

Dancing on the Saddles: A Geometric Framework for Stochastic Equilibrium Dynamics¹

Online Appendix

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A Sequential formulation of a canonical RBC

A representative household with temporal log utility is considered. Given initial condition (a_0, A_0) , the household maximizes life-time utility under stochastic aggregate TFP A_t , which is subject to a budget constraint as elaborated on below:

$$\max_{\{c_\tau(A^{(\tau)}), a_{\tau+1}(A^{(\tau)})\}_{\tau=0}^{\infty}} \mathbb{E}_0 \sum_{\tau=0}^{\infty} \beta^\tau \log(c_\tau(A^{(\tau)})) \quad (1)$$

$$\text{s.t. } c_\tau(A^{(\tau)}) + a_{\tau+1}(A^{(\tau)}) = a_\tau(A^{(\tau-1)})(1 + r(A^{(\tau)})) + w(A^{(\tau)}), \quad \text{for } \forall \tau, \forall A_t \quad (2)$$

$$a_{\tau+1}(A^{(\tau)}) \geq -\bar{a}, \quad \text{for } \forall \tau \quad (3)$$

where superscript τ inside a bracket denotes history of a variable up to period τ ; $-\bar{a}$ is the natural borrowing limit to preempt Ponzi scheme. Labor supply is exogenously fixed at unity. I consider the following competitive factor prices given CRS Cobb-Douglas production function:

$$r(A^{(\tau)}) = A_t \alpha (K(A^{(\tau)}))^{\alpha-1} - \delta \quad (4)$$

$$w(A^{(\tau)}) = A_t (1 - \alpha) (K(A^{(\tau)}))^{\alpha}, \quad (5)$$

K is capital stock, that satisfies $K(A^{(\tau)}) = a(A^{(\tau)})$ in equilibrium. With the regularity conditions given in [Stokey et al. \(1989\)](#), this sequential formulation yields the same optimality conditions as the recursive form in the main text.

B Individual conditional saddles in Aiyagari (1994)

I define individual-level conditional saddle path in the heterogeneous-household economy without aggregate uncertainty ($A = A' = 1$). The conditional saddle is defined for the SRCE as in [Aiyagari \(1994\)](#).

Definition 1 (Individual reachable equilibrium set).

Fix a stationary RCE of the Aiyagari (1994) economy, which delivers an individual asset law of motion

$$g_a : \mathbb{R}_+ \times \mathcal{Z} \rightarrow \mathbb{R}_+,$$

where \mathcal{Z} is a finite Markov set (e.g. $\mathcal{Z} = \{u, e\}$) with transition matrix Π . Fix an initial condition $(a_0, z_0) \in \mathbb{R}_+ \times \mathcal{Z}$.

The individual reachable equilibrium set $\mathcal{R}^{\text{ind}}(a_0, z_0) \subseteq \mathbb{R}_+ \times \mathcal{Z}$ is the smallest set

containing (a_0, z_0) that is closed under all feasible one-step equilibrium transitions:

$$(a, z) \in \mathcal{R}^{\text{ind}}(a_0, z_0) \quad \text{and} \quad \Pi_{zz'} > 0 \quad \implies \quad (g_a(a, z'), z') \in \mathcal{R}^{\text{ind}}(a_0, z_0).$$

Definition 2 (Individual conditional saddle).

Given (a_0, z_0) and $\mathcal{R}^{\text{ind}}(a_0, z_0)$, for each $z \in \mathcal{Z}$ define the z -slice

$$\mathcal{R}_z^{\text{ind}}(a_0, z_0) := \{a \in \mathbb{R}_+ : (a, z) \in \mathcal{R}^{\text{ind}}(a_0, z_0)\}.$$

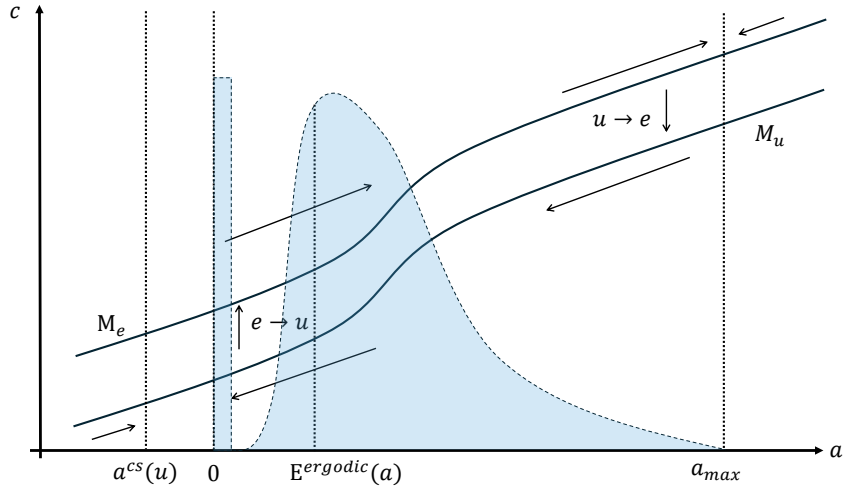
The individual conditional saddle under idiosyncratic state z is the largest subset $\mathcal{M}_z^{\text{ind}}(a_0, z_0) \subseteq \mathcal{R}_z^{\text{ind}}(a_0, z_0)$ that is forward invariant under the frozen- z mapping:

$$a \in \mathcal{M}_z^{\text{ind}}(a_0, z_0) \quad \implies \quad g_a(a, z) \in \mathcal{M}_z^{\text{ind}}(a_0, z_0).$$

Equivalently,

$$\mathcal{M}_z^{\text{ind}}(a_0, z_0) := \bigcup \left\{ \mathcal{S} \subseteq \mathcal{R}_z^{\text{ind}}(a_0, z_0) : g_a(\mathcal{S}, z) \subseteq \mathcal{S} \right\}.$$

Figure B.1: Individual conditional saddle paths in the stationary RCE



Notes: The figure illustrates the individual conditional saddle paths for $z = e$ and $z = u$ in Aiyagari (1994).

Figure B.1 illustrates individual-level conditional saddle paths in Aiyagari (1994). Because of the borrowing constraint, the conditional steady state associated with the unemployment state $z = u$ is not attained. Under standard calibrations, this generates a positive mass of agents at the borrowing limit. Analogous to the aggregate-level case, heterogeneity in the slopes of individual conditional saddle paths implies differential responses of individual consumption to idiosyncratic shocks.

C Proofs for the theoretical results

Proposition 2 (Aggregate uncertainty and the conditional steady states).

The following inequalities hold:

$$K_B^{cs} < K_B^{pf} < K_G^{pf} < K_G^{cs}, \quad c_B^{cs} < c_B^{pf} < c_G^{pf} < c_G^{cs}.$$

Proof.

Let $R(A, K) := 1 - \delta + \alpha AK^{\alpha-1}$ denote the gross return on capital and let the K -nullcline (feasibility locus) be

$$c(A, K) = AK^\alpha - \delta K.$$

The PF steady states solve the Euler equations under absorbing beliefs,

$$1 = \beta R(B, K_B^{pf}), \quad 1 = \beta R(G, K_G^{pf}).$$

Since $R(A, K)$ is strictly increasing in A and strictly decreasing in K (because $\alpha - 1 < 0$), it follows immediately that $K_B^{pf} < K_G^{pf}$.

Next define the frozen-regime CS Euler residuals evaluated on the K -nullcline by

$$\begin{aligned} F_B(K) &:= \beta \left[\pi_{BB} R(B, K) + \pi_{BG} \frac{c(B, K)}{c(G, K)} R(G, K) \right] - 1, \\ F_G(K) &:= \beta \left[\pi_{GG} R(G, K) + \pi_{GB} \frac{c(G, K)}{c(B, K)} R(B, K) \right] - 1. \end{aligned}$$

By construction, the conditional steady states satisfy $F_B(K_B^{cs}) = 0$ and $F_G(K_G^{cs}) = 0$.

Step 1: show $K_B^{cs} < K_B^{pf}$. Evaluate F_B at K_B^{pf} . Using $1 = \beta R(B, K_B^{pf})$ and $\pi_{BB} = 1 - \pi_{BG}$,

$$\begin{aligned} F_B(K_B^{pf}) &= \beta \left[(1 - \pi_{BG}) R(B, K_B^{pf}) + \pi_{BG} \frac{c(B, K_B^{pf})}{c(G, K_B^{pf})} R(G, K_B^{pf}) \right] - 1 \\ &= \pi_{BG} \left[\beta \frac{c(B, K_B^{pf})}{c(G, K_B^{pf})} R(G, K_B^{pf}) - 1 \right]. \end{aligned}$$

Thus $F_B(K_B^{pf}) < 0$ is equivalent to

$$\beta \frac{c(B, K_B^{pf})}{c(G, K_B^{pf})} R(G, K_B^{pf}) < 1 \quad \Longleftrightarrow \quad \frac{R(G, K_B^{pf})}{R(B, K_B^{pf})} < \frac{c(G, K_B^{pf})}{c(B, K_B^{pf})},$$

where we used $\beta R(B, K_B^{pf}) = 1$ to divide both sides by $R(B, K_B^{pf})$.

We now prove this strict inequality for any K with $c(B, K), c(G, K) > 0$. Write $x :=$

$\alpha K^{\alpha-1} > 0$ and $y := K^\alpha > 0$. Then

$$\frac{R(G, K)}{R(B, K)} = \frac{1 - \delta + Gx}{1 - \delta + Bx}, \quad \frac{c(G, K)}{c(B, K)} = \frac{Gy - \delta K}{By - \delta K}.$$

Since $1 - \delta > 0$, we have the strict bound

$$\frac{1 - \delta + Gx}{1 - \delta + Bx} < \frac{Gx}{Bx} = \frac{G}{B}.$$

Since $\delta K > 0$ and $Gy > By$, subtracting the same positive term from numerator and denominator enlarges the ratio, yielding

$$\frac{Gy - \delta K}{By - \delta K} > \frac{Gy}{By} = \frac{G}{B}.$$

Combining the two displays gives

$$\frac{R(G, K)}{R(B, K)} < \frac{G}{B} < \frac{c(G, K)}{c(B, K)},$$

and in particular the desired inequality holds at $K = K_B^{pf}$. Therefore $F_B(K_B^{pf}) < 0$.

Finally, note that F_B is strictly decreasing in K on the relevant region because both $R(B, K)$ and $R(G, K)$ are strictly decreasing in K and $c(B, K)/c(G, K)$ is also decreasing in K along the feasibility locus.² Hence, since $F_B(K_B^{cs}) = 0$ and $F_B(K_B^{pf}) < 0$, we must have $K_B^{cs} < K_B^{pf}$.

Step 2: show $K_G^{cs} > K_G^{pf}$. Similarly, evaluate F_G at K_G^{pf} . Using $1 = \beta R(G, K_G^{pf})$ and $\pi_{GG} = 1 - \pi_{GB}$,

$$\begin{aligned} F_G(K_G^{pf}) &= \beta \left[(1 - \pi_{GB}) R(G, K_G^{pf}) + \pi_{GB} \frac{c(G, K_G^{pf})}{c(B, K_G^{pf})} R(B, K_G^{pf}) \right] - 1 \\ &= \pi_{GB} \left[\beta \frac{c(G, K_G^{pf})}{c(B, K_G^{pf})} R(B, K_G^{pf}) - 1 \right]. \end{aligned}$$

Thus $F_G(K_G^{pf}) > 0$ is equivalent to

$$\beta \frac{c(G, K_G^{pf})}{c(B, K_G^{pf})} R(B, K_G^{pf}) > 1 \quad \Longleftrightarrow \quad \frac{R(B, K_G^{pf})}{R(G, K_G^{pf})} > \frac{c(B, K_G^{pf})}{c(G, K_G^{pf})}.$$

But the argument above applied with (B, G) swapped gives, for any K with positive con-

²This monotonicity is standard and can be verified by differentiation; it is also visually apparent in the (K, C) phase diagram.

sumption,

$$\frac{R(B, K)}{R(G, K)} > \frac{B}{G} > \frac{c(B, K)}{c(G, K)}.$$

Hence $F_G(K_G^{pf}) > 0$. Since F_G is strictly decreasing in K and $F_G(K_G^{cs}) = 0$, we conclude $K_G^{cs} > K_G^{pf}$.

Step 3: conclude the ordering and translate to consumption. We have shown $K_B^{cs} < K_B^{pf}$ and $K_G^{pf} < K_G^{cs}$, and already $K_B^{pf} < K_G^{pf}$, hence

$$K_B^{cs} < K_B^{pf} < K_G^{pf} < K_G^{cs}.$$

Finally, along the K -nullcline $c(A, K) = AK^\alpha - \delta K$ is strictly increasing in A and (on the relevant region) increasing in K , so the same ordering carries over to consumption:

$$c_B^{cs} < c_B^{pf} < c_G^{pf} < c_G^{cs}.$$

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D Continuous-time benchmark for Krusell and Smith (1998)

In discrete-time models, the possibility of "jumps" across a nullcline necessitates the Single-Crossing Condition (SC) to ensure that the path toward the steady state remains well-behaved. In continuous time, however, these pathologies are ruled out by the continuity of the equilibrium flow. This section demonstrates that on a one-dimensional conditional saddle, strict monotonicity, and thus the sufficiency of aggregate capital, is a topological consequence of stability and uniqueness.

In discrete-time models, the possibility of "jumps" across a nullcline necessitates the Single-Crossing Condition (SC) to ensure that the path toward the steady state remains well-behaved. In continuous time, however, these pathologies are ruled out by the continuity of the equilibrium path. This section demonstrates that on a one-dimensional conditional saddle, strict monotonicity, and thus the sufficiency of aggregate capital, is a topological consequence of stability and uniqueness.

Consider the continuous-time analogue of the stochastic economy where the exogenous aggregate state $A_t \in \{B, G\}$ follows a continuous-time Markov process. The equilibrium evolution of the endogenous state is characterized by the following law of motion for aggregate capital K_t :

$$\dot{K}_t = f(K_t, A_t) \tag{6}$$

where f denotes the net investment function derived from the recursive competitive equilib-

rium. Under a frozen-regime $A \in \{B, G\}$, the economy evolves as an autonomous dynamical system on the manifold \mathcal{M}_A :

$$\dot{K} = f_A(K) \quad (7)$$

Consistent with Assumption 1, I require the following regularity conditions:

- **Uniqueness:** There exists a unique conditional steady state K_A^{cs} where net investment is zero: $f_A(K_A^{cs}) = 0$.
- **Stability:** For any initial capital $K \in \mathcal{M}_A$, the equilibrium path $K(t)$ converges to K_A^{cs} as $t \rightarrow \infty$.

Proposition 3 (Monotonicity and sufficiency in continuous time).

In the continuous-time framework, aggregate capital K converges strictly monotonically to K_A^{cs} along any frozen-regime trajectory on the conditional saddle \mathcal{M}_A . Consequently, K is an exact sufficient statistic for the equilibrium state.

Proof.

Let $K(t)$ be an equilibrium path on the conditional saddle \mathcal{M}_A initialized at $K_0 \neq K_A^{cs}$.

1. **Sign of net investment:** By the uniqueness of the conditional steady state, net investment $f_A(K)$ is non-zero for all capital levels other than K_A^{cs} . Because f_A is a continuous function of K , it must maintain a constant sign for all capital levels between the initial state K_0 and the steady state K_A^{cs} .
2. **Direction of the path:** The stability requirement $\lim_{t \rightarrow \infty} K(t) = K_A^{cs}$ determines the required sign of net investment. If $K_0 < K_A^{cs}$, the economy must exhibit positive net investment ($f_A(K) > 0$) to reach the steady state; if $K_0 > K_A^{cs}$, net investment must be negative ($f_A(K) < 0$).
3. **Strict monotonicity:** Since the rate of change $\dot{K} = f_A(K)$ never reaches zero and cannot change sign without passing through a stationary point, the path $K(t)$ is strictly monotone.
4. **Injectivity and sufficiency:** Because capital moves strictly in one direction along the one-dimensional manifold \mathcal{M}_A , each capital level K corresponds to a unique distributional state Φ . Thus, knowing the scalar K is sufficient to identify the full state of the economy.

■

References

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- Stokey, N. L., R. E. Lucas, and E. C. Prescott (1989). *Recursive Methods in Economic Dynamics*. Harvard University Press.