Dancing on the Saddles: A Geometric Framework for Stochastic Equilibrium Dynamics[†]

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Abstract

This paper extends deterministic saddle-path analysis to stochastic environments by introducing *conditional saddle paths*: the equilibrium path under frozen exogenous states. Equilibrium fluctuations decompose into movements *along* (endogenous propagation) and *across* (exogenous state transitions) conditional saddle paths. The framework delivers two theoretical results. First, state-dependent impulse responses arise from differences in the slopes of conditional saddle paths. Second, if an aggregate equilibrium variable varies strictly monotonically along conditional saddle paths, it uniquely indexes equilibrium states, providing an exact one-dimensional sufficient statistic. I prove that aggregate capital is such a statistic in a canonical heterogeneous-household model (Krusell and Smith, 1998). This monotonicity is verifiable under perfect foresight, as it extends to rational expectations via homotopy.

Keywords: Conditional saddle path, business cycles, state-dependent dynamics, sufficient statistics, heterogeneous agents.

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1 Introduction

Understanding equilibrium dynamics in models with aggregate uncertainty remains a central challenge in macroeconomics. Unlike deterministic models, where saddle-path diagrams provide immediate geometric intuition, stochastic equilibrium models lack comparable geometric frameworks. This makes it difficult to develop intuition about how economies respond to shocks, how different states interact, and why certain computational methods work. The challenge becomes particularly acute in heterogeneous-agent models, where the natural state variable is an infinite-dimensional wealth distribution—yet computational work routinely achieves dimension reduction to low-dimensional aggregates. Can we extend geometric saddle-path analysis to stochastic environments? What does such a framework reveal about equilibrium dynamics, state-dependent responses, and the success of computational approximations?

This paper makes two contributions. First, it extends the saddle-path analysis of equilibrium dynamics from deterministic models to stochastic environments by introducing the notion of *conditional saddle path*: equilibrium trajectories of the regime-frozen economy (holding the exogenous state fixed at a level). This geometric object decomposes business-cycle fluctuations into movements *along* a conditional saddle (endogenous propagation) and *across* conditional saddles (transitions when the exogenous state changes), providing a unified framework for analyzing state-dependent equilibrium dynamics. The framework enables visualization of generalized impulse responses and nonlinear transition dynamics in phase diagrams, analogous to how saddle-path diagrams illuminate deterministic models. Based on this geometric framework, I establish that state-dependent shock responses arise essentially from differences in the local slopes along the conditional saddle paths.

Second, it provides a general dimension-reduction theorem on the conditional saddle paths. If an aggregate equilibrium variable is strictly monotone and convergent along the conditional saddle, then it is injective on the invariant equilibrium set and hence uniquely indexes all equilibrium allocations and prices. As a consequence, the equilibrium can be represented *exactly*—not merely approximately—with a one-dimensional sufficient statistic. Applying this theoretical result, I establish that aggregate capital is a sufficient statistic for canonical heterogeneous-agent models (Krusell and Smith, 1998). To be precise, K-sufficiency means that the distributional state Φ is uniquely recoverable from K along the equilibrium path—not that K follows a simple (e.g., log-linear or Markovian) law of motion. The former is an exact theoretical result; the latter remains a quantitative approximation. This provides a geometric foundation for the practical success of scalar state approximations in business-cycle applications and delivers economically interpretable sufficient conditions for exact one-dimensional representations on the relevant invariant set.

The key condition underlying the dimension-reduction result is strict monotonicity of aggregate capital along conditional saddle paths. I develop a perfect-foresight diagnostic: if K converges monotonically along the deterministic transition path, then monotonicity extends to the rational-expectations conditional saddle path via homotopy. This diagnostic is especially useful for models where the analytical primitive conditions (MPC bounds) are difficult to verify, such as those with CRRA preferences and endogenous labor supply.

These results extend beyond the canonical one-asset setting. In heterogeneous-agent models with multiple endogenous state variables, such as risky and riskless assets, liquid and illiquid assets, domestic and foreign bonds, the same logic applies (Krusell and Smith, 1997; Mendoza, 2010; Khan and Thomas, 2013; Kaplan and Violante, 2014; Berger and Vavra, 2015; Kaplan et al., 2018): whenever a monotone-convergent aggregate coordinate exists along conditional saddle paths, equilibrium dynamics remain exactly traceable by a scalar index.

Conditional saddle paths are equilibrium paths defined under frozen exogenous aggregate states: they describe counterfactual equilibrium continuations — how the economy would evolve if the aggregate state were held fixed at a given value — computed under the same decision rules that govern the stochastic equilibrium with regime switching. These counterfactual paths are economically meaningful because equilibrium decisions internalize the possibility of future regime changes, and the frozen-regime objects formalize the corresponding equilibrium "thought experiments."

Conditional saddle paths are closely connected to the random dynamical systems (RDS) notion of invariant manifolds for dynamics under exogenous forcing (Arnold, 1998; Schenk-Hoppé, 2001). In the present setting, the forcing is the aggregate Markov state, and a recursive competitive equilibrium (RCE) implies a time-homogeneous endogenous law of motion; conditional saddle paths are the discrete counterpart of the invariant manifolds from the frozen-regime equilibrium maps. Both perspectives organize stochastic dynamics using invariant geometric objects rather than local linearizations. The distinction in this paper is therefore not the mathematics of invariance, but the equilibrium discipline: the invariant-manifold structure is pinned down by optimality, market clearing, and equilibrium consistency. This is precisely what makes global transition functions and impulse responses well defined and comparable across aggregate states.

The framework applies to any recursive competitive equilibrium with a predetermined endogenous state. Companion work applies conditional saddle paths to New Keynesian models with an occasionally binding zero lower bound (Lee and Nomura, 2026), where the kink at $i_{-1} = 0$ provides a geometric characterization of the ZLB's effect on dynamics. Lee and Sun (2026) apply the framework to production network economies with endogenous linkage formation, where a sign reversal in conditional saddle slopes across regimes generates duration-dependent business cycle asymmetry. Section 5.1 illustrates the framework in a search-and-matching environment.

Related literature This paper contributes to three strands of literature in macroeconomics. The first is the literature studying global equilibrium dynamics and solution methods under aggregate uncertainty. The challenge of characterizing equilibrium dynamics in models with aggregate shocks has motivated extensive methodological development. Marcet (1988), Den Haan and Marcet (1990) and Krusell and Smith (1998) pioneered the use of bounded rationality approximations. In particular, Krusell and Smith (1998) discovered that a simple linear forecasting rule in aggregate capital achieves remarkable accuracy ($R^2 > 0.9999$) despite the infinite-dimensional state space. Thereafter, the literature has dramatically developed to sharpen the accuracy and improve the computational efficiency by incorporating moment-based approximations, exact aggregation, functional approximations, sequence-space approaches, and machine/deep learning (Den Haan, 1996, 1997; Reiter, 2001; Algan et al., 2008, 2010; Den Haan and Rendahl, 2010; Reiter, 2010; Ahn et al., 2018; Boppart et al., 2018; Elenev et al., 2021; Auclert et al., 2021; Cao et al., 2023; Azinovic et al., 2022; Fernández-Villaverde et al., 2023; Han et al., 2025; Payne et al., 2025).

This paper differs from these computational contributions by providing geometric foundations for *why* dimension reduction works. Rather than developing new algorithms, I introduce conditional saddle paths as a geometric framework for understanding equilibrium dynamics — analogous to how saddle-path diagrams provide intuition in deterministic models. The framework reveals that aggregate capital's sufficiency in a canonical heterogeneous-household model follows from geometric properties (nullcline invariance and monotonicity) rather than numerical happenstance.

The exact sufficiency result also has computational implications. Lee (2025) develops a repeated transition method that constructs conditional expectations for individual-level problems by identifying periods in simulations, where aggregate states are similar, enabling reuse of computed transitions across these similar states. The method requires a metric for determining when aggregate states are "close enough" to pool. My sufficiency result establishes that aggregate capital distance provides a theoretically justified metric — agents' problems are identical whenever capital stocks coincide, regardless of distributional differences. This validates distance-based pooling strategies and enables efficient implementation without requiring explicit distributional tracking.[1]

This paper builds on the traditional use of geometric methods to analyze economic dynamics in deterministic environments. Phase-diagram analyses of the Solow—Swan (Solow, 1956; Swan, 1956) and Ramsey—Cass—Koopmans (Ramsey, 1928; Cass, 1965; Koopmans, 1963) growth models provide foundational intuition about convergence and stability. I extend this geometric approach to stochastic environments by introducing conditional saddle paths. This framework offers a geometric interpretation of stochastic equilibrium dynamics, including nonlinear and state-dependent impulse responses. In particular, it provides a useful visual tool for understanding how microfounded frictions generate state-dependent dynamics, as documented in the recent literature (Kaplan and Violante, 2014; Vavra, 2014; Berger and Vavra, 2015; Basu and Bundick, 2017; Bloom et al., 2018; Kaplan et al., 2018; Petrosky-Nadeau et al., 2018; Baley and Blanco, 2019; Pizzinelli et al., 2020; Berger et al., 2021; Melcangi, 2024; Winberry, 2021; Lee, 2026).

Finally, in mathematics, the literature on random dynamical systems provides foundational tools for geometric analyses of stochastic dynamic processes. However, its focus has not been on history-invariant saddle paths of the type that arise in recursive competitive equilibrium (Arnold, 1998; Schenk-Hoppé, 1998; Schenk-Hoppé, 2001).[2] For example, Yannacopoulos (2011) introduces the notion of stochastic saddle

[1]The idea of solving heterogeneous-agent models by freezing the aggregate state and analyzing transitions appears in several computational approaches, including Bourany (2018), Section 4.3 of a working paper version of Achdou et al. (2021), and Lee (2025). The present paper provides a theoretical foundation: conditional saddle paths formalize the underlying geometric objects, and the sufficiency results explain when dimension reduction is exact.

[2]In the language of Random Dynamical Systems, the conditional steady state corresponds to a deterministic realization of the random fixed point (Schenk-Hoppé and Schmalfuß, 2001), while the

paths, which are conceptually distinct from conditional saddle paths in that they vary with the realized history of shocks.[3]

By contrast, conditional saddle paths furnish economists with a geometric representation of stochastic equilibrium dynamics that is directly analogous to the role of phase diagrams in deterministic models. While this framework relies on standard regularity conditions to ensure well-defined and bounded equilibrium paths (Kamihigashi, 2003, 2005), the emphasis here is not on establishing existence results, but rather on providing geometric tools for understanding and analyzing stochastic equilibrium dynamics.

2 Conditional saddle path

2.1 Definitions and assumptions

I consider a generic dynamic stochastic model where the corresponding recursive competitive equilibrium (RCE) is characterized by the following aggregate state S and the endogenous and exogenous law of motions $(\mathit{\Gamma}_{endo}, \mathit{\Gamma}_{exo})$:

$$S = [\mathit{\Phi}, A] \tag{1}$$

where $\mathit{\Phi}$ is the endogenous aggregate state variable, and A is the exogenous aggregate state variable. The latter admits a multivariate vector that follows a stochastic process. I assume the exogenous aggregate law of motion $\mathit{\Gamma}_{exo}$ is a Markov chain. For simplicity in the illustration, I assume $\mathit{\Gamma}_{exo}$ is a two-state Markov chain where $A \in \{B, G\}$, and $\mathit{\Gamma}_{exo}(A'|A) > 0$ for $\forall A', A$.

I consider a distributional state space $\mathcal{X}$, whose elements $\mathit{\Phi} \in \mathcal{X}$ summarize the cross-sectional distribution of idiosyncratic household states that are payoff-relevant for equilibrium (e.g., assets and employment/productivity types) together with any endogenous objects needed to evaluate equilibrium decision rules. Formally, one can take $\mathcal{X}$ to be a subset of a metric space of probability measures augmented, if needed, by a finite-dimensional vector of aggregate variables. Throughout, $\mathit{\Gamma}_{\text{endo}}(\cdot, A) : \mathcal{X} \to$

conditional saddle path represents the invariant manifold of the frozen-regime dynamics (Arnold, 1998).

[3] Stochastic saddle paths depend on the specific sequence of past realizations, whereas conditional saddle paths are invariant to history.

$\mathcal{X}$ denotes the recursive equilibrium law of motion mapping the current distributional state into the next-period state under frozen regime A.

Definition 1 (Conditional saddle path)**.**
Fix a regime $A \in \{B, G\}$ and an initial state $\Phi_0 \in \mathcal{X}$. Let $\{\Phi_t\}_{t\geq 0}$ denote the frozen-regime continuation under A, defined recursively by

$$\Phi_{t+1} = \Gamma_{\text{endo}}(\Phi_t, A), \qquad t \geq 0. \tag{2}$$

The conditional saddle path under A from Φ_0 is defined as the closure of the frozen-regime continuation:

$$\mathcal{M}(\Phi_0, A) := \overline{\{\Phi_t : t \geq 0\}}. \tag{3}$$

Throughout, the distributional state space is a metric space of probability measures $(\mathcal{X}, d)$, where d is compatible with weak convergence of distributions. The closure in Definition 1 is taken with respect to this metric. The frozen-regime continuation $\{\Phi_t\}_{t\geq 0}$ is a sequence in $\mathcal{X}$, and $\mathcal{M}(\Phi_0, A) := \overline{\{\Phi_t : t \geq 0\}}$ is its closure in $(\mathcal{X}, d)$. Based on the conditional saddle path, conditional steady state is defined as follows:

Definition 2 (Conditional steady state)**.**
Given (Φ_0, A_0), if the frozen-regime continuation $\{\Phi_t\}_{t\geq 0}$ converges, I denote its limit by

$$\Phi^{cs}(\Phi_0, A) \;:=\; \lim_{t\to\infty} \Phi_t. \tag{4}$$

Definition 1 defines the conditional saddle path directly from the frozen-regime continuation. Fixing A makes the equilibrium law of motion deterministic on the distribution space: starting from Φ_0, the sequence $\{\Phi_t\}_{t\geq 0}$ is generated by repeated application of the endogenous transition operator $\Gamma_{\text{endo}}(\cdot, A)$. In this sense, $\mathcal{M}(\Phi_0, A)$ is the natural analogue of the saddle arm in deterministic saddle-path analysis: it isolates the equilibrium states visited (and their limit points) along the economically relevant convergence dynamics under regime A.[4]

[4]Strictly speaking, $\mathcal{M}(\Phi_0, A) = \overline{\{\Phi_t : t \geq 0\}}$ as defined in discrete time is an orbit-closure (a countable set together with its limit point) and therefore need not be a smooth manifold. I nonethe-

Assumption 1 (Regularity of conditional saddle paths).
Fix a regime $A \in \{B, G\}$ and an initial state $\Phi_0 \in \mathcal{X}$. Let $\{\Phi_t\}_{t \geq 0}$ be defined by $\Phi_{t+1} = \Gamma_{\text{endo}}(\Phi_t, A)$.

(i) (Unique existence) The conditional steady state $\Phi^{cs}(\Phi_0, A)$ uniquely exists on $\mathcal{M}(\Phi_0, A)$.

(ii) (Continuity) The map $\Gamma_{\text{endo}}(\cdot, A) : \mathcal{X} \to \mathcal{X}$ is continuous. Moreover, the aggregate variables (e.g., aggregate capital K and consumption C) vary continuously in Φ when restricted to $\mathcal{M}(\Phi_0, A)$.

Assumption 1 collects the regularity properties of the conditional saddle path that are needed for the paper's geometric arguments. Assumption 1 (i) assumes that, conditional on a regime A and an initial state Φ_0, the frozen-regime continuation converges to a unique conditional steady state within the relevant invariant equilibrium set. This does not assert global uniqueness across all initial conditions, which would require stronger fixed-point arguments, nor does it rule out multiplicity across initial conditions or regimes.

In particular, Proehl (2025) establishes global existence and uniqueness of recursive equilibrium in heterogeneous-agent models under parametric restrictions, whereas Assumption 1 (i) allows for the possibility of global multiplicity as in Walsh and Young (2024).[5] A full characterization of primitives ensuring Assumption 1 (i) is beyond the scope of this paper.

Assumption 1 (ii) imposes continuity of the frozen-regime law of motion and of the relevant aggregate observables when restricted to the conditional saddle. This mild regularity ensures that small perturbations of the distributional state *within the equilibrium set under study* produce small changes in aggregate outcomes and subsequent states, which is used repeatedly when translating geometric properties of projected phase diagrams (e.g., nullclines) into restrictions on equilibrium dynamics along the saddle.

With this geometric regularity in place, the conditional saddle admits a natural notion of "position along the curve" toward the conditional steady state. Theorem 2

less use saddle-path language to emphasize its one-dimensional, path-like role for global dynamics. In a continuous-time formulation with a smooth flow on the state space, the corresponding stable branch is naturally represented as a one-dimensional invariant manifold under standard regularity conditions.

[5]Nevertheless, conditional on a given initial state and a fixed regime, the equilibrium path considered in the analysis is assumed to be uniquely determined.

formalizes the key implication: if some aggregate equilibrium variable moves strictly one-way along this curve, then it provides a valid coordinate for the entire conditional saddle. In that case, knowing the scalar value is equivalent to knowing the full distributional state within the relevant equilibrium set, so the stochastic equilibrium admits an exact one-dimensional representation on $\mathcal{M}(\Phi_0, A)$.

2.2 Representative-agent economy: A canonical RBC

A representative household with temporal log utility is considered. Given initial condition (a_0, A_0), the household maximizes life-time utility under stochastic aggregate TFP A_t. I present the economy in recursive form and work with a recursive competitive equilibrium. Under standard regularity conditions (including an appropriate transversality condition), this recursive formulation is equivalent to the sequential equilibrium; for completeness the sequential formulation is provided in Appendix A. The recursive form of the household's problem is as follows:

$$v(a; K, A) = \max_{c,a'} \quad log(c) + \beta \mathbb{E} v(a'; K', A') \tag{5}$$

$$c + a' = a(1 + r(X)) + w(X) \tag{6}$$

$$a' \geq -\bar{a} \tag{7}$$

$$K' = \varGamma_{endo}(K, A), \quad A' \sim \varGamma_{exo}(A'|A), \tag{8}$$

where v is the household's value function; a is the wealth in the beginning of a period; K is aggregate capital; A is aggregate TFP; $\varGamma_{endo}$ is the law of motion for K. For illustrative purposes, I assume that TFP A follows a Markov-switching process between the levels B and G, where $G > B$:

$$\varGamma_{exo} = \begin{bmatrix} \pi_{BB} & \pi_{BG} \\ \pi_{GB} & \pi_{GG} \end{bmatrix} \tag{9}$$

I consider the following competitive factor prices given CRS Cobb-Douglas production function:

$$r(K, A) = A\alpha K^{\alpha-1} - \delta \tag{10}$$

$$w(K, A) = A(1 - \alpha)K^{\alpha}, \tag{11}$$

where K is capital stock, which satisfies $K = a$ in equilibrium. The recursive competitive equilibrium (RCE, hereafter) is as follows:

Definition 3 (Recursive competitive equilibrium)**.**
$(c, a', v, r, w, \mathit{\Gamma}_{endo})$ is a recursive competitive equilibrium if these functions

1. *satisfy the individual optimality conditions*

2. *clear factor markets, resulting in the competitive prices.*

3. *satisfy the consistency:*

$$a'(K; K, A) = K' = \mathit{\Gamma}_{endo}(K, A) \tag{12}$$

Based on the endogenous law of motion $\mathit{\Gamma}_{endo}$ defined in the recursive competitive equilibrium of Definition 3, I define conditional K-nullcline.

Definition 4 (Conditional K-nullcline in RBC)**.**
Fix (K_0, A_0) and a regime $A \in \{B, G\}$. Let $\{K_t\}_{t\geq 0}$ be the frozen-regime continuation $K_{t+1} = \mathit{\Gamma}_{\mathrm{endo}}(K_t, A)$. The conditional K-nullcline along the continuation is

$$\mathcal{N}(A; K_0) := \Big\{(K_t, C_t) : \ t \geq 0, K_{t+1} - K_t = 0\Big\}, \ C_t := C(K_t, A) \tag{13}$$

where C is the aggregate equilibrium consumption variable. When $\mathcal{N}(A; K_0)$ is the graph of a function, define $C_A^{Kcnull}(\cdot; K_0)$ by $\mathcal{N}(A; K_0) = \{(K, C) : \ C = C_A^{Kcnull}(K; K_0)\}$.

Then, in the simple RBC model, I can explicitly characterize the conditional K-nullcline. In particular, the nullcline is independent of the initial condition K_0.

Proposition 1 (Characterizing the conditional K-nullcline in RBC)**.**
The conditional nullclines of aggregate capital K for $A \in \{B, G\}$ are as follows:

$$C_A^{Kcnull}(K) = AK^\alpha - \delta K, \tag{14}$$

thus, $C_A^{Kcnull}(K)$ is independent of K_0.

Proof. From the stationary condition for the capital stock ($\delta K = I$) and the national accounting identity ($Y = C + I$), the conditional nullclines of aggregate capital K for $A \in \{B, G\}$ are as $C_A^{Kcnull}(K) = AK^\alpha - \delta K$. ■

In optimal dynamics, the conditional saddle path can exhibit discrete-time pathologies in which the projected (K, C) trajectory develops folds or loops. In such cases the saddle may (i) touch the K-nullcline at an intermediate point $K \neq K_A^{cs}$ (a "turning point") or (ii) alternate sides of the nullcline across dates (a "spiral"). To rule out these pathologies, I impose the following geometric restriction on one-step transitions in the (K, C) plane.

Assumption 2 (No segment crossing of the conditional K-nullcline)**.**
A straight line segment in (K, C) connecting (K_t, C_t) and (K_{t+1}, C_{t+1}) does not intersect the conditional K-nullcline for $\forall A \in \{B, G\}$:

$$[(K_t, C_t), (K_{t+1}, C_{t+1})] \cap \mathcal{N}(A) = \emptyset, \ \forall t \geq 0. \tag{15}$$

Assumption 2 is a discrete-time "no side-switching" condition: along the frozen-regime continuation, consecutive points (K_t, C_t) and (K_{t+1}, C_{t+1}) remain in the same region of (K, C)-space separated by the K-nullcline. In particular, letting $H_t := C_t - C_A^{Kcnull}(K_t)$, Assumption 2 implies $H_t H_{t+1} > 0$ for all t prior to convergence, so net investment $\varDelta K_t := K_{t+1} - K_t = C_A^{Kcnull}(K_t) - C_t = -H_t$ cannot reverse sign along the conditional saddle.

In continuous time, reversals of $\dot{K}$ necessarily pass through $\dot{K} = 0$, so continuity of the drift often delivers a comparable one-crossing geometry along the stable branch in standard settings. In discrete time, by contrast, the time-t map can jump across the nullcline between dates without ever landing on it at an integer time. Assumption 2 rules out such nonlocal crossings and thereby ensures a well-behaved saddle geometry consistent with reporting global impulse responses and transition functions as single-valued equilibrium objects.[6]

Assumption 2 is not purely assumed: it can be verified from model primitives without solving the stochastic equilibrium. Sections 4.1 and 4.2 develop two complementary strategies for doing so. The first is a *perfect-foresight diagnostic*: if K converges monotonically along the deterministic perfect-foresight saddle path, a homotopy argument shows that monotonicity persists under rational expectations, im-

[6]Discrete time admits nonlocal movements that can generate folds or repeated nullcline intersections in the projected (K, C) dynamics even when the underlying saddle is one-dimensional. Assumption 2 rules out such pathologies along the conditional saddle. For related geometric restrictions in dynamic optimization, see Reddy et al. (2020) and, on threshold (Skiba) phenomena, Skiba (1978), Dechert and Nishimura (1983), and Wagener (2003).

plying Assumption 2. This reduces verification to checking a standard property of the Bewley–Aiyagari transition dynamics. The second route works directly under rational expectations from static allocation primitives: economically interpretable bounds on marginal propensities to consume and general-equilibrium price feedback ensure that net investment cannot reverse sign.

Proposition 2 (K monotonicity in RBC)**.**
Fix $A \in \{B, G\}$. Under Assumptions 1 and 2, aggregate capital along the frozen-regime continuation converges to K_A^{cs} strictly monotonically.

Proof. Fix A and let $\{\Phi_t\}_{t\geq 0}$ be the frozen-regime continuation, with $K_t := K(\Phi_t)$ and $C_t := C(\Phi_t, A)$. Define net investment

$$\Delta K_t := K_{t+1} - K_t. \tag{16}$$

In the RBC benchmark, the aggregate resource constraint implies the K-nullcline representation

$$\Delta K_t = C_A^{Kcnull}(K_t) - C_t, \tag{17}$$

so the sign of ΔK_t is the opposite of the nullcline gap $H_t := C_t - C_A^{Kcnull}(K_t)$.

By Assumption 2, for every t such that $\Phi_t \neq \Phi^{cs}(\Phi_0, A)$, the segment connecting (K_t, C_t) and (K_{t+1}, C_{t+1}) does not intersect the K-nullcline. Since the K-nullcline is the graph of a continuous function, this implies $H_t H_{t+1} > 0$, hence $\{H_t\}$ has a constant sign prior to convergence. Therefore $\Delta K_t = -H_t$ has a constant nonzero sign whenever $\Phi_t \neq \Phi^{cs}(\Phi_0, A)$, and $\{K_t\}$ is strictly monotone in t until convergence.

Finally, Assumption 1 (i) implies $\Phi_t \to \Phi^{cs}(\Phi_0, A)$, and continuity of K (Assumption 1 (ii)) yields $K_t \to K_A^{cs} := K(\Phi^{cs}(\Phi_0, A))$. ■

In the model, the household ex-ante takes into account the aggregate uncertainty in its decision. The conditional saddle path is the sequence of outcomes implied by such ex-ante decisions when the ex-post exogenous aggregate states are *frozen* at A. Figure 1 depicts the conditional saddle paths of the calibrated RBC model in the (K, C) plane under the two aggregate productivity regimes $A \in \{B, G\}$. The model is solved globally using the repeated transition method of Lee (2025) under the standard quarterly calibration. For each regime A, I construct the frozen-regime continuation by holding productivity fixed at A for 2,000 periods and iterating the equilibrium law of motion from two initial capital stocks: one sufficiently low to generate the upward-converging branch and one sufficiently high to generate the downward-converging branch.[7]

Panel (a) provides a zoom-in around the conditional steady states and reports a histogram of simulated (K_t, C_t) outcomes under the stochastic regime-switching economy, showing that realized dynamics concentrate near and move along the relevant saddle geometry. Panel (b) zooms out to emphasize the global partition of the (K, C) plane and the associated direction of net-investment dynamics under each regime. In panel (b), the conditional K-nullclines (labeled $\mathit{\Delta} K|_A = 0$) are overlaid; in the RBC benchmark they are sign-determining via $\mathit{\Delta} K = C_A^{Kcnull}(K) - C$. Direction arrows indicate the frozen-regime evolution along each conditional saddle.

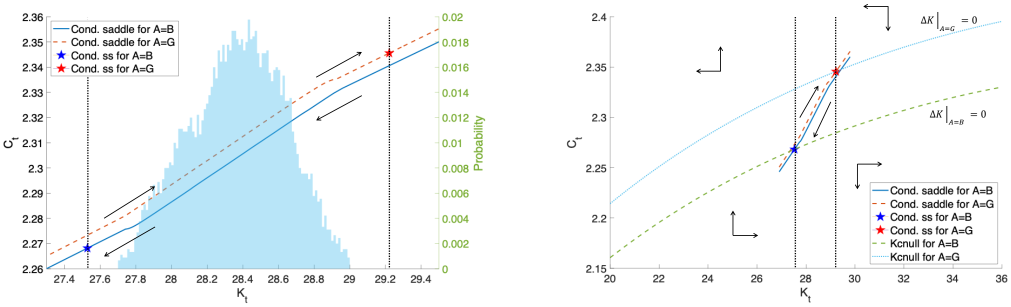
The global dynamics of the recursive competitive equilibrium decompose into movements along conditional saddle paths (endogenous propagation under constant A) and vertical jumps across them (regime switches, since capital is predetermined). Along each saddle, K strictly monotonically converges to the conditional steady state; this and the regime-switching dynamics jointly generate bounded stochastic fluctuations.

MIT shock on the saddle This decomposition yields a sharp characterization of unexpected transitory shocks (MIT shocks). Any unexpected transitory shock—whether TFP, monetary, fiscal, or otherwise—is necessarily mapped into a displacement along a conditional saddle path: there exists a magnitude of shift along the conditional saddle that is equivalent to the shock's impact, and all subsequent dynamics are along-path convergence. Consequently, the slope and curvature of the conditional saddle at the point of displacement fully determine the economy's propagation of the shock, including its persistence, amplification, and state dependence.

Generalized transition function The conditional saddle paths capture all equilibrium transitions—along saddles (endogenous propagation) and across them (exogenous regime shifts)—enabling generalized transition function (GTF) analysis (Lee, 2025), which encompasses generalized impulse response functions (Koop et al., 1996;

[7] This procedure traces the stable branch from both sides of the conditional steady state and is used only to recover the full conditional saddle in the phase diagram; the equilibrium continuation itself is single-valued given $(\mathit{\Phi}_0, A)$.

Figure 1: Conditional saddle paths and the phase diagram



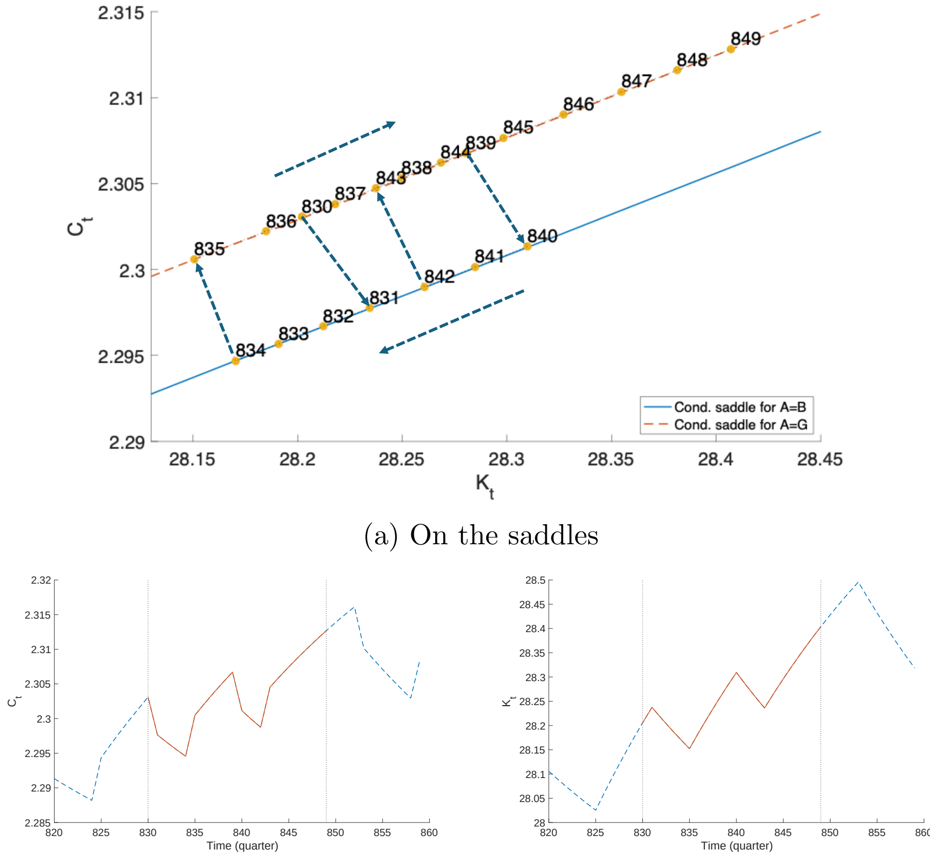
(a) Zoom-in (b) Zoom-out

Notes: Conditional K-nullclines ($\Delta K = 0$) partition the (K, C) plane. Direction arrows show frozen-regime convergence along each conditional saddle. Panel (a) zooms into the neighborhood of the conditional steady states shown in panel (b). The histogram plots the time-series distribution of aggregate capital.

Andreasen et al., 2017). The economy's response to a sequence of exogenous shocks is not confined to local dynamics around a steady state: its trajectory is sharply traced in the phase diagram as a sequence of along-saddle and across-saddle movements.

Figure 2 illustrates the equilibrium dynamics in the phase diagram (panel (a)) and in the time domain (panels (b) and (c)) for a 20-quarter subsample. Negative TFP shocks produce downward jumps in consumption across conditional saddle paths, followed by endogenous decline along them; positive shocks reverse the pattern. Compared to time-domain representations, the phase diagram more concisely illustrates the full equilibrium dynamics in a single figure and enables immediate consideration of counterfactual scenarios. For the GIRF illustration, I replace the two-state Markov chain with a continuous AR(1) process for productivity: $A' = (1-\rho)\mu + \rho A + \epsilon$, $\epsilon \sim N(0, \sigma)$. In this case, there is a continuum of frozen-A conditional saddles indexed by A, and the GIRF path moves both along a given conditional saddle (endogenous propagation) and across the foliation as A evolves.

Figures 3 and 4 illustrate post-shock dynamics following a positive TFP shock. Upon impact, consumption jumps upward across conditional saddles. In subsequent periods, forces along and across saddles jointly generate a bow-shaped trajectory in the phase diagram: capital exhibits a hump-shaped response as the shock mean-reverts, while consumption initially rises then falls as TFP crosses the threshold where the direction of along-saddle adjustment reverses.



(a) On the saddles

(b) C - time domain

(c) K - time domain

Figure 2: Equilibrium dynamics: saddle vs. time domain

Notes: Stochastic equilibrium dynamics on the conditional saddle paths (panel (a)) and in the time domain (panels (b) and (c)). Sample: 20 quarters (830–849).

Comparison with the perfect-foresight saddles In the *perfect-foresight (PF)* economy, agents believe the current regime persists forever; the PF steady states solve $1 = \beta(1 - \delta + \alpha A(K_A^{pf})^{\alpha-1})$. Along the *conditional saddle (CS)*, the regime is held fixed ex post, but agents correctly anticipate regime switches under Π, so the conditional steady states K_A^{cs} are pinned by the probability-weighted Euler equations.[8] PF and CS economies share the same K-nullcline (Proposition 3), but differ in their consumption dynamics.

Proposition 3 (K-nullcline invariance over beliefs)**.**

[8] Specifically, $\beta\pi_{AA}(1 + \alpha A(K_A^{cs})^{\alpha-1} - \delta) + \beta\pi_{AA'}\big(c(K_A^{cs}, A')/c_A^{cs}\big)^{-1}(1 + \alpha A'(K_A^{cs})^{\alpha-1} - \delta) = 1$ for each $A \in \{B, G\}$, where $A' \neq A$.

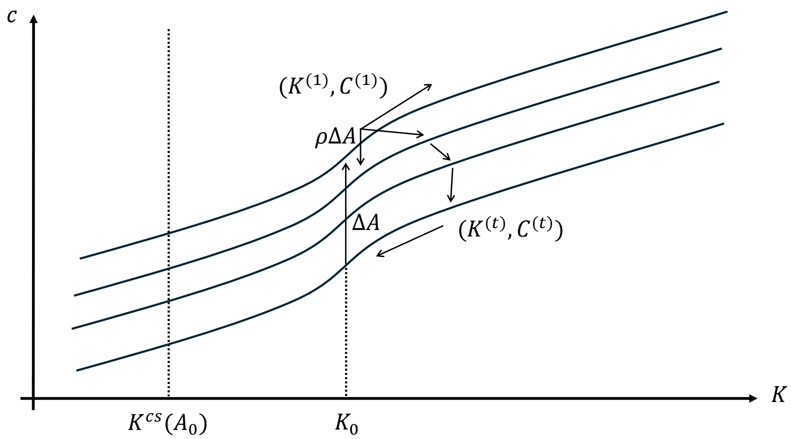


Figure 3: Generalized impulse response function (GIRF) at (K_0, A_0)

Notes: Generalized impulse responses of consumption and capital to a positive TFP shock on the conditional saddle paths.

Conditional K-nullclines are identical between the RBC model with the aggregate uncertainty and the perfect foresight counterpart.

Proof.
From the stationary condition $\delta K = I$, equation (14) are immediate for both models. Therefore, the conditional nullclines are the same. ∎

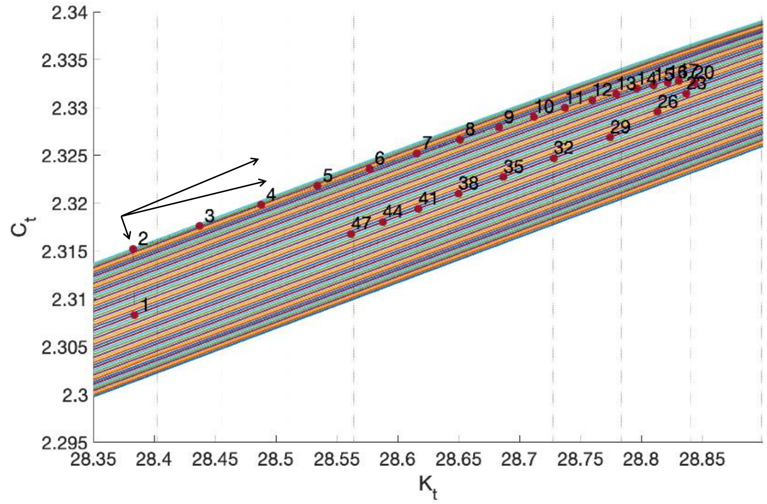
This invariance implies that PF–CS differences are driven entirely by consumption dynamics, not by the feasibility locus.[9] Proposition 4 formalizes this via steady-state orderings. The same invariance extends to the heterogeneous-household environment (Section 4), where it underlies both K-sufficiency and the perfect-foresight diagnostic (Section 4.1).

Proposition 4 (Aggregate uncertainty and the conditional steady states)**.**
The following inequalities hold:

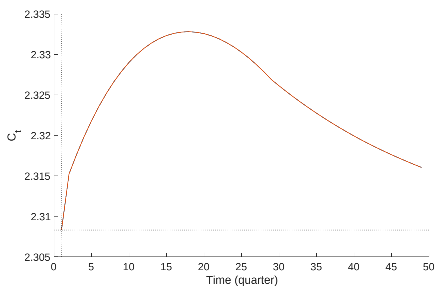
$$K_B^{cs} \;<\; K_B^{pf} \;<\; K_G^{pf} \;<\; K_G^{cs}, \qquad c_B^{cs} \;<\; c_B^{pf} \;<\; c_G^{pf} \;<\; c_G^{cs}. \tag{18}$$

Proof.
See Appendix C. ∎

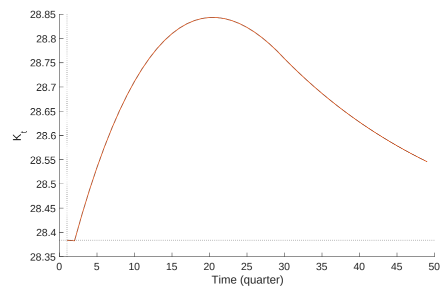
[9] There is no consumption nullcline under aggregate uncertainty; the Euler equations at the conditional steady states play that role.



(a) On the saddles



(b) C - time domain



(c) K - time domain

Figure 4: Impulse responses to a positive TFP shock: saddle vs. time domain

Notes: Impulse responses on conditional saddle paths (panel (a)) and in the time domain (panels (b) and (c)). Sample: 50 quarters.

Proposition 4 shows that PF steady states are nested within the CS steady states: aggregate uncertainty widens the spread, reflecting rational expectations of future regime shifts. The intuition is that agents in the conditional-saddle economy anticipate regime switches: during low TFP, the possibility of a future improvement raises the effective discount factor for investment, pushing K_B^{cs} below its PF counterpart. Symmetrically, during high TFP, the risk of a downturn depresses effective returns, pushing K_G^{cs} above K_G^{pf}. Figure 5 illustrates this nesting.

Conditional boundary condition I work directly with the recursive formulation, which implicitly imposes a stochastic analogue of the deterministic transversality

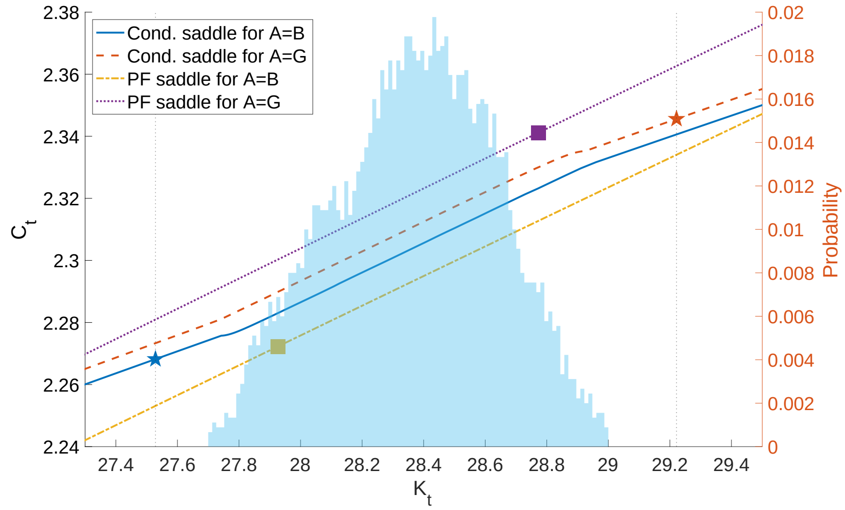


Figure 5: Conditional saddle path comparison: with and without uncertainty

Notes: Conditional saddle paths for $A = B$ (solid) and $A = G$ (dashed) and perfect-foresight saddle paths for $A = B$ (dash-dotted) and $A = G$ (dotted) under standard quarterly calibration.

condition.[10] Given conditional saddle paths, the natural counterpart is *conditional transversality*:

$$\lim_{t\to\infty} \beta^t u'(c_t)K_t = 0 \quad \text{under a frozen aggregate regime } A \in \{B, G\} \tag{19}$$
$$\text{(along the conditional saddle for each } A \in \{B, G\}).$$

Condition (19) rules out explosive continuations along each conditional saddle and ensures well-defined continuation values under frozen-regime dynamics. Under standard assumptions, (19) is equivalent to the usual stochastic transversality conditions (Kamihigashi, 2003, 2005) in the sense that both select the same bounded recursive competitive equilibrium. The conditional formulation makes explicit that the boundary condition operates regime-by-regime on $\mathcal{M}_A$, a perspective essential for the dimension-reduction results that follow.

Economies without conditional saddle paths A conditional saddle requires a predetermined endogenous state: without one, the economy jumps each period to the unique bounded allocation implied by the current exogenous state, leaving

[10]This condition is imposed at the aggregate level and should be distinguished from individual no-Ponzi constraints.

no nontrivial "along-the-saddle" dynamics. Formally, an economy is *endogenously stateless* if $\Phi_t = \phi(A_t)$ for some function ϕ—all allocations depend only on the current exogenous state—so the conditional "saddle" under frozen A collapses to a singleton.

The textbook three-equation New Keynesian model (Gali, 2008) is endogenously stateless under determinacy: $(x_t, \pi_t, i_t) = \Psi s_t$ for exogenous state $s_t := (r_t^n, u_t, \nu_t)$, so freezing s_t produces no nontrivial dynamics. Models with predetermined states—capital, habits, interest-rate smoothing, or distributional states in HANK—are endogenously stateful, and conditional saddle paths generically exist.

3 State dependence in a shock response

In this section, I analyze nonlinear shock responsiveness through the lens of conditional saddle paths. In any stochastic dynamic model that admits conditional saddle paths, the response of an aggregate variable to an exogenous shock is represented by a shift across different conditional saddle paths. If and only if all conditional saddle paths are parallel along the endogenous state, the response is state-independent.

Theorem 1 (State-(in)dependence as a geometric condition)**.**
Fix (Φ_0, A_0) and let $\mathcal{M}(\Phi_0, A_0)$ denote the conditional saddle path under the frozen regime A_0. Let $g(\Phi, A)$ be an aggregate equilibrium object (e.g. consumption) defined for $(\Phi, A) \in \mathcal{M}(\Phi_0, A_0) \times \{A_0, A_1\}$. Define the impact gap between regimes A_1 and A_0 at state Φ by

$$\Delta_g(\Phi; A_1, A_0) := g(\Phi, A_1) - g(\Phi, A_0). \tag{20}$$

Then the following are equivalent:

(i) (State-independent gap) $\Delta_g(\Phi; A_1, A_0)$ is constant on $\mathcal{M}(\Phi_0, A_0)$, i.e. there exists $c \in \mathbb{R}$ such that

$$g(\Phi, A_1) - g(\Phi, A_0) = c \qquad \forall\, \Phi \in \mathcal{M}(\Phi_0, A_0). \tag{21}$$

(ii) (Vertical-translation geometry) Viewed as subsets of $\mathcal{X} \times \mathbb{R}$,

$$\mathcal{G}_{A_j} := \{(\Phi, g(\Phi, A_j)) :\ \Phi \in \mathcal{M}(\Phi_0, A_0)\}, \qquad j \in \{0, 1\}, \tag{22}$$

satisfy $\mathcal{G}_{A_1} = \mathcal{G}_{A_0} + (0, c)$*, i.e.* $\mathcal{G}_{A_1}$ *is a constant vertical translation of* $\mathcal{G}_{A_0}$*.*

Proof. $(i) \Rightarrow (ii)$: if $g(\Phi, A_1) = g(\Phi, A_0) + c$ for all Φ, then $(\Phi, g(\Phi, A_1)) = (\Phi, g(\Phi, A_0) + c)$ for all Φ, hence $\mathcal{G}_{A_1} = \mathcal{G}_{A_0} + (0, c)$. $(ii) \Rightarrow (i)$: if $\mathcal{G}_{A_1} = \mathcal{G}_{A_0} + (0, c)$, then for each Φ we must have $g(\Phi, A_1) = g(\Phi, A_0) + c$, so the gap is constant. ■

It is worth noting that both $g(\Phi, A_0)$ and $g(\Phi, A_1)$ are evaluated at the same $\Phi \in M(\Phi_0, A_0)$, while only the regime label varies. Sharp state independence may be a knife-edge property of an RCE. However, Theorem 1 provides an insight regarding conditions under which state dependence becomes amplified. In particular, when a conditional saddle path is more steeply tilted with respect to the endogenous state, as illustrated in Figure 6, a shock responsiveness becomes state dependent. Such differential slopes may arise from various real (Winberry, 2021; Lee, 2025), financial (Melcangi, 2024), labor market frictions (Petrosky-Nadeau et al., 2018; Pizzinelli et al., 2020), and the scope of the relevant shocks include TFP shocks and fiscal/monetary policy shocks (Lee, 2025). The sufficiency and the necessity of tilt in the state dependence imply that models with such nature necessarily imply different slopes of the conditional saddle paths.

An example: asymmetric adjustment cost As an example where state dependence arises due to the tilt in the conditional saddle, I consider an extended RBC model with asymmetric adjustment cost. Specifically, the representative household's budget constraint is modified in the following way:

$$c + a' + \mathcal{C}(a', a) = a(1 + r(X)) + w(X) \tag{23}$$

$$\mathcal{C}(a', a) = \frac{\widetilde{\mu}}{2}\left(\frac{a' - a}{a}\right)^2 a \tag{24}$$

$$\widetilde{\mu} = \begin{cases} \mu_+ & \text{if } a' > a \\ \mu_- & \text{if } a' < a \end{cases} \tag{25}$$

where $\mathcal{C}$ is the wealth adjustment cost which indirectly reflects the frictional capital market. The adjustment cost is asymmetric between positive and negative investment, as specified in equation (25). In particular, I consider the case $\mu_+ > \mu_-$.[11]

[11] The adjustment-cost parameters are not calibrated. For illustrative purposes, I set $\mu_+ = 4$ and $\mu_- = 1$. All other parameters follow a standard quarterly RBC calibration.

Figure 6 plots the conditional saddle paths under asymmetric adjustment costs. Relative to the saddle path associated with A=G, the conditional saddle path for A=B is substantially steeper. As a result, a one—standard-deviation TFP shock generates a larger consumption response when the capital stock is low (3.42%) than when it is high (2.42%).

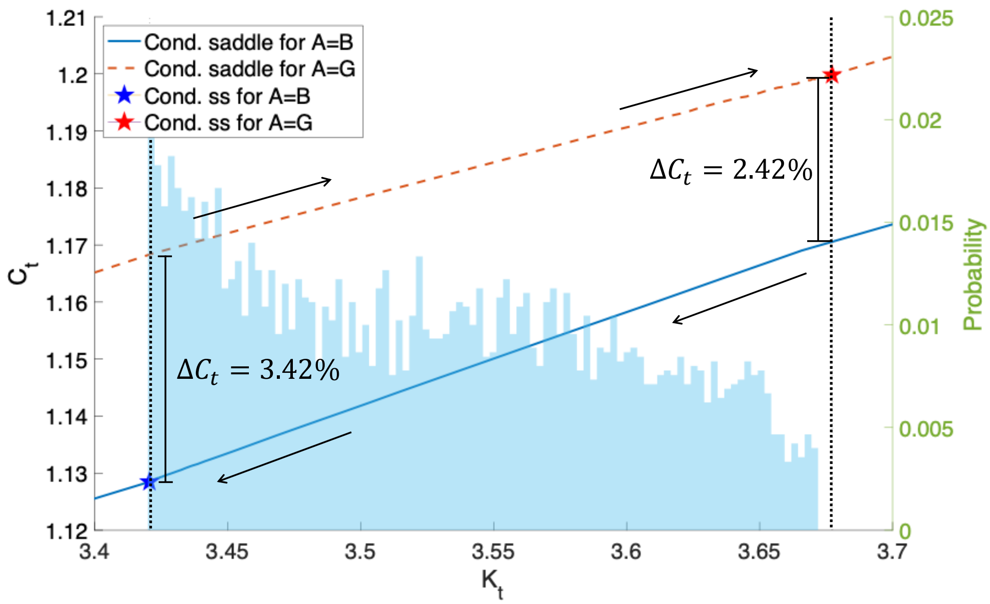


Figure 6: Differently tilted conditional saddle paths – endogenous state dependence

Notes: Conditional saddle paths for $A = B$ (solid) and $A = G$ (dashed) under asymmetric wealth adjustment cost ($\mu_+ = 4$, $\mu_- = 1$). The $A = B$ saddle is steeper, generating state-dependent shock responses.

This geometric representation formalizes the intuition that "climbing up is difficult, falling down is easy"—a recurring theme in the literature on asymmetric business cycles (Petrosky-Nadeau et al., 2018). By Theorem 1, state independence holds if and only if conditional saddles are parallel. Any deviation—whether from financial frictions (Melcangi, 2024), labor market frictions (Petrosky-Nadeau et al., 2018; Pizzinelli et al., 2020), or adjustment costs as above—manifests as differential tilt, the universal geometric signature of endogenous state dependence. This provides a structural interpretation of the growing empirical evidence that macroeconomic policy transmits asymmetrically over the cycle (Tenreyro and Thwaites, 2016; Auerbach and Gorodnichenko, 2012; Jo and Zubairy, 2025): if conditional saddle paths are steeper in one regime than another, any shock operating through the jump variable exhibits larger effects in that regime. Crucially, the asymmetry is *measurable*—for any globally solved model, the ratio of conditional saddle slopes at a given level of the predetermined state is a scalar that quantifies the degree of state dependence. This provides a sharp model-based diagnostic for state dependence across model specifications.

The framework also clarifies the distinction between endogenous and exogenous state dependence. Endogenous state dependence arises from slope differences across conditional saddles: at different levels of the predetermined state, the vertical gap between saddles differs, so the same exogenous shock produces responses of different magnitudes. Exogenous state dependence, by contrast, arises from the magnitude of vertical shifts across saddles and is present even when saddles are parallel.

4 Heterogeneous-household economy and dimension reduction

This section introduces conditional saddle path in a canonical heterogeneous-household economy. I consider a continuum of unit measure of ex-ante homogeneous households. The recursive formulation of the households' problem is as follows:

$$v(a, z; \Phi, A) = \max_{c,a'} \quad log(c) + \beta \mathbb{E} v(a', z'; \Phi', A') \tag{26}$$

$$c + a' = a(1 + r(X)) + w(X)z \tag{27}$$

$$a' \geq 0 \tag{28}$$

$$\Phi' = \Gamma_{endo}(\Phi, A), \quad A' \sim \Gamma_{exo}(A'|A). \tag{29}$$

The problem is the same as in the representative-household economy except for 1) uninsurable idiosyncratic labor productivity, which follows a Markov process $z \sim \Gamma_z(z'|z)$; 2) inclusion of distribution of individual states Φ in the aggregate endogenous state; and 3) the corresponding change in the law of motions for the endogenous aggregate state. The RCE is defined as in Krusell and Smith (1998).

The model includes two different stochastic exogenous processes: idiosyncratic productivity and aggregate TFP. Therefore, there are two layers of conditional saddle paths: one is individual conditional saddle path, and the other is aggregate conditional saddle path. The individual saddle path has its own cross-sectional implication which deserves a separate analysis, but it is out of this paper's focus. So, the corresponding analysis is included in Appendix B. I elaborate on the aggregate conditional saddle with stochastic TFP process, where the model closely follows Krusell and Smith (1998).[12] Consistent with the notations in the canonical RBC model in Section 2.2, I denote $K = K(\Phi)$ and $C = C(\Phi_t, A)$ as aggregate capital and consumption.

Based on the law of motion Γ_{endo}, I define conditional K-nullcline as follows:

Definition 5 (Conditional K-nullcline in Krusell and Smith (1998)).
Fix (Φ_0, A_0) and a regime $A \in \{B, G\}$. Let $\{\Phi_t\}_{t\geq 0}$ be the frozen-regime continuation $\Phi_{t+1} = \Gamma_{\text{endo}}(\Phi_t, A)$. The conditional K-nullcline along the continuation is

$$\mathcal{N}(\Phi_0, A) := \Big\{(K_t, C_t) : \ t \geq 0, \ K(\Gamma_{\text{endo}}(\Phi_t, A)) - K(\Phi_t) = 0\Big\}, \tag{30}$$

where $K_t := K(\Phi_t)$ and $C_t := C(\Phi_t, A)$. When $\mathcal{N}(\Phi_0, A)$ is the graph of a function, define $C_A^{Kcnull}(\cdot; \Phi_0)$ by $\mathcal{N}(\Phi_0, A) = \{(K, C) : \ C = C_A^{Kcnull}(K; \Phi_0)\}$.

Krusell and Smith (1998) posits an endogenous law of motion that tracks aggregate capital K rather than the full distribution Φ: $\log K' = \alpha(A) + \beta(A) \log K$, for $A \in \{B, G\}$, where α and β are state-dependent coefficients. This formulation embeds two distinct assumptions: (i) that aggregate capital K is sufficient to summarize the endogenous aggregate state, and (ii) that K follows a log-linear law of motion. Using the conditional-saddle framework, this paper establishes that (i) is exact: K uniquely indexes the distributional state on the conditional saddle, so conditioning on K entails no loss of information for equilibrium dynamics. By contrast, (ii) remains a parametric approximation—the true law of motion for K is generally non-Markovian and need not be log-linear. The sufficiency result concerns indexing, not the functional form of the law of motion. The following theorem provides the first step toward establishing the exact sufficiency of K.

Theorem 2 (Monotone aggregate variable as a coordinate).
Fix (Φ_0, A_0) and $A \in \{B, G\}$. Let $\{\Phi_t\}_{t\geq 0}$ be the frozen-regime continuation $\Phi_{t+1} = \Gamma_{\text{endo}}(\Phi_t, A)$ and let $\Phi^{cs} := \lim_{t\to\infty} \Phi_t$. Let $e : \mathcal{M}(\Phi_0, A) \to \mathbb{R}$ be an aggregate equilibrium variable. Suppose that the scalar sequence $\{e(\Phi_t)\}_{t\geq 0}$ is strictly monotone and

[12] In the original model of Krusell and Smith (1998), the exogenous individual labor supply comoves with the exogenous aggregate TFP. All the results stay unaffected after including this feature in the model, but for the expositional brevity, I assume the labor supply is exogenously fixed.

converges to $e(\Phi^{cs})$. Then e uniquely indexes states on the conditional saddle:

$$e(\Phi) = e(\Phi') \implies \Phi = \Phi' \qquad \forall\, \Phi, \Phi' \in \mathcal{M}(\Phi_0, A). \tag{31}$$

Proof. Define $\psi_A : \mathbb{N} \cup \{\infty\} \to \mathcal{M}(\Phi_0, A)$ by $\psi_A(t) := \Phi_t$ for $t \in \mathbb{N}$ and $\psi_A(\infty) := \Phi^{cs}$. By definition of $\mathcal{M}(\Phi_0, A)$ and convergence $\Phi_t \to \Phi^{cs}$, we have

$$\mathcal{M}(\Phi_0, A) = \{\Phi_t : t \geq 0\} \cup \{\Phi^{cs}\}, \tag{32}$$

so ψ_A is surjective. Strict monotonicity of $\{e(\Phi_t)\}$ implies that $e(\Phi_t) \neq e(\Phi_\tau)$ for $t \neq \tau$, hence $\Phi_t \neq \Phi_\tau$ for $t \neq \tau$; therefore ψ_A is injective on $\mathbb{N}$. Moreover, since $\{e(\Phi_t)\}$ is strictly monotone and converges to $e(\Phi^{cs})$, we also have $e(\Phi_t) \neq e(\Phi^{cs})$ for all finite t, so ψ_A is injective on $\mathbb{N} \cup \{\infty\}$. Thus ψ_A is a bijection between $\mathbb{N} \cup \{\infty\}$ and $\mathcal{M}(\Phi_0, A)$.

Now let $\Phi, \Phi' \in \mathcal{M}(\Phi_0, A)$. Pick $s, s' \in \mathbb{N} \cup \{\infty\}$ such that $\Phi = \psi_A(s)$ and $\Phi' = \psi_A(s')$. If $e(\Phi) = e(\Phi')$, then $e(\psi_A(s)) = e(\psi_A(s'))$. Since $e \circ \psi_A$ is injective, it follows that $s = s'$, hence $\Phi = \Phi'$. Therefore e is injective on $\mathcal{M}(\Phi_0, A)$, and an inverse map φ_A exists on $e(\mathcal{M}(\Phi_0, A))$. ■

Consequently, there exists a unique inverse map $\varphi_A : e(\mathcal{M}(\Phi_0, A)) \to \mathcal{M}(\Phi_0, A)$ such that $\varphi_A(e(\Phi)) = \Phi$ on $\mathcal{M}(\Phi_0, A)$, and any equilibrium object restricted to $\mathcal{M}(\Phi_0, A)$ can be written as a (single-valued) function of the scalar coordinate e.

The geometric intuition is that the conditional saddle is a one-dimensional curve of equilibrium states, and e acts as a "progress meter" along this curve: it moves strictly in one direction and never reverses. If two distinct states $\Phi \neq \Phi'$ shared the same value $e(\Phi) = e(\Phi')$, both would be at the same "progress" position, yet their subsequent histories could not merge on a one-dimensional path—a contradiction. Hence e uniquely labels states on the conditional saddle.

Remark 1 (Injectivity and sufficiency)**.**
Because e is injective on $\mathcal{M}_A$, any equilibrium object restricted to $\mathcal{M}_A$ can be written as a function of (e, A).

Specifically, the following variables can be defined:

$$v^e(\cdot,\cdot;\widetilde{e},A) \in V(\widetilde{e},A) := \{v(\cdot,\cdot;\Phi,A)|\forall\Phi \in \mathcal{M}_A(\Phi_0,A_0) \text{ s.t. } e(\Phi)=\widetilde{e}\} \tag{33}$$

$$r^e(\widetilde{e},A) \in R(\widetilde{e},A) := \{r(\Phi,A)|\forall\Phi \in \mathcal{M}_A(\Phi_0,A_0) \text{ s.t. } e(\Phi)=\widetilde{e}\} \tag{34}$$

$$w^e(\widetilde{e},A) \in W(\widetilde{e},A) := \{w(\Phi,A)|\forall\Phi \in \mathcal{M}_A(\Phi_0,A_0) \text{ s.t. } e(\Phi)=\widetilde{e}\} \tag{35}$$

$$\Gamma^e_{endo}(\widetilde{e},A) \in G(\widetilde{e},A) := \{e\left(\Gamma(\Phi,A)\right)|\forall\Phi \in \mathcal{M}_A(\Phi_0,A_0) \text{ s.t. } e(\Phi)=\widetilde{e}\}. \tag{36}$$

V, R, W and G are nonempty by Assumption 1 and singletons by Theorem 2. Therefore, the recursive problem below is equivalent to the original recursive formulation in equilibrium, as they yield the same equilibrium allocations.

$$v^e(a,z;e,A) = \max_{c,a'} \quad log(c) + \beta\mathbb{E}v^e(a',z';e',A') \tag{37}$$

$$c + a' = a(1+r^e(e,A)) + w^e(e,A)z \tag{38}$$

$$a' \geq 0 \tag{39}$$

$$e' = \Gamma^e_{endo}(e,A), \quad A' \sim \Gamma_{exo}(A'|A), \tag{40}$$

Now, I show that the conditional K nullclines are the same as in the representative agent model in the following proposition.

Proposition 5 (Conditional K-nullclines of Krusell and Smith (1998))**.**
The heterogeneous household model's conditional K nullclines are identical to the counterparts of the model with the representative household and invariant over the initial distribution Φ_0 and distributional dynamics:

$$C^{Kcnull}_A(K) = AK^\alpha - \delta K, \text{ for } A \in \{B,G\}. \tag{41}$$

Proof.
As in Proposition 3, the stationary condition $\delta K = I$ and the aggregate resource constraint $Y = C + I$ immediately imply the form of the conditional nullclines. ■

Proposition 5 makes the K-nullcline sign-determining: if $C_t < C^{Kcnull}_A(K_t)$ then $\Delta K_t > 0$ and hence $K_{t+1} > K_t$, whereas if $C_t > C^{Kcnull}_A(K_t)$ then $\Delta K_t < 0$ and hence $K_{t+1} < K_t$.

Proposition 6 (K monotonicity and injectivity)**.**
Fix $A \in \{B,G\}$. Under Assumption 1 and Assumption 2, aggregate capital converges

to K_A^{cs} strictly monotonically along the frozen-regime continuation on $\mathcal{M}(\Phi_0, A)$. Consequently, K is injective on $\mathcal{M}(\Phi_0, A)$.

Proof.
Fix A and let $\{\Phi_t\}_{t\geq 0}$ be the frozen-regime continuation, with $K_t := K(\Phi_t)$ and $C_t := C(\Phi_t, A)$. By Proposition 5, the conditional K-nullcline is

$$C_A^{Kcnull}(K) = AK^\alpha - \delta K, \tag{42}$$

so along the frozen-regime law of motion,

$$\Delta K_t := K_{t+1} - K_t = C_A^{Kcnull}(K_t) - C_t = -\, H_t, \tag{43}$$

$$H_t := C_t - C_A^{Kcnull}(K_t). \tag{44}$$

Assumption 2 implies that the line segment connecting (K_t, C_t) and (K_{t+1}, C_{t+1}) does not intersect the nullcline graph, hence

$$H_t H_{t+1} > 0 \qquad \text{for all } t \text{ such that } \Phi_t \neq \Phi^{cs}(\Phi_0, A). \tag{45}$$

Therefore $\{H_t\}$ has a constant sign until convergence, and thus $\Delta K_t = -H_t$ has a constant nonzero sign until convergence. It follows that $\{K_t\}$ is strictly monotone. By Assumption 1(i) and continuity of K,

$$K_t \to K_A^{cs} := K(\Phi^{cs}(\Phi_0, A)). \tag{46}$$

We now apply Theorem 2 with $e(\Phi) = K(\Phi)$ on

$$\mathcal{M}(\Phi_0, A) = \overline{\{\Phi_t : t \geq 0\}}. \tag{47}$$

Since $\{K(\Phi_t)\}_{t\geq 0} = \{K_t\}_{t\geq 0}$ is strictly monotone and converges to $K(\Phi^{cs})$, Theorem 2 implies

$$K(\Phi) = K(\Phi') \implies \Phi = \Phi' \qquad \forall\, \Phi, \Phi' \in \mathcal{M}(\Phi_0, A), \tag{48}$$

i.e. K is injective on $\mathcal{M}(\Phi_0, A)$. ■

Proposition 5 plays a key role in the proof above because it makes the K-nullcline *sign-determining*: the graph $C = C_A^{Kcnull}(K)$ partitions the (K, C) plane into $\Delta K > 0$ and $\Delta K < 0$ regions by $\Delta K = C_A^{Kcnull}(K) - C$. Assumption 2 rules out pathological discrete-time *crossings* of this partition between successive dates by requiring that the line segment connecting (K_t, C_t) and (K_{t+1}, C_{t+1}) does not intersect the nullcline. Together, these two ingredients imply that net investment cannot reverse sign along the conditional saddle, yielding strictly monotone capital dynamics and hence an exact one-dimensional representation indexed by K.

Proposition 6 implies that, conditional on (Φ_0, A), the mapping $K : \mathcal{M}(\Phi_0, A) \to \mathbb{R}$ is one-to-one. Hence there exists an inverse map $\varphi_{\Phi_0,A}$ such that $\Phi = \varphi_{\Phi_0,A}(K(\Phi))$ for all $\Phi \in \mathcal{M}(\Phi_0, A)$.[13] Equivalently, any equilibrium object restricted to $\mathcal{M}(\Phi_0, A)$ admits an exact single-valued representation as a function of (K, A), delivering a one-dimensional state representation along the conditional saddle.

To complement this theoretical result, Figure 7 plots the computed conditional saddle paths in the (K, C) phase diagram. The model is solved globally using the repeated transition method, and the dynamics under each frozen aggregate state are simulated for 2,000 periods. Although the true endogenous state of the model is the full distribution Φ rather than K alone, the figure shows that conditional saddle paths are strictly and monotonically ordered in K, providing clear computational support for the sufficiency result.

From spell-wise to global sufficiency. Proposition 6 is stated regime-by-regime: it establishes that K uniquely indexes the conditional saddle $\mathcal{M}(\Phi_0, A)$ under a single frozen regime. But the stochastic economy switches regimes. The key observation is that a regime switch does not break the reduction—it restarts it. Within any regime spell, the economy evolves along a conditional saddle monotonically indexed by K. When the regime switches at period τ, the current distributional state Φ_τ becomes the entry condition for the new frozen-regime continuation under A_τ. But Φ_τ is itself uniquely pinned by K_τ from the previous spell's indexing. By Proposition 6, the subsequent sequence $\{K_t\}_{t \geq \tau}$ is again strictly monotone along the new conditional saddle $\mathcal{M}(\Phi_\tau, A_\tau)$, so K provides an exact coordinate on every segment. Along any realized regime history, the equilibrium path is therefore a concatenation of one-dimensional segments, each exactly indexed by (K, A). No infinite-dimensional tracking is re-

[13] This is a path-specific notion of sufficiency: for a fixed (Φ_0, A), K uniquely indexes states on the entire conditional saddle $\mathcal{M}(\Phi_0, A)$ (i.e. globally *along the transition path*).

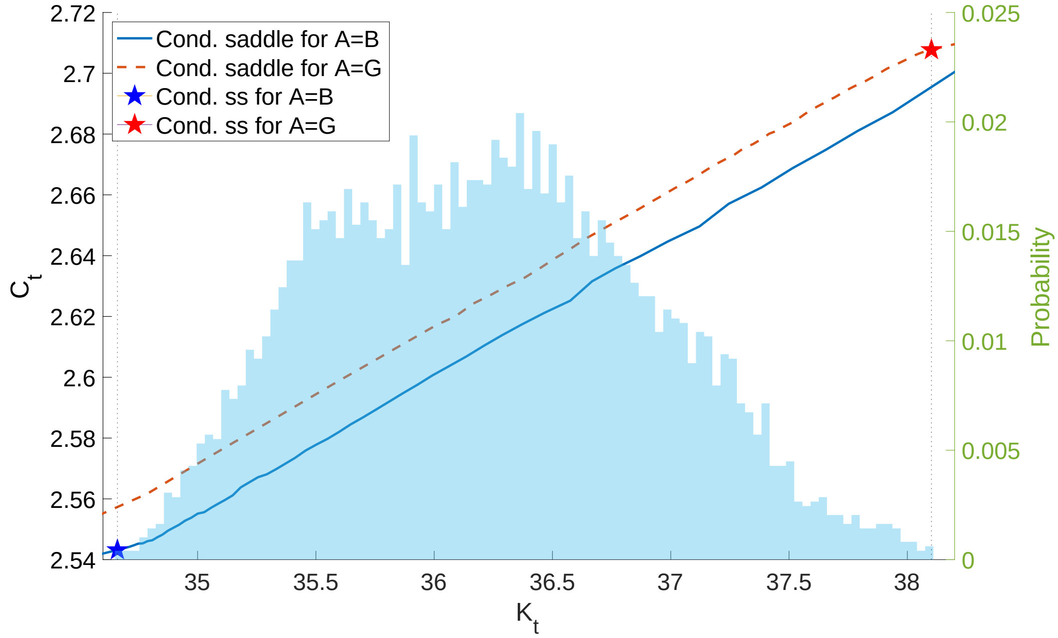


Figure 7: Conditional saddle paths in Krusell and Smith (1998)

Notes: The figure plots conditional saddle paths for $A = B$ (solid) and $A = G$ (dashed) implied by a canonical heterogeneous household business cycle model (Krusell and Smith, 1998). The histogram in the background plots the time-series distribution of the aggregate capital stock.

quired at any point: the "handoff" between spells is pinned down entirely by the scalar pair (K_τ, A_τ).

From near rationality to complete rationality Krusell and Smith (1998) assumes a specific parametric law of motion to compute the heterogeneous-household model, then confirms accuracy through the consistency between the realized and assumed dynamics. After this celebrated contribution, the approach is often labeled as near or bounded rational. The conditional-saddle framework shows that, restricted to the conditional saddle, conditioning on K can be exact (a sufficient statistic) for equilibrium objects. The additional log-linear functional form used by Krusell and Smith (1998) remains a quantitative approximation.

Parametric form of the conditional saddle paths Despite the indexing function K, specific conditional saddle paths and the form of the law of motion remain undetermined. This problem exists even for representative-agent business cycle models. Lee (2025) develops the repeated transition method (RTM) that utilizes the recurrence of the equilibrium allocations along the conditional saddle path. For heterogeneous-agent models, the existence of the indexing variable (sufficient statistic) starkly eases the implementation. Theorem 2 and Proposition 6 theoretically support the implementation feasibility of the RTM using the sufficient statistic.

Verifying monotonicity in practice. The sufficiency results above rely on strict monotonicity of K along frozen-regime continuations (Assumption 2). How should a researcher verify this in a given model? The following two subsections develop complementary strategies. Section 4.1 provides a *perfect-foresight diagnostic*: one solves the deterministic ($\mu = 0$) transition dynamics and checks whether K converges monotonically; if so, a homotopy to $\mu = 1$ guarantees monotonicity under rational expectations. Section 4.2 provides an independent route from static primitives—bounds on household MPCs and price feedback—that does not require solving any transition path.

4.1 Perfect-foresight diagnostic for monotonicity

This section shows that K-monotonicity under rational expectations can be verified from the deterministic perfect-foresight model alone. The key insight is that the K-nullcline—the "obstacle" that the path must not cross—is invariant over beliefs (Proposition 5), so monotonicity can be transported continuously from perfect foresight to rational expectations.

Belief family. For $\mu \in [0, 1]$, define the belief transition matrix $\mathit{\Pi}(\mu) := (1-\mu)\, I + \mu\, \mathit{\Pi}$, where I is the identity and $\mathit{\Pi}$ is the true Markov matrix for $A \in \{B, G\}$. At $\mu = 0$, agents assign probability one to the current regime persisting (perfect foresight); at $\mu = 1$, agents hold rational expectations. For each μ, the μ-*RCE* is the recursive competitive equilibrium under $A' \sim \mathit{\Pi}(\mu)(\cdot|A)$ with the same technology and market structure. Denote the frozen-regime continuation under (μ, A) by $\{(K_t(\mu), C_t(\mu))\}_{t\geq 0}$, with conditional steady state $K_A^{cs}(\mu)$.

Under standard regularity, the frozen-regime continuation and conditional steady state are continuous in μ.[14] The question is whether K-monotonicity, which may hold

[14]Since $\mathit{\Pi}(\mu)$ enters the Bellman equation linearly, the recursive equilibrium operator is continuous in μ. The result follows by the parametric contraction mapping theorem and continuous dependence

at $\mu = 0$ (perfect foresight), survives the passage to $\mu = 1$ (rational expectations). To formalize this, define the *monotonicity set*

$$\mathcal{S} := \big\{\mu \in [0,1] : \{K_t(\mu)\}_{t\geq 0} \text{ is strictly monotone along the frozen-}A\text{ continuation}\big\}.$$

Proposition 7 (Perfect-foresight diagnostic for K-monotonicity)**.**
Fix $A \in \{B, G\}$. If $0 \in \mathcal{S}$, then $\mathcal{S} = [0,1]$. In particular, strict K-monotonicity along perfect-foresight transition paths implies strict K-monotonicity under rational expectations, and Assumption 2 holds.

Proof.
Since $[0,1]$ is connected, it suffices to show $\mathcal{S}$ is nonempty, open, and closed.

Nonempty. $0 \in \mathcal{S}$ by assumption.

Open. Fix $\mu^* \in \mathcal{S}$ and define the nullcline gap $H_t(\mu) := C_t(\mu) - C_A^{Kcnull}(K_t(\mu))$, so that $\Delta K_t(\mu) = -H_t(\mu)$. At μ^*, strict monotonicity of K implies $H_t(\mu^*)$ has constant nonzero sign for all pre-limit t. By Proposition 5, the nullcline C_A^{Kcnull} is independent of μ. Combined with continuous dependence of $(K_t(\mu), C_t(\mu))$ on μ, for any finite T there exists $\varepsilon > 0$ such that $|\mu - \mu^*| < \varepsilon$ implies $H_t(\mu)\, H_t(\mu^*) > 0$ for all $t \leq T$.

For the tail: monotone convergence at μ^* implies the stable eigenvalue of the linearized frozen-regime dynamics at $K_A^{cs}(\mu^*)$ is real and lies in $(0,1)$—otherwise local convergence would be oscillatory. By continuity of the eigenvalue in μ, this property persists for nearby μ, extending monotonicity to the tail. See Appendix F for details.

Closed. Let $\mu_n \to \mu^*$ with $\mu_n \in \mathcal{S}$. By continuity, $H_t(\mu_n) \to H_t(\mu^*)$ for each t. If $H_t(\mu^*) \neq 0$ for all pre-limit t, the constant-sign property is inherited and $\mu^* \in \mathcal{S}$.

Suppose instead $H_{t^*}(\mu^*) = 0$ at some t^* with $K_{t^*}(\mu^*) \neq K_A^{cs}(\mu^*)$. Then $\Delta K_{t^*} = 0$, so $K_{t^*+1}(\mu^*) = K_{t^*}(\mu^*)$. But then $K_{t^*}(\mu^*)$ is a fixed point of the frozen-regime map $\Gamma(\cdot, A)$, so the path is constant from t^* onward. By Assumption 1(i), the unique fixed point on the conditional saddle is $K_A^{cs}(\mu^*)$—contradicting $K_{t^*}(\mu^*) \neq K_A^{cs}(\mu^*)$. So no such t^* exists and $\mu^* \in \mathcal{S}$. ■

Discussion. Proposition 7 reduces the verification of K-monotonicity under rational expectations to a single, checkable condition: monotonicity of the deterministic

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of the market-clearing fixed point. See Appendix E.

perfect-foresight transition path. For representative-agent models, this is a textbook property of the Ramsey saddle path (the stable eigenvalue is real and in $(0, 1)$ under standard calibrations). For heterogeneous-agent models at $\mu = 0$, the frozen-regime economy is a standard Bewley–Aiyagari economy under fixed TFP—a well-understood object whose monotone convergence follows from the same nullcline structure.

The proof strategy—connecting $\mu = 0$ to $\mu = 1$ through a continuous path of equilibria—is reminiscent of homotopy methods in general equilibrium theory. The crucial simplification here is that the nullcline is the common "obstacle" that is invariant along the homotopy (Proposition 5), so the topology of trajectories relative to the nullcline cannot change continuously.

More generally, the proof uses nullcline invariance only to ensure *unique stationarity* of K along the conditional saddle: the conditional steady state is the only point where $K_{t+1} = K_t$. The diagnostic therefore extends to any continuous aggregate variable e satisfying unique stationarity—that is, $e(\Phi_{t+1}) = e(\Phi_t)$ only at $\Phi_t = \Phi^{\text{cs}}$—even in settings where the nullcline is not distribution-free or not available. This is relevant for the extensions in Section 5. The bond economy and the CRRA case involve distributional dependence that blocks the direct analytical route of Section 4.2. However, at $\mu = 0$ there is no distributional state, so unique stationarity is automatically satisfied. Monotonicity need only be checked along the deterministic transition path.

4.2 Primitive conditions for Assumption 2

This section provides an independent route to Assumption 2 that works directly at $\mu = 1$ (rational expectations) from static allocation primitives, without the homotopy of Section 4.1. Assumption 2 rules out discrete-time "side-switching" across the K-nullcline between dates. The results below show that such side-switching is excluded under economically interpretable bounds on households' marginal propensities to consume and sufficiently weak general-equilibrium price feedback.

Household MPC bounds Let $c(a, z; r, w)$ denote the stationary equilibrium household consumption policy as a function of individual assets a, idiosyncratic state $z \in \mathcal{Z}$, and prices (r, w). Along a frozen-regime continuation, aggregate consumption can change with aggregate capital because K affects prices, which in turn affects contemporaneous income components wz and ra. A convenient "MPC bound" in this context is therefore a bound on the *marginal response of consumption to price-induced changes in current income*, rather than an MPC out of an exogenous transfer shock.

Assumption 3 (Bounded price-income MPC)**.**
Along the frozen-regime continuation under A, there exists $\bar{m} \in (0,1)$ such that for the stationary equilibrium policy $c(a,z;r,w)$,

$$0 \le \frac{\partial c}{\partial (wz)}(a,z;r,w) \le \bar{m}, \qquad 0 \le \frac{\partial c}{\partial (ra)}(a,z;r,w) \le \bar{m}, \tag{49}$$

for all (a,z) in the support of Φ_t and all t on the continuation.[15]

Under Assumption 3, a marginal increase in aggregate capital affects contemporaneous aggregate consumption through the induced changes in wages and returns. Define the (pathwise) bounds as

$$\bar{Z} := \sup_{t \ge 0} \int z \, d\Phi_t, \qquad \bar{a} := \sup_{t \ge 0} \int a \, d\Phi_t, \qquad \mathcal{K} := \{K_t : t \ge 0\}. \tag{50}$$

Then, the change in aggregate consumption induced by a marginal change in K can be bounded above by the aggregate MPC bound $\bar{m}$ times the induced change in contemporaneous labor and capital incomes.

Sufficient condition for Assumption 2. In the Krusell–Smith benchmark with Cobb–Douglas production, prices are smooth functions of K:

$$w_A(K) = (1-\alpha)AK^{\alpha}, \qquad r_A(K) = \alpha A K^{\alpha-1} - \delta, \tag{51}$$

with derivatives

$$w_A'(K) = (1-\alpha)\alpha A K^{\alpha-1}, \qquad r_A'(K) = \alpha(\alpha-1)AK^{\alpha-2}. \tag{52}$$

Moreover, the nullcline satisfies $\big(C_A^{Kcnull}\big)'(K) = \alpha A K^{\alpha-1} - \delta$. The next lemma shows that if general-equilibrium price feedback is sufficiently weak relative to the nullcline

[15] Assumption 3 bounds the marginal response of consumption to *price-induced* changes in contemporaneous income components. This local derivative bound differs from the standard MPC out of a transitory transfer. In incomplete-markets environments, precautionary saving and borrowing constraints typically prevent one-for-one consumption responses to marginal changes in current income, making an upper bound $\bar{m} < 1$ natural on the relevant region of the state space.

slope, then the nullcline gap H_t cannot change sign, which implies Assumption 2 along the continuation.

Lemma 1 (Sufficient condition for Assumption 2)**.**
Fix $A \in \{B, G\}$ and consider the frozen-A continuation $\{\Phi_t\}_{t\geq 0}$ with $K_t := K(\Phi_t)$ and $C_t := C(\Phi_t, A)$. Let $\bar{Z} := \sup_{t\geq 0} \int z\, d\Phi_t$ and $\bar{a} := \sup_{t\geq 0} \int a\, d\Phi_t$ denote the pathwise upper bounds on average productivity and average assets (both finite under the natural borrowing limit and compact idiosyncratic support). Under Assumption 3, suppose that along $\mathcal{K}$,

$$\sup_{K\in\mathcal{K}} \Big(\bar{m}\big(w'_A(K)\,\bar{Z} + |r'_A(K)|\,\bar{a}\big) - \big(C_A^{Kcnull}\big)'(K) \Big) \;<\; 1. \tag{53}$$

Then $H_t H_{t+1} > 0$ for all t such that $\Phi_t \neq \Phi^{cs}(\Phi_0, A)$, and hence Assumption 2 holds along the continuation. Consequently, $\{K_t\}$ is strictly monotone until convergence.

The proof, given in Appendix D, proceeds by showing that the nullcline gap $H_t := C_t - C_A^{Kcnull}(K_t)$ satisfies the recursion $H_{t+1} = [1 - (\frac{dC}{dK} - (C_A^{Kcnull})')]\, H_t$, and that (53) ensures the bracketed multiplier is strictly positive.

A local check near the conditional steady state. The global bound (53) can be weakened to a neighborhood condition near the conditional steady state. This is useful both for economic interpretation and for quantitative verification in standard calibrations.

Corollary 1 (A local check near K_A^{cs})**.**
Under Assumption 3, suppose there exists $\varepsilon > 0$ such that the continuation satisfies $K_t \in (K_A^{cs} - \varepsilon, K_A^{cs} + \varepsilon)$ for all sufficiently large t, and

$$\sup_{|K-K_A^{cs}|\leq\varepsilon} \Big(\bar{m}\big(w'_A(K)\,\bar{Z} + |r'_A(K)|\,\bar{a}\big) - \big(C_A^{Kcnull}\big)'(K) \Big) \;<\; 1. \tag{54}$$

Then there exists $T < \infty$ such that $H_t H_{t+1} > 0$ (and hence Assumption 2) holds for all $t \geq T$, implying that K_t is eventually strictly monotone and converges to K_A^{cs} without side-switching.

The proof restricts Lemma 1 to the neighborhood $|K - K_A^{cs}| \leq \varepsilon$.

Conditions (53)–(54) are directly checkable in quantitative implementations: one evaluates the left-hand side along a simulated frozen-regime continuation using the model-implied bound $\bar{m}$ and the moments $\bar{Z}$ and $\bar{a}$. In standard Krusell–Smith calibrations, K moves slowly and the general-equilibrium price feedback terms $w'_A(K)$ and $|r'_A(K)|$ are modest, so the inequalities are typically satisfied with slack.[16]

5 Extensions and applications

Theorem 2 provides verifiable conditions for dimension reduction: identify the K-nullcline and verify monotonicity. The perfect-foresight diagnostic of Proposition 7 and the primitive conditions of Lemma 1 provide two complementary routes to establish this monotonicity. This section applies these tools to several extensions, illustrating both the framework's scope and its limits.

Economies with multiple endogenous states Theorem 2 applies to an economy with the multivariate (distributional) endogenous state. As long as there is an aggregate equilibrium variable e that strictly monotonically converges to the conditional steady-state level, then e is a sufficient statistic.

For example, consider the following model that extends the heterogeneous-agent model above by adding endogenous bond holding. The corresponding budget constraint is:

$$c + a' + q(\Phi, A)b' = a(1 + r(\Phi, A)) + b + w(\Phi, A)z \tag{55}$$

where b is bond holding and q is the bond price competitively determined by

$$\int b'(a, z; \Phi, A)d\Phi = 0. \tag{56}$$

The key observation is that Proposition 5 continues to hold: the conditional K-nullcline remains $C_A^{K\text{-null}}(K) = AK^\alpha - \delta K$, determined solely by the aggregate resource constraint. Since bonds are in zero net supply, their inclusion does not alter the sign-determining property of the nullcline. Thus, the monotonicity argument of Proposition 6 applies, and K remains a sufficient statistic. Lee (2025) confirmed this prediction computationally: equilibrium allocations are strictly monotonically sorted

[16] In the baseline KS98 calibration, the left-hand side of (53) evaluated along the simulated frozen-A continuation is around 0.05 even under the conservative envelope $\bar{m} = 1$, well below the threshold 1.

along K in the globally solved model, consistent with Theorem 2.

Models with endogenous labor supply With GHH preferences $u(c, l_H) = \frac{1}{1-\sigma}(c - \frac{\eta}{1+1/\chi} l_H^{1+1/\chi})^{1-\sigma}$, labor supply depends only on wages and productivity, eliminating wealth effects. The conditional K-nullcline depends on technology parameters and the stationary productivity distribution but not on the wealth distribution (Appendix G), so the monotonicity argument of Theorem 2 applies and K remains a sufficient statistic.

Under separable CRRA utility $u(c, l_H) = \frac{c^{1-\sigma}}{1-\sigma} - \frac{\eta}{1+1/\chi} l_H^{1+1/\chi}$, individual labor supply depends on consumption through the wealth effect, so the K-nullcline acquires distributional dependence. Strict monotonicity cannot be verified analytically in this case. However, Lee (2025) provides computational evidence of strictly monotone convergence under standard calibrations, and the perfect-foresight diagnostic of Proposition 7 remains applicable since at $\mu = 0$ there is no distributional state. The derivations for both specifications are in Appendix G.

5.1 Further applications

The conditional saddle framework extends beyond the neoclassical setting.[17] I briefly illustrate the framework in a search-and-matching environment, where the predetermined state is unemployment and the conditional saddle paths recover the Beveridge curve as a regime-specific transition locus.

Search and matching Consider a discrete-time stochastic Diamond–Mortensen–Pissarides economy.[18] Aggregate productivity follows a two-state Markov chain $z \in \{G, B\}$. The aggregate state is $S = [u_{-1}, z]$, where unemployment $u_{-1} = 1 - n_{-1}$ is the sole predetermined endogenous state. Employment evolves according to

$$n = (1 - \lambda)\, n_{-1} + q(\theta)\, (1 - n_{-1}), \tag{57}$$

[17]Companion work applies conditional saddle paths to New Keynesian models with an occasionally binding zero lower bound (Lee and Nomura, 2026) and to production network economies with endogenous linkage formation (Lee and Sun, 2026).

[18]The full model specification is in Appendix H. I adopt a standard formulation following Shimer (2005) with Nash-bargained wages, exogenous separation, and Cobb–Douglas matching.

where $\lambda \in (0,1)$ is the exogenous separation rate, $\theta = v/u$ is labor market tightness, and $q(\theta) = m\theta^{-\xi}$ is the vacancy-filling probability implied by a Cobb–Douglas matching function. Market tightness $\theta(S)$—and hence vacancies $v(S) = \theta(S) \cdot u_{-1}$—is the jump variable, pinned down by the free-entry condition for vacancy posting:

$$\frac{\kappa}{q(S)} \;=\; (1-\lambda)\,\beta\,\mathbb{E}\left[\left(\frac{c(S)}{c(S')}\right)^{\sigma}\left(z' - w(S') + \frac{\kappa}{q(S')}\right)\,\middle|\, S\right], \tag{58}$$

where κ is the vacancy posting cost, $w(S)$ is the Nash-bargained wage, and $c(S) = n(S)\,z - \kappa\,v(S) + (1-n(S))\,b$ is aggregate consumption from the resource constraint.

The conditional saddle path $\mathcal{M}(z;\, u_0)$ is the orbit $\{(u_t, v_t)\}_{t\geq 0}$ generated by (57) and the equilibrium vacancy function $v(\cdot, z)$ under frozen productivity z, with limit point $(u_z^{\text{cs}}, v_z^{\text{cs}})$.

Slope asymmetry and the Beveridge curve. Figure 8 plots the conditional saddle paths in (u_{-1}, v) space. Both loci slope upward—higher inherited unemployment raises equilibrium vacancies—so state dependence is sign-preserving, as in the RBC benchmark. However, the slopes differ markedly: under G, matches are profitable and the saddle is steep; under B, the flow surplus is low and the saddle is nearly flat.

This asymmetry generates the Beveridge curve "loop" observed in empirical (u, v) data. Starting near the G steady state, a negative shock switches the economy onto the flat B saddle: vacancies collapse on impact and barely adjust as unemployment accumulates along the flat path. Upon recovery, the economy jumps back onto the steep G saddle at high inherited u_{-1}, driving a sharp vacancy spike and rapid job creation. The counterclockwise rotation through recessions and recoveries is thus the across-saddle jump dynamics of the stochastic equilibrium.

6 Concluding remarks

This paper develops a geometric framework for stochastic equilibrium dynamics by introducing *conditional saddle paths*: invariant equilibrium paths defined under frozen exogenous states. This object extends the familiar saddle-path intuition from deterministic models to environments under aggregate uncertainty. In the resulting phase-diagram representation, business-cycle fluctuations decompose into movements *along* a conditional saddle (endogenous propagation within a regime) and *across* condi-

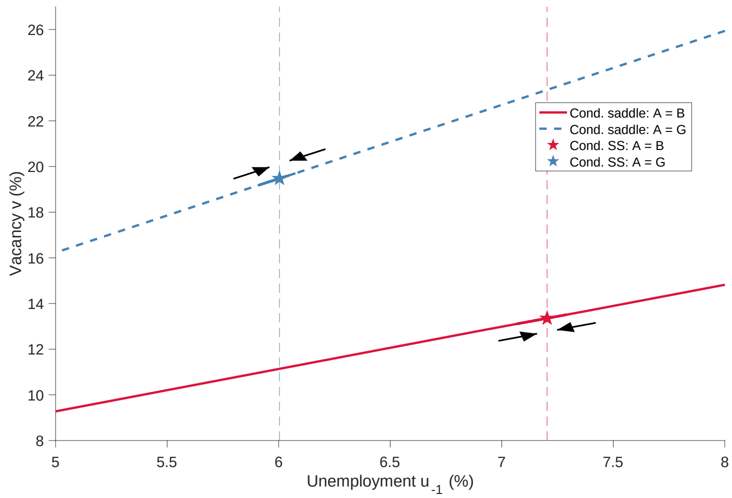

Figure 8: Conditional saddle paths in the DMP model

Notes: Conditional saddle paths in (u_{-1}, v) space under frozen productivity $z = G$ (dashed) and $z = B$ (solid). Stars mark conditional steady states. Both loci slope upward, but the G saddle is substantially steeper: inherited slack generates aggressive vacancy posting in expansions but elicits almost no response in recessions.

tional saddles (transitions in exogenous states). The framework clarifies why impulse responses can be state-dependent: such dependence is a geometric property of the stable equilibrium branch and arises precisely when conditional saddles differ in slope rather than by mere vertical translation.

When an aggregate equilibrium variable varies strictly monotonically along a conditional saddle, it provides a global coordinate: it uniquely indexes equilibrium states and therefore summarizes all equilibrium allocations and prices on the relevant invariant set. Applying this logic, I provide a theoretical proof of the sufficiency of aggregate capital in a canonical heterogeneous-household model. Beyond the Krusell–Smith benchmark, the same reasoning applies in multi-asset and richer heterogeneous-agent environments whenever a monotone-convergent aggregate coordinate exists. The perfect-foresight diagnostic developed in Section 4.1 makes verification practical: monotonicity need only be checked along deterministic transition paths, bypassing the need to solve the full stochastic equilibrium.

More broadly, conditional saddles offer a complementary lens on stochastic models: they provide a language for interpreting nonlinear dynamics and for assessing when scalar state approximations are exact rather than merely accurate. They also provide a natural geometry for *state-contingent* policy analysis: by making state dependence explicit in the phase diagram, the framework clarifies when the same intervention should be expected to have different quantitative effects across regimes and over the cycle. A promising direction for future work is to use these geometric objects to sharpen empirical restrictions on state dependence and to discipline the design of state-contingent stabilization policies in heterogeneous-agent economies under aggregate uncertainty.

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