

# Dancing on the Saddles: A Geometric Framework for Stochastic Equilibrium Dynamics<sup>1</sup>

## Online Appendix

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## A Sequential formulation of a canonical RBC

A representative household with temporal log utility is considered. Given initial condition  $(a_0, A_0)$ , the household maximizes life-time utility under stochastic aggregate TFP  $A_t$ , which is subject to a budget constraint as elaborated on below:

$$\max_{\{c_\tau(A^{(\tau)}), a_{\tau+1}(A^{(\tau)})\}_{\tau=0}^{\infty}} \mathbb{E}_0 \sum_{\tau=0}^{\infty} \beta^\tau \log(c_\tau(A^{(\tau)})) \quad (1)$$

$$\text{s.t. } c_\tau(A^{(\tau)}) + a_{\tau+1}(A^{(\tau)}) = a_\tau(A^{(\tau-1)})(1 + r(A^{(\tau)})) + w(A^{(\tau)}), \quad \text{for } \forall \tau, \forall A_t \quad (2)$$

$$a_{\tau+1}(A^{(\tau)}) \geq -\bar{a}, \quad \text{for } \forall \tau \quad (3)$$

where superscript  $\tau$  inside a bracket denotes history of a variable up to period  $\tau$ ;  $-\bar{a}$  is the natural borrowing limit to preempt Ponzi scheme. Labor supply is exogenously fixed at unity. I consider the following competitive factor prices given CRS Cobb-Douglas production function:

$$r(A^{(\tau)}) = A_t \alpha (K(A^{(\tau)}))^{(\alpha-1)} - \delta \quad (4)$$

$$w(A^{(\tau)}) = A_t (1 - \alpha) (K(A^{(\tau)}))^\alpha, \quad (5)$$

$K$  is capital stock, that satisfies  $K(A^{(\tau)}) = a(A^{(\tau)})$  in equilibrium. With the regularity conditions given in [Stokey et al. \(1989\)](#), this sequential formulation yields the same optimality conditions as the recursive form in the main text.

## B Individual conditional saddles in Aiyagari (1994)

I define individual-level conditional saddle path in the heterogeneous-household economy without aggregate uncertainty ( $A = A' = 1$ ). The conditional saddle is defined for the SRCE as in [Aiyagari \(1994\)](#).

**Definition 1** (Individual conditional saddle under frozen  $z$ ).

*Fix a stationary RCE of the Aiyagari (1994) economy, which delivers an individual asset policy*

$$g_a : \mathbb{R}_+ \times \mathcal{Z} \rightarrow \mathbb{R}_+,$$

*where  $\mathcal{Z}$  is a finite Markov set with transition matrix  $\Pi$ . Fix an initial condition  $(a_0, z_0) \in \mathbb{R}_+ \times \mathcal{Z}$ . For any frozen idiosyncratic state  $z \in \mathcal{Z}$ , define the frozen- $z$  continuation  $\{a_t(z)\}_{t \geq 0}$  recursively by*

$$a_{t+1}(z) = g_a(a_t(z), z), \quad a_0(z) = a_0. \quad (6)$$

The individual conditional saddle under frozen  $z$  is the orbit-closure

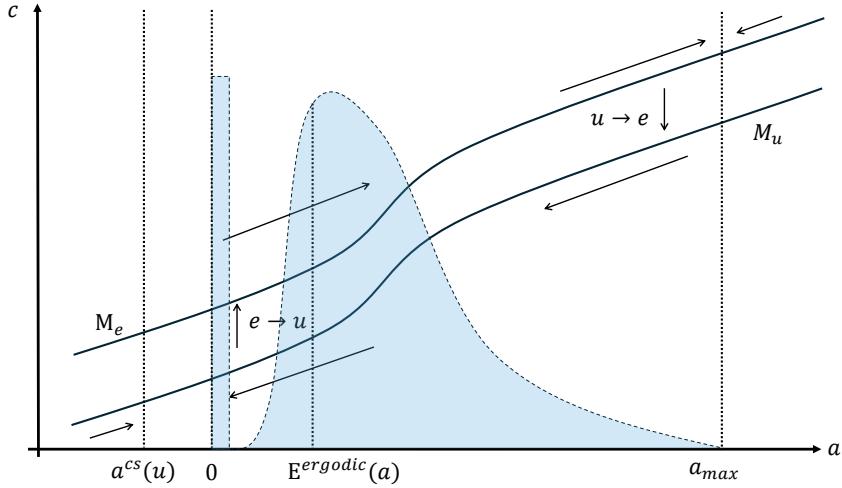
$$\mathcal{M}^{\text{ind}}(z; a_0) := \overline{\{a_t(z) : t \geq 0\}} \subseteq \mathbb{R}_+. \quad (7)$$

**Definition 2** (Individual conditional steady state under frozen  $z$ ).

Fix  $(a_0, z_0)$  and  $z \in \mathcal{Z}$ , and let  $\{a_t(z)\}_{t \geq 0}$  be the frozen- $z$  continuation. The individual conditional steady state under frozen  $z$  is defined by

$$a^{cs}(z; a_0) := \lim_{t \rightarrow \infty} a_t(z). \quad (8)$$

Figure B.1: Individual conditional saddle paths in the stationary RCE



*Notes:* The figure illustrates the individual conditional saddle paths for  $z = e$  and  $z = u$  in [Aiyagari \(1994\)](#).

Figure B.1 illustrates individual-level conditional saddle paths in [Aiyagari \(1994\)](#). Because of the borrowing constraint, the conditional steady state associated with the unemployment state  $z = u$  is not attained. Under standard calibrations, this generates a positive mass of agents at the borrowing limit. Analogous to the aggregate-level case, heterogeneity in the slopes of individual conditional saddle paths implies differential responses of individual consumption to idiosyncratic shocks.

## C Proofs for the theoretical results

**Proposition 2** (Aggregate uncertainty and the conditional steady states).

*The following inequalities hold:*

$$K_B^{cs} < K_B^{pf} < K_G^{pf} < K_G^{cs}, \quad c_B^{cs} < c_B^{pf} < c_G^{pf} < c_G^{cs}.$$

*Proof.*

Let  $R(A, K) := 1 - \delta + \alpha AK^{\alpha-1}$  denote the gross return on capital and let the  $K$ -nullcline (feasibility locus) be

$$c(A, K) = AK^\alpha - \delta K.$$

The PF steady states solve the Euler equations under absorbing beliefs,

$$1 = \beta R(B, K_B^{pf}), \quad 1 = \beta R(G, K_G^{pf}).$$

Since  $R(A, K)$  is strictly increasing in  $A$  and strictly decreasing in  $K$  (because  $\alpha - 1 < 0$ ), it follows immediately that  $K_B^{pf} < K_G^{pf}$ .

Next define the frozen-regime CS Euler residuals evaluated on the  $K$ -nullcline by

$$\begin{aligned} F_B(K) &:= \beta \left[ \pi_{BB} R(B, K) + \pi_{BG} \frac{c(B, K)}{c(G, K)} R(G, K) \right] - 1, \\ F_G(K) &:= \beta \left[ \pi_{GG} R(G, K) + \pi_{GB} \frac{c(G, K)}{c(B, K)} R(B, K) \right] - 1. \end{aligned}$$

By construction, the conditional steady states satisfy  $F_B(K_B^{cs}) = 0$  and  $F_G(K_G^{cs}) = 0$ .

*Step 1: show  $K_B^{cs} < K_B^{pf}$ .* Evaluate  $F_B$  at  $K_B^{pf}$ . Using  $1 = \beta R(B, K_B^{pf})$  and  $\pi_{BB} = 1 - \pi_{BG}$ ,

$$\begin{aligned} F_B(K_B^{pf}) &= \beta \left[ (1 - \pi_{BG}) R(B, K_B^{pf}) + \pi_{BG} \frac{c(B, K_B^{pf})}{c(G, K_B^{pf})} R(G, K_B^{pf}) \right] - 1 \\ &= \pi_{BG} \left[ \beta \frac{c(B, K_B^{pf})}{c(G, K_B^{pf})} R(G, K_B^{pf}) - 1 \right]. \end{aligned}$$

Thus  $F_B(K_B^{pf}) < 0$  is equivalent to

$$\beta \frac{c(B, K_B^{pf})}{c(G, K_B^{pf})} R(G, K_B^{pf}) < 1 \iff \frac{R(G, K_B^{pf})}{R(B, K_B^{pf})} < \frac{c(G, K_B^{pf})}{c(B, K_B^{pf})},$$

where we used  $\beta R(B, K_B^{pf}) = 1$  to divide both sides by  $R(B, K_B^{pf})$ .

We now prove this strict inequality for any  $K$  with  $c(B, K), c(G, K) > 0$ . Write  $x := \alpha K^{\alpha-1} > 0$  and  $y := K^\alpha > 0$ . Then

$$\frac{R(G, K)}{R(B, K)} = \frac{1 - \delta + Gx}{1 - \delta + Bx}, \quad \frac{c(G, K)}{c(B, K)} = \frac{Gy - \delta K}{By - \delta K}.$$

Since  $1 - \delta > 0$ , we have the strict bound

$$\frac{1 - \delta + Gx}{1 - \delta + Bx} < \frac{Gx}{Bx} = \frac{G}{B}.$$

Since  $\delta K > 0$  and  $Gy > By$ , subtracting the same positive term from numerator and denominator enlarges the ratio, yielding

$$\frac{Gy - \delta K}{By - \delta K} > \frac{Gy}{By} = \frac{G}{B}.$$

Combining the two displays gives

$$\frac{R(G, K)}{R(B, K)} < \frac{G}{B} < \frac{c(G, K)}{c(B, K)},$$

and in particular the desired inequality holds at  $K = K_B^{pf}$ . Therefore  $F_B(K_B^{pf}) < 0$ .

Finally, note that  $F_B$  is strictly decreasing in  $K$  on the relevant region because both  $R(B, K)$  and  $R(G, K)$  are strictly decreasing in  $K$  and  $c(B, K)/c(G, K)$  is also decreasing in  $K$  along the feasibility locus.<sup>2</sup> Hence, since  $F_B(K_B^{cs}) = 0$  and  $F_B(K_B^{pf}) < 0$ , we must have  $K_B^{cs} < K_B^{pf}$ .

*Step 2: show  $K_G^{cs} > K_G^{pf}$ .* Similarly, evaluate  $F_G$  at  $K_G^{pf}$ . Using  $1 = \beta R(G, K_G^{pf})$  and  $\pi_{GG} = 1 - \pi_{GB}$ ,

$$\begin{aligned} F_G(K_G^{pf}) &= \beta \left[ (1 - \pi_{GB}) R(G, K_G^{pf}) + \pi_{GB} \frac{c(G, K_G^{pf})}{c(B, K_G^{pf})} R(B, K_G^{pf}) \right] - 1 \\ &= \pi_{GB} \left[ \beta \frac{c(G, K_G^{pf})}{c(B, K_G^{pf})} R(B, K_G^{pf}) - 1 \right]. \end{aligned}$$

Thus  $F_G(K_G^{pf}) > 0$  is equivalent to

$$\beta \frac{c(G, K_G^{pf})}{c(B, K_G^{pf})} R(B, K_G^{pf}) > 1 \iff \frac{R(B, K_G^{pf})}{R(G, K_G^{pf})} > \frac{c(B, K_G^{pf})}{c(G, K_G^{pf})}.$$

But the argument above applied with  $(B, G)$  swapped gives, for any  $K$  with positive consumption,

$$\frac{R(B, K)}{R(G, K)} > \frac{B}{G} > \frac{c(B, K)}{c(G, K)}.$$

Hence  $F_G(K_G^{pf}) > 0$ . Since  $F_G$  is strictly decreasing in  $K$  and  $F_G(K_G^{cs}) = 0$ , we conclude  $K_G^{cs} > K_G^{pf}$ .

*Step 3: conclude the ordering and translate to consumption.* We have shown  $K_B^{cs} < K_B^{pf}$

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<sup>2</sup>This monotonicity is standard and can be verified by differentiation; it is also visually apparent in the  $(K, C)$  phase diagram.

and  $K_G^{pf} < K_G^{cs}$ , and already  $K_B^{pf} < K_G^{pf}$ , hence

$$K_B^{cs} < K_B^{pf} < K_G^{pf} < K_G^{cs}.$$

Finally, along the  $K$ -nullcline  $c(A, K) = AK^\alpha - \delta K$  is strictly increasing in  $A$  and (on the relevant region) increasing in  $K$ , so the same ordering carries over to consumption:

$$c_B^{cs} < c_B^{pf} < c_G^{pf} < c_G^{cs}.$$

■

## References

- Aiyagari, S. R. (1994). Uninsured Idiosyncratic Risk and Aggregate Saving. *The Quarterly Journal of Economics* 109(3), 659–684. Publisher: Oxford University Press.
- Stokey, N. L., R. E. Lucas, and E. C. Prescott (1989). *Recursive Methods in Economic Dynamics*. Harvard University Press.