Krusell and Smith (1997) with endogenous labor supply

Recursive formulation

We consider a continuum of unit measure of households who consumes, saves in two assets (capital and bond), and supplies labor, solving the following problem:

$$
\begin{aligned}
V(\omega, z; S) = \max_{c,n,a',b'} & \frac{c^{1-\sigma}}{1-\sigma} - \frac{\eta}{1+\frac{1}{\chi}} n^{1+\frac{1}{\chi}} + \beta \mathbb{E} V(z', \omega'; S') \\
\text{s.t. } & c + a' + q^b(S) b' = \omega + z w(S) n \\
& \omega' = a'(1 + r(S')) + b' \\
& a' \geq 0, \quad b' \geq \underline{b} \\
& S' = \Gamma_S(S) \quad \text{(Aggregate law of motion)} \\
& z' \sim \pi(z'|z)
\end{aligned}
$$

where c is consumption; ω is the capital income and stock in the beginning of a period; z is idiosyncratic labor productivity, n is endogenously determined labor supply, and w is wage to be competitively determined at the factor input market, which thereby indicates that $zw(S)n$ is the labor income; a is the capital stock that earns capital rent $r = r(S)$ in each period, where the rent is competitively determined in the factor input market; b is the risk-free bond holding of which the price is $q = q(S)$. Apostrophe indicates future allocation. σ is risk-aversion parameter; χ is the Frisch labor elasticity; η is the labor disutility parameter; β is the discount factor. $\underline{b} \leq 0$ is the borrowing limit for future bond holding, and future capital stock is also bound by zero borrowing limit.

S is the aggregate state defined as follows:

$$S := \{\Phi, A\}$$

where Φ is the joint distribution of the individual states; A is aggregate productivity. The stochastic processes for the aggregate productivity and the idiosyncratic productivity are as follows:

$$log(A') = \rho_A log(A) + \sigma_A \epsilon \quad \epsilon \sim_{iid} N(0,1)$$
$$log(z') = \rho_z log(z) + \sigma_z \epsilon \quad \epsilon \sim_{iid} N(0,1)$$

The production sector is as follows:

$$\max_{K,L} AK^{\alpha}L^{1-\alpha} - w(S)L - (r(S)+\delta)K$$

where K is capital factor demand and L is the labor factor demand. $\delta > 0$ is the capital depreciation rate.

Capital rent $r(S)$, wage $w(S)$, and the bond price $q^b(S)$ are determined at the competitive market:

$$[r]: \quad \int a'(\omega, z; S)d\Phi(S) = K'(S)$$
$$[w]: \quad \int zl(\omega, z; S)d\Phi(S) = L(S)$$
$$[q]: \quad \int b'(\omega, z; S)d\Phi(S) = 0$$

where we assume the aggregate net bond supply is zero as in Krusell and Smith (1997).

Optimality conditions

The Largrangian is as follows:

$$
\begin{aligned}
\mathcal{L} = & \frac{c^{1-\sigma}}{1-\sigma} - \frac{\eta}{1+\frac{1}{\chi}} n^{1+\frac{1}{\chi}} + \beta \mathbb{E} V(z', a'(1+r(S')) + b'; S') \\
& + \mu \left(\omega + zw(S)n - c - a' - q^b(S)b' \right) \\
& + \lambda a' \\
& + \phi(b' - \underline{b})
\end{aligned}
$$

where λ is the Lagrange multiplier. The first-order optimality conditions are as follows:

$$
\begin{aligned}
[c]: & \quad c^{-\sigma} = \mu \\
[a']: & \quad \mu = \beta \mathbb{E} V'_\omega (1 + r(S')) + \lambda \\
[b']: & \quad q^b(S)\mu = \beta \mathbb{E} V'_\omega + \phi \\
[\omega]: & \quad V_\omega = \mu \\
[n]: & \quad n = \left(\frac{zw}{\eta c^\sigma} \right)^\chi
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
c^{-\sigma} - \lambda &= \beta \mathbb{E} \left[(c')^{-\sigma} (1 + r(S')) \right] \\
q^b(S) c^{-\sigma} - \phi &= \beta \mathbb{E} \left[(c')^{-\sigma} \right] \\
n &= \left(\frac{zw}{\eta c^\sigma} \right)^\chi
\end{aligned}
$$

The equilibrium is computed by considering the following slackness conditions:

$$\lambda \begin{cases} > 0 & \text{if } a' = 0 \\ = 0 & \text{if } a' > 0 \end{cases}$$

$$\phi \begin{cases} > 0 & \text{if } b' = \underline{b} \\ = 0 & \text{if } b' > \underline{b} \end{cases}$$

$$\mu > 0$$