frac{25}{x}$ as the other one. Therefore, we are interested in maximizing the function $f(x) = x + \frac{25}{x}$.

Differentiating this, we obtain $f'(x) = 1 - \frac{25}{x^2}$. Equating this to zero, we obtain $x^2 = 25$, and so $x = \pm 5$. To obtain the maximum, we substitute these into f. So, $f(5) = 5 + \frac{25}{5} = 10$ and $f(-5) = -5 + \frac{25}{-5} = -10$. Since $10 > -10$, we choose the pair 5 and 5.

EXAMPLE 5: If 1000 cm^2 of cardboard is available to make a box with a square base, find the largest possible volume of the box.

Solution. If we let x be the length of a side of the square base and h be the height of the box, the total surface area of the box, S is given by $S = 2x^2 + 4xh$. Since the box is to be made from 1000cm^2 of cardboard, we have that $1000 = 2x^2 + 4xh$, or that $xh = \frac{500 - x^2}{2}$.

The volume of this box, V, is given by $V = x^2h = x(xh) = x\left(\frac{500 - x^2}{2}\right) = \frac{500x - x^3}{2}$. To maximize this, we obtain $V'(x) = \frac{500 - 3x^2}{2}$. To get the critical point, we solve $500 - 3x^2 = 0$.

Thus $x = \pm\dfrac{10\sqrt{15}}{3}$. Since x denotes the length of a side of the base, it must be a positive quantity. Thus, the maximum volume occurs at

$$\begin{aligned} V\left(\frac{10\sqrt{15}}{3}\right) &= \frac{10\sqrt{15}}{3}\left(\frac{500-\left(\frac{10\sqrt{15}}{3}\right)^2}{3}\right) \\ &= \frac{10\sqrt{15}}{3}\left(\frac{500-\frac{500}{3}}{3}\right) \\ &= \frac{10\sqrt{15}}{3}\left(\frac{1000}{9}\right) \\ &= \frac{10000\sqrt{15}}{27}\,\mathrm{cm}^3. \end{aligned}$$

EXAMPLE 6: Find the point on the line $y = 2x + 3$ that is closest to the origin.

Solution. An arbitrary point on the line $y = 2x+3$ is $(x, y) = (x, 2x+3)$. Thus, the distance, d, of this point from the origin can be written as

$$d\,(x) = \sqrt{(x-0)^2 + (2x+3-0)^2}.$$

Observe that since the square root function $f\,(x) = \sqrt{x}$ is increasing on $[0, +\infty)$, d is minimized if and only if d^2 is minimized. We do this to evaluate simpler derivatives. Thus, we minimize $d^2\,(x) = x^2 + (2x+3)^2$. We get the critical points of d^2.

$$\begin{aligned} \left(d^2\,(x)\right)' &= 2x + 2\,(2x+3)\,(2) \\ &= 2x + 8x + 12 \\ &= 10x + 12. \end{aligned}$$

Equating this to zero, we get a critical point $x = -\dfrac{6}{5}$. Clearly, d^2 becomes arbitrarily large as you input larger values of x. Thus, $x = -\dfrac{6}{5}$ would give the minimum. Getting the corresponding y-coordinate, we have $y = 2\left(-\dfrac{6}{5}\right) + 3 = \dfrac{3}{5}$. Thus, the point we are interested in is $\left(-\dfrac{6}{5}, \dfrac{3}{5}\right)$.

Supplementary Problems

1. Find the extrema of the given functions in the specified interval.

   (a) $f(x) = x^2 + 2$ on $[1, 10]$

   (b) $f(x) = x^2 - 5x + 6$ on $[-20, 20]$

   (c) $f(x) = \sin x + \cos x$ on $\left[0, \frac{\pi}{2}\right]$

   (d) $f(x) = e^x - x$ on $[0, 1]$

   (e) $f(x) = \frac{2}{3}x^3 - \frac{5}{2}x^2 - 3x + 1$ on $[-2, 5]$

   (f) $f(x) = \frac{3}{4}x^4 - \frac{1}{3}x^3 - \frac{3}{2}x^2 + x - 1$ on $[-1, 0]$

   (g) $f(x) = 3x^4 + 4x^3 - 12x^2 + 24$ on $[-1, 2]$

   (h) $f(x) = x^4 - 18x^2 + 4$ on $[-4, 5]$

   (i) $f(x) = e^{\sin x}$ on $\left[\frac{\pi}{2}, \pi\right]$

   (j) $f(x) = e^{\cos x}$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

2. Answer the following optimization problems systematically.

   (a) Find two numbers whose sum is 20 and whose product is a maximum.

   (b) Find two numbers whose difference is 50 and whose product is a minimum.

   (c) Find a positive number such that the sum of the number and twice its reciprocal is as small as possible.

   (d) Find a positive number such that the difference of the number and its reciprocal is as large as possible.

   (e) Find the dimensions of a rectangle with perimeter 200 m and whose area is as large as possible.

   (f) Find the dimensions of a rectangle with area is $625\,\mathrm{m}^2$ whose perimeter is as small as possible.

   (g) Find the point on the line $y = 3x - 1$ closest to the point $(-1, 1)$.

   (h) Find the point on the parabola, $y = x^2$ that is closest to the point $(0, 0)$.

   (i) Find the maximum profit given that the profit function is $P(x) = -x^4 + 2x^2 + 1$.

   (j) Find an equation of the tangent line to $y = x^3 - 4x^2 + 5x + 1$ having the least slope.

LESSON 8: Higher-Order Derivatives and the Chain Rule

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Compute higher-order derivatives of functions;
2. Illustrate the Chain Rule of differentiation; and
3. Solve problems using the Chain Rule.

TOPIC 8.1: Higher-Order Derivatives of Functions

Consider the function $y = f(x)$. The derivative $y', f'(x), D_x y$ or $\dfrac{dy}{dx}$ is called the **first derivative** of f with respect to x. The derivative of the first derivative is called the *second derivative* of f with respect to x and is denoted by any of the following symbols:

$$y'' \ , \ f''(x) \ , \ D_x^2 \ y \ , \ \frac{d^2y}{dx^2}$$

The **third derivative** of f with respect to x is the derivative of the second derivative and is denoted by any of the following symbols:

$$y''' \ , \ f'''(x) \ , \ D_x^3 \ y \ , \ \frac{d^3y}{dx^3}$$

In general, the n^{th} ***derivative*** of f with respect to x is the derivative of the $(n-1)^{st}$ derivative and is denoted by any of the following symbols:

$$y^{(n)} \ , \ f^{(n)}(x) \ , \ D_x^n \ y \ , \ \frac{d^ny}{dx^n}$$

Formally, we have the following definition.

Definition 4. *The n^{th}* ***derivative of the function*** *f is defined recursively by*

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad \text{for } n = 1, \text{ and}$$

$$f^{(n)}(x) = \lim_{\Delta x \to 0} \frac{f^{(n-1)}(x+\Delta x) - f^{(n-1)}(x)}{\Delta x} \qquad \text{for } n > 1,$$

provided that these limits exist. Thus, the nth derivative of f is just the derivative of the $(n-1)^{st}$ derivative of f.

Remark 1:

1. The function f can be written as $f^{(0)}(x)$.
2. In the notation $f^{(n)}(x)$, n is called the ***order*** of the derivative.

EXAMPLE 1:

1. Find the fourth derivative of the function $f(x) = x^5 - 3x^4 + 2x^3 - x^2 + 4x - 10$.

Solution. We differentiate the function repeatedly and obtain

$$\begin{aligned} f'(x) &= 5x^4 - 12x^3 + 6x^2 - 2x + 4 \\ f''(x) &= 20x^3 - 36x^2 + 12x - 2 \\ f'''(x) &= 60x^2 - 72x + 12 \\ f^{(4)}(x) &= 120x - 72. \end{aligned}$$

2. Find the first and second derivatives of the function defined by

$$y = (3x^2 - 4)(x^2 - 3x).$$

Solution. Using Product Rule, we compute the first derivative:

$$\begin{aligned} y' &= (3x^2 - 4)D_x(x^2 - 3x) + (x^2 - 3x)D_x(3x^2 - 4) \\ &= (3x^2 - 4)(2x - 3) + (x^2 - 3x)(6x) \\ &= 6x^3 - 9x^2 - 8x + 12 + 6x^3 - 18x^2 \\ &= 12x^3 - 27x^2 - 8x + 12. \end{aligned}$$

Similarly, we obtain the second derivative:

$$\begin{aligned} y'' &= D_x(12x^3 - 27x^2 - 8x + 12) \\ &= 36x^2 - 54x - 8. \end{aligned}$$

3. Let $y = \dfrac{3x+5}{x^2+4}$. Find $\dfrac{d^2y}{dx^2}$.

Solution. Using Quotient Rule twice, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{(x^2+4)D_x(3x+5) - (3x+5)D_x(x^2+4)}{(x^2+4)^2} \\ &= \frac{(x^2+4)(3) - (3x+5)(2x)}{(x^2+4)^2} \\ &= \frac{3x^2 + 12 - 6x^2 - 10x}{(x^2+4)^2} \\ &= \frac{12 - 10x - 3x^2}{(x^2+4)^2}, \end{aligned}$$

and

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{12-10x-3x^2}{(x^2+4)^2}\right)\\
&= \frac{d}{dx}\left(\frac{12-10x-3x^2}{x^4+8x^2+16}\right)\\
&= \frac{(x^4+8x^2+16)\frac{d}{dx}(12-10x-3x^2)-(12-10x-3x^2)\frac{d}{dx}(x^4+8x^2+16)}{(x^4+8x^2+16)^2}\\
&= \frac{(x^4+8x^2+16)(-10-6x)-(12-10x-3x^2)(4x^3+16x)}{(x^4+8x^2+16)^2}\\
&= \frac{6x^5+30x^4-48x^3+80x^2-288x-160}{(x^4+8x^2+16)^2}.
\end{aligned}
$$

4. Find the third derivative of the function defined by $g(x) = -4x^2e^x + 5xe^x - 10e^x$.

Solution. We differentiate repeatedly (applying the Product Rule) and obtain

$$
\begin{aligned}
g^{(1)}(x) &= [(-4x^2)(e^x)+(e^x)(-8x)]+[(5x)(e^x)+(e^x)(5)-10\cdot e^x]\\
&= -4x^2e^x-3xe^x-5e^x\\
&= e^x(-4x^2-3x-5).
\end{aligned}
$$

$$
\begin{aligned}
g^{(2)}(x) &= (e^x)(-8x-3)+(-4x^2-3x-5)(e^x)\\
&= e^x(-4x^2-11x-8).
\end{aligned}
$$

$$
\begin{aligned}
g^{(3)}(x) &= (e^x)(-8x-11)+(-4x^2-11x-8)(e^x)\\
&= e^x(-4x^2-19x-19).
\end{aligned}
$$

5. If $f(x) = e^x \sin x - 3e^x \cos x$, find $f^{(5)}(x)$.

Solution. We differentiate repeatedly (applying Product Rule) and obtain

$$
\begin{aligned}
f'(x) &= [(e^x)(\cos x) + (\sin x)(e^x)] - 3 \cdot [(e^x)(-\sin x) + (\cos x)(e^x)] \\
&= e^x \cos x + e^x \sin x + 3e^x \sin x - 3e^x \cos x \\
&= e^x(4 \sin x - 2 \cos x).
\end{aligned}
$$

$$
\begin{aligned}
f''(x) &= e^x[4(\cos x) - 2(-\sin x)] + (4 \sin x - 2 \cos x)(e^x) \\
&= e^x(2 \cos x + 6 \sin x).
\end{aligned}
$$

$$
\begin{aligned}
f'''(x) &= e^x[2(-\sin x) + 6(\cos x)] + (2 \cos x + 6 \sin x)(e^x) \\
&= e^x(8 \cos x + 4 \sin x).
\end{aligned}
$$

$$
\begin{aligned}
f^{(4)}(x) &= e^x[8(-\sin x) + 4(\cos x)] + (8 \cos x + 4 \sin x)(e^x) \\
&= e^x(12 \cos x - 4 \sin x).
\end{aligned}
$$

$$
\begin{aligned}
f^{(5)}(x) &= e^x[12(-\sin x) - 4(\cos x)] + (12 \cos x - 4 \sin x)(e^x) \\
&= e^x(8 \cos x - 16 \sin x).
\end{aligned}
$$

Solved Examples

EXAMPLE 1: Find the third derivative of the function $f(x) = x^4 - 3x^3 + 2x^2 + 2x + 1$.

Solution. We differentiate the function repeatedly to get the third derivative.

$$
\begin{aligned}
f'(x) &= 4x^3 - 9x^2 + 4x + 2 \\
f''(x) &= 12x^2 - 18x + 4 \\
f'''(x) &= 24x - 18.
\end{aligned}
$$

EXAMPLE 2: Find the first and second derivatives of $f(x) = \left(2x^2 - 3x + 3\right)\left(x^3 - 1\right)$.

Solution. To get the first derivative, we apply Product Rule.

$$
\begin{aligned}
f'(x) &= (4x - 3)\left(x^3 - 1\right) + \left(2x^2 - 3x + 3\right)\left(3x^2\right) \\
&= \left(4x^4 - 3x^3 - 4x + 3\right) + \left(6x^4 - 9x^3 + 9x^2\right) \\
&= 10x^4 - 12x^3 + 9x^2 - 4x + 3.
\end{aligned}
$$

We differentiate this to get the second derivative.

$$f''(x) = 40x^3 - 36x^2 + 18x - 4.$$

EXAMPLE 3: Let $y = \dfrac{5 - 6x^2}{x^2 - 1}$. Find $\dfrac{d^2y}{dx^2}$.

Solution. To get $\dfrac{dy}{dx}$, we apply Quotient Rule.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\left(x^2 - 1\right)(-12x) - \left(5 - 6x^2\right)(2x)}{\left(x^2 - 1\right)^2} \\
&= \frac{\left(-12x^3 + 12x\right) - \left(10x - 12x^3\right)}{\left(x^2 - 1\right)^2} \\
&= \frac{2x}{\left(x^2 - 1\right)^2}.
\end{aligned}$$

Applying Quotient Rule again, we obtain

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{\left(x^4 - 2x^2 + 1\right)(2) - (2x)\left(4x^3 - 4x\right)}{\left[(x^2 - 1)^2\right]^2} \\
&= \frac{2x^4 - 4x^2 + 2 - 8x^4 + 8x^2}{\left(x^2 - 1\right)^4} \\
&= \frac{-6x^4 + 6x^2 + 2}{\left(x^2 - 1\right)^4}.
\end{aligned}$$

EXAMPLE 4: Find the third derivative of $g(x) = xe^x + \sin x$.

Solution. We differentiate repeatedly to get

$$g'(x) = e^x + xe^x + \cos x$$

$$\begin{aligned}
g''(x) &= e^x + e^x + xe^x - \sin x \\
&= 2e^x + xe^x - \sin x
\end{aligned}$$

$$\begin{aligned}
g'''(x) &= 2e^x + e^x + xe^x - \cos x \\
&= 3e^x + xe^x - \cos x.
\end{aligned}$$

EXAMPLE 5: If $f(x) = e^x \sin x$, what is $f^{(5)}(x)$?

Solution.

$$f'(x) = e^x \sin x + e^x \cos x$$
$$= e^x (\sin x + \cos x)$$

$$f''(x) = e^x (\sin x + \cos x) + e^x (\cos x - \sin x)$$
$$= 2e^x \cos x$$

$$f'''(x) = 2e^x \cos x - 2e^x \sin x$$
$$= 2e^x (\cos x - \sin x)$$

$$f^{(4)}(x) = 2e^x (\cos x - \sin x) + 2e^x (-\sin x - \cos x)$$
$$= -4e^x \sin x$$

$$f^{(5)}(x) = (-4e^x \sin x) + (-4e^x \cos x)$$
$$= -4e^x (\sin x + \cos x).$$

EXAMPLE 6: Let $y = xe^x \sin x$. Find $\dfrac{d^2y}{dx^2}$.

Solution. Observe that if $f(x) = r(x)\, s(x)\, t(x)$, then we can regroup f as $f(x) = [r(x)\, s(x)]\, [t(x)]$. Applying now the product rule, we obtain,

$$\begin{aligned} f'(x) &= \left[r'(x)\, s(x) + r(x)\, s'(x)\right] [t(x)] + [r(x)\, s(x)] \left[t'(x)\right] \\ &= r'(x)\, s(x)\, t(x) + r(x)\, s'(x)\, t(x) + r(x)\, s(x)\, t'(x). \end{aligned}$$

We let $r(x) = x$, $s(x) = e^x$, and $t(x) = \sin x$. Applying the above formula, we get that

$$\frac{dy}{dx} = e^x \sin x + xe^x \sin x + xe^x \cos x.$$

Repeating the process, we obtain

$$\begin{aligned} \frac{d^2y}{dx^2} &= e^x \sin x + e^x \cos x + e^x \sin x + xe^x \sin x + xe^x \cos x \\ &\quad + e^x \cos x + xe^x \cos x - xe^x \sin x \\ &= 2e^x \sin x + 2e^x \cos x + 2xe^x \cos x \\ &= 2e^x (\sin x + (1 + x) \cos x). \end{aligned}$$

Supplementary Problems

1. Find y', y'', y''' and $y^{(4)}$ for the following expressions.

   (a) $y = 2x^4$
   (b) $y = x^{-3}$
   (c) $y = x^3$
   (d) $y = x^{\frac{2}{3}}$
   (e) $y = x^{1.2}$
   (f) $y = x^{e^2}$
   (g) $y = x^{\sqrt{\pi}}$

2. Find the first and second derivatives for the following:

   (a) $f(x) = (\sin x)\sqrt{x}$
   (b) $g(x) = x^2 e^x - 2\cos x$
   (c) $h(x) = x^3 + 3x^2 + xe^x$
   (d) $f(x) = \tan x - \sec x + \csc x$
   (e) $f(x) = x \sec x - 4x^{-2}$
   (f) $f(x) = 2x^5 - 4x^3 + 12x - 6$.
   (g) Let $f(x) = -\cos x$. Find $f'\left(\frac{\pi}{4}\right)$, $f''\left(\frac{\pi}{4}\right)$, and $f'''\left(\frac{\pi}{4}\right)$.
   (h) Find $f^{(5)}(1)$ if $f(x) = x^5 - 4x^3 + 2x - 1$.
   (i) Find $D_x^{2016}(\cos x + \sin x)$.
   (j) Find $D_x^{2016}(e^x)$.
   (k) Find $f''(\pi)$ if $f(x) = \sin x \cos x$.
   (l) Find $f'''(4)$ if $f(x) = \sqrt{x} - 2x^2$.
   (m) Consider $f(x) = x+1$, $g(x) = x^2+x+1$, $h(x) = x^3-2x+1$. What is $f''(x), g'''(x)$, and $h^{(4)}(x)$? In general, if $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_j \in \mathbb{R}$ for $j = 1, \ldots, n$, what is $f^{(n+1)}(x)$?

TOPIC 8.2: The Chain Rule

The Chain Rule below provides for a formula for the derivative of a composition of functions.

Theorem 9 (Chain Rule). *Let f be a function differentiable at c and let g be a function differentiable at $f(c)$. Then the composition $g \circ f$ is differentiable at c and*

$$D_x(g \circ f)(c) = g'(f(c)) \cdot f'(c).$$

Remark 1: Another way to state the Chain Rule is the following: If y is a differentiable function of u defined by $y = f(u)$ and u is a differentiable function of x defined by $u = g(x)$, then y is a differentiable function of x, and the derivative of y with respect to x is given by

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

In words, the derivative of a composition of functions is the derivative of the *outer* function evaluated at the inner function, times the derivative of the inner function.

EXAMPLE 1:

1. Recall our first illustration $f(x) = (3x^2 - 2x + 4)^2$. Find $f'(x)$ using Chain Rule.

   Solution. We can rewrite $y = f(x) = (3x^2 - 2x + 4)^2$ as $y = f(u) = u^2$ where $u = 3x^2 - 2x + 4$, a differentiable function of x. Using the Chain Rule, we have

$$\begin{aligned} f'(x) = y' &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= (2u)(6x - 2) \\ &= 2(3x^2 - 2x + 4)(6x - 2) \\ &= 36x^3 - 36x^2 + 56x - 16. \end{aligned}$$

2. For the second illustration, we have $y = \sin(2x)$. Find y' using Chain Rule.

   Solution. We can rewrite $y = \sin(2x)$ as $y = f(u)$ where $f(u) = \sin u$ and $u = 2x$. Hence,

$$\begin{aligned} y' &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \cos u \cdot 2 \\ &= 2\cos(2x). \end{aligned}$$

3. Find $\dfrac{dz}{dy}$ if $z = \dfrac{4}{(a^2 - y^2)^2}$, where a is a real number.

Solution. Notice that we can write $z = \dfrac{4}{(a^2 - y^2)^2}$ as $z = 4(a^2 - y^2)^{-2}$. Applying the Chain Rule, we have

$$\begin{aligned}\frac{dz}{dy} &= 4 \cdot -2(a^2 - y^2)^{-2-1} \cdot \frac{d}{dy}(a^2 - y^2) \\ &= -8(a^2 - y^2)^{-3} \cdot -2y \\ &= 16y(a^2 - y^2)^{-3} \\ &= \frac{16y}{(a^2 - y^2)^3}.\end{aligned}$$

Now, suppose we want to find the derivative of a power function of x. That is, we are interested in $D_x[f(x)^n]$. To derive a formula for this, we let $y = u^n$ where u is a differentiable function of x given by $u = f(x)$. Then by the Chain Rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= nu^{n-1} \cdot f'(x) \\ &= n[f(x)]^{n-1} \cdot f'(x)\end{aligned}$$

Thus, $D_x[f(x)]^n = n[f(x)]^{n-1} \cdot f'(x)$. This is called the GENERALIZED POWER RULE.

EXAMPLE 2:

1. What is the derivative of $y = (3x^2 + 4x - 5)^5$?

Solution.

$$\begin{aligned}D_x[(3x^2 + 4x - 5)^5] &= 5 \cdot (3x^2 + 4x - 5)^{5-1} \cdot D_x(3x^2 + 4x - 5) \\ &= 5(3x^2 + 4x - 5)^4(6x + 4).\end{aligned}$$

2. Find $\dfrac{dy}{dx}$ where $y = \sqrt{3x^3 + 4x + 1}$.

Solution.

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}(3x^3 + 4x + 1)^{\frac{1}{2}-1} D_x(3x^3 + 4x + 1) \\ &= \frac{1}{2}(3x^3 4x + 1)^{-\frac{1}{2}}(9x^2 + 4) \\ &= \frac{9x^2 + 4}{2\sqrt{3x^3 4x + 1}}. \end{aligned}$$

3. Find $\dfrac{dy}{dx}$ where $y = (\sin 3x)^2$.

Solution.

$$\begin{aligned} \frac{dy}{dx} &= 2 \cdot (\sin 3x)^{2-1} \cdot \frac{d}{dx}(\sin 3x) \\ &= 2(\sin 3x) \cdot \cos 3x \cdot \frac{d}{dx}(3x) \\ &= 2(\sin 3x)(\cos 3x) \cdot 3 \\ &= 6 \sin 3x \cos 3x. \end{aligned}$$

4. Differentiate $(3x^2 - 5)^3$.

Solution.

$$\begin{aligned} \frac{d(3x^2 - 5)^3}{dx} &= 3(3x^2 - 5)^{3-1} \cdot D_x(3x^2 - 5) \\ &= 3(3x^2 - 5)^2 \cdot 6x \\ &= 18x(3x^2 - 5)^2. \end{aligned}$$

5. Consider the functions $y = 3u^2 + 4u$ and $u = x^2 + 5$. Find $\dfrac{dy}{dx}$.

Solution. By Chain Rule, we have $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$ where $\dfrac{dy}{du} = 6u + 4$ and $\dfrac{du}{dx} = 2x$. Thus,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= (6u + 4)(2x) \\ &= \left[6(x^2 + 5) + 4\right](2x) \\ &= (6x^2 + 34)(2x) \\ &= 12x^3 + 68x. \end{aligned}$$

Solved Examples

EXAMPLE 1: Let $f(x) = \sqrt{x^2 + 3x + 1}$. Find $f'(x)$.

Solution. Note that we can view f as $f(x) = (g \circ h)(x)$, where $g(x) = \sqrt{x}$ and $h(x) = x^2 + 3x + 1$. Applying Chain Rule, we get

$$\begin{aligned} f'(x) &= g'(h(x)) \cdot h'(x) \\ &= \frac{1}{2\sqrt{x^2 + 3x + 1}}(2x + 3) \\ &= \frac{2x + 3}{2\sqrt{x^2 + 3x + 1}}. \end{aligned}$$

EXAMPLE 2: Suppose $y = 3\left(x^2 - 1\right)^2 + \sin\left(x^2 - 1\right)$. Find $\frac{dy}{dx}$.

Solution. We let $u = x^2 - 1$. Thus, we can write y as $y = 3u^2 + \sin u$. Applying Chain Rule, we obtain

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= (6u + \cos u)(2x) \\ &= \left(6\left(x^2 - 1\right) + \cos\left(x^2 - 1\right)\right)(2x) \\ &= 12\left(x^2 - 1\right) + 2x\cos\left(x^2 - 1\right). \end{aligned}$$

EXAMPLE 3: Let $g(x) = \sin(\cos x)$. What is $g'(x)$?

Solution. We note that g can be expressed as $g(x) = f(h(x))$, where $f(x) = \sin x$ and $h(x) = \cos x$. By Chain Rule, the derivative is

$$\begin{aligned} g'(x) &= f'(h(x)) \cdot h'(x) \\ &= \cos(\cos x) \cdot (-\sin x) \\ &= -(\sin x)(\cos(\cos x)). \end{aligned}$$

EXAMPLE 4: Find $\dfrac{dy}{dx}$ if $y = \sqrt[3]{\tan^2\left(x^3-1\right)}$.

Solution. In this problem, we make repeated use of Chain Rule, starting from the outermost function.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{3}\left(\tan^2\left(x^3-1\right)\right)^{\frac{1}{3}-1}\cdot\frac{d}{dx}\left(\tan^2\left(x^3-1\right)\right)\\
&= \frac{1}{3}\left(\tan^2\left(x^3-1\right)\right)^{-\frac{2}{3}}\cdot 2\tan\left(x^3-1\right)\cdot\frac{d}{dx}\left(\tan\left(x^3-1\right)\right)\\
&= \frac{1}{3}\left(\tan^2\left(x^3-1\right)\right)^{-\frac{2}{3}}\cdot 2\tan\left(x^3-1\right)\cdot\sec^2\left(x^3-1\right)\cdot\frac{d}{dx}\left(x^3-1\right)\\
&= \frac{1}{3}\left(\tan^2\left(x^3-1\right)\right)^{-\frac{2}{3}}\cdot 2\tan\left(x^3-1\right)\cdot\sec^2\left(x^3-1\right)\cdot\left(3x^2\right)\\
&= \frac{2x^2\tan\left(x^3-1\right)\sec^2\left(x^3-1\right)}{\sqrt[3]{\left(\tan^2\left(x^3-1\right)\right)^2}}\\
&= \frac{2x^2\tan\left(x^3-1\right)\sec^2\left(x^3-1\right)}{\sqrt[3]{\tan^4\left(x^3-1\right)}}.
\end{aligned}$$

EXAMPLE 5: Suppose $f(x) = (\sin(2x)+1)^7$. Find $f'\left(\frac{\pi}{2}\right)$.

Solution. We first obtain f'.

$$\begin{aligned}
f'(x) &= 7(\sin(2x)+1)^6\cdot\frac{d}{dx}(\sin(2x)+1)\\
&= 7(\sin(2x)+1)^6(2\cos(2x))\\
&= 14\cos(2x)(\sin(2x)+1)^6.
\end{aligned}$$

We now evaluate this expression at $x = \frac{\pi}{2}$.

$$\begin{aligned}
f'\left(\frac{\pi}{2}\right) &= 14\cos\left(2\cdot\frac{\pi}{2}\right)\left(\sin\left(2\cdot\frac{\pi}{2}\right)+1\right)^6\\
&= 14(-1)(0+1)^6\\
&= -14.
\end{aligned}$$

EXAMPLE 6: Let $f(x) = \tan\left(\dfrac{x+1}{x-1}\right)$. Find $f'(x)$.

Solution.

$$\begin{aligned} f'(x) &= \sec^2\left(\frac{x+1}{x-1}\right)\cdot\frac{d}{dx}\left(\frac{x+1}{x-1}\right)\\ &= \sec^2\left(\frac{x+1}{x-1}\right)\cdot\frac{(x-1)(1)-(x+1)(1)}{(x-1)^2}\\ &= \sec^2\left(\frac{x+1}{x-1}\right)\cdot\frac{-2}{(x-1)^2}\\ &= -\frac{2}{(x-1)^2}\sec^2\left(\frac{x+1}{x-1}\right). \end{aligned}$$

Supplementary Problems

1. Find $\dfrac{dy}{dx}$.

   (a) $y = \left(3x^2-4x+5\right)^3$

   (b) $y = \sqrt{\sin x+\cos x}$

   (c) $y = 2e^{x^2+5} - \cos\left(e^{2x}\right)$

   (d) $y = \sqrt{1+\sqrt{1+\sqrt{1+x}}}$

   (e) $y = (\tan\sqrt{x})^{\frac{1}{3}}$

   (f) $y = e^{\sec x}$

   (g) $y = e^{e^x}$

   (h) $y = \sin\left(\tan\left(e^{x^3-1}\right)\right)$

   (i) $y = \dfrac{\sin\left(x^4-1\right)}{\sqrt{1+2x^2}}$

   (j) $y = \dfrac{e^x}{\sqrt[3]{x^2-1}}$

2. Find the derivatives of the following functions:

   (a) $f(x) = \left(\csc\left(x^2+1\right)\right)\left(\cot\left(x^2+1\right)\right)$

   (b) $g(x) = \left(1-x^2\right)^{-2}$

   (c) $h(x) = \left(x^3-4x^2+1\right)^{\frac{5}{2}}$

   (d) $f(x) = \sqrt{x^2-e^x}$

   (e) $g(x) = e^{2x+1} - \sqrt{x^3+2}$

3. Evaluate the derivatives of the given function at the prescribed following values:

(a) $f(x) = \cos(e^x - 1)$, $x = 0$

(b) $f(x) = e^{x^2+2x+1}$, $x = 1$

(c) $g(x) = \tan\left(\cot\left(x^2\right)\right)$, $x = \dfrac{\sqrt{\pi}}{4}$

(d) $h(x) = \sin(e^x)$, $x = 0$

(e) $f(x) = \sqrt{3x^2 + 5x - 1}$, $x = 2$

LESSON 9: Implicit Differentiation

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Illustrate implicit differentiation;
2. Apply the derivatives of the natural logarithmic and inverse tangent functions; and
3. Use implicit differentiation to solve problems.

TOPIC 9.1: What is Implicit Differentiation?

We have seen that functions are not always given in the form $y = f(x)$ but in a more complicated form that makes it difficult or impossible to express y explicitly in terms of x. Such functions are called implicit functions, and y is said to be defined implicitly. In this lesson, we explain how these can be differentiated using a method called *implicit differentiation*.

Differentiating quantities involving only the variable x with respect to x is not a problem; for instance, the derivative of x is just 1. But if a function y is defined implicitly, then we need to apply the Chain Rule in getting its derivative. So, while the derivative of x^2 is $2x$, the derivative of y^2 is

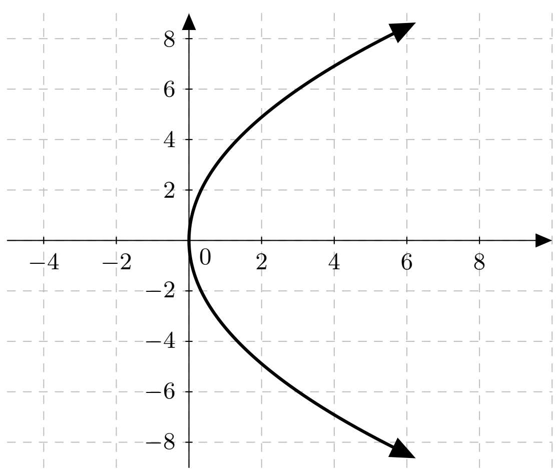
$$2y\frac{dy}{dx}.$$

More generally, if we have the expression $f(y)$, where y is a function of x, then

$$\frac{d}{dx}\left(f(y)\right) = \frac{d}{dy}\left(f(y)\right) \cdot \frac{dy}{dx}.$$

In order to master implicit differentiation, students need to review and master the application of the Chain Rule.

Consider a simple expression such as $y^2 = 4x$. Its graph is a parabola with vertex at the origin and opening to the right.



If we consider only the upper branch of the parabola, then y becomes a function of x. We can obtain the derivative dy/dx by applying the Chain Rule. When differentiating terms involving y, we are actually applying the Chain Rule, that is, we first differentiate with respect to y, and then multiply by dy/dx. Differentiating both sides with respect to x, we have

$$\begin{aligned} y^2 &= 4x, \\ \Longrightarrow \quad \frac{d}{dx}(y^2) &= \frac{d}{dx}(4x) \\ \Longrightarrow \quad 2y\frac{dy}{dx} &= 4. \end{aligned}$$

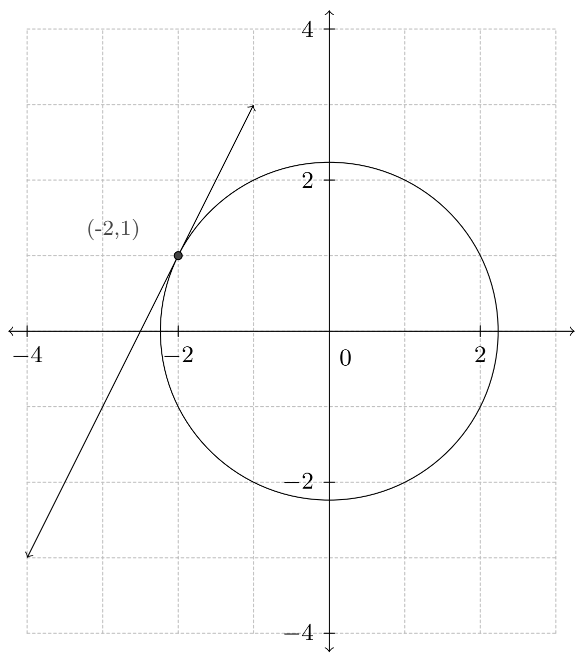
Solving for dy/dx, we obtain

$$\frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}.$$

Notice that the derivative contains y. This is typical in implicit differentiation.

Let us now use implicit differentiation to find the derivatives dy/dx in the following examples. Let us start with our original problem involving the circle.

EXAMPLE 1: Find the slope of the tangent line to the circle $x^2 + y^2 = 5$ at the point $(-2, 1)$.



$$\begin{aligned} x^2 + y^2 &= 5 \\ \Longrightarrow \quad \frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(5) \\ \Longrightarrow \quad \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= 0 \\ \Longrightarrow \quad 2x + 2y\frac{dy}{dx} &= 0. \end{aligned}$$

Solution. Solving for dy/dx, we obtain

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}.$$

Substituting $x = -2$ and $y = 1$, we find that the slope is

$$\frac{dy}{dx} = 2.$$

Notice that this is a faster and easier way to obtain the derivative.

EXAMPLE 2: Find $\dfrac{dy}{dx}$ for $y^3 + 4y^2 + 3x^2y + 10 = 0$.

Solution. Differentiating both sides of the equation gives

$$
\begin{aligned}
& \frac{d}{dx}(y^3 + 4y^2 + 3x^2y + 10) = \frac{d}{dx}(0) \\
\Longrightarrow \quad & \frac{d}{dx}(y^3) + \frac{d}{dx}(4y^2) + \frac{d}{dx}(3x^2y) + \frac{d}{dx}(10) = 0 \\
\Longrightarrow \quad & 3y^2\frac{dy}{dx} + 8y\frac{dy}{dx} + 3x^2\frac{dy}{dx} + 6xy + 0 = 0.
\end{aligned}
$$

We collect the terms involving dy/dx and rearrange to get

$$\frac{dy}{dx}(3y^2 + 8y + 3x^2) + 6xy = 0.$$

Thus,

$$\frac{dy}{dx} = \frac{-6xy}{3y^2 + 8y + 3x^2}.$$

Note that the derivative of the term $3x^2y$ is obtained by applying Product Rule. We consider $3x^2$ as one function and y as another function.

Implicit differentiation can be applied to any kind of function, whether they are polynomial functions, or functions that involve trigonometric and exponential quantitites.

EXAMPLE 3: Find $\dfrac{dy}{dx}$ for $x + y^3 = e^{xy^4}$.

Solution. Differentiating both sides with respect to x gives

$$
\begin{aligned}
& \frac{d}{dx}(x + y^3) = \frac{d}{dx}(e^{xy^4}) \\
\Longrightarrow \quad & 1 + 3y^2\frac{dy}{dx} = e^{xy^4}\frac{d}{dx}(xy^4) \\
\Longrightarrow \quad & 1 + 3y^2\frac{dy}{dx} = e^{xy^4}\left(4xy^3\frac{dy}{dx} + y^4\right)
\end{aligned}
$$

Collecting all terms with dy/dx gives

$$
\begin{aligned}
& \frac{dy}{dx}\left(3y^2 - e^{xy^4}4xy^3\right) = -1 + e^{xy^4}y^4 \\
\Longrightarrow \quad & \frac{dy}{dx} = \frac{-1 + e^{xy^4}y^4}{3y^2 - e^{xy^4}4xy^3}.
\end{aligned}
$$

DERIVATIVES OF THE NATURAL LOGARITHMIC AND INVERSE TANGENT FUNCTIONS

Derivatives of the Natural Logarithmic and Inverse Tangent Functions

Suppose u is a function of x. Then

- $\dfrac{d}{dx}(\ln u) = \dfrac{1}{u} \cdot \dfrac{du}{dx}$
- $\dfrac{d}{dx}(\tan^{-1} u) = \dfrac{1}{1+u^2} \cdot \dfrac{du}{dx}$

EXAMPLE 4: Find dy/dx.

1. $y = \ln(7x^2 - 3x + 1)$
2. $y = \tan^{-1}(2x - 3\cos x)$
3. $y = \ln(4x + \tan^{-1}(\ln x))$

Solution.

1. $\dfrac{dy}{dx} = \dfrac{1}{7x^2 - 3x + 1} \cdot (14x - 3)$
2. $\dfrac{dy}{dx} = \dfrac{1}{1 + (2x - 3\cos x)^2} \cdot (2 + 3\sin x)$
3. $\dfrac{dy}{dx} = \dfrac{1}{4x + \tan^{-1}(\ln x)} \cdot \left(4 + \dfrac{1}{1 + (\ln x)^2} \cdot \dfrac{1}{x}\right).$

EXAMPLE 5: Find $\dfrac{dy}{dx}$ for $\cos(y^2 - 3) = \tan^{-1}(x^3) + \ln y$.

Solution. Differentiating both sides gives

$$\begin{aligned} & \frac{d}{dx}(\cos(y^2 - 3)) = \frac{d}{dx}(\tan^{-1}(x^3) + \ln y) \\ \Longrightarrow \quad & -\sin(y^2 - 3) \cdot 2y \cdot \frac{dy}{dx} = \frac{1}{1 + (x^3)^2} \cdot 3x^2 + \frac{1}{y} \cdot \frac{dy}{dx}. \end{aligned}$$

Collecting terms with dy/dx:

$$\begin{aligned}
& \frac{dy}{dx}\left(-2y\sin(y^2-3)-\frac{1}{y}\right)=\frac{3x^2}{1+x^6} \\
\implies \quad & \frac{dy}{dx}=\frac{\dfrac{3x^2}{1+x^6}}{-2y\sin(y^2-3)-\dfrac{1}{y}}.
\end{aligned}$$

EXAMPLE 6: Find $\dfrac{dy}{dx}$ for $\tan^{-1} y = 3x^2y - \sqrt{\ln(x-y^2)}$.

Solution. Differentiating both sides with respect to x gives

$$\begin{aligned}
& \frac{d}{dx}\left(\tan^{-1} y\right)=\frac{d}{dx}\left(3x^2y-(\ln(x-y^2))^{1/2}\right) \\
\implies \quad & \frac{1}{1+y^2}\cdot\frac{dy}{dx}=3x^2\frac{dy}{dx}+6xy-\frac{1}{2}\left(\ln(x-y^2)\right)^{-1/2}\frac{1}{x-y^2}\left(1-2y\frac{dy}{dx}\right) \\
\implies \quad & \frac{1}{1+y^2}\cdot\frac{dy}{dx}=3x^2\frac{dy}{dx}+6xy-\frac{(\ln(x-y^2))^{-1/2}}{2(x-y^2)}+\frac{y(\ln(x-y^2))^{-1/2}}{x-y^2}\frac{dy}{dx}.
\end{aligned}$$

We now isolate dy/dx:

$$\begin{aligned}
& \frac{dy}{dx}\left(\frac{1}{1+y^2}-3x^2-\frac{y(\ln(x-y^2))^{-1/2}}{x-y^2}\right)=6xy-\frac{(\ln(x-y^2))^{-1/2}}{2(x-y^2)} \\
\implies \quad & \frac{dy}{dx}=\frac{6xy-\dfrac{(\ln(x-y^2))^{-1/2}}{2(x-y^2)}}{\dfrac{1}{1+y^2}-3x^2-\dfrac{y(\ln(x-y^2))^{-1/2}}{x-y^2}}.
\end{aligned}$$

Solved Examples

EXAMPLE 1: Find the equation of the tangent line to the curve of $4x^2+9y^2=36$ at the point $(0,2)$.

Solution.

$$
\begin{aligned}
4x^2 + 9y^2 &= 36 \\
\frac{d}{dx}\left(4x^2 + 9y^2\right) &= \frac{d}{dx}(36) \\
\frac{d}{dx}\left(4x^2\right) + \frac{d}{dx}\left(9y^2\right) &= 0 \\
8x + 18y\frac{dy}{dx} &= 0 \\
18y\frac{dy}{dx} &= -8x \\
\frac{dy}{dx} &= -\frac{4x}{9y}.
\end{aligned}
$$

To get the slope of the tangent line, we evaluate the derivative at the point $(0, 2)$. Thus,

$$
\frac{dy}{dx} = \frac{-4\,(0)}{9\,(2)} = 0.
$$

Using the point slope form, we get

$$
\begin{aligned}
y - 2 &= 0 \cdot (x - 0) \\
y &= 2.
\end{aligned}
$$

EXAMPLE 2: Find $\dfrac{dy}{dx}$ for $xy^2 + 4y^3 - 6x = 10$.

Solution.

$$
\begin{aligned}
\frac{d}{dx}\left(xy^2 + 4y^3 - 6x\right) &= \frac{d}{dx}(10) \\
\frac{d}{dx}\left(xy^2\right) + \frac{d}{dx}\left(4y^3\right) - \frac{d}{dx}(6x) &= 0 \\
\left(y^2 + 2xy\frac{dy}{dx}\right) + 12y^2\frac{dy}{dx} - 6 &= 0 \\
\left(2xy + 12y^2\right)\frac{dy}{dx} &= 6 - y^2 \\
\frac{dy}{dx} &= \frac{6 - y^2}{2xy + 12y^2}.
\end{aligned}
$$

EXAMPLE 3: Find $\dfrac{dy}{dx}$ for $\sin(xy) - \cos(x + y) = \ln(x - y)$.

Solution.

$$
\begin{aligned}
\frac{d}{dx}(\sin(xy)-\cos(x+y)) &= \frac{d}{dx}(\ln(x-y)) \\
\frac{d}{dx}(\sin(xy))-\frac{d}{dx}(\cos(x+y)) &= \left(\frac{1}{x-y}\right)\left(1-\frac{dy}{dx}\right) \\
\cos(xy)\cdot\left(y+x\frac{dy}{dx}\right)+\sin(x+y)\cdot\left(1+\frac{dy}{dx}\right) &= \left(\frac{1}{x-y}\right)\left(1-\frac{dy}{dx}\right) \\
\left(x\cos(xy)+\sin(x+y)+\frac{1}{x-y}\right)\frac{dy}{dx} &= \frac{1}{x-y}-y\cos(xy)-\sin(x+y)
\end{aligned}
$$

Therefore,

$$
\frac{dy}{dx}=\frac{\frac{1}{x-y}-y\cos(xy)-\sin(x+y)}{x\cos(xy)+\sin(x+y)+\frac{1}{x-y}}.
$$

EXAMPLE 4: Find $\frac{dy}{dx}$ if $y^2-3xe^{xy}=\sin(\cos(xy))$.

Solution.

$$
\begin{aligned}
\frac{d}{dx}\left(y^2-3xe^{xy}\right) &= \frac{d}{dx}(\sin(\cos(xy))) \\
\frac{d}{dx}\left(y^2\right)-3\frac{d}{dx}\left(xe^{xy}\right) &= \cos(\cos(xy))\cdot\frac{d}{dx}(\cos(xy)) \\
2y\frac{dy}{dx}-3\left(e^{xy}+x\frac{d}{dx}\left(e^{xy}\right)\right) &= \cos(\cos(xy))\cdot(-\sin(xy))\cdot\left(y+x\frac{dy}{dx}\right) \\
2y\frac{dy}{dx}-3\left(e^{xy}+xe^{xy}\left(y+x\frac{dy}{dx}\right)\right) &= -\cos(\cos(xy))\cdot(\sin(xy))\cdot\left(y+x\frac{dy}{dx}\right) \\
\left(2y-3x^2e^{xy}+x\sin(xy)\cos(\cos(xy))\right)\frac{dy}{dx} &= 3e^{xy}+3xye^{xy}-y\cos(\cos(xy))\sin(xy)
\end{aligned}
$$

Therefore,

$$
\frac{dy}{dx}=\frac{3e^{xy}+3xye^{xy}-y\cos(\cos(xy))\sin(xy)}{2y-3x^2e^{xy}+x\sin(xy)\cos(\cos(xy))}.
$$

EXAMPLE 5: Find $\frac{dy}{dx}$ for $x\tan y-\sqrt{y^2+2y}=e^y$.

Solution.

$$
\begin{aligned}
\frac{d}{dx}\left(x\tan y-\sqrt{y^2+2y}\right)&=\frac{d}{dy}\left(e^y\right)\\
\frac{d}{dx}\left(x\tan y\right)-\frac{d}{dx}\left(\sqrt{y^2+2y}\right)&=e^y\frac{dy}{dx}\\
\tan y+x\sec^2 y\frac{dy}{dx}-\frac{1}{2\sqrt{y^2+2y}}\left(2y+2\right)\frac{dy}{dx}&=e^y\frac{dy}{dx}\\
\left(x\sec^2 x-\frac{y+1}{\sqrt{y^2+2y}}-e^y\right)\frac{dy}{dx}&=-\tan y
\end{aligned}
$$

Therefore,

$$
\frac{dy}{dx}=\frac{-\tan y}{x\sec^2 x-\dfrac{y+1}{\sqrt{y^2+2y}}-e^y}.
$$

EXAMPLE 6: Find the equation of the normal line to the curve of $y^2-xe^{y-1}+x^3=7$ at the point $(2,1)$.

Solution. To get the equation of the normal line, we must first obtain its slope. We note that the tangent and normal lines are perpendicular. Hence, their slopes are negative reciprocals of each other. To get the slope of the tangent line, we apply implicit differentiation.

$$
\begin{aligned}
\frac{d}{dx}\left(y^2-xe^{y-1}+x^3\right)&=\frac{d}{dx}\left(7\right)\\
2y\frac{dy}{dx}-\left(e^{y-1}+xe^{y-1}\frac{dy}{dx}\right)+3x^2&=0\\
\left(2y-xe^{y-1}\right)\frac{dy}{dx}&=e^{y-1}-3x^2\\
\frac{dy}{dx}&=\frac{e^{y-1}-3x^2}{2y-xe^{y-1}}.
\end{aligned}
$$

Thus, the expression which would give us the slope of the normal line at any point on the curve is given by

$$
\begin{aligned}
m_{NL}&=-\frac{1}{\left(\dfrac{dy}{dx}\right)}\\
&=-\frac{2y-xe^{y-1}}{e^{y-1}-3x^2}\\
&=\frac{xe^{y-1}-2y}{e^{y-1}-3x^2}.
\end{aligned}
$$

At the point $(2,1)$, we get that the slope of the normal line is

$$\begin{aligned} m_{NL} &= \frac{2e^{1-1}-2}{e^{1-1}-3\,(2)^2} \\ &= \frac{0}{1-12} \\ &= 0. \end{aligned}$$

Observe that $(2,1)$ must also be on the normal line. Thus, the equation of the normal line is

$$\begin{aligned} y-1 &= 0\cdot(x-2) \\ y &= 1. \end{aligned}$$

EXAMPLE 7: Find dy/dx.

1. $y = \ln\left(3x^2-4x+2\right)$
2. $y = \tan^{-1}(3x-2\sin x)$
3. $y = \ln\left(2x-\tan^{-1}(\ln x)\right)$

Solution.

1. $\dfrac{dy}{dx} = \dfrac{1}{3x^2-4x+2}\cdot(6x-4) = \dfrac{6x-4}{3x^2-4x+2}.$
2. $\dfrac{dy}{dx} = \dfrac{1}{1+(3x-2\sin x)^2}\cdot(3-2\cos x) = \dfrac{3-2\cos x}{1+(3x-2\sin x)^2}.$
3. $\dfrac{dy}{dx} = \dfrac{1}{2x-\tan^{-1}(\ln x)}\cdot\left(2-\dfrac{1}{1+\ln^2 x}\cdot\dfrac{1}{x}\right).$

EXAMPLE 8: Find $\dfrac{dy}{dx}$ for $\sin\left(2-y^3\right) = \tan^{-1}\left(x^2\right) - \ln y$.

Solution.

$$\begin{aligned} \cos\left(2-y^3\right)\left(-3y^2\right)\frac{dy}{dx} &= \frac{1}{1+\left(x^2\right)^2}\cdot(2x) - \frac{1}{y}\cdot\frac{dy}{dx} \\ \left(\frac{1}{y}-3y^2\cos\left(2-y^3\right)\right)\frac{dy}{dx} &= \frac{2x}{1+x^4} \\ \frac{dy}{dx} &= \frac{2x}{(1+x^4)\left(\dfrac{1}{y}-3y^2\cos(2-y^3)\right)}. \end{aligned}$$

EXAMPLE 9: Find $\dfrac{dy}{dx}$ for $\ln y - 3xy^2 = \sqrt{\tan^{-1} y}$.

Solution.

$$\begin{aligned}
\frac{1}{y} \cdot \frac{dy}{dx} - 3\left(y^2 + 2xy\frac{dy}{dx}\right) &= \frac{1}{2\sqrt{\tan^{-1} y}} \cdot \frac{1}{1+y^2} \cdot \frac{dy}{dx} \\
\left(\frac{1}{y} - 6xy - \frac{1}{2\sqrt{\tan^{-1} y}} \cdot \frac{1}{1+y^2}\right)\frac{dy}{dx} &= 3y^2 \\
\frac{dy}{dx} &= \frac{3y^2}{\left(\dfrac{1}{y} - 6xy - \dfrac{1}{2\sqrt{\tan^{-1} y}} \cdot \dfrac{1}{1+y^2}\right)}.
\end{aligned}$$

EXAMPLE 10: Find the derivative with respect to x of

$$\tan^{-1}\left(y^2\right) + \ln(y+1) = \sin^3(x+1)$$

Solution.

$$\begin{aligned}
\frac{1}{1+\left(y^2\right)^2} \cdot 2y\frac{dy}{dx} + \frac{1}{y+1} \cdot \frac{dy}{dx} &= 3\sin^2(x+1)\cos(x+1) \\
\left(\frac{2y}{1+y^4} + \frac{1}{y+1}\right)\frac{dy}{dx} &= 3\sin^2(x+1)\cos(x+1) \\
\frac{dy}{dx} &= \frac{3\sin^2(x+1)\cos(x+1)}{\left(\dfrac{2y}{1+y^4} + \dfrac{1}{y+1}\right)}.
\end{aligned}$$

Supplementary Problems

1. Find $\dfrac{dy}{dx}$ for the following curves:

   (a) $x^2y - \sqrt{y} = 17 - \tan x$

   (b) $\sin(x+y) - e^{x-y} = \sec y$

   (c) $y^2 \cos x^2 - x^2 \sin y^2 = 1$

   (d) $(x-2)^2 + (y-3)^2 = 25$

   (e) $\dfrac{(y-3)^2}{7} - \dfrac{(x+1)^2}{4} = 1$

   (f) $ye^x + xe^y = 12$

   (g) $ye^x - xe^y = \ln(x+y)$

   (h) $\dfrac{x+y}{x-y} = 1$

   (i) $x \sec y = x - y$

   (j) $y^{\frac{3}{2}} e^x = 2$

   (k) $\sqrt{x + \sqrt{y}} = 1$

2. find the equation of the tangent and normal lines at the given points.

   (a) $x + y = 2$ at $(1, 1)$

   (b) $y^3 - xy + x^2 = 1$ at $(1, 0)$

   (c) $\cot(x + y) - \sin(x - y) = 1$ at $\left(0, \dfrac{\pi}{2}\right)$

   (d) $e^{-x} - y = 0$ at $(-1, 0)$

   (e) $3x^2 - 4y^3 = 8$ at $(-2, 1)$

LESSON 10: Related Rates

LEARNING OUTCOME: At the end of the lesson, the learner shall be able to solve situational problems involving related rates.

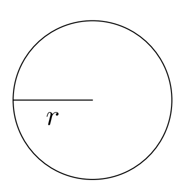
Suggestions in solving problems involving related rates:

1. If possible, provide an illustration for the problem that is valid for any time t.
2. Identify those quantities that change with respect to time, and represent them with variables. (Avoid assigning variables to quantities which are constant, that is, which do not change with respect to time. Label them right away with the values provided in the problem.)
3. Write down any numerical facts known about the variables. Interpret each rate of change as the derivative of a variable with respect to time. Remember that if a quantity decreases over time, then its rate of change is negative.
4. Identify which rate of change is being asked, and under what particular conditions this rate is being computed.
5. Write an equation showing the relationship of all the variables by an equation that is valid for any time t.
6. Differentiate the equation in (5) implicitly with respect to t.
7. Substitute into the equation, obtained in (6), all values that are valid at the particular time of interest. Sometimes, some quantities still need to be solved by substituting the particular conditions written in (4) to the equation in (6). Then, solve for what is being asked in the problem.
8. Write a conclusion that answers the question of the problem. Do not forget to include the correct units of measurement.

EXAMPLE 7: A water droplet falls onto a still pond and creates concentric circular ripples that propagate away from the center. Assuming that the area of a ripple is increasing at the rate of 2π cm^2/s, find the rate at which the radius is increasing at the instant when the radius is 10 cm.

Solution. We solve this step-by-step using the above guidelines.

(1) Illustration



(2) Let r and A be the radius and area, respectively, of a circular ripple at any time t.

(3) The given rate of change is $\dfrac{dA}{dt} = 2\pi$.

(4) We are asked to find $\dfrac{dr}{dt}$ at the instant when $r = 10$.

(5) The relationship between A and r is given by the formula for the area of a circle:

$$A = \pi r^2.$$

(6) We now differentiate implicitly with respect to time. (Be mindful that all quantities here depend on time, so we should always apply Chain Rule.)

$$\frac{dA}{dt} = \pi(2r)\frac{dr}{dt}.$$

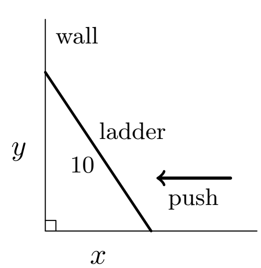
(7) Substituting $\dfrac{dA}{dt} = 2\pi$ and $r = 10$ gives

$$\begin{aligned} & 2\pi = \pi \cdot 2(10)\frac{dr}{dt} \\ \Longrightarrow \quad & \frac{dr}{dt} = \frac{1}{10}. \end{aligned}$$

(8) Conclusion: The radius of a circular ripple is increasing at the rate of $\dfrac{1}{10}$ cm/s.

EXAMPLE 8: A ladder 10 meters long is leaning against a wall. If the bottom of the ladder is being pushed horizontally towards the wall at 2 m/s, how fast is the top of the ladder moving when the bottom is 6 meters from the wall?

Solution. We first illustrate the problem.



Let x be the distance between the bottom of the ladder and the wall. Let y be the distance between the top of the ladder and the ground (as shown). Note that the length of the ladder is not represented by a variable as it is constant.

We are given that $\dfrac{dx}{dt} = -2$. (Observe that this rate is negative since the quantity x decreases with time.)

We want to find $\frac{dy}{dt}$ at the instant when $x = 6$.

Observe that the wall, the ground and the ladder determine a right triangle. Hence, the relationship between x and y is given by the the Pythagorean Theorem:

$$x^2 + y^2 = 100. \tag{2.4}$$

Differentiating both sides with respect to time t gives

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0. \tag{2.5}$$

Before we proceed to the next step, we ask ourselves if we already have everything we need. So, dx/dt is given, dy/dt is the quantity required, x is given, BUT, we still do not have y.

This is easy to solve by substituting the given condition $x = 6$ into the equation in (2.4). So,

$$6^2 + y^2 = 100 \quad \Longrightarrow \quad y = \sqrt{100 - 36} = \sqrt{64} = 8.$$

Finally, we substitute all the given values into equation (2.5):

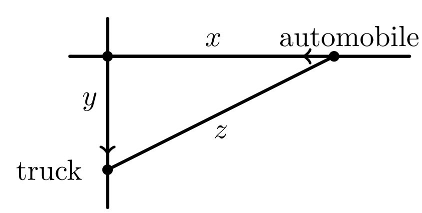
$$2(6)(-2) + 2(8)\frac{dy}{dt} = 0 \quad \Longrightarrow \quad \frac{dy}{dt} = \frac{24}{16} = \frac{3}{2}.$$

Thus, the distance between the top of the ladder and the ground is increasing at the rate of 1.5 m/s. Equivalently, we can also say that the top of the ladder is moving at the rate of 1.5 m/s.

EXAMPLE 9: An automobile traveling at the rate of 20 m/s is approaching an intersection. When the automobile is 100 meters from the intersection, a truck traveling at the rate of 40 m/s crosses the intersection. The automobile and the truck are on perpendicular roads. How fast is the distance between the truck and the automobile changing two seconds after the truck leaves the intersection?

Solution. Let us assume that the automobile is travelling west while the truck is travelling south as illustrated below.

Let x denote the distance of the automobile from the intersection, y denote the distance of the truck from the intersection, and z denote the distance between the truck and the automobile.



Then we have $\dfrac{dx}{dt} = -20$ (the negative rate is due to the fact that x decreases with time) and $\dfrac{dy}{dt} = 40$. We want to find $\dfrac{dz}{dt}$ when $t = 2$.

The equation relating x, y and z is given by the Pythagorean Theorem. We have

$$x^2 + y^2 = z^2. \tag{2.6}$$

Differentiating both sides with respect to t,

$$\begin{aligned} & \frac{d}{dt}\left[x^2 + y^2\right] &=& \frac{d}{dt}\left[z^2\right] \\ \Longrightarrow \quad & 2x\frac{dx}{dt} + 2y\frac{dy}{dt} &=& 2z\frac{dz}{dt} \\ \Longrightarrow \quad & x\frac{dx}{dt} + y\frac{dy}{dt} &=& z\frac{dz}{dt}. \end{aligned}$$

Before substituting the given values, we still need to find the values of x, y and z when $t = 2$. This is found by the distance-rate-time relationship:

$$distance \;=\; rate \times time.$$

For the automobile, after 2 seconds, it has travelled a distance equal to $(rate)(time) = 20(2) = 40$ from the 100 meter mark. Therefore, $x = 100 - 40 = 60$. On the other hand, for the truck, it has travelled $y = (rate)(time) = 40(2) = 80$. The value of z is found from (2.6):

$$z = \sqrt{x^2 + y^2} = \sqrt{60^2 + 80^2} = \sqrt{10^2(36 + 64)} = 100.$$

Finally,

$$40(-20) + 80(40) = 100\frac{dz}{dt} \quad \Longrightarrow \quad \frac{dz}{dt} = 20.$$

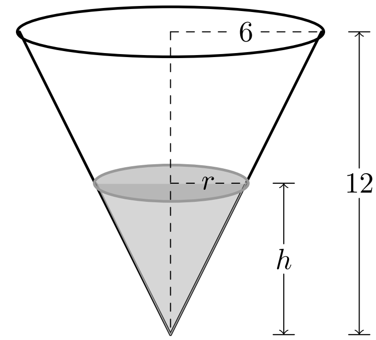
Thus, the distance between the automobile and the truck is increasing at the rate of 20 meters per second.

EXAMPLE 10: Water is pouring into an inverted cone at the rate of 8 cubic meters per minute. If the height of the cone is 12 meters and the radius of its base is 6 meters, how fast is the water level rising when the water is 4-meter deep?

Solution. We first illustrate the problem.

Let V be the volume of the water inside the cone at any time t. Let $h,\ r$ be the height and radius, respectively, of the cone formed by the volume of water at any time t.

We are given $\dfrac{dV}{dt} = 8$ and we wish to find $\dfrac{dh}{dt}$ when $h = 4$.



Now, the relationship between the three defined variables is given by the volume of the cone:

$$V = \frac{\pi}{3} r^2 h.$$

Observe that the rate of change of r is neither given nor asked. This prompts us to find a relationship between r and h. From the illustration, we see that by the proportionality relations in similar triangles, we obtain

$$\frac{r}{h} = \frac{6}{12}$$

or $r = \dfrac{h}{2}$. Thus,

$$V = \frac{\pi}{3} r^2 h = \frac{\pi}{3} \left(\frac{h}{2}\right)^2 h = \frac{\pi}{12} h^3.$$

Differentiating both sides with respect to t,

$$\begin{array}{rrcl} & \dfrac{d}{dt}(V) & = & \dfrac{d}{dt}\left(\dfrac{\pi}{12} h^3\right) \\ \Longrightarrow & \dfrac{dV}{dt} & = & \dfrac{\pi}{4} h^2 \dfrac{dh}{dt}. \end{array}$$

Thus, after substituting all given values, we obtain

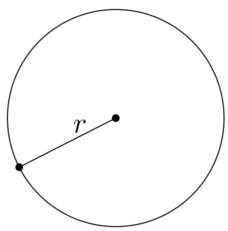
$$8 = \frac{\pi}{4}(4)^2 \frac{dh}{dt} \quad \Longrightarrow \quad \frac{dh}{dt} = \frac{32}{16\pi} = \frac{2}{\pi}.$$

Finally, we conclude that the water level inside the cone is rising at the rate of $\dfrac{2}{\pi}$ meters/minute.

Solved Examples

EXAMPLE 11: The radius of a circle is increasing at the rate of 5 cm/s. How fast is the area of the circle changing when the radius is 6 cm long?

Solution. Let r and A be the radius and the area of the circle respectively.



Given: $\frac{dr}{dt} = 5$

Asked: Find $\frac{dA}{dt}$ when $r = 6$.

The area of the circle is $A = \pi r^2$. Differentiate both sides of the equation with respect to t to obtain

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$$

Substitute $r = 6$ and $\frac{dr}{dt} = 5$:

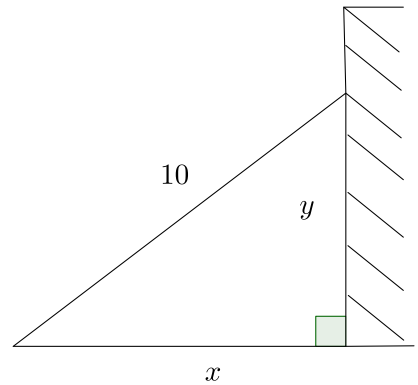
$$\begin{aligned}\frac{dA}{dt} &= 2\pi(6)(5) \\ &= 60\pi.\end{aligned}$$

So, the area of the circle is increasing at the rate of 60π cm^2/s.

EXAMPLE 12: A ladder 10 meters long leans against a wall. If the bottom of the ladder is being pulled away from the wall at the rate of 2 m/min, how fast is the ladder sliding down the wall when the top of the ladder is 3 meters from the ground?

Solution. Let x and y be the distance of the bottom and top of the ladder from the wall and the ground respectively. Given: $\frac{dx}{dt} = 2$. Find: $\frac{dy}{dt}$, when $y = 3$.

By Pythagorean Theorem, $x^2+y^2=100$. Differentiating with respect to t:



$$\begin{aligned} 2x\frac{dx}{dt}+2y\frac{dy}{dt} &= 0 \\ \Longrightarrow \frac{dy}{dt} &= -\frac{x}{y}\frac{dx}{dt}. \end{aligned}$$

Substitute $x=6$, $y=3$ and $\frac{dx}{dt}=2$:

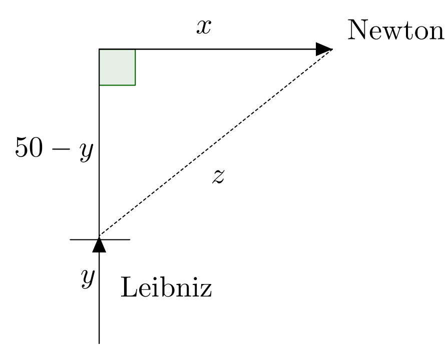
$$\begin{aligned} \frac{dy}{dt} &= -\frac{6\,(2)}{3} \\ &= -4. \end{aligned}$$

Therefore, the ladder is sliding down the wall at the rate of 4 m/min when the top of the ladder is 3 meters from the ground.

EXAMPLE 13: Newton is traveling eastward at 20 km/hr while Leibniz is traveling northward at 10 km/hr. At 1 pm, Leibniz is located 50 km south of Newton. At what rate is the distance between Newton and Leibniz changing at 3 pm?

Solution. Let x and y be the distance traveled by Newton and Leibniz, respectively, and let z be the distance from Newton to Lebniz.

Given: $\frac{dx}{dt}=20$. $\frac{dy}{dt}=10$.
Find: $\frac{dz}{dt}$ at 3 pm.



$$\begin{aligned} z^2 &= x^2+(50-y)^2 \\ \Longrightarrow z\frac{dz}{dt} &= x\frac{dx}{dt}+(50-y)\frac{dy}{dt}. \end{aligned}$$

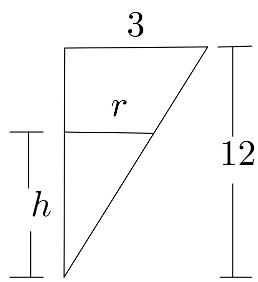
At 3 pm, $x=40$, $y=20$, $z=\sqrt{x^2+y^2}=\sqrt{40^2+30^2}=50$. Substitute these values to our equation to obtain

$$\begin{aligned} 50\frac{dz}{dt} &= 40\,(20)+(30)\,(10) \\ \frac{dz}{dt} &= 22. \end{aligned}$$

Hence, at 3 pm, the distance between Newton and Leibniz is increasing at the rate of 22 km/hr.

EXAMPLE 14: A water tank in the shape of an inverted right circular cone is 9 meters high. The top rim of the tank is a circle with a radius of 3 meters. If the tank is being filled with water at the rate of 2 cubic meters per minute, what is the rate of change of the water depth, in meters per minute, when the depth is 5 meters?

Solution. Let r, h and V be the radius, height and volume of the cone formed by the water inside the tank, respectively.



Given: $dr/dt = 2$, $r = 3$, $h = 5$.
Find: dh/dt when $h = 5$. The volume of the cone is $V = \frac{1}{3}\pi r^2 h$. We will express V is in terms of h only. Using similar triangles, we have

$$\begin{aligned} \frac{h}{r} &= \frac{9}{3} \\ \implies r &= \frac{h}{3}. \end{aligned}$$

Substitute this in V to get

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi \left(\frac{h}{3}\right)^2 h \\ &= \frac{1}{27}\pi h^3. \end{aligned}$$

Differentiate both sides of the equation with respect to t:

$$\frac{dV}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt}.$$

Substitute $dV/dt = 2$, $h = 5$ and solve for dh/dt :

$$\begin{aligned} 2 &= \frac{1}{9}\pi \left(5^2\right) \frac{dh}{dt} \\ \implies \frac{dh}{dt} &= \frac{18}{25\pi}. \end{aligned}$$

Hence, the rate of change of the water depth when the depth is 5 meters is $\frac{18}{25\pi}$ m^3/min.

Supplementary Problems

1. Let x and y be differentiable functions of t and suppose that they are related by the equation $xy - 1 = y^2$. Find dx/dt when $x = 2$, and $dy/dt = 1$.

2. Let r, h, and A be differentiable functions of t and $A = 2\pi r^2 + 2\pi rh$. Find dA/dt when $r = 2$, $h = 5$, $dr/dt = 2$ and $dh/dt = 1$.

3. Starting from the same point, Royden starts walking eastward at $\frac{1}{2}$ m/s while Nilo starts running towards southward at 2 m/s. How fast is the distance between Royden and Nilo increasing after 3 s?

4. A ladder 20 meters long leans against a wall. If the bottom of the ladder is being pushed towards the wall at the rate of 20 m/min, how fast is the top of the ladder moving up the wall when the top of the ladder is 6 meters from the ground?

5. Car A leaves a parking lot at 9 am and travels eastward at the rate of 60 kph. Another vehicle, car B, leaves the same parking lot at 10 am and travels southward at 40 kph. How fast is the distance between these two cars changing at 11 am?

6. A spherical balloon is being inflated in such a way that the radius is increasing at the rate of 1 m/s. What is the rate of change of its volume when its radius is 5 m?

7. A spherical ball shrinks at the constant rate of 5 cm^3/s. How fast is the radius changing when its volume is 20 cm^3?

8. A spherical ball is being pumped with air at the rate of 10 cm^3/s. How fast is its surface area changing when the radius is 30 cm?

9. Standing on top of a building and through a telescope, Jona observes Marco running towards the building. If the telescope is 400 m above the ground and Marco is running at a speed of 20 m/s, at what rate is the measure of the acute angle formed by the telescope and the vertical changing when Marco is 100 m from the building?

10. Standing on a cliff, Daniela is watching a motor boat through a telescope as the boat approaches the shoreline directly below her. If the telescope is 300 meters above the water level and if the boat is approaching the cliff at 30 m/s, at what rate is the acute angle, made by the telescope with the cliff, changing when the boat is $100\sqrt{3}$ meters from the shore?

CHAPTER 2 REVIEW

I. TRUE or FALSE. Write TRUE if the statement is always true. Otherwise, write FALSE.

1. If f is continuous at x, then f is differentiable at x.
2. $\lim\limits_{x \to x_0} \dfrac{(x+x_0)^{20} - x^{20}}{x - x_0} = 20x^{19}$.
3. If f is discontinuous at $x = 0$, then f is not differentiable at $x = 0$.

II. Solve for $\dfrac{dy}{dx}$. DO NOT SIMPLIFY.

1. $y = x^2 + 3x + 1$
2. $y = \dfrac{x^2+1}{x-1}$
3. $xy = \sin(x+y)$

III. Solve the following completely.

1. Find two numbers in [2,10] whose sum is 10 and whose product is maximum.
2. Starting from the same point, Lex started walking towards the east at the speed of 4 m/s and Rald ran towards the south at the speed of 3 m/s. Determine how fast is the distance between Lex and Rald changing after 1 second.
3. Find $f^{'}(-1)$ given that $f(x) = \dfrac{1}{x}$.
4. Find the slope of the tangent line on the curve given by $y = 3x^2 + x + 1$ at $x = 0$.

Chapter 3

Integration

LESSON 11: Integration

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Illustrate the antiderivative of a function;
2. Compute the general antiderivative of polynomial functions;
3. Compute the general antiderivative of root functions;
4. Compute the general antiderivative of exponential functions; and,
5. Compute the general antiderivative of trigonometric functions.

TOPIC 11.1: Illustration of an Antiderivative of a Function

Definition 5. *A function F is an* ***antiderivative*** *of the function f on an interval I if $F'(x) = f(x)$ for every value of x in I.*

EXAMPLE 1:

1. An antiderivative of $f(x) = 12x^2 + 2x$ is $F(x) = 4x^3 + x^2$. As we can see, the derivative of F is given by $F'(x) = 12x^2 + 2x = f(x)$.
2. An antiderivative of $g(x) = \cos x$ is $G(x) = \sin x$ because $G'(x) = \cos x = g(x)$.

EXAMPLE 2:

1. Other antiderivatives of $f(x) = 12x^2+2x$ are $F_1(x) = 4x^3+x^2-1$ and $F_2(x) = 4x^3+x^2+1$. In fact, any function of the form $F(x) = 4x^3 + x^2 + C$, where $C \in \mathbb{R}$ is an antiderivative of $f(x)$. Observe that $F'(x) = 12x^2 + 2x + 0 = 12x^2 + 2x = f(x)$.
2. Other antiderivatives of $g(x) = \cos x$ are $G_1(x) = \sin x + \pi$ and $G_2(x) = \sin x - 1$. In fact, any function $G(x) = \sin x + C$, where $C \in \mathbb{R}$ is an antiderivative of $g(x)$.

Theorem 10. *If F is an antiderivative of f on an interval I, then every antiderivative of f on I is given by $F(x) + C$, where C is an arbitrary constant.*

Terminologies and Notations:

- ***Antidifferentation*** is the process of finding the antiderivative.
- The symbol $\int$, also called the ***integral sign***, denotes the operation of antidifferentiation.
- The function f is called the ***integrand***.
- If F is an antiderivative of f, we write $\int f(x)\,dx = F(x) + C$.
- The symbols $\int$ and dx go hand-in-hand and dx helps us identify the variable of integration.
- The expression $F(x) + C$ is called the ***general antiderivative*** of f. Meanwhile, each antiderivative of f is called a ***particular antiderivative*** of f.

Solved Examples

EXAMPLE 1: Let $f(x) = -8x^3 - 10x + 5$ and $F(x) = -2x^4 - 5x^2 + 5x + 3$. Show that $F(x)$ is an antiderivative of $f(x)$.

Solution. $F'(x) = -8x^3 - 10x + 5 = f(x)$.

EXAMPLE 2: Determine if $F(x) = x^3 + x + 1$, $G(x) = x^3 + 2x + 1$ or $H(x) = x^3 + x + 3$ is/are antiderivatives of $f(x) = 3x^2 + 1$.

Solution. $F'(x) = 3x^2 + 1 = f(x)$. Thus, $F(x)$ is an antiderivative of $f(x)$.

$G'(x) = 3x^2 + 2 \neq f(x)$. Therefore, $G(x)$ is **not** an antiderivative of $f(x)$.

$H'(x) = 3x^2 + 1 = f(x)$. Hence, $H(x)$ is an antiderivative of $f(x)$.

Both $F(x)$ and $H(x)$ are antiderivatives of $f(x)$. In this example, we are able to illustrate that an antiderivative of a function is **not** unique.

Supplementary Problems

1. Determine the antiderivatives of the following functions.

   (a) $f(x) = 8x^7 + 2x^3 - 1$

   (b) $f(x) = -7$

   (c) $g(x) = 2x^3 - 2x - 1$

   (d) $h(x) = \sec x \tan x - 2 \sin x$

2. Match the functions in Column A with their corresponding antiderivatives in Column B.

Column A	Column B
1. $f(x) = -4x^2 - 9x - 1$	a. $F(x) = \frac{1}{4}x^4 + x - \frac{3}{2}$
2. $f(x) = 11x^2 - 121$	b. $F(x) = -\frac{4}{3}x^3 - \frac{9}{2}x^2 - x + 35$
3. $f(x) = 2x + 3$	c. $F(x) = \frac{2}{3}x^3 - \frac{5}{2}x^2 - 4x - 4$
4. $f(x) = x^3 + 1$	d. $F(x) = \frac{11}{3}x^3 - 121x - 2$
5. $f(x) = x(x^2 - 3)$	e. $F(x) = \frac{1}{4}x^4 - \frac{3}{2}x^2$
6. $f(x) = (x - 3)(2x + 1)$	f. $F(x) = x^2 + 3x - 1$

TOPIC 11.2: Antiderivatives of Algebraic Functions

Theorem 11. (Theorems on Antidifferentiation)

1. $\int dx = x + C.$
2. *If n is any real number and $n \neq -1$, then*
$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C.$$
3. *If a is any constant and f is a function, then*
$$\int af(x) \, dx = a \int f(x) \, dx.$$
4. *If f and g are functions defined on the same interval,*
$$\int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx.$$

EXAMPLE 1: Determine the following antiderivatives:

1. $\int 3 \, dx$
2. $\int x^6 \, dx$
3. $\int \frac{1}{x^6} \, dx$
4. $\int 4\sqrt{u} \, du$
5. $\int \left(12x^2 + 2x\right) \, dx$
6. $\int t \left(2t - 3\sqrt{t}\right) \, dt$
7. $\int \frac{x^2+1}{x^2} \, dx$

Solution. We will use the Theorems on Antidifferentiation to determine the antiderivatives.

1. Using (a) and (c) of the theorem, we have $\int 3 \, dx = 3 \int dx = 3x + C.$
2. Using (b) of the theorem, we have $\int x^6 \, dx = \frac{x^{6+1}}{6+1} + C = \frac{x^7}{7} + C.$
3. Using (b) of the theorem, we have
$$\int \frac{1}{x^6} \, dx = \int x^{-6} \, dx = \frac{x^{-6+1}}{-6+1} + C = \frac{x^{-5}}{-5} + C = -\frac{1}{5x^5} + C.$$
4. Using (b) and (c) of the theorem, we have
$$\int 4\sqrt{u} \, du = 4 \int u^{\frac{1}{2}} \, du = \frac{4u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{4u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{8u^{\frac{3}{2}}}{3} + C.$$

5. Using (b), (c) and (d) of the theorem, we have

$$\int \left(12x^2 + 2x\right)\, dx = 12\int x^2\, dx + 2\int x\, dx = \frac{12x^3}{3} + \frac{2x^2}{2} + C = 4x^3 + x^2 + C.$$

6. Using (b), (c) and (d), we have

$$\int t\left(2t - 3\sqrt{t}\right)\, dt = \int \left(2t^2 - 3t^{\frac{3}{2}}\right)\, dt = 2\int t^2\, dt - 3\int t^{\frac{3}{2}}\, dt = \frac{2t^3}{3} - \frac{6t^{\frac{5}{2}}}{5} + C.$$

7. Using (a), (b) and (d), we have

$$\int \frac{x^2+1}{x^2}\, dx = \int \left(1 + x^{-2}\right)\, dx = x + \frac{x^{-1}}{-1} + C = x - \frac{1}{x} + C.$$

Solved Examples

Evaluate the following integrals.

EXAMPLE 1: $\int 4\, dx$

Solution. $\int 4\, dx = 4\int dx = 4x + C.$

EXAMPLE 2: $\int (-4)\, dx$

Solution. $\int (-4)\, dx = (-4)\int dx = -4x + C.$

EXAMPLE 3: $\int x^8\, dx$

Solution. $\int x^8\, dx = \dfrac{x^{8+1}}{8+1} + C = \dfrac{x^9}{9} + C.$

EXAMPLE 4: $\int \dfrac{1}{x^9}\, dx$

Solution. $\int \dfrac{1}{x^9}\, dx = \int x^{-9}\, dx = \dfrac{x^{-9+1}}{-9+1} + C = \dfrac{x^{-8}}{-8} + C = -\dfrac{1}{8x^8} + C.$

EXAMPLE 5: $\int \sqrt{x}\, dx$

Solution. $\int \sqrt{x}\, dx = \int x^{\frac{1}{2}}\, dx = \dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \dfrac{2x^{\frac{3}{2}}}{3} + C.$

EXAMPLE 6: $\int 5\sqrt{u}\, dx$

Solution. $\int 5\sqrt{u}\, dx = 5\int u^{\frac{1}{2}}\, du = 5 \cdot \dfrac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = 5 \cdot \dfrac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \dfrac{10u^{\frac{3}{2}}}{3} + C.$

EXAMPLE 7: $\int \left(10x^4 + 2x^3\right)\, dx$

Solution. $\int \left(10x^4 + 2x^3\right)\, dx = \int 10x^4\, dx + \int 2x^3\, dx = \dfrac{10x^5}{5} + \dfrac{2x^4}{4} + C = 2x^5 + \dfrac{x^4}{2} + C.$

EXAMPLE 8: $\int \left(x^2 - 3\right)\left(\sqrt{x} + x\right) dx$

Solution.

$$\begin{aligned}
\int \left(x^2 - 3\right)\left(\sqrt{x} + x\right) dx &= \int \left(x^{\frac{5}{2}} + x^3 - 3x^{\frac{1}{2}} - 3x\right) dx \\
&= \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{x^4}{4} - \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{3x^2}{2} + C \\
&= \frac{2x^{\frac{7}{2}}}{7} - 2x^{\frac{3}{2}} + \frac{x^4}{4} - \frac{3x^2}{2} + C.
\end{aligned}$$

EXAMPLE 9: $\int \left(16x^3 - 3x^2 - 1\right) dx$

Solution.

$$\begin{aligned}
\int \left(16x^3 - 3x^2 - 1\right) dx &= \int 16x^3 dx - \int 3x^2 dx - \int dx \\
&= \frac{16x^4}{4} - \frac{3x^3}{3} - x + C \\
&= 4x^4 - x^3 - x + C.
\end{aligned}$$

EXAMPLE 10: $\displaystyle\int \frac{2x^4 - x}{x^3}\,dx$

Solution. $\displaystyle\int \frac{2x^4 - x}{x^3}\,dx = \int \left(2x - x^{-2}\right)\,dx = \frac{2x^2}{2} - \frac{x^{-1}}{-1} + C = x^2 + \frac{1}{x} + C.$

Supplementary Problems

Find the integrals of the following.

1. $\displaystyle\int 3\,dx$
2. $\displaystyle\int (-2)\,dx$
3. $\displaystyle\int x^4\,dx$
4. $\displaystyle\int x^{20}\,dx$
5. $\displaystyle\int 2\sqrt{x}\,dx$
6. $\displaystyle\int \sqrt[3]{x}\,dx$
7. $\displaystyle\int 7\sqrt[5]{x^2}\,dx$
8. $\displaystyle\int 11\sqrt[4]{x^{13}}\,dx$
9. $\displaystyle\int \left(x^5 - 16x^7\right)\,dx$
10. $\displaystyle\int \left(-2x^4 + x^3\right)\,dx$
11. $\displaystyle\int \left(-\frac{5}{4}x^9 + 10x^4 - 23\right)\,dx$
12. $\displaystyle\int \left(-\frac{2}{3}x - 17x^{16} + 10\right)\,dx$
13. $\displaystyle\int \frac{1}{x^5}\,dx$
14. $\displaystyle\int \frac{1}{x^{23}}\,dx$
15. $\displaystyle\int \left(\frac{1}{x^2} - 4x^{-5}\right)\,dx$
16. $\displaystyle\int \left(x^{-4} - \frac{3}{x^2} - 1\right)\,dx$
17. $\displaystyle\int \sqrt{x}\left(x^2 - 3x + 1\right)\,dx$
18. $\displaystyle\int x^2\left(2\sqrt{x} + 24x - 1\right)\,dx$
19. $\displaystyle\int (3w - w)^2\,dw$
20. $\displaystyle\int (1 - 5x)^2\,dx$
21. $\displaystyle\int \frac{t^4 - t^3 - 1}{t^2}\,dt$
22. $\displaystyle\int \frac{x^6 - x^2 + 1}{x^8}\,dx$
23. $\displaystyle\int \frac{y^3 - 4y^2 - 3y + 14}{y - 2}\,dy$
24. $\displaystyle\int \frac{2x^4 + 3x^3 - 2x^2 - x + 3}{2x + 3}\,dx$

TOPIC 11.3: Antiderivatives of Functions Yielding Exponential Functions and Logarithmic Functions

In this lesson, we will present the basic formulas for integrating functions that yield exponential and logarithmic functions. Let us first recall the following differentiation formulas:

1. $D_x(e^x) = e^x$.
2. $D_x(a^x) = a^x \ln a$.
3. $D_x(\ln x) = \dfrac{1}{x}$.

Because antidifferentiation is the inverse operation of differentiation, the following theorem should be immediate.

Theorem 12. (Theorems on integrals yielding the exponential and logarithmic functions)

1. $\displaystyle\int e^x\, dx = e^x + C.$

2. $\displaystyle\int a^x\, dx = \frac{a^x}{\ln a} + C.$ *Here,* $a > 0$ *with* $a \neq 1$.

3. $\displaystyle\int x^{-1}\, dx = \int \frac{1}{x}\, dx = \ln|x| + C.$

EXAMPLE 1: Find the integrals of the following functions.

1. $\displaystyle\int (e^x + 2^x)\, dx$
2. $\displaystyle\int 3^x\, dx$
3. $\displaystyle\int 3^{x+1}\, dx$
4. $\displaystyle\int \frac{2}{x}\, dx$

Solution. We will use the theorem to determine the integrals.

1. Using (1) and (2) of theorem, we have

$$\int (e^x + 2^x)\, dx = \int (e^x)\, dx + \int (2^x)\, dx = e^x + \frac{2^x}{\ln 2} + C.$$

2. Using (2) of the theorem, we have

$$\int 3^x\, dx = \frac{3^x}{\ln 3} + C.$$

3. Using (2) of the theorem, we have

$$\int 3^{x+1}\,dx = \int (3^x)(3^1)\,dx = 3\int (3^x)\,dx = 3\frac{3^x}{\ln 3} + C.$$

4. Using (3) of the theorem, we get

$$\int \frac{2}{x}\,dx = 2\int \frac{1}{x}\,dx = 2\ln|x| + C.$$

Solved Examples

Integrate the following.

EXAMPLE 1: $\int 2e^x dx$

Solution.

$$\int 2e^x dx = 2\int e^x dx = 2e^x + C.$$

EXAMPLE 2: $\int 5^x dx$

Solution.

$$\int 5^x dx = \frac{5^x}{\ln 5} + C.$$

EXAMPLE 3: $\int 7^{x+2} dx$

Solution.

$$\int 7^{x+2} dx = \int 7^2 \cdot 7^x dx = 7^2 \int 7^x dx = \frac{49}{\ln 7} \cdot 7^x + C.$$

EXAMPLE 4: $\int \frac{3}{x} dx$

Solution.

$$\int \frac{3}{x} dx = 3\int \frac{1}{x} dx = 3\ln|x| + C.$$

EXAMPLE 5: $\displaystyle\int \left(3e^x - \frac{2x+1}{x^2}\right) dx$

Solution.

$$\begin{aligned}
\int \left(3e^x - \frac{2x+1}{x^2}\right) dx &= \int \left(3e^x - \frac{2}{x} - x^{-2}\right) dx \\
&= 3\int e^x dx - 2\int \frac{1}{x} dx + \int x^{-2} dx \\
&= 3e^x - 2\ln|x| - x^{-1} + C \\
&= 3e^x - 2\ln|x| - \frac{1}{x} + C.
\end{aligned}$$

Supplementary Problems

Evaluate the following antiderivatives.

1. $\displaystyle\int 4e^x dx$
2. $\displaystyle\int (-3)\, e^x dx$
3. $\displaystyle\int 2^x dx$
4. $\displaystyle\int 3^x dx$
5. $\displaystyle\int e^{x+2} dx$
6. $\displaystyle\int 2^{x+4} dx$
7. $\displaystyle\int \frac{4}{x} dx$
8. $\displaystyle\int \frac{9}{7x} dx$
9. $\displaystyle\int \left(2e^x - \frac{x-1}{8x^2}\right) dx$
10. $\displaystyle\int \left(9^{2x} + \frac{x^2-3}{4x^2}\right) dx$
11. $\displaystyle\int \left(7^{3x+1} - \frac{3x^5 - x^3 - 1}{2x}\right) dx$
12. $\displaystyle\int \left(\frac{3x^2 - 27x^4}{3x^3} - \frac{3e^x}{\pi}\right) dx$
13. $\displaystyle\int \left(\frac{(2x+1)(x-3)}{x^3} - \frac{e^{x^2+x-1}}{e^{x^2+1}}\right) dx$
14. $\displaystyle\int \left(\frac{2^{x^3-x^2+x-3}}{2^{x^3-x^2+1}} - \frac{x\left(x^2-1\right)}{x^4}\right) dx$

TOPIC 11.4: Antiderivatives of Trigonometric Functions

Theorem 13. (Antiderivatives of Trigonometric Functions)

1. $\displaystyle\int \sin x\,dx = -\cos x + C$

2. $\displaystyle\int \cos x\,dx = \sin x + C$

3. $\displaystyle\int \sec^2 x\,dx = \tan x + C$

4. $\displaystyle\int \csc^2 x\,dx = -\cot x + C$

5. $\displaystyle\int \sec x \tan x\,dx = \sec x + C$

6. $\displaystyle\int \csc x \cot x\,dx = -\csc x + C$

EXAMPLE 1: Determine the antiderivatives of the following:

1. $\displaystyle\int (\cos x - \sin x)\ dx$
2. $\displaystyle\int \cot^2 x\,dx$
3. $\displaystyle\int \tan^2 v\,dv$
4. $\displaystyle\int \frac{\sin x}{\cos^2 x}\,dx$

Solution. We will use the theorem on antiderivatives of trigonometric functions.

1. Using (a) and (b) of the theorem, we have

$$\begin{aligned}\int (\cos x - \sin x)\ dx &= \int \cos x\,dx - \int \sin x\,dx\\ &= \sin x - (-\cos x) + C = \sin x + \cos x + C.\end{aligned}$$

2. Since we know that $\cot^2 x = \csc^2 -1$, then

$$\int \cot^2 x\,dx = \int \left(\csc^2 x - 1\right)\ dx = \int \csc^2 x - \int dx = -\cot^2 -x + C.$$

3. Since $\tan^2 v = \sec^2 v - 1$, we have

$$\int \tan^2 v\,dv = \int \left(\sec^2 v - 1\right)\ dv = \int \sec^2 v\,dv - \int dv = \tan v - v + C.$$

4. $\displaystyle\int \frac{\sin x}{\cos^2 x}\,dx = \int \frac{\sin x}{\cos x}\frac{1}{\cos x}\,dx = \int \tan x \sec x\,dx = \sec x + C.$

Solved Examples

Evaluate the following antiderivatives.

EXAMPLE 1: Evaluate $\displaystyle\int 4\sin x\,dx$.

Solution.

$$\begin{aligned}\int 4\sin x\,dx &= 4\int \sin x\,dx\\ &= -4\cos x + C.\end{aligned}$$

EXAMPLE 2: Evaluate $\displaystyle\int \cos\left(x+\frac{\pi}{3}\right)\,dx$.

Solution.

$$\begin{aligned}\int \cos\left(x+\frac{\pi}{3}\right)\,dx &= \int \left(\cos x\cos\frac{\pi}{3} - \sin x\sin\frac{\pi}{3}\right)\,dx\\ &= \int \frac{1}{2}\cos x\,dx - \int \frac{\sqrt{3}}{2}\sin x\,dx\\ &= \frac{1}{2}\int \cos x\,dx - \frac{\sqrt{3}}{2}\int \sin x\,dx\\ &= \frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x + C.\end{aligned}$$

EXAMPLE 3: $\displaystyle\int \frac{\cos x}{1-\cos^2 x}\,dx$

Solution.

$$\begin{aligned}\int \frac{\cos x}{1-\cos^2 x}\,dx &= \int \frac{\cos x}{\sin^2 x}\,dx\\ &= \int \frac{1}{\sin x}\cdot\frac{\cos x}{\sin x}\,dx\\ &= \int \csc x\cot x\,dx\\ &= -\csc x + C.\end{aligned}$$

Supplementary Problems

Evaluate the following integrals.

1. $\displaystyle\int 7\csc^2 x\,dx$
2. $\displaystyle\int 2\sec x\tan x\,dx$

3. $\displaystyle\int \left(1+\tan^2 x\right)\, dx$

4. $\displaystyle\int \left(1+\cot^2 x\right)\, dx$

5. $\displaystyle\int \sin\left(x-\frac{\pi}{6}\right)\, dx$

6. $\displaystyle\int \cos\left(x-\frac{\pi}{6}\right)\, dx$

7. $\displaystyle\int \frac{(1-\cos x)(1+\cos x)}{(1-\sin x)(1+\sin x)} \csc x\, dx$

8. $\displaystyle\int \frac{(1-\sin x)(1+\sin x)}{(1-\cos x)(1+\cos x)} \sec x\, dx$

LESSON 12: Techniques of Antidifferentiation

TIME FRAME: 7 hours

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Compute the antiderivative of a function using the substitution rule; and
2. Compute the antiderivative of a function using a table of integrals (including those whose antiderivatives involve logarithmic and inverse trigonometric functions).

TOPIC 12.1: Antidifferentiation by Substitution and by Table of Integrals

Antidifferentiation is more challenging than differentiation. To find the derivative of a given function, there are well-established rules that are always applicable to differentiable functions. For antidifferentiation, the antiderivatives given in the previous lesson may not suffice to integrate a given function.

A prerequisite is knowledge of the basic antidifferentiation formulas. Some formulas are easily derived, but most of them need to be memorized.

No hard and fast rules can be given as to which method applies in a given situation. In college, several techniques such as *integration by parts*, *partial fractions*, *trigonometric subtitution* will be introduced. This lesson focuses on the most basic technique - antidifferentiation by substitution - which is the inverse of the Chain Rule in differentiation.

There are occasions when it is possible to perform a difficult piece of integration by first making a ***substitution***. This has the effect of changing the variable and the integrand. The ability to carry out integration by substitution is a skill that develops with practice and experience, but sometimes a sensible substitution may not lead to an integral that can be evaluated. We must then be prepared to try out alternative substitutions.

Suppose we are given an integral of the form $\displaystyle\int f(g(x)) \cdot g'(x)\, dx$. We can transform this into another form by changing the independent variable x to u using the substitution $u = g(x)$. In this case, $\dfrac{du}{dx} = g'(x)\, dx$. Therefore,

$$\int f(g(x))g'(x)\, dx = \int f(u)\, du.$$

This change of variable is one of the most important tools available to us. This technique is called ***integration by substitution***. It is often important to guess what will be the useful substitution.

Usually, we make a substitution for a function whose derivative also occurs in the integrand.

EXAMPLE 1: Evaluate $\int (x+4)^5\,dx$.

Solution. Notice that the integrand is in the fifth power of the expression $(x+4)$. To tackle this problem, we make a ***substitution***. We let $u = x+4$. The point of doing this is to change the integrand into a much simpler u^5. However, we must take care to substitute appropriately for the term dx too.

Now, since $u = x+4$ it follows that $\frac{du}{dx} = 1$ and so $du = dx$. So, substituting $(x+4)$ and dx, we have

$$\int (x+4)^5\,dx = \int u^5\,du.$$

The resulting integral can be evaluated immediately to give $\frac{u^6}{6} + C$. Recalling that $u = x+4$, we have

$$\begin{aligned}\int (x+4)^5\,dx &= \int u^5\,du \\ &= \frac{u^6}{6} + C \\ &= \frac{(x+4)^6}{6} + C.\end{aligned}$$

An alternative way of finding the antiderivative above is to expand the expression in the integrand and antidifferentiate the resulting polynomial (of degree 5) term by term. We will NOT do this. Obviously, the solution above is simpler than the mentioned alternative.

EXAMPLE 2: Evaluate $\int (x^5+2)^9\,5x^4\,dx$.

Solution. If we let $u = x^5+2$, then $du = 5x^4\,dx$, which is precisely the other factor in the integrand. Thus, in terms of the variable u, this is essentially just a power rule integration.

That is,

$$\begin{aligned}\int (x^5+2)^9\,5x^4\,dx &= \int u^9\,du, \quad \text{where} \quad u = x^5+2 \\ &= \frac{u^{10}}{10} + C \\ &= \frac{(x^5+2)^{10}}{10} + C.\end{aligned}$$

Again, the alternative way is to expand out the expression in the integrand, and integrate the resulting polynomial (of degree 49) term by term. Again, we would rather NOT do this.

EXAMPLE 3: Evaluate $\displaystyle\int \frac{z^2}{\sqrt{1+z^3}}\,dz$.

Solution. In this example, we let $u = 1 + z^3$ so that $\dfrac{du}{dz} = 3z^2$. If $u = 1 + z^3$, then we need to express $z^2\,dz$ in the integrand in terms of du or a constant multiple of du.

From $\dfrac{du}{dz} = 3z^2$ it follows that $du = 3z^2\,dz$ and $z^2\,dz = \dfrac{1}{3}\,du$. Thus,

$$\begin{aligned}
\int \frac{z^2}{\sqrt{1+z^3}}\,dz &= \int \frac{1}{\sqrt{1+z^3}} \cdot z^2\,dz \\
&= \int \frac{1}{\sqrt{u}} \cdot \frac{1}{3}\,du \\
&= \frac{1}{3}\int u^{-\frac{1}{2}}\,du \\
&= \frac{1}{3}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + C \\
&= \frac{2}{3}u^{\frac{1}{2}} + C \\
&= \frac{2}{3}(1+z^3)^{\frac{1}{2}} + C
\end{aligned}$$

EXAMPLE 4: Evaluate $\displaystyle\int \frac{x}{\sqrt{x^2-1}}\,dx$.

Solution. Notice that if $u = x^2 - 1$, then $du = 2x\,dx$. This implies that $x\,dx = \dfrac{1}{2}\,du$, so we have

$$\begin{aligned}
\int \frac{x}{\sqrt{x^2-1}}\,dx &= \int \frac{1}{u^{\frac{1}{2}}} \cdot \frac{1}{2}\,du \\
&= \frac{1}{2}\int u^{-\frac{1}{2}}\,du \\
&= \frac{1}{2}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + C \\
&= (x^2-1)^{\frac{1}{2}} + C.
\end{aligned}$$

EXAMPLE 5: Evaluate $\displaystyle\int \frac{3x^5}{\sqrt{x^3+5}}\,dx$.

Solution. Let $u = x^3+5$. Then $x^3 = u-5$ and $du = 3x^2\,dx$. Thus,

$$\begin{aligned}
\int \frac{x^3}{\sqrt{x^3+5}}\,dx &= \int \frac{u-5}{u^{\frac{1}{2}}}\,dx \\
&= \int \left(\frac{u}{u^{\frac{1}{2}}} - \frac{5}{u^{\frac{1}{2}}}\right) dx \\
&= \int (u^{\frac{1}{2}} - 5u^{-\frac{1}{2}})\,dx \\
&= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{5u^{\frac{1}{2}}}{\frac{1}{2}} + C \\
&= \frac{2}{3}u^{\frac{3}{2}} - 10u^{\frac{1}{2}} + C \\
&= \frac{2}{3}(x^3+5)^{\frac{3}{2}} - 10(x^3+5)^{\frac{1}{2}} + C.
\end{aligned}$$

EXAMPLE 6: Evaluate $\displaystyle\int \frac{dx}{x\sqrt[3]{\ln x}}$.

Solution. We substitute $u = \ln x$ so that $du = \dfrac{1}{x}dx$, which occurs in the integrand. Thus,

$$\begin{aligned}
\int \frac{dx}{x\sqrt[3]{\ln x}} &= \int \frac{1}{\sqrt[3]{u}}\,du \\
&= \int u^{-\frac{1}{3}}\,du \\
&= \frac{u^{\frac{2}{3}}}{\frac{2}{3}} + C \\
&= \frac{3}{2}(\ln x)^{\frac{2}{3}} + C
\end{aligned}$$

We recall the theorem we stated in the previous lesson.

Theorem 14. (Theorems on integrals yielding the exponential and logarithmic functions)

1. $\displaystyle\int e^x\,dx = e^x + C.$

2. $\int a^x \, dx = \frac{a^x}{\ln a} + C$. *Here,* $a > 0$ *with* $a \neq 1$.

3. $\int x^{-1} \, dx = \int \frac{1}{x} \, dx = \ln |x| + C$.

EXAMPLE 7: Evaluate the following integrals.

1. $\int e^{3x} \, dx$

2. $\int 2^{4x} \, dx$

3. $\int \frac{1}{2x-1} \, dx$

Solution.

1. We let $u = 3x$. Then $du = 3dx$. Hence, $dx = \frac{du}{3}$. So,

$$\int e^{3x} \, dx = \int e^u \, \frac{du}{3} = \frac{1}{3} \int e^u \, du = \frac{1}{3} e^u + C = \frac{1}{3} e^{3x} + C.$$

2. Here, we let $u = 4x$ and so $du = 4dx$. Thus, $dx = \frac{du}{4}$. Hence, we have

$$\int 2^{4x} \, dx = \int 2^u \, \frac{du}{4} = \frac{1}{4} \int 2^u \, du = \frac{1}{4} \frac{2^u}{\ln 2} + C = \frac{1}{4 \ln 2} 2^{4x} + C.$$

3. Suppose we let $u = 2x - 1$. Then $du = 2dx$. Hence, $dx = \frac{du}{2}$. We have

$$\int \frac{1}{2x-1} \, dx = \int \frac{1}{u} \, \frac{du}{2} = \frac{1}{2} \int \frac{1}{u} \, du = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln |2x - 1| + C.$$

EXAMPLE 8: Evaluate $\int \cos(4x + 3) \, dx$.

Solution. Observe that if we make the substitution $u = 4x + 3$, the integrand will contain a much simpler form, $\cos u$, which we can easily integrate. So, if $u = 4x + 3$, then $du = 4 \, dx$ and $dx = \frac{1}{4} \, du$. So,

$$
\begin{aligned}
\int \cos(4x+3)\, dx &= \int \cos u \cdot \frac{1}{4}\, du \\
&= \frac{1}{4} \int \cos u\, du \\
&= \frac{1}{4} \sin u + C \\
&= \frac{1}{4} \sin(4x+3) + C.
\end{aligned}
$$

EXAMPLE 9: Evaluate the integral $\int \sin x \cos x\, dx$.

Solution. Note that if we let $u = \sin x$, its derivative is $du = \cos x\, dx$ which is the other factor in the integrand and our integral becomes

$$
\begin{aligned}
\int \sin x \cos x\, dx &= \int u\, du \\
&= \frac{u^2}{2} + C_1 \\
&= \frac{\sin^2 x}{2} + C_1.
\end{aligned}
$$

Alternative solution to the problem: If we let $u = \cos x$, then $du = -\sin x\, dx$ which is also the other factor in the integrand. Doing a similar technique as above, we get

$$\int \sin x \cos x\, dx = -\frac{\cos^2 x}{2} + C_2.$$

Even if both answers are different, the presence of the constants avoids any contradiction, for example, letting $C_2 = \frac{1}{2} + C_1$ shows that the answers are equivalent.

EXAMPLE 10: Evaluate the integral $\int e^{\sin x} \cos x \, dx$.

Solution. We let $u = \sin x$ so that the other factor in the integrand $\cos x \, dx = du$. Thus, the integral becomes

$$\begin{aligned} \int e^{\sin x} \cos x \, dx &= \int e^u \, du \\ &= e^u + C \\ &= e^{\sin x} + C. \end{aligned}$$

Recall that we had earlier presented the integrals $\int \sin x \, dx$ and $\int \cos x \, dx$. Now that we already know integration by substitution, we can now present the integrals of other trigonometric functions: $\tan x$, $\cot x$, $\sec x$, and $\csc x$.

First, let us use substitution technique to find $\int \tan x \, dx$. Note that $\tan x = \dfrac{\sin x}{\cos x}$. Hence,

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx.$$

Now, if we let $u = \cos x$, then $du = -\sin x \, dx$. Hence, we have

$$\begin{aligned} \int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx \\ &= \int \frac{1}{u} \, (-du) \\ &= -\int \frac{1}{u} \, du \\ &= -\ln |u| + C \\ &= -\ln |\cos x| + C. \end{aligned}$$

Equivalently, $-\ln |\cos x| + C = \ln |\cos x|^{-1} = \ln |\sec x| + C$.

Similarly, we can use substitution technique to show $\int \cot x \, dx = \ln |\sin x| + C$. Here, we use $\cot x = \dfrac{\cos x}{\sin x}$ and choose $u = \sin x$.

Let us now find $\int \sec x \, dx$. The usual trick is to multiply the numerator and the denominator by $\sec x + \tan x$.

$$\int \sec x \, dx = \int (\sec x) \frac{\sec x + \tan x}{\sec x + \tan x} \, dx = \int \frac{1}{\sec x + \tan x} (\sec x \tan x + \sec^2 x) \, dx.$$

Now, if we let $u = \sec x + \tan x$, then $du = \sec x \tan x + \sec^2 x$. Thus, we have

$$\begin{aligned}
\int \sec x \, dx &= \int \frac{1}{\sec x + \tan x} (\sec x \tan x + \sec^2 x) \, dx. \\
&= \int \frac{1}{u} \, du \\
&= \ln |u| + C \\
&= \ln |\sec x + \tan x| + C.
\end{aligned}$$

Similarly, we can show $\displaystyle\int \csc x \, dx = \ln |\csc x - \cot x| + C$.

Hence, we have the following formulas:

1. $\displaystyle\int \tan x \, dx = -\ln |\cos x| + C = \ln |\sec u| + C$.

2. $\displaystyle\int \cot x \, dx = \ln |\sin x| + C$.

3. $\displaystyle\int \sec x \, dx = \ln |\sec x + \tan x| + C$.

4. $\displaystyle\int \csc x \, dx = \ln |\csc x - \cot x| + C$.

EXAMPLE 11: Evaluate $\displaystyle\int x^4 \sec(x^5) \, dx$.

Solution. We let $u = x^5$. Then $du = 5x^4 dx$. Thus, $x^4 dx = \dfrac{du}{5}$. We have

$$\begin{aligned}
\int x^4 \sec(x^5) \, dx &= \int \sec u \frac{du}{5} \\
&= \frac{1}{5} \int \sec u \, du \\
&= \frac{1}{5} \ln |\sec u + \tan u| + C \\
&= \frac{1}{5} \ln |\sec x^5 + \tan x^5| + C.
\end{aligned}$$

EXAMPLE 12: Evaluate $\displaystyle\int \frac{4 + \cos \frac{x}{4}}{\sin \frac{x}{4}} \, dx$.

Solution. Let $u = \frac{x}{4}$. Then $du = \frac{1}{4}\,dx$ and so $4\,du = dx$. Thus, we have

$$\begin{aligned}
\int \frac{4+\cos\frac{x}{4}}{\sin\frac{x}{4}}\,dx &= \int \frac{4+\cos u}{\sin u}\,4du \\
&= 4\int \left[\frac{4}{\sin u} + \frac{\cos u}{\sin u}\right] du \\
&= 4\int \Big[4\csc u + \cot u\Big]\,du \\
&= 4\int \cot u\,du + 16\int \csc u\,du \\
&= 4\ln|\sin u| + 16\ln|\csc u - \cot u| + C \\
&= 4\ln\left|\sin\frac{x}{4}\right| + 16\ln\left|\csc\frac{x}{4} - \cot\frac{x}{4}\right| + C.
\end{aligned}$$

INTEGRALS OF INVERSE CIRCULAR FUNCTIONS

1. $\displaystyle\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$
2. $\displaystyle\int \frac{du}{1+u^2} = \tan^{-1} u + C$
3. $\displaystyle\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} u + C$

If the constant 1 in these integrals is replaced by some other positive number, one can use the following generalizations:

Let $a > 0$. Then

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C \tag{3.1}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \tag{3.2}$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C \tag{3.3}$$

EXAMPLE 13: Evaluate $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx$.

Solution. From Formula (3.1) with $a=3$, we write this into

$$\int \frac{1}{\sqrt{9-x^2}}\,dx = \int \frac{1}{\sqrt{3^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{3}\right) + C.$$

EXAMPLE 14: $\displaystyle\int \frac{dx}{9x^2+36}$

Solution. Let $u=3x$, $du=3\,dx$. Then from Formula (3.2),

$$\begin{aligned}
\int \frac{dx}{9x^2+36} &= \frac{1}{3}\int \frac{du}{u^2+36} \\
&= \frac{1}{3}\cdot\frac{1}{6}\tan^{-1}\left(\frac{u}{6}\right) + C \\
&= \frac{1}{18}\tan^{-1}\left(\frac{3x}{6}\right) + C \\
&= \frac{1}{18}\tan^{-1}\left(\frac{x}{2}\right) + C.
\end{aligned}$$

EXAMPLE 15: Evaluate $\displaystyle\int \frac{dx}{x\ln x\sqrt{(\ln x)^2-9}}$.

Solution. Let $u=\ln x$, $du=\dfrac{1}{x}\,dx$. Then from Formula (3.3),

$$\begin{aligned}
\int \frac{dx}{x\ln x\sqrt{(\ln x)^2-9}} &= \int \frac{du}{u\sqrt{u^2-9}} \\
&= \frac{1}{3}\sec^{-1}\left(\frac{u}{3}\right) + C \\
&= \frac{1}{3}\sec^{-1}\left(\frac{\ln x}{3}\right) + C.
\end{aligned}$$

EXAMPLE 16: $\displaystyle\int \frac{dx}{\sqrt{9+8x-x^2}}$.

Solution. Observe that by completing the squares, and Formula (3.1),

$$\begin{aligned}\int \frac{dx}{\sqrt{9+8x-x^2}} &= \int \frac{dx}{\sqrt{9-(x^2-8x)}}\\ &= \int \frac{dx}{\sqrt{9-(x^2-8x+16-16)}}\\ &= \int \frac{dx}{\sqrt{25-(x-4)^2}}.\end{aligned}$$

Let $u = x-4$, $du = dx$. Then

$$\begin{aligned}\int \frac{dx}{\sqrt{9+8x-x^2}} &= \int \frac{du}{\sqrt{25-u^2}}\\ &= \sin^{-1}\frac{u}{5} + C\\ &= \sin^{-1}\left(\frac{x-4}{5}\right) + C.\end{aligned}$$

EXAMPLE 17: $\displaystyle\int \frac{18x+3}{9x^2+6x+2}\,dx$.

Solution. Let $u = 9x^2+6x+2$, $du = (18x+6)\,dx$. Then

$$\begin{aligned}\int \frac{18x+3}{9x^2+6x+2}\,dx &= \int \frac{18x+6}{9x^2+6x+2}\,dx - \int \frac{3}{9x^2+6x+2}\,dx\\ &= \int \frac{du}{u} - 3\int \frac{dx}{9x^2+6x+1+1}\\ &= \ln|u| - 3\int \frac{dx}{(3x+1)^2+1}.\end{aligned}$$

Let $v = 3x+1$, $dv = 3\,dx$. Then by Formula (3.2),

$$\begin{aligned}\int \frac{18x+3}{9x^2+6x+2}\,dx &= \ln|u| - \frac{3}{3}\int \frac{1}{v^2+1}\,dv\\ &= \ln|u| - \tan^{-1} v + C\\ &= \ln|9x^2+6x+2| - \tan^{-1}(3x+1) + C.\end{aligned}$$

SUMMARY/TABLE OF INTEGRALS

1. $\int dx = x + C$

2. $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + C, \text{ if } n \neq -1$

3. $\int af(x) \, dx = a \int f(x) \, dx$

4. $\int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx$

5. $\int e^x \, dx = e^x + C$

6. $\int a^x \, dx = \dfrac{a^x}{\ln a} + C$

7. $\int x^{-1} \, dx = \int \dfrac{1}{x} \, dx = \ln|x| + C$

8. $\int \sin x \, dx = -\cos x + C$

9. $\int \cos x \, dx = \sin x + C$

10. $\int \sec^2 x \, dx = \tan x + C$

11. $\int \csc^2 x \, dx = -\cot x + C$

12. $\int \sec x \tan x \, dx = \sec x + C$

13. $\int \csc x \cot x \, dx = -\csc x + C$

14. $\int \tan x \, dx = -\ln|\cos x| + C$

15. $\int \cot x \, dx = \ln|\sin x| + C$

16. $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

17. $\int \csc x \, dx = \ln|\csc x - \cot x| + C$

18. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right) + C$

19. $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a} \tan^{-1}\left(\dfrac{x}{a}\right) + C$

20. $\int \dfrac{dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a} \sec^{-1}\left(\dfrac{x}{a}\right) + C$

Solved Examples

Evaluate the following antiderivatives.

EXAMPLE 1: $\int (3x+1)^{15} \, dx$

Solution. Let $u = 3x + 1$. Then $du = 3\,dx$.

$$\begin{aligned} \int (3x+1)^{15} \, dx &= \frac{1}{3} \int u^{15} \, du \\ &= \frac{1}{3} \cdot \frac{u^{16}}{16} + C \\ &= \frac{u^{16}}{48} + C \\ &= \frac{(3x+1)^{16}}{48} + C. \end{aligned}$$

EXAMPLE 2: $\int (8x^3 + 2x^2 - 1)^7 (6x^2 + x)\, dx$

Solution. Let $u = 8x^3 + 2x^2 - 1$. Then $du = (24x^2 + 4x)\, dx = 4(6x^2 + x)\, dx$.

$$\begin{aligned}
\int (8x^3 + 2x^2 - 1)^7 (6x^2 + x)\, dx &= \frac{1}{4} \int u^7\, du \\
&= \frac{1}{4} \cdot \frac{u^8}{8} + C \\
&= \frac{\left(8x^3 + 2x - 1\right)^8}{32} + C.
\end{aligned}$$

EXAMPLE 3: $\int \frac{x^2}{\sqrt{50 - 5x^3}}\, dx$

Solution. Let $u = 50 - 5x^3$. Then $du = -15x^2\, dx$ which implies $x^2\, dx = \frac{du}{-15}$. Therefore,

$$\begin{aligned}
\int \frac{x^2}{\sqrt{50 - 5x^3}}\, dx &= \int \frac{1}{\sqrt{u}} \cdot \frac{du}{-15} \\
&= -\frac{1}{15} \int u^{-\frac{1}{2}}\, du \\
&= -\frac{1}{15} \cdot \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C \\
&= -\frac{2}{15} \sqrt{50 - 5x^3} + C.
\end{aligned}$$

EXAMPLE 4: $\int e^{-9z+11} dz$

Solution. Let $u = -9z + 11$. Then $du = -9\, dz$. So,

$$\begin{aligned}
\int e^{-9z+11} dz &= -\frac{1}{9} \int \mathrm{e}^u\ du \\
&= -\frac{1}{9} \mathrm{e}^u + C \\
&= -\frac{1}{9} \mathrm{e}^{-9z+11} + C.
\end{aligned}$$

EXAMPLE 5: $\int x2^{x^2-4}\,dx$

Solution. Let $u = x^2 - 4$. Then $du = 2x\,dx$.

$$\begin{aligned}\int x2^{x^2-4}\,dx &= \frac{1}{2}\int 2^u \;\, du \\ &= \frac{1}{2}\cdot\frac{2^u}{\ln 2} + C \\ &= \frac{2^{x^2-4}}{2\ln 2} + C.\end{aligned}$$

EXAMPLE 6: $\int \frac{dx}{3x+7}$

Solution. Let $u = 3x + 7$. Then $du = 3\,dx$.

$$\begin{aligned}\int \frac{dx}{3x+7} &= \frac{1}{3}\int \frac{du}{u} \\ &= \frac{1}{3}\ln|u| + C \\ &= \frac{1}{3}\ln|3x+7| + C.\end{aligned}$$

EXAMPLE 7: $\int \frac{e^{2x}}{e^{2x}+3}\,dx$

Solution. Let $u = e^{2x} + 3$. Then $du = 2e^{2x}\,dx$.

$$\begin{aligned}\int \frac{e^{2x}}{e^{2x}+3}\,dx &= \frac{1}{2}\int \frac{du}{u} \\ &= \frac{1}{2}\ln|u| + C \\ &= \frac{1}{2}\ln|e^{2x}+3| + C.\end{aligned}$$

EXAMPLE 8: $\int \frac{t}{(t^2+1)\sqrt{\ln(t^2+1)}}\,dt$

Solution. Let $u = \ln\left(t^2+1\right)$. Then $du = \dfrac{2t}{t^2+1}\,dt$.

$$\begin{aligned}
\int \frac{t}{\left(t^2+1\right)\sqrt{\ln\left(t^2+1\right)}}\,dt &= \int \frac{1}{2\sqrt{u}}\,du \\
&= \frac{1}{2}\int u^{-\frac{1}{2}}\,du \\
&= \frac{1}{2}\cdot\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C \\
&= u^{\frac{1}{2}} + C \\
&= \sqrt{\ln\left(t^2+1\right)} + C.
\end{aligned}$$

EXAMPLE 9: $\displaystyle\int \tan^2(2x)\,dx$

Solution. Let $u = 2x$ then $du = 2\,dx$.

$$\begin{aligned}
\int \tan^2(2x)\,dx &= \frac{1}{2}\int \tan^2 u\,du \\
&= \frac{1}{2}\int \left(\sec^2 u - 1\right)\,du \\
&= \frac{1}{2}\int \sec^2 u\,du - \frac{1}{2}\int\,du \\
&= \frac{\tan u}{2} - \frac{u}{2} + C \\
&= \frac{\tan(2x)}{2} - x + C.
\end{aligned}$$

EXAMPLE 10: $\displaystyle\int \sin^4 x\cos x\,dx$

Solution. Let $u = \sin x$. Then $du = \cos x\,dx$.

$$\begin{aligned}
\int \sin^4 x\cos x\,dx &= \int u^4\,du \\
&= \frac{u^5}{5} + C \\
&= \frac{\sin^5 x}{5} + C.
\end{aligned}$$

EXAMPLE 11: $\displaystyle\int \left(\sec^2(\tan x)\right)\sec^2 x\,dx$

Solution. Let $u = \tan x$. Then $du = \sec^2 x\,dx$.

$$\begin{aligned}\int \left(\sec^2(\tan x)\right)\sec^2 x\,dx &= \int \sec^2 u\,du \\ &= \tan u + C \\ &= \tan(\tan x) + C.\end{aligned}$$

EXAMPLE 12: $\displaystyle\int 3^{\pi+\cos\theta}\sin\theta\,d\theta$

Solution. Let $u = \pi + \cos\theta$. Then $du = -\sin\theta d\theta$.

$$\begin{aligned}\int 3^{\pi+\cos\theta}\sin\theta\,dx &= -\int 3^u\,du \\ &= -\frac{3^u}{\ln 3} + C \\ &= -\frac{3^{\pi+\cos\theta}}{\ln 3} + C.\end{aligned}$$

EXAMPLE 13: $\displaystyle\int \tan\left(\frac{\theta}{2}\right)d\theta$

Solution. Let $u = \dfrac{\theta}{2}$. Then $du = \dfrac{1}{2}\,d\theta$.

$$\int \tan\left(\frac{\theta}{2}\right)d\theta = 2\int \tan u\ d\theta.$$

From the table of integrals,

$$\begin{aligned}\int \tan\left(\frac{\theta}{2}\right)d\theta &= -2\ln\left|\cos u\right| + C \\ &= -2\ln\left|\cos\left(\frac{\theta}{2}\right)\right| + C.\end{aligned}$$

EXAMPLE 14: $\displaystyle\int \frac{\cot(3x)\csc^2(3x)}{(1-\cot(3x))^3}\,dx$

Solution. Let $u = 1 - \cot(3x)$. Then $du = 3\csc^2(3x)\ dx$.

$$\begin{aligned}
\int \frac{\cot(3x)\csc^2(3x)}{(1-\cot(3x))^3}\,dx &= \frac{1}{3}\int \frac{(1-u)}{u^3}\,du \\
&= \frac{1}{3}\int \left(u^{-3} - u^{-2}\right)\,du \\
&= \frac{1}{3}\left(\frac{u^{-2}}{-2} - \frac{u^{-1}}{-1}\right) + C \\
&= -\frac{1}{6u^2} + \frac{1}{3u} + C \\
&= -\frac{1}{6(1-\cot(3x))^2} + \frac{1}{3(1-\cot(3x))} + C.
\end{aligned}$$

EXAMPLE 15: $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx$

Solution. Since $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$, taking $a = 2$ gives,

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \int \frac{1}{\sqrt{2^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{2}\right) + C.$$

EXAMPLE 16: $\displaystyle\int \frac{dx}{9x^2+25}$

Solution. Let $u = 3x$. Then $du = 3\,dx$. Using formula (19) of the Table of Integrals,

$$\int \frac{dx}{9x^2+25} = \frac{1}{3}\int \frac{du}{u^2+5^2} = \frac{1}{3}\cdot\frac{1}{5}\tan^{-1}\left(\frac{u}{5}\right) + C = \frac{1}{15}\tan^{-1}\left(\frac{3x}{5}\right) + C.$$

EXAMPLE 17: $\displaystyle\int \frac{dx}{(2x-5)^2+9}$

Solution. Let $u = 2x - 5$. Then $du = 2\,dx$. Formula (19) implies

$$\begin{aligned}
\int \frac{dx}{(2x-5)^2+9} &= \frac{1}{2}\int \frac{dx}{u^2+3^2} \\
&= \frac{1}{2}\cdot\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right)+C \\
&= \frac{1}{6}\tan^{-1}\left(\frac{2x-5}{3}\right)+C.
\end{aligned}$$

EXAMPLE 18: $\displaystyle\int \frac{dx}{\sqrt{20+8x-x^2}}$

Solution. Completing the squares,

$$\begin{aligned}
\int \frac{dx}{\sqrt{20+8x-x^2}} &= \int \frac{dx}{\sqrt{20-(x^2-8x)}} \\
&= \int \frac{dx}{\sqrt{20-(x^2-8x+16-16)}} \\
&= \int \frac{dx}{\sqrt{36-(x-4)^2}}
\end{aligned}$$

Let $u = x - 4$. Then $du = dx$. Formula (18) implies

$$\int \frac{dx}{\sqrt{20-8x+x^2}} = \int \frac{du}{\sqrt{36-u^2}} = \sin^{-1}\left(\frac{u}{6}\right)+C = \sin^{-1}\left(\frac{x-4}{6}\right)+C.$$

EXAMPLE 19: $\displaystyle\int \frac{x+2}{x^2+2x+5}\,dx$

Solution. Let $u = x^2+2x+5$. Then $du = (2x+2)\,dx = 2\,(x+1)\,dx$.

$$\begin{aligned}
\int \frac{x+2}{x^2+2x+5}\,dx &= \int \frac{x+1}{x^2+2x+5}\,dx + \int \frac{dx}{x^2+2x+5} \\
&= \frac{1}{2}\int \frac{du}{u} + \int \frac{dx}{x^2+2x+5} \\
&= \frac{1}{2}\int \frac{du}{u} + \int \frac{dx}{(x+1)^2+4}
\end{aligned}$$

Let $v = x + 1$. Then $dv = dx$.

$$\begin{aligned}
\int \frac{x+2}{x^2+2x+5}\,dx &= \frac{1}{2}\ln|u| + \int \frac{dv}{v^2+4} + C \\
&= \frac{1}{2}\ln|u| + \frac{1}{2}\tan^{-1}\left(\frac{v}{2}\right) + C \\
&= \frac{1}{2}\ln|x^2+2x+5| + \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C.
\end{aligned}$$

EXAMPLE 20: $\displaystyle\int \frac{dx}{\sqrt{36^x-36}}$

Solution. Observe that

$$\int \frac{dx}{\sqrt{36^x-36}} = \int \frac{dx}{\sqrt{(6^x)^2-6^2}}.$$

Let $u = 6^x$. Then $du = 6^x \ln 6\, dx$. Hence $dx = \dfrac{du}{6^x \ln 6} = \dfrac{du}{u \ln 6}$. So,

$$\begin{aligned}
\int \frac{dx}{\sqrt{36^x-36}} &= \frac{1}{\ln 6}\int \frac{du}{u\sqrt{u^2-6^2}} \\
&= \frac{1}{\ln 6}\cdot\frac{1}{6}\sec^{-1}\left(\frac{u}{6}\right) + C \\
&= \frac{1}{6\ln 6}\sec^{-1}\left(\frac{6^x}{6}\right) + C \\
&= \frac{1}{6\ln 6}\sec^{-1}\left(6^{x-1}\right) + C.
\end{aligned}$$

EXAMPLE 21: $\displaystyle\int \frac{1}{e^x+e^{-x}}\,dx$

Solution. $\displaystyle\int \frac{1}{e^x+e^{-x}}\,dx = \int \frac{1}{e^x+e^{-x}}\cdot\frac{e^x}{e^x}\,dx = \int \frac{e^x}{(e^x)^2+1}\,dx.$

Let $u = e^x$. Then $du = e^x\ dx$. So,

$$\begin{aligned}
\int \frac{1}{e^x+e^{-x}}\,dx &= \int \frac{du}{u^2+1} \\
&= \tan^{-1} u + C \\
&= \tan^{-1}(e^x) + C.
\end{aligned}$$

Supplementary Problems

1. $\int (5x-3)^{100}\,dx$

2. $\int (x-3)^{1/4}\,dx$

3. $\int \left(x^2+\frac{1}{2}\right)\left(\frac{2}{3}x^3+x\right)^5\,dx$

4. $\int (8x-3)\left(4x^2-3x\right)^{\frac{3}{4}}\,dx$

5. $\int \frac{x}{\sqrt{1-4x^2}}\,dx$

6. $\int \frac{x}{\sqrt{3-5x^2}}\,dx$

7. $\int x\sqrt{x-1}\,dx$

8. $\int (x)\sqrt{x^2-1}\,dx$

9. $\int \frac{1}{5x+1}\,dx$

10. $\int \frac{dx}{3x+9}$

11. $\int \frac{4x-6}{(x-1)(x-2)}\,dx$

12. $\int \frac{4x-4}{x(x-2)}\,dx$

13. $\int \frac{2}{x\sqrt{(\ln(x))^3}}\,dx$

14. $\int \frac{3}{(x-1)\sqrt{(\ln(x+1))^7}}\,dx$

15. $\int \frac{1}{5}e^{2-5x}\,dx$

16. $\int (2x-3)\,e^{x^2-3x}\,dx$

17. $\int \frac{e^{4x+1}}{e^{2x}-1}\,dx$

18. $\int \frac{e^{-2x}}{e^{-2x}-1}\,dx$

19. $\int 2^{7x+3}\,dx$

20. $\int 7^{7x-7}\,dx$

21. $\int 3^{x^3-x+1}(12x^2-4)\,dx$

22. $\int 11^{x^7-3x+\cos x}\left(7x^6-3-\sin x\right)\,dx$

23. $\int \cos^4 x\sin x\,dx$

24. $\int \csc^4 x\cot^3 x\,dx$

25. $\int x\sec^3(x^2)\tan(x^2)\,dx$

26. $\int \sin^2 x\,dx$ $\left(\text{Hint} : \cos(2x)=1-2\sin^2 x\right)$

27. $\int \frac{1}{\sqrt{4-(x-1)^2}}\,dx$

28. $\int \frac{1}{9x^2+25}\,dx$

29. $\int \frac{dx}{x\ln x\sqrt{(\ln x)^2-36}}$

30. $\int \frac{1}{\sqrt{4^x-4}}\,dx$

31. $\int \frac{dx}{\sqrt{7+6x-x^2}}$

32. $\int \frac{8x+1}{4x^2+12x+10}\,dx$

LESSON 13: Application of Antidifferentiation to Differential Equations

LEARNING OUTCOME: At the end of the lesson, the learner shall be able to solve separable differential equations using antidifferentiation.

TOPIC 13.1: Separable Differential Equations

A ***differential equation (DE)*** is an equation that involves x, y and the derivatives of y. The following are examples of differential equations:

1. $\dfrac{dy}{dx} = 2x + 5$
2. $\dfrac{dy}{dx} = -\dfrac{x}{y}$
3. $y'' + y = 0$

The ***order*** of a differential equation pertains to the highest order of the derivative that appears in the differential equation.

The first two examples above are first-order DEs because they involve only the first derivative, while the last example is a second-order DE because y'' appears in the equation.

A ***solution*** to a differential equation is a function $y = f(x)$ or a relation $f(x, y) = 0$ that satisfies the equation.

For example, $y = x^2 + 5x + 1$ is a solution to $\dfrac{dy}{dx} = 2x + 5$ since

$$\frac{d}{dx}(y) = \frac{d}{dx}(x^2 + 5x + 1) = 2x + 5.$$

The relation $x^2 + y^2 = 1$ is a solution to $\dfrac{dy}{dx} = -\dfrac{x}{y}$ because if we differentiate the relation implicitly, we get

$$2x + 2y\frac{dy}{dx} = 0 \qquad \Longrightarrow \qquad \frac{dy}{dx} = -\frac{x}{y}.$$

Finally, $y = \sin x$ solves the differential equation $y'' + y = 0$ since $y' = \cos x$ and $y'' = -\sin x$, and therefore

$$y'' + y = (-\sin x) + \sin x = 0.$$

Solving a differential equation means finding *all* possible solutions to the DE.

A differential equation is said to be ***separable*** if it can be expressed as

$$f(x)\,dx = g(y)\,dy,$$

where f and g are functions of x and y, respectively. Observe that we have separated the variables in the sense that the left-hand side only involves x while the right-hand side is purely in terms of y.

If it is possible to separate the variables, then we can find the solution of the differential equation by simply integrating:

$$\int f(x)\,dx = \int g(y)\,dy,$$

and applying appropriate techniques of integration. Note that the left-hand side yields a function of x, say $F(x) + C_1$, while the right-hand side yields a function of y, say $G(y) + C_2$. We thus obtain

$$F(x) = G(y) + C \qquad \text{(Here, } C = C_2 - C_1\text{)}$$

which we can then express into a solution of the form $y = H(x) + C$, if possible.

We will now look at some examples of how to solve separable differential equations.

EXAMPLE 1: Solve the differential equation $\dfrac{dy}{dt} = \dfrac{1}{4}\,y$.

Solution. Observe that $y = 0$ is a solution to the differential equation. Suppose that $y \neq 0$. We divide both sides of the equation by y to separate the variables:

$$\frac{dy}{y} = \frac{1}{4}\,dt.$$

Integrating the left-hand side with respect to y and integrating the right-hand side with respect to t yield

$$\int \frac{dy}{y} = \int \frac{1}{4}\,dt \qquad \Longrightarrow \qquad \ln|y| = \frac{1}{4}t + C.$$

Taking the exponential of both sides, we obtain

$$|y| = e^{\frac{1}{4}t+C} = e^{C}\,e^{\frac{1}{4}t}.$$

Therefore, $y = \pm A e^{\frac{1}{4}t}$, where $A = e^{C}$ is any positive constant. Therefore, along with the solution $y = 0$ that we have found at the start, the solution to the given differential equation is $y = Ae^{\frac{1}{4}t}$, where A is any real number.

EXAMPLE 2: Solve the differential equation $2y\,dx - 3x\,dy = 0$.

Solution. If $y = 0$, then $\dfrac{dy}{dx} = 0$ which means that the function $y = 0$ is a solution to the differential equation $2y - 3x\dfrac{dy}{dx} = 0$. Assume that $y \neq 0$. Separating the variables, we get

$$2y\,dx = 3x\,dy$$

$$\frac{2}{x}\,dx = \frac{3}{y}\,dy.$$

Integrating both sides, we obtain

$$\begin{aligned}
\int \frac{2}{x}\,dx &= \int \frac{3}{y}\,dy \\
2\ln|x| &= 3\ln|y| + C \\
\ln|x|^2 - C &= \ln|y|^3 \\
|y|^3 &= e^{-C}\,e^{\ln|x|^2} = e^{-C}\,|x|^2.
\end{aligned}$$

Therefore, the solutions are $y = \pm\sqrt[3]{A}\,x^{2/3}$ where $A = e^{-C}$ is any positive constant. Along with the trivial solution $y = 0$, the set of solutions to the differential equation is $y = Bx^{2/3}$ where B is any real number. *Note: The absolute value bars are dropped since x^2 is already nonnegative.*

EXAMPLE 3: Solve the equation $xy^3\,dx + e^{x^2}\,dy = 0$.

Solution. First, we separate the variables. We have

$$\begin{aligned}
-xy^3\,dx &= e^{x^2}\,dy \\
-xe^{-x^2}\,dx &= \frac{1}{y^3}\,dy.
\end{aligned}$$

Integrating both sides of the equation with respect to their variables, we have

$$-\int xe^{-x^2}\,dx = \int y^{-3}\,dy.$$

Meanwhile, if $u = -x^2$, then $du = -2x\,dx$ so that $-\dfrac{du}{2} = x\,dx$. Hence,

$$-\int xe^{-x^2}\,dx = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{-x^2} + C.$$

Therefore,

$$\begin{aligned}
-\int xe^{-x^2}\,dx &= \int y^{-3}\,dy \\
\frac{1}{2}e^{-x^2} &= -\frac{y^{-2}}{2} + C.
\end{aligned}$$

If we solve for y in terms of x, we get $y = \pm\left(A - e^{-x^2}\right)^{-1/2}$, where $A = 2C$ is any constant.

EXAMPLE 4: Solve the equation $3(y+2)\;dx - xy\;dy = 0$.

Solution. Separating the variables of the differential equation gives us

$$\begin{aligned}3\,(y+2)\ dx &= xy\ dy \\ \frac{3}{x}\ dx &= \frac{y}{y+2}\ dy \\ \frac{3}{x}\ dx &= \frac{(y+2)-2}{y+2}\ dy \\ \frac{3}{x}\ dx &= \left(1-\frac{2}{y+2}\right)\ dy.\end{aligned}$$

Now that we have separated the variables, we now integrate the equation term by term:

$$\begin{aligned}\int \frac{3}{x}\ dx &= \int \left(1-\frac{2}{y+2}\right)\ dy \\ 3\ln|x| &= y-2\ln|y+2|+C.\end{aligned}$$

Note that in the previous examples, a constant of integration is always present. If there are initial conditions, or if we know that the solution passes through a point, we can solve this constant and get a ***particular solution*** to the differential equation.

EXAMPLE 5: Find the particular solutions of the following given their corresponding initial conditions:

1. $\dfrac{dy}{dt}=\dfrac{1}{4}y$ when $y=100$ and $t=0$
2. $2y\,dx-3x\,dy=0$ when $x=1$ and $y=1$
3. $xy^3\,dx+e^{x^2}\,dy=0$ when $x=0$ and $y=1$
4. $3(y+2)\,dx-xy\,dy=0$ when $x=1$ and $y=-1$

Solution. We will use the general solutions from the previous examples.

1. The solution to Example 1 is $y=A\,e^{1/4t}$. Using the conditions $y=100$ and $t=0$, we get $100=Ae^0$. Hence, $A=100$ and therefore the particular solution is $y=100e^{\frac{1}{4}t}$.
2. The solution to Example 2 is $y=B\,x^{2/3}$. Substituting $(x,y)=(1,1)$ gives $1=B\,1^{2/3}=B$. Hence, the particular solution is $y=\sqrt[3]{x^2}$.

3. From the Example 3, the general solution is $y = \pm \dfrac{1}{\sqrt{A - e^{-x^2}}}$. Substituting $(x, y) = (1, 0)$ yields $1 = \pm \dfrac{1}{\sqrt{A-1}}$. Since the square root of a real number is never negative, $\sqrt{A-1} = +1$ and so $A = 2$. Finally, the particular solution is $y = \pm \dfrac{1}{\sqrt{2 - e^{-x^2}}}$.

4. From Example 4, the general solution is $3 \ln|x| = y - 2\ln|y+2| + C$. Substituting the given values $(x, y) = (1, -1)$, we obtain $3 \ln|1| = -1 - 2\ln|-1+2| + C$. Simplifying this gives $C = 1$. Hence, the particular solution is the relation $3\ln|x| = y - 2\ln|y+2| + 1$.

Solved Examples

EXAMPLE 1: Find the general solution to the differential equation $\dfrac{dy}{dx} = \dfrac{x^2}{y+3}$.

Solution*.*

$$\begin{aligned}
\frac{dy}{dx} &= \frac{x^2}{y+3} \\
(y+3)\ dy &= x^2\, dx \\
\int (y+3)\ dy &= \int x^2\, dx \\
\frac{y^2}{2} + 3y &= \frac{x^3}{3} + C.
\end{aligned}$$

EXAMPLE 2: Find the general solution to the differential equation $\dfrac{dy}{dx} = \dfrac{1}{(e^y+1)(1+x^2)}$.

Solution*.*

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{(e^y+1)(1+x^2)} \\
(e^y+1)\ dy &= \frac{1}{1+x^2}\, dx \\
\int (e^y+1)\ dy &= \int \frac{1}{1+x^2}\, dx \\
e^y + y &= \tan^{-1} x + C.
\end{aligned}$$

EXAMPLE 3: Find the general solution to the differential equation

$$\frac{dy}{dx} = \frac{y\left(x^3+x\right)}{y-4}.$$

.

Solution.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{y\left(x^3+x\right)}{y-4} \\
\frac{dy}{dx} &= \frac{y}{y-4} \cdot \frac{x^3+x}{1} \\
\left(\frac{y-4}{y}\right) dy &= \left(x^3+x\right)\, dx \\
\int \frac{y-4}{y}\, dy &= \int \left(x^3+x\right)\, dx \\
\int \left(1-\frac{4}{y}\right) dy &= \int \left(x^3+x\right)\, dx \\
y - 4\ln|y| &= \frac{x^4}{4} + \frac{x^2}{2} + C.
\end{aligned}$$

EXAMPLE 4: Solve the initial value problem $xy + 2x + \dfrac{dy}{dx} = x^2\left(y+2\right) + y + 2$ and $y(0) =$ 4.

Solution.

$$\begin{aligned}
xy + 2x + \frac{dy}{dx} &= x^2\left(y+2\right) + y + 2 \\
x\left(y+2\right) + \frac{dy}{dx} &= x^2\left(y+2\right) + \left(y+2\right) \\
\frac{dy}{dx} &= x^2\left(y+2\right) - x\left(y+2\right) + \left(y+2\right) \\
\frac{dy}{dx} &= \left(y+2\right)\left(x^2-x+1\right) \\
\frac{1}{y+2}\, dy &= \left(x^2-x+1\right)\, dx \\
\int \frac{1}{y+2}\, dy &= \int \left(x^2-x+1\right)\, dx \\
\ln|y+2| &= \frac{x^3}{3} - \frac{x^2}{2} + x + C.
\end{aligned}$$

Since $y = 4$ when $x = 0$, we have

$$\begin{aligned}
\ln\left(4+2\right) &= C \\
\ln 6 &= C.
\end{aligned}$$

The solution to the initial value problem is $\ln|y+2| = \dfrac{x^3}{3} - \dfrac{x^2}{2} + x + \ln 6$.

EXAMPLE 5: Solve the initial value problem $\dfrac{dy}{dx} = 6e^{3x} + 1$ and $y(0) = 0$.

Solution.

$$\begin{aligned}
\frac{dy}{dx} &= 6e^{3x} + 1 \\
dy &= \left(6e^{3x} + 1\right) dx \\
\int dy &= \int \left(6e^{3x} + 1\right) dx \\
y &= \frac{6e^{3x}}{3} + x + C \\
y &= 2e^{3x} + x + C.
\end{aligned}$$

Substituting $x = 0$ and $y = 0$, we get

$$\begin{aligned}
0 &= 2e^{0} + 0 + C \\
-2 &= C
\end{aligned}$$

Hence the solution to the initial value problem is $y = 2e^{3x} + x - 2$.

Supplementary Problems

1. Determine whether each of the following differential equations is separable or not, if it is separable, rewrite the equation in the form $g(y)\,dy = f(x)\,dx$.

   (a) $\dfrac{dy}{dx} = y + 3$

   (b) $\dfrac{dy}{dx} = x^2y - xy + 3y$

   (c) $\dfrac{dy}{dx} = \cos(3x - y)$

   (d) $\dfrac{dy}{dx}x - 3y + 2$

   (e) $\dfrac{1}{\sin x} \cdot \dfrac{dy}{dx} = \cos\left(3y^2 - y - 1\right)$

   (f) $-xy^3 + 3x + \dfrac{dy}{dx} = 3 - y^3$

   (g) $\dfrac{dy}{dx} = \sqrt{4 + y - x}$

2. Find the general solution of the following differential equations.

   (a) $\dfrac{dy}{dx} = \dfrac{3}{3y}$

   (b) $\dfrac{dy}{dx} = y^2(1 + e^x)$

   (c) $\dfrac{dy}{dx} = 16x^3(y - 1)$

   (d) $\dfrac{dy}{dx} = -\dfrac{x+1}{2(y+3)}$

   (e) $(\tan y)\dfrac{dy}{dx} = \csc^2(x + 3)$

3. Solve the following initial value problems.

   (a) $\dfrac{dy}{dx} = x^2 - 2 \quad \text{and} \quad y(3) = 7$

   (b) $\dfrac{dy}{dx} = \dfrac{x}{y-4} \quad \text{and} \quad y\,(2) = 9$

   (c) $\dfrac{dy}{dx} = \dfrac{2x^3}{y^2} \quad \text{and} \quad y(2) = 1$

   (d) Find the particular solution of the differential equation $\dfrac{d^2y}{dx^2} = -\dfrac{1}{x^5}$ determined by the conditions $y = -\dfrac{1}{12}$ and $\dfrac{dy}{dx} = \dfrac{5}{4}$ when $x = 1$.

LESSON 14: Application of Differential Equations in Life Sciences

LEARNING OUTCOME: At the end of the lesson, the learner shall be able to solve situational problems involving: exponential growth and decay, bounded growth, and logistic growth.

TOPIC 14.1: Situational Problems Involving Growth and Decay Problems

EXPONENTIAL GROWTH AND DECAY

Let $y = f(t)$ be the size of a certain population at time t, and let the birth and death rates be the positive constants b and d, respectively. The rate of change $\frac{dy}{dt}$ in the population y with respect to time t is given by

$$\frac{dy}{dt} = ky, \text{ where } k = b - d.$$

If k is positive, that is when $b > d$, then there are more births than deaths and $\frac{dy}{dt}$ denotes **growth**. If k is negative, that is when $b < d$, then there are more deaths than births and $\frac{dy}{dt}$ denotes **decay**

EXAMPLE 1: Suppose that a colony of lice grows exponentially. After 1 day, 50 lice are counted. After 3 days, 200 were counted. How many are there originally? What is the exponential growth equation for the colony?

Solution. Recall the exponential growth equation and identify information given in the problem that will help answer the question.

- $y_1 = 50$ means that $50 = y_0 e^{k \cdot 1}$.
- $y_3 = 200$ means that $200 = y_0 e^{k \cdot 3}$.

Note that these two equations will give us the values for the two unknowns, y_0 and e^k.

$$\begin{aligned} 50 &= y_0 e^k \\ 200 &= y_0 e^{3k}. \end{aligned}$$

From the first equation, $y_0 = 50e^{-k}$. Using this in the second equation,

$$\begin{aligned} 200 &= (50\, e^{-k}) e^{3k} \\ 200 &= 50\, e^{2k} \\ 4 &= e^{2k} \\ 4 &= (e^k)^2 \\ \text{or } \ 2 &= e^k. \end{aligned}$$

Substituting this in the first equation,

$$\begin{aligned} 50 &= y_0 \cdot 2 \\ \text{or } y_0 &= 25. \end{aligned}$$

We now have the answers to the two questions given. First, there were originally 25 lice in the colony. Second, the exponential growth equation for the given word problem is

$$y = 25 \cdot 2^t.$$

Now, let us take a decay problem.

EXAMPLE 2: The rate of decay of radium is said to be proportional to the amount of radium present. If the half-life of radium is 1690 years and there are 200 grams on hand now, how much radium will be present in 845 years?

Solution. The exponential decay equation again starts off as $y = Ce^{kt}$.

Since there are 200 grams present at the start, the equation immediately evolves to

$$y = 200e^{kt}.$$

A half-life of 1690 years means that the initial amount of 200 grams of radium will reduce to half, or just 100 grams, in 100 years. Thus,

$$100 = 200e^{k \cdot 1690}.$$

This gives

$$e^k = \left(\frac{1}{2}\right)^{1/1690},$$

and consequently,

$$y = 200\left(\frac{1}{2}\right)^{t/1690}.$$

To answer the problem,

$$\begin{aligned} y &= 200\left(\frac{1}{2}\right)^{845/1690} \\ &= 200\left(\frac{1}{2}\right)^{1/2} \\ &= 200\left(\frac{1}{\sqrt{2}}\right) \\ &\approx 200\left(\frac{1}{0.707}\right) \\ &= 141.4. \end{aligned}$$

Therefore, after 845 years, there will be approximately 141.4 grams of radium left.

BOUNDED GROWTH

Let $y = f(t)$ be the size of a certain population at time t. We say that the type of growth y has is called *bounded growth* if y satisfies the differential equation $\frac{dy}{dt} = k(K - y)$, where $K > 0$ is the *carrying capacity(limiting quatity)*.

EXAMPLE 3: A certain *pawikan* breeding site is said to be able to sustain 5000 *pawikans*. One thousand *pawikans* are brought there initially. After a year, this increased to 1100 *pawikans*. How many *pawikans* will there be after 5 years? Assume that *pawikans* follow the limited growth model.

Solution. We recall the bounded growth equation and identify parts given in the word problem.

- $K = 5000$.
- $y_0 = 1000$. This means that $C = 5000 - 1000 = 4000$ and the equation becomes

$$y = 5000 - 4000 \cdot e^{-kt}.$$

The population after 1 year, $y_1 = 1100$, means we can substitute y with 1100 and t with 1 to obtain e^{-k}.

$$\begin{aligned} 1100 &= 5000 - 4000 \cdot e^{-k} \\ 4000 \cdot e^{-k} &= 5000 - 1100 \\ &= 3900 \\ e^{-k} &= \frac{3900}{4000} \\ &= 0.975. \end{aligned}$$

With the values we have enumerated and solved, the bounded equation is now of the form

$$y = 5000 - 4000 \cdot (0.975)^t.$$

We can now find the required population, y_5.

$$\begin{aligned} y &= 5000 - 4000 \cdot (0.975)^5 \\ &\approx 5000 - 4000 \cdot (0.881) \\ &= 5000 - 3524 \\ &= 1476. \end{aligned}$$

Therefore, there will be approximately 1476 *pawikans* in the breeding site.

The next example illustrates a sort of "decay." Remember we said earlier that there are occasions when $y > K$? This is one instance.

EXAMPLE 4: Suppose that newly-baked cupcakes are taken out of the oven which is set at 100 degrees. Room temperature is found to be 25 degrees, and in 15 minutes the cupcakes are found to have a temperature of 50 degrees. Determine the approximate temperature of the cupcakes after 30 minutes.

Solution. Newton's Law of Cooling states that the rate of change of the temperature of an object is equal to the difference between the object's temperature and that of the surrounding air. This gives the differential equation

$$\frac{dy}{dt} = -k(y - 25).$$

Since the situation anticipates that the temperature of an object, y, will decrease towards that of the surrounding air, y_a. Thus, y is assumed to be greater than y_a. Furthermore, to denote the decrease, the constant of proportionality is written as $-k$, with $k > 0$. t in this problem is measured in minutes.

By separation of variables, this becomes

$$y = 25 + Ce^{-kt}.$$

- $y_0 = 100$, we get $C = 75$ and the equation becomes

$$y = 25 + 75e^{-kt}.$$

- The 50-degree temperature after 15 minutes gives

$$e^{-k} = \left(\frac{1}{3}\right)^{1/15},$$

and the equation changes further to

$$y = 25 + 75\left(\frac{1}{3}\right)^{t/15}.$$

We can now proceed to approximate the temperature after 30 minutes:

$$\begin{aligned} y &= 25 + 75\left(\frac{1}{3}\right)^{30/15} \\ &= 25 + 75\left(\frac{1}{3}\right)^{2} \\ &= 25 + 75\left(\frac{1}{9}\right) \\ &\approx 25 + 8.33 \\ &= 33.33. \end{aligned}$$

Hence, after 30 minutes, the cupcakes' temperature will be approximately 33 degrees.

LOGISTIC GROWTH

Let $y = f(t)$ be the size of a certain population at time t. We say that the type of growth y has is called *logistic growth* if y satisfies the differential equation $\frac{dy}{dt} = ky(K - y)$, where y is the size of the population.

EXAMPLE 5: Ten Philippine eagles were introduced to a national park 10 years ago. There are now 23 eagles in the park. The park can support a maximum of 100 eagles. Assuming a logistic growth model, when will the eagle population reach 50?

Solution. To solve the problem, we first recognize how the given information will fit into and improve our equation.

- Since $K = 100$, we have $y = \dfrac{100}{1 + C \cdot e^{-100kt}}$
- Since $y_0 = 23$, we can solve for C.

$$\begin{aligned}
10 &= \frac{100}{1 + C \cdot e^0} \\
10 &= \frac{100}{1 + C} \\
10 + 10C &= 100 \\
10C &= 90 \\
\text{or } C &= 9.
\end{aligned}$$

Hence, the equation becomes

$$y = \frac{100}{1 + 9 \cdot e^{-100kt}}.$$

The current population of 23 eagles is equal to the population after 10 years, or $y_{10} = 23$. This piece of information allows us to solve for the exponential term.

$$\begin{aligned}
23 &= \frac{100}{1 + 9e^{-100k \cdot 10}} \\
23 &= \frac{100}{1 + 9e^{-1000k}} \\
1 + 9e^{-1000k} &= \frac{100}{23} \\
9e^{-1000k} &= \frac{100}{23} - 1 \\
9e^{-1000k} &= \frac{77}{23} \\
e^{-1000k} &= \frac{77}{23 \cdot 9} \\
\text{or} \quad e^{-1000k} &\approx 0.37.
\end{aligned}$$

Instead of solving for k, it will suffice to find a substitute for $e^{-Kk} = e^{-100k}$. Clearly, if $e^{-1000k} \approx 0.37$, then $e^{-100k} \approx (0.37)^{1/10}$. So,

$$y = \frac{100}{1 + 9 \cdot (0.37)^{t/10}}.$$

We are now ready to answer the question, "When will the eagle population reach 50?" Given the most recent version of our logistic equation, we just substitute y with 50 and solve for t, the time required to have 50 eagles in the population.

$$
\begin{aligned}
50 &= \frac{100}{1+9\cdot(0.37)^{t/10}}\\
50(1+9\cdot(0.37)^{t/10}) &= 100\\
50+450\cdot(0.37)^{t/10} &= 100\\
(0.37)^{t/10} &= \frac{100-50}{450}\\
(0.37)^{t/10} &= \frac{1}{9}\\
(0.37)^{t} &= \left(\frac{1}{9}\right)^{10}\\
\ln(0.37)^{t} &= \ln\left(\frac{1}{9}\right)^{10}\\
t\cdot\ln(0.37) &= 10\cdot\ln\left(\frac{1}{9}\right)\\
t &= 10\cdot\frac{\ln(1/9)}{\ln(0.37)}\\
t &\approx 10(2.2)\;=\;22.
\end{aligned}
$$

The eagle population in the said national park will reach 50 in approximately 22 years.

Solved Examples

Solve each item completely.

EXAMPLE 1: It is observed that the number of bacteria present in the portion of the small intestine of an animal grows exponentially. Initially, there are 10,000 bacteria present. Three hours later, the bacteria grow to 500,000. Assuming no bacteria die in the process, how many bacteria will there be after one day? After how many hours will the bacteria double its number?

Solution. Let $y(t)$ be the number of bacteria in the intestine after t hours. Since the number of bacteria follows an exponential growth model then

$$y(t) \;=\; y_0e^{kt}$$

for some constant y_0 and k. Using the assumptions that there are 10,000 bacteria present initially and 500,000 bacteria are present after 3 hours then

$$10,000 = y(0) = y_0e^{k\cdot 0} \implies y_0 = 10,000$$

and

$$500,000 = y(3) = 10,000e^{3k} \implies e^{3k=50} \implies e^{3k} = 50^{1/3}.$$

Hence

$$y(t) \quad = \quad 10,000\,(50)^{(1/3)t}.$$

So after 1 day = 24 hours, there will be $y(24) = 10,000\,(50)^{(1/3)(24)} = 3.9 \times 10^{17}$ cells in the intestine and the cells will double its number after

$$\begin{aligned}
2y(0) &= y(t) \\
20\ 000 &= 10\ 000\,(50)^{(1/3)t} \\
50^{(1/3)t} &= 2 \\
\ln\left(50^{(1/3)t}\right) &= \ln 2 \\
t\ln\left(50^{(1/3)t}\right) &= \ln 2 \\
t &= \frac{\ln 2}{\ln\left(50^{1/3}\right)} \\
t &= 0.53 \text{ hours.}
\end{aligned}$$

EXAMPLE 2: The half-life of a radioactive substance is defined to be the amount of time it takes for the substace to decay 50% of its amount. If substance X has a half-life of 3,600 years, what part of substance X will remain after 4,500 years?

Solution. Let $y(t)$ be the amount of substance X that remains after t years. $y(t)$ follows an exponential decay model. Let y_0 be the amount of substance X at $t = 0$. Since substance X has a half life of 3,600 years, then

$$\frac{y_0}{2} = y(3600) = y_0 e^{3600k} \implies e^{3600k} = \frac{1}{2} \implies e^k = \left(\frac{1}{2}\right)^{(1/3600)}.$$

Hence,

$$y(t) = y_0\left(\frac{1}{2}\right)^{t(1/3600)}.$$

and when $t = 4,500$,

$$\begin{aligned}
y(4500) &= y_0\left(\frac{1}{2}\right)^{4500(1/3600)} \\
&\approx 0.42y_0.
\end{aligned}$$

Approximately 42% of the substance will remain after 4,500 years.

EXAMPLE 3: In a community with 300 families, it was found out that 50 families were infected by an incurable disease. After one week, the number of families that were infected by the disease was increased by 10. Assuming that the spread of disease follows a bounded growth model, when will 90% of the community be infected?

Solution. Let $y(t)$ be the number of infected families at week t. Since the spread of disease follows a bounded growth model, then

- $K = 300$
- $y_0 = 50$
- $y(1) = 60$
- $C = K - y_0 = 300 - 50 = 250$

So,

$$y(t) = 300 - 250e^{-kt}.$$

Since 60 families were infected after one week, then $y(1) = 60$. We can solve for k using this information.

$$\begin{aligned} 300 - 250e^{-k} &= 60 \\ -250e^{-k} &= -240 \\ e^{-k} &= \frac{24}{25}. \end{aligned}$$

Then our model becomes

$$y(t) = 300 - 250\left(\frac{24}{25}\right)^t.$$

We want to find t such that $y(t) = 300(0.9)$.

$$\begin{aligned} 300 - 250\left(\frac{24}{25}\right)^t &= 300(0.9) \\ 250\left(\frac{24}{25}\right)^t &= 300(0.1) \\ \left(\frac{24}{25}\right)^t &= \frac{3}{25} \\ \ln\left(\frac{24}{25}\right)^t &= \ln\left(\frac{3}{25}\right) \\ t\ln\left(\frac{24}{25}\right) &= \ln\left(\frac{3}{25}\right) \\ t &= \frac{\ln\left(\frac{3}{25}\right)}{\ln\left(\frac{24}{25}\right)} \\ t &\approx 51.94 \end{aligned}$$

Hence, 90% of the community will be infected by the disease in 51.94 weeks.

EXAMPLE 4: A thermometer reads 18°C inside an airconditioned house. It is placed outside the house where the air temperature is 30°C. Three minutes later, it is found that the thermometer reading is 21°C. Find the thermometer reading after 6 minutes.

Solution. Let $y(t)$ be the temperature reading of the thermometer at time t minutes after it was placed outside. Then by Newton's Law of Cooling,

$$\frac{dy}{dt} = -k\left(y-30\right).$$

Solving by separation of variables we have

$$\begin{aligned} -\frac{1}{k}\cdot\frac{dy}{y-30} &= dt \\ -\frac{1}{k}\ln|y-30| &= t+C \\ y-30 &= e^{-k(t+C)} \\ y &= Ke^{-kt}+30, \end{aligned}$$

where $K=e^{-kC}$. Since $y(0)=18$,

$$\begin{aligned} 18 &= Ke^{-k\cdot 0}+30 \\ K &= -12, \end{aligned}$$

and $y(3)=21$ implies

$$\begin{aligned} 21 &= -12e^{-3k}+30 \\ -9 &= -12e^{-3k} \\ e^{-3k} &= 0.75 \\ e^{-k} &= 0.75^{1/3}. \end{aligned}$$

Hence

$$y(t) = -12\left(0.75\right)^{(1/3)t}+30.$$

In 6 minutes, the temperature reading of the thermometer will be

$$\begin{aligned} y(6) &= -12\left(0.75\right)^{(1/3)(6)}+30 \\ &= 23.25^{\circ}\text{C}. \end{aligned}$$

EXAMPLE 5: Scientists stock a very large aquarium with 500 shrimps for an experiment and the aquarium can accommodate up to 10,000 shrimps. The number of shrimps tripled during the first quarter of the year.

1. Assuming that the size of the shrimp population satisfies the logistic equation, find an expression for the size of the population after t quarters.

2. How long will it take the population to reach 3,000?

Solution. Let $y(t)$ be the shrimp population after t quarters. Since the population satisfies the logistic equation, then

$$y(t) = K - Ce^{-kt}$$

where $K = 10,000$, $y(0) = 500$ and $y(1) = 1\ 500$. We substitute these values to obtain the following

$$500 = 10,000 - C \implies C = 9,500$$

and

$$1,500 = 10,000 - 9,500e^{-k} \implies 9,500e^{-k} = 8,500 \implies e^{-k} = \frac{17}{19}.$$

Hence

$$y(t) = 10,000 - 9,500\left(\frac{17}{19}\right)^t.$$

We now solve the time when the population becomes 3,000.

$$\begin{aligned}
y(t) &= 3,000 \\
10,000 - 9,500\left(\frac{17}{19}\right)^t &= 3,000 \\
9,500\left(\frac{17}{19}\right)^t &= 7,000 \\
\left(\frac{17}{19}\right)^t &= \frac{14}{19} \\
\ln\left(\frac{17}{19}\right)^t &= \ln\frac{14}{19} \\
t\ln\left(\frac{17}{19}\right) &= \ln\frac{14}{19} \\
t &= \frac{\ln\frac{14}{19}}{\ln\frac{17}{19}} \\
t &= 2.7456 \text{ quarters}
\end{aligned}$$

Supplementary Problems

1. A petri dish has 10,000 bacteria in it. After 4 hours the bacteria's population increased to 20,000. If the number of colonies grows exponentially, write a formula for the number of bacteria in the dish at any time t, where t is in hours.

2. In a farm, there initally were 10 rabbits. After three months, there are already 20 rabbits. Assuming that the growth is exponential and no rabbit dies in the process, at what time will there be 100 rabbits?

3. Suppose that the value of a ₱100,000 asset decreases at a constant percentage rate of 10% per year.

   (a) Find its worth after 20 years.

   (b) When will the asset be half of its original value?

4. A certain culture is able to sustain 100 colonies of bacteria. Initially, there were 10 colonies in the culture. After 5 hours, the colonies increased to 25. How many colonies will there be in 24 hours? Assume that the bacteria follow the limited growth model.

5. A bowl of soup at 95°C (too hot) is placed in a 22°C room. One minute later, the soup has cooled to 83°C. When will the temperature be 49°C (just right)?

6. A bacterial population is known to have a logistic growth pattern with initial population 1000 and an equilibrium population of 10,000. A count shows that at the end of 1 hour there are 2000 bacteria present. Determine the population as a function of time.

7. The Department of Animal Welfare released 100 deer into a game preserve. During the first 5 years, the population increases to 450 deer. Find a model for the population growth assuming logistic growth with a limit of 5000 deer. What will be the predicted population of the deer in 10 years? In 20 years?

LESSON 15: Riemann Sums and the Definite Integral

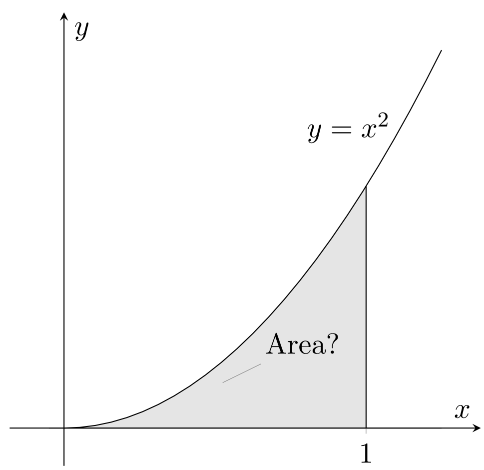
LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Approximate the area of a region under a curve using Riemann sums: (a) left, (b) right, and (c) midpoint; and
2. Define the definite integral as the limit of the Riemann sums.

TOPIC 15.1: Approximation of Area using Riemann Sums

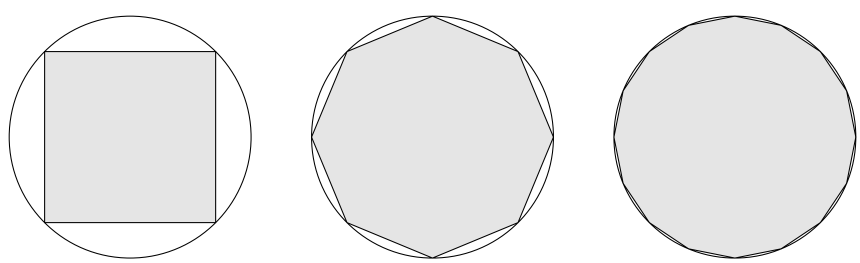
METHOD OF EXHAUSTION

Notice that geometry provides formulas for the area of a region bounded by straight lines. However, it does not provide formulas to compute the area of a general region. For example, it is quite impossible to compute for the area of the region below the parabola $y = x^2$ using geometry alone.



Even the formula for the area of the circle $A = \pi r^2$ uses a limiting process. Before, since people only knew how to find the area of polygons, they tried to cover the area of a circle by inscribing n-gons until the error was very small. This is called the *Method of Exhaustion*.

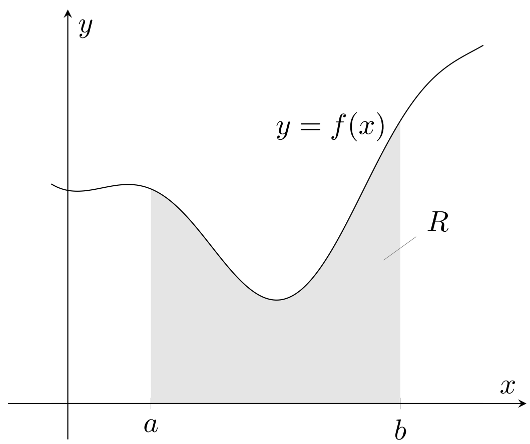
The method of exhaustion is attributed to the ancient Greek mathematician Antiphon of Athens (ca. 5th century BCE), who thought of inscribing a sequence of regular polygons, each with double the number of sides than the previous one, to approximate the area of a circle.



Our method for approximating the area of a region uses the same technique. However, instead of inscribing regular n-gons, we use the simplest polygon – rectangles.

RIEMANN SUMS

Throughout this lesson, we will assume that function f is positive (that is, the graph is above the x-axis), and continuous on the closed and bounded interval $[a, b]$. The **goal** of this lesson is to approximate the area of the region R bounded by $y = f(x)$, $x = a$, $x = b$, and the x-axis.



We first partition $[a, b]$ regularly, that is, into congruent subintervals. Similar to the method of exhaustion, we fill R with rectangles of equal widths. The *Riemann sum of f* refers to the number equal to the combined area of these rectangles. Notice that as the number of rectangles increases, the Riemann sum approximation of the exact area of R becomes better and better.

Of course, the Riemann sum depends on how we construct the rectangles and with how many rectangles we fill the region. We will discuss three basic types of Riemann sums: Left, Right, and Midpoint.

PARTITION POINTS

First, we discuss how to divide equally the interval $[a, b]$ into n subintervals. To do this, we compute the *step size* Δx, the length of each subinterval:

$$\Delta x = \frac{b - a}{n}.$$

Next, we let $x_0 = a$, and for each $i = 1, 2, \ldots, n$, we set the ith intermediate point to be $x_i = a + i\Delta x$. Clearly, the last point is $x_n = a + n\Delta x = a + n\left(\frac{b-a}{n}\right) = b$. Please refer to the following table:

x_0	x_1	x_2	x_3	$\cdots$	x_i	$\cdots$	x_{n-1}	x_n
a	$a + \Delta x$	$a + 2\Delta x$	$a + 3\Delta x$	$\cdots$	$a + i\Delta x$	$\cdots$	$a + (n-1)\Delta x$	b

We call the collection of points $\mathcal{P}_n = \{x_0, x_1, \ldots, x_n\}$ a set of *partition points* of $[a, b]$. Note that to divide an interval into n subintervals, we need $n + 1$ partition points.

EXAMPLE 1: Find the step size Δx and the partition points needed to divide the given interval into the given number of subintervals.

Example: $[0,1]$; 6 $\Rightarrow$ $\Delta x = \frac{1}{6}$, $\quad \mathcal{P}_6 = \left\{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1\right\}$.

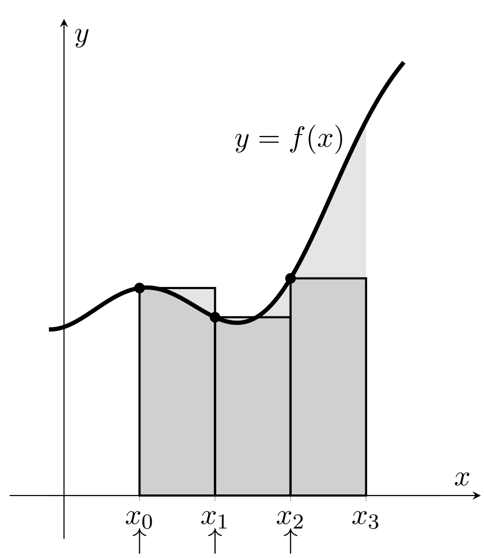
- $[0,1]$; 7 Answer: $\Delta x = \frac{1}{7}$, $\quad \mathcal{P}_7 = \{0, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, 1\}$
- $[2,5]$; 6 Answer: $\Delta x = \frac{1}{2}$, $\quad \mathcal{P}_6 = \{2, \frac{5}{2}, 3, \frac{7}{2}, 4\frac{9}{2}, 5\}$
- $[-3,4]$; 4 Answer: $\Delta x = \frac{7}{4}$, $\quad \mathcal{P}_4 = \{-3, -\frac{5}{4}, \frac{1}{2}, \frac{9}{4}, 4\}$
- $[-5,-1]$; 5 Answer: $\Delta x = \frac{4}{5}$, $\quad \mathcal{P}_5 = \{-5, -\frac{21}{5}, -\frac{17}{5}, -\frac{13}{5}, -\frac{9}{5}, -1\}$
- $[-10,-3]$; 8 Answer: $\Delta x = \frac{7}{8}$, $\quad \mathcal{P}_8 = \{-10, -\frac{73}{8}, -\frac{33}{4}, -\frac{59}{8}, -\frac{13}{2}, -\frac{45}{8}, -\frac{19}{4}, -\frac{31}{8}, -3\}$

Assume that the interval $[a,b]$ is already divided into n subintervals. We then cover the region with rectangles whose bases correspond to a subinterval. The three types of Riemann sum depend on the heights of the rectangles we are covering the region with.

LEFT RIEMANN SUM

The nth left Riemann sum L_n is the sum of the areas of the rectangles whose heights are the functional values of the left endpoints of each subinterval.

For example, we consider the following illustration. We subdivide the interval into three subintervals corresponding to three rectangles. Since we are considering left endpoints, the height of the first rectangle is $f(x_0)$, the height of the second rectangle is $f(x_1)$, and the height of the third rectangle is $f(x_2)$.



Therefore, in this example, the 3rd left Riemann sum equals

$$L_3 = f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1) + f(x_2)(x_3 - x_2) = (f(x_0) + f(x_1) + f(x_2))\,\Delta x.$$

In general, if $[a, b]$ is subdivided into n intervals with partition points $\{x_0, x_1, \ldots, x_n\}$, then the nth left Riemann sum equals

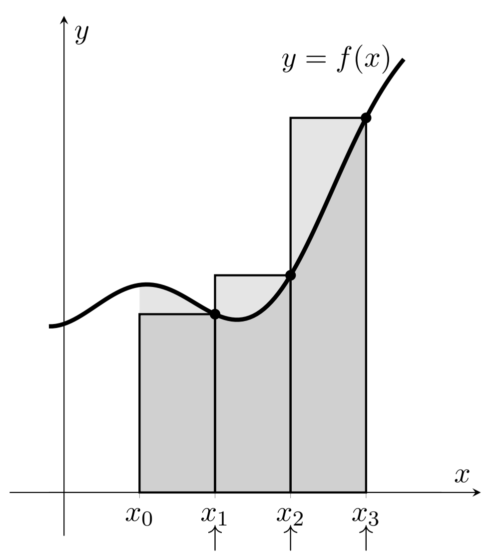
$$L_n = (f(x_0) + f(x_1) + \ldots + f(x_{n-1}))\,\Delta x = \sum_{k=1}^{n} f(x_{k-1})\Delta x.$$

We define the right and midpoint Riemann sums in a similar manner.

RIGHT RIEMANN SUM

The nth right Riemann sum R_n is the sum of the areas of the rectangles whose heights are the functional values of the right endpoints of each subinterval.

For example, we consider the following illustration. We subdivide the interval into three subintervals corresponding to three rectangles. Since we are considering right endpoints, the height of the first rectangle is $f(x_1)$, the height of the second rectangle is $f(x_2)$, and the height of the third rectangle is $f(x_3)$.



Therefore, in this example, the 3rd right Riemann sum equals

$$R_3 = f(x_1)(x_1 - x_0) + f(x_2)(x_2 - x_1) + f(x_3)(x_3 - x_2) = (f(x_1) + f(x_2) + f(x_3))\,\Delta x.$$

In general, if $[a, b]$ is subdivided into n intervals with partition points $\{x_0, x_1, \ldots, x_n\}$, then the nth right Riemann sum equals

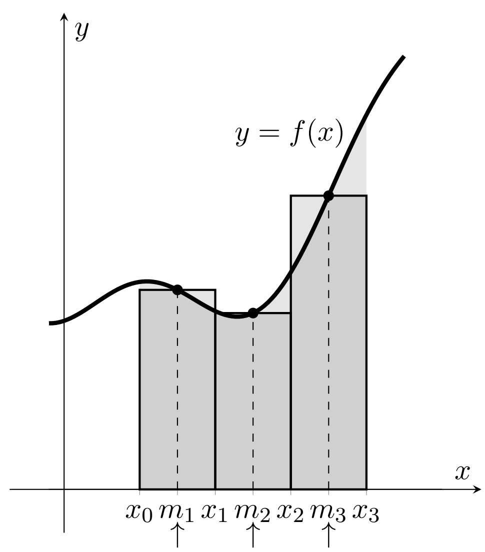
$$R_n = (f(x_1) + f(x_2) + \ldots + f(x_n))\,\Delta x = \sum_{k=1}^{n} f(x_k)\Delta x.$$

MIDPOINT RIEMANN SUM

The nth midpoint Riemann sum M_n is the sum of the areas of the rectangles whose heights are the functional values of the midpoints of the endpoints of each subinterval. For the sake of notation, we denote by m_k the midpoint of two consecutive partition points x_{k-1} and x_k; that is,

$$m_k = \frac{x_{k-1} + x_k}{2}.$$

We now consider the following illustration. We subdivide the interval into three subintervals corresponding to three rectangles. Since we are considering midpoints of the endpoints, the height of the first rectangle is $f(m_1)$, the height of the second rectangle is $f(m_2)$, and the height of the third rectangle is $f(m_3)$.



Therefore, in this example, the 3rd midpoint Riemann sum equals

$$M_3 = f(m_1)(x_1 - x_0) + f(m_2)(x_2 - x_1) + f(m_3)(x_3 - x_2) = (f(m_1) + f(m_2) + f(m_3))\,\Delta x.$$

In general, if $[a, b]$ is subdivided into n intervals with partition points $\{x_0, x_1, \ldots, x_n\}$, then the nth midpoint Riemann sum equals

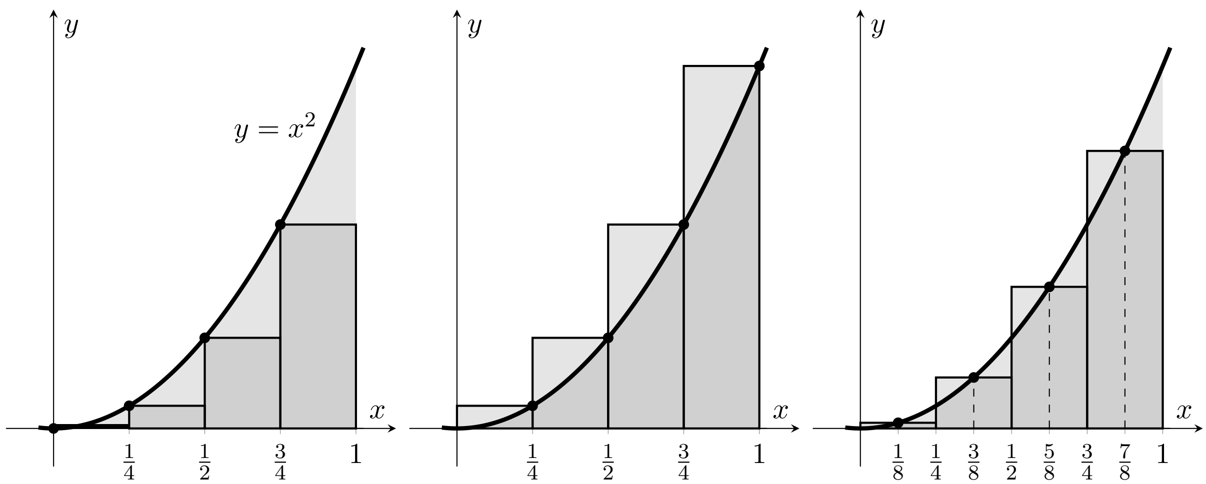
$$L_n = (f(m_1) + f(m_2) + \ldots + f(m_n))\,\Delta x = \sum_{k=1}^{n} f(m_k)\Delta x.$$

EXAMPLE 2: Find the 4th left, right, and midpoint Riemann sums of the following functions with respect to a regular partitioning of the given intervals.

1. $f(x) = x^2$ on $[0, 1]$
2. $f(x) = \sin x$ on $[0, \pi]$.

Solution.

1. First, note that $\Delta x = \frac{1-0}{4} = \frac{1}{4}$. Hence, $\mathcal{P}_4 = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$. We then compute the midpoints of the partition points: $\left\{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\right\}$.



The 4th left Riemann sum equals

$$\begin{aligned} L_4 &= \left(f(0) + f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right)\right) \cdot \Delta x \\ &= \left(0 + \frac{1}{16} + \frac{1}{4} + \frac{9}{16}\right) \cdot \frac{1}{4} = 0.21875. \end{aligned}$$

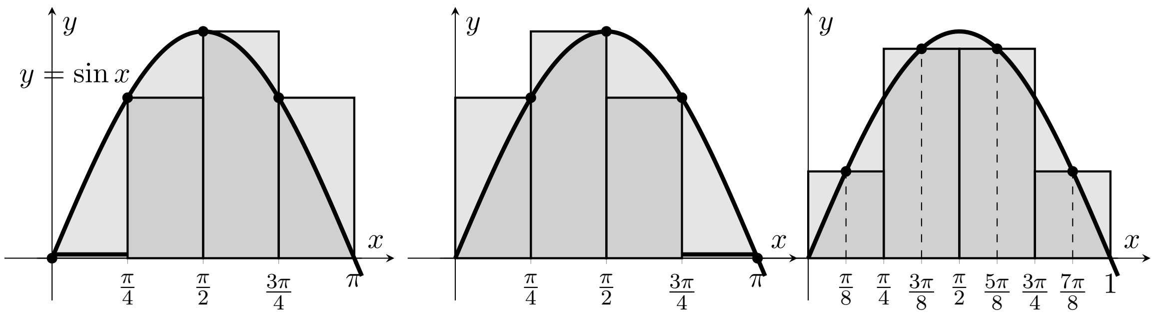
The 4th right Riemann sum equals

$$\begin{aligned} R_4 &= \left(f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right) + f(1)\right) \cdot \Delta x \\ &= \left(\frac{1}{16} + \frac{1}{4} + \frac{9}{16} + 1\right) \cdot \frac{1}{4} = 0.46875. \end{aligned}$$

Lastly, the 4th midpoint Riemann sum equals

$$\begin{aligned} M_4 &= \left(f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right) \cdot \Delta x \\ &= \left(\frac{1}{64} + \frac{9}{64} + \frac{25}{64} + \frac{49}{64}\right) \cdot \frac{1}{4} = 0.328125. \end{aligned}$$

2. First, note that $\Delta x = \dfrac{\pi - 0}{4} = \dfrac{\pi}{4}$. Hence, $\mathcal{P}_4 = \left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, 1\right\}$. We then compute the midpoints of the partition points: $\left\{\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}\right\}$.



The 4th left Riemann sum equals

$$\begin{aligned} L_4 &= \left(f(0) + f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{2}\right) + f\left(\frac{3\pi}{4}\right)\right) \cdot \Delta x \\ &= \left(\sin 0 + \sin\frac{\pi}{4} + \sin\frac{\pi}{2} + \sin\frac{3\pi}{4}\right) \cdot \frac{\pi}{4} = 1.896... \end{aligned}$$

The 4th right Riemann sum equals

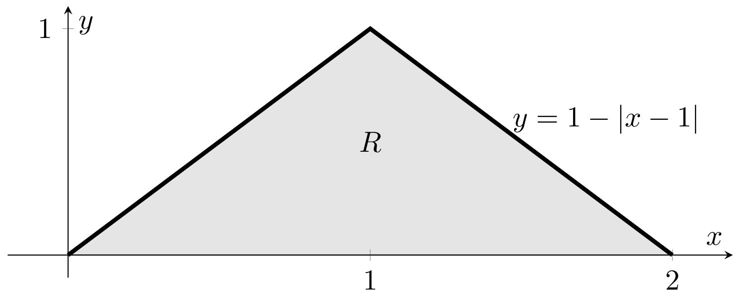
$$\begin{aligned} R_4 &= \left(f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{2}\right) + f\left(\frac{3\pi}{4}\right) + f(\pi)\right) \cdot \Delta x \\ &= \left(\sin\frac{\pi}{4} + \sin\frac{\pi}{2} + \sin\frac{3\pi}{4} + \sin\pi\right) \cdot \frac{\pi}{4} = 1.896... \end{aligned}$$

Finally, the 4th midpoint Riemann sum equals

$$\begin{aligned} M_4 &= \left(f\left(\frac{\pi}{8}\right) + f\left(\frac{3\pi}{8}\right) + f\left(\frac{5\pi}{8}\right) + f\left(\frac{7\pi}{8}\right)\right) \cdot \Delta x \\ &= \left(\sin\frac{\pi}{8} + \sin\frac{3\pi}{8} + \sin\frac{5\pi}{8} + \sin\frac{7\pi}{8}\right) \cdot \frac{\pi}{4} = 2.052... \end{aligned}$$

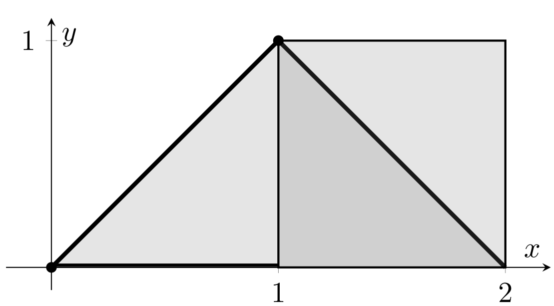
The following example shows that arbitrarily increasing the number of partition points does not necessarily give a better approximation of the true area of the region.

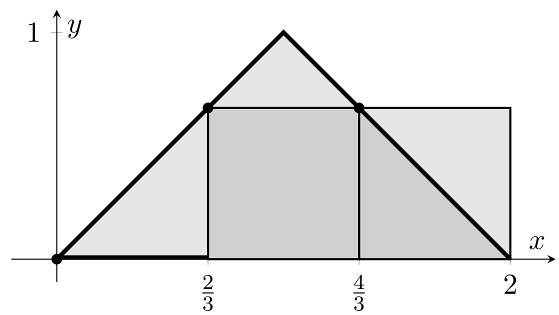
EXAMPLE 3: Let $f(x) = 1 - |x - 1|$ and consider the closed region R bounded by $y = f(x)$ and the x-axis on the interval $[0, 2]$.



Show that relative to regular partitioning, the second left Riemann sum L_2 is a better approximation of the area of R than the third left Riemann sum L_3 of f on $[0, 2]$.

Solution. First, observe that the exact area of R is $\frac{1}{2}(1)(2) = 1$.





Now, the step size and partition points, repectively, are $\Delta x = 1$ and $\mathcal{P}_2 = \{0, 1, 2\}$ for L_2 and $\Delta x = \frac{2}{3}$ and $\mathcal{P}_3 = \{0, \frac{2}{3}, \frac{4}{3}, 2\}$ for L_3. Using the formula for the left Riemann sum, we have the following computations:

$$L_2 = (f(0) + f(1)) \cdot \Delta x = (0 + 1) \cdot 1 = 1$$

while

$$L_3 = \left(f(0) + f\left(\frac{2}{3}\right) + f\left(\frac{4}{3}\right)\right) \cdot \Delta x = \left(0 + \frac{2}{3} + \frac{2}{3}\right) \cdot \frac{2}{3} = \frac{4}{3} \cdot \frac{2}{3} = \frac{8}{9}.$$

Clearly, L_2 is closer (and in fact, equal) to the exact value of 1, than L_3.

REFINEMENT

To deal with the question above, we define the concept of a *refinement* of a partition:

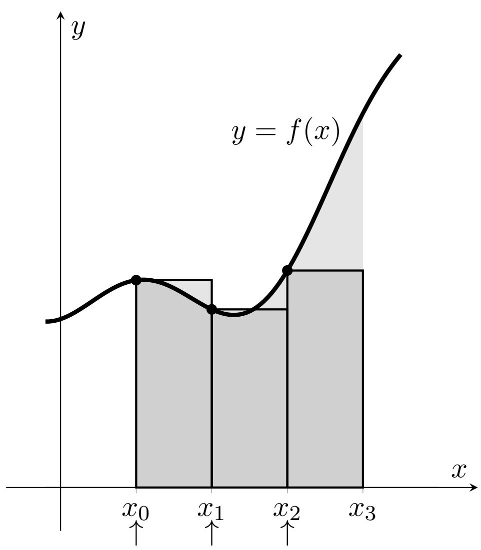
Definition 6. *A partition $\mathcal{Q}$ of an interval I is a refinement of another partition $\mathcal{P}$ of I if $\mathcal{P} \subseteq \mathcal{Q}$, meaning, $\mathcal{Q}$ contains all partition points of $\mathcal{P}$ and more.*

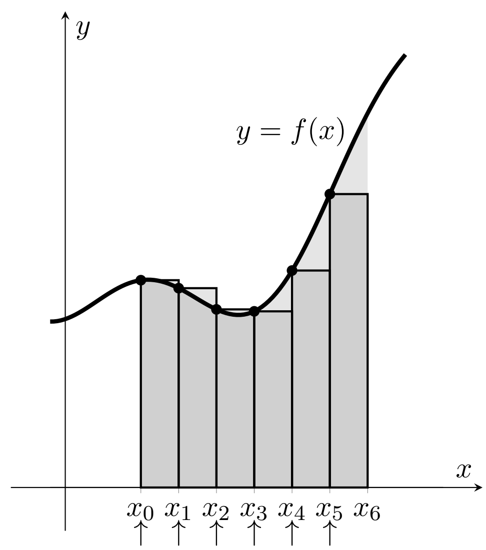
EXAMPLE 4: For the interval $I = [0, 2]$, $\mathcal{P}_3 = \{0, \frac{2}{3}, \frac{4}{3}, 2\}$ is not a refinement of $\mathcal{P}_2 = \{0, 1, 2\}$. However, $\mathcal{P}_4 = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$ is a refinement of $\mathcal{P}_2$ while $\mathcal{P}_6 = \{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2\}$ is a refinement of $\mathcal{P}_3$.

EXAMPLE 5: $\mathcal{P}_{2n}$ is always a refinement of $\mathcal{P}_n$. In fact $\mathcal{P}_{2n}$ contains all partition points of $\mathcal{P}_n$ and all the midpoints therein.

Theorem 15. *Suppose $\mathcal{Q}$ is a refinement of $\mathcal{P}$. Then, any (left, midpoint, right) Riemann sum approximation of a region R relative to $\mathcal{Q}$ is equal to or is better than the same kind of Riemann sum approximation relative to $\mathcal{P}$.*

It is not hard to convince ourselves about the validity of the above theorem. Consider the diagrams below. The one on the left illustrates L_3 while the right one illustrates L_6 relative to the regular partition of the given interval.





Remark 1: The theorem also conforms to the procedure in the classical method of exhaustion wherein they use the areas of inscribed regular n-gons to approximate the area of a circle. The sequence of n-gons the Greeks considered was such that the next n-gon would have twice the number of sides as the previous one.

Remark 2: A consequence of the theorem is that for any positive integer n, the sequence

$$L_n, L_{2n}, L_{4n}, L_{8n}, L_{16n}, \ldots$$

is a monotone sequence converging to the exact area of the region. This means that if A is the exact area of the region and $L_n \leq A$, then

$$L_n \leq L_{2n} \leq L_{4n} \leq L_{8n} \leq L_{16n} \leq \ldots \leq A.$$

For instance, in Example 3, $A = 1$ and $L_3 = \frac{8}{9}$. The above inequalities imply that $L_3 \leq L_6 \leq A$. Indeed,

$$\begin{aligned} L_6 &= \left(f(0) + f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right) + f(1) + f\left(\frac{4}{3}\right) + f\left(\frac{5}{3}\right) \right) \cdot \Delta x \\ &= \left(0 + \frac{1}{3} + \frac{2}{3} + 1 + \frac{2}{3} + \frac{1}{3} \right) \cdot \frac{1}{3} = 1. \end{aligned}$$

IRREGULAR PARTITION

Sometimes, the partition of an interval is irregular, that is, the lengths of the subintervals are not equal. This kind of partitioning is usually used when you want to obtain a refinement of a partition (and thereby get a better approximation) without computing for a lot more points.

For example, suppose that you think that a Riemann sum relative to the partition $\mathcal{P} = \{0, \frac{2}{3}, \frac{4}{3}, 2\}$ is already close to the exact value, then you can simply insert one more point, say 1, to get the partition $\mathcal{P}' = \{0, \frac{2}{3}, 1, \frac{4}{3}, 2\}$. Since $\mathcal{P}'$ is a refinement of $\mathcal{P}$, then a Riemann sum relative to it should have a better value than that of $\mathcal{P}$ and you just have to compute for one more functional value, $f(1)$, rather than 3 more values.

To get Riemann sums relative to irregular partitions, the idea is the same, you just have to be careful about the variable step sizes.

Consider an irregular partition $\mathcal{P} = \{x_0, x_1, x_2, x_3, \ldots, x_n\}$ of an interval. In general it may not be the case that $x_1 - x_0 = x_2 - x_1$. So, we define the step sizes Δx_k.

For each $k \in \{1, 2, \ldots, n\}$, define the kth step size Δx_k to be the length of the kth subinterval, i.e.

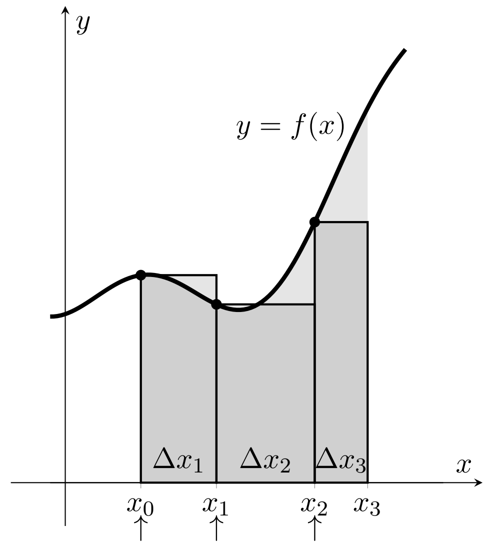
$$\Delta x_k = x_k - x_{k-1}.$$

With this notation, the left Riemann sum with respect to the partition $\mathcal{P}$ is

$$\begin{aligned} L_{\mathcal{P}} &= f(x_0)\Delta x_1 + f(x_1)\Delta x_2 + \ldots + f(x_{n-2})\Delta x_{n-1} + f(x_{n-1})\Delta x_n \\ &= \sum_{k=1}^{n} f(x_{k-1})\Delta x_k. \end{aligned}$$

Now, suppose we are given $y = f(x)$ and a partition $\mathcal{P} = \{x_0, x_1, x_2, x_3\}$. Then the left Riemann sum with respect to the partition $\mathcal{P}$ is

$$L_{\mathcal{P}} = f(x_0)\Delta x_1 + f(x_1)\Delta x_2 + f(x_2)\Delta x_3.$$



Very similarly, the right Riemann sum is given by

$$\begin{aligned} R_{\mathcal{P}} &= f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \ldots + f(x_{n-1})\Delta x_{n-1} + f(x_n)\Delta x_n \\ &= \sum_{k=1}^{n} f(x_k)\Delta x_k, \end{aligned}$$

and the midpoint Riemann sum is given by

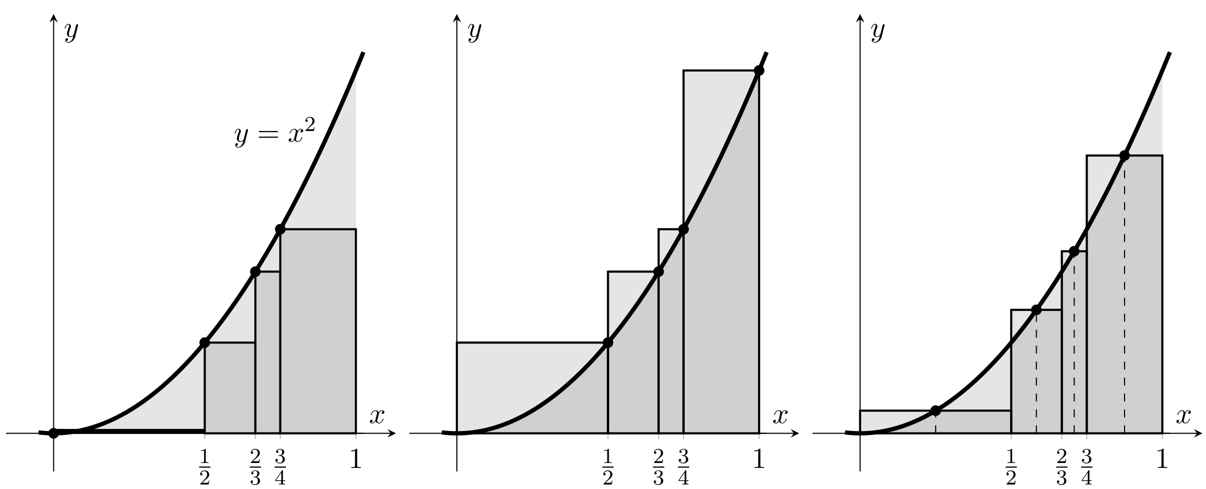
$$\begin{aligned} M_{\mathcal{P}} &= f(m_1)\Delta x_1 + f(m_2)\Delta x_2 + \ldots + f(m_{n-1})\Delta x_{n-1} + f(m_n)\Delta x_n \\ &= \sum_{k=1}^{n} f(m_k)\Delta x_k, \end{aligned}$$

with the same convention that $m_k = \dfrac{x_{k-1} + x_k}{2}$, the midpoint of the kth interval.

EXAMPLE 6: Relative to the partition $\mathcal{P} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1\}$, find the left, right, and midpoint Riemann sums of $f(x) = x^2$ on the interval $[0, 1]$.

Solution. Observe that $\mathcal{P}$ partitions $[0, 1]$ into 4 irregular subintervals: $[0, \frac{1}{2}]$, $[\frac{1}{2}, \frac{2}{3}]$, $[\frac{2}{3}, \frac{3}{4}]$, $[\frac{3}{4}, 1]$. The step sizes are $\Delta x_1 = \frac{1}{2}$, $\Delta x_2 = \frac{1}{6}$, $\Delta x_3 = \frac{1}{12}$, $\Delta x_4 = \frac{1}{4}$. This implies that the Riemann sums are

$$
\begin{aligned}
L_{\mathcal{P}} &= f(0)\Delta x_1 + f\left(\frac{1}{2}\right)\Delta x_2 + f\left(\frac{2}{3}\right)\Delta x_3 + f\left(\frac{3}{4}\right)\Delta x_4 \\
&= 0\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{6} + \frac{4}{9}\cdot\frac{1}{12} + \frac{9}{16}\cdot\frac{1}{4} = 0.2193. \\
R_{\mathcal{P}} &= f\left(\frac{1}{2}\right)\Delta x_1 + f\left(\frac{2}{3}\right)\Delta x_2 + f\left(\frac{3}{4}\right)\Delta x_3 + f\left(1\right)\Delta x_4 \\
&= \frac{1}{4}\cdot\frac{1}{2} + \frac{4}{9}\cdot\frac{1}{6} + \frac{9}{16}\cdot\frac{1}{12} + 1\cdot\frac{1}{4} = 0.4959. \\
M_{\mathcal{P}} &= f\left(\frac{1}{4}\right)\Delta x_1 + f\left(\frac{7}{12}\right)\Delta x_2 + f\left(\frac{17}{24}\right)\Delta x_3 + f\left(\frac{7}{8}\right)\Delta x_4 \\
&= \frac{1}{16}\cdot\frac{1}{2} + \frac{49}{144}\cdot\frac{1}{6} + \frac{289}{576}\cdot\frac{1}{12} + \frac{49}{64}\cdot\frac{1}{4} = 0.3212.
\end{aligned}
$$



Solved Examples

EXAMPLE 1: Find the step size Δx and the partitioning needed to divide the given interval into given number of regular subintervals.

a. $[0, 2]; 8$ b. $[0, \pi] : 4$ c. $[-1, 4] : 5$

Solution.

a. First, we solve for Δx. For $a = 0$, $b = 2$, and $n = 8$,

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{8} = \frac{2}{8} = \frac{1}{4}.$$

Next, we compute for the partition $\mathcal{P}_6$:

$$\mathcal{P}_6 = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2\right\}.$$

b. Similar to (a), we compute for the values of Δx and $\mathcal{P}_4$. For Δx we have,

$$\Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4},$$

and for the partition, we have

$$\mathcal{P}_4 = \left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\right\}.$$

c. Note that, $\Delta x = 1$. Hence, $\mathcal{P}_5 = \{-1, 0, 1, 2, 3, 4\}$.

EXAMPLE 2: Find the 5th left, right, and midpoint Riemann sums of the following given functions with respect to the regular partitioning of the given intervals.

a. $f(x) = 2x$ on $[0, 1]$

b. $g(x) = (x + 1)^2$ on $[-1, 4]$

Solution.

a. First, note that $\Delta x = \frac{1-0}{5} = \frac{1}{5}$ and $\mathcal{P}_5 = \left\{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\right\}$. Also, we compute for the midpoints of the partition points: $\left\{\frac{1}{10}, \frac{3}{10}, \frac{1}{2}, \frac{7}{10}, \frac{9}{10}\right\}$.

The 5th left Riemann sum is

$$\begin{aligned} L_5 &= \left[f(0) + f\left(\frac{1}{5}\right) + f\left(\frac{2}{5}\right) + f\left(\frac{3}{5}\right) + f\left(\frac{4}{5}\right)\right] \cdot \Delta x \\ &= \left(0 + \frac{2}{5} + \frac{4}{5} + \frac{6}{5} + \frac{8}{5}\right) \cdot \frac{1}{5} = \frac{4}{5} = 0.80. \end{aligned}$$

Next, the 5th right Riemann sum is

$$\begin{aligned} R_5 &= \left[f\left(\frac{1}{5}\right) + f\left(\frac{2}{5}\right) + f\left(\frac{3}{5}\right) + f\left(\frac{4}{5}\right) + f(1)\right] \cdot \Delta x \\ &= \left(\frac{2}{5} + \frac{4}{5} + \frac{6}{5} + \frac{8}{5} + 2\right) \cdot \frac{1}{5} = 1.20. \end{aligned}$$

Lastly, the 5th midpoint Riemann sum is

$$\begin{aligned} M_5 &= \left[f\left(\frac{1}{10}\right) + f\left(\frac{3}{10}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{7}{10}\right) + f\left(\frac{9}{10}\right)\right] \cdot \Delta x \\ &= \left(\frac{2}{10} + \frac{3}{5} + 1 + \frac{7}{5} + \frac{9}{5}\right) \cdot \frac{1}{5} = 1. \end{aligned}$$

b. Note that $\Delta x = \dfrac{4-(-1)}{5} = \dfrac{5}{5} = 1$. Then, $\mathcal{P}_5 = \{-1, 0, 1, 2, 3, 4\}$. Furthermore, the midpoints of the partition points are $\left\{-\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{3}{2}, \dfrac{5}{2}, \dfrac{7}{2}\right\}$.

The 5th left Riemann sum is

$$\begin{aligned} L_5 &= [g(-1)+g(0)+g(1)+g(2)+g(3)] \cdot \Delta x \\ &= (0+1+4+9+16) \cdot 1 = 30. \end{aligned}$$

The 5th right Riemann sum is

$$\begin{aligned} R_5 &= [g(0)+g(1)+g(2)+g(3)+g(4)] \cdot \Delta x \\ &= (1+4+9+16+25) \cdot 1 = 55. \end{aligned}$$

Finally, the 5th midpoint Riemann sum is

$$\begin{aligned} M_5 &= \left[g\left(-\frac{1}{2}\right)+g\left(\frac{1}{2}\right)+g\left(\frac{3}{2}\right)+g\left(\frac{5}{2}\right)+g\left(\frac{7}{2}\right)\right] \cdot \Delta x \\ &= \left(\frac{1}{4}+\frac{9}{4}+\frac{25}{4}+\frac{49}{4}+\frac{81}{4}\right) \cdot 1 = \frac{165}{4} = 41.25. \end{aligned}$$

EXAMPLE 3: Find the 6th left, right, and midpoint Riemann sums of the following given functions with respect to the regular partitioning of the given intervals.

a. $h(x) = \cos x$ on $[0, \frac{\pi}{2}]$

b. $k(x) = \dfrac{1}{x}$ on $[1, 4]$

Solution.

a. Computing for $\Delta x = \dfrac{\frac{\pi}{2}-0}{6} = \dfrac{\pi}{12}$. Hence, $\mathcal{P}_6 = \left\{0, \dfrac{\pi}{12}, \dfrac{\pi}{6}, \dfrac{\pi}{4}, \dfrac{\pi}{3}, \dfrac{5\pi}{12}, \dfrac{\pi}{2}\right\}$. We further compute for the midpoints of the the partition points:

$$\left\{\frac{\pi}{24}, \frac{\pi}{8}, \frac{5\pi}{24}, \frac{7\pi}{24}, \frac{3\pi}{8}, \frac{11\pi}{24}\right\}.$$

The $6th$ left Riemann sum is

$$\begin{aligned} L_6 &= \left[h(0)+h\left(\frac{\pi}{12}\right)+h\left(\frac{\pi}{6}\right)+h\left(\frac{\pi}{4}\right)+h\left(\frac{\pi}{3}\right)+h\left(\frac{5\pi}{12}\right)\right] \cdot \Delta x \\ &= \left[\cos 0+\cos\left(\frac{\pi}{12}\right)+\cos\left(\frac{\pi}{6}\right)+\cos\left(\frac{\pi}{4}\right)+\cos\left(\frac{\pi}{3}\right)+\cos\left(\frac{5\pi}{12}\right)\right] \cdot \frac{\pi}{12} \\ &= 1.125... \end{aligned}$$

The $6th$ right Riemann sum is

$$\begin{aligned}
R_6 &= \left[h\left(\frac{\pi}{12}\right)+h\left(\frac{\pi}{6}\right)+h\left(\frac{\pi}{4}\right)+h\left(\frac{\pi}{3}\right)+h\left(\frac{5\pi}{12}\right)+h\left(\frac{\pi}{2}\right)\right]\cdot\Delta x\\
&= \left[\cos\left(\frac{\pi}{12}\right)+\cos\left(\frac{\pi}{6}\right)+\cos\left(\frac{\pi}{4}\right)+\cos\left(\frac{\pi}{3}\right)+\cos\left(\frac{5\pi}{12}\right)+\cos\left(\frac{\pi}{2}\right)\right]\cdot\frac{\pi}{12}\\
&= 0.863...
\end{aligned}$$

Lastly, the 6th midpoint Riemann sum is

$$\begin{aligned}
M_6 &= \left[h\left(\frac{\pi}{24}\right)+h\left(\frac{\pi}{8}\right)+h\left(\frac{5\pi}{24}\right)+h\left(\frac{7\pi}{24}\right)+h\left(\frac{3\pi}{8}\right)+h\left(\frac{11\pi}{24}\right)\right]\cdot\Delta x\\
&= \left[\cos\left(\frac{\pi}{24}\right)+\cos\left(\frac{\pi}{8}\right)+\cos\left(\frac{5\pi}{24}\right)+\cos\left(\frac{7\pi}{24}\right)+\cos\left(\frac{3\pi}{8}\right)+\cos\left(\frac{11\pi}{24}\right)\right]\cdot\frac{\pi}{12}\\
&= 1.002...
\end{aligned}$$

b. Note that $\Delta x = \dfrac{4-1}{6} = \dfrac{1}{2}$. Hence, $\mathcal{P}_6 = \left\{1, \dfrac{3}{2}, 2, \dfrac{5}{2}, 3, \dfrac{7}{2}, 4\right\}$ and the midpoints of this partition are $\left\{\dfrac{5}{4}, \dfrac{7}{4}, \dfrac{9}{4}, \dfrac{11}{4}, \dfrac{13}{4}, \dfrac{15}{4}\right\}$.

The 6th left Riemann sum is

$$\begin{aligned}
L_6 &= \left[k(1)+k\left(\frac{3}{2}\right)+k(2)+k\left(\frac{5}{2}\right)+k(3)+k\left(\frac{7}{2}\right)\right]\cdot\Delta x\\
&= \left(1+\frac{2}{3}+\frac{1}{2}+\frac{2}{5}+\frac{1}{3}+\frac{2}{7}\right)\cdot\frac{1}{2} = \frac{223}{140} = 1.592...
\end{aligned}$$

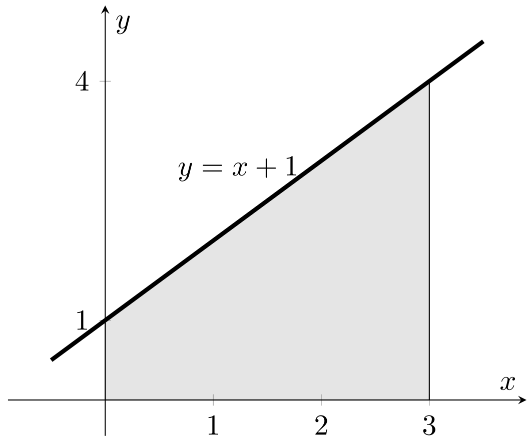
The 6th right Riemann sum is

$$\begin{aligned}
R_6 &= \left[k\left(\frac{3}{2}\right)+k(2)+k\left(\frac{5}{2}\right)+k(3)+k\left(\frac{7}{2}\right)+k(4)\right]\cdot\Delta x\\
&= \left(\frac{2}{3}+\frac{1}{2}+\frac{2}{5}+\frac{1}{3}+\frac{2}{7}+\frac{1}{4}\right)\cdot\frac{1}{2} = \frac{341}{280} = 1.217...
\end{aligned}$$

The 6th midpoint Riemann sum is

$$\begin{aligned}
M_6 &= \left[k\left(\frac{5}{4}\right)+k\left(\frac{7}{4}\right)+k\left(\frac{9}{4}\right)+k\left(\frac{11}{4}\right)+k\left(\frac{13}{4}\right)+k\left(\frac{15}{4}\right)\right]\cdot\Delta x\\
&= \left(\frac{4}{5}+\frac{4}{7}+\frac{4}{9}+\frac{4}{11}+\frac{4}{13}+\frac{4}{15}\right)\cdot\frac{1}{2} = 1.376...
\end{aligned}$$

EXAMPLE 4: Let $f(x) = x + 1$ and consider the closed region bounded by $y = f(x)$ and the x-axis on the interval $[0, 3]$.



Show that relative to the regular partitioning, the third right Riemann sum R_3 is a better approximation to the area of the closed region than the second right Riemann sum R_2 of f on $[0, 3]$.

Solution. The closed region is a trapezoid with bases 1 and 4 and height 3. Therefore, the area is equal to $\left(\dfrac{1+4}{2}\right) \cdot 3 = \dfrac{15}{2} = 7.5$.

The step size and the partition are $\Delta x = 1.5$ and $\mathcal{P}_2 = \{0, 1.5, 3\}$, respectively, for R_2 and $\Delta x = 1$ and $\mathcal{P}_3 = \{0, 1, 2, 3\}$, respectively, for R_3 . We can now compute for the values of R_2 and R_3.

$$R_2 = (f(1.5) + f(3)) \cdot \Delta x = (2.5 + 4) \cdot 1.5 = 9.75,$$

while

$$R_3 = (f(1) + f(2) + f(3)) \cdot \Delta x = (2 + 3 + 4) \cdot 1 = 9.$$

Comparing the values of R_2 and R_3 to the original area of the trapezoid, R_3 is indeed a better approximation than R_2.

EXAMPLE 5: Let $\mathcal{P}_1 = \{0, 1, 2, 3\}$, $\mathcal{P}_2 = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{3}{2}, 2, 3\}$ and $\mathcal{P}_3 = \{0.8, 1, 2, 3\}$ be partitions on interval $[0, 3]$. Explain why $\mathcal{P}_2$ is a refinement of $\mathcal{P}_1$ but not $\mathcal{P}_3$.

Solution. $\mathcal{P}_2$ is a refinement of $\mathcal{P}_1$ because $\mathcal{P}_1 \subseteq \mathcal{P}_2$. On the other hand,$\mathcal{P}_3$ is not a refinement of $\mathcal{P}_1$ since $\mathcal{P}_3 \nsubseteq \mathcal{P}_1$.

EXAMPLE 6: Given the irregular partition $\mathcal{P} = \{0, \frac{3}{4}, 1, \frac{3}{2}, 2\}$, find the left, right, and midpoint Riemann sums of $f(x) = \sqrt{x}$ on the interval $[0, 2]$.

Solution. Note that $\mathcal{P}$ partitions $[0,2]$ into 4 subintervals: $[0,\frac{3}{4}]$, $[\frac{3}{4},1]$, $[1,\frac{3}{2}]$, and $[\frac{3}{2},2]$. The step sizes of each subare $\Delta x_1 = \frac{3}{4}$, $\Delta x_2 = \frac{1}{4}$, $\Delta x_3 = \frac{1}{2}$, and $\Delta x_4 = \frac{1}{2}$. Therefore, the Riemann sums relative to the partition $\mathcal{P}$ are given:

$$\begin{aligned} L_{\mathcal{P}} &= f(0)\Delta x_1 + f\left(\frac{3}{4}\right)\Delta x_2 + f(1)\Delta x_3 + f\left(\frac{3}{2}\right)\Delta x_4 \\ &= 0 \cdot \frac{3}{4} + \sqrt{\frac{3}{4}} \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + \sqrt{\frac{3}{2}} \cdot \frac{1}{2} = 2.078... \end{aligned}$$

$$\begin{aligned} R_{\mathcal{P}} &= f\left(\frac{3}{4}\right)\Delta x_1 + f(1)\Delta x_2 + f\left(\frac{3}{2}\right)\Delta x_3 + f(2)\Delta x_4 \\ &= \sqrt{\frac{3}{4}} \cdot \frac{3}{4} + 1 \cdot \frac{1}{4} + \sqrt{\frac{3}{2}} \cdot \frac{1}{2} + \sqrt{2} \cdot \frac{1}{2} = 2.218... \end{aligned}$$

$$\begin{aligned} M_{\mathcal{P}} &= f\left(\frac{3}{8}\right)\Delta x_1 + f\left(\frac{7}{8}\right)\Delta x_2 + f\left(\frac{5}{4}\right)\Delta x_3 + f\left(\frac{7}{4}\right)\Delta x_4 \\ &= \sqrt{\frac{3}{8}} \cdot \frac{3}{4} + \sqrt{\frac{7}{8}} \cdot \frac{1}{4} + \sqrt{\frac{5}{4}} \cdot \frac{1}{2} + \sqrt{\frac{7}{4}} \cdot \frac{1}{2} = 1.913... \end{aligned}$$

Supplementary Problems

1. Let $f(x) = 3x$ be defined on $[0,2]$. Find the left Riemann sum relative to the regular partition $\mathcal{P}_3$.

2. Let $f(x) = \sqrt{4-x^2}$ be defined on $[-1,1]$. Find the right Riemann sum relative to the regular partition $\mathcal{P}_4$.

3. Let $f(x) = 3x+1$ be defined on $[1,3]$. Find the right Riemann sum relative to the regular partition $\mathcal{P}_3$.

4. Let $f(x) = x^3+2$ be defined on $[-1,1]$. Find the left Riemann sum relative to the regular partition $\mathcal{P}_2$.

5. Let $f(x) = x^2+2x+4$ be defined on $[0,1]$. Find the right Riemann sum relative to the regular partition $\mathcal{P}_3$.

6. Let $f(x) = 1+\sqrt{x+1}$ be defined on $[-1,1]$. Find the midpoint Riemann sum relative to the regular partition $\mathcal{P}_4$.

7. Let $f(x) = \dfrac{1}{x+1}$ be defined on $[1, 3]$. Find the left Riemann sum relative to the regular partition $\mathcal{P}_4$.

8. Let $f(x) = \sin(2x)$ be defined on $[0, \pi]$. Find the right Riemann sum relative to the regular partition $\mathcal{P}_5$.

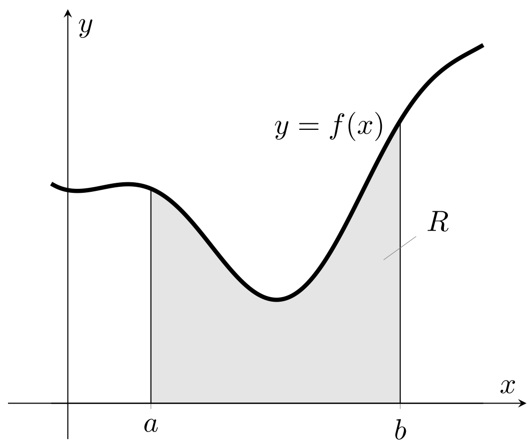
9. Let $f(x) = 2\tan(x)$ be defined on $[0, \frac{\pi}{4}]$. Find the midpoint Riemann sum relative to the regular partition $\mathcal{P}_3$.

10. Let $f(x) = x^2 - 4$ be defined on $[0, 5]$. Find the left Riemann sum relative to the partition $\mathcal{P} = \left\{0, \frac{1}{4}, \frac{1}{2}, 1, \frac{4}{3}, 2\right\}$

TOPIC 15.2: The Formal Definition of the Definite Integral

DEFINITE INTEGRAL

We work with a continuous positive function $y = f(x)$ defined on a closed and bounded interval $[a, b]$. The objective of this lesson is to find the area of the region R bounded by $y = f(x)$ from above, the x-axis from below, the line $x = a$ from the left and $x = b$ from the right.



To avoid complications, we just consider the case where the partition on the interval is regular. We recall that $\mathcal{P}_n = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ (where $x_0 = a$ and $x_k = x_{k-1} + \Delta x$ with $\Delta x = \frac{b-a}{n}$) partitions $[a, b]$ into n congruent subintervals.

For each subinterval $k = 1, 2, \ldots, n$, let x_k^* be any point in the kth subinterval $[x_{k-1}, x_k]$. Then, the Riemann sum, defined by this choice of points, relative to the partition $\mathcal{P}$ is

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_{n-1}^*)\Delta x + f(x_n^*)\Delta x = \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

- If this is a left Riemann sum, then $x_k^* = x_{k-1}$;
- If this is a right Riemann sum, then $x_k^* = x_k$; and finally,
- If this is a midpoint Riemann sum, then $x_k^* = \frac{1}{2}(x_{k-1} + x_k)$.

In any case, we know that the above Riemann sum is only an approximation of the exact area of R. To make this estimate exact, we let n approach infinity. This limit of the Riemann sum is what we call the ***definite integral*** of f over $[a, b]$:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x,$$

if this limit exists. The value of this integral does not depend on the kind (left, right, or midpoint) of Riemann sum being used.

Remember that Δx actually depends on n: $\Delta x = \dfrac{b-a}{n}$. So, this term cannot be taken out from the limit operator. After taking the limit, this Δx becomes our differential dx.

The integral sign $\int$ and the differential dx act as delimiters, which indicate that everything between them is the *integrand* - the upper boundary of the region whose area is what this integral is equal to. The numbers a and b are called the *lower and upper limits of integration*, respectively. Recall that the integral sign $\int$ was earlier used to denote the process of antidifferentiation. There is a reason why the same symbol ($\int$) is being used – we shall see later that antiderivatives are intimately related to finding areas below curves.

Geometric Interpretation of the Definite Integral

Let f be a positive continuous function on $[a, b]$. Then the definite integral

$$\int_a^b f(x)\,dx$$

is the area of the region bounded by $y = f(x)$, the x-axis, $x = a$, and $x = b$. Consequently, the definite integral **does not** depend on the variable x. Changing this variable only changes the name of the x-axis but not the area of the region. Therefore,

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du.$$

COMPUTING DEFINITE INTEGRALS BY APPEALING TO GEOMETRIC FORMULAS

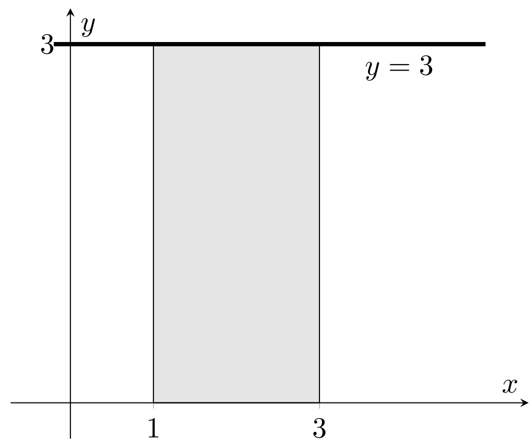
Using the geometric interpretation of the definite integral, we can always think of a definite integral as an area of a region. If we are lucky that the region has an area that is easy to compute using elementary geometry, then we are able to solve the definite integral without resorting to its definition.

EXAMPLE 1: Find the exact values of the following definite integrals:

1. $\displaystyle\int_1^2 3\,dx$
2. $\displaystyle\int_0^2 (1 - |x-1|)\,dx$
3. $\displaystyle\int_1^3 (3x+1)\,dx$
4. $\displaystyle\int_{-1}^1 \sqrt{1-x^2}\,dx$

Solution. Using the above definition of the definite integral, we just draw the region and find its area using elementary geometry.

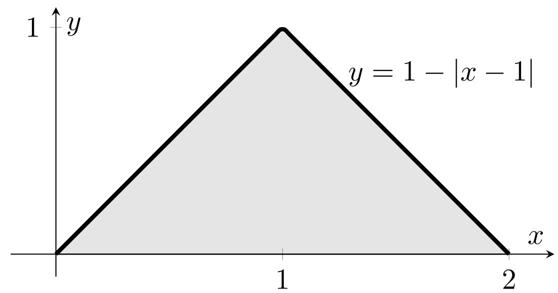
1. The graph of $y = 3$ is a horizontal line.



Since the region is a rectangle, its area equals $L \times W = 2 \times 3 = 6$. Therefore,

$$\int_1^3 3\,dx = 6.$$

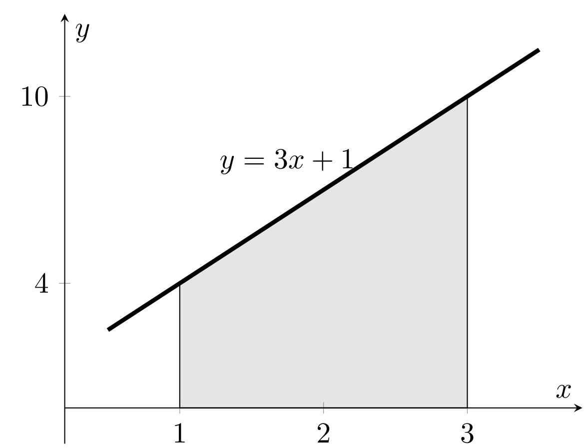
2. The graph of $y = 1 - |x - 1|$ is as given.



The shaded region is a triangle with base $b = 2$ and height $h = 1$. Its area equals $\frac{1}{2}bh = 1$. Therefore,

$$\int_0^2 (1 - |x - 1|)\,dx = 1.$$

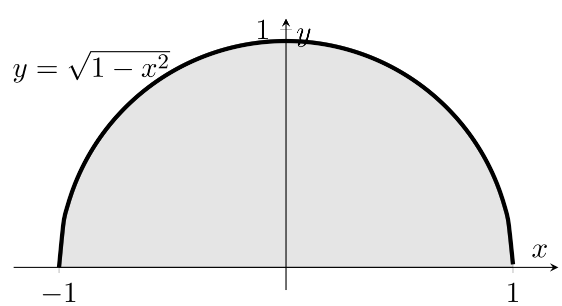
3. The graph of $y = 3x + 1$ is a line slanting to the right.



The shaded region is a trapezoid with bases $b_1 = 4$ and $b_2 = 10$ and height $h = 2$. Its area equals $\frac{1}{2}(b_1 + b_2)h = \frac{1}{2}(4 + 10)2 = 14$. Therefore,

$$\int_1^3 (3x + 1)\,dx = 14.$$

4. The graph of $y = \sqrt{1 - x^2}$ is a semicircle centered at the origin with radius 1.



The area of the shaded region is $\dfrac{1}{2}\pi(1)^2$. Therefore,

$$\int_{-1}^1 \sqrt{1 - x^2}\,dx = \frac{\pi}{2}.$$

Consider the rational function $f(x) = \dfrac{3x+1}{x+3}$. We describe the behavior of this function for large values of x using a table of values:

x	$f(x) = \frac{3x+1}{x+3}$
100	2.92233
$1,000$	2.99202
$10,000$	2.99920
$100,000$	2.99992

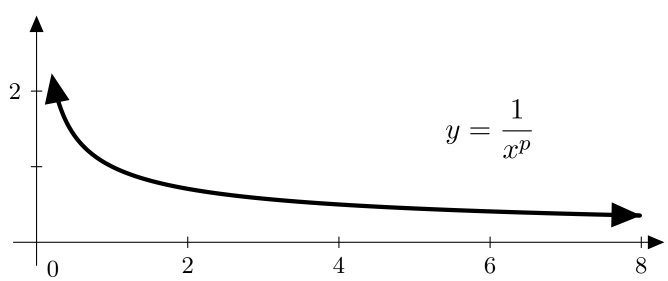
Clearly, as x goes very large, the value of y approaches the value of 3. We say that the limit of $f(x) = \dfrac{3x+1}{x+3}$ as x approaches infinity is 3, and we write

$$\lim_{x\to\infty} \frac{3x+1}{x+3} = 3.$$

To compute limits at infinity of rational functions, it is very helpful to know the following theorem:

Theorem 16. *If* $p > 0$, *then* $\displaystyle\lim_{x\to\infty} \frac{1}{x^p} = 0$.

We illustrate the theorem using the graph of $y = \dfrac{1}{x^p}$.



Observe that as x takes large values, the graph approaches the x-axis, or the $y = 0$ line. This means that the values of y can take arbitrarily small values by making x very large. This is what we mean by $\displaystyle\lim_{x\to\infty} \frac{1}{x^p} = 0$.

The technique in solving the limit at infinity of rational functions is to *divide by the largest power of* x in the rational function and apply the above theorem.

EXAMPLE 2: Compute the limits of the following rational functions.

(a) $\lim\limits_{x\to\infty} \dfrac{2x+4}{5x+1}$

(b) $\lim\limits_{x\to\infty} \dfrac{4-x+x^2}{3x^2-2x+7}$

(c) $\lim\limits_{x\to\infty} \dfrac{20x+1}{3x^3-5x+1}$

(d) $\lim\limits_{x\to\infty} \dfrac{3x^2+4}{8x-1}$

Solution.

(a) The highest power of x here is 1. So,

$$\frac{2x+4}{5x+1} = \frac{2x+4}{5x+1} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \frac{2+\frac{4}{x}}{5+\frac{1}{x}}.$$

Using limit theorems and Theorem 16,

$$\lim_{x\to\infty} \frac{2x+4}{5x+1} = \frac{\lim\limits_{x\to\infty} 2 + 4 \lim\limits_{x\to\infty} \frac{1}{x}}{\lim\limits_{x\to\infty} 5 + \lim\limits_{x\to\infty} \frac{1}{x}} = \frac{2+0}{5+0} = \frac{2}{5}.$$

(b) The highest power of x here is 2. So, by Theorem 16,

$$\lim_{x\to\infty} \frac{4-x+x^2}{3x^2-2x+7} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{4}{x^2}+\frac{1}{x}+1}{3-\frac{2}{x}+\frac{7}{x^2}} = \frac{0-0+1}{3-0+0} = \frac{1}{3}.$$

(c) The highest power of x here is 3. Again, using Theorem 16,

$$\lim_{x\to\infty} \frac{20x+1}{3x^3-5x+1} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{20}{x^2}+\frac{1}{x^3}}{3-\frac{5}{x^2}+\frac{1}{x^3}} = \frac{0+0}{3-0+0} = 0.$$

(d) The highest power of x here is 2. So,

$$\lim_{x\to\infty} \frac{3x^2+4}{8x-1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{3+\frac{4}{x^2}}{\frac{8}{x}-\frac{1}{x^2}}.$$

Notice that the numerator approaches 3 but the denominator approaches 0. Therefore, the limit does not exist.

Limits at Infinity of Rational Functions

Suppose the degrees of the polynomials $p(x)$ and $q(x)$ are m and n, respectively. There are only three cases that could happen in solving for $\lim_{x\to\infty} \frac{p(x)}{q(x)}$.

- If $m = n$, then $\lim_{x\to\infty} \frac{p(x)}{q(x)}$ exists. In fact, the limit is nonzero and equals the ratio of the leading coefficient of $p(x)$ to the leading coefficient of $q(x)$.
- If $m < n$, then $\lim_{x\to\infty} \frac{p(x)}{q(x)}$ exists and is equal to 0.
- If $m > n$, then $\lim_{x\to\infty} \frac{p(x)}{q(x)}$ DNE.

COMPUTING AREAS USING THE FORMAL DEFINITION OF THE DEFINITE INTEGRAL

The problem with the definition of the definite integral is that it contains a summation, which needs to be evaluated completely before we can apply the limit. For simple summations like $\sum_{k=1}^{n} k^p$, where p is a positive integer, formulas exist which can be verified by the principle of mathematical induction. However, in general, formulas for complex summations are scarce.

The next simple examples illustrate how a definite integral is computed by definition.

EXAMPLE 3: Show that $\int_1^3 3x + 1\,dx = 14$ using the definition of the definite integral as a limit of a sum.

Solution. Let us first get the right (the choice of "right" here is arbitrary) Riemann sum of $f(x) = 3x + 1$ relative to the regular partition $\mathcal{P}_n$ of $[1, 3]$ into n subintervals. Note that $\Delta x = \frac{3-1}{n} = \frac{2}{n}$. Since we are looking for the *right* Riemann sum, the partition points are $x_k = x_0 + k\Delta x = 1 + k\frac{2}{n} = 1 + \frac{2k}{n}$. Thus, by the formula of the right Riemann sum, we have

$$R_n = \sum_{k=1}^{n} f(x_k)\Delta x_k = \sum_{k=1}^{n} (3x_k + 1)\Delta x_k.$$

Using the definitions of Δx and x_k above, we obtain

$$R_n = \sum_{k=1}^{n} \left(3\left(1 + \frac{2k}{n}\right) + 1\right)\frac{2}{n} = \sum_{k=1}^{n} \left(4 + \frac{6k}{n}\right)\frac{2}{n} = \sum_{k=1}^{n} \left(\frac{8}{n} + \frac{12k}{n^2}\right).$$

We apply properties of the summation: distributing the summation symbol over the sum, factoring out those which are independent of the index k, and finally, applying the formulas,

$$\sum_{k=1}^{n} 1 = n \quad \text{and} \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

This gives

$$R_n = \frac{8}{n}\sum_{k=1}^{n} 1 + \frac{12}{n^2}\sum_{k=1}^{n} k = \frac{8}{n} \cdot n + \frac{12}{n^2} \cdot \frac{n(n+1)}{2} = 8 + 6\left(1 + \frac{1}{n}\right).$$

Finally, by definition

$$\int_1^3 3x + 1\, dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left(8 + 6\left(1 + \frac{1}{n}\right)\right).$$

Using Theorem 16, $\displaystyle\lim_{n\to\infty}\frac{1}{n} = 0$. Therefore, it follows that $\displaystyle\int_1^3 3x + 1\, dx = 14$, as desired.

For the next example, we need the following formula for the sum of the first n perfect squares:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \tag{3.4}$$

EXAMPLE 4: Use the definition of the definite integral as a limit of a Riemann sum to show that $\displaystyle\int_0^1 x^2\, dx = \frac{1}{3}$.

Solution. For convenience, let us again use the right Riemann sum relative to the partition $\mathcal{P}_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$ of $[0, 1]$. Clearly, $\Delta x = \frac{1}{n}$ and $x_k = x_0 + k\Delta x = 0 + k \cdot \frac{1}{n} = \frac{k}{n}$. By the formula of the right Riemann sum, we have

$$R_n = \sum_{k=1}^{n} f(x_k)\Delta x_k = \sum_{k=1}^{n}\left(\frac{k}{n}\right)^2 \frac{1}{n}.$$

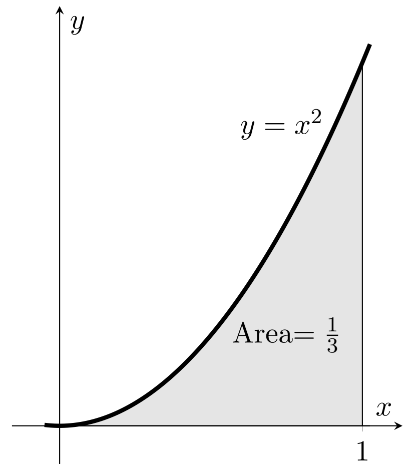
Simplifying, and using formula (3.4), we obtain

$$R_n = \frac{1}{n^3}\sum_{k=1}^{n} k^2 = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{1}{n^3} \cdot \frac{2n^3 + 3n^2 + n}{6} = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}.$$

The definite integral is just the limit of the above expression as n tends to infinity. We use Theorem 16 to evaluate the limits of the last two terms in the expression. Therefore, we have

$$\int_0^1 x^2\, dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{3}.$$

This illustrates the power of the definite integral in computing areas of non-polygonal regions.



PROPERTIES OF THE DEFINITE INTEGRAL

As a limit of a sum, the definite integral shares the common properties of the limit and of the summation.

Theorem 17 (Linearity of the Definite Integral)**.** *Let f and g be positive continuous functions on $[a, b]$ and let $c \in \mathbb{R}$. Then*

1. $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
2. $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

EXAMPLE 5: Suppose we are given that $\displaystyle\int_a^b f(x)\,dx = 2$ and $\displaystyle\int_a^b g(x)\,dx = 7$. Find the exact value of the following:

1. $\displaystyle\int_a^b 3f(x)\,dx$
2. $\displaystyle\int_a^b f(x) - g(x)\,dx$
3. $\displaystyle\int_a^b 2f(x) + g(x)\,dx$
4. $\displaystyle\int_a^b 3f(x) - 2g(x)\,dx$

Solution. Using the properties of the integral,

1. $\displaystyle\int_a^b 3f(x)\,dx = 3\int_a^b f(x)\,dx = 3(2) = 6.$
2. $\displaystyle\int_a^b f(x) - g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = 2 - 7 = -5.$

3. $\displaystyle\int_a^b 2f(x) + g(x)\,dx = 2\int_a^b f(x) + \int_a^b g(x)\,dx = 2(2) + 7 = 11.$

4. $\displaystyle\int_a^b 3f(x) - 2g(x)\,dx = 3\int_a^b f(x)\,dx - 2\int_a^b g(x)\,dx = 3(2) - 2(7) = -8.$

Another important property of the definite integral is called *additivity*. Before we proceed, we first define the following:

Definition 7. *Let f be a continuous positive function on $[a, b]$. Then*

1. $\displaystyle\int_a^a f(x)\,dx = 0,$ *and*

2. $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$

The first one is very intuitive if you visualize the definite integral as an area of a region. Since the left and right boundaries are the same ($x = a$), then there is no region and the area therefore is 0 . The second one gives meaning to a definite integral whenever the lower limit of integration is bigger than the upper limit of integration. We will see later that this is needed so the property of additivity will be consistent with our intuitive notion.

Theorem 18 (Additivity of the Definite Integral)**.** *Let f be a positive continuous function on a closed and bounded interval I containing distinct numbers a, b and c. Then*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

no matter how a, b and c are ordered on the interval I.

EXAMPLE 6: Suppose that $\displaystyle\int_0^2 f(x)\,dx = I$ and $\displaystyle\int_0^1 f(x)\,dx = J$. Find $\displaystyle\int_1^2 f(x)\,dx$.

Solution. By additivity,

$$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx.$$

Substituting the given values yields

$$I = \int_1^2 f(x)\,dx + J.$$

This implies that $\displaystyle\int_1^2 f(x)\,dx = I - J.$

EXAMPLE 7: Suppose we are given that $\int_a^b f(x)\,dx = 3$, $\int_d^c f(x) = 10$ and $\int_d^b f(x)\,dx = 4$. Find $\int_c^a f(x)\,dx$.

Solution. By additivity,

$$\begin{aligned}\int_c^a f(x)\,dx &= \int_c^d f(x)\,dx + \int_d^b f(x)\,dx + \int_b^a f(x)\,dx \\ &= -\int_d^c f(x)\,dx + \int_d^b f(x)\,dx - \int_a^b f(x)\,dx \\ &= -10 + 4 - 3 = -9.\end{aligned}$$

THE DEFINITE INTEGRAL AS A NET SIGNED AREA

We always assumed that the function that we are considering is always positive. What happens if the function has a negative part? How do we interpret this geometrically?

General Geometric Interpretation of a Definite Integral as a Net Signed Area
Suppose that f is a continuous function on $[a, b]$. (Notice that we dropped the assumption that f must be positive.) Then the definite integral

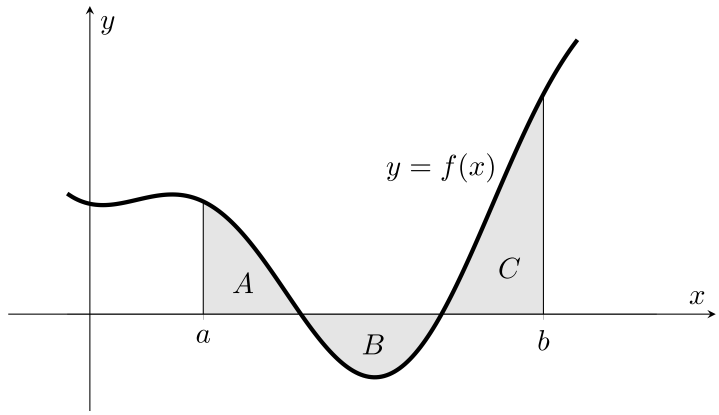
$$\int_a^b f(x)\,dx$$

is the net signed area of the region with boundaries $y = f(x)$, $x = a$, $x = b$, and the x-axis.

The *net signed* area equals the sum of all the areas above the x-axis minus the sum of all the areas below the x-axis. In effect, we are associating positive areas for the regions above the x-axis and negative areas for the regions below the x-axis.

For example, consider the following graph of $y = f(x)$ on $[a, b]$. If the areas of the shaded regions are A, B and C, as shown, then

$$\int_a^b f(x)\,dx = A + C - B.$$



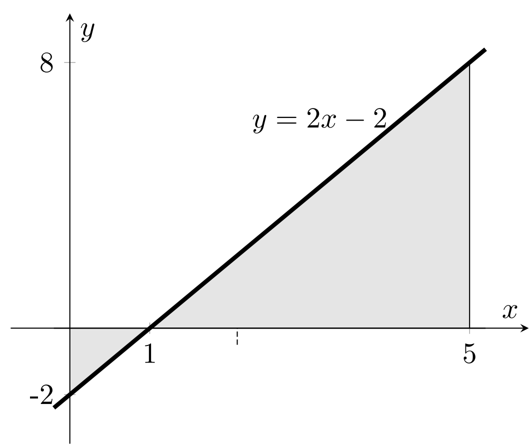
EXAMPLE 8: Interpret the following integrals as signed areas to obtain its value.

1. $\displaystyle\int_0^5 2x - 2\,dx$

2. $\displaystyle\int_{-\pi/2}^{\pi/2} \sin x\,dx$

Solution.

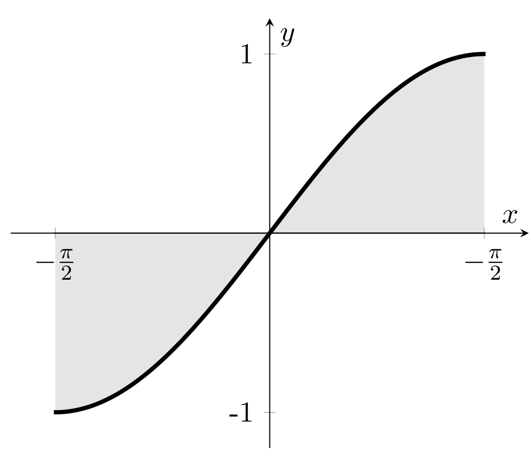
1. The graph of $y = 2x - 2$ is shown on the right.



From 0 to 1, the area of the triangle (below the x-axis) is $\frac{1}{2}(1)(2) = 1$. From 1 to 5, the area of the triangle (above the x-axis) is $\dfrac{1}{2}(4)(8) = 16$. Therefore, the net signed area equals

$$\int_0^5 2x - 2\,dx = 16 - 1 = 15.$$

2. The graph of $y = \sin x$ is shown on the right.



Observe that because $y = \sin x$ is symmetric with respect to the origin, the region (below the x-axis) from $-\pi/2$ to 0 is congruent to the region (above the x-axis) from 0 to $\pi/2$. Therefore, the net signed area equals

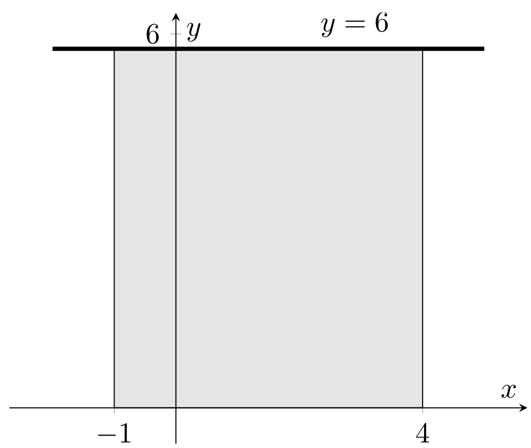
$$\int_{-\pi/2}^{\pi/2} \sin x\,dx = 0.$$

Solved Examples

EXAMPLE 1: Find the exact values of the following definite integrals.

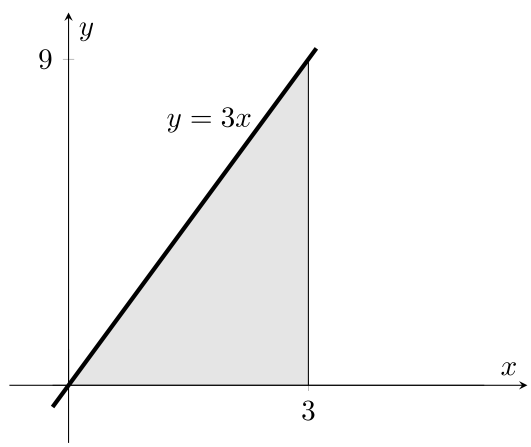
1. $\int_{-1}^{4} 6\,dx$
2. $\int_{0}^{3} 3x\,dx$
3. $\int_{1}^{4} (2x+1)\,dx$
4. $\int_{-3}^{3} \sqrt{9-x^2}dx$

Solution.



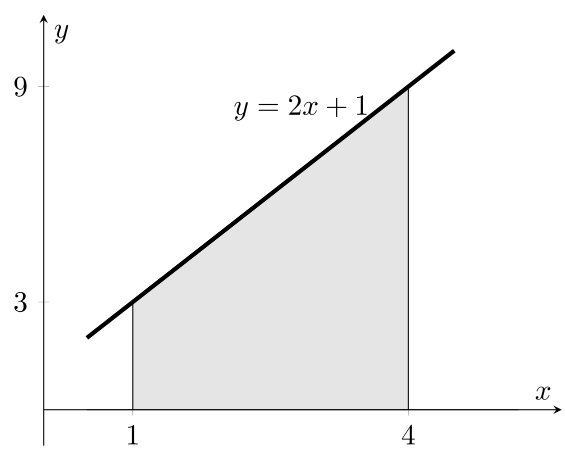
1. Observe that the region is a rectangle. Hence, the area is equal to $L \times W = 5 \times 6 = 30$. Therefore,

$$\int_{-1}^{4} 6\,dx = 30$$



2. From geometry, the area of triangle is $\frac{1}{2}$ times the base, 3, and a height, 9. Therefore,

$$\int_{0}^{3} 3x\,dx = \frac{27}{2}.$$

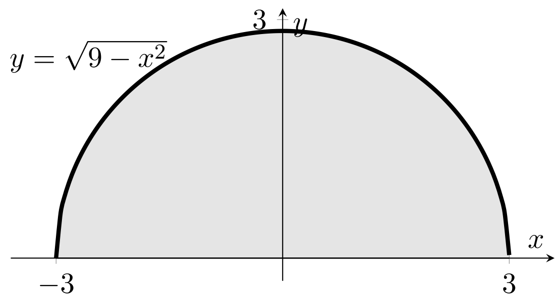


3. The required area is covered by a trapezoid. Note that the area of a trapezoid is given by $A = \left(\frac{b_1+b_2}{2}\right)h = \left(\frac{3+9}{2}\right)3 = 18$. Therefore,

$$\int_{1}^{4} (2x+1)\,dx = 18.$$

4. Observe that the shaded region is a semicircle centered at the origin with radius 3. Hence, $A = \pi(3)^2 = 9\pi$. Therefore,

$$\int_{-3}^{3} \sqrt{9-x^2}dx = 9\pi.$$



EXAMPLE 2: Use the definition of definite integral as a limit of the Riemann sum to show that

$$\int_{2}^{4} (2x-1)\,dx = 10.$$

Solution. First, we get the *right* Riemann sum (note that the choice of right here is arbitrary) of $f(x) = 2x-1$ relative to the regular partition $\mathcal{P}_n$ of $[2,4]$. This gives $\Delta x = \frac{4-2}{n} = \frac{2}{n}$. Hence, the partition points are $x_k = x_0 + k\Delta x = 2 + \frac{2k}{n}$. Thus, we have

$$R_n = \sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n} (2x_k - 1)\Delta x.$$

We take note of the following formulas:

$$\sum_{k=1}^{n} 1 = n \quad \text{and} \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Thus,

$$\begin{aligned} R_n &= \sum_{k=1}^{n} \left(2\left(2+\frac{2k}{n}\right)-1\right)\frac{2}{n} = \sum_{k=1}^{n}\left(4+\frac{4k}{n}-1\right)\frac{2}{n} = \sum_{k=1}^{n}\left(3+\frac{4k}{n}\right)\frac{2}{n} \\ &= \sum_{k=1}^{n}\left(\frac{6}{n}+\frac{8k}{n^2}\right) = \frac{6}{n}\sum_{k=1}^{n}1 + \frac{8}{n^2}\sum_{k=1}^{n}k = \frac{6}{n}n + \frac{8}{n^2}\left(\frac{n(n+1)}{2}\right) \\ &= 6 + 4\left(\frac{n^2+n}{n^2}\right) = 6 + 4\left(1+\frac{1}{n}\right) = 10 + \frac{4}{n}. \end{aligned}$$

By definition,

$$\int_{2}^{4} (2x-1)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} 10 + \frac{4}{n} = 10.$$

EXAMPLE 3:

Show that $\displaystyle\int_0^4 (3x^2+3)\,dx = 76$ using the definition of the definite integral as a limit sum.

Solution. We again use the *right* Riemann sum of $f(x) = 3x^2+3$ relative to the regular partition $\mathcal{P}_n$ of $[0,4]$. Note that $\Delta x = \frac{4}{n}$. Thus, the points are $x_k = x_0 + k\Delta x = \frac{4k}{n}$.

Another useful formula in evaluating the Riemann sum is

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} = \left(\frac{2n^3+3n^2+n}{6}\right).$$

Using the formula for the right Riemann sum, we have

$$\begin{aligned}
R_n = \sum_{k=1}^{n} f(x_k)\Delta x &= \sum_{k=1}^{n}\left(3\left(\frac{4k}{n}\right)^2+3\right)\frac{4}{n} = \sum_{k=1}^{n}\left(3\left(\frac{16k^2}{n^2}\right)+3\right)\frac{4}{n}\\
&= \sum_{k=1}^{n}\left(\frac{48k^2}{n^2}+3\right)\frac{4}{n} = \sum_{k=1}^{n}\left(\frac{192k^2}{n^2}+\frac{12}{n}\right)\\
&= \frac{192}{n^2}\sum_{k=1}^{n}k^2 + \frac{12}{n}\sum_{k=1}^{n}1 = \frac{192}{n}\left(\frac{2n^3+3n^2+n}{6}\right)+\frac{12}{n}(n)\\
&= 192\left(\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right)+12.
\end{aligned}$$

Therefore, by the definition of definite integral,

$$\int_0^4 (3x^2+3)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left(192\left(\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right)+12\right) = 192\left(\frac{1}{3}\right)+12 = 76.$$

EXAMPLE 4: Use the definition of definite integral to show $\displaystyle\int_2^5 (4x-x^2)\,dx = 3$.

Solution. Using the *right* Riemann sum relative to the partition $\mathcal{P}_n$ of $[2,5]$, it is clear that $\Delta x = \frac{3}{n}$ and $x_k = 2+\frac{3k}{n}$.

Thus,

$$
\begin{aligned}
R_n = \sum_{k=1}^{n} (4x_k - x_k^2)\,\Delta x &= \sum_{k=1}^{n} \left(4\left(2 + \frac{3k}{n}\right) - \left(2 + \frac{3k}{n}\right)^2 \right) \frac{3}{n} \\
&= \sum_{k=1}^{n} \left(\left(8 + \frac{12k}{n}\right) - \left(4 + \frac{12k}{n} + \frac{9k^2}{n^2}\right) \right) \frac{3}{n} \\
&= \sum_{k=1}^{n} \left(4 - \frac{9k^2}{n^2}\right) \frac{3}{n} \\
&= \sum_{k=1}^{n} \left(\frac{12}{n} - \frac{27k^2}{n^3}\right) \\
&= \frac{12}{n} \sum_{k=1}^{n} 1 - \frac{27}{n^3} \sum_{k=1}^{n} k^2 \\
&= 12 - 27\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right).
\end{aligned}
$$

Hence, by the definition of definite integral,

$$
\int_2^5 (4x - x^2)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \left(12 - 27\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right)\right) = 12 - 9 = 3.
$$

EXAMPLE 5: Suppose that $\int_a^b f(x)\,dx = 9$ and $\int_a^b g(x)\,dx = 4$. Find the exact value of the following:

1. $\int_a^b 2f(x)\,dx$
2. $\int_a^b (f(x) - g(x))\,dx$
3. $\int_a^b (f(x) + 2g(x))\,dx$
4. $\int_a^b (3f(x) - 2g(x))\,dx$

Solution.

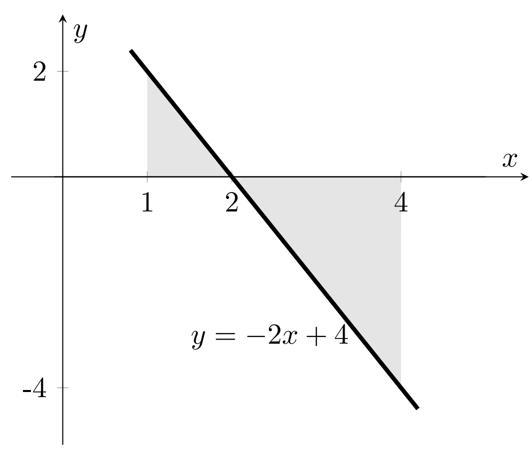
1. $\displaystyle\int_a^b 2f(x)\,dx = 2\int_a^b f(x)\,dx = 2(9) = 18.$
2. $\displaystyle\int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = 9 - 4 = 5.$
3. $\displaystyle\int_a^b (f(x) + 2g(x))\,dx = \int_a^b f(x)\,dx + 2\int_a^b g(x)\,dx = 9 + 2(4) = 9 + 8 = 17.$
4. $\displaystyle\int_a^b (3f(x) - 2g(x))\,dx = 3\int_a^b f(x)\,dx - 2\int_a^b g(x)\,dx = 3(9) - 2(4) = 19.$

EXAMPLE 6: Interpret the following integrals as net signed areas to obtain its values.

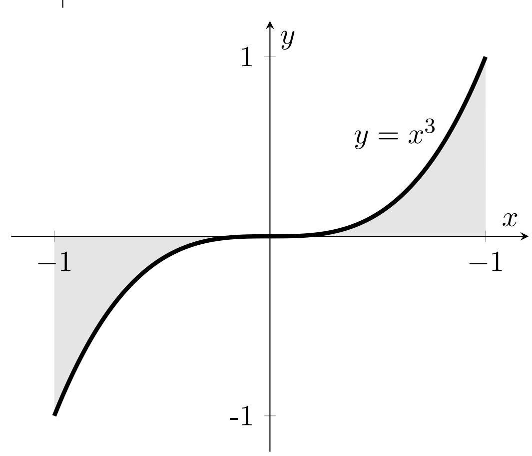
1. $\displaystyle\int_1^4 (-2x + 4)\,dx$
2. $\displaystyle\int_{-1}^1 x^3\,dx$

Solution.

1. The area of the triangle from 1 to 2 is 1, while the area of triangle from 2 to 4 is 4. Therefore, the net signed area is
$$\int_1^4 (-2x + 4)\,dx = -3.$$



2. Notice that the graph of $y = x^3$ is symmetric with respect to the origin. Hence, the region from -1 to 0 is the same as the region from 0 to 1. Thus, the net signed area is
$$\int_{-1}^1 x^3\,dx = 0.$$



Supplementary Problems

1. Evaluate each integral by interpreting it in terms of signed areas.

(a) $\int_{-1}^{1} 4\,dx$

(b) $\int_{0}^{3} 5\,dx$

(c) $\int_{1}^{3} 2\,dx$

(d) $\int_{-2}^{2} |x|\,dx$

(e) $\int_{-4}^{6} (4-|x|)\,dx$

(f) $\int_{-3}^{1} (2-|x+1|)\,dx$

(g) $\int_{1}^{4} (2x+2)\,dx$

(h) $\int_{2}^{5} (3x-4)\,dx$

(i) $\int_{-4}^{-1} (-x+2)\,dx$

(j) $\int_{-6}^{6} \sqrt{36-x^2}\,dx$

(k) $\int_{0}^{3} \sqrt{9-x^2}\,dx$

(l) $\int_{-4}^{4} \sqrt{16-x^2}\,dx$

2. Given: $\int_{1}^{2} f(x)\,dx = 5$, $\int_{1}^{4} f(x)\,dx$, $\int_{1}^{2} g(x)\,dx$, and $\int g(x)\,dx = 1$. Find the following.

(a) $\int_{1}^{1} f(x)\,dx$

(b) $\int_{-3}^{2} g(x)\,dx$

(c) $\int_{1}^{2} (4g(x)-3f(x))\,dx$

(d) $\int_{4}^{1} f(x)\,dx$

(e) $\int_{2}^{4} f(x)\,dx$

3. Use the definition of definite integral as a limit of Riemann sum to evaluate the following definite integrals.

(a) $\int_{0}^{5} 4x\,dx$

(b) $\int_{0}^{3} (5x^2-2x)\,dx$

(c) $\int_{-2}^{0} (3x^2+2x)\,dx$

(d) $\int_{0}^{2} (x^2+10)\,dx$

(e) $\int_{-3}^{0} (4x^2-5x-1)\,dx$

(f) $\int_{-5}^{-1} (x^2+3x+5)\,dx$

LESSON 16: The Fundamental Theorem of Calculus

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Illustrate the Fundamental Theorem of Calculus; and
2. Compute the definite integral of a function using the Fundamental Theorem of Calculus.

TOPIC 16.1: Illustration of the Fundamental Theorem of Calculus

Fundamental Theorem of Calculus (FTOC)

Let f be a continuous function on $[a, b]$ and let F be an antiderivative of f, that is, $F'(x) = f(x)$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

EXAMPLE 1: Note that $F(x) = \dfrac{x^3}{3}$ is an antiderivative of $f(x) = x^2$ (since $F'(x) = f(x)$.) Hence, by FTOC,

$$\int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right|_0^1 = F(1) - F(0) = \frac{1}{3} - 0 = \frac{1}{3},$$

as we have seen before.

Vertical Bar Notation

We adopt the following notation for evaluating $F(x)$ from $x = a$ to $x = b$:

$$F(x)\Big|_a^b = F(b) - F(a).$$

For example,

$$(1 + x - x^2)\Big|_1^2 = (1 + 2 - 2^2) - (1 + 1 - 1^2) = (-1) - (1) = -2,$$

and

$$\sin x\Big|_{\pi/4}^{\pi/2} = \sin(\pi/2) - \sin(\pi/4) = 1 - \frac{\sqrt{2}}{2} = \frac{2 - \sqrt{2}}{2}.$$

Using the above notation, the FTOC now states: If F is an antiderivative of f, then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b.$$

The constant of integration that was necessary for indefinite integration will now just cancel out:

$$\big(F(x) + C\big)\Big|_a^b = \big(F(b) + C\big) - \big(F(a) + C\big) = F(b) - F(a) = F(x)\Big|_a^b.$$

The next examples will validate that FTOC works by redoing the examples in the previous section.

EXAMPLE 2: Without referring to the graphs of the integrands, find the exact values of the following definite integrals:

1. $\displaystyle\int_1^2 3\,dx$
2. $\displaystyle\int_0^2 (1-|x-1|)\,dx$
3. $\displaystyle\int_1^3 (3x+1)\,dx$
4. $\displaystyle\int_{-1}^1 \sqrt{1-x^2}\,dx$

Solution. We integrate using the Fundamental Theorem of Calculus.

1. $\displaystyle\int_1^2 3\,dx = 3x\Big|_1^2 = 3(2-1) = 3.$
2. The solution for this problem takes a few more steps because the absolute value function is essentially a piecewise function. Recall that if E is any expression, then $|E| = E$ if $E \geq 0$, while $|E| = -E$ if $E < 0$. With this in mind,

$$\begin{aligned} 1-|x-1| &= \begin{cases} 1-(x-1) & \text{if } x-1 \geq 0 \\ 1-[-(x-1)] & \text{if } x-1 < 0 \end{cases} \\ &= \begin{cases} 2-x & \text{if } x \geq 1 \\ x & \text{if } x < 1. \end{cases} \end{aligned}$$

   Therefore, by additivity,

$$\begin{aligned} \int_0^2 (1-|x-1|)\,dx &= \int_1^2 2-x\,dx + \int_0^1 x\,dx \\ &= \left(2x-\frac{x^2}{2}\right)\Bigg|_1^2 + \left(\frac{x^2}{2}\right)\Bigg|_0^1 \\ &= \left[\left(4-\frac{4}{2}\right)-\left(2-\frac{1}{2}\right)\right] + \left[\frac{1}{2}-0\right] = 1. \end{aligned}$$

3. $\displaystyle\int_1^3 (3x+1)\,dx = \left(\frac{3x^2}{2}+x\right)\Bigg|_1^3 = \left(\frac{27}{2}+3\right)-\left(\frac{3}{2}+1\right) = 14.$
4. There is a technique of integration needed to integrate $\sqrt{1-x^2}$. This is called ***trigonometric substitution***, and the student will learn this in college. For now, we convince ourselves that

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\left(\sin^{-1}x + x\sqrt{1-x^2}\right) + C$$

by differentiating the right-hand side and observing that it yields the integrand of the left-hand side. Hence, by FTOC,

$$\int_{-1}^{1} \sqrt{1-x^2}\, dx = \frac{1}{2}\left(\sin^{-1} x + x\sqrt{1-x^2}\right)\Bigg|_{-1}^{1} = \frac{1}{2}\left(\frac{\pi}{2}+0\right) - \frac{1}{2}\left(-\frac{\pi}{2}-0\right) = \frac{\pi}{2}.$$

Because FTOC evaluates the definite integral using antiderivatives and not by "net signed area," we do not have to look at the graph and look at those regions below the x-axis, that is, FTOC works even if the graph has a "negative part."

EXAMPLE 3: Evaluate the following integrals using FTOC.

1. $\displaystyle\int_{0}^{5} (2x-2)\, dx$

2. $\displaystyle\int_{-\pi/2}^{\pi/2} \sin x\, dx$

Solution.

1. $\displaystyle\int_{0}^{5} (2x-2)\, dx = \left(2\frac{x^2}{2} - 2x\right)\Bigg|_{0}^{5} = (25-10)-(0-0) = 15.$

2. $\displaystyle\int_{-\pi/2}^{\pi/2} \sin x\, dx = (-\cos x)\Bigg|_{-\pi/2}^{\pi/2} = (0-0) = 0.$

These answers are the same as when we appealed to the geometrical solution of the integral in the previous section.

Solved Examples

EXAMPLE 1: Using FTOC, find $\displaystyle\int_{0}^{\pi} \cos x\, dx$.

Solution. Note that $F(x) = \sin x$ is an antiderivative of $f(x) = \cos x$ since $F'(x) = \cos x$. Therefore, by FTOC,

$$\int_{0}^{\pi} \cos x\, dx = \sin x\Bigg|_{0}^{\pi} = \sin \pi - \sin 0 = 0.$$

Without referring to the graphs of the integrands, find the exact value of the following definite integrals.

EXAMPLE 2: $\displaystyle\int_1^3 6\,dx$

Solution. $\displaystyle\int_1^3 6\,dx = 6x\Big|_1^3 = 6(3) - 6(1) = 12.$

EXAMPLE 3: $\displaystyle\int_0^3 3x\,dx$

Solution. $\displaystyle\int_0^3 3x\,dx = \frac{3x^2}{2}\Big|_0^3 = \frac{3}{2}(9-0) = \frac{27}{2}.$

EXAMPLE 4: $\displaystyle\int_1^4 (2x+1)\,dx$

Solution. $\displaystyle\int_1^4 (2x+1)\,dx = (x^2+x)\Big|_1^4 = (16+4) - (1+1) = 20 - 2 = 18.$

Supplementary Problems

Evaluate the following integrals using FTOC.

1. $\displaystyle\int_0^1 5dx$
2. $\displaystyle\int_{-2}^{-1} \frac{1}{3}dx$
3. $\displaystyle\int_1^2 (|x+1|+2)\,dx$
4. $\displaystyle\int_{-1}^1 |x|\,dx$
5. $\displaystyle\int_2^3 (2x-1)\,dx$
6. $\displaystyle\int_2^4 (3x+7)\,dx$
7. $\displaystyle\int_{-1}^1 \left(x^2+1\right)dx$
8. $\displaystyle\int_0^2 \left(1+x^2\right)dx$
9. $\displaystyle\int_{10}^{20} xdx$
10. $\displaystyle\int_1^3 (2+2x)\,dx$

TOPIC 16.2: Computation of Definite Integrals using the Fundamental Theorem of Calculus

In the previous section, we illustrated how the Fundamental Theorem of Calculus works. If f is a continuous function on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b)\Big|_a^b = F(b) - F(a),$$

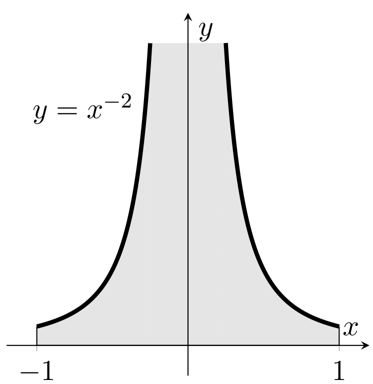
where $f(x) = F'(x)$.

For this lesson, we start with a remark about the applicability of the FTOC and proceed with answering some exercises, either individually or by group.

Remark 1: If the function is not continuous on its interval of integration, the FTOC will not guarantee a correct answer. For example, we know that an antiderivative of $f(x) = x^{-2}$ is $F(x) = -x^{-1}$. So, if we apply the FTOC to f on $[-1, 1]$, we get

$$\int_{-1}^1 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{-1}^1 = (-1 - (+1)) = -2.$$

This is absurd as the region described by $\int_{-1}^1 x^{-2}\,dx$ is given below:



Clearly, the area should be positive. The study of definite integrals of functions which are discontinuous on an interval (which may not be closed nor bounded) is called *improper integration* and will be studied in college.

Table of Integrals

Observe that for FTOC to work, the student must be able to produce an antiderivative for the integrand. This is why the student must be comfortable with the first few lessons in this chapter. A very common and indispensable formula is the Power Rule:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1.$$

For other cases, we recall the table of integrals for reference.

1. $\displaystyle\int dx = x + C$
2. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad \text{if } n \neq -1$
3. $\displaystyle\int af(x)\,dx = a\int f(x)\,dx$
4. $\displaystyle\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
5. $\displaystyle\int e^x\,dx = e^x + C.$
6. $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C.$
7. $\displaystyle\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C.$
8. $\displaystyle\int \sin x\,dx = -\cos x + C$
9. $\displaystyle\int \cos x\,dx = \sin x + C$
10. $\displaystyle\int \sec^2 x\,dx = \tan x + C$
11. $\displaystyle\int \csc^2 x\,dx = -\cot x + C$
12. $\displaystyle\int \sec x \tan x\,dx = \sec x + C$
13. $\displaystyle\int \csc x \cot x\,dx = -\csc x + C$
14. $\displaystyle\int \tan x\,dx = -\ln|\cos x| + C$
15. $\displaystyle\int \cot x\,dx = \ln|\sin x| + C$
16. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C$
17. $\displaystyle\int \csc x\,dx = \ln|\csc x - \cot x| + C$
18. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
19. $\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
20. $\displaystyle\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$

EXAMPLE 1: Using FTOC, evaluate the following definite integrals:

1. $\displaystyle\int_1^4 \sqrt{x}\,dx$
2. $\displaystyle\int_1^2 \frac{x^3 - 2x^2 + 4x - 2}{x}\,dx$
3. $\displaystyle\int_0^{\pi/4} \cos x + \tan x\,dx$
4. $\displaystyle\int_0^{1/2} \frac{dx}{\sqrt{1 - x^2}}$

Solution.

1. $\displaystyle\int_1^4 \sqrt{x}\,dx = \int_1^4 x^{1/2}\,dx = \frac{x^{3/2}}{3/2}\Bigg|_1^4 = \frac{2}{3}(8-1) = \frac{14}{3}.$

2. We first divide the numerator by the denominator to express the fraction as a sum.

$$\begin{aligned}\int_1^2 \frac{x^3-2x^2+4x-2}{x}\,dx &= \int_1^2 \left(x^2-2x+4-\frac{2}{x}\right)dx\\ &= \left(\frac{x^3}{3}-x^2+4x-2\ln|x|\right)\Bigg|_1^2\\ &= \left(\frac{8}{3}-4+8-2\ln 2\right)-\left(\frac{1}{3}-1+4-0\right)\\ &= \frac{10}{3}-2\ln 2.\end{aligned}$$

3. $\displaystyle\int_0^{\pi/4} \cos x + \tan x\,dx = (\sin x + \ln|\sec x|)\Bigg|_0^{\pi/4} = \frac{\sqrt{2}}{2} + \ln(\sqrt{2})$

4. $\displaystyle\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x\Bigg|_0^{1/2} = \sin^{-1}(1/2) - \sin^{-1}(0) = \frac{\pi}{6} - 0 = \frac{\pi}{6}.$

INTEGRANDS WITH ABSOLUTE VALUES

Solving definite integrals with absolute values in the integrands has been discussed in passing in previous examples. We will now give a more in-depth discussion.

As has been said, for any continuous expression E, its absolute value can always be written in piecewise form:

$$|E| = \begin{cases} E & \text{if } E \geq 0 \\ -E & \text{if } E < 0. \end{cases}$$

Therefore, the first step in solving this kind of integral is to eliminate the absolute value bars. This is done by dividing the interval of integration $[a, b]$ into two subintervals according to the piecewise version of the function. Hence, one integrand is either purely positive (or zero) and the other is purely negative.

EXAMPLE 2: Evaluate the following definite integrals:

1. $\displaystyle\int_{-1}^{2} |x-3|\,dx$
2. $\displaystyle\int_{3}^{7} |x-3|\,dx$
3. $\displaystyle\int_{1}^{4} |x-3|\,dx$
4. $\displaystyle\int_{0}^{2} 4x+|2x-1|\,dx$
5. $\displaystyle\int_{0}^{2\pi/3} |\cos x|\,dx$
6. $\displaystyle\int_{-2}^{3} |x^2-1|\,dx$

Solution. For items 1-3, observe that by definition,

$$|x-3| \quad = \quad \begin{cases} x-3, & \text{if } x-3 \geq 0 \\ -(x-3), & \text{if } x-3 < 0, \end{cases} \quad = \quad \begin{cases} x-3, & \text{if } x \geq 3, \\ -x+3, & \text{if } x < 3. \end{cases}$$

1. Since $x-3$ is always negative on the interval of integration $[-1,2]$, we replace $|x-3|$ with $-(x-3)$. Hence,

$$\int_{-1}^{2} |x-3|\,dx = \int_{-1}^{2} -x+3\,dx = \left(-\frac{x^2}{2}+3x\right)\Bigg|_{-1}^{2} = \left(-\frac{4}{2}+6\right) - \left(-\frac{1}{2}-3\right) = \frac{15}{2}.$$

2. Since $x-3$ is always nonnegative on $[3,7]$, we replace $|x-3|$ with $x-3$. So,

$$\int_{3}^{7} |x-3|\,dx = \int_{3}^{7} x-3\,dx = \left(\frac{x^2}{2}-3x\right)\Bigg|_{3}^{7} = \left(\frac{49}{2}-21\right) - \left(\frac{9}{2}-9\right) = 8.$$

3. Since $x-3$ is neither purely positive nor purely negative on $[1,4]$, we need to divide this interval into $[1,3]$ and $[3,4]$. On the first, we replace $|x-3|$ with $-x+3$, while on the second, we replace $|x-3|$ with $x-3$.

$$\begin{aligned} \int_{1}^{4} |x-3|\,dx &= \int_{1}^{3} -x+3\,dx + \int_{3}^{4} x-3\,dx \\ &= \left(-\frac{x^2}{2}+3x\right)\Bigg|_{1}^{3} + \left(\frac{x^2}{2}-3x\right)\Bigg|_{3}^{4} \\ &= \left[\left(-\frac{9}{2}+9\right) - \left(-\frac{1}{2}+3\right)\right] + \left[\left(\frac{16}{2}-12\right) - \left(\frac{9}{2}-9\right)\right] \\ &= \left[\frac{9}{2}-\frac{5}{2}\right] + \left[-\frac{8}{2}+\frac{9}{2}\right] = \frac{5}{2}. \end{aligned}$$

4. We split the interval of integration into $[0,\frac{1}{2}]$ and $[\frac{1}{2},2]$ since

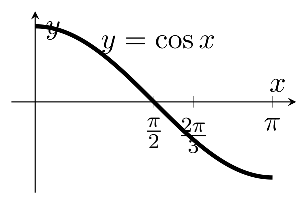
$$|2x-1| = \begin{cases} 2x-1 & \text{if } 2x-1 \geq 0 \\ -(2x-1) & \text{if } 2x-1 < 0. \end{cases}$$

Hence, $4x + |2x-1| = \begin{cases} 4x + (2x-1) & \text{if } x \geq \frac{1}{2} \\ 4x - (2x-1) & \text{if } x < \frac{1}{2}. \end{cases}$

Therefore,

$$\begin{aligned} \int_0^2 4x + |2x-1|\, dx &= \int_0^{\frac{1}{2}} 2x + 1\, dx + \int_{\frac{1}{2}}^2 6x - 1\, dx \\ &= (x^2 + x)\Big|_0^{1/2} + (3x^2 - x)\Big|_{1/2}^2 \\ &= \left[\left(\frac{1}{4} + \frac{1}{2}\right) - (0+0)\right] + \left[(12-2) - \left(\frac{3}{4} - \frac{1}{2}\right)\right] = \frac{21}{2}. \end{aligned}$$

5. The cosine function is nonnegative on $[0, \pi/2]$ and negative on $[\pi/2, 2\pi/3]$. See graph below.

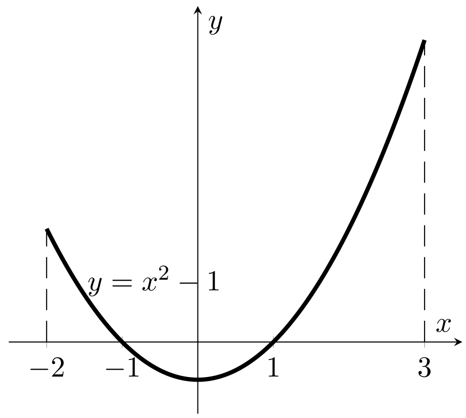


Hence, $|\cos x| = \begin{cases} \cos x, & \text{if } \cos x \geq 0, \\ -\cos x, & \text{if } \cos x < 0 \end{cases} = \begin{cases} \cos x, & \text{if } x \in [0, \pi/2], \\ -\cos x, & \text{if } x \in [\pi/2, 2\pi/3]. \end{cases}$

Therefore,

$$\begin{aligned} \int_0^{2\pi/3} |\cos x|\, dx &= \int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{2\pi/3} -\cos x\, dx \\ &= \sin x\Big|_0^{\pi/2} + (-\sin x)\Big|_{\pi/2}^{2\pi/3} \\ &= (1-0) + [(-\sqrt{3}/2) - (-1)] = 2 - \frac{\sqrt{3}}{2}. \end{aligned}$$

6. The quadratic function $y = x^2 - 1$ is below the x-axis only when $x \in (-1, 1)$. This conclusion can be deduced by solving the inequality $x^2 - 1 < 0$ or by referring to the graph below.



This implies that we have to divide $[-2,3]$ into the three subintervals $[-2,-1]$, $[-1,1]$, and $[1,3]$. On $[-1,1]$, we replace $|x^2-1|$ with $-x^2+1$ while on the other subintervals, we simply replace $|x^2-1|$ with x^2-1. Hence,

$$
\begin{aligned}
\int_{-2}^{3} |x^2-1|\,dx &= \int_{-2}^{-1} x^2-1\,dx + \int_{-1}^{1} -x^2+1\,dx + \int_{1}^{3} x^2-1\,dx \\
&= \left(\frac{x^3}{3}-x\right)\Bigg|_{-2}^{-1} + \left(-\frac{x^3}{3}+x\right)\Bigg|_{-1}^{1} + \left(\frac{x^3}{3}-x\right)\Bigg|_{1}^{3} \\
&= \left[\left(-\frac{1}{3}+1\right)-\left(-\frac{8}{3}+2\right)\right] + \left[\left(-\frac{1}{3}+1\right)-\left(\frac{1}{3}-1\right)\right] \\
&\quad + \left[\left(\frac{27}{3}-3\right)-\left(\frac{1}{3}-1\right)\right] \\
&= \frac{28}{3}.
\end{aligned}
$$

Solved Examples

Using FTOC, evaluate the following definite integrals.

EXAMPLE 1: $\displaystyle\int_4^9 \frac{1}{\sqrt{x}}\,dx$

Solution. $\displaystyle\int_4^9 \frac{1}{\sqrt{x}}\,dx = \int_4^9 x^{-\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}}\Bigg|_4^9 = 2(3-2) = 2.$

EXAMPLE 2: $\displaystyle\int_0^2 \frac{12x^3+6x^2+4x}{2x}\,dx$

Solution.

$$\begin{aligned}
\int_0^2 \frac{12x^3+6x^2+4x}{2x}\,dx &= \int_0^2 (6x^2+3x+2)\,dx \\
&= \left(\frac{6x^3}{3}+\frac{3x^2}{2}+2x\right)\Bigg|_0^2 \\
&= \left(2(2)^3+\frac{3(2)^2}{2}+2(2)\right)-\left(2(0)^3+\frac{3(0)^2}{2}+2(0)\right) \\
&= 26.
\end{aligned}$$

EXAMPLE 3: $\displaystyle\int_0^{\frac{\pi}{4}} (\sin x + \sec x \tan x)\,dx$

Solution.

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} (\sin x + \sec x \tan x)\,dx &= (-\cos x + \sec x)\Big|_0^{\frac{\pi}{4}} \\
&= \left(-\cos\frac{\pi}{4}+\sec\frac{\pi}{4}\right)-(-\cos 0+\sec 0) \\
&= \left(-\frac{\sqrt{2}}{2}+\sqrt{2}\right)-(-1+1) \\
&= \frac{\sqrt{2}}{2}.
\end{aligned}$$

EXAMPLE 4: $\displaystyle\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$

Solution. $\displaystyle\int_1^2 \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x\Big|_1^2 = \sec^{-1}(2)-\sec^{-1}(1) = \frac{\pi}{3}.$

EXAMPLE 5: $\displaystyle\int_{-1}^{1} |2x-1|\,dx$

Solution.

$$|2x-1| = \begin{cases} 2x-1 & \text{if } x \geq \frac{1}{2} \\ -2x+1 & \text{if } x < \frac{1}{2} \end{cases}$$

Therefore,

$$\begin{aligned} \int_{-1}^{1} |2x-1|\,dx &= \int_{-1}^{\frac{1}{2}} (-2x+1)\,dx + \int_{\frac{1}{2}}^{1} (2x-1)\,dx \\ &= \left(-x^2+x\right)\Big|_{-1}^{\frac{1}{2}} + (x^2-x)\Big|_{\frac{1}{2}}^{1} \\ &= \left[\left(-\frac{1}{4}+\frac{1}{2}\right) - (-1-1)\right] + \left[0 - \left(\frac{1}{4}-\frac{1}{2}\right)\right] \\ &= \frac{5}{2}. \end{aligned}$$

EXAMPLE 6: $\displaystyle\int_{-2}^{3} |x+1|\,dx$

Solution.

$$|x+1| = \begin{cases} x+1 & \text{if } x \geq -1 \\ -x-1 & \text{if } x < -1. \end{cases}$$

Therefore,

$$\begin{aligned} \int_{-2}^{3} |x+1|\,dx &= \int_{-2}^{-1} (-x-1)\,dx + \int_{-1}^{3} (x+1)\,dx \\ &= \left(\frac{-x^2}{2} - x\right)\Bigg|_{-2}^{-1} + \left(\frac{x^2}{2}+x\right)\Bigg|_{-1}^{3} \\ &= \left[\left(-\frac{1}{2}+1\right) - \left(-\frac{4}{2}+2\right)\right] + \left[\left(\frac{9}{2}+3\right) - \left(\frac{1}{2}-1\right)\right] \\ &= \frac{17}{2}. \end{aligned}$$

EXAMPLE 7: $\displaystyle\int_{-6}^{0} |2x+4|\,dx$

Solution.

$$|2x+4| = \begin{cases} 2x+4 & \text{if } x \geq -2 \\ -2x-4 & \text{if } x < -2. \end{cases}$$

Therefore,

$$\begin{aligned}
\int_{-6}^{0} |2x+4|\,dx &= \int_{-6}^{-2} (-2x-4)\,dx + \int_{-2}^{0} (2x+4)\,dx \\
&= (-x^2-4x)\Big|_{-6}^{-2} + (x^2+4x)\Big|_{-2}^{0} \\
&= [(-4+8)-(-36+24)] + [0-(4-8)] \\
&= 20.
\end{aligned}$$

EXAMPLE 8: $\displaystyle\int_{-1}^{1} (|3x+1|+3x)\,dx$

Solution.

$$|3x+1| = \begin{cases} 3x+1 \text{ if } x \geq -\frac{1}{3} \\ -3x-1 \text{ if } x < -\frac{1}{3}. \end{cases}$$

Now,

$$\begin{aligned}
\int_{-1}^{1} (|3x+1|+3x)\,dx &= \int_{-1}^{-\frac{1}{3}} [(-3x-1)+3x]\,dx + \int_{-\frac{1}{3}}^{1} [(3x+1)+3x]\,dx \\
&= \int_{-1}^{-\frac{1}{3}} -1\,dx + \int_{-\frac{1}{3}}^{1} (6x+1)\,dx \\
&= (-x)\Big|_{-1}^{-\frac{1}{3}} + (3x^2+x)\Big|_{-\frac{1}{3}}^{1} \\
&= \left[-\left(-\frac{1}{3}+1\right)\right] + \left[(3+1)-\left(\frac{1}{3}-\frac{1}{3}\right)\right] \\
&= \frac{10}{3}.
\end{aligned}$$

EXAMPLE 9: $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} |\sin x|\,dx$

Solution.

$$|\sin x| = \begin{cases} \sin x & \text{if } x \in [0, \frac{\pi}{4}] \\ -\sin x & \text{if } x \in [-\frac{\pi}{4}, 0). \end{cases}$$

Therefore,

$$\begin{aligned}
\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} |\sin x|\, dx &= \int_{-\frac{\pi}{4}}^{0} -\sin x\, dx + \int_{0}^{\frac{\pi}{4}} \sin x\, dx \\
&= \cos x \Big|_{-\frac{\pi}{4}}^{0} - \cos x \Big|_{0}^{\frac{\pi}{4}} \\
&= \left[\cos 0 - \cos\left(-\frac{\pi}{4}\right)\right] - \left[\cos\left(\frac{\pi}{4}\right) - \cos 0\right] \\
&= \left(1 - \frac{\sqrt{2}}{2}\right) + \left(-\frac{\sqrt{2}}{2} + 1\right) \\
&= 2 - \sqrt{2}.
\end{aligned}$$

EXAMPLE 10: $\displaystyle\int_0^4 |x^2 - 4|\, dx$

Solution.

$$|x^2 - 4| = \begin{cases} x^2 - 4 & \text{if } x \geq 2 \text{ or } x \leq -2 \\ -x^2 + 4 & \text{if } -2 < x < 2. \end{cases}$$

Thus,

$$\begin{aligned}
\int_0^4 |x^2 - 4|\, dx &= \int_0^2 (-x^2 + 4)\, dx + \int_2^4 (x^2 - 4)\, dx \\
&= \left(-\frac{x^3}{3} + 4x\right)\Big|_0^2 + \left(\frac{x^3}{3} - 4x\right)\Big|_2^4 \\
&= \left[\left(-\frac{8}{3} + 8\right) - 0\right] + \left[\left(\frac{64}{3} - 16\right) - \left(\frac{8}{3} - 8\right)\right] \\
&= 16.
\end{aligned}$$

Supplementary Problems

Evaluate the following definite integrals.

1. $\displaystyle\int_1^2 2x\,dx$
2. $\displaystyle\int_{-2}^2 (x+5)\,dx$
3. $\displaystyle\int_0^1 (x^2+3x+1)\,dx$
4. $\displaystyle\int_0^3 (5-2x^2)\,dx$
5. $\displaystyle\int_1^3 (x^3-6x^2+2x-7)\,dx$
6. $\displaystyle\int_1^{27} x^{-4/3}\,dx$
7. $\displaystyle\int_1 6^8 1x^{1/4}\,dx$
8. $\displaystyle\int_0^1 \frac{x^6+x^4+x^2}{x^2}\,dx$
9. $\displaystyle\int_1^2 \frac{4}{x^2}\,dx$
10. $\displaystyle\int_1^4 \frac{1}{x\sqrt{x}}\,dx$
11. $\displaystyle\int_1^e \frac{2}{x}\,dx$
12. $\displaystyle\int_0^2 e^{-x}\,dx$
13. $\displaystyle\int_0^2 x|1-x|\,dx$
14. $\displaystyle\int_{-2}^3 (5-2|x-2|)\,dx$
15. $\displaystyle\int_0^1 (|2-3x|+1)\,dx$
16. $\displaystyle\int_{-1}^3 (|x-2|+x^2)\,dx$
17. $\displaystyle\int_0^\pi \cos x\,dx$
18. $\displaystyle\int_0^{\frac{\pi}{3}} \sec^2 x\,dx$
19. $\displaystyle\int_0^\pi (\sin x+\cos x)\,dx$
20. $\displaystyle\int_0^\pi (4x^4+\sin x)\,dx$

LESSON 17: Integration Technique: The Substitution Rule for Definite Integrals

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Illustrate the substitution rule; and
2. Compute the definite integral of a function using the substitution rule

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TOPIC 17.1: Illustration of the Substitution Rule for Definite Integrals

SUBSTITUTION RULE

Method 1.

We first consider the definite integral as an indefinite integral and apply the substitution technique. The answer (antiderivative of the function) is expressed in terms of the original variable and the FTOC is applied using the limits of integration $x = a$ and $x = b$.

For example, if we want to integrate $\int_1^3 (x-2)^{54}\,dx$, we first apply the substitution technique to the indefinite integral using the substitution $y = x - 2$ and express the antiderivative in terms of x :

$$\int (x-2)^{54}\,dx = \frac{1}{55}(x-2)^{55} + C.$$

We apply the FTOC using the original limits of integration $x = 1$ and $x = 3$, so we have

$$\int_1^3 (x-2)^{54}\,dx = \frac{1}{55}(x-2)^{55} + C\Big|_1^3 = \frac{1}{55}(1)^{55} - \frac{1}{55}(-1)^{55} = \frac{2}{55}.$$

Note that for definite integrals, we can omit the constant of integration C in the antiderivative since this will cancel when we evaluate at the limits of integration.

Method 2.

In the second method, the substitution is applied directly to the definite integral and the limits or bounds of integration are also changed according to the substitution applied. How is this done? If the substitution $y = g(x)$ is applied, then the limits of integration $x = a$ and $x = b$ are changed to $g(a)$ and $g(b)$, respectively. The FTOC is then applied to the definite integral where the integrand is a function of y and using the new limits of integration $y = g(a)$ and $y = g(b)$.

To illustrate this method, let us consider the same definite integral $\int_1^3 (x-2)^{54}\,dx$. Applying the substitution technique, we let $y = (x-2)$ so $dy = dx$. For the limits of integration in the given definite integral, these are changed in accordance to the substitution $y = x - 2$:

$$\text{If } \ x = 1, \text{ then } y = 1 - 2 = -1 \ \text{ and if } \ x = 3 \text{ then } y = 3 - 2 = 1.$$

We then apply the FTOC to the definite integrable involving the new variable y yielding:

$$\begin{aligned}\int_1^3 (x-2)^{54}\,dx &= \int_{-1}^1 y^{54}\,dy \\ &= \frac{y^{55}}{55} + C\Big|_{-1}^{1} \\ &= \frac{(1)^{55}}{55} - \frac{(-1)^{55}}{55} = \frac{2}{55}.\end{aligned}$$

Let us summarize:

In applying the substitution technique of integration to the definite integral

$$\int_a^b f(x)\,dx,$$

the integrand "$f(x)\,dx$" is replaced by an expression in terms of y where y is a function of x, say $y = g(x)$ which implies $dy = g'(x)dx$. The antiderivative, say $F(y)$ is thus expressed as a function of y.

In the first method, the variable y is then expressed in terms of x, giving the antiderivative $F(g(x))$ and this is evaluated with the original bounds $x = a$ and $x = b$.

In the second method, we proceed with the substitution as above and the new bounds are computed through the same equation used to perform the substitution. Thus, if $y = g(x)$, then the new bounds are

$$y = g(a) \text{ and } y = g(b)$$

and the definite integral is now expressed as

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(y)\,dy.$$

This is known as the Substitution Rule for Definite Integrals (SRDI).

EXAMPLE 1: Compute $\displaystyle\int_0^2 (2x-1)^3\,dx$.

Solution. Method 1. Let $y = 2x - 1$. It follows that $dy = 2\,dx$. Hence, $dx = \dfrac{1}{2}\,dy$. Evaluating the definite integral, we have

$$\begin{aligned}
\int (2x-1)^3 \, dx &= \int y^3 \cdot \frac{1}{2} \, dy \\
&= \int \frac{1}{2} y^3 \, dy \\
&= \frac{1}{2} \int y^3 \, dy \\
&= \frac{1}{2} \cdot \frac{y^4}{4} + C \\
&= \frac{1}{8} y^4 + C \\
&= \frac{1}{8}(2x-1)^4 + C.
\end{aligned}$$

So, by FTOC,

$$\int (2x-1)^3 \, dx = \frac{1}{8}(2x-1)^4 \Big|_0^2 = \frac{1}{8}(3)^4 - \frac{1}{8}(-1)^4 = 10.$$

Method 2. Let $y = 2x - 1$, and so $dy = 2\,dx$. Hence, $dx = \frac{1}{2}\,dy$. The bounds are then transformed as follows:

If $x = 0$, then

$$y = 2(0) - 1 = -1.$$

If $x = 2$, then

$$y = 2(2) - 1 = 3.$$

The substitution yields the transformed definite integral

$$\int_{-1}^{3} y^3 \cdot \frac{1}{2} \, dy.$$

Evaluating the above definite integral,

$$
\begin{aligned}
\int_{-1}^{3} y^3 \cdot \frac{1}{2} dy &= \int_{-1}^{3} \frac{1}{2} y^3 \, dy \\
&= \frac{1}{2} \int_{-1}^{3} y^3 \, dy \\
&= \frac{1}{2} \cdot \frac{1}{4} y^4 \Big|_{-1}^{3} \\
&= \frac{1}{8}[(3)^4 - (-1)^4] \\
&= \frac{1}{8}(80) \\
&= 10.
\end{aligned}
$$

EXAMPLE 2: Compute $\displaystyle\int_{-2}^{-1} \sqrt{2-7x}\, dx$.

Solution. Let $y = 2 - 7x$. It follows that $dy = -7\, dx$ or $dx = -\dfrac{1}{7}\, dy$. For the transformed bounds: If $x = -2$, then

$$y = 2 - 7(-2) = 16.$$

If $x = -1$, then

$$y = 2 - 7(-1) = 9.$$

The substitution yields the transformed definite integral

$$
\begin{aligned}
\int_{16}^{9} \sqrt{y} \cdot -\frac{1}{7}\, dy &= \int_{16}^{9} y^{1/2} \cdot -\frac{1}{7}\, dy \\
&= -\frac{1}{7} \int_{16}^{9} y^{1/2}\, dy \\
&= \frac{1}{7} \int_{9}^{16} y^{1/2}\, dy.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\int_{-2}^{-1} \sqrt{2-7x}\, dx &= \frac{1}{7} \int_{9}^{16} y^{1/2}\, dy \\
&= \frac{1}{7} \cdot \frac{2}{3} y^{3/2} \Big|_{9}^{16} \\
&= \frac{2}{21}[16^{3/2} - 9^{3/2}] \\
&= \frac{2}{21}(64 - 27) \\
&= \frac{74}{21}.
\end{aligned}
$$

EXAMPLE 3: Evaluate $\displaystyle\int_0^1 14\sqrt[3]{1+7x}\,dx$.

Solution. Let $y = 1+7x$. Then $dy = 7\,dx$, and $14dx = 2\,dy$. If $x=0$, then $y=1$. If $x=1$, then $y=8$. Hence,

$$\int_0^1 14\sqrt[3]{1+7x}\,dx = \int_1^8 2y^{1/3}\,dy = \frac{2}{4/3}\,y^{4/3}\Big|_1^8 = \frac{3}{2}\,(8^{4/3}-1^{4/3}) = \frac{3}{2}\,(16-1) = \frac{45}{2}.$$

EXAMPLE 4: Evaluate $\displaystyle\int_0^2 \frac{9x^2}{(x^3+1)^{3/2}}\,dx$.

Solution. Let $y = x^3+1$. Then $dy = 3x^2\,dx$, and $9x^2dx = 3\,dy$. If $x=0$, then $y=1$. If $x=2$, then $y=9$. Hence,

$$\begin{aligned}
\int_0^2 \frac{9x^2}{(x^3+1)^{3/2}}\,dx &= \int_1^9 \frac{3}{y^{3/2}}\,dy \\
&= \int_1^9 3y^{-3/2}\,dy \\
&= \frac{3}{-1/2}\,y^{-1/2}\Big|_1^9 \\
&= -6(9^{-1/2}-1) \\
&= -6\left(\frac{1}{3}-1\right) \\
&= 4.
\end{aligned}$$

EXAMPLE 5: Evaluate the integral $\displaystyle\int_4^9 \frac{\sqrt{x}}{(30-x^{\frac{3}{2}})^2}\,dx$.

Solution. Notice that if we let $y = 30-x^{\frac{3}{2}}$, then we have $dy = -\frac{3}{2}x^{\frac{1}{2}}\,dx$ so that $-\frac{2}{3}\,dy = \sqrt{x}\,dx$, which is the numerator of the integrand. Converting the limits of integration, we have $x=4$ implying $y=22$ and $x=9$ implying $y=3$. Thus,

$$
\begin{aligned}
\int_4^9 \frac{\sqrt{x}}{(30 - x^{\frac{3}{2}})^2}\, dx &= \int_{22}^{3} \left(\frac{1}{y^2}\right)\left(-\frac{2}{3}\, dy\right) \\
&= -\frac{2}{3}\int_{22}^{3} y^{-2}\, dy \\
&= -\frac{2}{3}\left(\frac{y^{-1}}{-1}\right)\Bigg|_{22}^{3} \\
&= \left[\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\right] - \left[\left(\frac{2}{3}\right)\left(\frac{1}{22}\right)\right] \\
&= \left(\frac{2}{3}\right)\left[\frac{1}{3} - \frac{1}{22}\right] \\
&= \frac{19}{99}.
\end{aligned}
$$

EXAMPLE 6: Evaluate the integral $\displaystyle\int_0^{\frac{\pi}{4}} \sin^3 2x \cos 2x\, dx$.

Solution. Let $y = \sin 2x$. Differentiating both sides we have, $dy = 2\cos 2x\, dx$ and $\cos 2x\, dx = \dfrac{dy}{2}$. Now, when $x = 0$ it implies that $y = \sin 0 = 0$ and when $x = \dfrac{\pi}{4}$ implies $y = \sin\dfrac{\pi}{2} = 1$. Thus,

$$
\int_0^{\frac{\pi}{4}} \sin^3 2x \cos 2x\, dx = \int_0^{\frac{\pi}{4}} (\sin 2x)^3 \cos 2x\, dx = \int_0^1 y^3 \cdot \frac{dy}{2} = \frac{1}{2}\left(\frac{y^4}{4}\right)\Bigg|_0^1 = \frac{1}{8}.
$$

EXAMPLE 7: Evaluate $\displaystyle\int_{-1}^{0} x^2\, e^{x^3+1}\, dx$.

Solution. Recall that $D_t(e^t) = e^t$, and therefore $\displaystyle\int e^t dt = e^t + c$. In other words, the derivative and the antiderivative of the exponential are both the exponential itself.

Now, let $y = x^3 + 1$. Then $dy = 3x^2\, dx$, so that $x^2 dx = \dfrac{1}{3}\, dy$.
If $x = -1$, then $y = 0$. If $x = 0$, then $y = 1$. Hence,

$$
\int_{-1}^{0} x^2\, e^{x^3+1}\, dx = \int_0^1 \frac{1}{3} e^y\, dy = \frac{1}{3} e^y \Bigg|_0^1 = \frac{1}{3}(e^1 - e^0) = \frac{1}{3}(e - 1) = \frac{e-1}{3}.
$$

If we are dealing with an integral whose lower limit of integration is greater than the upper limit, we can use the property that says

$$\int_a^b f(x)\,dx \;=\; -\int_b^a f(x)\,dx.$$

EXAMPLE 8: Evaluate the definite integral $\displaystyle\int_2^0 2x(1+x^2)^3\,dx$.

Solution. First, note that

$$\int_2^0 2x(1+x^2)^3\,dx = -\int_0^2 2x(1+x^2)^3\,dx$$

Let $u = 1+x^2$ so that $du = 2x\,dx$ which is the other factor. Changing the $x-$limits, we have when $x = 0$, then $u = 1+0^2 = 1$ and when $x = 2$, then $u = 1+2^2 = 5$. Thus,

$$-\int_0^2 2x(1+x^2)^3\,dx = -\int_1^5 u^3\,du = -\left(\frac{u^4}{4}\right)\Bigg|_1^5 = -\frac{5^4}{4} - \frac{1^4}{4} = -\frac{625-1}{4} = -\frac{624}{4} = -156.$$

Solved Examples

Evaluate the following integrals.

EXAMPLE 1: $\displaystyle\int_0^1 2x\sqrt{x^2+1}\,dx$

Solution. Let $u = x^2+1$. This implies $du = 2x\,dx$. Computing for the $u-$ limits: if $x = 0$, then $u = 1$; if $x = 1$, then $u = 2$. Hence,

$$\begin{aligned}\int_0^1 2x\sqrt{x^2+1}\,dx &= \int_1^2 u^{\frac{1}{2}}\,du\\ &= \frac{2u^{\frac{3}{2}}}{3}\Bigg|_1^2\\ &= \frac{2}{3}\left(2^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)\\ &= \frac{2}{3}(2\sqrt{2}-1)\\ &= \frac{4\sqrt{2}-2}{3}.\end{aligned}$$

EXAMPLE 2: $\displaystyle\int_0^1 \frac{8x}{(4x^2+2)^2}\,dx$

Solution. Let $u = 4x^2 + 2$. Hence, $du = 8x\,dx$. Computing for the $u-$ limits: if $x = 0$, then $u = 2$; if $x = 1$, then $u = 6$.
Therefore,

$$\begin{aligned}\int_0^1 \frac{8x}{(4x^2+2)^2}\,dx &= \int_2^6 \frac{1}{u^2}\,du \\ &= \int_2^6 u^{-2}\,du \\ &= -u^{-1}\Big|_2^6 \\ &= -\frac{1}{6} + \frac{1}{2} \\ &= \frac{1}{3}.\end{aligned}$$

EXAMPLE 3: $\displaystyle\int_{\frac{\pi}{3}}^{\pi} \sin^2\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\,dx$

Solution. Let $u = \sin\left(\dfrac{x}{2}\right)$. This implies that $du = \dfrac{1}{2}\cos\left(\dfrac{x}{2}\right)\,dx$ or $2\,du = \cos\left(\dfrac{x}{2}\right)\,dx$.
If $x = \dfrac{\pi}{3}$, then $u = \dfrac{1}{2}$; if $x = \pi$, then $u = 1$. Hence,

$$\begin{aligned}\int_{\frac{\pi}{3}}^{\pi} \sin^2\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\,dx &= \int_{\frac{1}{2}}^{1} 2u^2\,du \\ &= \frac{2u^3}{3}\Bigg|_{\frac{1}{2}}^{1} \\ &= \frac{2}{3}\left(1 - \frac{1}{8}\right) \\ &= \frac{7}{12}.\end{aligned}$$

EXAMPLE 4: $\displaystyle\int_1^4 \frac{1}{\sqrt{x}(\sqrt{x}+2)^3}\,dx$

Solution. Let $u = \sqrt{x} + 2$. Then, $du = \dfrac{1}{2\sqrt{x}}\,dx$ or $2du = \dfrac{1}{\sqrt{x}}\,dx$. Moreover, if $x = 1$, then

$u = 3$ and if $x = 4$, then $u = 4$. Hence,

$$\begin{aligned}
\int_1^4 \frac{1}{\sqrt{x}(\sqrt{x}+2)^3}\,dx &= \int_3^4 \frac{2}{u^3}\,du \\
&= \int_3^4 2u^{-3}\,du \\
&= \left.\frac{2u^{-2}}{-2}\right|_3^4 \\
&= -\left(4^{-2} - 3^{-2}\right) \\
&= \frac{7}{144}.
\end{aligned}$$

EXAMPLE 5: $\displaystyle\int_2^{\sqrt{5}} x^3(x^2-4)^3\,dx$

Solution. Let $u = x^2 - 4$. This implies $du = 2x\,dx$ or $\dfrac{du}{2} = x\,dx$.
If $x = -2$, then $u = 0$; if $x = \sqrt{5}$, then $u = 1$. Note that $x^2 = u + 4$. Therefore,

$$\begin{aligned}
\int_2^{\sqrt{5}} x^3(x^2-4)^3\,dx &= \int_2^{\sqrt{5}} x^2(x^2-4)^3 x\,dx \\
&= \int_0^1 (u+4)(u^3)\left(\frac{1}{2}\right)\,du \\
&= \int_0^1 \frac{1}{2}(u^4+4u^3)\,du \\
&= \frac{1}{2}\left.\left(\frac{u^5}{5}+u^4\right)\right|_0^1 \\
&= \frac{1}{2}\left[\left(\frac{1^5}{5}+1^4\right) - \left(\frac{0^5}{5}+\frac{(0)^4}{5}\right)\right] \\
&= \frac{3}{5}.
\end{aligned}$$

EXAMPLE 6: $\displaystyle\int_{-2}^{1} (x+1)\sqrt{x+3}\,dx$

Solution. Let $u = x + 3$. Hence, $du = dx$. When $x = -2$, then $u = 1$; when $x = 1$, then $u = 4$.

Note that $u-2=x+1$. Therefore,

$$\begin{aligned}
\int_{-2}^{1}(x+1)\sqrt{x+3}\,dx &= \int_{1}^{4}(u-2)u^{\frac{1}{2}}\,du\\
&= \int_{1}^{4}(u^{\frac{3}{2}}-2u^{\frac{1}{2}})\,du\\
&= \left(\frac{2u^{\frac{5}{2}}}{5}-\frac{4u^{\frac{3}{2}}}{3}\right)\Bigg|_{1}^{4}\\
&= \left(\frac{2(4)^{\frac{5}{2}}}{5}-\frac{4(4)^{\frac{3}{2}}}{3}\right)-\left(\frac{2(1)^{\frac{5}{2}}}{5}-\frac{4(1)^{\frac{3}{2}}}{3}\right)\\
&= \frac{46}{15}.
\end{aligned}$$

EXAMPLE 7: $\displaystyle\int_{0}^{\frac{\pi}{6}}\frac{\sin 2x}{\cos^2 2x}\,dx$

Solution. Let $u=\cos 2x$. Hence, $du=-2\sin 2x\,dx$ or $-\dfrac{du}{2}=\sin 2x\,dx$. If $x=0$, then $u=1$ and if $x=\dfrac{\pi}{6}$, then $u=\dfrac{1}{2}$. Therefore,

$$\begin{aligned}
\int_{0}^{\frac{\pi}{6}}\frac{\sin 2x}{\cos^2 2x}\,dx &= \int_{1}^{\frac{1}{2}}-\frac{1}{2u^2}\,du\\
&= \int_{\frac{1}{2}}^{1}\frac{u^{-2}}{2}\,du\\
&= -\frac{u^{-1}}{2}\Bigg|_{\frac{1}{2}}^{1}\\
&= -\frac{1}{2}\left[1^{-1}-\left(\frac{1}{2}\right)^{-1}\right]\\
&= \frac{1}{2}.
\end{aligned}$$

EXAMPLE 8: $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos x\cos(\sin x)\,dx$

Solution. Let $u=\sin x$. Then, $du=\cos x\,dx$. If $x=-\dfrac{\pi}{4}$ and $x=\dfrac{\pi}{4}$, then $u=-\dfrac{\sqrt{2}}{2}$ and

$u = \dfrac{\sqrt{2}}{2}$, respectively. Thus,

$$\begin{aligned}
\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos x \cos(\sin x)\, dx &= \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \cos u\, du \\
&= \sin x \Big|_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \\
&= \sin\left(\frac{\sqrt{2}}{2}\right) - \sin\left(-\frac{\sqrt{2}}{2}\right) \\
&= 2\sin\left(\frac{\sqrt{2}}{2}\right).
\end{aligned}$$

EXAMPLE 9: $\displaystyle\int_0^1 xe^{x^2}\, dx$

Solution. Let $u = x^2$. Then $du = 2x\, dx$ or $\dfrac{du}{2} = x\, dx$. In addition, if $x = 0$, then $u = 0$, and if $x = 1$, then $u = 1$. Therefore,

$$\begin{aligned}
\int_0^1 xe^{x^2}\, dx &= \int_0^1 \frac{1}{2} e^u\, du \\
&= \frac{1}{2} e^u \Big|_0^1 \\
&= \frac{1}{2}(e^1 - e^0) \\
&= \frac{e-1}{2}.
\end{aligned}$$

Supplementary Problems

Evaluate the following definite integrals.

1. $\displaystyle\int_0^1 (x+1)^3\, dx$
2. $\displaystyle\int_0^1 (x-2)^3\, dx$
3. $\displaystyle\int_0^3 x\sqrt{x+1}\, dx$
4. $\displaystyle\int_0^{\pi} 4\sin(x/4)\, dx$
5. $\displaystyle\int_0^{\ln 5} e^x(3-e^x)\, dx$
6. $\displaystyle\int_{-\pi/4}^{\pi} \cos\theta \cos(\sin\theta)\, d\theta$
7. $\displaystyle\int_{\pi^2/36}^{\pi^2/16} \frac{\sec^2\sqrt{x}}{\sqrt{x}}\, dx$

8. $\displaystyle\int_1^2 x^2\sqrt{x^3+1}\,dx$

9. $\displaystyle\int_{-2}^0 3x\sqrt{4-x^2}\,dx\text{xs}$

10. $\displaystyle\int_1^3 \frac{x}{(3x^2-1)^3}\,dx$

11. $\displaystyle\int_{\pi/8}^{\pi/4} 3\csc^2 2x\,dx$

12. $\displaystyle\int_1^{\sqrt{2}} \frac{4x^3}{3+x^4}\,dx$

13. $\displaystyle\int_{-\ln 3}^{\ln 3} \frac{e^x}{e^x+4}\,dx$

14. $\displaystyle\int_{\pi/2}^{\pi} 6\sin x(\cos x+1)^5\,dx$

15. $\displaystyle\int_{\pi^2}^{4\pi^2} \frac{1}{\sqrt{x}}\sin\sqrt{x}\,dx$

16. $\displaystyle\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx$

17. $\displaystyle\int_0^1 \frac{e^x+1}{e^x+x}\,dx$

18. $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2}\,dx$

19. $\displaystyle\int_0^1 x^2(1+2x^3)^3\,dx$

20. $\displaystyle\int_0^1 x(x^2-1)^3\,dx$

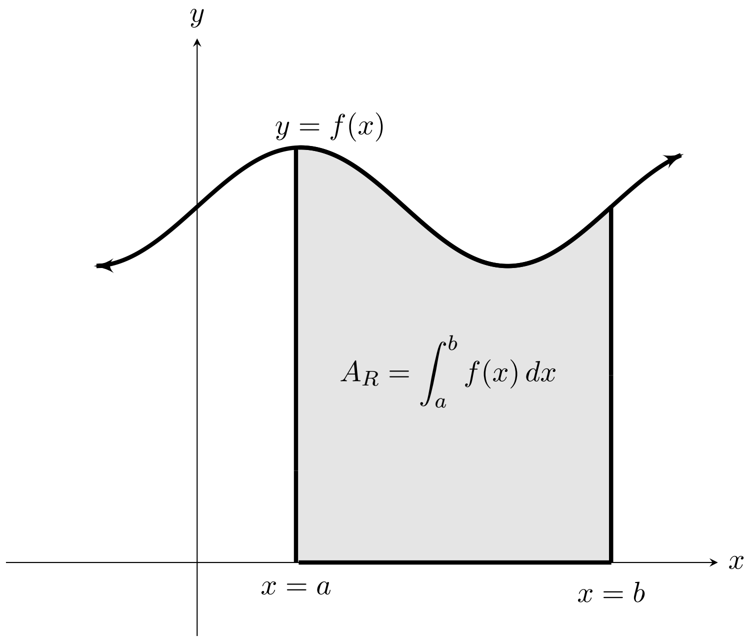
LESSON 18: Application of Definite Integrals in the Computation of Plane Areas

LEARNING OUTCOMES: At the end of the lesson, the learner shall be able to:

1. Compute the area of a plane region using the definite integral; and
2. Solve problems involving areas of plane regions.

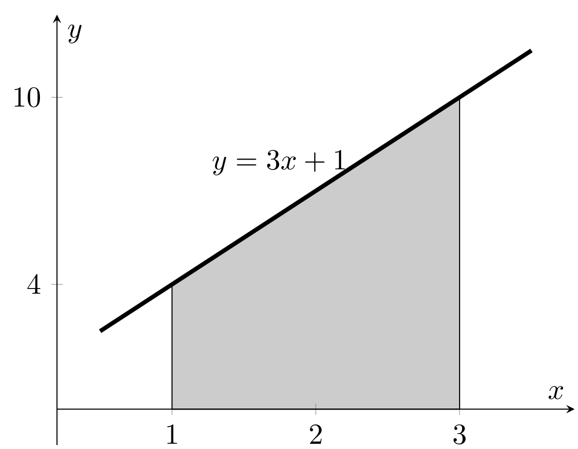
TOPIC 18.1: Areas of Plane Regions Using Definite Integrals

Consider a continuous function f. If the graph of $y = f(x)$ over the interval $[a, b]$ lies entirely above the x-axis, then $\int_a^b f(x)\,dx$ gives the area of the region bounded by the curves $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$. This is illustrated in the figure below:



EXAMPLE 1: Find the area of the plane region bounded by $y = 3x + 1$, $x = 1$, $x = 3$, and the x-axis.

Solution. The graph of the plane region is shown in the figure below.



This plane region is clearly in the first quadrant of the Cartesian plane (see figure above) and hence immediately from the previous discussion, we obtain

$$\begin{aligned} A_R &= \int_a^b f(x)\,dx \\ &= \int_1^3 (3x+1)\,dx. \end{aligned}$$

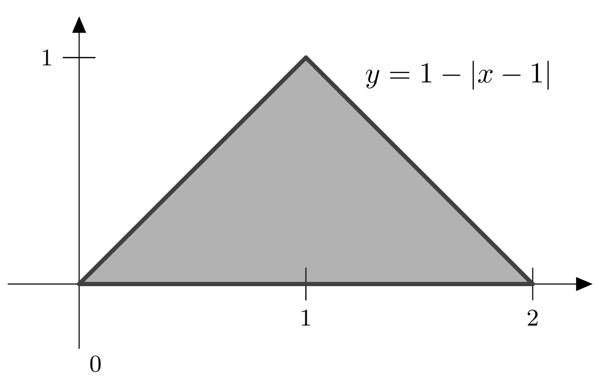
Evaluating the integral and applying the Fundamental Theorem of Calculus, we get

$$\begin{aligned} A_R &= \int_1^3 (3x+1)\,dx \\ &= \left(\frac{3x^2}{2}+x\right)\Bigg|_1^3 \\ &= \left(\frac{27}{3}+3\right)-\left(\frac{3}{2}+1\right) \\ &= 14 \text{ square units.} \end{aligned}$$

Recall that in the previous discussion, we evaluated $\int_1^3 (3x+1)\,dx$ and got the value 14. As we previously mentioned, this is the reason why we use the same symbol since antiderivatives are intimately related to finding the areas below curves.

EXAMPLE 2: Find the area of the plane region bounded above by $y = 1 - |x-1|$ and below by the x-axis.

Solution. The graph of the plane region is shown below.



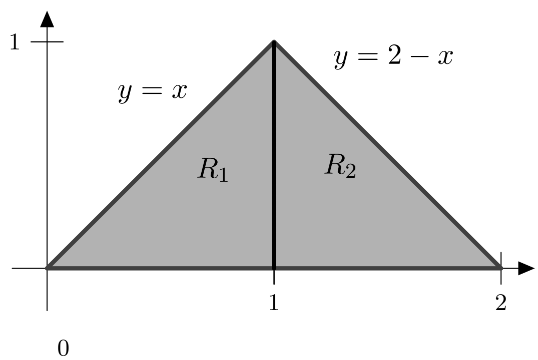
Observe that the line from the point $(0, 0)$ to $(1, 1)$ is given by

$$y = 1 - [-(x - 1)] = x$$

and the line from the point $(1, 1)$ to $(0, 2)$ is given by

$$y = 1 - (x - 1) = 2 - x.$$

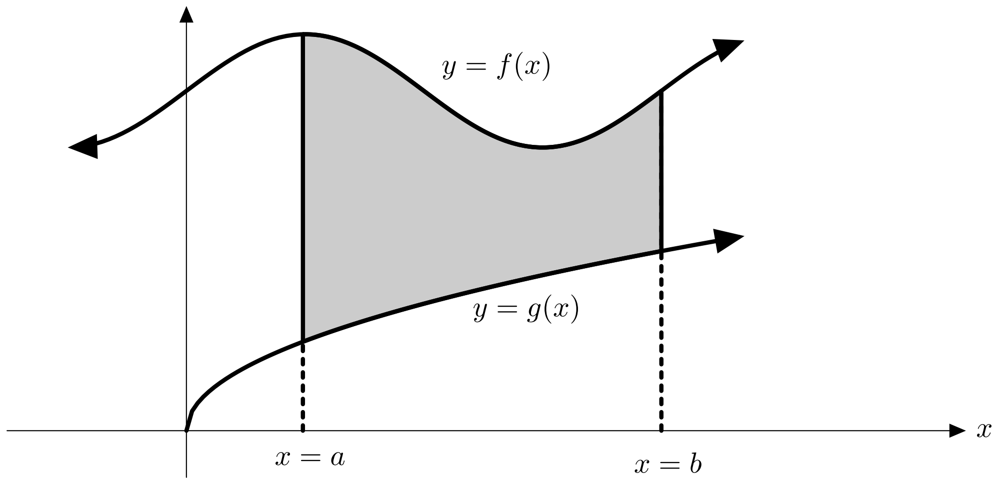
Clearly, we have two subregions here, Region 1 (R_1) which is bounded above by $y = x$, and Region 2 (R_2) which is bounded above by $y = 2 - x$.



Hence, the area of the entire plane region is given by

$$\begin{aligned} \mathrm{A} &= \mathrm{A}_{R_1} + \mathrm{A}_{R_2} \\ &= \int_0^1 x\,dx + \int_1^2 (2 - x)\,dx \\ &= \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2 \\ &= \frac{1^2}{2} - \frac{0^2}{2} + \left[2(2) - \frac{2^2}{2}\right] - \left[2(1) - \frac{1^2}{2}\right] \\ &= 1 \text{ square unit.} \end{aligned}$$

We now generalize the problem from finding the area of the region bounded by above by a curve and below by the x-axis to finding the area of a plane region bounded by several curves (such as the one shown below).



The height or distance between two curves at x is

$$h = (y\text{-coordinate of the upper curve}) - (y\text{-coordinate of the lower curve}).$$

Now, if $y = f(x)$ is the upper curve and $y = g(x)$ is the lower curve, then

$$h = f(x) - g(x).$$

Area between two curves

If f and g are continuous functions on the interval $[a, b]$ and $f(x) \geq g(x)$ for all $x \in [a, b]$, then the area of the region R bounded above by $y = f(x)$, below by $y = g(x)$, and the vertical lines $x = a$ and $x = b$ is

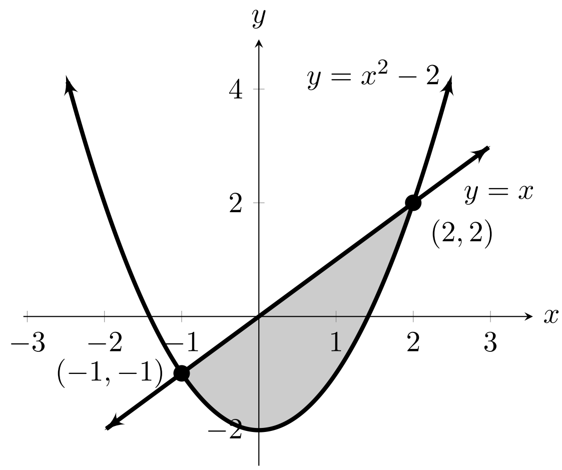
$$A_R = \int_a^b [f(x) - g(x)]\, dx.$$

EXAMPLE 3: Find the area of the plane region bounded by the curves $y = x^2 - 2$ and $y = x$.

Solution. We start by finding the points of intersection of the two curves. Substituting $y = x$ into $y = x^2 - 2$, we obtain

$$\begin{aligned} x &= x^2 - 2 \\ \implies \quad 0 &= x^2 - x - 2 \\ \implies \quad 0 &= (x-2)(x+1). \end{aligned}$$

Thus, we have $x = 2$ or $x = -1$. When $x = 2$, $y = 2$ while when $x = -1$, $y = -1$. Hence, we have the points of intersection $(2, 2)$ and $(-1, -1)$. The graphs of the two curves, along with their points of intersection, are shown below.



The function $f(x) - g(x)$ will be $x - (x^2 - 2)$. Our interval is $I = [-1, 2]$ and so $a = -1$ and $b = 2$. Therefore, the area of the plane region is

$$\begin{aligned}
A_R &= \int_a^b [f(x) - g(x)]\, dx \\
&= \int_{-1}^{2} [x - (x^2 - 2)]\, dx \\
&= \left[\frac{x^2}{2} - \frac{x^3}{3} + 2x\right]_{-1}^{2} \\
&= \left[\frac{2^2}{2} - \frac{2^3}{3} + 2(2)\right] - \left[\frac{-1^2}{2} - \frac{(-1)^3}{3} + 2(-1)\right] \\
&= \left[2 - \frac{8}{3} + 4\right] - \left[\frac{1}{2} - \frac{-1}{3} - 2\right] \\
&= \frac{9}{2} \text{ square units.}
\end{aligned}$$

EXAMPLE 4: Find the area of the plane region bounded by the curves $y = x^2$, $x = -1$, $x = 2$, and $y = -1$.

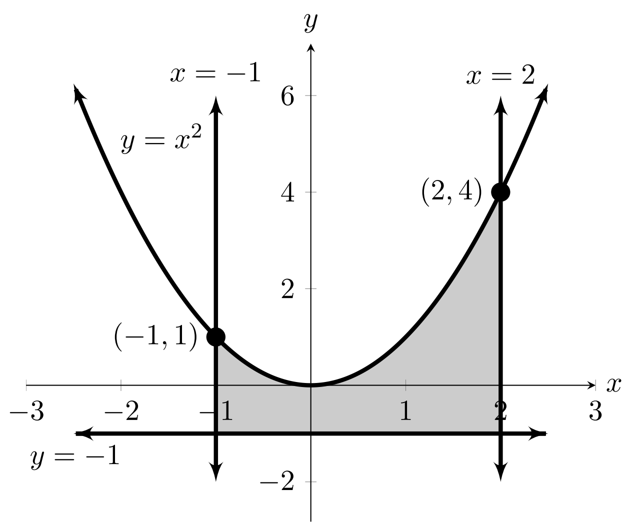
Solution. First, we find the points of intersection of the curves. With respect to the curves $y = x^2$ and $x = -1$, we have

$$y = (-1)^2 = 1.$$

Hence, these curves intersect at the point $(-1, 1)$. For the curves $y = x^2$ and $x = 2$, we have

$$y = 2^2 = 4.$$

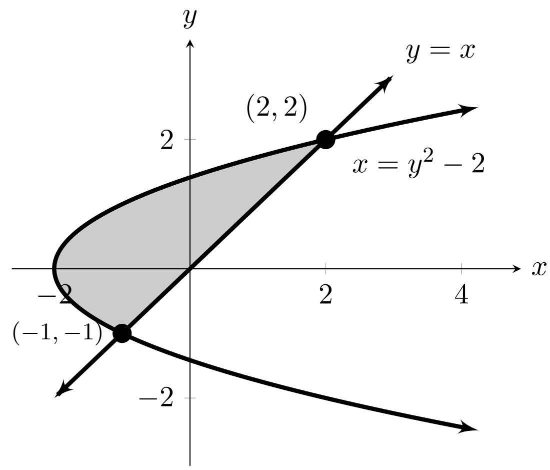
Thus, they intersect at the point $(2, 4)$. Now, for the curves $x = -1$ and $y = -1$, they intersect at $(-1, -1)$. While for $x = 2$ and $y = -1$, they intersect at $(2, -1)$. The graphs of these curves are shown below and the required region is shaded.



The function $f(x) - g(x)$ will be $x^2 - (-1) = x^2 + 1$. Our interval is $I = [-1, 2]$ and so $a = -1$ and $b = 2$. Therefore, the area of the plane region is

$$\begin{aligned}
A_R &= \int_a^b [f(x) - g(x)]\,dx \\
&= \int_{-1}^{2} (x^2 + 1)\,dx \\
&= \left(\frac{x^3}{3} + x\right)\Bigg|_{-1}^{2} \\
&= \left[\frac{2^3}{3} + 2\right] - \left[\frac{-1^3}{3} - 1\right] \\
&= \left[\frac{8}{3} + 2\right] - \left[\frac{-1}{3} - 1\right] \\
&= 6 \text{ square units.}
\end{aligned}$$

In the formula for the area of a plane region, the upper curve $y = f(x)$ is always above the lower curve $y = g(x)$ on $[a, b]$. Hence, the height of any vertical line on the region will always have the same length that is given by the function $f(x) - g(x)$. What if this is not true anymore? Consider the figure:



To the left of $x = -1$, the upper curve is the part of the parabola located above the x-axis while the lower curve is the part of the parabola below the x-axis. On the other hand, to the right of $x = 1$, the upper curve is the parabola while the lower curve is the line $y = x$. Hence, in this case, we need to split the region into subregions in such a way that in each subregion the difference of the upper and lower curves is the same throughout the subregion.

EXAMPLE 5: Set up the definite integral for the area of the region bounded by the parabola $x = y^2 - 2$ and the the line $y = x$. (Refer to the above figure.)

Solution. First, we find the points of intersection of the parabola and the line. Substituting $x = y$ in $x = y^2 - 2$, we have

$$\begin{aligned} y &= y^2 - 2 \\ \implies \quad 0 &= y^2 - y - 2 \\ \implies \quad 0 &= (y-2)(y+1). \end{aligned}$$

Hence, we have $y = 2$ or $y = -1$. If $y = 2$, $x = 2$, and $x = -1$ when $y = -1$. Thus, we have two points of intersection $(2, 2)$ and $(-1, -1)$.

Note that the equation of the parabola $x = y^2 - 2$ gives us two expressions for y: $y = \sqrt{x+2}$ (points on the parabola located above the x-axis) and $y = -\sqrt{x+2}$ (points on the parabola located below the x-axis).

We first set up the integral of subregion R_1 (part of the region located on the left of $x = -1$). The upper curve here as we have observed earlier is $y = \sqrt{x+2}$ and the lower curve is $y = -\sqrt{x+2}$.

Thus, the difference of the upper and lower curve is given by

$$\textit{upper curve} - \textit{lower curve} = \sqrt{x+2} - [-\sqrt{x+2}] = 2\sqrt{x+2}.$$

The interval I is $[-2,-1]$. Hence, the area of R_1 is

$$A_{R_1} = \int_{-2}^{-1} 2\sqrt{x+2}\,dx.$$

For subregion R_2 (part of the region located to the right of $(-1,-1)$, the upper curve is $y=\sqrt{x+2}$ and the lower curve is $y=x$. So, we have

$$\textit{upper curve} - \textit{lower curve} = \sqrt{x+2} - x.$$

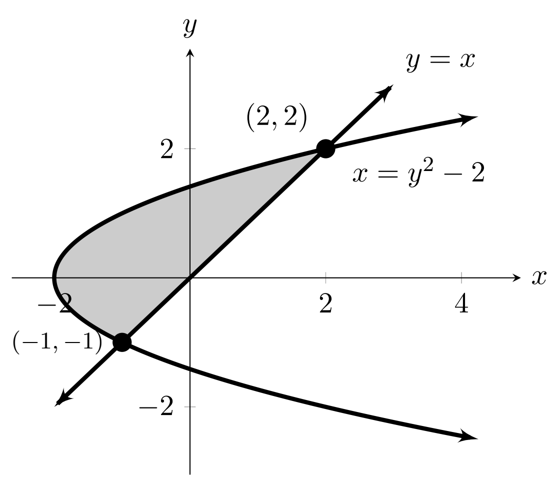
The interval I is $[-1,2]$. Hence, the area of R_2 is

$$A_{R_2} = \int_{-1}^{2} [\sqrt{x+2} - x]\,dx.$$

Finally, to get the area of the region R, we simply add the areas of subregions R_1 and R_2, i.e.,

$$\begin{aligned} A_R &= A_{R_1} + A_{R_2} \\ &= \int_{-2}^{-1} 2\sqrt{x+2}\,dx + \int_{-1}^{2} [\sqrt{x+2} - x]\,dx. \end{aligned}$$

Consider again the figure below.



Observe that if we use horizontal rectangles, the length of a rectangle at an arbitrary point y would be the same throughout the region. Indeed, using horizontal rectangles is an alternative method of solving the problem.

If u and v are continuous functions on the interval $[c, d]$ and $v(y) \geq u(y)$ for all $y \in [c, d]$, then the area of the region R bounded on the left by $x = u(y)$, on the right by $x = v(y)$, and the horizontal lines $y = c$ and $y = d$ is

$$A_R = \int_c^d [v(y) - u(y)]\, dy.$$

Observe that the function $v(y) - u(y)$ is always given by

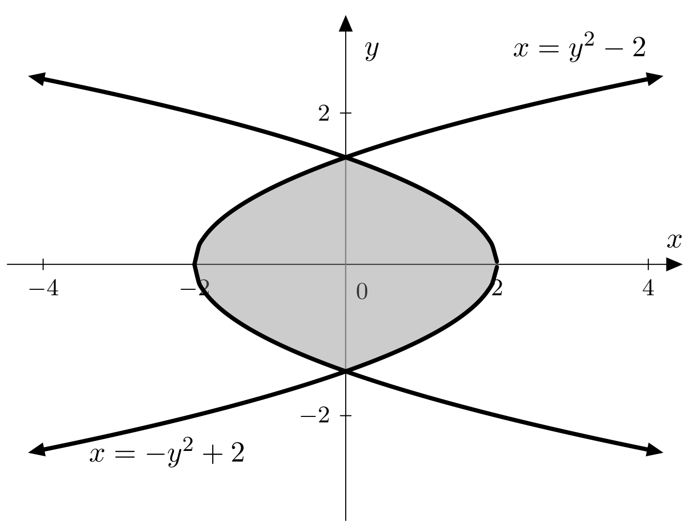
right curve – left curve.

EXAMPLE 6: Set up the integral of the area of the region bounded by the curves $x = -y^2 + 2$ and $x = y^2 - 2$.

Solution. We first find the points of intersection of the two curves. We have

$$\begin{aligned} -y^2 + 2 &= y^2 - 2 \\ \Longrightarrow \quad 0 &= 2y^2 - 4 \\ \Longrightarrow \quad 0 &= 2(y^2 - 2) \\ \Longrightarrow \quad 0 &= 2(y - \sqrt{2})(y + \sqrt{2}). \end{aligned}$$

We get $y = -\sqrt{2}$ and $y = \sqrt{2}$. Using the curve $x = y^2 - 2$, we have the points of intersection: $(0, -\sqrt{2})$ and $(0, \sqrt{2})$. The graphs of the two curves are shown below.



Let us now determine the height of a vertical rectangle. Observe that to the left the y-axis the upper curve is $x = y^2 - 2$ while the upper curve to the right of the y-axis is $x = -y^2 + 2$. Hence, we have to split the region into two subregions to get the area of the shaded region.

However, notice that if we use a horizontal rectangle, we have the same curve on the right and the same curve on the left. The function $v(y) - u(y)$ is given by

$$\textit{right curve} - \textit{left curve} = (y^2 - 2) - (-y^2 + 2).$$

Our interval $I = [-\sqrt{2}, \sqrt{2}]$. Therefore, the area of the plane region is given by the integral

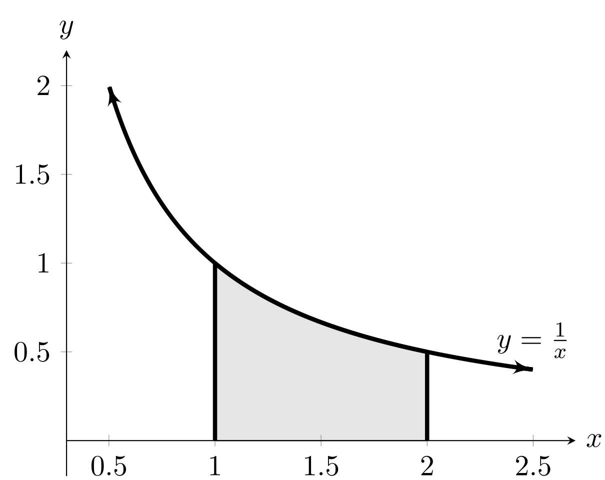
$$A_R = \int_c^d [v(y) - u(y)]\, dy = \int_{-\sqrt{2}}^{\sqrt{2}} [(y^2 - 2) - (-y^2 + 2)]\, dy = \int_{-\sqrt{2}}^{\sqrt{2}} [2y^2 - 4]\, dy.$$

Solved Examples

Find the area of the region enclosed by the given curves.

EXAMPLE 1: $y = \dfrac{1}{x}$, $y = 0$, $x = 1$, $x = 2$

Solution. The region bounded by the curves is shown below.

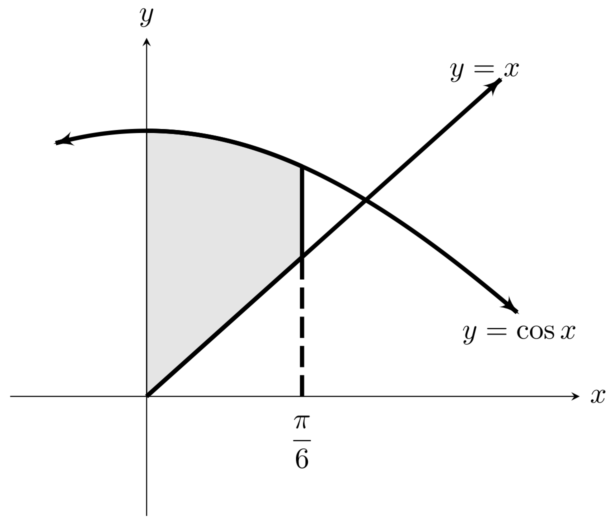


Hence, the area bounded by the region is

$$\int_1^2 \frac{1}{x}\, dx = \ln x \Big|_1^2 = \ln 2 - \ln 1 = \ln 2.$$

EXAMPLE 2: $y = x$, $y = \cos x$, $x = 0$, $x = \dfrac{\pi}{6}$

Solution. The graph of the curves is shown below.



Therefore, the area is

$$\begin{aligned}\int_0^{\pi/6} (\cos x - x)\, dx &= \left(\sin x - \frac{x^2}{2}\right)\Big|_0^{\pi/6} \\ &= \left[\sin\left(\frac{\pi}{6}\right) - \frac{\pi^2}{72}\right] - [\sin 0 - 0] \\ &= \frac{1}{2} - \frac{\pi^2}{76}.\end{aligned}$$

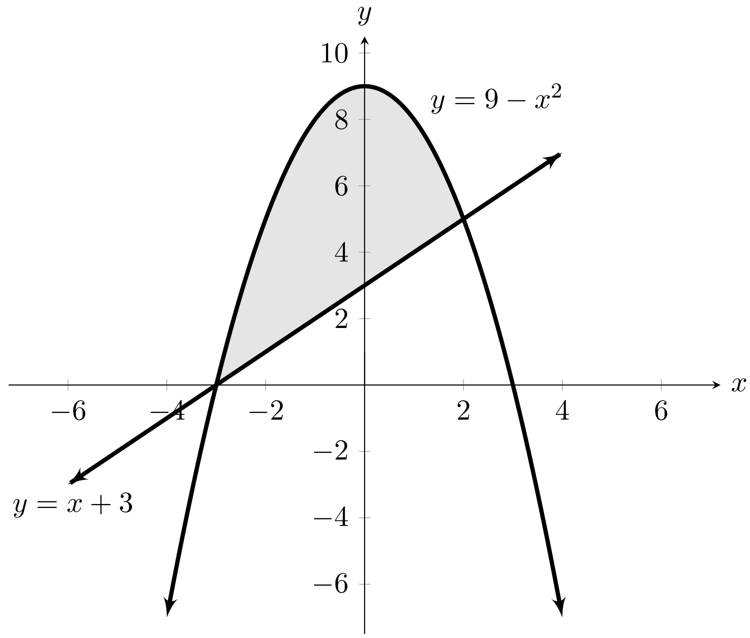
EXAMPLE 3: $y = 9 - x^2$, $y = x + 3$

Solution. We first solve the intersection points of the two curves.

$$\begin{aligned}9 - x^2 &= x + 3 \\ x^2 + x - 6 &= 0 \\ (x+3)(x-2) &= 0 \\ x = -3 &\text{ or } x = 2\end{aligned}$$

The intersection points are $(-3, 0)$ and $(2, 5)$.

The graph of the curves is shown and the shaded portion is the region.



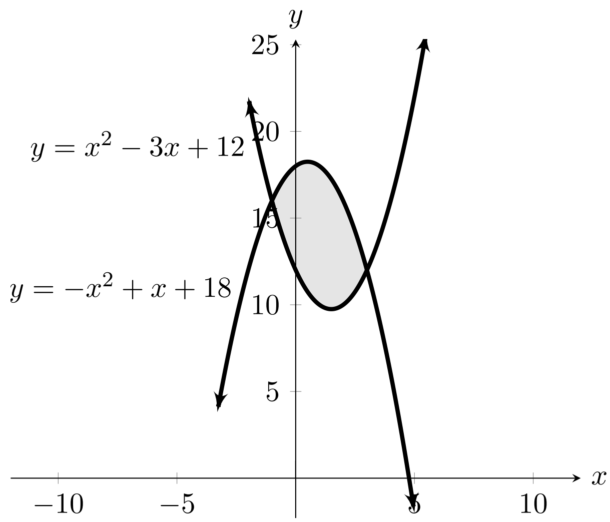
Therefore, the area is

$$\begin{aligned}\int_{-3}^{2} [(9 - x^2) - (x + 3)]\, dx &= \int_{-3}^{2} (-x^2 - x + 6)\, dx \\ &= \left(-\frac{x^3}{3} - \frac{x^2}{2} + 6x\right)\Bigg|_{-3}^{2} \\ &= \left(-\frac{2^3}{3} - \frac{2^2}{2} + 6(2)\right) - \left(-\frac{(-3)^3}{3} - \frac{(-3)^2}{2} + 6(-3)\right) \\ &= \frac{125}{6}.\end{aligned}$$

Use vertical rectangles to set up the definite integral for the area of the region bounded by the given curves.

EXAMPLE 4: $y = x^2 - 3x + 12$, $y = -x^2 + x + 18$

Solution. Note that the points of intersection of the curves are $(-1, 16)$ and $(3, 12)$. The graph of the curve is given.



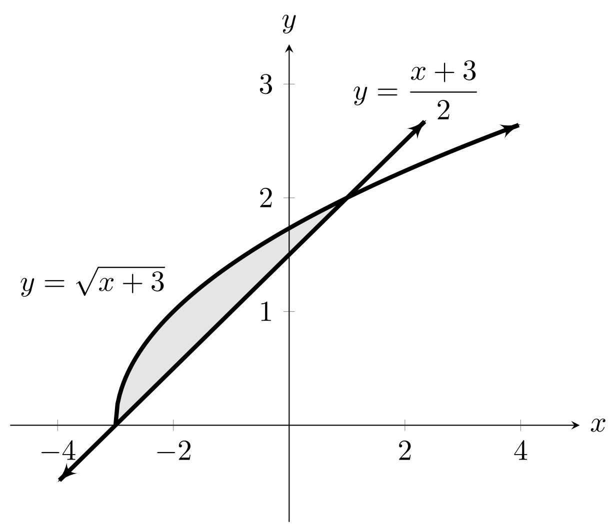
Setting up the area, we have

$$\int_{-3}^{1} [(-x^2 + x + 18) - (x^2 - 3x + 12)]\, dx = \int_{-3}^{1} (-2x^2 + 4x + 6)\, dx.$$

EXAMPLE 5: $y = \sqrt{x+3}$, $y = \dfrac{x+3}{2}$

Solution. Note that the intersections of the curves are at $x = -3$ and $x = 1$.

The graph of the curves is shown.



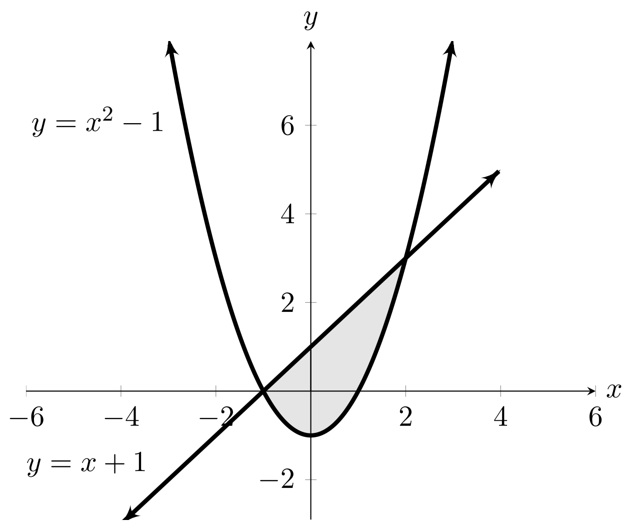
Hence, the area is

$$\int_{-3}^{1} \left(\sqrt{x+3} - \frac{x+2}{2} \right) dx.$$

EXAMPLE 6: $y = x^2 - 1$, $y = x + 1$

Solution. It is easy to show that the two curves intersect at $x = -1$ and $x = 2$.

The graph of the curves is shown below.

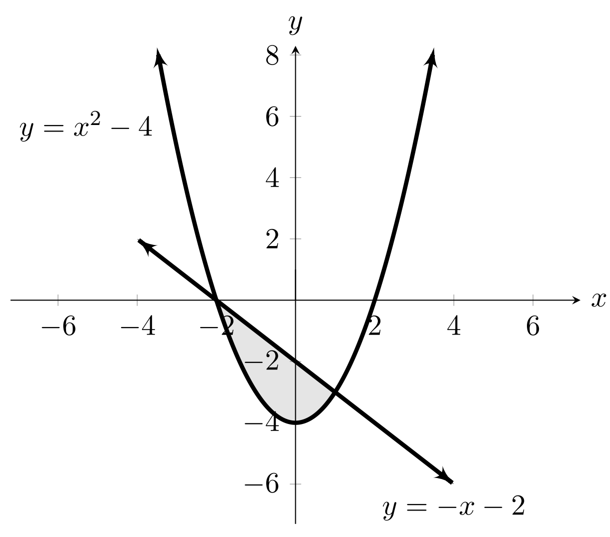


Hence, the area is

$$\int_{-1}^{2} (x+1) - (x^2 - 1)\,dx = \int_{-1}^{2} (-x^2 + x + 2)\,dx.$$

EXAMPLE 7: $y = x^2 - 4$, $y = -x - 2$

Solution. The graph of the curve is shown below.

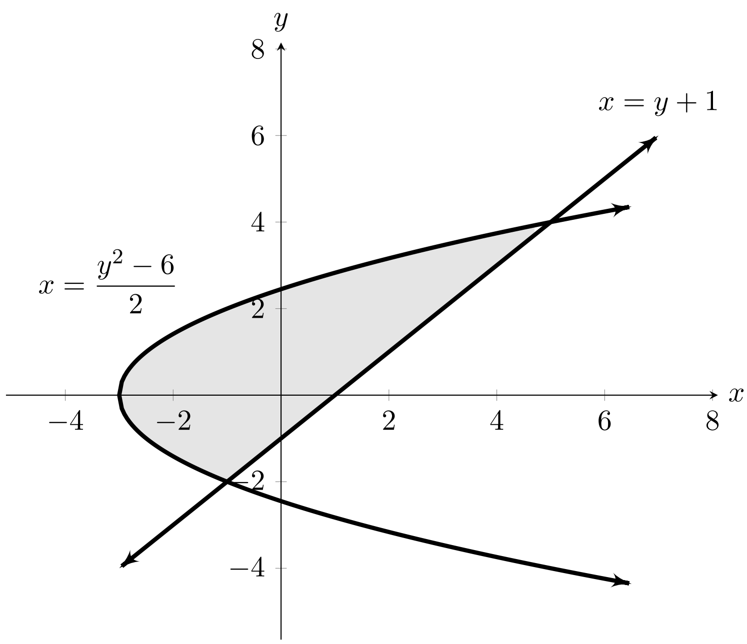


Note that curves intersect at $x=-2$ and $x=1$. Hence, the area is

$$\int_{-2}^{1}[(-x-2)-(x^2-4)]\,dx = \int_{-2}^{1}(-x^2-x+2)\,dx.$$

EXAMPLE 8: Using horizontal rectangles, set up the integral of the area of the region bounded by the curves $x=y+1$ and $x=\dfrac{y^2-6}{2}$.

Solution. The graph of the curves is given below. The curves intersect at $y=-2$ and $y=4$.



Therefore, the area is

$$\int_{-2}^{4}\left[(y+1)-\left(\frac{y^2-6}{2}\right)\right]\,dy.$$

Supplementary Problems

1. Find the area of the region bounded by the graphs of:

   (a) $y = x^2$, $y = \sqrt{x}$, $x = \frac{1}{4}$, $x = 1$

   (b) $y = x^3 - 4x$, $y = 0$, $x = 0$, $x = 2$

   (c) $y = \cos 2x$, $y = 0$, $x = \frac{\pi}{4}$, $x = \frac{\pi}{2}$

2. Set up the definite integral of the region (using the most convenient rectangles) bounded by the graphs of:

   (a) $x^2 = y$, $x = y - 2$

   (b) $y = e^x$, $y = e^{2x}$, $x = 0$, $x = \ln 2$

   (c) $x = \frac{1}{y}$, $x = 0$, $y = 1$, $y = e$

   (d) $y = 5x - x^2$, $y = x$

   (e) $y = \sqrt{x + 2}$, $y = x + 2$, $x = 2$, $x = 4$

   (f) $y = x^2$, $y = \sqrt{9 - x^2}$

   (g) $x = y^2 - 2y$, $x = 4 - y^2$

TOPIC 18.2: Application of Definite Integrals: Word Problems

Parcels of land are shaped in the form of regular polygons – usually triangles and quadrilaterals. However, there are possibilities that one can acquire a piece of land with an irregular shape. This can happen in places where the property being acquired is near a river. Rivercurrents normally erode the soil, changing the shape of the riverbank. Sometimes, land is divided irregularly resulting in irregular shapes of the land parcels.

WORD PROBLEMS

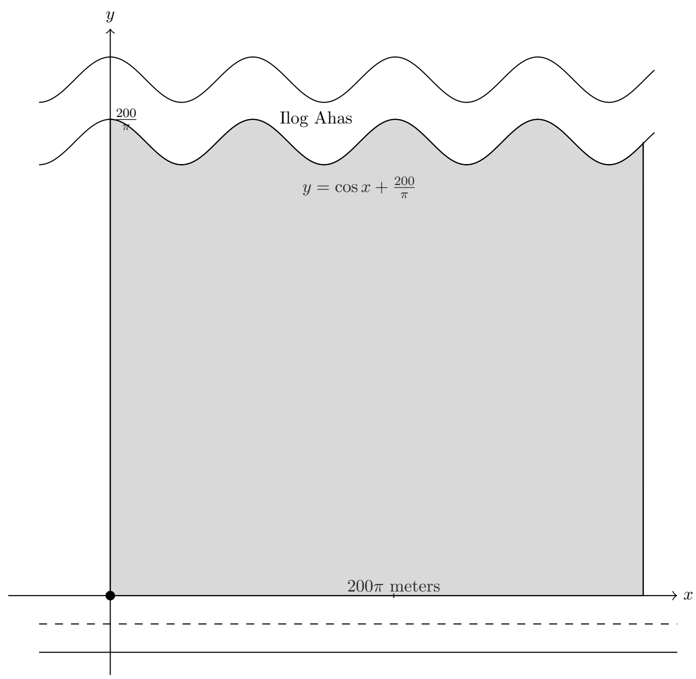
EXAMPLE 1: Juan wants to acquire a lot 200π meters wide and with length bounded from the road side to the banks of "Ilog Ahas", which follows the equation

$$y = \cos x + \frac{200}{\pi}.$$

(Refer to the figure.)

a. Find the area of the lot.

b. If the price per square meter is ₱500, how much is the cost of land?



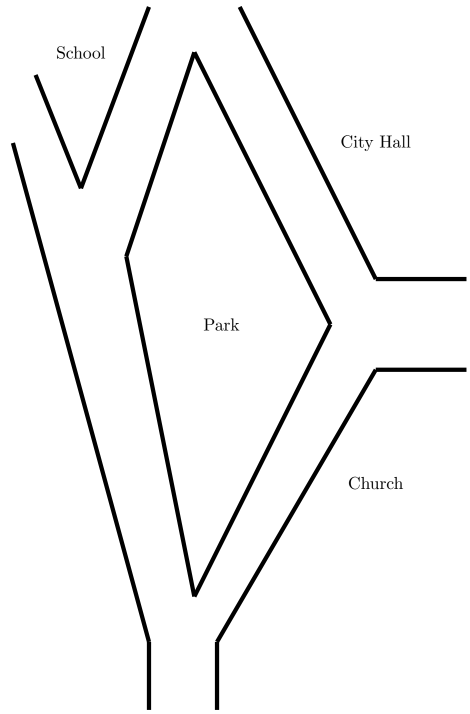
Solution.

a. Suppose we place the x-axis along the side of the road and the y-axis on one side of the lot, as shown. Note that the y-axis is placed such that it runs along the farthest side of "Ilog Ahas". We can now apply definite integrals to find the area of the region. (Refer to the figure.)

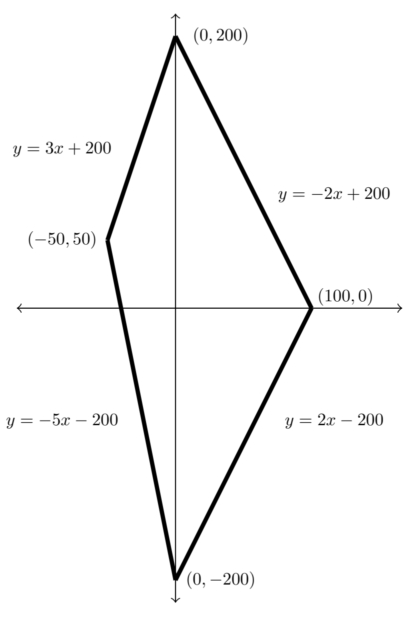
$$\begin{aligned} A &= \int_0^{200\pi} \left(\cos x + \frac{200}{\pi}\right) dx \\ &= \left(\sin x + \frac{200}{\pi}\right)\Big|_0^{200\pi} \\ &= \sin(200\pi) + \frac{200}{\pi}(200\pi) - \sin 0 - \frac{200}{\pi}(0) \\ &= 40,000 \text{ square meters.} \end{aligned}$$

b. The price of the lot is $500(40,000) = ₱20,000,000$.

EXAMPLE 2: Consider the figure on the right which shows the shape of a park in a certain city. The Mayor of the city asked the city engineer to cover the entire park with frog grass that costs ₱150 per square foot. Determine how much budget the Mayor should allocate to cover the entire park with frog grass.



Solution. To determine the area, the city engineer first places the $x-$axis and $y-$axis accordingly, as shown. The points of the park's vertices are then determined. He discovered that the lines are $y = -2x + 200$, $y = 2x - 200$, $y = 3x + 200$ and $y = -5x - 200$. (Refer to the figure.)



Using vertical rectangles, the city engineer has to split the region into two subregions. Subregion R_1 is the one to the left of the y-axis whose upper curve is $y = 3x + 200$ and $y = -5x - 200$ as the lower curve. Subregion R_2 is the one to the right of the y-axis whose upper curve is $y = -2x + 200$ and $y = 2x - 200$ as the lower curve. The length of the vertical rectangles on R_1 is

$$3x + 200 - (-5x - 200) = 8x + 400$$

while the height of the vertical rectangles on R_2 is

$$-2x + 200 - (2x - 200) = -4x + 400.$$

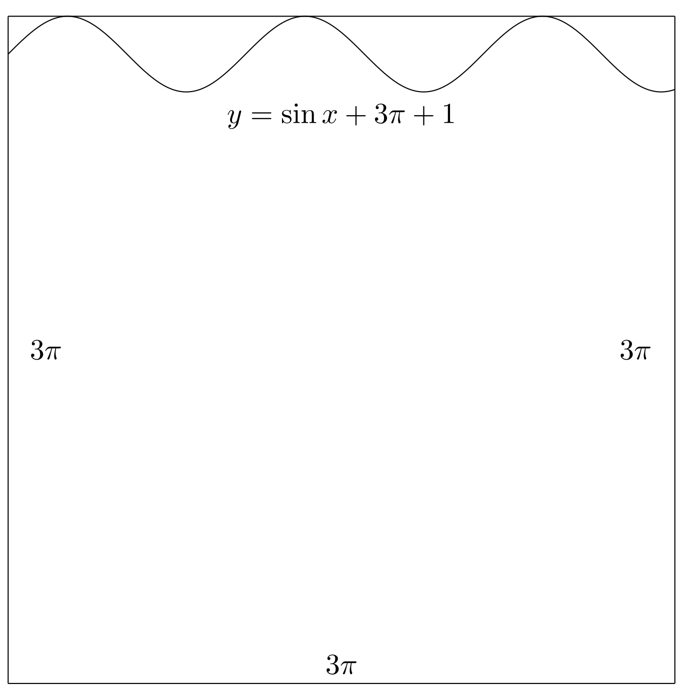
Hence, the area of the region will be

$$\begin{aligned} A_R = A_{R_1} + A_{R_2} &= \int_{-50}^{0} (8x+400)\,dx + \int_{0}^{100} (-4x+400)\,dx \\ &= \left.(4x^2+400x)\right|_{-50}^{0} + \left.(-2x^2+400x)\right|_{0}^{100} \\ &= 30,000 \text{ square feet.} \end{aligned}$$

Therefore, the cost of covering the entire park with frog grass that costs ₱ 150 per square foot is ₱ $150(30,000) =$ ₱ $4,500,000$.

Solved Examples

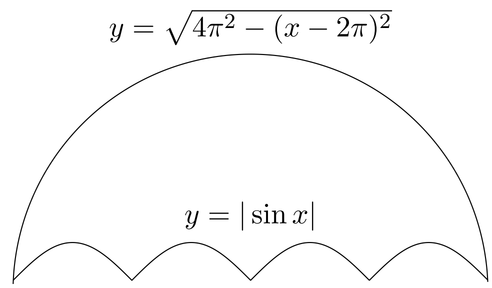
EXAMPLE 1: Jake wants new carpeting for his bed room. However, he wants one of the sides of the carpet to follow the graph of $y = \sin x + 3\pi + 1$ as shown. If the dimension of his bedroom is $3\pi \times 3\pi$ meters, how much carpeting does he need to cover his bed room?



Solution. The area of the carpet is given by

$$
\begin{aligned}
A &= \int_0^{3\pi} (\sin x + 3\pi + 1)\, dx \\
&= (\cos x + 3\pi x + x)\Big|_0^{3\pi} \\
&= (9\pi^2 + 3\pi - 2) \text{ square meter of carpet.}
\end{aligned}
$$

EXAMPLE 2: Camille, the best architect in her city, wants to build a special type of room for the mayor. The floor plan is illustrated below. Using definite integral, set up the floor area of the room.



Solution. The equation for the semicircle is $y = \sqrt{4\pi^2 - (x - 2\pi)^2}$ and for the base, $y = |\sin x|$. Note that the radius of the semicircle is 4π. Hence, the area of the region is

$$
A = \int_0^{4\pi} \left(\sqrt{4\pi^2 - (x - 2)^2} - |\sin x| \right) dx.
$$

Supplementary Problems

1. Raiza has a quadrilateral vegetable garden that is enclosed by the x and $y-$axes, and equations $y = 10 - x$ and $y = x + 2$. She wants to fertilize the entire garden. If one bag of fertilizer can cover $17m^2$. How many bags of fertilizer does she need?

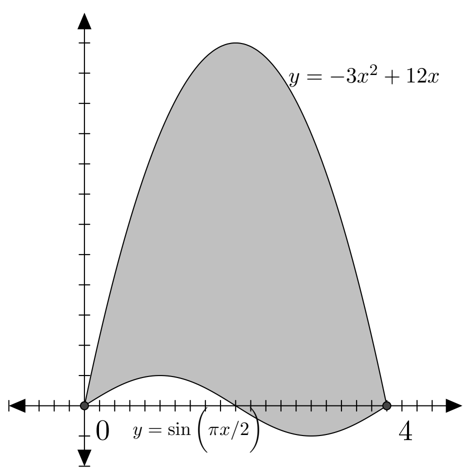
2. A certain dining room can be described by the region bounded by the y axis, x axis and the lines $y = 25 - 5x$ and $y = \frac{1}{2}x + 3$. The dining room has to be tiled by linoleum, which costs ₱100.00/m^2. Find the cost of linoleum needed to cover the dining room.

CHAPTER 3 REVIEW

I. Evaluate the following integrals

1. $\int \left(3x^2 - x + 1\right) dx$
2. $\int \frac{e^{3x} + 4}{e^x} dx$
3. $\int \frac{32}{16 + 16x^2} dx$
4. $\int \frac{dy}{3y - 2}$
5. $\int_0^3 (2x + 4)\, dx$
6. $\int_{-1}^0 (3x + 1)\left(3x^2 + 2x - 6\right) dx$

II. Set up the integral of the area of the region in the figure. $y = -3x^2 + 12x$ and $y = \sin\left(\frac{\pi x}{2}\right)$. Use vertical rectangles.



III. Find the particular solution to the differential equation $\sec x \tan x = \frac{y^1 - 1}{y} dy$ where $y = 0$ when $x = 0$.

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