Dynamically Consistent Global Nonlinear Solutions in the Sequence Space: Theory and Applications*

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January 30, 2025
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Abstract

This paper develops a novel global nonlinear solution method for dynamic stochastic general equilibrium models that achieves high accuracy and computational efficiency in sequence space. The method computes conditional expectations of future value or policy functions by contingently combining the previous iteration's realizations without requiring a parametric law of motion or parametrized expectation. The method efficiently handles rich heterogeneity and occasionally binding constraints while providing theoretical foundations for dimensional reduction through sufficient statistics. Despite its simple implementation, the computation is highly efficient, bypassing fixed-point problems in each iteration, including non-trivial market clearing, as demonstrated by applications to various macroeconomic models.

Keywords: Global nonlinear solution, sequence space, dynamic consistency, sufficient statistic, business cycle.

JEL codes: C63, E32, C61

*This paper is based on my Ph.D. dissertation at the University of Pennsylvania. I am extremely grateful to Jesús Fernández-Villaverde, Dirk Krueger, Andrew Abel, and Frank Schorfheide for their invaluable guidance and support. I also thank Kosuke Aoki, Florin Bilbiie, Christopher Carroll, Vasco Carvalho, Chris Edmond, Martin Ellison, Wouter Den Haan, Xiang Fang, Joel Flynn, James Graham, Zhen Huo, In Hwan Jo, Miles Kimball, Sagiri Kitao, Dongya Koh, Eunseong Ma, Albert Marcet, Yusuf Mercan, Makoto Nirei, Alessandro Peri, Bruce Preston, José-Víctor Ríos-Rull, Kjetil Storesletten, Ludwig Straub, Stephen Terry, Yucheng Yang, Donghai Zhang, and seminar participants at the KEA, SCE, RES Symposium, Keio University, University of Colorado Boulder, Midwest Macro, University of Tokyo, Osaka University ISER, University of Sydney, Monash University, University of Melbourne, Queen Mary University of London, University of Cambridge, CEA, Tinbergen Institute, HKUST/Jinan Macro Workshop, and NUS/Dynare conference for their insightful comments and discussions. All errors are my own.

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1 Introduction

Modern macroeconomic analysis increasingly requires solving models with rich heterogeneity and complex nonlinear dynamics. These features are crucial for understanding state-dependent policy effects, the interplay between inequality and business cycles, and distributional responses to aggregate shocks. However, existing solution methods face a challenging trade-off: they require an increasingly intricate algorithm as model complexity grows, while more simple and efficient methods rely on approximations that may miss important nonlinear dynamics.

This paper develops a solution method that resolves this trade-off, enabling accurate global analysis of complex models while maintaining implementation simplicity. The approach computes state-contingent expectations without parameterizing laws of motion, enabling accurate analysis of nonlinear dynamics in heterogeneous agent models. The method is particularly valuable for analyzing state-dependent policy effects and distributional outcomes that standard linearization-based approaches may miss.

Moreover, this paper provides theoretical foundations for using sufficient statistics to solve dynamic stochastic general equilibrium (DSGE, hereafter) models with complex aggregate states, including infinite-dimensional objects. Although sufficient statistics have proven valuable for summarizing aggregate states, the precise conditions justifying their use in solving dynamic models have remained an open question. This paper fills the gap by establishing theoretical conditions that guarantee exact solutions when using sufficient statistics in the new solution method. This theoretical advance enables the method to efficiently solve a broad spectrum of macroeconomic models – from a standard real business cycle model to cutting-edge heterogeneous-agent models with multidimensional aggregate states – all within a unified computational approach.[1]

The key innovation of the method lies in computing conditional expectations directly from realized equilibrium allocations of previous iterations. In particular, the method, which I name the *repeated transition method* (RTM, hereafter), exploits a fundamental property of DSGE models' recursive competitive equilibria - their *recursivity*. That is, if a simulated path of a stationary aggregate shock process is long enough, an endogenous aggregate state is almost surely revisited. Along with this recursion of the endogenous aggregate state, different exogenous aggregate states are stochastically realized on the path, forming an ergodic set of aggregate states. This ergodic set encompasses all the possible combinations of endogenous and exogenous aggregate states in

[1] All the sample MATLAB codes for various DSGE models with heterogeneous agents, frictions, or multiple aggregate states are available at https://sites.google.com/sas.upenn.edu/hanbaeklee/computation-lab?authuser=0.

equilibrium. This property implies that all state-contingent future allocations are obtainable as realized equilibrium outcomes somewhere on the sufficiently long simulated path.

Therefore, by identifying periods with equilibrium outcomes corresponding to each contingent future state from the previous iteration, the RTM characterizes an agent's conditional expectations at any point on the simulated path. The identifiability of such period is guaranteed by the *recursivity* (or by the *ergodicity*) of the recursive competitive equilibrium. Then, the conditional expectation converges to the true level (function) as iterations proceed, ensured by the *stability* and the *uniqueness* of the recursive competitive equilibrium. Notably, this approach eliminates the need to specify parametric laws of motion or expectations - the method requires only a metric to assess similarity between aggregate states across periods.

For example, consider an agent's infinite-horizon dynamic problem under aggregate uncertainty with two possible exogenous aggregate states: G (Good) or B (Bad). To solve the agent's problem at period t, a researcher needs to construct the expected value (policy) function of period $t+1$. The RTM accomplishes this by first identifying, for each possible future state $S_{t+1} \in \{G, B\}$, a period in the previous iteration's allocation path where the endogenous aggregate state most closely matches that of period t+1. The expected future value function is then constructed by combining the time-specific value functions from these identified periods. This approach works because the ergodicity of a sufficiently long simulation ensures the existence of periods where endogenous aggregate allocations (such as the distribution of individual states) match those of period t+1 under each possible shock realization. Consequently, the expected future value (policy) function at each period can be accurately constructed by combining these realized outcomes from the simulation path.[2]

The method iteratively updates the guessed (predicted) allocation path by using the realized allocation path in the past iteration until they converge to each other.[3] In these steps, each iteration passes over the information of the whole sequence of the realized allocations to the next iteration, utilizing the maximal set of information regarding the transition dynamics. This approach fundamentally differs from existing state-space methods, which restrict transition dynamics to functional relationships between current and future states. By avoiding such parametric specifications, the RTM accurately computes the equilibrium dynamics even in highly nonlinear models. Further, the

[2]The method's name - repeated transition method - reflects its key feature of utilizing *repeated transitions* between the same (similar) endogenous states with different exogenous states.

[3]The terminology "predicted" means predicted from the perspective of researcher outside the model. It is equivalent to n^{th} guess for the allocation paths.

required simulation length for the RTM is not longer than that for the existing methods, as the update based on the whole sequence utilizes all information about the dynamics, minimizing the information waste per marginal increase in the number of periods in a simulation.

When a model includes multidimensional endogenous aggregate states, the step to compare such high-dimensional objects across the periods can be a computational bottleneck.[4] To address this challenge, I establish theoretical foundations for using sufficient statistics in the RTM. Specifically, I prove that when time-specific value functions are strictly monotone in an aggregate equilibrium variable (conditional on individual and exogenous aggregate states), this variable serves as a sufficient statistic that delivers exact solutions. The variable (sufficient statistic) effectively plays a role as an indexing function of the value (policy) function's ranking across the time and thus as a similarity index.[5] I demonstrate the power of this approach in a leading application to an extended version of Krusell and Smith (1997), where a univariate sufficient statistic successfully captures the dynamics of the multi-dimensional endogenous distribution through this monotonicity property.

When the accuracy of the RTM is compared with the existing solution methods for a highly nonlinear RBC model with occasionally binding investment irreversibility constraint (McGrattan, 1996; Christiano and Fisher, 2000), the RTM achieves superior accuracy in terms of Euler equation errors and dynamic consistency metrics compared to existing approaches.[6] In terms of computational efficiency, the RTM offers substantial advantages over Krusell and Smith (1997), particularly for models with period-by-period fixed-point problems like market clearing conditions. While their method requires an internal loop to solve market-clearing conditions in each period, the RTM avoids this computationally intensive step. Instead of finding exact fixed points, it computes period-specific implied allocations (prices) from market clearing conditions and update these along with other allocations across iterations. Due to the *stability* and *uniqueness* of the recursive competitive equilibrium, the implied allocation path converges to the true fixed point path in the limit. The computational gains are particularly striking for models with complex aggregate states. For instance, in the Khan and Thomas (2008) framework with non-closed-form market clearing prices, the RTM achieves more than a tenfold speed improvement. Similarly, in the leading appli-

[4]Saving and updating the sequence of the distributions are also computationally burdensome tasks.

[5]The existence of such indexing variable is an important issue. However, the theoretical investigation of the existence is beyond this paper's scope. The author conjectures that the first moment of the endogenous individual state's distribution serves as a sufficient statistic if all individual inter-temporal policy functions are weakly monotone in each exogenous state, *and* the strict monotonicity holds for non-zero measures of individuals in any aggregate state realizations. Several applications in the online supplement support this conjecture, though I leave formal proof for future research.

[6]The RTM achieves this accuracy using standard MATLAB code, without requiring lower-level programming languages.

cation to the extended Krusell and Smith (1997) model with endogenous labor supply, the RTM efficiently handles multiple non-trivial market clearing conditions that typically create significant computational challenges for traditional approaches.

The RTM provides an integrated framework for solving a wide range of macroeconomic models. The online supplement demonstrates this versatility through sample codes that address various computational challenges, including 1) heterogeneous agents, 2) nonlinear aggregate dynamics including occasionally-binding constraints, 3) non-trivial market clearing conditions, 4) multiple aggregate shocks (including aggregate uncertainty shocks (Bloom et al., 2018)), 5) multi-dimensional endogenous aggregate states, 6) frictional labor markets, and 7) sticky prices (New Keynesian models).[7] I also validate the sufficient statistic approach in these models by testing the monotonicity condition.

This paper's first leading application extends the canonical heterogeneous-household real business cycle (RBC, hereafter) model of Krusell and Smith (1998) by incorporating endogenous labor supply, investment irreversibility, and fiscal spending shocks. This model combines several computational challenges: heterogeneous households with idiosyncratic labor productivity, multiple exogenous aggregate states, non-trivial market clearing conditions, and occasionally binding constraints. Despite these features generating highly nonlinear aggregate dynamics, the RTM delivers an accurate and efficient solution.

The global nonlinear solution of this model reveals two important insights about how micro-level nonlinearities shape macroeconomic outcomes. First, it shows that nonlinearities cause aggregate dynamics in the heterogeneous-household model to substantially deviate from those of the representative-household model, despite sharing identical parameters except for idiosyncratic productivity. This finding suggests that representative-agent approximations can significantly mischaracterize dynamics away from steady state under nonlinear equilibrium. Notably, investment and output volatilities are significantly muted in the heterogeneous-household model. This divergence disappears when I remove the occasionally binding constraint, with both models' dynamics converging to log-linear dynamics. Second, it shows that the fiscal spending shock leads to state-dependent multipliers due to the endogenous nonlinear variation in the portion of hand-to-mouth households, complementing recent findings in the literature (Kaplan and Violante, 2014; Kaplan

[7] Also, Lee et al. (2024) applies the RTM to Diamond-Pissarides-Mortensen (DMP) models with exogenous and endogenous job separation to analyze nonlinear labor market dynamics. The RTM also solves nonlinear New Keynesian models globally and accurately. Lee and Nomura (2024) applies the method to analyze the nonlinear inflation dynamics and Phillips curve outside and at the zero lower bound (ZLB). The method's ability to capture nonlinear dynamics enables analysis of state and history dependence that directly maps to empirical observations (Pizzinelli et al., 2020).

et al., 2018). The model generates an empirically realistic share of wealthy hand-to-mouth households through occasionally binding saving irreversibility constraints. The global nonlinear solution reveals that the fiscal spending multiplier is significantly greater during the periods of the high portion of hand-to-mouth households. This analysis demonstrates how the RTM enables study of endogenous state dependencies within a single equilibrium framework, avoiding the need to compare different steady states or parameter values under linearized dynamics.

The second leading application is a heterogeneous-household RBC model with portfolio choice (Krusell and Smith, 1997) and endogenous labor supply. This complex setting generates rich endogenous aggregate states and features non-trivial market clearing conditions in both labor and bond markets, along with two occasionally binding constraints. The RTM efficiently handles these computational challenges while maintaining accuracy. Importantly, the marginal benefits in the multiple inter-temporal optimality conditions exhibit strict monotonicity in aggregate capital stock K, validating the sufficient statistics approach. These findings highlight promising avenues for future research, as the macroeconomic implications of heterogeneous portfolio choice over the business cycle remain understudied despite their significance for macroeconomic policy. By overcoming the computational barriers that have historically limited research in this area, the RTM opens new possibilities for analyzing these critical but previously intractable questions.

Related literature The state space-based approach developed by Krusell and Smith (1997, 1998), which uses parametric laws of motion, represents a foundational contribution to global solution methods. While powerful for models with linear aggregate dynamics, this approach faces challenges with nonlinear dynamics due to difficulties in correctly specifying the law of motion. These specification challenges can lead to inaccurate computation of conditional expectations and dynamic inconsistency in equilibrium paths. Alternative approaches have emerged to address these limitations. Marcet (1988) and den Haan and Marcet (1990) introduced parameterized expectations, computing conditional expectations through parameterized functions with optimized coefficients. Den Haan and Rendahl (2010) advanced this literature by characterizing laws of motion through explicit aggregation of Taylor-approximated individual policy functions, enabling analysis of nonlinear dynamics. These methods share a common feature: they approximate conditional expectations through combinations of basis functions. The RTM takes a fundamentally different approach. Rather than approximating functional forms, it computes conditional expectations by identifying periods with matching future state realizations and combining the corresponding value or policy functions. This approach leverages three fundamental properties of DSGE models' recursive competitive equilibria - their recursivity, stability, and uniqueness - to deliver accurate solutions without functional approximation.

The RTM shares key features with recent global solution approaches developed by Cao et al. (2023) and Elenev et al. (2021), who achieve dynamic consistency by simultaneously solving transition equations and individual policy functions in state space. Like the RTM, their methods update transition equations using implied dynamics. However, their approaches require functional approximation when transition equations lack explicit forms. The RTM avoids this limitation by operating directly in sequence space, utilizing the complete realized equilibrium path to update transition dynamics without approximation or fitting. This fundamental difference leads to both greater efficiency and higher accuracy. The RTM's computational advantages extend beyond this core innovation. As I demonstrate in subsequent sections, this approach delivers particular gains in solving models with non-trivial market clearing conditions, significantly reducing computational burden while maintaining solution accuracy.

While also operating in sequence space, the RTM differs fundamentally from Auclert et al. (2021), which achieve remarkable computational efficiency through sequences of Jacobians. Their approach enables rapid likelihood-based estimation but requires perfect foresight. The RTM, in contrast, handles aggregate uncertainty while maintaining computational efficiency. This capability allows the method to accurately compute period-specific expected outcomes under uncertainty, distinguishing it from perfect-foresight approaches in the literature (Fair and Taylor, 1983; Juillard, 1996; Judd, 2002; Cai et al., 2017; Boppart et al., 2018). The RTM further differentiates itself by computing aggregate allocations and market-clearing prices directly on the simulated path without requiring law of motion specifications. This approach contrasts with perturbation and linearization methods (Reiter, 2009; Boppart et al., 2018; Ahn et al., 2018; Winberry, 2018; Childers, 2018), providing a more direct route to equilibrium solutions.

The RTM shares important features with simulation-based methods developed by Judd et al. (2011) and Maliar et al. (2011), which achieve computational efficiency by focusing on the realized ergodic state space. The RTM builds on this insight while making a crucial advance: we use the information contained in the realized state space to construct agents' conditional expectations at each point on the simulated path, significantly improving solution accuracy.

The RTM's reliance on a single, sufficiently long simulated path of aggregate shocks connects to recent work by Kahou et al. (2021). They show that aggregate dynamics can be characterized by solving a finite set of agents' problems using a single Monte Carlo draw of individual shocks under permutation invariance, computing the law of motion through deep learning. The RTM takes a different approach, using the long simulation to identify state-contingent outcomes directly, avoiding the need for functional approximation.

Roadmap Section 2 explains the repeated transition method. Section 3 explains the sufficient statistic approach. Section 4 explains how the RTM bypasses non-trivial market clearing conditions. Section 5 validates the accuracy and speed of the RTM through a comparison with the existing methods in the literature. Section 6 explains the RTM applications to the leading examples. Section 7 concludes.

2 The repeated transition method

2.1 A generic model framework

This section develops the repeated transition method (RTM) by first introducing a generic model framework that encompasses a broad class of dynamic stochastic general equilibrium (DSGE) models. The framework's flexibility allows it to accommodate both heterogeneous and representative agent specifications. I denote the individual state as x and the aggregate state as X. The individual state x is composed of the endogenous individual state a and the exogenous individual state s (the idiosyncratic shocks). The aggregate state X is composed of the endogenous aggregate states Φ and the exogenous aggregate state S (the aggregate shocks). The endogenous aggregate state Φ takes different forms depending on the model class. In heterogeneous agent models, it represents the distribution of individual states x, while in representative agent models, it captures the set of aggregate allocations.

$$\text{[Individual state]}: \qquad x = \{a, s\} \tag{1}$$

$$\text{[Aggregate state]}: \qquad X = \{\Phi, S\} \tag{2}$$

The idiosyncratic and aggregate shock processes are assumed to follow a Markov process with a transition matrix Π^s and Π^S, respectively. The value function is denoted as V. Following standard notation, variables with apostrophes indicate future period values. The objective function of an economic agent is composed of the contemporaneous part $f(y, x', x; X)$ and the expected future value. The agent maximizes the objective function by choosing (y, a'), where y is a vector of control variables that affects only the contemporaneous period. Then, the recursive formulation of an agent's problem is as follows:

$$V(x;X) = \max_{y,a'} \; f(y,a',x;X) + \mathbb{E}m(X,X')V(a',s';X') \tag{3}$$

$$\text{s.t.} \quad (y,x') \in \mathcal{B}(x;X,X',q), \quad \Phi' = F(X) \tag{4}$$

where $m(X,X')$ is the stochastic discount factor; $q(X,X')$ is a price bundle; $\mathcal{B}(x;X,X',q)$ is the budget constraint; $F(X)$ is the law of motion known to an individual agent.[8] For notational convenience, I combine the price bundle (m,q) into p. The following market clearing condition pins down the price p:[9]

$$\text{[Market clearing]}: \quad p(X,X') = \arg_{\widetilde{p}}\{Q^D(\widetilde{p},X,X') - Q^S(\widetilde{p},X,X') = 0\}, \tag{5}$$

where Q^D and Q^S are the functions of demand and supply, which are endogenously determined by the model. For expositional clarity, I consider a simple case where the exogenous aggregate state S can take two possible values $\{G,B\}$ with a 2×2 transition matrix Π^S.[10]

In the following sections, I explain the method based on the recursive form in value functions for the comprehensiveness of exposition. However, the method is seamlessly applied even if the value function is replaced by the first-order derivative or the policy functions. In the online supplement, I provide multiple applications where the expected policy function is computed instead of the expected value function.

The RTM achieves convergence in sequence space. Therefore, despite the converged equilibrium allocations being fully describable in a recursive form, I denote the equilibrium object in a sequential expression, such as $\{V_t\}$, for the sake of a coherent explanation. Hereafter, given a realized state $\{x_t, X_t\}$ for an individual (or representative) agent in a given period t, the value function in the sequential expression V_t and the value function in the recursive form $V(\cdot;X_t)$ are used interchangeably.

[8] The stochastic discount factor can be a constant, for example β, as in a canonical dynamic household's problems. In a dynamic firm problem, the stochastic discount factor needs to be included.

[9] Any period-specific fixed point problem can be considered in the RTM. For brevity, I only include the non-trivial market clearing condition.

[10] The RTM's applicability is not limited to a certain number of grid points for the aggregate shocks. Moreover, multiple aggregate shocks can be considered an exogenous state.

2.2 Assumptions

In this section, I discuss the necessary features of a model for the application of the RTM. The method relies on the a) stability and the b) uniqueness of the recursive competitive equilibrium. If a model violates these two conditions, convergence cannot be guaranteed. Also, the c) recursivity of the equilibrium is a necessary condition. Without recursivity, there is a set of equilibrium allocations in a period that will never become an equilibrium allocation again in the following periods. In this case, the computation of the expectation of such allocations is not feasible in the RTM.[11] From this point on, I focus only on the models that satisfy these three conditions.

To ensure a well-defined equilibrium, I assume there is no redundancy in the representation of the aggregate state X. Specifically, I require a one-to-one mapping between the economy's fundamental state and the aggregate state X, formalized as:

$$V(x; X) = V(x; X') \text{ for } \forall x \quad \Longleftrightarrow \quad X = X'. \tag{6}$$

This condition rules out redundant state variables that could artificially generate equilibrium multiplicity through superfluous expansions of the state space. Put differently, X must contain exactly the information necessary to characterize the aggregate economy, no more and no less.

2.3 The methodology

I start from explaining the basic structure of the methodology. Suppose T periods of aggregate exogenous states $\{S_t\}_{t=0}^T$ are simulated, and hypothetically the simulated path is long enough to make almost all the possible equilibrium allocations happen on the path.[12] The solution process starts by conjecturing three time series: 1) value functions, $\{V_t^{(0)}\}_{t=0}^T$, 2) endogenous states $\{\Phi_t^{(0)}\}_{t=0}^T$, and 3) prices $\{p_t^{(0)}\}_{t=0}^T$. Using these guesses, I solve the allocations backward from the terminal period T to obtain the implied value function solution $\{V_t^*\}_{t=0}^T$, and simulate the economy forward using the solution. The forward simulation generates the time series of the endogenous states $\{\Phi_t^*\}_{t=0}^T$, and implied prices $\{p_t^*\}_{t=0}^T$ from the market-clearing conditions. Here the price p_t^* is the price implied by the market clearing condition, rather than the market clearing price. This distinction is discussed in detail in Section 4. The guess is then updated through convex combinations of prior guesses and realized allocations to form $\{V_t^{(1)}, \Phi_t^{(1)}, p_t^{(1)}\}_{t=0}^T$. While this broad approach

[11]It is also conceptually challenging to let an agent form the expectation of such allocation.

[12]In theory, an infinitely long simulation needs to be considered, but for illustrative purposes, I consider a T-period long simulation. Later in the application, a long-enough finite simulation is used as an approximation for the infinitely long ergodic path.

shares similarities with perfect-foresight methods (Fair and Taylor, 1983), it differs fundamentally in the backward solution step due to its treatment of conditional expectations.

To clarify this point, consider period t in iteration $n+1$, after solving backward from T to $t+1$. Suppose the exogenous state at period $t+1$ is G ($S_{t+1} = G$). To solve an agent's problem at t, one needs to construct an expected future value function $\mathbb{E}_t \widetilde{V}_{t+1}$.[13] This presents a challenge: while $V_{t+1}(\cdot, S = G)$ available from the backward solution, $V_{t+1}(\cdot, S = B)$ is not, as only one exogenous state realizes in each period. I define this unobserved $V_{t+1}(\cdot, S = B)$ as a *counterfactual* conditional value function.

The standard state space-based approach addresses this challenge by replacing time indices with endogenous and exogenous aggregate states, interpolating endogenous states through an assumed law of motion. The solution's accuracy thus critically depends on correctly specifying this law of motion. However, verifying the specification's accuracy is impossible before solving the equilibrium. An incorrect specification requires restarting the solution process with a new guess, presenting two fundamental challenges: determining which statistics to include and selecting appropriate functional forms. This problem cannot be easily resolved unless the aggregate dynamics are known to be log-linear, as in Krusell and Smith (1998).

The RTM takes a fundamentally different approach. Instead of specifying a law of motion, it obtains the counterfactual conditional value function from another period $\widetilde{t}+1$ where the endogenous aggregate state matches that of period t+1 but realizes the counterfactual exogenous state:

$$\Phi^{(n)}_{\widetilde{t}+1} = \Phi^{(n)}_{t+1} \tag{7}$$

$$S_{\widetilde{t}+1} = B \neq G = S_{t+1}. \tag{8}$$

Under these conditions, all aggregate states in period $\widetilde{t}+1$ match those of the *counterfactual state* in period $t+1$, implying

$$V^{(n)}_{\widetilde{t}+1}(\cdot, S = B) = V^{(n)}_{t+1}(\cdot, S = B). \tag{9}$$

Importantly, $V^{(n)}_{\widetilde{t}+1}(\cdot, S = B)$ is the observed *factual* conditional value function available in the n^{th} iteration. With both $V^{(n)}_{t+1}(\cdot, S = G)$ and $V^{(n)}_{t+1}(\cdot, S = B) (= V^{(n)}_{\widetilde{t}+1}(\cdot, S = B))$ available from iteration n, the expected future value function $\mathbb{E}_t \widetilde{V}_{t+1}$ can be consistently computed. This approach extends

[13]The method can potentially accommodate various expectation formations beyond rational expectations. The conditional expectation computation step can be adjusted to any well-defined expectation structure.

naturally to finer discretizations of the aggregate shock process.[14] The recursivity of the recursive competitive equilibrium ensures that such a period $\widetilde{t}+1$ exists almost surely in a sufficiently long simulation.

This approach eliminates the need to specify a law of motion for computing expected future value functions. Instead, the critical step becomes identifying period $\widetilde{t}+1$ that replicates the counterfactual conditions of period $t+1$. This identification relies on tracking the sequence of endogenous aggregate states $\{\Phi_t^{(n)}\}_{t=0}^{T}$, which serves as the key criterion for locating appropriate matching periods. In the Online Appendix A, I elaborate on the detailed steps to implement the RTM and the required length of the simulated path. Due to the method's ability to utilize the full information from the entire sequence, it requires no additional simulation length beyond what traditional methods demand.

3 A sufficient statistic approach

When a model includes a complex endogenous aggregate state, identifying the period $\widetilde{t}+1$ that is sharing the same aggregate states to period $t+1$ can be the most demanding step. For example, if a distribution of individual state is an aggregate state as in heterogeneous agent models, each period on the simulated path needs to be compared with respect to an infinite-dimensional distribution. However, if there is a sufficient statistic that can perfectly represent a period's endogenous aggregate state, the computational efficiency can be substantially improved. This enables to locate the target period $\widetilde{t}+1$ by only comparing the distance between these sufficient statistics instead of the distributions. In the following section, I theoretically analyze the condition under which a variable can be used as a sufficient statistic.

3.1 The qualification for the sufficient statistic

While a large body of literature has employed sufficient statistics to address the curse of dimensionality in DSGE computation, the theoretical foundations justifying this approach have remained unclear. Proposition 1 fills this gap by establishing precise conditions under which a variable can serve as a valid sufficient statistic in the Repeated Transition Method.

Proposition 1 (The qualification for the sufficient statistic)**.**
For a sufficiently large T, if there exists a time series of a variable $\{e_t\}_{t=0}^{T}$ such that for each time

[14] Most applications in the online supplement employ finer grids than two points.

partition $\mathcal{T}_S = \{t | S_t = S\}$, $\forall S \in \{B, G\}$ *and for* $\forall (a, s)$,

$$(i) \quad e_{\tau_0} < e_{\tau_1} \iff V_{\tau_0}(a,s) < V_{\tau_1}(a,s) \text{ for any } \tau_0, \tau_1 \in \mathcal{T}_S$$

$$\text{or}$$

$$(ii) \quad e_{\tau_0} < e_{\tau_1} \iff V_{\tau_0}(a,s) > V_{\tau_1}(a,s) \text{ for any } \tau_0, \tau_1 \in \mathcal{T}_S,$$

then e_t *is the sufficient statistic of the endogenous aggregate state* Φ_t *for* $\forall t$*. That is,*

$$\mathbb{E}_S V(x; \Phi', S') = \int V(x; \Phi', S') d\Gamma_{S,S'} = \int V_{\widetilde{\tau}(S')+1}(x) d\Gamma_{S,S'}$$

where $\widetilde{\tau}(S') + 1 = \arg\inf_{\tau \in \mathcal{T}_{S'}} ||e_\tau - e_{t+1}||_\infty$.

Proof.

See Online Appendix E. ■

Proposition 1 states that when a time series $\{e_t\}_{t=0}^T$ *monotonically ranks* the level of the corresponding period's value function for each individual state, e_t qualifies as a sufficient statistic for period t in the RTM. The intuition behind the proposition is as follows.Consider a researcher searching for the appropriate value function to compute conditional expectations. If the correct counterfactual period τ were known explicitly, identifying the appropriate value function would be trivial since all value functions are indexed by time - V_τ would simply be the correct choice.

Now instead of τ, suppose the level of e_τ is known to the researcher. Then, similar to the prior situation where τ is known, the researcher can identify which value function to use because the ranking information of e_τ uniquely pins down the corresponding value function due to the strict monotonicity. For example, if two periods τ_0 and τ_1 share the same level of e_t, thus $e_{\tau_0} = e_{\tau_1}$, then the strict monotonicity implies that $V_{\tau_0} = V_{\tau_1}$. Were this equality to fail ($V_{\tau_0} \neq V_{\tau_1}$), it would lead to a contradiction by violating either the strict monotonicity condition or the validity of the endogenous aggregate state Φ, which is the proof's key idea.

This strict monotonicity establishes a bijection between sufficient statistics and target periods, ensuring unique identification. The framework extends naturally to policy functions when solving models through first-order optimality conditions. The key insight is that knowledge of the ranking across different periods' value functions enables precise identification of the appropriate value function for any given period.

This theoretical result provides rigorous foundations for using sufficient statistics in the RTM.

In Section 6, I show how the monotonicity can be validated for converged solutions. Importantly, a sufficient statistic that satisfies these conditions for the RTM may not qualify as a sufficient statistic for laws of motion in state space-based approaches. This is because the statistic may not necessarily include sufficient information about the inter-temporal dynamics of the endogenous aggregate state variables. For example, in the nonlinear model explained in Section 6, if I fit the nonlinear aggregate dynamics of the sufficient statistic obtained from the RTM to the nonlinear specifications of the single sufficient statistic, R^2 remains well below unity, indicating that one variable cannot adequately capture the full law of motion. Nevertheless, the variable alone serves perfectly as a sufficient statistic in the RTM by satisfying the monotonicity condition. This distinction highlights a key advantage of the RTM: it can achieve exact solutions with simpler sufficient statistics because it does not need to capture the full complexity of inter-temporal dynamics.

4 Non-trivial market clearing conditions

In this section, I explain how the RTM efficiently handles non-trivial market clearing conditions and why this approach is infeasible in state space-based methods.

Consider the following market clearing condition:

$$\begin{aligned} &Q^D(p_t, X_t, X_{t+1}) - Q^S(p_t, X_t, X_{t+1}) = 0. \\ &p_t := \arg_{\widetilde{p}}\{Q^D(\widetilde{p}, X_t, X_{t+1}) - Q^S(\widetilde{p}, X_t, X_{t+1}) = 0\}. \end{aligned} \tag{10}$$

where Q^D and Q^S are demand and supply functions; p_t is the market clearing price; X_t and X_{t+1} are the current and future aggregate states. The market clearing condition is non-trivial when either demand, supply, or both lack closed-form characterization. For this problem, the RTM utilizes the implied price p_t^* instead of the exact clearing price p_t, where

$$\begin{aligned} p_t^* &:= \arg_{\widetilde{p}}\{Q^D(p_t^{(n)}, X_t, X_{t+1}) - Q^S(\widetilde{p}, X_t, X_{t+1}) = 0\} \text{ or} \\ &:= \arg_{\widetilde{p}}\{Q^D(\widetilde{p}, X_t, X_{t+1}) - Q^S(p_t^{(n)}, X_t, X_{t+1}) = 0\}. \end{aligned} \tag{11}$$

This approach fixes either demand or supply using the n^{th} iteration's guessed price, then finds the price that clears the remaining side. Computing this implied price is substantially simpler than finding the market clearing price, which requires solving a fixed-point problem where price simultaneously affects both supply and demand.

During the iteration, the implied price does not clear the market at each period, as it's only the implied price. However, as iteration goes by, the predicted path of prices $\{p_t^{(n)}\}_{t=0}^T$ converges to the equilibrium prices $\{p_t\}_{t=0}^T$. This convergence makes the implied price clear the market in the limit for the following reasons:[15]

$$\lim_{n\to\infty} Q^D(p_t^{(n)}, X_t, X_{t+1}) - Q^S(p_t^*, X_t, X_{t+1}) = 0 \tag{12}$$

$$\Longrightarrow\ Q^D(\lim_{n\to\infty} p_t^{(n)}, X_t, X_{t+1}) - Q^S(p_t^*, X_t, X_{t+1}) = 0 \tag{13}$$

$$\Longrightarrow\ Q^D(p_t, X_t, X_{t+1}) - Q^S(p_t^*, X_t, X_{t+1}) = 0 \tag{14}$$

$$\Longrightarrow\ p_t^* = p_t \quad (\because \text{ uniqueness of the equilibrium}). \tag{15}$$

Thus, the RTM's converged solution delivers exact market clearing prices alongside other equilibrium allocations.

In contrast, the implied price cannot replace the market clearing price in the state space-based approach (Krusell and Smith, 1997) in general. In their approach, the price dynamics is approximated by the parametric function of the aggregate state or the sufficient statistic, and the coefficients of the function carry the information about the relationship between the price and the aggregate state. Given that the number of coefficients cannot technically exceed the number of periods, the coefficients can only carry the summarized information. If the coefficients are updated based on the implied price rather than the market clearing price, the update is based on inaccurate levels, thus leading to a biased coefficient.[16] Then, the wrongly updated coefficients often lead to a divergent path, as there is no theoretical guarantee that the coefficient of the functional form features *stability*. The RTM, by contrast, preserves complete information about price-state relationships by carrying entire sequences through iterations. This approach enables uniform convergence in sequence space, guaranteed by equilibrium stability, without requiring functional approximations or coefficient estimation.

[15] The local continuity of demand or supply is necessary to proceed from the first to the second line. This should be true except for the knife-edge case where the unique equilibrium is at the discontinuity point, which DSGE models are barely subject to.

[16] While Bakota (2023) develops a method to update pricing rules approximately without exact market clearing, improving state space computation speed, the RTM bypasses the need for such approximations entirely.

5 Accuracy: Comparison with the existing methods

This section compares the RTM's performance with existing nonlinear solution methods. The comparison is based on a real business cycle model with irreversible investment (McGrattan, 1996; Christiano and Fisher, 2000), where I benchmark the RTM against three alternatives: linearized solution, the OccBin method of Guerrieri and Iacoviello (2015) and the GDSGE solution of Cao et al. (2023). In addition, Appendix B evaluates the RTM's performance in two additional settings: the heterogeneous-household model of Krusell and Smith (1998) comparing against Maliar et al. (2010), and the heterogeneous-firm model of Khan and Thomas (2008), benchmarking against their original Krusell and Smith (1997) solution approach.[17]

Consider a representative household solving the following problem:

$$V(a;X) = \max_{c,a'} \frac{c^{1-\sigma}}{1-\sigma} + \beta \mathbb{E} V(a';X') \tag{16}$$

$$\text{s.t. } c + a' - (1-\delta)a = Aa^{\alpha} \tag{17}$$

$$a' - (1-\delta)a \geq \phi I_{ss} \tag{18}$$

where V is the value function of a household. The value function's arguments are wealth a and the aggregate state X. c is consumption and σ is the risk-aversion parameter. I_{ss} is the steady-state investment level. ϕ is the parameter for the degree of the irreversibility. δ is the depreciation rate, and α is the capital share in the production function $F(a;A) := Aa^{\alpha}$. Apostrophes denote next-period variables. The aggregate state X is as follows

$$X = [K, A]. \tag{19}$$

K is the aggregate capital stock, satisfying $a = K$ in equilibrium, as the capital market clears. A is TFP that follows the log AR(1) process:

$$log(A') = \rho log(A) + \sigma\epsilon, \quad \sigma \sim N(0,1). \tag{20}$$

The model features highly nonlinear aggregate dynamics due to the occasionally binding irreversibility constraint for capital investment. Therefore, besides the macroeconomic implications, the model serves as an ideal testing ground for the accuracy of the different methods for the non-

[17] All computations use a MacBook Pro 2021 with M1 Pro chip.

linear solutions. For precise comparison, I generate a single TFP path using the Tauchen method (7 grid points covering three standard deviations) and apply this path to all solution methods. The simulation runs for 5,000 periods with 500 burn-in periods. Each method exhibits a trade-off between accuracy and computational efficiency depending on convergence criteria. In this comparison, I tune the repeated transition method to stop after around 90 seconds, matching the speed of the GDSGE toolkit.

The first four rows of Table 1 compare solution accuracy across methods, with columns presenting results for the RTM, GDSGE, OccBin, and linearized solutions, respectively. I evaluate accuracy using two criteria: dynamic consistency error ($Error_t$) and Euler equation error (EE_t) following Judd (1992).[18] The dynamic consistency error is defined as:

$$Error_t = K_t^{(n)} - K_t^* \tag{21}$$

where $\{K_t^{(n)}\}_{t=1}^T$ represents the capital stock sequence from each solution method, and K_t^* is the implied capital path assuming agents expect $\{K_t^{(n)}\}_{t=1}^T$. The RTM constructs the period-specific expected future allocations by properly combining allocations in the predicted path (the previous iteration). Then, it provides the realized allocations implied by the prediction path. Dynamic consistency requires these predicted and realized paths to coincide. Thus, the RTM to serve as a diagnostic tool for other solution methods - by feeding their simulated paths as predicted paths into the RTM algorithm, the RTM can evaluate their dynamic consistency.

The RTM displays a higher accuracy than other methodology in terms of the four statistics: the absolute maxima of the dynamic inconsistency (first row) and the Euler equation error (third row); the square roots of the mean-squared dynamic inconsistency (second row) and the Euler equation error (fourth row).

The RTM achieves its accuracy and speed using standard MATLAB code, without relying on lower-level languages like C++ or advanced econometric techniques. The method's dynamic consistency error can theoretically be reduced to any arbitrary level by adjusting convergence tolerance, suggesting potential for further improvement through integration with lower-level programming languages or modern machine learning techniques (Azinovic et al., 2022; Fernández-Villaverde et al., 2023; Han et al., 2025).[19]

[18] The Euler equation error specification follows Guerrieri and Iacoviello (2015).

[19] One potential synergy between the RTM and the advanced machine learning techniques is in the identification step for the target period from the previous iteration.

Table 1: Comparison across the solution methods

	RTM	GDSGE	OccBin	Linear
Accuracy				
$max(\lvert Error_t \rvert)$ (% of steady-state K)	0.003	0.735	1.317	2.019
$\sqrt{mean(Error_t^2)}$ (% of steady-state K)	0.001	0.025	0.217	0.559
$max(\lvert EE_t \rvert)$ (% of contemp. C_t)	0.014	0.057	2.854	2.323
$\sqrt{mean(EE_t^2)}$ (% of contemp. C_t)	0.002	0.059	0.775	0.707
Business cycle stat.				
$mean(I)$	0.363	0.363	0.365	0.363
$mean(C)$	1.166	1.166	1.164	1.160
$vol(I)$	0.022	0.022	0.023	0.023
$vol(C)$	0.052	0.052	0.052	0.052
$skewness(I)$	1.363	1.320	1.307	1.407
$skewness(C)$	-0.225	-0.213	-0.322	-0.095
$kurtosis(I)$	4.447	4.578	4.513	4.255
$kurtosis(C)$	2.776	2.546	2.858	2.796

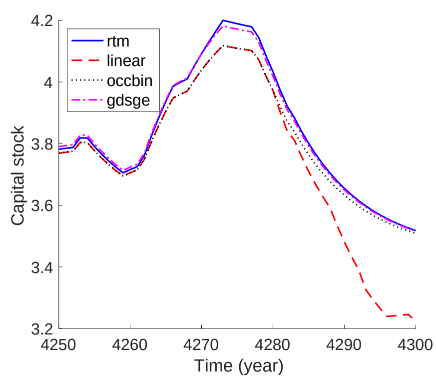


Figure 1: The equilibrium path of different solutions

Table 1's lower panel reports business cycle statistics across solutions. While lower-order moments show negligible differences, higher-order moments (skewness and kurtosis) reveal significant variations across methods. Figure 1 illustrates these differences by plotting capital stock paths from each solution method. The RTM solution most closely matches the GDSGE toolkit, while the OccBin solution shows notable deviations, particularly when aggregate capital is high. The absolute difference between RTM and OccBin solutions correlates positively with output (correlation 0.631), indicating pro-cyclical computation error. Conversely, differences between RTM and linear solutions show counter-cyclical patterns (correlation -0.640). These patterns reflect each method's relative strength: OccBin provides more accurate solutions during downturns when constraints bind but less accurately captures precautionary behavior in normal times. The linear solution, by contrast, fundamentally struggles with occasionally binding constraints, leading to large errors during downturns.

In Online Appendix B, I compare the RTM performance with state-space-based methods based on heterogeneous-agent models. When a heterogeneous-agent model features non-trivial market clearing conditions, the RTM demonstrates substantial computational advantages over state space-based methods. A striking example comes from solving the Khan and Thomas (2008) model, where the RTM achieves convergence approximately ten times faster than the Krusell and Smith (1997) algorithm. This efficiency gain stems from a fundamental methodological difference: while the state-space-based approach requires an additional computational loop to find exact market clearing prices in each period, the RTM employs implied prices that naturally converge to market clearing values through iteration. This approach eliminates the need for nested fixed-point calculations while maintaining solution accuracy. The detailed comparison is available in Online Appendix B.

6 Applications

The subsequent sections present two leading applications that extend Krusell and Smith (1997, 1998). These applications address important macroeconomic questions that have remained unexplored due to computational barriers. The RTM's methodological breakthrough enables efficient solution of these challenging problems.

6.1 The leading application I: Krusell and Smith (1998) with endogenous labor supply, investment irreversibility, and fiscal spending shock

The first leading application extends the Krusell and Smith (1998) heterogeneous-household RBC model by incorporating endogenous labor supply, investment irreversibility, and both aggregate TFP and fiscal spending shocks. The model features a continuum of ex-ante identical households of unit measure in an infinite-horizon discrete-time economy.

The model environment is characterized as follows. Each household faces uninsurable idiosyncratic labor productivity shocks and makes endogenous labor supply decisions n. The temporal utility is assumed to be a log utility with a future discount factor $\beta > 0$. At the beginning of each period, households observe their individual states (wealth a and productivity z) and the aggregate state X, forming rational expectations about future aggregate conditions X'. Apostrophes denote next-period variables.

The recursive formulation of a household's problem is as follows:

$$V(a,z;X) = \max_{c,n,a'} \; log(c) - \frac{\eta}{1+\frac{1}{\chi}} n^{1+\frac{1}{\chi}} + \beta \mathbb{E} V(a',z';X') \tag{22}$$

$$\text{s.t.} \quad c + a' = (1+r(X))a + w(X)zn - T(X) \tag{23}$$

$$a' - (1-\delta)a \geq \phi I^{ss} \tag{24}$$

$$X' = \Gamma_X(X) \quad \text{(Aggregate law of motion)} \tag{25}$$

$$z' \sim \pi(z'|z) \tag{26}$$

where V is the value function of a household; r and w are capital rent and wage that are determined at the competitive input factor markets. I_{ss} is the steady-state aggregate saving (investment) level. T is the lump-sum tax. χ is the Frisch elasticity parameter, and η is the labor disutility parameter. ϕ is the parameter that governs the degree of the saving irreversibility. Γ_X is the aggregate law of motion. The idiosyncratic productivity z follows a Markov process, where $\pi(z'|z)$ governs the transition probability.

We consider a production sector that operates using a CRS Cobb-Douglas production function:

$$\max_{K,L} \; AK^{\alpha}L^{1-\alpha} - w(X)L - (r(X)+\delta)K, \tag{27}$$

where A is the aggregate TFP, K and L are capital and labor input demands.

The aggregate state X includes following three components:

$$X = \{\Phi, A, G\}. \tag{28}$$

where Φ is the distribution of the individual states, A is TFP, and G is government demand. The first is endogenous aggregate state, and the others follow exogenous log AR(1) processes specified as follows:

$$log(A') = \rho_A log(A) + \sigma_A \epsilon \quad \epsilon \sim_{iid} N(0,1) \tag{29}$$

$$log(G') = (1-\rho_G) log(\overline{G}) + \rho_G log(G) + \sigma_G \epsilon \quad \epsilon \sim_{iid} N(0,1) \tag{30}$$

where $\overline{G}$ is the steady-state government demand. For $j \in \{A, G\}$, ρ_j is the persistence parameter for the exogenous processes, and σ_j is the volatility parameter. These processes are discretized by the Tauchen method in the computation. I assume the simplest government setup where the budget is balanced by lump-sum tax collection: $T(X) = G$. By assuming this, the symmetric lump-sum tax is collected from heterogeneous households. For computation, I use the standard parameter levels in the literature, which are available in Appendix C.

The recursive competitive equilibrium is defined based on the following market-clearing conditions:

$$\text{(Labor market)} \quad L(X) = \int zn(a, z; X)d\Phi \tag{31}$$

$$\text{(Capital market)} \quad K(X) = \int ad\Phi. \tag{32}$$

The market clearing is non-trivial as the wage determines individual labor supply, which needs to be aggregated instead of directly pinning down the aggregate labor supply. To illustrate this, I introduce an aggregate labor demand curve $\widetilde{L}$ and an individual labor supply curve $\widetilde{n}$, all of which include a price level $\widetilde{w}$ as an argument:

$$\begin{aligned} w(S) &= \arg_{\widetilde{w}} \left\{ \widetilde{L}(\widetilde{w}, X) - \int \widetilde{n}(a, z; \widetilde{w}, X)d\Phi = 0 \right\} \\ &= \arg_{\widetilde{w}} \left\{ \left(\frac{(1-\alpha)A}{\widetilde{w}} \right)^{\frac{1}{\alpha}} K - \int \left(\frac{z\widetilde{w}}{\eta c(a, z; \widetilde{w}, X)} \right)^{\chi} d\Phi = 0 \right\}. \end{aligned} \tag{33}$$

Line (33) follows from the first-order optimality conditions from the production sector (demand) and the household (supply). Given each wage level $\widetilde{w}$, the optimal consumption of the household needs to be specified, which requires an internal loop for clearing.[20] This fixed-point problem is computationally costly to solve. However, instead of the market clearing price, the repeated transition method uses the implied price w^*, as follows:

$$\begin{aligned} w^* &= \arg_{\widetilde{w}} \left\{ \widetilde{L}(\widetilde{w}, X) - \int \widetilde{n}(a, z; w^{(n)}, X)d\Phi = 0 \right\} \\ &= \arg_{\widetilde{w}} \left\{ \left(\frac{(1-\alpha)A}{\widetilde{w}} \right)^{\frac{1}{\alpha}} K - \int \left(\frac{zw^{(n)}}{\eta c(a, z; w^{(n)}, X)} \right)^{\chi} d\Phi = 0 \right\}. \end{aligned} \tag{34}$$

where $w^{(n)}$ is the guessed wage (predicted wage) in the n^{th} iteration. By bypassing the costly fixed-point problem, the method dramatically improves the speed of the computation.

From the first-order condition, the following inter-temporal optimality condition is obtained:

$$\frac{1}{c(a, z; X)} = \beta \mathbb{E}_{z,X} \left[\left(\frac{1}{c(a', z'; X')} \right) (1 + r(X')) - (1-\delta)\lambda(a', z'; X') \right] + \lambda(a, z; X) \tag{35}$$

[20] If a GHH utility is considered, this problem becomes trivial due to the missing wealth effect coming through the consumption in the denominator. However, the wealth effect is the key channel for the fiscal policy as will be demonstrated in the following section.

where λ is the Lagrange multiplier of the irreversibility constraint. The left-hand side of the equation above is the marginal cost of saving in the unit of utility, and the right-hand side is the expected marginal benefit of saving. The marginal benefit includes contemporaneous gain out of relaxing today's constraint $(+\lambda(a, z; X))$ and the cost of tightening the future constraints $(-(1-\delta)\lambda(a', z'; X'))$.

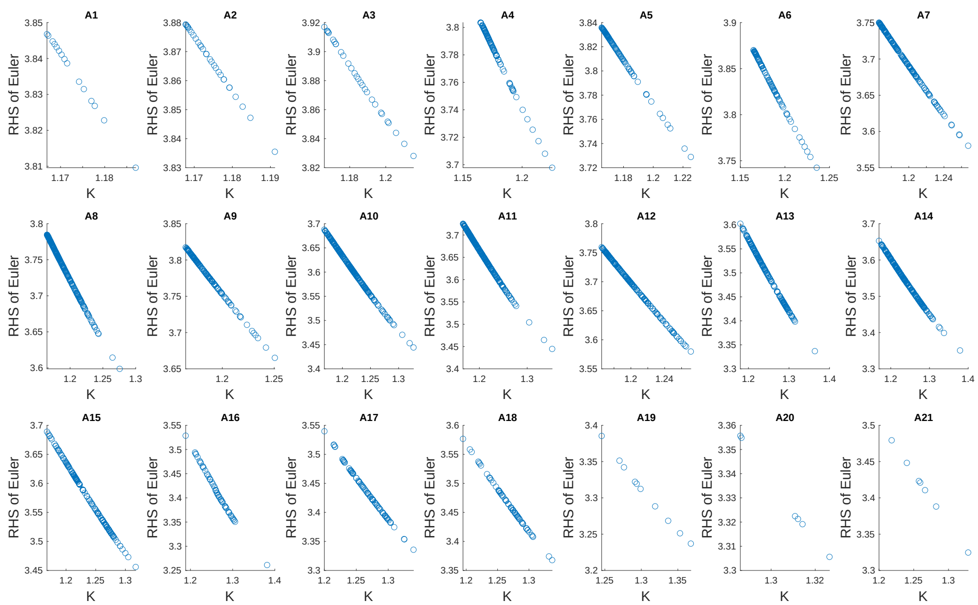


Figure 2: Strict monotonicity of the marginal benefit in the aggregate capital stock

The expected marginal benefit requires the computation of state-contingent allocations ($X'-$ *dependent*) of marginal utility $1/c$, capital rent r, and Lagrange multiplier λ.[21] In the computation of these terms, I employ the sufficient statistic approach described in Section 3.1, using aggregate capital stock $K(X)$ - the first moment of the individual wealth distribution - as the sufficient statistic. To validate this approach, I demonstrate that each individual's marginal benefit satisfies the strict monotonicity condition required by Proposition 1 across all aggregate exogenous state realizations. Figure 2 provides graphical evidence, plotting marginal benefits against aggregate capital stock across different TFP and government demand levels $(A1, A2, A3, ..., A21)$. The plots fix individual states at median levels of wealth and labor productivity, though the monotonicity property holds robustly across the entire distribution of individual characteristics.

[21]The RTM computes the exact level of the Lagrange multiplier for the occasionally binding constraint at the individual level, which enables the accurate computation. The path of the lagrange multipliers is computed by the residuals using the Euler equation as in Rendahl (2014).

For the computation, I use a 3,000-period simulation with independent aggregate TFP and government demand shocks. The TFP process is discretized using the Tauchen method with 7 grid points spanning three standard deviations, while the government demand shock uses 3 grid points covering one standard deviation. This yields 21 (= 3 × 7) total grid points for exogenous aggregate state variations. The solution reveals highly nonlinear dynamics in the aggregate capital path, driven by the occasionally binding constraint. Figure 5 compares three capital paths: the predicted path $\{K_t^{(n)}\}_{t=0}^T$, the realized (implied) path $\{K_t^*\}_{t=0}^T$ from the RTM, and a simulated path using a fitted log-linear law of motion. While the predicted and realized paths coincide to form the equilibrium solution, the log-linear approximation shows significant deviations.[22]

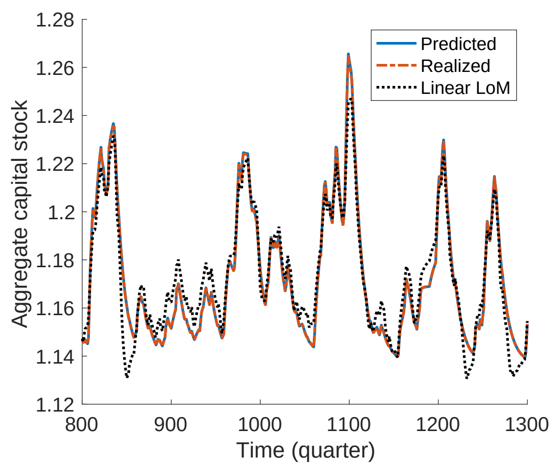


Figure 3: The equilibrium path of aggregate capital stock

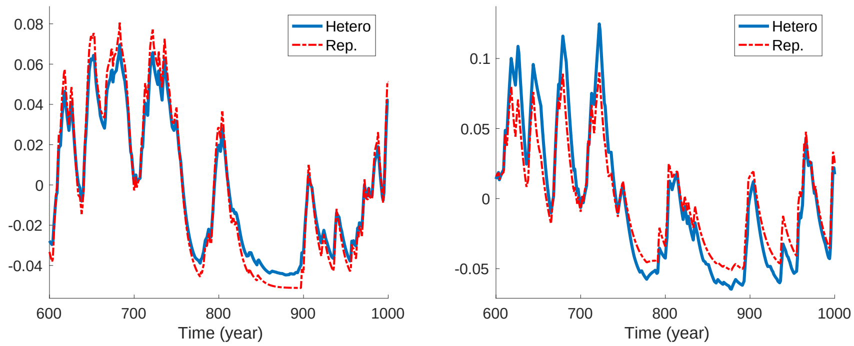
6.1.1 Nonlinearity and aggregation

This section examines how micro-level nonlinearities shape aggregate dynamics by comparing heterogeneous and representative agent versions of the model. I contrast the heterogeneous-household model (HH) with a representative-household variant (RH) that eliminates labor productivity heterogeneity while maintaining all other parameter values. To ensure precise comparison, I apply identical aggregate TFP paths to both specifications and solve them using the RTM.

Figure 4 presents comparative dynamics in two contexts. Panel (a) displays equilibrium capital paths for the HH model (solid line) and RH model (dash-dotted line), expressed as log deviations

[22] Fitting the aggregate capital dynamics to a log-linear $AR(1)$ specification with exogenous shock controls yields an R^2 of 0.996. However, as Den Haan (2010) demonstrates, such seemingly high R^2 values can mask substantial inaccuracies in aggregate dynamics as shown in Figure 3. Achieving high accuracy requires considerably more complex specifications than log-linear forms (details available upon request).

from their respective steady states. For broader perspective, Panel (b) provides an analogous comparison between heterogeneous-firm and representative-firm models subject to the same investment irreversibility constraint.[23]



(a) Heterogeneous households

(b) Heterogeneous firms

Figure 4: Equilibrium dynamics comparison: Heterogeneous households vs. firms

The analysis reveals striking differences in aggregate dynamics between heterogeneous and representative agent specifications. HH exhibits substantially lower capital stock volatility than RH, driven by reduced investment volatility. However, this volatility reduction does not stem merely from the presence of micro-level heterogeneity per se. Indeed, panel (b) demonstrates that heterogeneous-firm models display markedly higher capital stock volatility, stemming from increased investment volatility. This contrast highlights how the specific nature of heterogeneity shapes aggregate nonlinear dynamics.

The business cycle statistics reported in Table 2 quantify these differences. While the HH and RH models generate similar time-series averages, they differ notably in higher moments. The HH model produces lower volatility in both output (5% reduction) and investment (18% reduction), though consumption volatility remains similar between the specifications. The models also differ in asymmetry: the HH model exhibits more negative skewness in output and consumption, but more positive skewness in investment.[24]

Therefore, the representative-household model fails to adequately capture business cycle dynamics present in the heterogeneous-household model. The key driver of this misalignment is nonlinearity in household-level wealth dynamics.[25] To see this, I compute the same heterogeneous and

[23] Online Appendix D provides detailed specifications for the heterogeneous-firm model with irreversible investment shown in Panel (b).

[24] Detailed business cycle statistics for the firm-side comparisons are provided in Appendix D.

[25] This result is specific to this model. For example, Khan and Thomas (2008) shows that the general equilibrium effect washes out the firm-level nonlinearity in their model.

Table 2: Business cycle statistics: heterogeneous (HH) vs. representative household (RH)

	Heterogeneous	Representative
Mean		
Output	0.512	0.508
Consumption	0.288	0.287
Investment	0.122	0.120
Volatility		
$log(Output)$	0.042	0.044
$log(Consumption)$	0.057	0.058
$log(Investment)$	0.049	0.060
Skewness		
$log(Output)$	-0.026	0.024
$log(Consumption)$	-0.363	-0.349
$log(Investment)$	0.936	0.874

representative household models without the occasionally binding irreversibility constraint (fully reversible investment), which is the source of the nonlinearity. Specifically, for the HH benchmark, the irreversibility constraint is replaced by zero borrowing limit constraint as in Krusell and Smith (1998) and Aiyagari (1994). Figure 5 plots equilibrium capital dynamics for these 'reversible benchmark' cases, showing log deviations from steady state alongside a fitted log-linear law of motion. The perfect alignment of all three paths demonstrates that when heterogeneous household decisions are (near-)linear, the representative agent model provides an almost exact characterization of aggregate dynamics.[26]

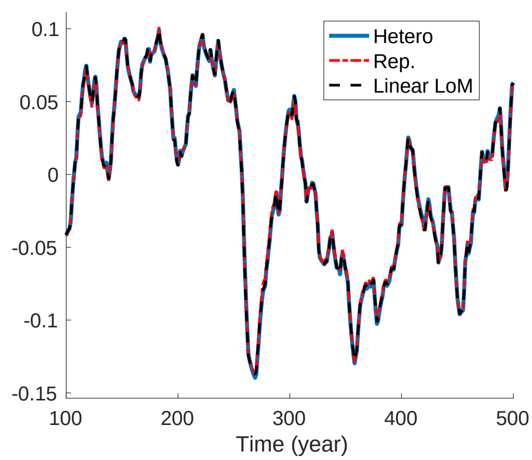


Figure 5: Equilibrium dynamics comparison - frictionless: Heterogeneous vs. representative

[26]The hand-to-mouth households in Krusell and Smith (1998) and Aiyagari (1994) display kinked saving policy. However, their contribution to the aggregate capital dynamics is almost negligible.

6.1.2 Policy implication: endogenous state dependence of the fiscal multiplier

State-dependent policy effects are a crucial feature of models with nonlinear aggregate fluctuations, yet analyzing these effects requires solution methods that can accurately capture global nonlinearities. The model in this paper provides an ideal laboratory for studying such state dependence through the lens of household borrowing constraints. The key mechanism operates through an asymmetric wealth adjustment constraint: households face greater friction when attempting to reduce their wealth positions than when increasing them. This asymmetry creates state-dependent marginal propensities to consume (MPCs), as households near their constraint exhibit nearly one-to-one consumption responses to negative income shocks while maintaining more modest responses to positive shocks.

The model's calibration generates patterns of household financial constraints that align remarkably well with empirical evidence. Using an irreversibility parameter of $\phi = 0.975$ used in Guerrieri and Iacoviello (2015), the model produces a steady-state share of 33.9% hand-to-mouth households, which is highly consistent with the empirical estimates from Kaplan and Violante (2014). Importantly, the model captures a key feature of household financial constraints: they affect not only low-wealth households but also wealthy ones who face temporary difficulties adjusting their wealth. This feature emerges naturally from specifying the constraint in terms of the wealth adjustment rather than the level of wealth. As a result, 21.6% of hand-to-mouth households in the model hold above-average wealth, consistent with the "wealthy hand-to-mouth" phenomenon documented in the literature.

A positive fiscal demand shock generates heterogeneous output responses that depend crucially on the distribution of financially constrained households. When a large fraction of households face binding constraints, the economy exhibits a powerful amplification mechanism: The lump-sum taxes levied to finance government spending trigger substantial declines in household consumption, which in turn induce a strong positive labor supply response through a wealth effect. This amplification mechanism leads to a significant increase in aggregate output, as illustrated in Equation (36). In contrast, when most households operate away from their constraints, the same fiscal shock induces a more muted response. These unconstrained households can smooth consumption through saving adjustments, resulting in a smaller labor supply response and consequently a more modest output expansion, as shown in Equation (37).

$$[\textit{Mostly constrained}]: \quad \underbrace{C_t}_{\downarrow\downarrow\downarrow} + \underbrace{I_t}_{\downarrow} + \underbrace{G_t}_{\uparrow\uparrow} = Y_t \ \upuparrows \quad (\because \text{Large wealth effect}) \tag{36}$$

$$[\textit{Mostly unconstrained}]: \quad \underbrace{C_t}_{\downarrow\downarrow} + \underbrace{I_t}_{\downarrow\downarrow} + \underbrace{G_t}_{\uparrow\uparrow} = Y_t \ \uparrow \quad (\because \text{Small wealth effect}) \tag{37}$$

To quantify this state dependence empirically, I estimate the following regression using simulated data from the model's global solution:

$$Y_t = \beta_0 + \beta_1 G_t + \beta_2 G_t \times \Lambda_t + \beta_3 log(K_t) + \beta_3 log(A_t) + \epsilon_t, \tag{38}$$

where Y_t is aggregate output; G_t is government demand measured in the unit of output; Λ_t is the aggregated Lagrange multiplier defined by $\Lambda_t = \Lambda(X_t) := \int \lambda(a, z; X_t) d\Phi_t$. Λ_t captures both the portion of constrained households and the average binding intensity. The coefficient of primary interest is β_2, which captures how the fiscal multiplier varies with the prevalence of binding constraints.

The baseline HH specification without interaction terms yields a fiscal multiplier of approximately 0.8 over a two-year horizon, aligning with empirical estimates from Ramey (2020). However, incorporating state dependence through the interaction term reveals that this average effect masks substantial variation: the direct effect β_1 becomes notably smaller, while the interaction term accounts for roughly 23.5% of output variation ($= std(\widehat{\beta}_2 G_t A_t)/std(Y) \approx 23.5\%$). The model achieves remarkable fit after including state dependence, with mean squared prediction errors below 10^{-6}.

The result indicates that the fiscal multiplier is greater when a greater portion of households are constrained, which endogenously fluctuate over the business cycle. Importantly, this implies strong counter-cyclical variation in fiscal policy effectiveness in equilibrium: the negative correlation (-0.788) between output Y_t and the constraint intensity measure Λ_t indicates that borrowing constraints bind more frequently during economic downturns. This pattern implies that fiscal stimulus becomes particularly potent precisely when the economy is weak, providing a natural stabilization mechanism through state-dependent multipliers.

The third column presents the regression coefficients when the data is simulated from the RH model. Notably, the degree of state dependence in HH remains largely unchanged in its representative-agent counterpart. The fourth column reports the coefficients under the GHH utility specification, which eliminates the wealth effect arising from tax-driven consumption reductions.

Table 3: State-dependent fiscal spending multipliers

	Dependent variable: Y_t (\$)			
	Hetero. (HH)		Rep. (RH)	GHH
G_t (\$)	0.402	0.182	0.206	0.000
	(0.005)	(0.002)	(0.000)	(0.001)
G_t (\$) $\times \Lambda_t$		0.533	0.534	0.000
		(0.003)	(0.002)	(0.000)
$log(K_t)$	0.143	0.105	0.104	0.496
	(0.002)	(0.001)	(0.000)	(0.000)
$log(A_t)$	0.463	0.598	0.591	1.504
	(0.001)	(0.001)	(0.001)	(0.000)
Constant	Yes	Yes	Yes	Yes
Observations	3,000	3,000	3,000	3,000
R^2	0.992	0.999	0.999	0.999
Adjusted R^2	0.992	0.999	0.999	0.999

When using contemporaneous consumption C_t as the dependent variable in the same regression setup with the GHH utility, the results confirm that a fiscal demand shock significantly reduces consumption, consistent with both the HH and RH models. However, this does not translate into any effect on aggregate output, as indicated by the near-zero estimates for β_1 and β_2. These findings highlight that the state-dependent fiscal multiplier operates through variations in the state-dependent MPC and the resulting wealth effect. In this context, the key equilibrium property of interest is global nonlinearity, which the RTM effectively captures by providing an accurate solution in sequence space.

6.2 The leading application II: A heterogeneous-household RBC model of portfolio choice (Krusell and Smith, 1997) with endogenous labor supply

In this section, I apply the RTM to the heterogeneous household's portfolio choice problem over the business cycle (Krusell and Smith, 1997), where households endogenously determine labor supply. A continuum of unit measure of households who consumes, saves in two assets (capital and bond), and supplies labor, which is summarized by following recursive formulation:

$$V(a,b,z;X) = \max_{c,n,a',b'} \frac{c^{1-\sigma}}{1-\sigma} - \frac{\eta}{1+\frac{1}{\chi}} n^{1+\frac{1}{\chi}} + \beta \mathbb{E} V(a',b',z';X') \tag{39}$$

$$\text{s.t. } c + a' + q^b(X)b' = a(1+r(X)) + b + zw(X)n \tag{40}$$

$$a' \geq 0, \quad b' \geq \underline{b} \tag{41}$$

$$X' = \Gamma_X(X) \quad \text{(Aggregate law of motion)} \tag{42}$$

$$z' \sim \pi(z'|z) \tag{43}$$

where c is consumption; z is idiosyncratic labor productivity, which follows a log AR(1) process; n is endogenously chosen labor supply; w is wage to be competitively determined at the factor input market, which thereby indicates that $zw(X)n$ is the labor income; a is the risky asset (capital) that earns capital rent $r = r(X)$ in each period, which is competitively determined at the capital market; b is the risk-free bond holding of which the price is $q = q(X)$. The bond price is competitively determined at the bond market. Apostrophe indicates future allocation. σ is risk-aversion parameter; χ is the Frisch labor elasticity; η is the labor disutility parameter; β is the discount factor. $\underline{b} \leq 0$ is the borrowing limit for future bond holding, and future risky asset is bound by zero borrowing limit.

The aggregate state X is defined as follows:

$$X := \{\Phi, A\} \tag{44}$$

where Φ is the joint distribution of the individual states; A is aggregate productivity that follows the same two-state Markov chain as in Krusell and Smith (1997). The rest of the model ingredients are identical to the leading application I (Section 6.1) except for the following bond market clearing condition: The bond price $q^b(X)$ is determined at the competitive market as follows:

$$[q]: \quad \int b'(\omega,z;X)d\Phi(X) = 0 \tag{45}$$

where I assume the aggregate net bond supply is zero as in Krusell and Smith (1997). I use the standard parameter levels in the literature, which are available in Appendix C.

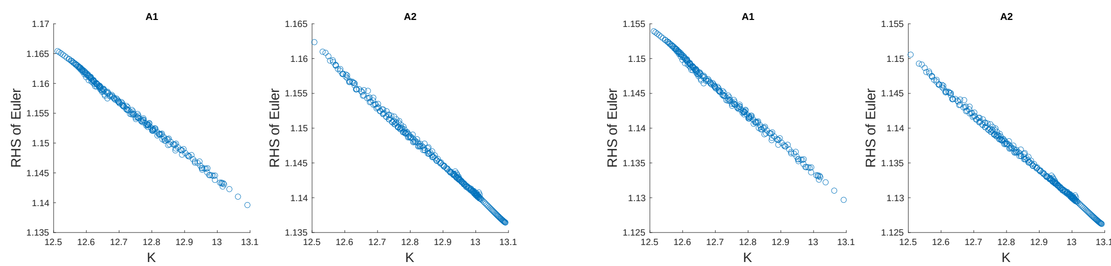
The model includes two inter-temporal assets, which necessarily leads to a highly complex endogenous aggregate state in equilibrium. Moreover, the model includes two occasionally binding constraints and the two non-trivial market clearing conditions for labor and bond market, which exponentially increases the computational burdens. Nevertheless, the RTM efficiently computes the global solution.

The first-order conditions lead to two inter-temporal optimality conditions:

$$
\begin{aligned}
\text{[Risky asset]}: \quad \frac{1}{c(k,z;X)} &= \beta \mathbb{E}_{z,X}\left[\left(\frac{1}{c(k',z';X')}\right)(1+r(X'))\right] + \lambda(k,z;X) && (46) \\
\text{[Safe asset]}: \quad \frac{1}{c(k,z;X)} &= \beta \mathbb{E}_{z,X}\left[\left(\frac{1}{c(k',z';X')}\right)\right] + \psi(k,z;X) && (47)
\end{aligned}
$$

where λ and ψ are Largrange multipliers for risky and safe assets' borrowing limit conditions, respectively.

I take the sufficient statistic approach in Section 3.1, using the aggregate capital stock $K(X)$ - the first moment of the individual risky asset distribution - as the sufficient statistic. I validate this approach by checking the monotonicity condition of Proposition 1 for both of the inter-temporal optimality conditions (46) and (47), as reported in Figure 6. Panel (a) and (b) are scatter plots of aggregate capital stock in the horizontal axis and the marginal benefit terms in the optimality condition in the vertical axis for different aggregate productivity realizations, given the median levels of individual wealth and labor productivity. According to panel (a), the marginal benefit of the risky asset strictly monotonically decreases in aggregate capital K for each aggregate productivity, validating the sufficient statistic approach. Similarly, panel (b) reports the strict monotonicity for the safe asset. The monotonicity property holds robustly across the entire cross-section of the individual states.



(a) Risky asset's inter-temporal optimality

(b) Risk-free asset's inter-temporal optimality

Figure 6: Strict monotonicity of the marginal benefit in the aggregate capital stock

The bond market presents a unique computational challenge beyond standard market clearing issues: zero net supply ($B'(X) = 0$) creates difficulties in characterizing the implied bond price using the RTM. The clearing condition reduces to a non-invertible identity: $q^b(X) \times B'(X) = B \iff q^b(X) \times 0 = 0$. This challenge becomes apparent when examining the national accounting identity derived from aggregating individual budget constraints:

$$C(X) + K'(X) + q^b(X)B'(X) = K(1 + r(X)) + B + w(X)N \tag{48}$$

$$\iff C(X) + I(X) + q^b(X)B' - B = Y(X) \tag{49}$$

$$\implies q^b(X)B' = B \iff q^b(X) \times 0 = 0, \tag{50}$$

where (C, K, I, Y, B) are aggregate consumption, capital stock, investment, output, and bond holdings. To overcome this computational challenge, I introduce a dummy bond term $\overline{B} > 0$, which remains fixed over the iterations. This allows characterization of the implied price q^{b*} through

$$q_t^{b*}\overline{B} := Y_t^{(n)} - C_t^* - I_t^* + B_t^* - q_t^{b(n)}\overline{B} \implies q_t^{b*} = \frac{Y_t^{(n)} - C_t^* - I_t^* + B_t^* - q_t^{b(n)}\overline{B}}{\overline{B}}, \tag{51}$$

where asterisks denote aggregations of individual optimal choices given the n^{th} iteration's guessed price path. Through iteration, this approach achieves two convergence results: the implied bond price sequence $\{q_t^{b*}\}_{t=0}^T$ converges to market-clearing levels, and the aggregate net bond supply $B_t^* = \int b_t^* d\Phi$ converges to 0 for $\forall$ t.[27]

6.2.1 Nonlinear bond price and heterogeneous portfolio adjustment over the business cycle

The RTM computes the equilibrium paths of the price bundle $(r(X), w(X), q(X))$. Among the prices, the bond price dynamics is particularly nonlinear, as can be seen from Figure 7. The predicted path (solid line) and the realized path (dash-dotted line) are indistinguishably close, which demonstrates the solution's accuracy. Its dynamics significantly deviate from the log-linear prediction (dotted line) based on the sufficient statistic $K(X)$. The fitted line's R^2 is only around 0.50, underscoring the inadequacy of linear approximations.

This finding has important implications for solution methods. The bond price's true law of motion is too complex for conventional state-space approaches, which would require correctly specifying the functional form before solving the model. Yet notably, despite this complexity in price dynamics, the inter-temporal optimality conditions maintain strict monotonicity in the sufficient statistic K, as demonstrated in Figure 6. This result demonstrates a key advantage of the RTM: its sufficient statistic approach remains valid even when linear prediction rules fail, enabling accurate

[27] The equilibrium path shows high sensitivity to bond price adjustments, necessitating conservative updating (weight = 0.999) in the implementation. An alternative specification for the implied bond price is $q_t^{b*} = \frac{B_{t+1}^{(n)} - q_t^{b(n)}\overline{B}}{\overline{B}}$, which also leads to the convergent outcomes.

solutions to models with complex nonlinear dynamics without requiring explicit functional forms. The heterogeneous agent model generates bond price dynamics that sharply contrast with repre-

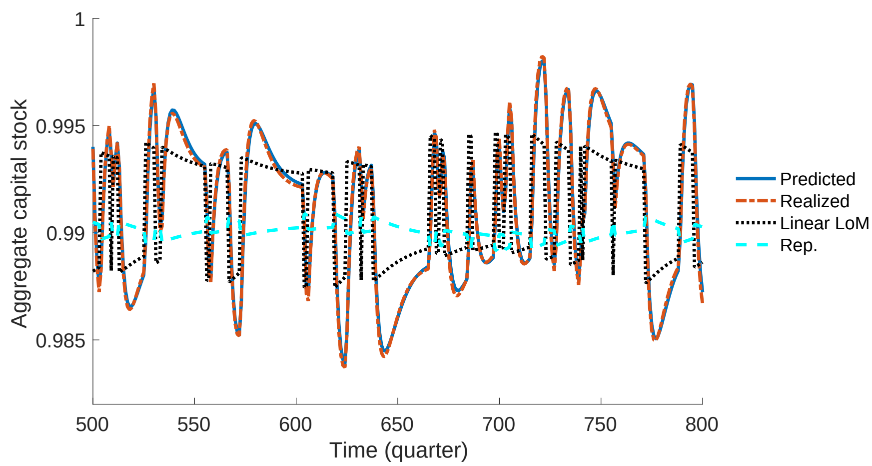


Figure 7: The equilibrium bond price path

sentative agent predictions.[28] The difference manifests in both cyclicality and volatility. While the heterogeneous agent model predicts pro-cyclical bond prices ($corr(Y_t, q_t^b) = 0.64$), while the representative agent framework implies counter-cyclical prices ($corr(Y_t, q_t^b) = -0.43$). The volatility difference is also striking: the heterogeneous agent model produces bond price volatility more than 9 times greater than its representative agent counterpart. These substantial differences stem from the rich heterogeneity in households' hedging motives across individual states, an equilibrium feature that representative agent models necessarily abstract from.

The global nonlinear solution of the model reveals a novel prediction about heterogeneous portfolio adjustment across households. Figure 8 illustrates this heterogeneity by tracking the household-level average of the risky asset weight in the portfolio for two groups: high-productivity households (defined as those in the top tercile of the productivity distribution) and low-productivity households (those in the bottom tercile), plotted against output deviations from steady state.

Both groups' households display counter-cyclical portfolio adjustments, increasing their risky asset allocations with a one-period lag relative to output fluctuations. However, the magnitude of these adjustments differs markedly across productivity levels: low-productivity households exhibit substantially more aggressive rebalancing behavior, with portfolio volatility more than 4 times greater than that of high-productivity households.

[28] In the representative agent model, the bond price is derived from the inverse of the risk-free rate. This remains an implied price since the zero net bond supply precludes actual transaction by the representative agent.

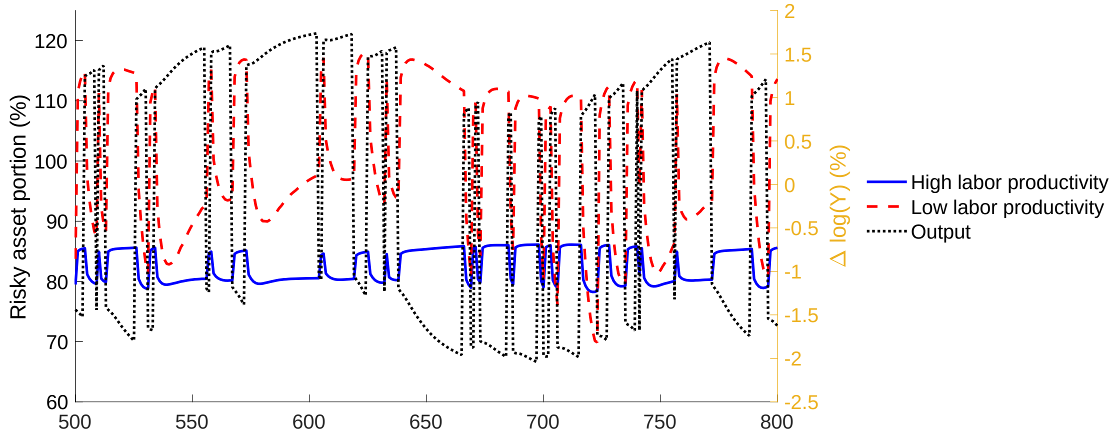
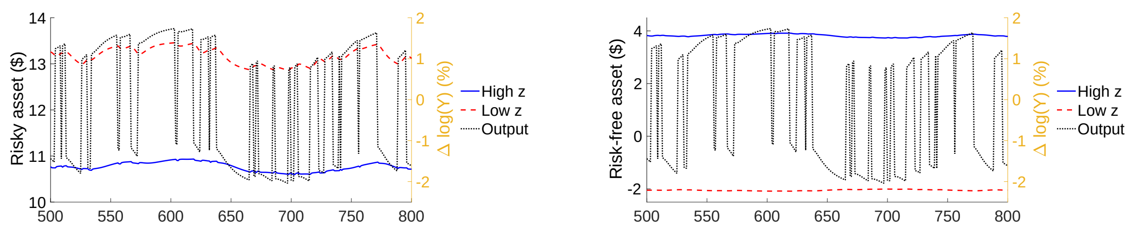


Figure 8: The equilibrium paths of risky asset portion: high vs. low productivity households

The group-level total asset composition dynamics are also starkly different. Figure 9 contrasts the asset holdings of high and low productivity groups against output deviations from steady state, with panel (a) showing risky asset holdings and panel (b) displaying risk-free asset positions. Panel (a) demonstrates that low productivity households maintain larger risky asset positions than their high productivity counterparts, with more pronounced pro-cyclical variation. Panel (b) reveals the financing strategy behind these positions: low productivity households achieve their large risky asset holdings through aggressive leverage, maintaining risk-free borrowing positions consistently near the constraint ($\underline{b} = -2.4$). In contrast, high productivity households hold large and stable risk-free asset positions throughout the business cycle.

These patterns offer important insights for both inequality dynamics and asset pricing theory. The RTM solution reveals how the bond market mediates heterogeneous hedging motives across household types, generating highly nonlinear and volatile bond prices. This interaction between household heterogeneity and financial markets provides new perspectives on both inequality transmission and asset pricing mechanisms.



(a) Risky asset dynamics

(b) Risk-free asset dynamics

Figure 9: Risky and risk-free asset dynamics: high and low productivity

7 Concluding remarks

This paper develops the repeated transition method (RTM), a powerful approach for solving dynamic stochastic general equilibrium models that achieves both global accuracy and computational efficiency. Working in sequence space, the method delivers dynamically consistent solutions while maintaining implementation simplicity. The RTM's fundamental innovation lies in computing conditional expectations by combining realized equilibrium outcomes, eliminating the need for parametric approximations of laws of motion.

A central theoretical contribution of this paper is establishing precise conditions under which sufficient statistics deliver exact solutions in models with complex aggregate states. This advance provides researchers with a rigorous foundation for dimensional reduction in sophisticated economic models, supported by readily testable conditions. The method's ability to accurately capture nonlinear dynamics without relying on specific functional forms represents another key advancement in solving modern macroeconomic models.

The applications in this paper demonstrate the practical value of these methodological advances. By accurately capturing nonlinear dynamics and state-dependent relationships, the RTM reveals important insights about fiscal policy effectiveness and heterogeneous portfolio choice behavior that might be missed by traditional solution methods. This capability opens new possibilities for investigating complex economic relationships and policy effects in modern macroeconomic models.

A Appendix: proposition and proof

Proposition 1 (The qualification for the sufficient statistic)**.**
For a sufficiently large T, if there exists a time series of a variable $\{e_t\}_{t=0}^{T}$ such that for each time partition $\mathcal{T}_S = \{t | S_t = S\}$, $\forall S \in \{B, G\}$ and for $\forall (a, s)$,

$$
\begin{aligned}
&(i) \quad e_{\tau_0} < e_{\tau_1} \iff V_{\tau_0}(a,s) < V_{\tau_1}(a,s) \text{ for any } \tau_0, \tau_1 \in \mathcal{T}_S \\
&or \\
&(ii) \quad e_{\tau_0} < e_{\tau_1} \iff V_{\tau_0}(a,s) > V_{\tau_1}(a,s) \text{ for any } \tau_0, \tau_1 \in \mathcal{T}_S,
\end{aligned}
$$

then e_t is the sufficient statistic of the endogenous aggregate state Φ_t for $\forall t$. That is,

$$
\mathbb{E}_S V(x; \Phi', S') = \int V(x; \Phi', S') d\Gamma_{S,S'} = \int V_{\widetilde{\tau}(S')+1}(x) d\Gamma_{S,S'}
$$

where $\widetilde{\tau}(S') + 1 = \arg\inf_{\tau \in \mathcal{T}_{S'}} ||e_\tau - e_{t+1}||_\infty$.

Proof.
Given the recessivity of the recursive competitive equilibrium over the sufficiently long path, there exists $\widetilde{\tau}(S') + 1$ with unit probability such that $e_{t+1} = e_{\widetilde{\tau}(S')+1}$.

Then, it is sufficient to show that the following equivalence holds:

$$
\{t \in \mathcal{T}_S | e_t = e_\tau\} = \{t \in \mathcal{T}_S | \Phi_t = \Phi_\tau\} \quad \text{for } \forall \tau \text{ and } \forall S \in \{B, G\},
$$

as it implies that $V(x; \Phi, S') = V_t(x)$ if Φ is from the set in the right-hand side of the equation and t is from the left-hand side.

First, the following direction holds:

$$
\{t \in \mathcal{T}_S | e_t = e_\tau\} \supseteq \{t \in \mathcal{T}_S | \Phi_t = \Phi_\tau\} \quad \text{for } \forall \tau \text{ and } \forall S \in \{B, G\}.
$$

It is because if two periods share the same aggregate states (both endogenous and exogenous), the level of the time-specific value function is the same. This implies, $e_{\widetilde{t}} = e_\tau$. That is,

$$
\text{For } \forall \widetilde{t} \in \{t \in \mathcal{T}_S | \Phi_t = \Phi_\tau\}, \quad V_{\widetilde{t}} = V_\tau \implies \text{For } \forall \widetilde{t} \in \{t \in \mathcal{T}_S | \Phi_t = \Phi_\tau\}, \quad e_{\widetilde{t}} = e_\tau.
$$

Otherwise, the strict monotonicity condition (i) or (ii) is violated.

Second, we need to show

$$\{t \in \mathcal{T}_S | e_t = e_\tau\} \subseteq \{t \in \mathcal{T}_S | \Phi_t = \Phi_\tau\} \quad \text{for } \forall \tau \text{ and } \forall S \in \{B, G\}.$$

From the monotonicity condition, the following is true:

$$\text{For } \forall \widetilde{t} \in \{t \in \mathcal{T}_S | e_t = e_\tau\}, \quad V_{\widetilde{t}} = V_\tau \quad \text{for } \forall \tau \text{ and } \forall S \in \{B, G\}.$$

Then, it is sufficient to show that

$$V_{\widetilde{t}} = V_\tau \implies \Phi_{\widetilde{t}} = \Phi_\tau.$$

Suppose it is not true. Then, there exists $\widetilde{t}$ such that

$$V_{\widetilde{t}} = V_\tau \text{ and } \Phi_{\widetilde{t}} \neq \Phi_\tau.$$

This contradicts to the assumed non-redundancy of the aggregate state and equilibrium uniqueness in Section 2.2. Therefore, the following holds:

$$\text{For } \forall \widetilde{t} \in \{t \in \mathcal{T}_S | e_t = e_\tau\}, \quad \Phi_{\widetilde{t}} = \Phi_\tau.$$

This implies

$$\{t \in \mathcal{T}_S | e_t = e_\tau\} \subseteq \{t \in \mathcal{T}_S | \Phi_t = \Phi_\tau\}$$

for any τ and $S \in \{B, G\}$.

$\blacksquare$

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