Dancing on the Saddles: A Geometric Framework for Stochastic Equilibrium Dynamics[1]

Online Appendix

Contents

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A Sequential formulation of a canonical RBC

A representative household with temporal log utility is considered. Given initial condition (a_0, A_0), the household maximizes life-time utility under stochastic aggregate TFP A_t, which is subject to a budget constraint as elaborated on below:

$$\max_{\{c_\tau(A^{(\tau)}),a_{\tau+1}(A^{(\tau)})\}_{\tau=0}^{\infty}} \mathbb{E}_0 \sum_{\tau=0}^{\infty} \beta^\tau log(c_\tau(A^{(\tau)})) \tag{1}$$

$$\text{s.t. } c_\tau(A^{(\tau)}) + a_{\tau+1}(A^{(\tau)}) = a_\tau(A^{(\tau-1)})(1 + r(A^{(\tau)})) + w(A^{(\tau)}), \quad \text{for } \forall\tau, \forall A_t \tag{2}$$

$$a_{\tau+1}(A^{(\tau)}) \geq -\overline{a}, \quad \text{for } \forall\tau \tag{3}$$

where superscript τ inside a bracket denotes history of a variable up to period τ; $-\overline{a}$ is the natural borrowing limit to preempt Ponzi scheme. Labor supply is exogenously fixed at unity. I consider the following competitive factor prices given CRS Cobb-Douglas production function:

$$r(A^{(\tau)}) = A_t\alpha(K(A^{(\tau)}))^{(\alpha-1)} - \delta \tag{4}$$

$$w(A^{(\tau)}) = A_t(1 - \alpha)(K(A^{(\tau)}))^\alpha, \tag{5}$$

K is capital stock, that satisfies $K(A^{(\tau)}) = a(A^{(\tau)})$ in equilibrium. With the regularity conditions given in Stokey et al. (1989), this sequential formulation yields the same optimality conditions as the recursive form in the main text.

B Individual conditional saddles in Aiyagari (1994)

I define individual-level conditional saddle path in the heterogeneous-household economy without aggregate uncertainty ($A = A' = 1$). The conditional saddle is defined for the SRCE as in Aiyagari (1994).

Definition 1 (Individual conditional saddle under frozen z)**.**
Fix a stationary RCE of the Aiyagari (1994) economy, which delivers an individual asset policy

$$g_a : \mathbb{R}_+ \times \mathcal{Z} \to \mathbb{R}_+,$$

where $\mathcal{Z}$ is a finite Markov set with transition matrix Π*. Fix an initial condition $(a_0, z_0) \in \mathbb{R}_+ \times \mathcal{Z}$. For any frozen idiosyncratic state $z \in \mathcal{Z}$, define the frozen-z continuation $\{a_t(z)\}_{t\geq 0}$ recursively by*

$$a_{t+1}(z) = g_a(a_t(z), z), \qquad a_0(z) = a_0. \tag{6}$$

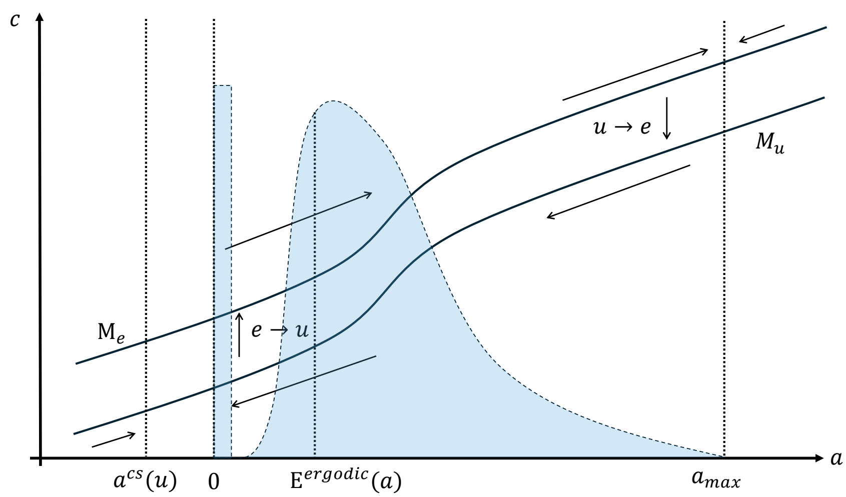
The individual conditional saddle under frozen z is the orbit-closure

$$\mathcal{M}^{\text{ind}}(z; a_0) := \overline{\{a_t(z) : t \geq 0\}} \;\subseteq\; \mathbb{R}_+. \tag{7}$$

Definition 2 (Individual conditional steady state under frozen z)**.**
Fix (a_0, z_0) and $z \in \mathcal{Z}$, and let $\{a_t(z)\}_{t\geq 0}$ be the frozen-z continuation. The individual conditional steady state *under frozen z is defined by*

$$a^{cs}(z; a_0) := \lim_{t\to\infty} a_t(z). \tag{8}$$

Figure B.1: Individual conditional saddle paths in the stationary RCE



Notes: The figure illustrates the individual conditional saddle paths for $z = e$ and $z = u$ in Aiyagari (1994).

Figure B.1 illustrates individual-level conditional saddle paths in Aiyagari (1994). Because of the borrowing constraint, the conditional steady state associated with the unemployment state $z = u$ is not attained. Under standard calibrations, this generates a positive mass of agents at the borrowing limit. Analogous to the aggregate-level case, heterogeneity in the slopes of individual conditional saddle paths implies differential responses of individual consumption to idiosyncratic shocks.

C Proofs for the theoretical results

Proposition 4 (Aggregate uncertainty and the conditional steady states)**.**
The following inequalities hold:

$$K_B^{cs} \;<\; K_B^{pf} \;<\; K_G^{pf} \;<\; K_G^{cs}, \qquad c_B^{cs} \;<\; c_B^{pf} \;<\; c_G^{pf} \;<\; c_G^{cs}.$$

Proof.
Let $R(A,K) := 1-\delta+\alpha A K^{\alpha-1}$ denote the gross return on capital and let the K-nullcline (feasibility locus) be

$$c(A,K) = AK^{\alpha} - \delta K.$$

The PF steady states solve the Euler equations under absorbing beliefs,

$$1 = \beta R(B, K_B^{pf}), \qquad 1 = \beta R(G, K_G^{pf}).$$

Since $R(A,K)$ is strictly increasing in A and strictly decreasing in K (because $\alpha - 1 < 0$), it follows immediately that $K_B^{pf} < K_G^{pf}$.

Next define the frozen-regime CS Euler residuals evaluated on the K-nullcline by

$$F_B(K) := \beta\Big[\pi_{BB} R(B,K) + \pi_{BG} \frac{c(B,K)}{c(G,K)}\, R(G,K)\Big] - 1,$$
$$F_G(K) := \beta\Big[\pi_{GG} R(G,K) + \pi_{GB} \frac{c(G,K)}{c(B,K)}\, R(B,K)\Big] - 1.$$

By construction, the conditional steady states satisfy $F_B(K_B^{cs}) = 0$ and $F_G(K_G^{cs}) = 0$.

Step 1: show $K_B^{cs} < K_B^{pf}$. Evaluate F_B at K_B^{pf}. Using $1 = \beta R(B, K_B^{pf})$ and $\pi_{BB} = 1 - \pi_{BG}$,

$$\begin{aligned} F_B(K_B^{pf}) &= \beta\Big[(1-\pi_{BG}) R(B, K_B^{pf}) + \pi_{BG} \frac{c(B,K_B^{pf})}{c(G,K_B^{pf})} R(G, K_B^{pf})\Big] - 1 \\ &= \pi_{BG}\Big[\beta \frac{c(B,K_B^{pf})}{c(G,K_B^{pf})} R(G,K_B^{pf}) - 1\Big]. \end{aligned}$$

Thus $F_B(K_B^{pf}) < 0$ is equivalent to

$$\beta \frac{c(B,K_B^{pf})}{c(G,K_B^{pf})} R(G,K_B^{pf}) < 1 \quad\Longleftrightarrow\quad \frac{R(G,K_B^{pf})}{R(B,K_B^{pf})} < \frac{c(G,K_B^{pf})}{c(B,K_B^{pf})},$$

where we used $\beta R(B, K_B^{pf}) = 1$ to divide both sides by $R(B, K_B^{pf})$.

We now prove this strict inequality for any K with $c(B,K), c(G,K) > 0$. Write $x := \alpha K^{\alpha-1} > 0$ and $y := K^{\alpha} > 0$. Then

$$\frac{R(G,K)}{R(B,K)} = \frac{1-\delta+Gx}{1-\delta+Bx}, \qquad \frac{c(G,K)}{c(B,K)} = \frac{Gy-\delta K}{By - \delta K}.$$

Since $1-\delta > 0$, we have the strict bound

$$\frac{1-\delta+Gx}{1-\delta+Bx} < \frac{Gx}{Bx} = \frac{G}{B}.$$

Since $\delta K > 0$ and $Gy > By$, subtracting the same positive term from numerator and denominator enlarges the ratio, yielding

$$\frac{Gy - \delta K}{By - \delta K} > \frac{Gy}{By} = \frac{G}{B}.$$

Combining the two displays gives

$$\frac{R(G,K)}{R(B,K)} < \frac{G}{B} < \frac{c(G,K)}{c(B,K)},$$

and in particular the desired inequality holds at $K = K_B^{pf}$. Therefore $F_B(K_B^{pf}) < 0$.

Finally, note that F_B is strictly decreasing in K on the relevant region because both $R(B,K)$ and $R(G,K)$ are strictly decreasing in K and $c(B,K)/c(G,K)$ is also decreasing in K along the feasibility locus.[2] Hence, since $F_B(K_B^{cs}) = 0$ and $F_B(K_B^{pf}) < 0$, we must have $K_B^{cs} < K_B^{pf}$.

Step 2: show $K_G^{cs} > K_G^{pf}$. Similarly, evaluate F_G at K_G^{pf}. Using $1 = \beta R(G, K_G^{pf})$ and $\pi_{GG} = 1 - \pi_{GB}$,

$$\begin{aligned} F_G(K_G^{pf}) &= \beta\Big[(1-\pi_{GB})R(G,K_G^{pf}) + \pi_{GB}\frac{c(G,K_G^{pf})}{c(B,K_G^{pf})}R(B,K_G^{pf})\Big] - 1 \\ &= \pi_{GB}\Big[\beta\frac{c(G,K_G^{pf})}{c(B,K_G^{pf})}R(B,K_G^{pf}) - 1\Big]. \end{aligned}$$

Thus $F_G(K_G^{pf}) > 0$ is equivalent to

$$\beta\frac{c(G,K_G^{pf})}{c(B,K_G^{pf})}R(B,K_G^{pf}) > 1 \quad\iff\quad \frac{R(B,K_G^{pf})}{R(G,K_G^{pf})} > \frac{c(B,K_G^{pf})}{c(G,K_G^{pf})}.$$

But the argument above applied with (B,G) swapped gives, for any K with positive consumption,

$$\frac{R(B,K)}{R(G,K)} > \frac{B}{G} > \frac{c(B,K)}{c(G,K)}.$$

Hence $F_G(K_G^{pf}) > 0$. Since F_G is strictly decreasing in K and $F_G(K_G^{cs}) = 0$, we conclude $K_G^{cs} > K_G^{pf}$.

Step 3: conclude the ordering and translate to consumption. We have shown $K_B^{cs} < K_B^{pf}$

[2] This monotonicity is standard and can be verified by differentiation; it is also visually apparent in the (K,C) phase diagram.

and $K_G^{pf} < K_G^{cs}$, and already $K_B^{pf} < K_G^{pf}$, hence

$$K_B^{cs} < K_B^{pf} < K_G^{pf} < K_G^{cs}.$$

Finally, along the K-nullcline $c(A, K) = AK^\alpha - \delta K$ is strictly increasing in A and (on the relevant region) increasing in K, so the same ordering carries over to consumption:

$$c_B^{cs} < c_B^{pf} < c_G^{pf} < c_G^{cs}.$$

■

D Search and matching model

This appendix provides the full specification of the Diamond–Mortensen–Pissarides economy used in Section 4.3. The model is a discrete-time stochastic version of the canonical search-and-matching framework with exogenous separation, Nash-bargained wages, and Cobb–Douglas matching, solved at monthly frequency.

Environment. A representative household comprises a unit measure of homogeneous workers. Employed workers produce output at productivity z and earn a bilaterally determined wage w. Unemployed workers engage in home production with flow value $b > 0$. Firms post vacancies at per-unit cost $\kappa > 0$.

Aggregate states. The aggregate state is

$$S \;=\; [u_{-1},\, z], \tag{9}$$

where $u_{-1} = 1 - n_{-1}$ is the inherited unemployment rate (the predetermined endogenous state) and z is aggregate TFP. Log productivity follows an AR(1) process,

$$\ln z_t \;=\; \rho_z \ln z_{t-1} \;+\; \varepsilon_t, \qquad \varepsilon_t \sim \mathcal{N}(0,\, \sigma_z^2), \tag{10}$$

discretized into an 11-state Markov chain using the Tauchen method with ± 3 standard deviation bounds.

Matching. A Cobb–Douglas matching function governs the flow of new hires:

$$M(u, v) \;=\; \bar{m}\, u^\xi\, v^{1-\xi}, \tag{11}$$

where $u = 1 - n_{-1}$ is the unemployment level, v is aggregate vacancies, $\bar{m} > 0$ is matching efficiency, and $\xi \in (0,1)$ is the matching elasticity. Labor market tightness is defined as $\theta := v/u$. The vacancy-filling probability is

$$q(\theta) \;=\; \frac{M(u,v)}{v} \;=\; \bar{m}\,\theta^{-\xi}, \tag{12}$$

and the job-finding probability is $f(\theta) = \bar{m}\,\theta^{1-\xi}$. Because the Cobb–Douglas specification can imply $q(\theta) > 1$ for low θ, we truncate q at unity in the numerical implementation.

Employment dynamics. Employment evolves according to

$$n \;=\; (1-\lambda)\,n_{-1} \;+\; q(\theta)\,v, \tag{13}$$

where $\lambda \in (0,1)$ is the exogenous job separation rate.

Resource constraint. Aggregate consumption equals market output net of vacancy posting costs plus home production by the unemployed:

$$c(S) \;=\; n(S)\,z \;-\; \kappa\,v(S) \;+\; \big(1 - n(S)\big)\,b. \tag{14}$$

Household preferences. The household has temporal CRRA utility with risk aversion $\sigma > 0$ and discount factor $\beta \in (0,1)$. The stochastic discount factor between aggregate states S and S' is

$$\Lambda(S,S') \;=\; \beta\left(\frac{c(S)}{c(S')}\right)^{\sigma}. \tag{15}$$

Wage determination. Wages are determined by Nash bargaining with worker bargaining weight $\eta \in (0,1)$:

$$w(S) \;=\; (1-\eta)\,b \;+\; \eta\big(z + \kappa\,\theta(S)\big). \tag{16}$$

The first term reflects the worker's outside option (home production), and the second captures the worker's share of the match surplus inclusive of the firm's saved vacancy costs.

Free entry. Firms post vacancies until the expected value of a filled vacancy equals the posting cost. The free-entry condition is

$$\frac{\kappa}{q(S)} \;=\; (1-\lambda)\,\beta\,\mathbb{E}\!\left[\left(\frac{c(S)}{c(S')}\right)^{\sigma}\left(z' - w(S') + \frac{\kappa}{q(S')}\right)\,\middle|\; S\right]. \tag{17}$$

The left-hand side is the expected cost of filling a vacancy. The right-hand side is the discounted continuation value of a filled job: next-period flow profit $z' - w(S')$ plus the option value of the continuing match $\kappa/q(S')$, discounted at the stochastic rate $\Lambda(S, S')$ and weighted by the survival probability $(1-\lambda)$.

Recursive competitive equilibrium. A recursive competitive equilibrium consists of policy functions $\{\theta(S), v(S), n(S), w(S), c(S)\}$ such that:

1. The free-entry condition (17) holds for all S.
2. Wages satisfy the Nash bargaining outcome (16).
3. Employment evolves according to (13).
4. The resource constraint (14) holds.
5. Expectations are consistent with the induced law of motion for S.

The model is solved globally using the repeated transition method (Lee, 2025).

Calibration. Table D.1 reports the parameter values. The model is calibrated at monthly frequency. The discount factor $\beta = 0.99^{1/3} \approx 0.9967$ corresponds to a quarterly discount factor of 0.99. The matching elasticity $\xi = 0.6353$ falls within the range surveyed by Petrongolo and Pissarides (2001). The worker's bargaining weight satisfies the Hosios condition $\eta = \xi$, ensuring constrained efficiency of the decentralized equilibrium absent aggregate risk. The separation rate $\lambda = 0.0283$ per month is consistent with JOLTS data. Log TFP follows an AR(1) with persistence $\rho_z = 0.983$ and innovation standard deviation $\sigma_z = 0.007$, discretized into 11 states via the Tauchen method.

Table D.1: DMP model: calibrated parameters

Parameter	Symbol	Value	Target / Source
Discount factor	β	$0.99^{1/3} \approx 0.9967$	Quarterly $\beta = 0.99$
Risk aversion	σ	1.000	Log utility
Matching efficiency	$\bar{m}$	0.291	Steady-state job-finding rate
Matching elasticity	ξ	0.635	Petrongolo and Pissarides (2001)
Separation rate	λ	0.0283	JOLTS monthly separation rate
Vacancy posting cost	κ	0.0667	Steady-state vacancy rate
Home production	b	0.672	Hagedorn and Manovskii (2008)
Bargaining weight	η	0.635	Hosios condition ($\eta = \xi$)
TFP persistence	ρ_z	0.983	Monthly AR(1)
TFP innovation s.d.	σ_z	0.009	Output volatility
TFP grid points		11	Tauchen, ± 3 s.d.

References

Aiyagari, S. R. (1994). Uninsured Idiosyncratic Risk and Aggregate Saving. *The Quarterly Journal of Economics 109*(3), 659–684. Publisher: Oxford University Press.

Hagedorn, M. and I. Manovskii (2008, September). The cyclical behavior of equilibrium unemployment and vacancies revisited. *American Economic Review 98*(4), 1692–1706.

Lee, H. (2025). Global nonlinear solutions in sequence space and the generalized transition function. Working Paper.

Petrongolo, B. and C. A. Pissarides (2001, June). Looking into the black box: A survey of the matching function. *Journal of Economic Literature 39*(2), 390–431.

Stokey, N. L., R. E. Lucas, and E. C. Prescott (1989). *Recursive Methods in Economic Dynamics*. Harvard University Press.