A Dynamically Consistent Global Nonlinear Solution Method in the Sequence Space and Applications*

Hanbaek Lee†

University of Tokyo

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Abstract

This paper develops a global nonlinear solution method in the sequence space for dynamic stochastic general equilibrium models. The method utilizes the equilibrium's ergodicity: if a simulated path of the aggregate shock is long enough, all the possible equilibrium outcomes are realized somewhere on the path. Then, the conditional expectation at each period can be completely characterized by identifying the periods of each possible future state realization and combining the corresponding time-specific value (policy) functions without a law of motion or parametrized expectation. I theoretically show that a sufficient statistic can be used for a complex aggregate state, including distributions, when the individual value (policy) functions are strictly monotone in the statistic. Despite its simple implementation, the computation is highly efficient, bypassing fixed-point problems in each iteration, including non-trivial market clearing.

Keywords: Global nonlinear solution, sequence space, dynamic consistency, sufficient statistic, business cycle.

JEL codes: C61, C63, E32.

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†University of Tokyo. Graduate School of Economics, 7-3-1, Hongo, Bunkyo-ku, Tokyo 113-0033, Japan. Email: hanbaeklee1@gmail.com

1 Introduction

How can we accurately compute the global solution for the dynamic stochastic general equilibrium (DSGE, hereafter) models? It is a fundamental question in macroeconomics, and its answer is crucial for macroeconomic policies referencing the DSGE model predictions. This paper develops a novel method to solve the dynamically consistent global solution in the sequence space for the DSGE models.

Recent studies in the literature have developed the global nonlinear solution method in the state space, boosting the speed and accuracy through advanced econometric techniques or optimizing the speed of computing machines. However, there have been relatively few studies on the global nonlinear solution method in the sequence space, which this paper shows to have great advantages over the state space approaches in terms of accuracy and speed despite its simple implementation. Moreover, this method provides a novel angle to understand how a sufficient statistic can be used for the computation of DSGE models with a complex aggregate state, possibly including infinite-dimensional objects. Despite its importance and usefulness, the conditions under which a sufficient statistic can be used for the computation remain an open question. This paper bridges the gap by theoretically characterizing the condition for using sufficient statistics in implementing the new method. The method is easily applicable to a broad class of models regardless of whether the model is heterogeneous-agent-based or representative-agent-based. In particular, it is useful for models with highly nonlinear aggregate fluctuations, as it does not require specifying the unknown nonlinear form of the law of motions.

The novelty of the method lies in the accurate computation of the sequence of the conditional expectation of the future value (policy) functions. In particular, the method, which I name the *repeated transition method*, utilizes the ergodicity of DSGE models: if a simulated path of the stationary aggregate shock process is long enough, all the possible equilibrium allocations should be realized on the simulated path. This characteristic implies that state-contingent future allocations are obtainable somewhere on the simulated path as a realized equilibrium outcome. Then, by properly identifying the period that has the corresponding

outcome to each expected future state, an agent's conditional expectation can be completely characterized at any period on the simulated path. In the identifying step for the corresponding periods for expected future outcomes, a law of motion (transition equation) or parameterized expectation does not need to be specified: the only information needed for this step is a metric of similarity among the aggregate states across the periods.

For example, suppose an agent is at time t, and a researcher needs to come up with an expected value (policy) function of period $t+1$ for a dynamics problem. Suppose there are two possible exogenous aggregate state realizations: G (Good) or B (Bad). For each exogenous state realization $s \in \{G, B\}$ in $t+1$, the method finds a period in the simulated path where the endogenous aggregate states are the closest to the ones in period $t+1$, and the aggregate shock realization is s. Then, it combines the time-specific value (policy) functions from these two periods to construct the expected future value (policy) function of period $t+1$. Due to the ergodicity, there almost surely exists such a period where the endogenous aggregate allocations (e.g., the distribution of individual states) are identical to the ones in period $t+1$ among the periods of the shock realization s if the simulated path is long enough. Therefore, the expected future value (policy) function at each period can be accurately constructed by combining the time-specific value (policy) functions.[1]

During the implementation, the method iteratively updates the predicted allocation path using the realized allocation path until both converge in the sequence space. To be specific, the identifying step of the similar periods and the computation step of the expectation are based on the allocation path from the last iteration (predicted path), and I update the predicted path of the next iteration based on the realized allocation implied by the predicted path. Therefore, each iteration passes over the information of the whole sequence of the realized allocations to the next iteration, which is the maximal set of information regarding the transition dynamics. This is sharply contrasted with the state space-based approach, where the transitional dynamics are summarized by the functional relationship between the current and the future states. Due to the feasibility of updating the predicted dynamics based on the rich information set, it accurately computes the equilibrium dynamics even

[1] As explained in the main text, the method utilizes *repeated transitions* between the same (similar) endogenous states with different exogenous states, so the method is named based on this computation step.

for the highly nonlinear models. The required simulation length for the repeated transition method is not particularly longer than the existing methods, but the way an update is made for each iteration boosts the accuracy of the solution.

When a model includes heterogeneous agents or multidimensional endogenous aggregate states, the step to compare high-dimensional objects across the periods can be a computational bottleneck.[2] For this, I suggest a sufficient statistic approach and provide a theoretical condition under which the approach leads to the exact solution. Specifically, when the time-specific value (policy) functions on the simulated path are strictly monotone in an aggregate equilibrium allocation given an individual state and an exogenous aggregate state, the aggregate allocation can be used as a sufficient statistic to obtain the exact solution: the allocation is playing a role as an indexing function of the value (policy) function's ranking across the time.[3]

This approach contrasts with the existing sufficient-statistic approaches in the literature (Krusell and Smith, 1997, 1998): the repeated transition method uses sufficient statistics to identify the target period with the same statistic level (ranking) for forming the expected value (policy) function, unlike using the statistic as an argument of a parametric law of motion. The repeated transition method does not assume a specific (parametric/non-parametric) functional form of the law of motion, as the whole sequence of the realized sufficient statistics is utilized to update the predicted path. So, it achieves the convergence of the allocation paths between the predicted and the realized in the sequence space rather than the convergence of the coefficients in the functional form of a law of motion or the fixed point in a transition equation. Therefore, the repeated transition method is highly accurate in terms of dynamic consistency.

In terms of speed, the repeated transition method substantially outperforms the method of Krusell and Smith (1997) when a model features period-by-period fixed-point problems such as non-trivial market-clearing conditions. In Krusell and Smith (1997), the market needs

[2]Saving and updating the sequence of the distributions are also computationally burdensome tasks.

[3]The existence of such indexing allocation is an important issue. However, the theoretical investigation of the existence is beyond this paper's scope. The author conjectures that the first moment of the endogenous individual state's distribution serves as a sufficient statistic if all individual inter-temporal policy functions are weakly monotone in each exogenous state, *and* the strict monotonicity holds for non-zero measures of individuals in any aggregate state realizations, as evidenced by several applications provided in the online appendix. I leave the further theoretical investigation of the existence for future research.

to be cleared in each period on the simulated path to let the updated coefficients of the law of motion converge to the true level. This step requires extra loops in the algorithm, resulting in a high computation cost. In contrast, the repeated transition method does not require solving the fixed-point problem for each period. Instead, it computes the period-specific implied allocation (implied price) by the fixed-point problem (market clearing condition), which is distinguished from the exact fixed point. Then, it updates the allocation (price) path using the implied level symmetrically with the other predicted allocation path. Therefore, the method bypasses the solution step for the fixed-point problems over the iteration. However, as the iteration goes by, the implied allocation path converges with the sequence of the fixed points. This convergence is not due to the specialty in the repeated transition method but due to the stability and the uniqueness of the recursive competitive equilibrium of the model. The repeated transition method fully utilizes such natures of the models that are readily available to help the computation.[4] By bypassing the period-by-period fixed-point problem, the method is substantially faster than the method based on the law of motion. According to my computation of Khan and Thomas (2008), the repeated transition method is more than 10 times faster than the latter.

One advantage of the repeated transition method is its applicability to a broad class of DSGE models. One example is an extended version of Krusell and Smith (1998), where an individual household endogenously chooses the quantity of the labor supply and is subject to an aggregate uncertainty shock (Bloom et al., 2018). Despite the difficulties arising from the infinite-dimensional aggregate state, the non-trivial market clearing condition, the occasionally binding constraint, and the uncertainty shock, the method accurately solves the model. I also verify that the sufficient statistic approach is valid by testing the theoretical monotonicity condition. I provide sample codes for multiple applications in the Online Appendix. Also, Lee et al. (2024) applies the repeated transition method to Diamond-Pissarides-Mortensen (DMP) models with exogenous and endogenous job separation to analyze nonlinear labor market dynamics. The repeated transition method also solves nonlinear New Keynesian models globally and accurately. Lee and Nomura (2024) applies the method to analyze the

[4]The convergence is not guaranteed if an equilibrium does not feature stability and uniqueness. However, the computational difficulty arising from not having these natures is not specific to the repeated transition method.

nonlinear inflation dynamics through the lens of a canonical Calvo pricing model.[5].

In this paper, I present real business cycle (RBC, hereafter) models of irreversible investment with a representative firm and heterogeneous firms as a set of leading applications.[6] The irreversible investment provides an interesting environment to the RBC model, where a firm cannot easily reduce its capital size, constrained by the investment lower bound. The investment size constraint is occasionally binding, leading to highly nonlinear dynamics of capital stock even for the representative agent model. As other methodology papers in the literature have extensively analyzed the solution of this model, the comparison across the solutions is convenient. Specifically, I compare the solution of the representative-firm model from the repeated transition method with the solution from the OccBin toolkit by Guerrieri and Iacoviello (2015) and the solution of the GDSGE toolkit by Cao et al. (2023). Despite its simple implementation, the repeated transition method outperforms the others in terms of accuracy with respect to dynamic consistency at a similar computation speed.[7]

Then, I illustrate the computation of the heterogeneous-firm business cycle model with irreversibility. In this model, each firm is subject to the irreversibility constraint, so the firm-level capital dynamics are highly nonlinear. Due to this micro-level nonlinearity, it is difficult to specify the correct aggregate law of motion for the capital dynamics in the state space approach.[8] On top of this, the presence of a non-trivial market clearing condition for the labor market adds another layer of computational hurdle, as the traditional state space-based approach takes a long time to clear the market in each period. Nevertheless, the repeated transition method solves the equilibrium path quickly and accurately.

The accurate global nonlinear solution enables rich macroeconomic analysis, including the role of micro-level nonlinearity on macro-level nonlinearity. In the leading application, the nonlinearity makes the representative-firm dynamics substantially deviate from the

[5]Likewise, the method's applicability to the broad class of nonlinear models enables equilibrium-based analysis on the state dependence (or history dependence), which is directly comparable to the observed state dependence (Pizzinelli et al., 2020)

[6]In online appendix, I provide multiple applications of the methodology including an extension of Krusell and Smith (1998) with an endogenous labor supply and aggregate uncertainty, the model of Khan and Thomas (2008), and variations of the real business cycle models. I briefly discuss them in the main text.

[7]The repeated transition method does not rely on external source codes in a lower-level computational language such as C++. The sample codes are all based on MATLAB.

[8]The nonlinear micro-level dynamics do not always result in the nonlinear aggregate dynamics as studied in Khan and Thomas (2008).

heterogeneous-firm dynamics despite the same parameter levels shared by the two models, except for the idiosyncratic productivity. Both are nonlinear, but the two models display significantly different nonlinearity. In particular, the heterogeneous firm displays significantly greater volatility in investment and output. These differences in the aggregate dynamics completely disappear when the occasionally binding constraint is lifted (perfect reversibility). Therefore, it is the nonlinearity at the micro level stemming from the occasionally binding constraint that makes the macro-level dynamics more volatile and nonlinear.

To summarize the advantages of the repeated transition method, 1) it computes the accurate global DSGE solution even for the highly nonlinear models such as the ones with an occasionally binding constraint; 2) it achieves high computational efficiency by bypassing the period-by-period fixed point problem; 3) a nonlinear model with a complex aggregate state (e.g., heterogeneous-agent models) can be solved using the sufficient statistic approach, and the validity of the approach can be tested based on a theory; 4) multiple aggregate shocks can be easily accommodated; 5) the implementation is simple and does not rely on any external source code.[9] Regarding the last point, there is room for the prototype repeated transition method to be further improved by adopting the recent machine learning and deep learning techniques (Han et al., 2021; Azinovic et al., 2022; Fernández-Villaverde et al., 2023), the adaptive sparse grids (Winschel and Krätzig, 2010; Brumm and Scheidegger, 2017), and the lower-level computing languages as in Cao et al. (2023). I leave this possibility to future research.

Related literature One of the most popular global solution algorithms is the state space-based approach using a parametric law of motion developed by Krusell and Smith (1997, 1998). While the methods of these papers are powerful in solving models with linear aggregate dynamics, they often fail to solve models with nonlinear dynamics due to the difficulty in correctly specifying the law of motion. Then, the computation of the conditional expectations of future value (policy) functions is possibly inaccurate, leading to a dynamic inconsistency in the computed equilibrium path. Alternatively, the method of parametrized

[9] All the sample codes are written in MATLAB, but the computational efficiency is still at the top-notch, which will be discussed in Section 5.

expectation by Marcet (1988) and den Haan and Marcet (1990) compute the conditional expectations of future variables using the parametrized function and finding the optimal parameter levels of approximation. Den Haan and Rendahl (2010) characterizes the law of motion by explicitly aggregating the Taylor-approximated individual policy functions, which can handle the nonlinear law of motion. These methods approximate the conditional expectation based on a combination of the basis functions. In contrast, the repeated transition method computes the conditional expectation without a functional fitting or approximation by identifying the period with each possible future state realization and combining the value or policy functions in the identified period. This is by utilizing the ergodicity of the DSGE models.

The global solution methods developed by Cao et al. (2023) and Elenev et al. (2021) are closely related to the repeated transition method. Their methods solve the transition equation and the individual policy function simultaneously on the state space, which efficiently achieves dynamic consistency. Then, the transition equation is updated by the implied dynamics as in the repeated transition method. In solving the transition equation, a functional fitting or approximation is a necessary step if there is no explicit form of the transition equations. In contrast, the repeated transition method does not require such steps, as the whole realized equilibrium path is utilized for updating transition dynamics on the sequence space without fitting or approximating the inter-temporal transitions, leading to a high degree of efficiency and accuracy in the solution. Other computational gains of the repeated transition method, including the speed gain for the non-trivial market clearing conditions, are investigated further in detail in the following sections.

The recent method utilizing the sequence of Jacobians to compute the equilibrium path by Auclert et al. (2021) is also related to my paper, as both methodologies are based on the sequence space. Their methodology features high speed by efficiently reducing the number of computation steps, enabling rapid likelihood-based estimation. However, the method assumes perfect foresight, while the repeated transition method does not require the assumption. Specifically, my method accurately recovers the expected future outcomes at each period under the aggregate uncertainty. Therefore, the repeated transition method is distinguished from the other computation methods with the perfect foresight (Fair and Taylor,

1983; Juillard, 1996; Judd, 2002; Cai et al., 2017; Boppart et al., 2018).

Also, it directly computes aggregate allocations and market-clearing prices in each period on the simulated path without specifying the law of motion. Therefore, the repeated transition method is distinguished from the solution methods based on perturbation and linearization (Reiter, 2009; Boppart et al., 2018; Ahn et al., 2018; Winberry, 2018; Childers, 2018).

The repeated transition method is also related to the simulation-based approach of Judd et al. (2011) and Maliar et al. (2011). Their method solves a model only in the realized ergodic state space in the equilibrium. This method significantly saves computation time, as it focuses only on a part of the entire state space. On top of this gain, the repeated transition method utilizes the information contained in the realized state space to form the agent's rational expectation in each period on the simulated path, which significantly improves the accuracy of the solution.

As the repeated transition method utilizes a single path of simulated aggregate shock that is long enough to represent the stochastic process completely, the approach is related to Kahou et al. (2021). Kahou et al. (2021) utilizes the fact that a whole economy's dynamics can be characterized by solving a finite number of agents' problems on a single Monte Carlo draw of individual shocks under the permutation-invariance condition. Then, the law of motion is computed using the deep-learning algorithm.

Roadmap Section 2 explains the repeated transition method. Section 3 explains the sufficient statistic approach. Section 4 explains how the repeated transition method bypasses non-trivial market clearing conditions. Section 5 validates the accuracy and the speed of the repeated transition method through a comparison with the existing methods in the literature. Section 6 explains the application to a leading example. Section 7 concludes.

2 The repeated transition method

2.1 A generic model framework

To explain the repeated transition method, I introduce a generic model framework that can nest a broad class of general equilibrium models. I denote the individual state as x and the aggregate state as X. The individual state x is composed of the endogenous individual state a and the exogenous individual state s (the idiosyncratic shocks). The aggregate state X is composed of the endogenous aggregate states Φ and the exogenous aggregate state S (the aggregate shocks). Φ can be a distribution of the individual states x in a heterogeneous-agent model or a set of aggregate allocations in a representative-agent model. For comprehensive explanation, I refer to Φ as the distribution of the individual states, which is an infinite-dimensional object.

$$\begin{aligned}
&[\text{Individual state}]: \quad x = \{a, s\} \\
&[\text{Aggregate state}]: \quad X = \{\Phi, S\}
\end{aligned}$$

The idiosyncratic and aggregate shock processes are assumed to follow a Markov process represented by a transition matrix Π^s and Π^S, respectively. I denote the value function as V. All the variables with an apostrophe indicate the variables of the future period. The objective function of an economic agent is composed of the contemporaneous part $f(y, x', x; X)$ and the expected future value. The agent maximizes the objective function by choosing (y, a'), where y is a vector of control variables that affects only the contemporaneous period. Then, the recursive formulation of an agent's problem is as follows:

$$\begin{aligned}
V(x; X) = \max_{y,a'} \; & f(y, a', x; X) + \mathbb{E}m(X, X')V(a', s'; X') \\
\text{s.t.} \quad & (y, x') \in \mathcal{B}(x; X, X', q), \quad \Phi' = F(X)
\end{aligned}$$

where $m(X, X')$ is the stochastic discount factor; $q(X, X')$ is a price bundle;
$\mathcal{B}(x; X, X', q)$ is the budget constraint; $F(X)$ is the law of motion known to to an individual

agent.[10] I denote the combined price bundle (m, q) as p. The following market clearing condition pins down the price p:[11]

$$[\text{Market clearing}]: \quad p(X, X') = \arg_{\widetilde{p}}\{Q^D(\widetilde{p}, X, X') - Q^S(\widetilde{p}, X, X') = 0\},$$

where Q^D and Q^S are the functions of demand and supply, which are endogenously determined by the model. For simplicity of the illustration, I assume the aggregate shock S can take two possible values $\{G, B\}$, and the transition matrix Π^S is a 2×2 matrix.[12]

The generic model framework nests the cases where the value function and the constraint allow the analytical expression of the first-order optimality conditions. In such cases, the first-order conditions and the envelope condition are characterized as follows:

$$\begin{aligned}
[a']: \quad & -\frac{\partial f}{\partial a'} = \mathbb{E}m(X, X')\frac{\partial V}{\partial a'} + \lambda\frac{\partial}{\partial a'}\mathcal{B} \\
[y]: \quad & -\frac{\partial f}{\partial y} = \mathbb{E}m(X, X')\frac{\partial V}{\partial y} + \lambda\frac{\partial}{\partial a'}\mathcal{B} \\
[a]: \quad & \frac{\partial V}{\partial a} = \frac{\partial f}{\partial a} + \lambda\frac{\partial}{\partial a}\mathcal{B}
\end{aligned}$$

where the lagrange multiplier λ satisfies the slackness condition. In the following sections, I explain the method based on the recursive form in value functions for a comprehensive explanation. However, the method is seamlessly applied even if the value function is replaced by the first-order derivative or the policy functions in the explained algorithm below. In the Online Appendix, I provide multiple applications where the expected policy function is computed instead of the expected value function.

The repeated transition method achieves convergence on the sequence space. Therefore, despite the converged equilibrium allocations being fully describable in a recursive form, I

[10] The stochastic discount factor can be a constant, for example β, as in a canonical dynamic household's problems. In a dynamic firm problem, the stochastic discount factor needs to be included.

[11] Any period-specific fixed point problem can be considered in the repeated transition method. For brevity, I only include the non-trivial market clearing condition.

[12] The applicability of the repeated transition method is not limited to a certain number of grid points for the aggregate shocks. The choice of two grid points is purely for an easy illustration. Moreover, multiple aggregate shocks can be considered as an exogenous state. In the online sample code, I include the application for the RBC model with both the aggregate TFP shock and convex adjustment cost shock. Similarly, an aggregate uncertainty shock can be incorporated on top of the aggregate TFP shock.

denote the equilibrium object in a sequential expression, such as $\{V_t\}$, for the sake of a coherent explanation. Hereafter, given a realized state $\{x_t, X_t\}$ for an individual (or representative) agent in a given period t, the value function in the sequential expression V_t and the value function in the recursive form $V(x_t; X_t)$ are interchangeably used.

2.2 Assumptions

In this section, I discuss the necessary features of the models for the application of the repeated transition method. The method relies on the a) stability and the b) uniqueness of the equilibrium. If a model violates these two conditions, it is hard to expect the convergence of the method. Also, the c) ergodicity of the equilibrium is a necessary condition. That is, equilibrium allocations are recurrent and aperiodic on the infinitely long simulated path in the equilibrium. Without ergodicity, there is a set of equilibrium allocations in a period that will never become an equilibrium allocation again in the following periods. In this case, the computation of the expectation of such allocations is not feasible in the repeated transition method.[13] From this point on, I focus only on the models that satisfy these three conditions.

2.3 Intuition behind the method

I start from explaining the basic structure of the methodology. Suppose I simulate T periods of aggregate exogenous state $\{S_t\}_{t=0}^T$, and hypothetically the simulated path is long enough to make almost all the possible equilibrium allocations happen on the simulated path.[14] Then, I start from guessing the following three time series: 1) value functions, $\{V_t^{(0)}\}_{t=0}^T$, 2) distributions of individual states $\{\Phi_t^{(0)}\}_{t=0}^T$, and 3) prices $\{p_t^{(0)}\}_{t=0}^T$. Using these guesses, I solve the allocations backward from the terminal period T to obtain the implied value function solution $\{V_t^*\}_{t=0}^T$, and simulate the economy forward using the solution. The forward simulation generates the time series of the distribution of individual states $\{\Phi_t^*\}_{t=0}^T$ and prices $\{p_t^*\}_{t=0}^T$ from the market-clearing conditions. Using these, I update the guess to

[13] It is also conceptually challenging to let an agent form the expectation of such allocation.

[14] In theory, an infinitely long simulation needs to be considered, but for illustrative purposes, I consider a T-period long simulation. Later in the application, a long-enough finite simulation is used as an approximation for the infinitely long ergodic simulation.

move on to the next iteration, $\{V_t^{(1)}, \Phi_t^{(1)}, p_t^{(1)}\}_{t=0}^{T}$. The entire structure of the algorithm looks similar to the perfect-foresight solution method (Fair and Taylor, 1983), but the backward solution step is sharply distinguished.

To clarify this point, suppose that I've run the n^{th} iteration and that I am now at the $(n+1)^{th}$ iteration at period t after solving the problem backward from the terminal period T until period $(t+1)$. On the simulated aggregate state path, suppose that the exogenous state realization at period $t+1$ is G: $S_{t+1} = G$. For the problem of an agent at t, a macroeconomist needs to construct an expected future value function denoted as $\mathbb{E}_t\widetilde{V}_{t+1}$.[15] However, this is a difficult task because only $V_{t+1}(\cdot, S = G)$ is available from the backward solution, while $V_{t+1}(\cdot, S = B)$ is not. This is natural as only one exogenous aggregate state can be realized in a period. I define this unobserved values $V_{t+1}(\cdot, S = B)$ as a *counterfactual* conditional value function.

In the standard state space-based approach, this problem is handled by replacing the time index with the endogenous and exogenous aggregate states and by interpolating the endogenous states using the assumed law of motion. Therefore, the accuracy of the specified law of motion strongly affects the accuracy of the solution. However, before obtaining the solving the equilibrium, it is hardly known whether the law of motion is correctly specified or not. Then, if the law of motion turns out to be incorrect, a researcher needs to restart solving the problem from scratch, coming up with a new guess about the law of motion. The proper guess is difficult to obtain, as there is an infinite degree of freedom in the new guess. In particular, there are two types of difficulties in this step. One is about which statistics to include in the law of motion; the other is about what functional forms to choose for the law of motion. Unless the aggregate dynamics are well-known to be log-linear, as in Krusell and Smith (1998), this problem cannot be easily resolved.

Then, the repeated transition method takes a different route where the counterfactual conditional value function is obtained from the value function of another period $\widetilde{t}+1$ in which the endogenous aggregate state is exactly the same as the period $t+1$, but the counterfactual

[15] The rational expectation is not necessary for the application of the methodology. As long as the specific form of the expectation is given, the step for computing the conditional expectation can be flexibly adjusted.

exogenous state is realized:[16]

$$\Phi^{(n)}_{\widetilde{t}+1} = \Phi^{(n)}_{t+1}$$
$$S_{\widetilde{t}+1} = B \neq G = S_{t+1}.$$

Then, all the aggregate states of the realized state of period $\widetilde{t}+1$ are identical to the ones in the *counterfactual state* of period $t+1$. Thus, the following equation holds:

$$V^{(n)}_{\widetilde{t}+1}(\cdot, S = B) = V^{(n)}_{t+1}(\cdot, S = B).$$

Importantly, $V^{(n)}_{\widetilde{t}+1}(\cdot, S = B)$ is the observed *factual* conditional value function available in the n^{th} iteration. As both $V^{(n)}_{t+1}(\cdot, S = G)$ and $V^{(n)}_{t+1}(\cdot, S = B)(= V^{(n)}_{\widetilde{t}+1}(\cdot, S = B))$ are available (n^{th} iteration outcome), the expected future value function $\mathbb{E}_t\widetilde{V}_{t+1}$ can be consistently computed. Even when the aggregate shock process is discretized finer than two grid points, the rationally expected future value function can be obtained using the same procedures.[17] Due to the ergodicity of the dynamic stochastic general equilibrium, if a simulated path is long enough, the existence of such period $\widetilde{t}+1$ is almost surely guaranteed.

In this new approach, a law of motion does not need to be specified to construct the rational expected future value function. As long as the period $\widetilde{t}+1$ that mimics the counterfactual realization of $t+1$ is identified, the expectation is computed. For this step, it is necessary to track the endogenous aggregate states $\{\Phi^{(n)}_t\}^T_{t=0}$, as it is the key identifier to locate the period $\widetilde{t}+1$. In the following section, I elaborate on the detailed steps to implement the repeated transition method and how to handle the curse of the dimensionality using a sufficient statistic approach.

2.4 Algorithm

I simulate a single path of exogenous aggregate shocks for a long-enough period T, $\mathbb{S} = \{S_t\}^T_{t=0}$, using the aggregate transition matrix Π^S. I define a time partition $\mathcal{T}(S)$ that groups

[16] Note that the counterfactual conditional value function is obtained from the n^{th} guess.

[17] Most of the sample codes provided in the Online Appendix use finer grids than two points.

periods with the same shock realization as follows.

$$\mathcal{T}_S := \{\tau | S_\tau = S\} \subseteq \{0, 1, 2, ..., T\} \text{ for } S \in \{B, G\}.$$

The pseudo algorithm of the repeated transition method is as follows:

Step 1. Guess on the paths of the value functions, the state distributions, and the prices. $\{V_t^{(n)}, \Phi_t^{(n)}, p_t^{(n)}\}_{t=0}^T$.[18]

Step 2. Solve the model backward from the terminal period T in the following sub-steps. The explanation is based on an arbitrary period t. Without a loss of generality, I assume $S_t = G$ and $S_{t+1} = G$:

2-a. Find $\widetilde{t}+1$ where the endogenous aggregate allocation in period is identical to the one in period $t+1$, but the shock realization is different from period $t+1$ ($S_{\widetilde{t}+1} = B$):[19]

$$\widetilde{t}+1 = \arg \inf_{\tau \in \mathcal{T}_B} ||\Phi_\tau^{(n)} - \Phi_{t+1}^{(n)}||_\infty.$$

2-b. Compute the expected future value function as follows:

$$\mathbb{E}_t \widetilde{V}_{t+1} = \pi_{G,G} V_{t+1}^{(n)} + \pi_{G,B} V_{\widetilde{t}+1}^{(n)}.$$

2-c. Using $\mathbb{E}_t \widetilde{V}_{t+1}$ and $p_t^{(n)}$, solve the individual agent's problem at the period t. Then, I obtain the solution $\{V_t^*, a_{t+1}^*\}$.

After the taking these sub-steps for $\forall t$, $\{V_t^*, a_{t+1}^*\}_{t=0}^T$ are available.

Step 3. Using $\{a_{t+1}^*\}_{t=0}^T$, simulate forward the time series of the distribution of the individual states $\{\Phi_t^*\}_{t=0}^T$ starting from $\Phi_0^* = \Phi_0^{(n)}$.[20]

Step 4. Using $\{\Phi_t^*\}_{t=0}^T$, all the aggregate allocations over the whole path such as $\{K_t^*\}_{t=0}^T$ can be obtained. Using the market-clearing condition, compute the time series of the market-

[18] In practice, I use the stationary equilibrium allocations for all periods as the initial guess.
[19] Such $\widetilde{t}+1$ might not be unique. However, any of such $\widetilde{t}+1$ is equally good to be used in the next step.
[20] In this step, I use the non-stochastic simulation method (Young, 2010).

clearing price. If the model features a non-trivial market clearing condition, compute the time series of the implied prices $\{p_t^*\}_{t=0}^T$.[21]

Step 5. Check the distance between the implied prices and the guessed prices.

$$\sup_{BurnIn \leq t \leq T-BurnIn} ||p_t^* - p_t^{(n)}||_\infty < tol$$

Note that the distance is measured after excluding the burn-in periods at the beginning and the end of the simulated path. This is an adjustment to handle a potential bias from the imperfect guesses on the terminal period's value function $V_T^{(n)}$ and the initial period's distribution $\Phi_0^{(n)}$. The convergence criterion can be augmented by including the distance in other allocations, such as value functions or distributions.

If the distance is smaller than the tolerance level, the algorithm is converged. Otherwise, I make the following updates on the guess:[22]

$$p_t^{(n+1)} = p_t^{(n)}\psi_1 + p_t^*(1-\psi_1)$$

$$V_t^{(n+1)} = V_t^{(n)}\psi_2 + V_t^*(1-\psi_2)$$

$$\Phi_t^{(n+1)} = \Phi_t^{(n)}\psi_3 + \Phi_t^*(1-\psi_3)$$

[21] It is worth noting that the prices here are not the market-clearing prices that are determined from the interactions between demand and supply. Rather, they are the prices implied by the market-clearing condition given either demand or supply fixed at the n^{th} iteration:

$$p_t^* = \arg_{\widetilde{p}}\{Q^D(p_t^{(n)}, X_t, X_{t+1}) - Q^S(\widetilde{p}, X_t, X_{t+1}) = 0\} \text{ or}$$
$$p_t^* = \arg_{\widetilde{p}}\{Q^D(\widetilde{p}, X_t, X_{t+1}) - Q^S(p_t^{(n)}, X_t, X_{t+1}) = 0\}.$$

In Section 6, I use this method to solve the model in Khan and Thomas (2008). In the computation method used in Khan and Thomas (2008), a market-clearing price needs to be computed in an additional loop due to the non-trivial market-clearing condition. The implied price cannot replace the market-clearing price in this method, as the misspecified price prediction rule can lead to a divergent law of motion of the aggregate allocation. In contrast, due to the missing market clearing step, the repeated transition method significantly saves computation time. I discuss the non-trivial market clearing condition further in Section 4.

[22] In highly nonlinear aggregate dynamics, I have found that the log-convex combination updating rule marginally dominates the standard convex combination updating rule in terms of convergence speed. The log-convex combination rule is as follows:

$$log(p_t^{(n+1)}) = log(p_t^{(n)})\psi_1 + log(p_t^*)(1-\psi_1).$$

for $\forall t \in \{0, 1, 2, 3, ..., T\}$. With the updated guess $\{V_t^{(n+1)}, \Phi_t^{(n+1)}, p_t^{(n+1)}\}_{t=0}^{T}$, I go back to Step 1.

(ψ_1, ψ_2, ψ_3) are the parameters of convergence speed in the algorithm. If ψ_i is high, then the algorithm conservatively updates the guess, leaving the algorithm to converge slowly. If the equilibrium dynamics are almost linear, as in Krusell and Smith (1998), uniformly setting ψ_i at around 0.9 guarantees convergence at a fairly high convergence speed. However, if a model is highly nonlinear, the convergence speed needs to be controlled to be substantially slower than the one in the linear models. This is because the nonlinearity can lead to a sudden jump in the realized allocations during the iteration if a new guess is too dramatically changed from the last guess. A heterogeneous updating rule $\psi_i \neq \psi_j$ $(i \neq j)$ is also helpful in cases where the dynamics of certain allocations are particularly more nonlinear than the others.

As can be seen from the convergence criterion in Step 5, the algorithm stops when the predicted allocation paths (n^{th} iteration) are close enough to the realized allocation paths (with asterisks). Therefore, once the convergence is achieved, the solution is guaranteed to be dynamically consistent: the predicted path coincides with the realized path. If the accuracy is measured in R^2 or in the mean-squared errors, as in Krusell and Smith (1998), the repeated transition method features R^2 at 1, and its mean-squared error becomes negligibly different than zero.

2.5 The required length of simulation path

In this section, I discuss how long a simulation needs to be for the repeated transition method. First, one of the most crucial determinants of the desired length is the assumed Markov process of the exogenous aggregate states. That is, the number of realizations of each exogenous state during the simulation is the key information. As in Krusell and Smith (1998), if only two aggregate states are realized based on a symmetric transition probability of a moderate level (0.875), the simulation of 500 periods is good enough to make the solution stay unaffected by further lengthening. For the example of the RBC model with irreversible

investment, which will be discussed in Sections 5 and 6, if an aggregate TFP process is discretized by the Tauchen method, covering the three standard deviation ranges, then at least 3,000 periods are needed to have enough realizations (at least 30) for both ends of the grids. Still, depending on the persistence of the exogenous state process, the required number of periods might vary.

The repeated transition method is not the only one of which the precision is affected by the length of the simulation. For example, in the state space-based approach, the regression or law of motion also needs to have enough number of observations in each exogenous state realization to guarantee the accuracy.

Second, the nonlinearity of the model is another critical factor. As will be introduced in Section 3, in practice, the group of the periods with similar endogenous aggregate states is identified instead of the period with exactly identical states. Then, piecewise interpolation makes up for the missing observations. If a DSGE model is highly nonlinear, having additional observation in each exogenous state realization might substantially improve the accuracy of the solution, adding more nodes for the interpolation. For the example of the RBC model with irreversible investment, which is highly nonlinear, an increase in the length of the simulation gradually changes the solutions until adding the 4,000 periods, after which the solution stays almost unaffected.

In summary, the required simulation length is not substantially longer than the number of periods often used in the state space-based approach. However, it varies depending on the shock's persistence and the nonlinearity of the model. The accuracy of the repeated transition method comes from updating the entire predicted path of allocations based on the entire realized ones, which passes over the maximal information (period-by-period) for the dynamics to the next iteration. It is contrasted with the state space-based approach, where the update is based on a functional relationship between the current and future period that only summarily captures the information on the dynamics. This is why the repeated transition method does not have to use a particularly longer simulation period than the alternative, while the accuracy is high.

3 A sufficient statistic approach

In the algorithm explained in the previous section, Step 2-a is the most demanding step for heterogeneous-agent models, as it needs to find a period $\widetilde{t}+1$ that is identical to period $t+1$ in terms of distribution. Therefore, the similarity of the distributions across the periods needs to be measured, which is a computationally costly process.

However, if there is a sufficient statistic that can perfectly represent a period's endogenous aggregate state, the computational efficiency can be substantially improved. This enables to locate the target period $\widetilde{t}+1$ by only comparing the distance between these sufficient statistics instead of the distributions. For example, in Krusell and Smith (1998), if the aggregate capital is the sufficient statistic, Step 2-a becomes easier as follows:

$$\widetilde{t}+1 = \arg \inf_{\tau \in \mathcal{T}_B} ||K_{\tau}^{(n)} - K_{t+1}^{(n)}||_{\infty}.$$

As the algorithm relies on the ergodicity, a sufficiently long period of simulation is needed for accurate computation. However, in practice, the simulation still ends in finite periods. Therefore, the period $\widetilde{t}+1$ that shares exactly identical sufficient statistic as period t+1 might not exist. For this hurdle, the following adjusted versions of Step 2-a and Step 2-b help improve the accuracy of the solution:

2-a′. Find $\widetilde{t}^{up}+1$ where the sufficient statistic of the endogenous aggregate state is closest to the one in period $t+1$ from above, but the shock realization is different from period $t+1$:

$$\widetilde{t}^{up}+1 = \arg \inf_{\tau \in \mathcal{T}_B \text{ s.t. } e_{\tau}^{(n)} \geq e_{t+1}^{(n)}} ||e_{\tau}^{(n)} - e_{t+1}^{(n)}||_{\infty},$$

where e_{τ} denotes the sufficient statistic of the endogenous aggregate state in period τ. Similarly, find $\widetilde{t}^{dn}+1$ where the sufficient statistic of the endogenous aggregate state is closest to the one in period $t+1$ from below, but the shock realization is different

from period $t+1$:

$$\widetilde{t}^{dn}+1=\arg\inf_{\tau\in\mathcal{T}_B \text{ s.t. } e_{\tau}^{(n)}<e_{t+1}^{(n)}}||e_{\tau}^{(n)}-e_{t+1}^{(n)}||_{\infty}.$$

Then, I have $e_{\widetilde{t}^{up}+1}^{(n)}$ and $e_{\widetilde{t}^{dn}+1}^{(n)}$ that are closest to $e_{t+1}^{(n)}$ from above and below, respectively. Using these two, I compute the weight ω to be used in the convex combination of value functions in the next step:

$$\omega=\frac{e_{t+1}^{(n)}-e_{\widetilde{t}^{dn}+1}^{(n)}}{e_{\widetilde{t}^{up}+1}^{(n)}-e_{\widetilde{t}^{dn}+1}^{(n)}}.$$

2-b′. Compute the expected future value function as follows:

$$\mathbb{E}_t\widetilde{V}_{t+1}=\pi_{G,G}V_{t+1}^{(n)}+\pi_{G,B}\left(\omega V_{\widetilde{t}^{up}+1}^{(n)}+(1-\omega)V_{\widetilde{t}^{dn}+1}^{(n)}\right).$$

Step 2-a' and Step 2-b' construct a synthetic counterfactual conditional value function by the convex combination of the two value functions that are for the most similar periods to period $t+1$. These adjusted steps help accurately solve the problem in relatively short periods of simulation. For example, the model in Krusell and Smith (1998) can be accurately solved using only $T=500$ periods of simulation (except for 100 burn-in periods at the beginning and the end of the simulated path).

The step of interpolation after finding the closest periods in terms of sufficient statistics can be understood as a piecewise interpolation, in contrast to the unconditional linear interpolation used in the state space-based approach based on the regression coefficients.

3.1 The qualification for the sufficient statistic

In this section, I analyze under which condition a variable can serve as a sufficient statistic that replaces the entire distribution for implementing the repeated transition method. A large body of the literature has considered sufficient statistics to overcome the curse of dimensionality for the computation of DSGE, but there has been little theoretical explanation

of when such an approximation can be used. Proposition 1 provides the qualification for a variable to be used as a sufficient statistic in the repeated transition method.

Proposition 1 (The qualification for the sufficient statistic)**.**
Suppose a time series of the value functions is given from a recursive competitive equilibrium that is unique. For a sufficiently large T, if there exists a time series of a variable $\{e_t\}_{t=0}^{T}$ such that for each time partition $\mathcal{T}_S = \{t|S_t = S\}$, $\forall S \in \{B, G\}$ and for $\forall(a, z)$,

$$(i) \quad e_{\tau_0} < e_{\tau_1} \iff V_{\tau_0}(a, z) < V_{\tau_1}(a, z) \text{ for any } \tau_0, \tau_1 \in \mathcal{T}_S$$

or

$$(ii) \quad e_{\tau_0} < e_{\tau_1} \iff V_{\tau_0}(a, z) > V_{\tau_1}(a, z) \text{ for any } \tau_0, \tau_1 \in \mathcal{T}_S,$$

then e_t is the sufficient statistic of the endogenous aggregate state Φ_t for $\forall t$.

Proof.
See Online Appendix. ■

Proposition 1 states that if a time series $\{e_t\}_{t=0}^{T}$ *monotonically ranks* the level of the corresponding period's value function for each individual state, e_t is the sufficient statistic of time period t in the repeated transition method. The intuition behind the proposition is as follows. Suppose a situation where a researcher is searching for a value function to compute the conditional expectation. If the time index of the correct counterfactual period to use is explicitly given as τ, then the researcher can easily identify which value function to use, as all value functions are indexed by time. So, in this case, V_τ is trivially the one to use.

Now instead of τ, suppose the level of e_τ is known to the researcher. Then, similar to the prior situation where τ is known, the researcher can identify which value function to use because the ranking information uniquely pins down the corresponding value function due to the strict monotonicity. For example, if two periods τ_0 and τ_1 share the same level of e_t, thus $e_{\tau_0} = e_{\tau_1}$, then the strict monotonicity says $V_{\tau_0} = V_{\tau_1}$. If this is not the case ($V_{\tau_0} \neq V_{\tau_1}$), then either the equilibrium's uniqueness or strict monotonicity is violated, which is the proof's key idea. The same argument is seamlessly applied when the value function is replaced by a

policy function in case the model is solved by the first-order optimality condition.

To summarize the theoretical result in this section, once the ranking information across the different periods' value functions is known, one can exactly pin down which period's value function to use. The qualification provides a theoretical ground to understand how a sufficient statistic approach works in the repeated transition method. In Section 6, I show how the monotonicity is quantitatively validated for the converged solution. One important note is that a sufficient statistic in the repeated transition method is not always qualified as a sufficient statistic for the law of motion in the state space-based approach. This is because the former may not include sufficient information about the nonlinear inter-temporal dynamics of the endogenous aggregate state variables. For example, in the nonlinear model explained in Section 6, if I fit the nonlinear aggregate dynamics of the sufficient statistic obtained from the repeated transition method to the nonlinear specifications of the single sufficient statistic, R^2 is still significantly lower than unity. However, the single-dimensional aggregate allocation (the aggregate capital stock) perfectly serves as a sufficient statistic in the repeated transition method, satisfying the monotonicity condition.

4 Non-trivial market clearing conditions

In this section, I explain how the repeated transition method efficiently handle the non-trivial market clearing conditions and why such approach is not feasible in the state space-based model.

Consider the following market clearing condition:

$$
\begin{aligned}
&Q^D(p_t, X_t, X_{t+1}) - Q^S(p_t, X_t, X_{t+1}) = 0. \\
&p_t := \arg_{\widetilde{p}}\{Q^D(\widetilde{p}, X_t, X_{t+1}) - Q^S(\widetilde{p}, X_t, X_{t+1}) = 0\}.
\end{aligned}
$$

where Q^D and Q^S are demand and supply functions; p_t is the market clearing price; X_t and X_{t+1} are the current and future aggregate states. The non-trivial market clearing condition indicates the case where there is no closed-form characterization of either demand side or the supply side (or both). For this problem, the repeated transition method utilizes the implied

price p_t^* instead of the exact clearing price p_t, where

$$\begin{aligned} p_t^* &:= \arg_{\widetilde{p}}\{Q^D(p_t^{(n)}, X_t, X_{t+1}) - Q^S(\widetilde{p}, X_t, X_{t+1}) = 0\} \text{ or} \\ &:= \arg_{\widetilde{p}}\{Q^D(\widetilde{p}, X_t, X_{t+1}) - Q^S(p_t^{(n)}, X_t, X_{t+1}) = 0\} \end{aligned}$$

That is, either the demand side or the supply side is fixed by assuming the guessed price level from the n^{th} iteration, and the price implied by the remaining side satisfying the equation is utilized. The computation of such implied price is substantially easier than the computation of the market clearing price, as the latter is a fixed point problem: both supply and demand are simultaneously affected by the price.

During the iteration, the implied price does not clear the market at each period, as it's only the implied price. However, as iteration goes by with the gradual updates, the predicted path of prices $\{p_t^{(n)}\}_{t=0}^T$ converges to the equilibrium prices $\{p_t\}_{t=0}^T$. This convergence makes the implied price clear the market in the limit for the following reasons:[23]

$$\begin{aligned} &\lim_{n\to\infty} Q^D(p_t^{(n)}, X_t, X_{t+1}) - Q^S(p_t^*, X_t, X_{t+1}) = 0 \\ \implies\ & Q^D(\lim_{n\to\infty} p_t^{(n)}, X_t, X_{t+1}) - Q^S(p_t^*, X_t, X_{t+1}) = 0 \\ \implies\ & Q^D(p_t, X_t, X_{t+1}) - Q^S(p_t^*, X_t, X_{t+1}) = 0 \\ \implies\ & p_t^* = p_t \quad (\because \text{uniqueness of the equilibrium}). \end{aligned}$$

We need the local continuity of demand or supply to proceed from the first to the second line. This should be true except for the knife-edge case where the unique equilibrium is at the discontinuity point, which DSGE models are not often subject to, so I assume the continuity. Thus, the converged solution in the repeated transition method computes the exact market clearing price in the limit simultaneously with other allocations.

In contrast, the implied price cannot replace the market clearing price in the state space-based approach (Krusell and Smith, 1997) in general. In this approach, the price dynamics is approximated by the parametric function of the aggregate state or the sufficient statistic, and the coefficients of the function carry the information about the relationship between

[23] For brevity, I suppose the case where the demand side Q^D does not have a closed-form characterization.

the price and the aggregate state. Given that the number of coefficients cannot technically exceed the number of periods, the coefficients can only carry the summarized information. If the coefficients are updated based on the implied price rather than the market clearing price, the update is based on inaccurate levels, thus in an inaccurate relationship. Then, the wrongly updated coefficients easily lead to a divergent path, as there is no theoretical guarantee that the coefficient of the functional form features *stability*. On the other hand, the repeated transition method preserves all the information about the relationship between the aggregate state and the price, as the method carries the whole sequence (time paths) over the iteration. Then, it achieves the uniform convergence of the paths in the sequence space due to the stability of the equilibrium.

5 Accuracy and speed: Comparison with the existing methods

In this section, I compare the repeated transition method with the other nonlinear solution methods in the literature for three DSGE models. The first is the real business cycle model with irreversible investment. The repeated transition method is compared with three different methods: linearized solution, OccBin solution by Guerrieri and Iacoviello (2015), and GDSGE solution by Cao et al. (2023). The second model is the heterogeneous-household business cycle model in Krusell and Smith (1998). I compare my method with the algorithm introduced in Maliar et al. (2010). Lastly, I compare the heterogeneous-firm business cycle model in Khan and Thomas (2008). The repeated transition method is compared directly to the paper's solution which utilizes the method of Krusell and Smith (1997). The computing machine used for the analysis is MacBook Pro 2021 with M1 Pro chip.

The first comparison is using the real business cycle model with the irreversible investment. A detailed description of the model is available in the appendix and also in Guerrieri and Iacoviello (2015). The model features highly nonlinear aggregate dynamics due to the occasionally binding irreversibility constraint for capital investment. Therefore, besides the macroeconomic implications, the model serves as a testing ground for the accuracy of the dif-

ferent methods for the nonlinear solutions. For an accurate comparison, I feed the same path of the simulated aggregate exogenous state (TFP) for each solution to each algorithm. The persistent TFP process is discretized by the Tauchen method using 7 grid points covering 3 standard deviation ranges with a uniform distance between the grid points. The simulation is 5,000 periods long with 500 extra burn-in periods. Depending on the convergence criterion, each solution displays the trade-off between the accuracy and the computation efficiency (time). In this comparison, I tune the repeated transition method to stop after around 90 seconds, matching the speed of the GDSGE toolkit.

First four of the Table 1 compares the accuracy of the different solution methods; the column with RTM label is for the repeated transition method; GDSGE, OccBin, and the linearized method follow in order. Two criteria of the accuracy measure are considered: the dynamic consistency error $Error_t$ and the Euler equation error EE_t following Judd (1992). Especially, the former criterion is defined as follows:[24]

$$Error_t = K_t^{(n)} - K_t^*$$

where $\{K_t^{(n)}\}_{t=1}^T$ is the dynamics of the capital stock from each solution method, and K_t^* is the implied capital path assuming the agent is expecting the path of $\{K_t^{(n)}\}_{t=1}^T$. The repeated transition method constructs the period-specific expected future allocations by properly combining allocations in the predicted path. Then, it provides the realized allocations implied by the prediction path. The dynamic consistency is achieved only when these two paths coincide. Thus, the method provides a useful tool to test the dynamic consistency in other solution methods by feeding the simulated path based on other solution methods as the predicted path to the algorithm.

The repeated transition method displays a higher accuracy than other methodology in terms of the four statistics: the absolute maxima of the dynamic inconsistency (first row) and the Euler equation error (third row); the square roots of the mean-squared dynamic inconsistency (second row) and the Euler equation error (fourth row).

It is worth noting that the repeated transition method's accuracy and speed do not rely on

[24]The form of the EE_t is identical to the one in Guerrieri and Iacoviello (2015).

Table 1: Comparison across the solution methods

	RTM	GDSGE	OccBin	Linear
Accuracy				
$max(\|Error_t\|)$ (% of steady-state K)	0.003	0.735	1.317	2.019
$\sqrt{mean(Error_t^2)}$ (% of steady-state K)	0.001	0.025	0.217	0.559
$max(\|EE_t\|)$ (% of contemp. C_t)	0.014	0.057	2.854	2.323
$\sqrt{mean(EE_t^2)}$ (% of contemp. C_t)	0.002	0.059	0.775	0.707
Business cycle stat.				
$mean(I)$	0.363	0.363	0.365	0.363
$mean(C)$	1.166	1.166	1.164	1.160
$vol(I)$	0.022	0.022	0.023	0.023
$vol(C)$	0.052	0.052	0.052	0.052
$skewness(I)$	1.363	1.320	1.307	1.407
$skewness(C)$	-0.225	-0.213	-0.322	-0.095
$kurtosis(I)$	4.447	4.578	4.513	4.255
$kurtosis(C)$	2.776	2.546	2.858	2.796

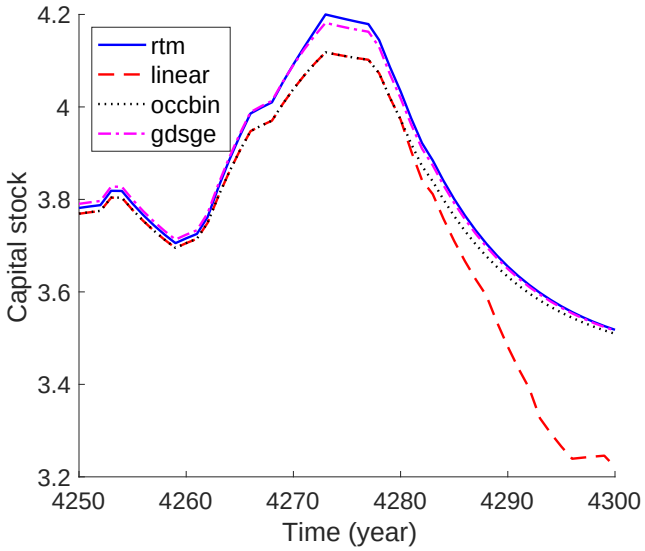


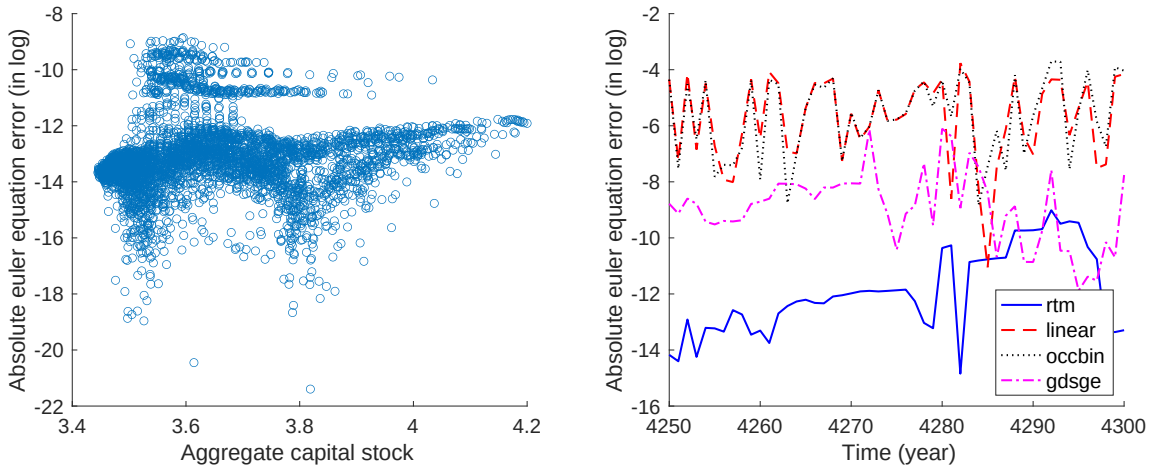
Figure 1: The equilibrium path of different solutions

computational boost from the lower-level computation language such as C++ or advanced econometric techniques. In theory, the dynamic consistency error can be lowered to an arbitrarily small level depending on the tolerance, using the repeated transition method. Therefore, there is room for the algorithm to be further improved in terms of speed and accuracy once it is combined with the lower-level language or the recently developed machine-learning/deep-learning techniques.

The bottom part of Table 1 reports the business cycle statistics implied in each solution. There are little differences in the lower-order moments but significant differences in the higher-order moments, such as skewness and kurtosis. Figure 1 plots a part of the equilibrium path of the capital stock obtained by each solution. The difference between the solutions of the repeated transition method and the GDSGE toolkit is the smallest; the OccBin solution displays a significant deviation from the repeated transition method when the aggregate capital stock is large. Over the 5,000 simulation periods, the correlation between the aggregate output (from the repeated transition method) and the absolute difference between the solutions of repeated transition method and OccBin toolkit is 0.631, implying a pro-cyclicality of the computation error. On the other hand, the absolute difference between the linear solution and the repeated transition method is counter-cyclical, with an output correlation of -0.640. This is because the solution by the OccBin toolkit provides a relatively more accurate solution during the downturn when the constraint is binding, while the nonlinearity stemming from the precautionary motivation during the normal periods is relatively inaccurately captured. In contrast, the linear solution cannot properly handle the occasionally binding constraint during the downturn, which leads to a large computation error during these periods.

Panel (a) of Figure 2 plots the absolute log Euler equation errors of the repeated transition method along with the equilibrium capital stock in the horizontal axis. Panel (b) plots the time series of the absolute log Euler equation errors in comparison with the other methods. Consistent with the statistics in Table 1, the Euler equation error is significantly lower in the repeated transition method than in the others.

Next, I compare the equilibrium allocations obtained from the repeated transition method and the ones from the method in Maliar et al. (2010) for the model of Krusell and Smith



(a) Over the equilibrium capital stock (RTM) (b) Over the equilibrium time path

Figure 2: Euler equation errors

(1998).[25] The repeated transition method computes the exact level of the Lagrange multiplier for the occasionally binding constraint at the individual level, which enables the accurate computation. The path of the lagrange multipliers is computed by the residuals using the Euler equation as in Rendahl (2014).[26] Both of the algorithms are tuned to satisfy the convergence criteria after running for approximately two minutes. The result is that the accuracy of two algorithms in the metric of the square root dynamic consistency is negligibly different (less than 10^{-5}) despite Maliar et al. (2010) abstracting from computing the Lagrange multiplier for the occasionally binding constraint. This is due to the model characteristic that the total amount of wealth held by constrained households is negligibly small. The similar performance between the two algorithms is due to the linearity of the capital dynamics, which makes the state space-based approach and the sequence space-based approach only negligibly different.

Figure 3 plots a part of the predicted path $\{K_t^{(n)}\}_{t=0}^T$ and the realized (implied) path $\{K_t^*\}_{t=0}^T$ of aggregate capital K_t obtained from the repeated transition method and the

[25]The parameters are set as in the benchmark model in Krusell and Smith (1998) without idiosyncratic shocks in the patience parameter β.

[26]For some models where an occasionally binding constraint includes the current individual/aggregate state (e.g., irreversible investment), it is necessary to compute the expected future Lagrange multipliers. The repeated transition method can also accurately compute this expectation. See the sample codes in the Online Appendix.

simulated path from the fitted log-linear law of motion (Krusell and Smith, 1998).[27] As can be seen from all three lines hardly distinguished from each other, the repeated transition method computes almost identical equilibrium allocations as the log-linear law of motion by Krusell and Smith (1998). This is because the log-linear specification almost perfectly captures the actual law of motion in the model.

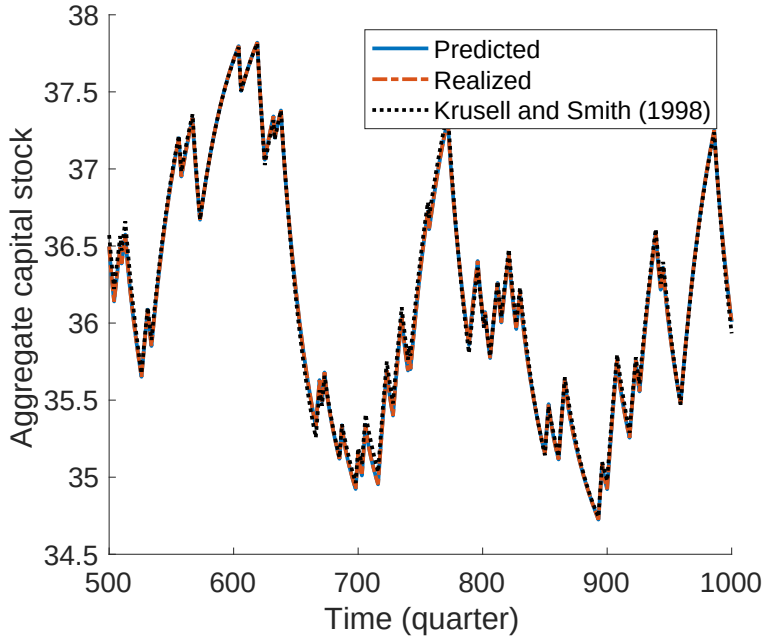


Figure 3: Equilibrium dynamics of the aggregate capital stock (Krusell and Smith, 1998)

Notes: The figure plots the time series of the aggregate wealth (capital) K_t in the model of Krusell and Smith (1998). The line with a round tick mark is the predicted wealth time series (n^{th} guess) $\{K_t^{(n)}\}_{t=500}^{1000}$. The line with the square tick mark is the realized wealth time series $\{K_t^*\}_{t=500}^{1000}$. The dashed line is the predicted wealth time series implied by the law of motion in Krusell and Smith (1998).

However, when a model features a non-trivial market-clearing condition, as in the model of Khan and Thomas (2008), the repeated transition method substantially outperforms the state space-based approach based on the law of motion Krusell and Smith (1997).[28] This is because the non-trivial market-clearing condition requires the state space-based approach to include an extra loop to find an exact market-clearing condition in each period, while the issue does not exist in the repeated transition method. Instead, the repeated transition

[27] This figure is motivated from the fundamental accuracy plot suggested in Den Haan (2010).

[28] Krusell and Smith (1997) algorithm is a variant of the algorithm in Krusell and Smith (1998), which is applicable to models with non-trivial market-clearing conditions. Khan and Thomas (2008) uses this algorithm.

method utilizes implied price by the market clearing condition, which coincides with the market clearing price only in the limit with respect to the number of iteration.

I solve the model in Khan and Thomas (2008) using both the repeated transition method and the Krusell and Smith (1997) algorithm with an external loop for the non-trivial market-clearing condition. Due to the substantial difference in the computational efficiency, it is tricky to compare the accuracy given the same implementation time. Therefore, instead of fixing the time, I impose the same termination condition for both methods in terms of the dynamic consistency.[29] Figure 4 plots the dynamics of price p_t (panel (a)) and aggregate capital stock K_t (panel (b)) computed from the repeated transition method and Krusell and Smith (1997) algorithm. For the allocations computed from the repeated transition method, both the predicted time series and the realized time series are plotted. As shown in the figure, all three lines display almost identical dynamics of the price and the aggregate allocations. The mean squared difference in the solutions between the repeated transition method and Khan and Thomas (2008) is less than 10^{-5}.

In terms of the computational efficiency, the two methods display a substantial discrepancy. the repeated transition method takes around 9 minutes to converge, while Krusell and Smith (1997) algorithm converge in around 5 to 6 hours on average in MATLAB.[30]

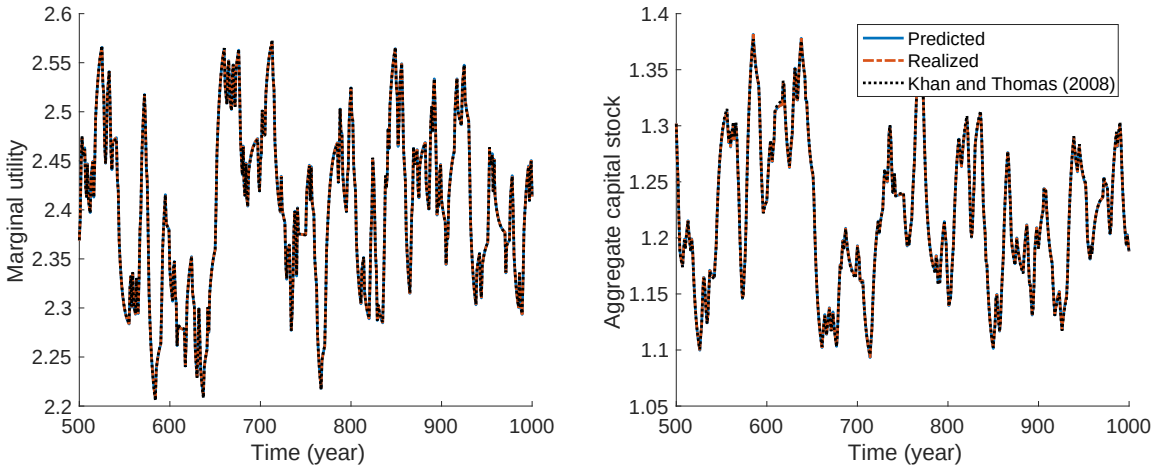
For both models of Krusell and Smith (1998) and Khan and Thomas (2008), I use the repeated transition method using the sufficient statistic approach where the aggregate capital stock (the first moment) is used as the sufficient statistic. For this approach, it is necessary to check whether the firm's individual value (policy) function is strictly monotone in the aggregate capital stock. In Appendix B, I show the strict monotonicity holds. Therefore, the aggregate capital stock functions as a sufficient statistic. Moreover, the inter-temporal dynamics of this statistic are approximately log-linear, which also guarantees the accuracy of the state space-based approach using the log-linear law of motion.

[29] Both of the algorithms are designed to stop when the following criterion is satisfied:

$$\max\{\sup_t\{||p_t^* - p_t^{(n)}||\}, \sup_t\{||K_t^* - K_t^{(n)}||\}\} < 10^{-7}.$$

The terminal condition is slightly different from the one in Step 5 of Section 2.4. Likewise, the terminal condition can be flexibly adjusted based on different combinations of $V_t^{(n)}, \Phi_t^{(n)}$, and $p_t^{(n)}$.

[30] The convergence speed might change depending on the updating weight.



(a) Price p_t $(= 1/C_t)$ (b) Aggregate capital stocks K_t

Figure 4: Computed dynamics of aggregate allocations (Khan and Thomas, 2008)

Notes: The figure plots the time series of the price p_t the aggregate wealth (capital) K_t in the model of Khan and Thomas (2008). In both panels, the line with a round tick mark is the predicted time series (n^{th} guess) $\{p_t^{(n)}, K_t^{(n)}\}_{t=500}^{1000}$; the line with the square tick mark is the realized time series $\{p_t^*, K_t^*\}_{t=500}^{1000}$; the dashed line is the predicted time series implied by the law of motion.

6 Applications

6.1 Overview of the applications

In this section, I overview the list of the applications I provide in the Online Appendix. The repeated transition method is applicable to a broad class of DSGE models. The application is not limited by the inclusion of 1) heterogeneous agents, 2) occasionally binding constraints, 3) multiple aggregate shocks, and 4) non-trivial market clearing conditions.

For example, the sample codes include the application for an extension of Krusell and Smith (1998) model by the endogenous labor supply with aggregate uncertainty shock, where the aggregate uncertainty shock is from Bloom et al. (2018).[31] Solving this extended model is demanding as it features all the hurdles mentioned above. Nevertheless, the repeated transition method accurately solves the model efficiently using the sufficient statistic approach, and I verify the monotonicity condition required for the sufficient statistic. However, the

[31]The extension by the endogenous labor supply is introduced in the appendix of Krusell and Smith (1998). The paper's appendix explains that the aggregate law of motion is still highly log-linear in aggregate capital, while the wage-capital relationship is nonlinear. This nonlinear relationship is accurately solved in the provided sample code using the repeated transition method.

equilibrium aggregate dynamics of the model are still log-linear, as in Krusell and Smith (1998), due to only a negligible mass of households occasionally constrained by the borrowing limit. Therefore, from the perspective of computation, it is not significantly different from the result of Krusell and Smith (1998) and Khan and Thomas (2008), where the micro-level non-linearity washes out after aggregation: the model can be accurately solved by linear state space approach.[32] Instead, I present a heterogeneous-firm real business cycle model where individual firms are subject to the occasionally binding capital irreversibility constraint in the following section, which leads to highly nonlinear aggregate dynamics. The model extends the canonical real business cycle model with irreversible investment used in Section 5 by including 1) heterogeneous firms that invest and are subject to the irreversibility constraint and 2) endogenous labor supply and demand.

6.2 The leading application

In this section, I show that the repeated transition method is powerful in solving nonlinear DSGE models with heterogeneous agents. Specifically, I analyze a heterogeneous-firm real business cycle model where individual firms are subject to the occasionally binding capital irreversibility constraint.

The representative household maximizes the following lifetime utility in the recursive form:

$$\begin{aligned} V(a;X) = \max_{c,a',N} \; & log(c) - \eta N + \beta \mathbb{E} V(a';X') \\ \text{s.t.} \quad & c + \int a'(X') q(X') d\Gamma_{X'} = a + w(X) N \\ & X' = G_H(X) \end{aligned}$$

where c is consumption; a is the asset (equity) value before dividend payment; N is labor supply; q is the Arrow-Debreu state price; w is wage; η is labor disutility parameter.[33] The

[32] The extended version of Krusell and Smith (1998) is available in the online appendix.

[33] Following Khan and Thomas (2008), we assume the Frisch elasticity of the labor supply is infinity. However, the repeated transition method also solves the case of the finite Frisch elasticity seamlessly. In the online appendix such examples are provided.

aggregate state X is a bundle of the distribution of individual firms Φ and aggregate TFP A: $X = \{\Phi, A\}$. The representative agent rationally expects the law of motion G_H of the aggregate states. I use the apostrophe to indicate future allocations.

Heterogeneous firms solve the following maximization problem of the present value of the sum of the dividend stream in the recursive form:

$$\begin{aligned} J(k, z; X) = \max_{k'} \; & d + \mathbb{E}_{z,X} M(X, X') J(k', z'; X') \\ \text{s.t. } & d = \pi(k, z; X) + (1 - \delta)k - k' \\ & k' \geq \phi I_{ss} + (1 - \delta)k \\ & \pi(k, z; X) = \max_{k,n} \; Ak^{\alpha} n^{\gamma} - w(X)n \\ & X' = G_F(X) \end{aligned}$$

where d is dividend; k is individual capital stock; z is the idiosyncratic productivity; n is the labor demand; π is a temporal profit; δ is the depreciation rate; I_{ss} is the steady-state aggregate investment level; ϕ is the irreversibility parameter. G_F is the law of motion of the aggregate state from a firm's perspective, which coincides with G_H under the rational expectation and satisfies the dynamic consistency in the recursive competitive equilibrium. The stochastic discount factor M is determined in the competitive market as follows:

$$M(X, X') = \beta \frac{c(X)}{c(X')}$$

The individual and aggregate log productivities follow AR(1) processes, which are discretized by the standard Tauchen method.

The recursive competitive equilibrium is defined based on the following market-clearing conditions:

$$\begin{aligned} \text{(Labor market)} \quad & N(X) = \int n(k, z; X) d\Phi \\ \text{(Equity market)} \quad & a(X) = \int J(k, z; X) d\Phi. \end{aligned}$$

In the market clearing condition, the supply of equity meets the demand in the form of

household wealth.

For computation, I use the standard parameter levels in the literature, which are available in Appendix C. For easier computation, I normalize the firm's value function by contemporaneous consumption $c(S)$ following Khan and Thomas (2008). Then, I define a price $p(S) := 1/c(S)$ and the normalized value function $\tilde{J}(k, z; S) := p(S)J(k, z; S)$. From the intra-temporal and inter-temporal optimality conditions of households, $w(S) = \eta/p(S)$ and $M(S, S') = \beta p(S')/p(S)$. Thus, $p(S)$ is the only price to characterize the equilibrium. Then, the equilibrium price $p(S)$ is determined from the following variant of the non-trivial market clearing condition:[34]

$$p = \arg_{\widetilde{p}} \left\{ 1/\widetilde{p} - \int \left[d(x; X, \widetilde{p}) + w(X, \widetilde{p}) n(x; X, \widetilde{p}) \right] d\Phi = 0 \right\}.$$

where the consumption is $c = 1/\widetilde{p}$. In the problem above, the individual dividend policy d, wage w, and labor demand n functions are augmented by an argument $\widetilde{p}$, indicating that these objects are calculated assuming the price is at the level of $\widetilde{p}$. This is a fixed-point problem and computationally costly to solve, as consumption $c = 1/\widetilde{p}$ affects the firms' inter-temporal decision, which in turn affects the dividend, consumption, and thus, $\widetilde{p}$. Instead of the market clearing price, the repeated transition method uses the implied price p^*, which is obtained as follows:

$$\begin{aligned} p^* &= \arg_{\widetilde{p}} \left\{ 1/\widetilde{p} - \int \left[d(x; X, p^{(n)}) + w(X, p^{(n)}) l(x; X, p^{(n)}) \right] d\Phi = 0 \right\} \\ &= 1/ \int \left[d(x; X, p^{(n)}) + w(X, p^{(n)}) l(x; X, p^{(n)}) \right] d\Phi, \end{aligned}$$

where $p^{(n)}$ is the guessed price in the n^{th} iteration. By bypassing the costly fixed-point problem, the method dramatically improves the speed of the computation.

[34] This condition is derived from combining the household's budget constraint and the equity market clearing condition.

The first-order condition of a firm's problem is as follows:[35]

$$1 = \mathbb{E}_{z,X} M(X, X') J_1(k', z'; X') + \lambda(k, z; X) \tag{1}$$

where λ is the Lagrange multiplier of the occasionally binding constraint. The envelope condition of a firm's problem is as follows:

$$\begin{aligned} J_1(k, z; X) &= \pi_1(k, z; X) + (1 - \delta) - \lambda(k, z; X)(1 - \delta) \\ &= \pi_1(k, z; X) + (1 - \delta)(1 - \lambda(k, z; X)) \end{aligned} \tag{2}$$

Combining equations (1) and (2), we have the following inter-temporal optimality condition:

$$1 - \lambda(k, z; X) = \underbrace{\mathbb{E}_{z,X} M(X, X')(\pi_1(k', z'; X') + (1 - \delta)(1 - \lambda(k', z'; X')))}_{\text{Expected marginal benefit}},$$

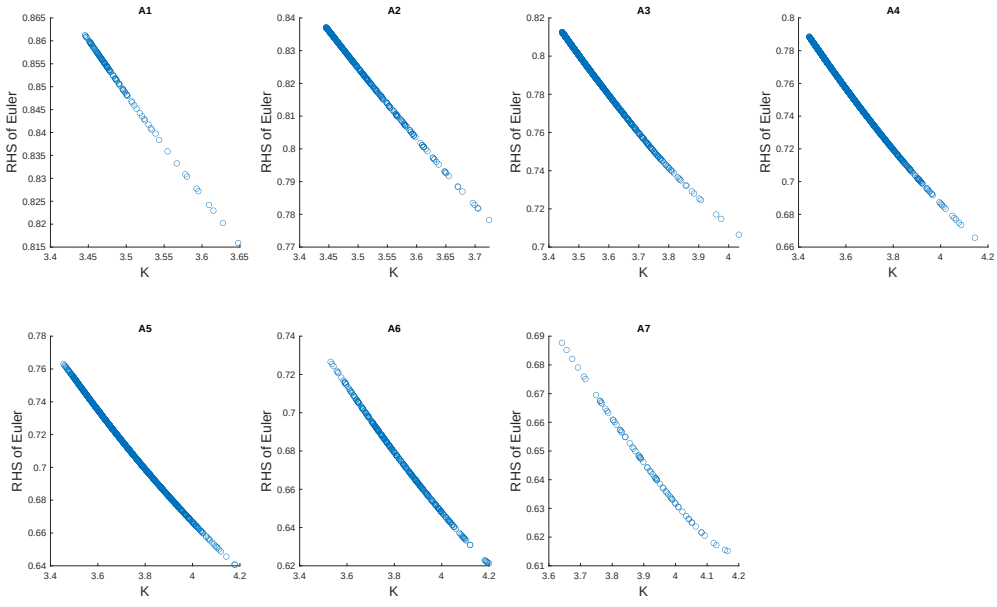


Figure 5: Strict monotonicity of the marginal benefit in the aggregate capital stock

The expected marginal benefit requires the computation of state-contingent allocations

[35] The subscript denotes the partial derivative with respect to the argument in the corresponding argument order.

$(X' - dependent)$ of marginal profit π_1 and Lagrange multiplier λ. For this problem, I take the sufficient statistic approach described in Section 3.1, and the aggregate capital stock $K(S)$ (the first moment of the distribution of the firm-level capital stocks) is the sufficient statistic. I validate this approach by showing the monotonicity condition of Proposition 1 is satisfied. In particular, I show that each individual's marginal benefit $\pi_1 + (1-\delta)(1-\lambda)$ is strictly monotone in K for all aggregate exogenous state realization (TFP level). Figure 5 plots the marginal benefit in the vertical axis and the corresponding aggregate equilibrium capital stock in the horizontal axis for different TFP levels ($A1$, $A2$, $A3$, ..., $A7$) given the individual state fixed at the median levels of capital stock and the firm-level productivity. In the unreported tests, which are available in the sample code, I confirm that monotonicity holds regardless of the choice of an individual firm.

For the computation, I run a simulation of 1,000 periods of aggregate TFP shock, where the TFP is discretized by the Tauchen method, covering the two standard deviation ranges. The computed aggregate capital path is highly nonlinear due to the occasionally binding constraint. Figure 6 plots a part of the predicted path $\{K_t^{(n)}\}_{t=0}^T$ and the realized (implied) path $\{K_t^*\}_{t=0}^T$ of aggregate capital K_t obtained from the repeated transition method and the simulated path from the fitted log-linear law of motion. While the predicted path and the realized path coincide at the equilibrium path, the log-linear prediction significantly deviates from the others.

6.3 Nonlinearity and aggregation

In this section, I compare the nonlinearity implied in the heterogeneous firm model with the one in the representative firm model, which is obtained by simply muting the heterogeneous firm-level productivity.[36] For a valid comparison, I feed the same exogenous aggregate TFP path for both models and compute the equilibrium using the repeated transition method. Figure 7 plots a part of the equilibrium capital path of the heterogeneous firm model (solid line) and the representative firm model (dash-dotted line) in log deviation from

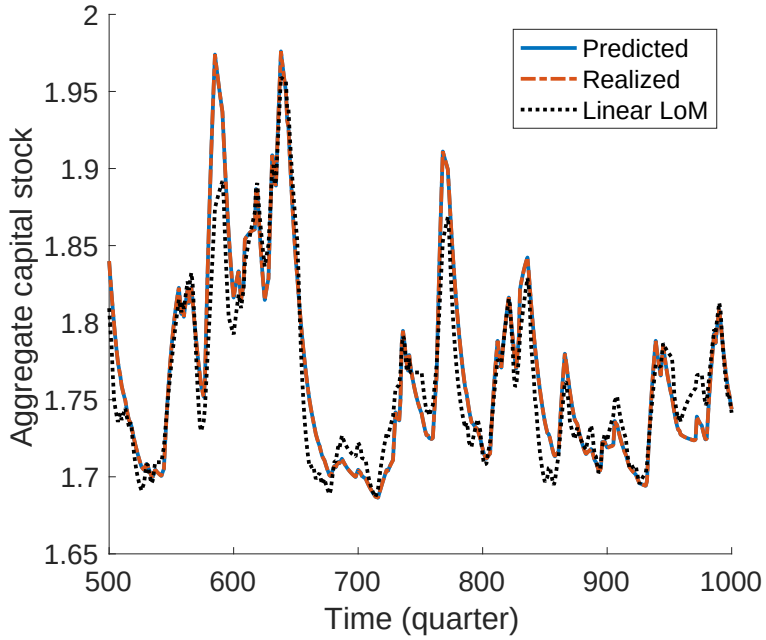


Figure 6: The equilibrium path of aggregate capital stock

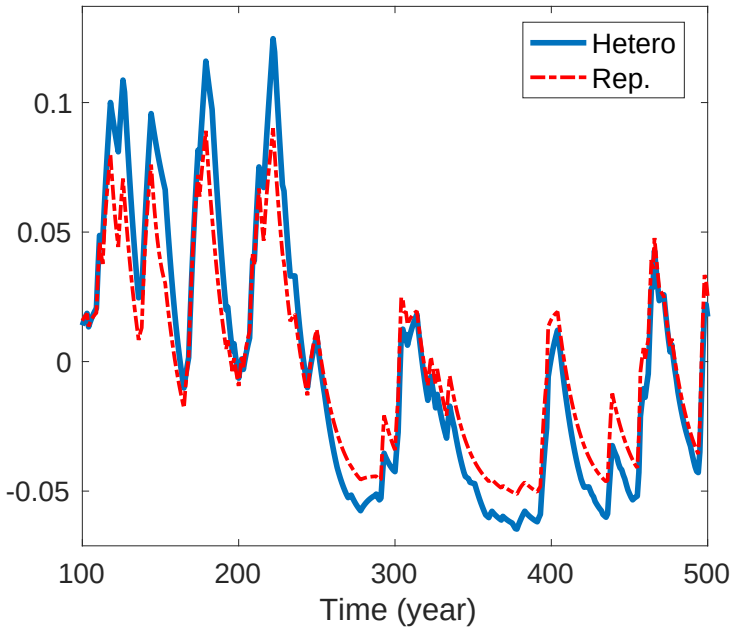


Figure 7: Equilibrium dynamics comparison: Heterogeneous vs. representative

each model's steady state.

The volatility of the aggregate capital stock is significantly greater in the heterogeneous

[36] All the parameters are assumed at the same level except for the firm-level productivity.

firm model, which is the by-product of the greater volatility in the aggregate investment. Table 2 reports the business cycle statistics of the two models. The output and investment are around 10 percent more volatile in the heterogeneous firm model than the other, while consumption volatilities are at a similar level. The skewness of output is greater, and the skewness of consumption and investment is lower in the heterogeneous firm model.

Table 2: Business cycle statistics: Heterogeneous vs. representative

	Heterogeneous	Representative
Volatility		
$log(Output)$	0.042	0.039
$log(Consumption)$	0.034	0.034
$log(Investment)$	0.083	0.077
Skewness		
$log(Output)$	0.672	0.638
$log(Consumption)$	-0.049	-0.02
$log(Investment)$	1.757	1.926

The representative firm model fails to represent the heterogeneous-firm model over the business cycle. The reason is the nonlinearity at the firm-level capital dynamics.[37] To see this, I compute the same heterogeneous and representative firm models without the occasionally binding constraint (fully reversible investment), which is the source of the nonlinearity. I refer to this version as the frictionless benchmark. Figure 8 plots the aggregate capital dynamics of the frictionless benchmark of both models in log deviation from the steady state and the predicted path by the log-linear law of motion. These three lines perfectly coincide indicating that firm-level linearity makes the perfect representation that is also linear.

[37]This result is specific to this model. For example, Khan and Thomas (2008) shows that the general equilibrium effect washes out the firm-level nonlinearity in their model, as can be also seen from Figure 4.

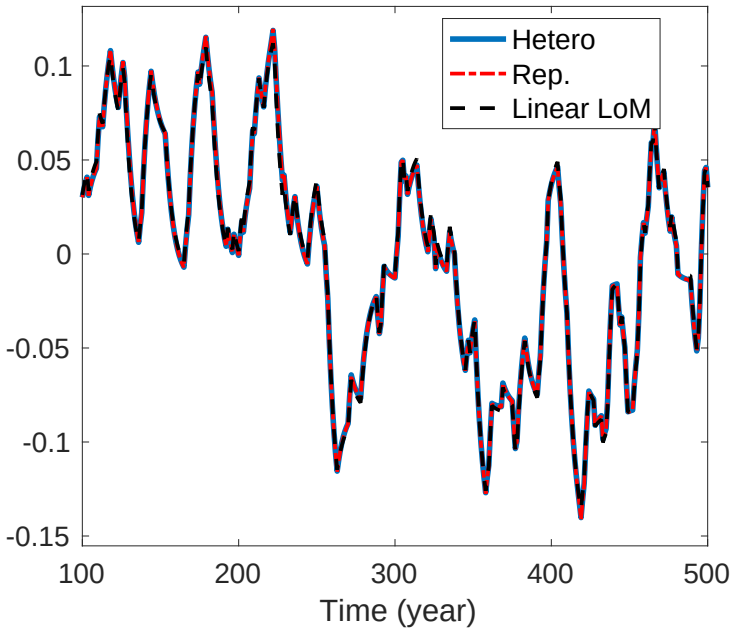


Figure 8: Equilibrium dynamics comparison - frictionless: Heterogeneous vs. representative

7 Concluding remarks

The repeated transition method globally, accurately, and efficiently solves DSGE models in the sequence space, and the method is simple and applicable to a broad class of DSGE models. The solution is dynamically consistent by construction. The method also provides a new angle to a sufficient statistic approach in the computation of models with complex aggregate states. Based on this method, I provide a theoretical condition under which the sufficient statistic approach works for the global solution, and the condition is easily testable. This method's novelty lies in the computation of the expected future value (policy) functions, which can be flexibly combined with existing methods in the literature. As the method does not rely on a particular form of the law of motion, it is a particularly useful tool for analyzing nonlinear DSGE models with complex aggregate states and history-dependent (or state-dependent) equilibrium dynamics.

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