Dancing on the Saddles: A Geometric Framework for Stochastic Equilibrium Dynamics[1]

Online Appendix

Contents

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A Sequential formulation of a canonical RBC

I consider a representative household with temporal log utility. Given initial condition (a_0, A_0), the household maximizes lifetime utility under stochastic aggregate TFP A_t, subject to a budget constraint:

$$\max_{\{c_\tau(A^{(\tau)}), a_{\tau+1}(A^{(\tau)})\}_{\tau=0}^{\infty}} \mathbb{E}_0 \sum_{\tau=0}^{\infty} \beta^\tau \log(c_\tau(A^{(\tau)})) \tag{1}$$

$$\text{s.t. } c_\tau(A^{(\tau)}) + a_{\tau+1}(A^{(\tau)}) = a_\tau(A^{(\tau-1)})(1 + r(A^{(\tau)})) + w(A^{(\tau)}), \quad \forall \tau, \forall A^{(\tau)} \tag{2}$$

$$a_{\tau+1}(A^{(\tau)}) \geq -\overline{a}, \quad \forall \tau \tag{3}$$

where the superscript τ inside a bracket denotes the history of a variable up to period τ; $-\overline{a}$ is the natural borrowing limit to preempt a Ponzi scheme. Labor supply is exogenously fixed at unity. I consider competitive factor prices given a CRS Cobb–Douglas production function:

$$r(A^{(\tau)}) = A_\tau \alpha (K(A^{(\tau)}))^{(\alpha-1)} - \delta \tag{4}$$

$$w(A^{(\tau)}) = A_\tau (1 - \alpha)(K(A^{(\tau)}))^{\alpha}, \tag{5}$$

K is the capital stock that satisfies $K(A^{(\tau)}) = a(A^{(\tau)})$ in equilibrium. With the regularity conditions given in Stokey et al. (1989), this sequential formulation yields the same optimality conditions as the recursive form in the main text.

B Individual conditional saddles in Aiyagari (1994)

I define individual-level conditional saddle paths in the heterogeneous-household economy without aggregate uncertainty ($A = A' = 1$). The conditional saddle is defined for the stationary RCE as in Aiyagari (1994).

Definition 1 (Individual conditional saddle under frozen z)**.**
Fix a stationary RCE of the Aiyagari (1994) economy, which delivers an individual asset policy

$$g_a : \mathbb{R}_+ \times \mathcal{Z} \to \mathbb{R}_+,$$

where $\mathcal{Z}$ is a finite Markov set with transition matrix Π. *Fix an initial condition $(a_0, z_0) \in \mathbb{R}_+ \times \mathcal{Z}$. For any frozen idiosyncratic state $z \in \mathcal{Z}$, define the frozen-z* *continuation* $\{a_t(z)\}_{t\geq 0}$ *recursively by*

$$a_{t+1}(z) = g_a(a_t(z), z), \qquad a_0(z) = a_0. \tag{6}$$

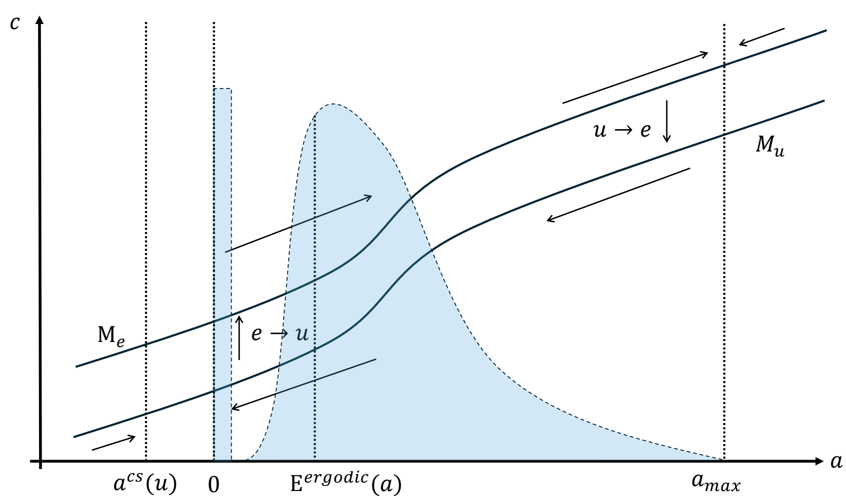
The individual conditional saddle under frozen z *is the orbit-closure*

$$\mathcal{M}^{\text{ind}}(z; a_0) := \overline{\{a_t(z) : t \geq 0\}} \;\subseteq\; \mathbb{R}_+. \tag{7}$$

Definition 2 (Individual conditional steady state under frozen z)**.**
Fix (a_0, z_0) *and* $z \in \mathcal{Z}$, *and let* $\{a_t(z)\}_{t\geq 0}$ *be the frozen-*z *continuation. The* individual conditional steady state *under frozen* z *is defined by*

$$a^{cs}(z; a_0) := \lim_{t\to\infty} a_t(z). \tag{8}$$

Figure B.1: Individual conditional saddle paths in the stationary RCE



Notes: The figure illustrates the individual conditional saddle paths for $z = e$ and $z = u$ in Aiyagari (1994).

Figure B.1 illustrates individual-level conditional saddle paths in Aiyagari (1994). Because of the borrowing constraint, the conditional steady state associated with the unemployment state $z = u$ is not attained. Under standard calibrations, this generates a positive mass of agents at the borrowing limit. Analogous to the aggregate-level case, heterogeneity in the slopes of individual conditional saddle paths implies differential responses of individual consumption to idiosyncratic shocks.

C Proof of Proposition 4

Proposition 4 (Aggregate uncertainty and the conditional steady states)**.**
The following inequalities hold:

$$K_B^{cs} \ < \ K_B^{pf} \ < \ K_G^{pf} \ < \ K_G^{cs}, \qquad c_B^{cs} \ < \ c_B^{pf} \ < \ c_G^{pf} \ < \ c_G^{cs}.$$

Proof.
Let $R(A,K) := 1 - \delta + \alpha A K^{\alpha-1}$ denote the gross return on capital and let the K-nullcline (feasibility locus) be

$$c(A,K) = AK^{\alpha} - \delta K.$$

The PF steady states solve the Euler equations under absorbing beliefs,

$$1 = \beta R(B, K_B^{pf}), \qquad 1 = \beta R(G, K_G^{pf}).$$

Since $R(A,K)$ is strictly increasing in A and strictly decreasing in K (because $\alpha - 1 < 0$), it follows immediately that $K_B^{pf} < K_G^{pf}$.

Next define the frozen-regime CS Euler residuals evaluated on the K-nullcline by

$$\begin{aligned}
F_B(K) &:= \beta\Big[\pi_{BB} R(B,K) + \pi_{BG} \frac{c(B,K)}{c(G,K)}\, R(G,K)\Big] - 1, \\
F_G(K) &:= \beta\Big[\pi_{GG} R(G,K) + \pi_{GB} \frac{c(G,K)}{c(B,K)}\, R(B,K)\Big] - 1.
\end{aligned}$$

By construction, the conditional steady states satisfy $F_B(K_B^{cs}) = 0$ and $F_G(K_G^{cs}) = 0$.

Step 1: show $K_B^{cs} < K_B^{pf}$. Evaluate F_B at K_B^{pf}. Using $1 = \beta R(B, K_B^{pf})$ and $\pi_{BB} = 1 - \pi_{BG}$,

$$\begin{aligned}
F_B(K_B^{pf}) &= \beta\Big[(1-\pi_{BG}) R(B, K_B^{pf}) + \pi_{BG} \frac{c(B, K_B^{pf})}{c(G, K_B^{pf})} R(G, K_B^{pf})\Big] - 1 \\
&= \pi_{BG}\Big[\beta \frac{c(B, K_B^{pf})}{c(G, K_B^{pf})} R(G, K_B^{pf}) - 1\Big].
\end{aligned}$$

Thus $F_B(K_B^{pf}) < 0$ is equivalent to

$$\beta \frac{c(B, K_B^{pf})}{c(G, K_B^{pf})} R(G, K_B^{pf}) < 1 \quad \Longleftrightarrow \quad \frac{R(G, K_B^{pf})}{R(B, K_B^{pf})} < \frac{c(G, K_B^{pf})}{c(B, K_B^{pf})},$$

where I used $\beta R(B, K_B^{pf}) = 1$ to divide both sides by $R(B, K_B^{pf})$.

I now prove this strict inequality for any K with $c(B, K), c(G, K) > 0$. Write $x := \alpha K^{\alpha-1} > 0$ and $y := K^{\alpha} > 0$. Then

$$\frac{R(G, K)}{R(B, K)} = \frac{1 - \delta + Gx}{1 - \delta + Bx}, \qquad \frac{c(G, K)}{c(B, K)} = \frac{Gy - \delta K}{By - \delta K}.$$

Since $1 - \delta > 0$, the strict bound

$$\frac{1 - \delta + Gx}{1 - \delta + Bx} < \frac{Gx}{Bx} = \frac{G}{B}.$$

Since $\delta K > 0$ and $Gy > By$, subtracting the same positive term from numerator and denominator enlarges the ratio, yielding

$$\frac{Gy - \delta K}{By - \delta K} > \frac{Gy}{By} = \frac{G}{B}.$$

Combining the two displays gives

$$\frac{R(G, K)}{R(B, K)} < \frac{G}{B} < \frac{c(G, K)}{c(B, K)},$$

and in particular the desired inequality holds at $K = K_B^{pf}$. Therefore $F_B(K_B^{pf}) < 0$.

Finally, note that F_B is strictly decreasing in K on the relevant region because both $R(B, K)$ and $R(G, K)$ are strictly decreasing in K and $c(B, K)/c(G, K)$ is also decreasing in K along the feasibility locus.[2] Hence, since $F_B(K_B^{cs}) = 0$ and $F_B(K_B^{pf}) < 0$, it follows that $K_B^{cs} < K_B^{pf}$.

Step 2: show $K_G^{cs} > K_G^{pf}$. Similarly, evaluate F_G at K_G^{pf}. Using $1 = \beta R(G, K_G^{pf})$ and

[2] This monotonicity is standard and can be verified by differentiation; it is also visually apparent in the (K, C) phase diagram.

$\pi_{GG} = 1 - \pi_{GB}$,

$$\begin{aligned} F_G(K_G^{pf}) &= \beta\Big[(1-\pi_{GB})R(G,K_G^{pf}) + \pi_{GB}\frac{c(G,K_G^{pf})}{c(B,K_G^{pf})}R(B,K_G^{pf})\Big] - 1 \\ &= \pi_{GB}\Big[\beta\frac{c(G,K_G^{pf})}{c(B,K_G^{pf})}R(B,K_G^{pf}) - 1\Big]. \end{aligned}$$

Thus $F_G(K_G^{pf}) > 0$ is equivalent to

$$\beta\frac{c(G,K_G^{pf})}{c(B,K_G^{pf})}R(B,K_G^{pf}) > 1 \quad \iff \quad \frac{R(B,K_G^{pf})}{R(G,K_G^{pf})} > \frac{c(B,K_G^{pf})}{c(G,K_G^{pf})}.$$

But the argument above applied with (B, G) swapped gives, for any K with positive consumption,

$$\frac{R(B,K)}{R(G,K)} > \frac{B}{G} > \frac{c(B,K)}{c(G,K)}.$$

Hence $F_G(K_G^{pf}) > 0$. Since F_G is strictly decreasing in K and $F_G(K_G^{cs}) = 0$, it follows that $K_G^{cs} > K_G^{pf}$.

Step 3: conclude the ordering and translate to consumption. Steps 1 and 2 have shown $K_B^{cs} < K_B^{pf}$ and $K_G^{pf} < K_G^{cs}$, and already $K_B^{pf} < K_G^{pf}$, hence

$$K_B^{cs} < K_B^{pf} < K_G^{pf} < K_G^{cs}.$$

Finally, along the K-nullcline $c(A,K) = AK^\alpha - \delta K$ is strictly increasing in A and increasing in K on the dynamically efficient region ($\alpha AK^{\alpha-1} > \delta$, which holds at all four steady states under standard calibrations), so the same ordering carries over to consumption:

$$c_B^{cs} < c_B^{pf} < c_G^{pf} < c_G^{cs}.$$

■

D Proof of Proposition 5

Proof.

The proof derives the peak time in closed form and applies the implicit function theorem to sign the comparative static in initial capital.

Step 1 (Blanchard–Kahn solution on the saddle branch). Log-linearize the RBC dynamics around the non-stochastic steady state. Let $\hat{k}_t := \ln K_t - \ln K^*$ and $\hat{a}_t := \ln A_t - \ln A^*$ denote log-deviations. The linearized capital law of motion and saddle-path consumption policy are

$$\hat{k}_{t+1} = \lambda\,\hat{k}_t + \theta\,\hat{a}_t, \tag{9}$$

$$\hat{c}_t = \psi_k\,\hat{k}_t + \psi_a\,\hat{a}_t, \tag{10}$$

where $\lambda \in (0,1)$ is the stable eigenvalue of the capital dynamics, $\theta > 0$ is the capital-TFP elasticity, and $\psi_k > 0$, $\psi_a > 0$ are the consumption-capital and consumption-TFP elasticities on the saddle branch.

Consider a one-time shock $\hat{a}_0 = \varepsilon > 0$ with deterministic mean-reversion $\hat{a}_t = \rho^t \varepsilon$, $\rho \in (0,1)$. Iterating (9) gives

$$\hat{k}_t = \lambda^t \hat{k}_0 + \theta\varepsilon \sum_{s=0}^{t-1} \lambda^{t-1-s} \rho^s = \lambda^t \hat{k}_0 + \frac{\theta\varepsilon}{\lambda - \rho}\big(\lambda^t - \rho^t\big), \tag{11}$$

where the closed form uses $\lambda \neq \rho$. Substituting into (10):

$$\hat{c}_t = \psi_k \left[\lambda^t \hat{k}_0 + \frac{\theta\varepsilon}{\lambda - \rho}\big(\lambda^t - \rho^t\big)\right] + \psi_a \rho^t \varepsilon = \mathcal{P}\,\lambda^t + \mathcal{Q}\,\rho^t, \tag{12}$$

where

$$\mathcal{P}(\hat{k}_0) := \psi_k \left[\hat{k}_0 + \frac{\theta\varepsilon}{\lambda - \rho}\right], \qquad \mathcal{Q} := \left[\psi_a - \frac{\psi_k \theta}{\lambda - \rho}\right]\varepsilon. \tag{13}$$

Crucially, $\mathcal{P}$ depends on $\hat{k}_0$ while $\mathcal{Q}$ does not.

Step 2 (Peak time in closed form). Treating t as continuous, the peak satisfies $d\hat{c}_t/dt = 0$:

$$\mathcal{P} \ln\lambda \cdot \lambda^{t^*} + \mathcal{Q} \ln\rho \cdot \rho^{t^*} = 0. \tag{14}$$

Rearranging:

$$\left(\frac{\lambda}{\rho}\right)^{t^*} = -\frac{\mathcal{Q}\ln\rho}{\mathcal{P}\ln\lambda}. \tag{15}$$

Taking logarithms yields

$$t^* = \frac{1}{\ln(\lambda/\rho)}\,\ln\!\Big(\frac{-\mathcal{Q}\ln\rho}{\mathcal{P}\ln\lambda}\Big), \tag{16}$$

which is well-defined when $-\mathcal{Q}\ln\rho/(\mathcal{P}\ln\lambda) > 0$, i.e., $\mathcal{Q}/\mathcal{P} < 0$ (since $\ln\lambda < 0$ and $\ln\rho < 0$ make the two negative signs cancel only when $\mathcal{Q}$ and $\mathcal{P}$ have opposite signs). With $\mathcal{P} > 0$, this requires $\mathcal{Q} < 0$: the direct TFP effect on consumption is smaller than the capital-accumulation channel ($\psi_a < \psi_k\theta/(\lambda-\rho)$), which is exactly what generates the hump.

Step 3 (Comparative static via the implicit function theorem). Differentiating (16) with respect to $\hat{k}_0$:

$$\frac{\partial t^*}{\partial \hat{k}_0} = \frac{1}{\ln(\lambda/\rho)}\cdot\frac{-1}{\mathcal{P}}\cdot\frac{\partial\mathcal{P}}{\partial\hat{k}_0} = \frac{-\psi_k}{\mathcal{P}(\hat{k}_0)\cdot\ln(\lambda/\rho)}, \tag{17}$$

where $\partial\mathcal{P}/\partial\hat{k}_0 = \psi_k > 0$ from (13). Under $\lambda > \rho$, we have $\ln(\lambda/\rho) > 0$. Combined with $\psi_k > 0$ and $\mathcal{P} > 0$ (which holds when $\hat{k}_0 > -\theta\varepsilon/(\lambda-\rho)$, a condition easily satisfied within the ergodic capital range), this gives

$$\frac{\partial t^*}{\partial\hat{k}_0} < 0. \tag{18}$$

That is, the consumption peak arrives strictly later when initial capital is farther below the steady state. ■

E Proof of Proposition 8

Proof.
The proof computes the expected contraction factor over a geometric spell length.

Step 1 (Within-spell contraction). By assumption, the endogenous transition operator $\Gamma_{\text{endo}}(\cdot, A)$ contracts at rate $\lambda_c \in (0,1)$ on the conditional saddle $\mathcal{M}(\Phi_0, A)$:

$$d\big(\Gamma^t_{\text{endo}}(\Phi, A),\, \Gamma^t_{\text{endo}}(\Phi', A)\big) \le \lambda_c^t\, d(\Phi,\Phi'), \qquad \forall\, \Phi,\Phi' \in \mathcal{M}(\Phi_0, A), \tag{19}$$

where Γ^t_{endo} denotes the t-fold iterate. After a spell of length τ, the distributional discrepancy between two paths sharing the same K but originating from different initial conditions is reduced by the factor λ_c^τ.

Step 2 (Geometric spell length). The regime spell length τ is the first time the exogenous state switches away from A. Under the Markov transition matrix Γ_{exo}, τ is geometrically distributed with parameter $1-\rho_A$ (the probability of switching), where $\rho_A := \pi_{AA}$:

$$\Pr(\tau = k) = (1-\rho_A)\rho_A^{k-1}, \qquad k = 1, 2, \dots \tag{20}$$

Step 3 (Expected discount factor). The expected contraction factor over one spell is

$$\mathbb{E}[\lambda_c^\tau] = \sum_{k=1}^{\infty}(1-\rho_A)\rho_A^{k-1}\lambda_c^k = (1-\rho_A)\lambda_c \sum_{k=0}^{\infty}(\rho_A\lambda_c)^k = \frac{(1-\rho_A)\lambda_c}{1-\rho_A\lambda_c}, \tag{21}$$

where the geometric series converges because $\rho_A\lambda_c < 1$ (both factors are strictly less than one). Multiplying by the maximal same-K distributional discrepancy $D := \sup\{d(\Phi_1, \Phi_2) : K(\Phi_1) = K(\Phi_2)\}$ gives the expected across-spell discrepancy bound

$$\mathbb{E}_\tau\left[\lambda_c^\tau\right] \cdot D = \frac{(1-\rho_A)\lambda_c}{1-\rho_A\lambda_c}\, D. \tag{22}$$

The bound vanishes as $\rho_A \to 1$ (long spells allow more within-spell contraction) or $\lambda_c \to 0$ (fast convergence within each spell). ■

F Primitive conditions for Assumption 2

This appendix provides an independent route to Assumption 2 from static allocation primitives, working directly at $\mu = 1$ (rational expectations) without the homotopy of Appendix G. The key idea is that side-switching across the K-nullcline is excluded under economically interpretable bounds on households' marginal propensities to consume and sufficiently weak general-equilibrium price feedback.

Household MPC bounds. Let $c(a, z; r, w)$ denote the stationary equilibrium household consumption policy as a function of individual assets a, idiosyncratic state $z \in \mathcal{Z}$, and prices (r, w). Along a frozen-regime continuation, aggregate consumption can change with aggregate capital because K affects prices, which in turn affects contemporaneous income components wz and ra. A convenient "MPC bound" in this context is therefore a bound on the *marginal response of consumption to price-induced changes in current income*, rather than an MPC out of an exogenous transfer shock.

Assumption 3 (Bounded price-income MPC)**.**
Along the frozen-regime continuation under A, there exists $\bar{m} \in (0,1)$ such that for the stationary equilibrium policy $c(a,z;r,w)$,

$$0 \le \frac{\partial c}{\partial (wz)}(a,z;r,w) \le \bar{m}, \qquad 0 \le \frac{\partial c}{\partial (ra)}(a,z;r,w) \le \bar{m}, \tag{23}$$

for all (a,z) in the support of Φ_t and all t on the continuation.[3]

Under Assumption 3, a marginal increase in aggregate capital affects contemporaneous aggregate consumption through the induced changes in wages and returns. Define the (pathwise) bounds as

$$\bar{Z} := \sup_{t\ge 0} \int z \, d\Phi_t, \qquad \bar{a} := \sup_{t\ge 0} \int a \, d\Phi_t, \qquad \mathcal{K} := \{K_t : t \ge 0\}. \tag{24}$$

Then, the change in aggregate consumption induced by a marginal change in K can be bounded above by the aggregate MPC bound $\bar{m}$ times the induced change in contemporaneous labor and capital incomes.

Sufficient condition for Assumption 2. In the Krusell–Smith benchmark with Cobb–Douglas production, prices are smooth functions of K:

$$w_A(K) = (1-\alpha)AK^{\alpha}, \qquad r_A(K) = \alpha A K^{\alpha-1} - \delta, \tag{25}$$

with derivatives

$$w_A'(K) = (1-\alpha)\alpha A K^{\alpha-1}, \qquad r_A'(K) = \alpha(\alpha-1)AK^{\alpha-2}. \tag{26}$$

Moreover, the nullcline satisfies $\left(C_A^{Kcnull}\right)'(K) = \alpha A K^{\alpha-1} - \delta$. The next lemma shows that if general-equilibrium price feedback is sufficiently weak relative to the nullcline

[3] Assumption 3 bounds the marginal response of consumption to *price-induced* changes in contemporaneous income components. This local derivative bound differs from the standard MPC out of a transitory transfer. In incomplete-markets environments, precautionary saving and borrowing constraints typically prevent one-for-one consumption responses to marginal changes in current income, making an upper bound $\bar{m} < 1$ natural on the relevant region of the state space.

slope, then the nullcline gap H_t cannot change sign, which implies Assumption 2 along the continuation.

Lemma 1 (Sufficient condition for Assumption 2)**.**
Fix $A \in \{B, G\}$ and consider the frozen-A continuation $\{\Phi_t\}_{t\geq 0}$ with $K_t := K(\Phi_t)$ and $C_t := C(\Phi_t, A)$. Let $\bar{Z} := \sup_{t\geq 0} \int z\, d\Phi_t$ and $\bar{a} := \sup_{t\geq 0} \int a\, d\Phi_t$ denote the pathwise upper bounds on average productivity and average assets (both finite under the natural borrowing limit and compact idiosyncratic support). Under Assumption 3, suppose that along $\mathcal{K}$,

$$\sup_{K\in\mathcal{K}} \Big(\bar{m}\big(w'_A(K)\,\bar{Z} + |r'_A(K)|\,\bar{a}\big) - \big(C_A^{Kcnull}\big)'(K)\Big) \;<\; 1. \tag{27}$$

Then $H_t H_{t+1} > 0$ for all t such that $\Phi_t \neq \Phi^{cs}(\Phi_0, A)$, and hence Assumption 2 holds along the continuation. Consequently, $\{K_t\}$ is strictly monotone until convergence.

Proof.
Define the nullcline gap $H_t := C_t - C_A^{Kcnull}(K_t)$ so that $\Delta K_t := K_{t+1} - K_t = -H_t$. By the mean-value theorem applied to each term separately, there exist ξ_t^C and ξ_t^N, each between K_t and K_{t+1}, such that

$$\begin{aligned} H_{t+1} - H_t &= \big(C_{t+1} - C_t\big) - \Big(C_A^{Kcnull}(K_{t+1}) - C_A^{Kcnull}(K_t)\Big) &(28)\\ &= \Big(\frac{dC}{dK}(\xi_t^C) - \big(C_A^{Kcnull}\big)'(\xi_t^N)\Big)\,(K_{t+1} - K_t). &(29)\end{aligned}$$

Here dC/dK denotes the total derivative of aggregate consumption with respect to aggregate capital *along the frozen-regime continuation*—that is, the derivative accounting for the induced change in the distributional state Φ_t as K varies along the conditional saddle, not a partial derivative holding the distribution fixed.
By Assumption 3, the induced change in aggregate consumption from a marginal increase in K is bounded above by the aggregate MPC bound times the induced change in contemporaneous incomes:

$$\frac{dC}{dK}(\xi_t^C) \leq \bar{m}\Big(w'_A(\xi_t^C)\int z\, d\Phi_t + |r'_A(\xi_t^C)|\int a\, d\Phi_t\Big) \leq \bar{m}\big(w'_A(\xi_t^C)\bar{Z} + |r'_A(\xi_t^C)|\bar{a}\big). \tag{30}$$

Using $K_{t+1} - K_t = -H_t$ gives

$$H_{t+1} = \Big[1 - \Big(\frac{dC}{dK}(\xi_t^C) - \big(C_A^{Kcnull}\big)'(\xi_t^N)\Big)\Big] H_t. \tag{31}$$

The MPC bound condition from Lemma 1,

$$\sup_{K \in \mathcal{K}} \Big(\bar{m}\big(w_A'(K)\,\bar{Z} + |r_A'(K)|\,\bar{a}\big) - \big(C_A^{Kcnull}\big)'(K)\Big) \;<\; 1, \tag{32}$$

implies—since both $\xi_t^C, \xi_t^N \in \mathcal{K}$—that the bracketed multiplier is strictly positive for all t, hence $\{H_t\}$ cannot change sign: $H_t H_{t+1} > 0$ for all pre-limit t. Therefore $\Delta K_t = -H_t$ has constant nonzero sign, so $\{K_t\}$ is strictly monotone until convergence. ∎

A local check near the conditional steady state. The global bound (27) can be weakened to a neighborhood condition near the conditional steady state. This is useful both for economic interpretation and for quantitative verification in standard calibrations.

Corollary 1 (A local check near K_A^{cs}).
Under Assumption 3, suppose there exists $\varepsilon > 0$ such that the continuation satisfies $K_t \in (K_A^{cs} - \varepsilon, K_A^{cs} + \varepsilon)$ for all sufficiently large t, and

$$\sup_{|K - K_A^{cs}| \le \varepsilon} \Big(\bar{m}\big(w_A'(K)\,\bar{Z} + |r_A'(K)|\,\bar{a}\big) - \big(C_A^{Kcnull}\big)'(K)\Big) \;<\; 1. \tag{33}$$

Then there exists $T < \infty$ such that $H_t H_{t+1} > 0$ (and hence Assumption 2) holds for all $t \ge T$, implying that K_t is eventually strictly monotone and converges to K_A^{cs} without side-switching.

Proof. Restrict Lemma 1 to the neighborhood $|K - K_A^{cs}| \le \varepsilon$. ∎

Conditions (27)–(33) are directly checkable in quantitative implementations: one evaluates the left-hand side along a simulated frozen-regime continuation using the model-implied bound $\bar{m}$ and the moments $\bar{Z}$ and $\bar{a}$. In the baseline Krusell and Smith (1998) calibration, the left-hand side of (27) evaluated along the simulated frozen-A continuation is below 0.04 even under the conservative bound $\bar{m} = 1$, well below the threshold 1, confirming that the condition holds with substantial slack.

G Continuous dependence on beliefs

This appendix establishes continuous dependence of the frozen-regime continuation on the belief parameter μ.

Proof of continuous dependence on μ.
The argument proceeds in three steps: household optimization, market clearing, and equilibrium composition.

Step 1 (Household optimization). For given prices (r, w) and belief parameter $\mu \in [0, 1]$, the household's recursive problem is

$$v(a, z; \mu) = \max_{c,a'} \; u(c) + \beta \sum_{A'} \Pi(\mu)(A'|A) \, \mathbb{E}_{z'|z} \, v(a', z'; \mu),$$

subject to the budget constraint and borrowing limit. On the bounded asset domain $[\underline{a}, \bar{a}]$ induced by the natural borrowing limit and stationary prices, and under strict concavity and Inada conditions, T_μ is a contraction in the sup-norm for each μ (Stokey et al., 1989, Theorem 9.6). Since $\Pi(\mu) = (1 - \mu)I + \mu\Pi$ enters linearly in the expectation operator, T_μ is continuous in μ, so its unique fixed point $v(\cdot; \mu)$ depends continuously on μ by the parametric contraction mapping theorem. Continuous dependence of the policy function $c(\cdot; \mu)$ follows by the theorem of the maximum.

Step 2 (Market clearing). For each μ, the stationary distribution $\Phi(\mu)$ and aggregate capital $K(\mu)$ are determined by market clearing. Since the individual policy function $a'(\cdot; \mu)$ is continuous in μ, the induced Markov transition kernel on (a, z) is Feller. On the compact state space $[\underline{a}, \bar{a}] \times \mathcal{Z}$, the kernel admits a unique ergodic distribution $\Phi(\mu)$ (by geometric ergodicity under the standard mixing conditions of the idiosyncratic process), and uniqueness together with the Feller property implies that $\Phi(\mu)$ varies continuously in μ in the weak topology.

Step 3 (Equilibrium composition). The frozen-regime continuation $\{(K_t(\mu), C_t(\mu))\}_{t \geq 0}$ is obtained by iterating $\Gamma_{\text{endo}}(\cdot, A; \mu)$ from the initial condition. Since each iterate is a continuous function of μ (by composition of continuous maps from Steps 1–2), the entire sequence depends continuously on μ pointwise in t. The conditional steady state $K_A^{cs}(\mu)$ is the fixed point of $K \mapsto K(\Gamma_{\text{endo}}(\cdot, A; \mu))$, which varies continuously in μ by the implicit function theorem (or equivalently, by continuity of the Euler equation residual evaluated at the steady state). ■

H Details for the openness step in Proposition 9

This appendix provides the tail argument used in the openness step of Proposition 9.

Tail monotonicity via eigenvalue continuity.
Fix $\mu^* \in \mathcal{S}$. Strict monotone convergence of $\{K_t(\mu^*)\}$ to $K_A^{cs}(\mu^*)$ implies that, near the conditional steady state, the linearized frozen-regime dynamics

$$K_{t+1} - K_A^{cs}(\mu^*) \;\approx\; \lambda_s(\mu^*)\,(K_t - K_A^{cs}(\mu^*)) \tag{34}$$

have a stable eigenvalue $\lambda_s(\mu^*) \in (0, 1)$. If $\lambda_s(\mu^*)$ were negative or complex, convergence near the steady state would be oscillatory, contradicting monotonicity of $\{K_t(\mu^*)\}$ in a neighborhood of its limit.

The eigenvalue $\lambda_s(\mu)$ is the stable root of the linearized Euler equation–resource constraint system at $K_A^{cs}(\mu)$. Both the steady state $K_A^{cs}(\mu)$ and the coefficients of the linearized system depend continuously on μ (by the continuous dependence established in Appendix G). Since eigenvalues of a matrix depend continuously on its entries, $\lambda_s(\mu)$ is continuous in μ.

Therefore, there exists $\varepsilon_1 > 0$ such that $|\mu - \mu^*| < \varepsilon_1$ implies $\lambda_s(\mu) \in (0, 1)$, ensuring monotone convergence near $K_A^{cs}(\mu)$.

Choose T large enough that $K_t(\mu^*)$ lies within a neighborhood of $K_A^{cs}(\mu^*)$ where the local map is monotone with slope in $(0, 1)$, hence order-preserving. By continuity of $(K_t(\mu), C_t(\mu))$ in μ for each finite t, for μ sufficiently close to μ^*, the path $\{K_t(\mu)\}_{t\geq T}$ also enters this neighborhood and inherits monotone convergence from the eigenvalue condition. Combined with the finite-horizon continuity argument (which preserves the sign of H_t for $t \leq T$), monotonicity holds for all t, proving $\mu \in \mathcal{S}$. ∎

I Endogenous labor supply extensions

This appendix provides the derivations for the endogenous labor supply extensions discussed in Section 5.

GHH preferences. Consider preferences $u(c, l_H) = \frac{1}{1-\sigma}(c - \frac{\eta}{1+1/\chi} l_H^{1+1/\chi})^{1-\sigma}$ with budget constraint $c + a' = a(1 + r(\Phi, A)) + w(\Phi, A) z l_H$, where z denotes idiosyncratic labor productivity. The individual labor-supply optimality condition yields:

$$l_H(a, z; \Phi, A) = \left(\frac{z}{\eta}\right)^{\chi} w(\Phi, A)^{\chi}, \qquad L(\Phi, A) = w(\Phi, A)^{\chi} \int \frac{z^{\chi+1}}{\eta^{\chi}} d\Phi_z, \tag{35}$$

where Φ_z is the marginal distribution of productivity. Defining $M = \int \frac{z^{\chi+1}}{\eta^{\chi}} d\Phi_z$, the labor market clearing condition $(1-\alpha)AK^{\alpha}L^{-\alpha} = w = (L/M)^{1/\chi}$ implies:

$$c_A^{Kcnull} = A\big(M^{\frac{1}{1+\alpha\chi}}((1-\alpha)A)^{\frac{\chi}{1+\alpha\chi}}\big)^{1-\alpha}(K_A^{Kcnull})^{\alpha+(1-\alpha)\frac{\alpha\chi}{1+\alpha\chi}} - \delta K_A^{Kcnull}. \tag{36}$$

Since M depends only on the stationary productivity distribution (not the wealth distribution), the nullcline is distribution-free and the monotonicity argument applies.

CRRA preferences. Under $u(c, l_H) = \frac{c^{1-\sigma}}{1-\sigma} - \frac{\eta}{1+1/\chi} l_H^{1+1/\chi}$, the individual labor supply becomes:

$$l_H(a, z; \Phi, A) = \left(\frac{z}{\eta c(a, z; \Phi, A)^{\sigma}}\right)^{\chi} w(\Phi, A)^{\chi}. \tag{37}$$

Following the same derivation, the conditional K-nullcline is:

$$c_A^{Kcnull} = A\big(M(\Phi^{Kcnull})^{\frac{1}{1+\alpha\chi}}((1-\alpha)A)^{\frac{\chi}{1+\alpha\chi}}\big)^{1-\alpha}(K_A^{Kcnull})^{\alpha+(1-\alpha)\frac{\alpha\chi}{1+\alpha\chi}} - \delta K_A^{Kcnull}, \tag{38}$$

where $M(\Phi^{Kcnull}) = \int \frac{z^{\chi+1}}{\eta^{\chi} c^{\sigma\chi}} d\Phi^{Kcnull}$ now depends on the wealth distribution through individual consumption. This distributional dependence prevents analytical verification of monotonicity.

J Search and matching model

This appendix provides the full specification of the Diamond–Mortensen–Pissarides economy used in Section 5.1. The model is a discrete-time stochastic version of the canonical search-and-matching framework with exogenous separation, Nash-bargained wages, and Cobb–Douglas matching, solved at monthly frequency.

Environment. A representative household comprises a unit measure of homogeneous workers. Employed workers produce output at productivity z and earn a bilaterally determined wage w. Unemployed workers engage in home production with flow value $b > 0$. Firms post vacancies at per-unit cost $\kappa > 0$.

Aggregate states. The aggregate state is

$$S \;=\; [u_{-1},\, z], \tag{39}$$

where $u_{-1} = 1 - n_{-1}$ is the inherited unemployment rate (the predetermined endogenous state) and z is aggregate TFP. Log productivity follows an AR(1) process,

$$\ln z_t \;=\; \rho_z \ln z_{t-1} \;+\; \varepsilon_t, \qquad \varepsilon_t \sim \mathcal{N}(0,\, \sigma_z^2), \tag{40}$$

discretized into an 11-state Markov chain using the Tauchen method with ± 3 standard deviation bounds.

Matching. A Cobb–Douglas matching function governs the flow of new hires:

$$M(u,v) \;=\; \bar{m}\, u^{\xi}\, v^{1-\xi}, \tag{41}$$

where $u = 1 - n_{-1}$ is the unemployment level, v is aggregate vacancies, $\bar{m} > 0$ is matching efficiency, and $\xi \in (0,1)$ is the matching elasticity. Labor market tightness is defined as $\theta := v/u$. The vacancy-filling probability is

$$q(\theta) \;=\; \frac{M(u,v)}{v} \;=\; \bar{m}\, \theta^{-\xi}, \tag{42}$$

and the job-finding probability is $f(\theta) = \bar{m}\, \theta^{1-\xi}$. Because the Cobb–Douglas specification can imply $q(\theta) > 1$ for low θ, I truncate q at unity in the numerical implementation.

Employment dynamics. Employment evolves according to

$$n \;=\; (1 - \lambda)\, n_{-1} \;+\; q(\theta)\, v, \tag{43}$$

where $\lambda \in (0,1)$ is the exogenous job separation rate.

Resource constraint. Aggregate consumption equals market output net of vacancy posting costs plus home production by the unemployed:

$$c(S) \;=\; n(S)\,z \;-\; \kappa\,v(S) \;+\; \big(1 - n(S)\big)\,b. \tag{44}$$

Household preferences. The household has temporal CRRA utility with risk aversion $\sigma > 0$ and discount factor $\beta \in (0,1)$. The stochastic discount factor between aggregate states S and S' is

$$\Lambda(S,S') \;=\; \beta\left(\frac{c(S)}{c(S')}\right)^{\sigma}. \tag{45}$$

Wage determination. Wages are determined by Nash bargaining with worker bargaining weight $\eta \in (0,1)$:

$$w(S) \;=\; (1-\eta)\,b \;+\; \eta\big(z + \kappa\,\theta(S)\big). \tag{46}$$

The first term reflects the worker's outside option (home production), and the second captures the worker's share of the match surplus inclusive of the firm's saved vacancy costs.

Free entry. Firms post vacancies until the expected value of a filled vacancy equals the posting cost. The free-entry condition is

$$\frac{\kappa}{q(S)} \;=\; (1-\lambda)\,\beta\,\mathbb{E}\left[\left(\frac{c(S)}{c(S')}\right)^{\sigma}\left(z' - w(S') + \frac{\kappa}{q(S')}\right)\,\middle|\, S\right]. \tag{47}$$

The left-hand side is the expected cost of filling a vacancy. The right-hand side is the discounted continuation value of a filled job: next-period flow profit $z' - w(S')$ plus the option value of the continuing match $\kappa/q(S')$, discounted at the stochastic rate $\Lambda(S,S')$ and weighted by the survival probability $(1-\lambda)$.

Recursive competitive equilibrium. A recursive competitive equilibrium consists of policy functions $\{\theta(S),\, v(S),\, n(S),\, w(S),\, c(S)\}$ such that:

1. The free-entry condition (47) holds for all S.
2. Wages satisfy the Nash bargaining outcome (46).

3. Employment evolves according to (43).

4. The resource constraint (44) holds.

5. Expectations are consistent with the induced law of motion for S.

The model is solved globally using the repeated transition method (Lee, 2025).

Calibration. Table J.1 reports the parameter values. The model is calibrated at monthly frequency. The discount factor $\beta = 0.99^{1/3} \approx 0.9967$ corresponds to a quarterly discount factor of 0.99. The matching elasticity $\xi = 0.635$ falls within the range surveyed by Petrongolo and Pissarides (2001). The worker's bargaining weight satisfies the Hosios condition $\eta = \xi$, ensuring constrained efficiency of the decentralized equilibrium absent aggregate risk. The separation rate $\lambda = 0.028$ per month is consistent with JOLTS data. Log TFP follows an AR(1) with persistence $\rho_z = 0.950$ and innovation standard deviation $\sigma_z = 0.050$, discretized into 11 states via the Tauchen method with ± 3 standard deviation bounds.

Table J.1: DMP model: calibrated parameters

Parameter	Symbol	Value	Target / Source
Discount factor	β	$0.99^{1/3} \approx 0.9967$	Quarterly $\beta = 0.99$
Risk aversion	σ	1.000	Log utility
Matching efficiency	$\bar{m}$	0.543	Steady-state job-finding rate
Matching elasticity	ξ	0.635	Petrongolo and Pissarides (2001)
Separation rate	λ	0.028	JOLTS monthly separation rate
Vacancy posting cost	κ	0.309	Steady-state vacancy rate
Home production	b	0.400	
Bargaining weight	η	0.635	Hosios condition ($\eta = \xi$)
TFP persistence	ρ_z	0.950	Monthly AR(1)
TFP innovation s.d.	σ_z	0.050	Output volatility
TFP grid points		11	Tauchen, ± 3 s.d.

Benefit extensions. The main text considers two extensions of the benefit structure. The first extension introduces independently varying stochastic benefits $b \in \{b_L, b_H\}$ following a two-state Markov chain with persistence $\rho_b = 0.98$. The low benefit equals the baseline value $b_L = b = 0.400$, and the high benefit is $b_H =$

$b + 0.50\, w^{ss}$, where w^{ss} is the steady-state wage. The exogenous state expands to $(z, b) \in \{B, G\} \times \{b_L, b_H\}$ with a Kronecker-product transition matrix, producing four conditional saddle paths.

The second extension implements counter-cyclical benefits. Benefits increase linearly with unemployment between thresholds $u_{\text{low}} = 0.055$ and $u_{\text{high}} = 0.070$:

$$b(u) = b + \min\Big(1,\ \max\big(0,\ \tfrac{u - u_{\text{low}}}{u_{\text{high}} - u_{\text{low}}}\big)\Big) \cdot 1.50\, w^{ss}, \tag{48}$$

so that benefits are at the baseline level when $u \leq u_{\text{low}}$ and reach the maximum top-up of $1.50\, w^{ss}$ when $u \geq u_{\text{high}}$. Since unemployment is high precisely when productivity is low, this produces effectively counter-cyclical benefits.

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