# Global Nonlinear Solutions in the Sequence Space and the Generalized Transition Function

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# This paper

## Research question

How can we solve the nonlinear models under the aggregate uncertainty (with heterogeneous agents) *globally* and *accurately*?

What do we learn from the nonlinear heterogeneous-agent models?

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## This paper

develops a global solution framework in the sequence space.

introduces the generalized transition function (*GTF*): a versatile tool for nonlinear macro analysis.

studies key nonlinear implications from heterogeneous-household business cycle models with *endogenous labor supply*, *irreversible investment*, and *portfolio choice*.

- endogenous disaster/ Interaction between uncertainty and growth/ state-dependent fiscal multiplier
- heterogeneous portfolio adjustment over business cycle/ state-dependent risk premium dynamics

# Overview of the sequence-space global solution framework (RTM)

- solves nonlinear business cycle models accurately in the sequence space.
- global method, not requiring law of motion, without assuming perfect foresight.
- efficiently handles non-trivial market clearing conditions and occasionally binding constraints.
- provides theoretical foundations for the sufficient statistic approach.
- easy.
- recovers the full RCE, which immediately computes the GTF:
  - enables to fully utilize the equilibrium dynamics: state dependence within a unified RCE framework.
  - where are we on the path? It allows the analysis *away from the steady state*.

# Related papers

- **Solution methods for the heterogeneous (representative) agent models**:
  - Global method: Marcet (1988); Den Haan and Marcet (1990); Den Haan (1996); Krusell and Smith (1997,1998); Rios-Rull (1997); Den Haan and Rendahl (2010); Reiter (2010); Cao et al. (2023)
  - Policy function iteration: Coleman (1990); Judd (1992); Cao et al. (2023)
  - Perfect foresight: Fair and Taylor (1983); Juillard (1996); Judd (2002); Cai et al. (2017); Boppart et al. (2018); Auclert et al. (2021)
  - Approximation (fast): Reiter (2009); Childers (2018); Bayer and Luetticke (2019); Auclert et al. (2021); Gross and Hansen (2021)
  - Machine/Deep learning: Fernandez-Villaverde et al. (2021); Kahou et al. (2021); Han, Yang, and E (2022); Aznovic, Gaegauf, and Scheidegger (2022); Kase, Melosi, and Rottner (2024)
  - Simulation-based: Judd et al. (2011)
- **Short-run equilibrium dynamics**:
  - Andreasen et al., (2017); Petrosky-Nadeau and Zhang (2021)
  - Theoretical analysis: Cao (2020)
- **Nonlinear business cycle models / Heterogeneous portfolio choice**:
  - Christiano et al. (2011); Kaplan and Violante (2014); Guerrieri and Iacoviello (2015); Petrosky-Nadeau et al. (2018); Fernandez-Villaverde et al. (2021)
  - Den Haan (1996); Krusell and Smith (1997); Heaton and Lucas (2000); Gomes and Michaelides (2007); Calvet et al. (2009); Bayer et al. (2019); Luetticke (2021); Auclert et al. (2024,2025)

# A nonlinear global solution in the sequence space

# A generic model framework

We consider a problem where the *recursive formulation* (bounded lifetime return, *SLP* 1989) is as follows:

$$V(x;X) = \max_{y,a'} f(y,a',x;X) + \mathbb{E}q(X,X')V(a',s';X')$$
$$\text{s.t.} \quad (y,a') \in \mathcal{B}(x;X,X',m), \quad X' = M(X)$$

$$[\text{Individual state}]: \quad x = \{a, s\}$$
$$[\text{Aggregate state}]: \quad X = \{\Phi, S\}$$

Denote the price bundle $p = (m, q)$. The following market clearing condition pins down the price $p$:

$$[\text{Market clearing}]: \quad p(X,X') = \arg_{\tilde{p}}\{Q^D(\tilde{p},X,X') - Q^S(\tilde{p},X,X') = 0\},$$

<u>*Note*</u>: A usual household's dynamic problem is with $q(X,X') = \beta$. cf. A firm dynamic problem with SDF.

- For expositional clarity, I consider an exogenous Markov chain $S \in \{G, B\}$ that evolves by $\Gamma_{SS'}$.
- Define a *standard RCE* with individual optimality + market clearing + consistency.
- Assume: unique RCE + regularity conditions in SLP (1989) and Meyn & Tweedie (1993): (Feller property + Lyapunov drift condition + Irreducibility + Minorization) + Aperiodicity

# Sequence-space global solution method (RTM)

- It relies on the *Recursivity* of RCE: history repeats itself.
  - Harris recurrence + Aperiodicity $\implies$ Ergodicity.
  - Ergodicity: If a simulation path is long enough, the simulation path captures all possible aggregate state realizations, and then the realizations repeat themselves.

- In each period $t$, we need the conditional expectation: $\mathbb{E}V_{t+1} = \Gamma_{SG}V(\cdot;\Phi_{t+1},G) + \Gamma_{SB}V(\cdot;\Phi_{t+1},B)$. Suppose $S_{t+1} = G$.

- If the simulation path is long enough, there exists a period $\tilde{t}+1$ such that
  1. $\Phi_{t+1} = \Phi_{\tilde{t}+1}$.
  2. $S_{\tilde{t}+1} = B$.

- We can use $V_{\tilde{t}+1}$ to fill up the missing counterfactual value function at period $t+1$, $V(\cdot;\Phi_{t+1},B)$.

- The method relies on the *similarity* of the aggregate states across the periods and the recursive nature of RCE: history repeats itself.

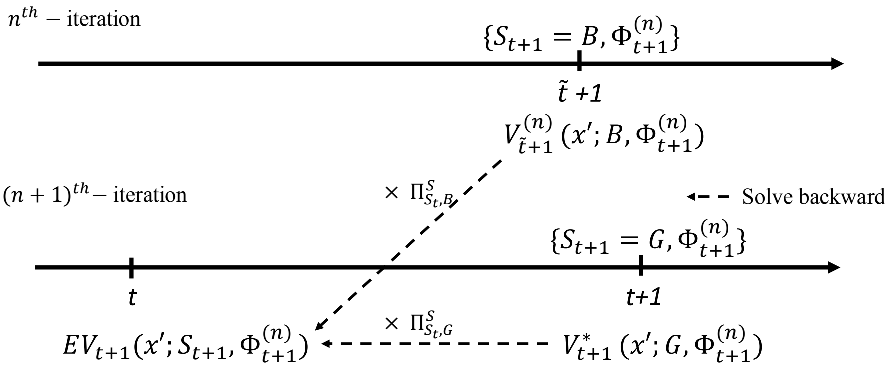



Figure: The computing step for conditional expectation based on the RTM

*Notes:* The conditional expectation in period $t$ can be computed by

$$\mathbb{E}V_{t+1}(x'; S_{t+1}, \Phi_{t+1}^{(n)}) = \Pi_{S_t,B}^{S} \times V_{\tilde{t}+1}^{(n)}(x'; B, \Phi_{t+1}^{(n)}) + \Pi_{S_t,G}^{S} \times V_{t+1}^{*}(x'; G, \Phi_{t+1}^{(n)}).$$

# Repeated transition method – implementation

**Step 1** **(Initialization)** Simulate a long exogenous state path $\{S_t\}_{t=0}^T$. Conjecture $\{V_t^{(0)}, \Phi_t^{(0)}, p_t^{(0)}\}_{t=0}^T$

**Step 2** **(Backward solution)** Starting from the terminal period $T$, solve agents' problems backward using expectations based on $\{V_t^{(n)}\}_{t=0}^T$. For each exogenous state in $t+1$:

(a) Find periods $\tau$ such that $\Phi_\tau^{(n)} \approx \Phi_{t+1}^{(n)}$ and collect realized state-contingent value functions.
(b) Use these values to construct expectations $E_t[V_{t+1}]$.

Then, obtain the optimal value function $V_t^*$ and the policy function $g_t^{a*}$.

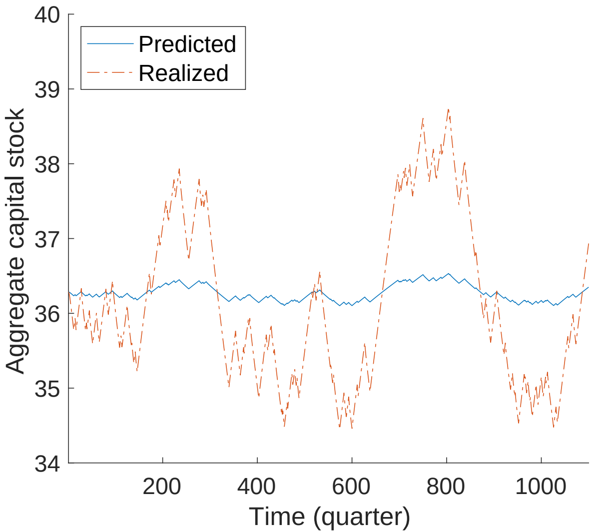
**Step 3** **(Forward simulation)** Simulate the model forward using $\{g_t^{a*}\}_{t=0}^T$ to generate $\{\Phi_t^*\}_{t=0}^T$ and *implied* prices $\{p_t^*\}_{t=0}^T$.

**Step 4** **(Update)** Update the guessed sequences via convex combination:

$$V_t^{(n+1)} = \lambda V_t^* + (1-\lambda) V_t^{(n)}, \quad \text{similarly for } \Phi_t,\ p_t.$$

**Step 5** **(Iteration)** Repeat Steps 2–4 until convergence:

$$\sup_t \left\| p_t^{(n+1)} - p_t^{(n)} \right\| < \varepsilon.$$



Predicted
Realized
Aggregate capital stock
40
39
38
37
36
35
34
200
400
600
800
1000
Time (quarter)

# A simple theory of sufficient statistics

# The theoretical ground for the sufficient statistic approach in RTM

## Definition 1 (Sufficient statistics)

Consider a function $e : \Omega \to \mathbb{R}^n$. An equilibrium object $e_t := e(\Phi_t)$ is a sufficient statistic if for $\mathcal{T}_S = \{t | S_t = S\}, \forall S \in \{B, G\}$,

$$e_t = e_{\tilde{t}} \implies V_t = V_{\tilde{t}}, \quad t, \tilde{t} \in \mathcal{T}_S. \tag{1}$$

- Two different sufficient statistics: *state space* vs. *sequence space*
  - *State space*: a variable that can literally replace or summarize $\Phi_t$ in the price determination or in the law of motion (Alvarez et al., 2016; Baley and Blanco, 2021).
  - *Sequence space*: an indexing variable where the same level of the variable indicates the periods with the same aggregate state.

$$\arg\min_{t} ||e_t - e_{\tilde{t}}|| = \arg\min_{t} ||\Phi_t - \Phi_{\tilde{t}}||, \tag{2}$$

- In the end, RHS is not too costly.
- When can we take LHS?

# Monotonicity condition

**Proposition 1 (The monotonicity condition for a univariate sufficient statistic)**

*For each time partition $\mathcal{T}_S = \{t|S_t = S\}$, $S \in \{B, G\}$, if $e_t$ is strictly monotone in $V_t$ for $\forall t \in \mathcal{T}_S$ and all individual states $(a, s)$, then, $e_t$ is a sufficient statistic.*

- If $e_t$ conditionally ranks the value functions, it's a sufficient statistic.
  - the ranking information = the target period's location
- A natural candidate for $e_t$: the first moment (a necessary condition for the similarity).
- The monotonicity is tested using the Spearman's coefficient.
- Proposition 1 can be extended to a multivariate sufficient statistic based on the monotonicity within the partitioned state space. ▸ extension

# Taming curse of dimensionality

Why is the RTM particularly efficient for taming curse of dimensionality?

$$[\text{Aggregate state}]: \quad X = \{\Phi, S\} \tag{3}$$

- State space: As $X$'s dimensionality increases, the value function $V$ and policy function $g_a$ needs to accommodate a greater dimensions.
- Sequence space: As $X$'s dimensionality increases, the value function $V$ and policy function $g_a$'s dimensions do not need to increase. It's just labeled by time index $t$: No curse of dimensionality.

Thus, the model can be **flexibly** solved based on different bounded rationality setup: compare the similarity based on a few sufficient statistics vs. full distribution.

- It can systematically test the existence of the "*self-fulfilling multiple equilibria*" (Krusell and Smith, 2006; Cozzi, 2015).
- I've found no evidence for the model of Krusell and Smith (1998).

# Non-trivial market clearing conditions in RTM
(subject to time constraint)

Another computational hurdle in the literature: "non-trivial market clearing conditions."
Let's revisit the market clearing condition:

$$Q^D(p_t, X_t, X_{t+1}) - Q^S(p_t, X_t, X_{t+1}) = 0.$$
$$p_t := \arg_{\tilde{p}}\{Q^D(\tilde{p}, X_t, X_{t+1}) - Q^S(\tilde{p}, X_t, X_{t+1}) = 0\}.$$

where $Q^D$ and $Q^S$ are demand and supply functions; $p_t$ is the market clearing price; $X_t$ and $X_{t+1}$ are the current and future aggregate states. (dependence on $X_{t+1}$ is *optional*.)

- **why?** *non-trivial aggregation* of demand or supply functions (or both) leads to *non-invertibility*.
- **how to handle?** an internal loop to clear the market in each period to get $p_t^*$.
  - a significant computational bottle neck. What if two or three non-trivial markets?

# RTM efficiently handles the non-trivial market clearing conditon

- The RTM utilizes the implied price $p_t^*$ instead of $p_t$, where

$$p_t^* := \arg_{\tilde{p}}\{Q^D(p_t^{(n)}, X_t, X_{t+1}) - Q^S(\tilde{p}, X_t, X_{t+1}) = 0\} \text{ or}$$
$$:= \arg_{\tilde{p}}\{Q^D(\tilde{p}, X_t, X_{t+1}) - Q^S(p_t^{(n)}, X_t, X_{t+1}) = 0\}.$$

- Note that this is *NOT* a fixed-point problem: the market is not cleared.
- However, as iteration goes by with the gradual updates, the predicted path of prices $\{p_t^{(n)}\}_{t=0}^T$ converges to the equilibrium prices $\{p_t\}_{t=0}^T$.

$$\lim_{n\to\infty} Q^D(p_t^{(n)}, X_t, X_{t+1}) - Q^S(p_t^*, X_t, X_{t+1}) = 0$$
$$\implies Q^D(\lim_{n\to\infty} p_t^{(n)}, X_t, X_{t+1}) - Q^S(p_t^*, X_t, X_{t+1}) = 0$$
$$\implies Q^D(p_t, X_t, X_{t+1}) - Q^S(p_t^*, X_t, X_{t+1}) = 0$$
$$\implies p_t = p_t^* \quad (\because \text{uniqueness of the equilibrium}).$$

# The Generalized Transition Function (GTF)

# The generalized transition function (GTF)

The RCE offers a *powerful* conceptual framework, but it has traditionally been difficult to leverage this object. How can we fully utilize the RCE?

**Definition 2 (Generalized transition function)**

Given an aggregate state realization $(\Phi_0, S_0)$ in the RCE, the generalized transition function $g$ of the variable of interest $v$ as follows:

$$g(v; \Phi_0, S_0) = \int v(x; \Phi_j, S_j) d\Phi_j, \quad S_j \sim \Gamma^j(S_j; S_0), \quad j \geq 1 \tag{4}$$

where $S_j$ is a random variable of the exogenous aggregate state, which follows a j-length Markov chain $\Gamma^j$ from the initial realization of $S_0$.

- The global solution already includes all the possible GTFs: the computation is immediate.
- The GTF nests GIRF (Koop et al., 1996; Andreasen et al., 2017) and stochastic growth path (Justiniano and Primiceri, 2008; Hansen et al., 2008). ▸ GIRF
- An extended version-$\Phi_0$ not belonging to RCE: Equilibrium exists (Cao, 2020). I assume an immediate jump to the *nearest* $\Phi^{RCE}$, like required by Ramsey-Kass-Coopmans model's TVC.

# What do we learn from the nonlinear heterogeneous-agent models?

## – Leading applications

## Households

***Heterogeneous*** households consume, save, and *endogenously* supply labor.

***(i) The irreversible investment constraint is occasionally binding: Illiquid nature***.

Krusell and Smith (1998) extended by endogenous labor supply

$+$ Guerrieri and Iacoviello (2015)

$+$ Multiple aggregate shocks (TFP & Government demand)

***(ii) Households form portfolios between risky and riskless assets over the business cycle***.

Krusell and Smith (1997) extended by endogenous labor supply

## Firm

A production sector operates using a CRS Cobb-Douglas production function.

## Competitive market

***A non-trivial labor and bond market clearing conditions***

# Recursive formulation

- A heterogeneous household model's recursive form:

$$V(x;X) = \max_{c,n,x'} \; log(c) - \frac{\eta}{1+\frac{1}{\chi}} n^{1+\frac{1}{\chi}} + \beta \mathbb{E} V(x';X') \tag{5}$$

$$\text{s.t.} \quad (c,n,x') \in \mathcal{B}(x;X,X',m) \tag{6}$$

$$\Phi' = \Gamma_X(X) \quad \text{(Aggregate law of motion)}$$

$$S' \sim \pi(S'|S), \quad z' \sim \pi(z'|z)$$

(Application I)

$$x = [a,z], \quad X = [\Phi, A, G] \tag{7}$$

$$c + a' = (1 + r(X))a + w(X)zn - T(X) \tag{8}$$

$$a' - (1-\delta)a \geq \phi I^{ss} \tag{9}$$

(Application II)

$$x = [a,b,z], \quad X = [\Phi, A] \tag{10}$$

$$c + a' + q^b(X)b' = (1 + r(X))a + b + w(X)zn \tag{11}$$

$$a' \geq 0, \quad b' \geq \underline{b} \tag{12}$$

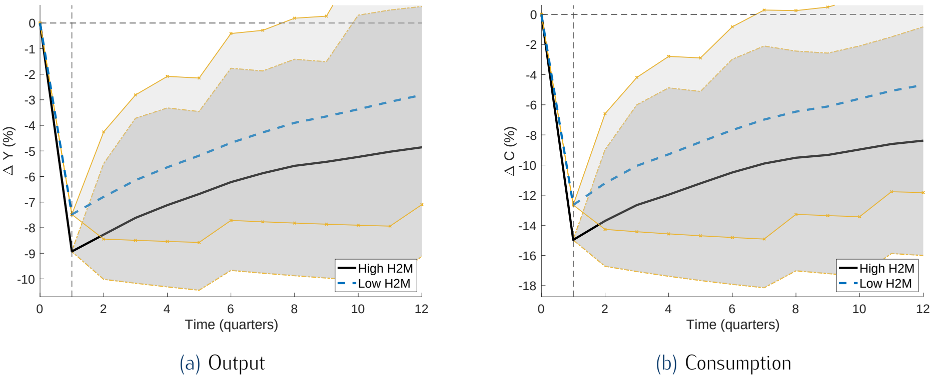



Figure: State-dependent responsiveness: high vs low hand-to-mouth portion

*Notes:* Panel (a) plots the generalized impulse response functions (GIRF) of output when the aggregate states before the shock hits were with a high portion hand-to-mouth households (solid line) and with a low portion of hand-to-mouth households (dashed line). Panel (b) plots the GIRFs of consumption. The shaded areas indicate the 95% confidence interval.

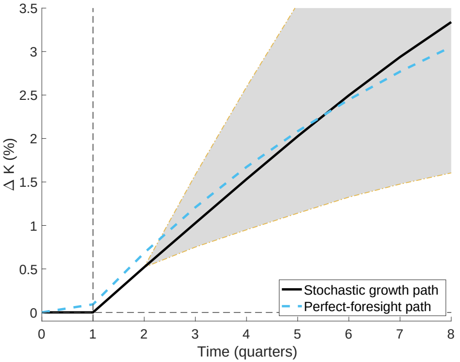



Figure: Stochastic growth path vs. perfect-foresight growth path

*Notes:* The figure plots the GTF-based stochastic growth path (solid line) in comparison with the perfect-foresight growth path (dashed line). The shaded areas indicate the 95% confidence interval. The transition is initiated from the steady state of an economy with a 5% lower aggregate TFP productivity.

$$Y_t = \beta_0 + \beta_1 G_t + \beta_2 G_t \times \Lambda_t + \beta_3 log(K_t) + \beta_3 log(A_t) + \epsilon_t$$

Table: State-dependent fiscal spending multipliers

| | Dependent variable: $Y_t$ ($) | | | |
|---|---|---|---|---|
| | Hetero. (HH) | | Rep. (RH) | GHH |
| $G_t$ ($) | 0.402 | 0.182 | 0.206 | 0.000 |
| | (0.005) | (0.002) | (0.000) | (0.001) |
| $G_t$ ($) $\times \Lambda_t$ | | 0.533 | 0.534 | 0.000 |
| | | (0.003) | (0.002) | (0.000) |
| Constant | Yes | Yes | Yes | Yes |
| Observations | 3,000 | 3,000 | 3,000 | 3,000 |
| $R^2$ | 0.992 | 0.999 | 0.999 | 0.999 |
| Adjusted $R^2$ | 0.992 | 0.999 | 0.999 | 0.999 |

*Notes:* The table reports the regression results based on the specification above. The first two columns are results based on the heterogeneous household baseline model. The next column is based on the representative-household counterpart. The last column is based on the representative household model with GHH utility, where the wealth effect is muted.

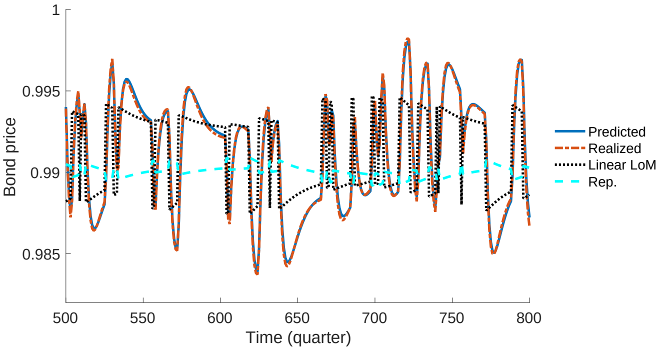



Figure: The equilibrium bond price path

*Notes:* The figure plots the time series of the bond price $q_t^b$ in the extended model of Krusell and Smith (1997). The solid line is the predicted bond price ($n^{th}$ guess) $\{q_t^{b(n)}\}_{t=500}^{800}$. The dashed line is the implied bond price $\{q_t^{b*}\}_{t=500}^{800}$. The dotted line is the bond price predicted by the linear law of motion.

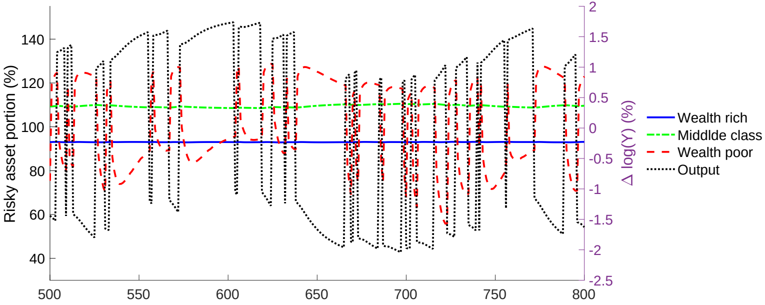
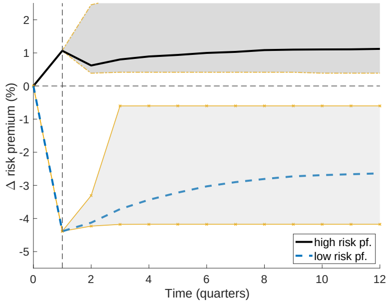
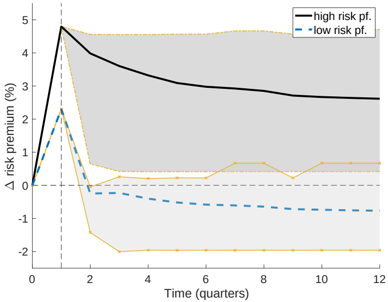



Figure: The time series of average risky asset portion across different wealth groups

*Notes:* The figure plots the time series of the risky asset portion (%) in the wealth portfolio for different households in the extended model of Krusell and Smith (1997). The solid line represents households in the top wealth tercile, while the dashed and the dash-dotted lines show households in the bottom wealth tercile and the remaining middle class. The dotted line depicts output (measured as percentage deviation from steady state), with values shown on the secondary vertical axis at the right side of the figure.




(a) Negative TFP shock




(b) Positive TFP shock

Figure: State-dependent risk premium dynamics

*Notes:* The figure plots the GTFs of the risk premium for a negative (panel (a)) and positive (panel (b)) 2% TFP shocks. The solid line represents the economy with the highest portion of risky asset before the shock hits and the dashed line does it for the economy with the lowest portion of risky asset.

# Other applications with sample codes

- Sample codes for 20+ models are available, and more are coming soon.

  **Representative-agent RBC models**

  - Representative RBC models with adjustment costs, asset price, Frisch-driven labor supply, etc.

  **Heterogenenos-agent RBC models**

  - Krusell and Smith (1997;1998) and its variants
  - Khan and Thomas (2008) and its variants
  - Heterogeneous-agent RBC models with firm-level frictions
  - Uncertainty shocks

  **SaM models**

  - A canonical DMP model with exogenous separation
  - [*Coming soon*] A DMP model with endogenous separation (joint with Francesco Zanetti and Philip Schnattinger)

  **NK models**

  - A canonical NK model with three shocks (Rotemberg)
  - [*Coming soon*] A canonical NK model with ZLB (joint with Kao Nomura)

# Conclusion

# Conclusion

- The *global nonlinear solution in the sequence space* efficiently solves DSGE models accurately utilizing RCE's recursivity, uniqueness, and stability.
- It proposes an effective way of taming curse of dimensionality.
- The GTF enables to fully utilize powerful equilibrium framework RCE.
- The global nonlinear solution framework reveals novel nonlinear implications:
  - Endogenous disaster
  - Interaction between uncertainty and growth
  - State-dependent fiscal multiplier and risk premium dynamics
  - Heterogeneous portfolio adjustment over the business cycle

# Appendix

"What drives the business cycles?"

"What drives the business cycles?"

Define the aggregate state $S := (\Phi, A)$, where $\Phi$ is endogenous, and $A$ is exogenous. Suppose the variable of interest is $X = X(\Phi, A)$.

- State dependence

$$\frac{\partial X}{\partial A} = G(\Phi, A)$$

- Nonlinear propagation

$$\frac{\partial X}{\partial A} = G(\Phi, A)$$

# AN RBC MODEL WITH INVESTMENT IRREVERSIBILITY ▸ back

The representative household solves the following problem:

$$V(a;S) = \max_{c,a'} \frac{c^{1-\sigma}}{1-\sigma} + \beta \mathbb{E} V(a';S')$$
$$\text{s.t. } c + a' - (1-\delta)a = Aa^{\alpha}$$
$$a' - (1-\delta)a \geq \phi I_{ss}$$

where $I_{ss}$ is the steady-state investment level. The aggregate state $S$ is as follows

$$S = [K, A].$$

$K$ is the aggregate capital stock. $A$ is TFP that follows the log AR(1) process:

$$log(A') = \rho log(A) + \sigma\epsilon, \quad \sigma \sim N(0,1).$$

$c$ is consumption, $a$ is the wealth in the beginning of a period. $\phi$ is the parameter that governs the degree of the irreversibility.

Each firm with an individual state $(k, z)$ solves the following problem:

$$J(k,z;X) = \max_{k'} \pi(k,z;X) + (1-\delta)k - k' + \mathbb{E}_{z,X} M(X,X') J(k',z';X')$$
$$\text{s.t. } k' \geq \phi I_{ss} + (1-\delta)k$$
$$\pi(k,z;X) = \max_{k,n} \; Ak^{\alpha} n^{\gamma} - w(X)n$$

where $k$ is individual capital stock; $z$ is the idiosyncratic productivity; $n$ is the labor demand; $I_{ss}$ is the steady-state aggregate investment level.
The household-side problem is as follows:

$$V(a;X) = \max_{c,a',N} \; log(c) - \eta N + \beta \mathbb{E} V(a';X')$$
$$\text{s.t.} \quad c + \int a'(X') q(X,X') d\Gamma_{X'} = a + w(X)N$$

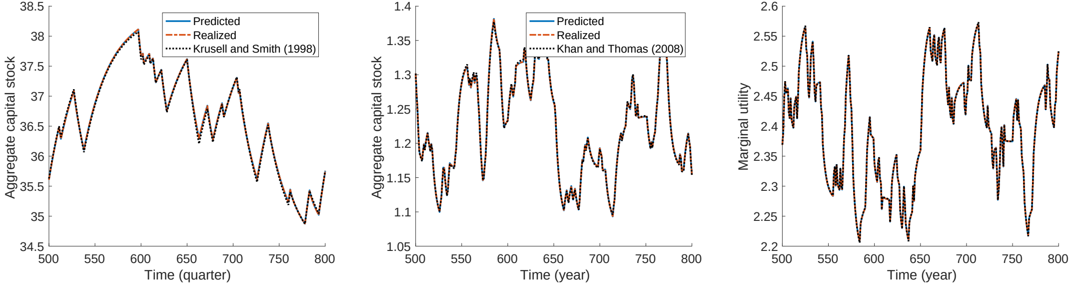
where $q$ is the stochastic discount factor; $a'$ is the future equity portfolio.

# Comparison

- The model is a representative-agent RBC model with irreversible investment (occ. bin. const.).

| | RTM | GDSGE | OccBin | Linear |
|---|---|---|---|---|
| **Accuracy** | | | | |
| $max(\|Error_t\|)$ (% of steady-state $K$) | 0.003 | 0.735 | 1.317 | 2.019 |
| $\sqrt{mean(Error_t^2)}$ (% of steady-state $K$) | 0.001 | 0.025 | 0.217 | 0.559 |
| $max(\|EE_t\|)$ (% of contemp. $C_t$) | 0.014 | 0.057 | 2.854 | 2.323 |
| $\sqrt{mean(EE_t^2)}$ (% of contemp. $C_t$) | 0.002 | 0.059 | 0.775 | 0.707 |
| **Business cycle stat.** | | | | |
| $mean(I)$ | 0.363 | 0.363 | 0.365 | 0.363 |
| $mean(C)$ | 1.166 | 1.166 | 1.164 | 1.160 |
| $vol(I)$ | 0.022 | 0.022 | 0.023 | 0.023 |
| $vol(C)$ | 0.052 | 0.052 | 0.052 | 0.052 |
| $skewness(I)$ | 1.363 | 1.320 | 1.307 | 1.407 |
| $skewness(C)$ | -0.225 | -0.213 | -0.322 | -0.095 |
| $kurtosis(I)$ | 4.447 | 4.578 | 4.513 | 4.255 |
| $kurtosis(C)$ | 2.776 | 2.546 | 2.858 | 2.796 |

# Comparison with other methods II




(a) Krusell and Smith (1998): $K_t$ (b) Khan and Thomas (2008): $K_t$ (c) Khan and Thomas (2008): $p_t (= 1/C_t)$

Figure: Dynamically consistent equilibrium dynamics in heterogeneous-agent models

- The accuracy is the same as the log-linear state-space approach. ($R^2 = 0.9999$)
- For Khan and Thomas (2008), the speed gain at the non-trivial market clearing condition is substantial: more than 10 times faster.

*What if an aggregate shock is relevant for a part of the economy? or differently relevant?*

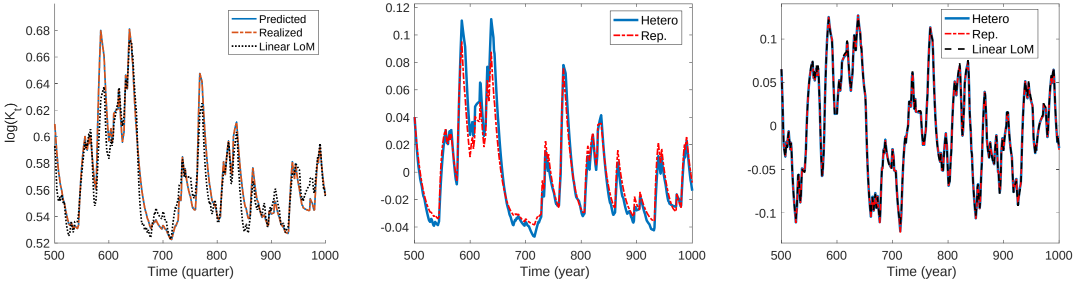
– e.g., *Tariffs*. Energy-saving technologies.

Let $\mathcal{X}$ denote the set of all individual states $x = (a, s)$, and let $\{\mathcal{P}_j\}_{j=1}^n$ be a cross-sectional partition of $\mathcal{X}$ such that $\cup \mathcal{P}_j = \mathcal{X}$ and $\mathcal{P}_j$'s are pairwise disjoint.

## Proposition 2 (The qualification for the multivariate sufficient statistic)

*For each time partition $\mathcal{T}_S = \{t | S_t = S\}$ $S \in \{B, G\}$ and for cross-sectional partition $\mathcal{P}_j$, $j \in \{1, 2, \dots, n\}$, if $e_t^j$ is strictly monotone in $V_t$ for $\forall t \in \mathcal{T}_S$ and $\forall (a, s) \in \mathcal{P}_t^j$, then, $e_t = (e_t^j)_{j=1}^n \in \mathbb{R}^n$ is a sufficient statistic.*

- In the end, the RTM is about *clustering* periods with *similar* aggregate states.
- Sequence space can *efficiently* **tame curse of dimensionality**




(a) Baseline (b) Hetero vs. Rep. (c) Without the constraint

Figure: Equilibrium capital dynamics

(a): The solution is dynamically consistent while significantly deviates from log-linear LoM.

(b): Both hetero. and rep. models are differently nonlinear.

– Hetero. model features a greater volatility and a greater skewness.

(c): When the occ. bin. constraint is lifted, the perfect log-linear representation holds.

- The fiscal spending multiplier is state-dependent: nonlinear ***MPC*** channel

## Definition 3 (A refined generalized impulse response function)

Given an aggregate state realization $(\Phi_0, S_0)$ in the RCE, the refined generalized impulse response $g^{girf}$ of the variable of interest $v$ to an exogenous state realization $S_1$ in the following period is as follows:

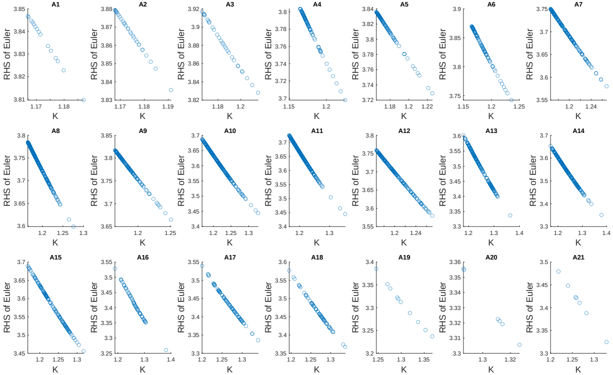
$$g^{girf}(v; \Phi_0, S_0, S_1, 1) = \int v(x; \Phi_1, S_1) d\Phi_1 \tag{13}$$

$$g^{girf}(v; \Phi_0, S_0, S_1, j) = \int v(x; \Phi_j, S_j) d\Phi_j, \quad S_j \sim \Gamma^j(S_j; S_1), \quad j > 1 \tag{14}$$

where $S_j$ is a random variable of the exogenous aggregate state, which follows a j-length Markov chain from the initial realization of $S_1$.

- Notably, $\mathcal{G} := \{g_1, g_2, \dots\}$ is a sub-path of the RCE, where $g_j$ is a GTF for $S_1$.
- The first component of the $g^{girf}$ is deterministic upon the impact of the aggregate shock, as the magnitude of the shock $|S_1 - S_0|$ is set by a researcher from choosing $S_1$.
- The average path with the 95% CIs can be characterized based on simulated shock paths.

- Despite the nonlinearity, the strict monotonicity still holds: $K_t$ qualifies as a sufficient statistic.
  - Vertical axes: expected policy functions, horizontal axes: $K_t$.




  - A similar monotonicity holds for models with two endogenous aggregate states.

The bond market presents a unique computational challenge:

x

- The clearing condition reduces to a non-invertible identity: $q^b(X) \times B'(X) = B \iff q^b(X) \times 0 = 0$.
- It becomes clear from the national accounting identity:

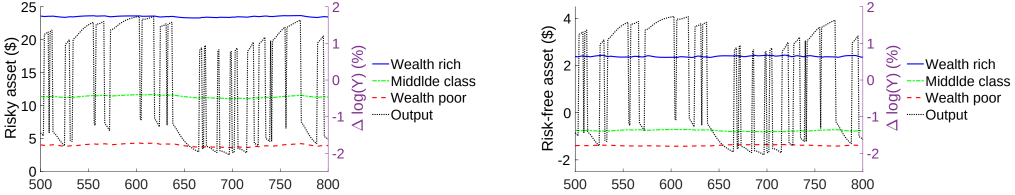
$$C(X) + K'(X) + q^b(X)B'(X) = K(1 + r(X)) + B + w(X)N \tag{15}$$

$$\iff C(X) + I(X) + q^b(X)B' - B = Y(X) \tag{16}$$

$$\implies q^b(X)B' = B \iff q^b(X) \times 0 = 0, \tag{17}$$

- I introduce a dummy bond variable trick to handle this problem:

$$q_t^{b*}\overline{B} := Y_t^{(n)} - C_t^* - I_t^* + B_t^* - q_t^{b(n)}\overline{B} \implies q_t^{b*} = \frac{Y_t^{(n)} - C_t^* - I_t^* + B_t^* - q_t^{b(n)}\overline{B}}{\overline{B}}, \tag{18}$$




(a) Risky asset dynamics

(b) Risk-free asset dynamics

Figure: Risky and risk-free asset holding dynamics across different wealth groups

*Notes:* The figure plots the time series of the different asset holdings ($) by household types in the extended model of Krusell and Smith (1997). Panel (a) is for the risky asset, and panel (b) is for the risk-free asset. The solid line represents households in the top productivity tercile, while the dashed line shows households in the bottom productivity tercile. The dotted line depicts output (measured as percentage deviation from steady state), with values shown on the secondary vertical axis at the right side of the figure.

# DMP MODELS AND NK MODELS
(subject to time constraint)

# The key inter-termporal equations

The key inter-temporal optimality conditions are as follows:

- [DMP] The vacancy posting decision:

$$\frac{\kappa}{q(S)} = (1-\lambda)\beta\mathbb{E}\left[\left(\frac{c(S)}{c(S')}\right)^{\sigma}\left(z' - w(S') + \frac{\kappa}{q(S')}\right)\right]$$

- [NK-Rotemberg] The price adjustment decision (NKPC):

$$\begin{aligned}&\epsilon - 1 \\ &= \epsilon mc(S) - \psi(1+\pi(S))(\pi(S)-\overline{\pi}) + \beta\psi\mathbb{E}\left[\left(\frac{c(S)}{c(S')}\right)^{\sigma}(1+\pi(S'))(\pi(S')-\overline{\pi})\frac{Y(S')}{Y(S)}\right]\end{aligned}$$

- The RTM sharply computes the conditional expectations.