Dancing on the Saddles: A Geometric Framework for Stochastic Equilibrium Dynamics[†]

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Abstract

This paper extends deterministic saddle-path analysis to stochastic environments by introducing *conditional saddle paths*: the equilibrium path under frozen exogenous states. This concept yields a global geometric representation of stochastic equilibrium dynamics, in which equilibrium fluctuations decompose into movements *along* (endogenous propagation) and *across* (exogenous state transitions) conditional saddle paths. The framework delivers two theoretical results. First, state-dependent impulse responses arise from differences in the slopes of conditional saddle paths. Second, if an aggregate equilibrium variable varies strictly monotonically along conditional saddle paths, it uniquely indexes equilibrium states and thus provides an exact one-dimensional sufficient statistic. Applying this result, I prove that aggregate capital is a sufficient statistic in a canonical heterogeneous-household model (Krusell and Smith, 1998).

Keywords: Conditional saddle path, business cycles, state-dependent dynamics, sufficient statistics, heterogeneous agents.

JEL codes: C62, D31, E32.

1 Introduction

Understanding equilibrium dynamics in models with aggregate uncertainty remains a central challenge in macroeconomics. Unlike deterministic models, where saddle-path diagrams provide immediate geometric intuition, stochastic equilibrium models lack comparable geometric frameworks. This makes it difficult to develop intuition about how economies respond to shocks, how different states interact, and why certain computational methods work. The challenge becomes particularly acute in heterogeneous-agent models, where the natural state variable is an infinite-dimensional wealth distribution—yet computational work routinely achieves dimension reduction to low-dimensional aggregates. Can we extend geometric saddle-path analysis to stochastic environments? What does such a framework reveal about equilibrium dynamics, state-dependent responses, and the success of computational approximations?

This paper makes two contributions. First, it extends the saddle-path analysis of equilibrium dynamics from deterministic models to stochastic environments by introducing the notion of *conditional saddle path*: equilibrium trajectories of the regime-frozen economy (holding the exogenous state fixed at a level). This geometric object decomposes business-cycle fluctuations into movements *along* a conditional saddle (endogenous propagation) and *across* conditional saddles (transitions when the exogenous state changes), providing a unified framework for analyzing state-dependent equilibrium dynamics. The framework enables visualization of generalized impulse responses and nonlinear transition dynamics in phase diagrams, analogous to how saddle-path diagrams illuminate deterministic models. Based on this geometric framework, I establish that state-dependent shock responses arise essentially from differences in the local slopes along the conditional saddle paths.

Second, it provides a general dimension-reduction theorem on the conditional saddle paths. If an aggregate equilibrium variable is strictly monotone and convergent along the conditional saddle, then it is injective on the invariant equilibrium set and hence uniquely indexes all equilibrium allocations and prices. As a consequence, the equilibrium can be represented *exactly*—not merely approximately—with a one-dimensional sufficient statistic. Applying this theoretical result, I establish that aggregate capital is a sufficient statistic for canonical heterogeneous-agent models (Krusell and Smith, 1998). This provides a geometric foundation for the practical success of scalar state approximations in business-cycle applications and delivers economically

interpretable sufficient conditions for exact one-dimensional representations on the relevant invariant set.

These results extend beyond the canonical one-asset setting. In heterogeneous-agent models with multiple endogenous state variables, such as risky and riskless assets, liquid and illiquid assets, domestic and foreign bonds, the same logic applies (Krusell and Smith, 1997; Mendoza, 2010; Khan and Thomas, 2013; Kaplan and Violante, 2014; Berger and Vavra, 2015; Kaplan et al., 2018): whenever a monotone-convergent aggregate coordinate exists along conditional saddle paths, equilibrium dynamics remain exactly traceable by a scalar index.

Conditional saddle paths are equilibrium paths defined under frozen exogenous aggregate states: they describe counterfactual equilibrium continuations — how the economy would evolve if the aggregate state were held fixed at a given value — computed under the same decision rules that govern the stochastic equilibrium with regime switching. These counterfactual paths are economically meaningful because equilibrium decisions internalize the possibility of future regime changes, and the frozen-regime objects formalize the corresponding equilibrium "thought experiments."

Conditional saddle paths are closely connected to the random dynamical systems (RDS) notion of invariant manifolds for dynamics under exogenous forcing (Arnold, 1998; Schenk-Hoppé, 2001). In the present setting, the forcing is the aggregate Markov state, and a recursive competitive equilibrium (RCE) implies a time-homogeneous endogenous law of motion; conditional saddle paths are the discrete counterpart of the invariant manifolds from the frozen-regime equilibrium maps. Both perspectives organize stochastic dynamics using invariant geometric objects rather than local linearizations. The distinction in this paper is therefore not the mathematics of invariance, but the equilibrium discipline: the invariant-manifold structure is pinned down by optimality, market clearing, and equilibrium consistency, and it is precisely what makes global transition functions and impulse responses well defined and comparable across aggregate states within the recursive equilibrium framework.

Related literature This paper contributes to three strands of literature in macroeconomics. The first is the literature studying global equilibrium dynamics and solution methods under aggregate uncertainty. The challenge of characterizing equilibrium dynamics in models with aggregate shocks has motivated extensive methodological development. Marcet (1988), Den Haan and Marcet (1990) and Krusell and

Smith (1998) pioneered the use of bounded rationality approximations. In particular, Krusell and Smith (1998) discovered that a simple linear forecasting rule in aggregate capital achieves remarkable accuracy ($R^2 > 0.9999$) despite the infinite-dimensional state space. Thereafter, the literature has dramatically developed to sharpen the accuracy and improve the computational efficiency by incorporating moment-based approximations, exact aggregation, functional approximations, sequence-space approaches, and machine/deep learning (Den Haan, 1996, 1997; Reiter, 2001; Algan et al., 2008, 2010; Den Haan and Rendahl, 2010; Reiter, 2010; Ahn et al., 2018; Boppart et al., 2018; Elenev et al., 2021; Auclert et al., 2021; Cao et al., 2023; Azinovic et al., 2022; Fernández-Villaverde et al., 2023; Han et al., 2025; Payne et al., 2025).

This paper differs from these computational contributions by providing geometric foundations for *why* dimension reduction works. Rather than developing new algorithms, I introduce conditional saddle paths as a geometric framework for understanding equilibrium dynamics — analogous to how saddle-path diagrams provide intuition in deterministic models. The framework reveals that aggregate capital's sufficiency in a canonical heterogeneous-household model follows from geometric properties (nullcline invariance and monotonicity) rather than numerical happenstance.

The exact sufficiency result also has computational implications. Lee (2025) develops a repeated transition method that constructs conditional expectations for individual-level problems by identifying periods in simulations, where aggregate states are similar, enabling reuse of computed transitions across these similar states. The method requires a metric for determining when aggregate states are "close enough" to pool. My sufficiency result establishes that aggregate capital distance provides a theoretically justified metric — agents' problems are identical whenever capital stocks coincide, regardless of distributional differences. This validates distance-based pooling strategies and enables efficient implementation without requiring explicit distributional tracking.

This paper builds on the traditional use of geometric methods to analyze economic dynamics in deterministic environments. Phase-diagram analyses of the Solow–Swan (Solow, 1956; Swan, 1956) and Ramsey–Cass–Koopmans (Ramsey, 1928; Cass, 1965; Koopmans, 1963) growth models provide foundational intuition about convergence and stability. I extend this geometric approach to stochastic environments by introducing conditional saddle paths. This framework offers a geometric interpretation of stochastic equilibrium dynamics, including nonlinear and state-dependent impulse

responses. In particular, it provides a useful visual tool for understanding how microfounded frictions generate state-dependent dynamics, as documented in the recent literature (Kaplan and Violante, 2014; Vavra, 2014; Berger and Vavra, 2015; Basu and Bundick, 2017; Bloom et al., 2018; Kaplan et al., 2018; Petrosky-Nadeau et al., 2018; Baley and Blanco, 2019; Pizzinelli et al., 2020; Berger et al., 2021; Melcangi, 2024; Winberry, 2021; Lee, 2026).

Finally, in mathematics, the literature on random dynamical systems provides foundational tools for geometric analyses of stochastic dynamic processes. However, its focus has not been on history-invariant saddle paths of the type that arise in recursive competitive equilibrium (Arnold, 1998; Schenk-Hoppé, 1998; Schenk-Hoppé, 2001).[1] For example, Yannacopoulos (2011) introduces the notion of stochastic saddle paths, which are conceptually distinct from conditional saddle paths in that they vary with the realized history of shocks.[2]

By contrast, conditional saddle paths furnish economists with a geometric representation of stochastic equilibrium dynamics that is directly analogous to the role of phase diagrams in deterministic models. While this framework relies on standard regularity conditions to ensure well-defined and bounded equilibrium paths (Kamihigashi, 2003, 2005), the emphasis here is not on establishing existence results, but rather on providing geometric tools for understanding and analyzing stochastic equilibrium dynamics.

2 Conditional saddle path

2.1 Definitions and assumptions

I consider a generic dynamic stochastic model where the corresponding recursive competitive equilibrium (RCE) is characterized by the following aggregate state S and the endogenous and exogenous law of motions ($\Gamma_{endo}, \Gamma_{exo}$):

$$S = [\Phi, A] \tag{1}$$

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[1]In the language of Random Dynamical Systems, the conditional steady state corresponds to a deterministic realization of the random fixed point (Schenk-Hoppé and Schmalfuß, 2001), while the conditional saddle path represents the invariant manifold of the frozen-regime dynamics (Arnold, 1998).

[2]Stochastic saddle paths depend on the specific sequence of past realizations, whereas conditional saddle paths are invariant to history.

where Φ is the endogenous aggregate state variable, and A is the exogenous aggregate state variable. The latter admits a multivariate vector that follows a stochastic process. I assume the exogenous aggregate law of motion Γ_{exo} is a Markov chain. For simplicity in the illustration, I assume Γ_{exo} is a two-state Markov chain where $A \in \{B, G\}$, and $\Gamma_{exo}(A'|A) > 0$ for $\forall A', A$.

I consider a distributional state space $\mathcal{X}$, whose elements $\Phi \in \mathcal{X}$ summarize the cross-sectional distribution of idiosyncratic household states that are payoff-relevant for equilibrium (e.g., assets and employment/productivity types) together with any endogenous objects needed to evaluate equilibrium decision rules. Formally, one can take $\mathcal{X}$ to be a subset of a metric space of probability measures augmented, if needed, by a finite-dimensional vector of aggregate variables. Throughout, $\Gamma_{\text{endo}}(\cdot, A) : \mathcal{X} \to \mathcal{X}$ denotes the recursive equilibrium law of motion mapping the current distributional state into the next-period state under frozen regime A.

Definition 1 (Conditional saddle path).
Fix a regime $A \in \{B, G\}$ and an initial state $\Phi_0 \in \mathcal{X}$. Let $\{\Phi_t\}_{t \geq 0}$ denote the frozen-regime continuation under A, defined recursively by

$$\Phi_{t+1} = \Gamma_{\text{endo}}(\Phi_t, A), \qquad t \geq 0. \tag{2}$$

The conditional saddle path under A from Φ_0 is defined as the closure of the frozen-regime continuation:

$$\mathcal{M}(A; \Phi_0) := \overline{\{\Phi_t : t \geq 0\}} \tag{3}$$

.

Based on the conditional saddle path, conditional steady state can be defined as follows:

Definition 2 (Conditional steady state).
Given (Φ_0, A_0), if the frozen-regime continuation $\{\Phi_t\}_{t \geq 0}$ converges, we denote its limit by

$$\Phi^{cs}(A; \Phi_0) := \lim_{t \to \infty} \Phi_t. \tag{4}$$

Definition 1 defines the conditional saddle path directly from the frozen-regime

continuation. Fixing A makes the equilibrium law of motion deterministic on the distribution space: starting from Φ_0, the sequence $\{\Phi_t\}_{t\geq 0}$ is generated by repeated application of the endogenous transition operator $\Gamma_{\text{endo}}(\cdot, A)$. In this sense, $\mathcal{M}(A; \Phi_0)$ is the natural analogue of the saddle arm in deterministic saddle-path analysis: it isolates the equilibrium states visited (and their limit points) along the economically relevant convergence dynamics under regime A.[3]

Assumption 1 (Regularity of conditional saddle paths).
Fix a regime $A \in \{B, G\}$ and an initial state $\Phi_0 \in \mathcal{X}$. Let $\{\Phi_t\}_{t\geq 0}$ be defined by $\Phi_{t+1} = \Gamma_{\text{endo}}(\Phi_t, A)$.

(i) *(Unique existence) The conditional steady state $\Phi^{cs}(A; \Phi_0)$ uniquely exists on $\mathcal{M}(A; \Phi_0)$.*

(ii) *(Continuity) The map $\Gamma_{\text{endo}}(\cdot, A) : \mathcal{X} \to \mathcal{X}$ is continuous. The aggregator $K : \mathcal{X} \to \mathbb{R}$ is continuous. Moreover, the aggregate variables (e.g., aggregate capital K and consumption C) vary continuously in Φ when restricted to $\mathcal{M}(A; \Phi_0)$.*

Assumption 1 collects the regularity properties of the conditional saddle path that are needed for the paper's geometric arguments. Part (i) posits that, conditional on a regime A and an initial state Φ_0, the frozen-regime continuation converges to a unique conditional steady state within the relevant invariant equilibrium set. This requirement is deliberately local-to-the-experiment: it does not rule out global multiplicity across initial conditions or regimes.

In particular, Proehl (2025) establishes global existence and uniqueness of recursive equilibrium in heterogeneous-agent models under parametric restrictions, whereas Assumption 1 (i) allows for the possibility of global multiplicity as in Walsh and Young (2024).[4] A full characterization of primitives ensuring Assumption 1 (i) is beyond the scope of this paper.

Part (ii) imposes continuity of the frozen-regime law of motion and of the relevant aggregate observables when restricted to the conditional saddle. This mild regularity

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[3]Strictly speaking, $\mathcal{M}(A; \Phi_0) = \overline{\{\Phi_t : t \geq 0\}}$ as defined in discrete time is an orbit-closure (a countable set together with its limit point) and therefore need not be a smooth manifold. We nonetheless use saddle-path language to emphasize its one-dimensional, path-like role for global dynamics. In a continuous-time formulation with a smooth flow on the state space, the corresponding stable branch is naturally represented as a one-dimensional invariant manifold under standard regularity conditions.

[4]Nevertheless, conditional on a given initial state and a fixed regime, the equilibrium path considered in the analysis is assumed to be uniquely determined.

ensures that small perturbations of the distributional state *within the equilibrium set under study* produce small changes in aggregate outcomes and subsequent states, which is used repeatedly when translating geometric properties of projected phase diagrams (e.g., nullclines) into restrictions on equilibrium dynamics along the saddle.

With this geometric regularity in place, the conditional saddle admits a natural notion of "position along the curve" toward the conditional steady state. Theorem 2 formalizes the key implication: if some aggregate equilibrium variable moves strictly one-way along this curve, then it provides a valid coordinate for the entire conditional saddle. In that case, knowing the scalar value is equivalent to knowing the full distributional state within the relevant equilibrium set, so the stochastic equilibrium admits an exact one-dimensional representation on $\mathcal{M}(A; \Phi_0)$.

2.2 Representative-agent economy: A canonical RBC

A representative household with temporal log utility is considered. Given initial condition (a_0, A_0), the household maximizes life-time utility under stochastic aggregate TFP A_t. I present the economy in recursive form and work with a recursive competitive equilibrium. Under standard regularity conditions (including an appropriate transversality condition), this recursive formulation is equivalent to the sequential equilibrium; for completeness the sequential formulation is provided in Appendix A. The recursive form of the household's problem is as follows:

$$v(a; K, A) = \max_{c, a'} \quad log(c) + \beta \mathbb{E} v(a'; K', A') \tag{5}$$

$$c + a' = a(1 + r(X)) + w(X) \tag{6}$$

$$a' \geq -\overline{a} \tag{7}$$

$$K' = \Gamma_{endo}(K, A), \quad A' \sim \Gamma_{exo}(A'|A), \tag{8}$$

where v is the household's value function; a is the wealth in the beginning of a period; K is aggregate capital; A is aggregate TFP; Γ_{endo} is the law of motion for K. For illustrative purposes, I assume that TFP A follows a Markov-switching process between the levels B and G, where $G > B$:

$$\Gamma_{exo} = \begin{bmatrix} \pi_{BB} & \pi_{BG} \\ \pi_{GB} & \pi_{GG} \end{bmatrix} \tag{9}$$

I consider the following competitive factor prices given CRS Cobb-Douglas production function:

$$r(K, A) = A\alpha K^{\alpha-1} - \delta \tag{10}$$

$$w(K, A) = A(1-\alpha)K^{\alpha}, \tag{11}$$

where K is capital stock, which satisfies $K = a$ in equilibrium. The recursive competitive equilibrium (RCE, hereafter) is as follows:

Definition 3 (Recursive competitive equilibrium).
$(c, a', v, r, w, \Gamma_{endo})$ *is a recursive competitive equilibrium if these functions*

1. *satisfy the individual optimality conditions*

2. *clear factor markets, resulting in the competitive prices.*

3. *satisfy the consistency:*

$$a'(K; K, A) = K' = \Gamma_{endo}(K, A) \tag{12}$$

In optimal dynamics, the conditional saddle path can exhibit discrete-time pathologies in which the projected (K, C) trajectory develops folds or loops. In such cases the saddle may (i) touch the K-nullcline at an intermediate point $\Phi \neq \Phi_A^{cs}$ (a "turning point") or (ii) alternate sides of the nullcline across dates (a "spiral"). Either behavior defeats a single-valued global representation indexed by K, because distinct distributional states on the relevant equilibrium set can share the same aggregate capital while implying different future allocations. To rule out these pathologies, I impose the following geometric restriction on one-step transitions in the (K, C) plane.

Assumption 2 (No segment crossing of the conditional K-nullcline).
Fix (Φ_0, A_0) and $A \in \{B, G\}$. Let $\{\Phi_t\}_{t \geq 0}$ be the frozen-regime continuation $\Phi_{t+1} = \Gamma_{endo}(\Phi_t, A)$ and write $K_t := K(\Phi_t)$ and $C_t := C(\Phi_t, A)$. For every t such that $\Phi_t \neq \Phi^{cs}(A; \Phi_0)$, the straight line segment in (K, C) connecting (K_t, C_t) and (K_{t+1}, C_{t+1}) does not intersect the conditional K-nullcline $\{(K, C) : C = C_A^{Kcnull}(K)\}$.

Assumption 2 is a discrete-time "no side-switching" condition: along the frozen-regime continuation, consecutive points (K_t, C_t) and (K_{t+1}, C_{t+1}) remain in the same

region of (K,C)-space separated by the K-nullcline. In particular, letting $H_t :=$ $C_t - C_A^{Kcnull}(K_t)$, Assumption 2 implies $H_t H_{t+1} > 0$ for all t prior to convergence, so net investment $\Delta K_t := K_{t+1} - K_t = C_A^{Kcnull}(K_t) - C_t = -H_t$ cannot reverse sign along the conditional saddle.

In continuous time, reversals of $\dot{K}$ necessarily pass through $\dot{K} = 0$, so continuity of the drift often delivers a comparable one-crossing geometry along the stable branch in standard settings. In discrete time, by contrast, the time-t map can jump across the nullcline between dates without ever landing on it at an integer time. Assumption 2 rules out such nonlocal crossings and thereby ensures a well-behaved saddle geometry consistent with reporting global impulse responses and transition functions as single-valued equilibrium objects.[5]

In the following proposition, I show that Assumption 2 rules out sign reversals of net investment along the frozen-regime continuation, implying strictly monotone convergence of aggregate capital.

Proposition 1 (K monotonicity in RBC).
Fix $A \in \{B, G\}$. Under Assumptions 1 and 2, aggregate capital along the frozen-regime continuation converges to K_A^{cs} strictly monotonically.

Proof. Fix A and let $\{\Phi_t\}_{t \geq 0}$ be the frozen-regime continuation, with $K_t := K(\Phi_t)$ and $C_t := C(\Phi_t, A)$. Define net investment

$$\Delta K_t := K_{t+1} - K_t. \tag{13}$$

In the RBC benchmark, the aggregate resource constraint implies the K-nullcline representation

$$\Delta K_t = C_A^{Kcnull}(K_t) - C_t, \tag{14}$$

so the sign of ΔK_t is the opposite of the nullcline gap $H_t := C_t - C_A^{Kcnull}(K_t)$.

By Assumption 2, for every t such that $\Phi_t \neq \Phi^{cs}(A; \Phi_0)$, the segment connecting (K_t, C_t) and (K_{t+1}, C_{t+1}) does not intersect the K-nullcline. Since the K-nullcline

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[5]Discrete time admits nonlocal movements that can generate folds or repeated nullcline intersections in the projected (K, C) dynamics even when the underlying saddle is one-dimensional. Assumption 2 rules out such pathologies along the conditional saddle. For related geometric restrictions in dynamic optimization, see Reddy et al. (2020) and, on threshold (Skiba) phenomena, Skiba (1978), Dechert and Nishimura (1983), and Wagener (2003).

is the graph of a continuous function, this implies $H_t H_{t+1} > 0$, hence $\{H_t\}$ has a constant sign prior to convergence. Therefore $\Delta K_t = -H_t$ has a constant nonzero sign whenever $\Phi_t \neq \Phi^{cs}(A; \Phi_0)$, and $\{K_t\}$ is strictly monotone in t until convergence.

Finally, Assumption 1 (i) implies $\Phi_t \to \Phi^{cs}(A; \Phi_0)$, and continuity of K (Assumption 1 (ii)) yields $K_t \to K_A^{cs} := K(\Phi^{cs}(A; \Phi_0))$. ∎

In the model, the household ex-ante takes into account the aggregate uncertainty in its decision. The conditional saddle path is the sequence of outcomes implied by such ex-ante decisions when the ex-post exogenous aggregate states are *frozen* at A. Figure 1 illustrates two conditional saddle paths corresponding to the aggregate productivity states B and G. The figure is based on a standard quarterly calibration of the RBC model described above. The model is solved globally using the repeated transition method of Lee (2025). Conditional saddle paths are then recovered by fixing the aggregate productivity state for 2,000 periods and tracing the resulting equilibrium trajectories of capital and consumption.[6]

The global dynamics of the recursive competitive equilibrium are fully characterized by movements along and across these conditional saddle paths. When aggregate productivity remains constant, the economy evolves along the corresponding saddle path. If the economy is initialized between the two conditional saddle paths and $A = G$, capital converges strictly monotonically upward toward the conditional steady state K_G^{cs}; if $A = B$, capital converges strictly monotonically downward toward K_B^{cs} consistent with Proposition 1. If the initial state lies outside this region, the economy gradually transitions into it while moving along the relevant saddle paths.

Along the conditional saddle, the economy evolves according to the law of motion Γ_{endo} for the endogenous state. In particular, the aggregate capital K strictly monotonically converges to the conditional steady state — if K is lower than a conditional steady state of the conditional saddle the economy is in, it increases, and vice versa for the case of K higher than it. This strictly monotonically converging pre-determined variable is the key to link the heterogeneous-agent model and the representative-agent model as in the current setup — the variable becomes the sufficient statistic for the heterogeneous-agent model.

When the level of TFP exogenously changes, the economy jumps from one saddle

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[6]For each productivity state, the dynamics are computed twice: once starting from a sufficiently low initial capital stock to obtain the upward-converging path, and once from a sufficiently high initial capital stock to obtain the downward-converging path.
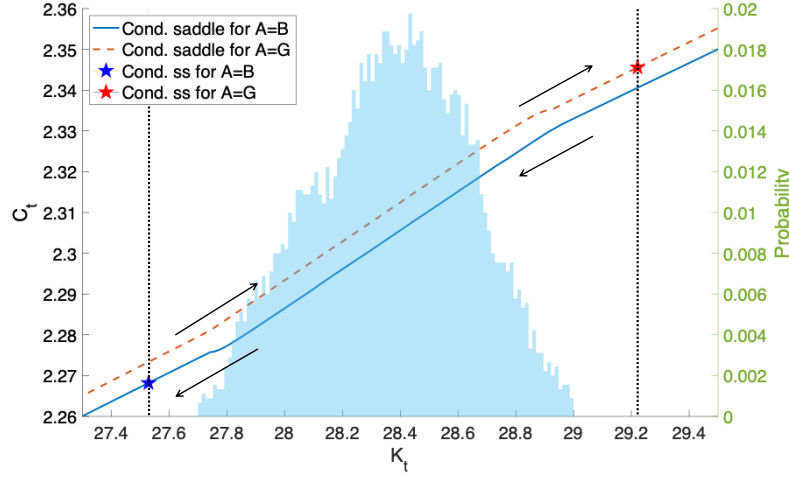
Figure 1: Conditional saddle paths and steady states

Notes: The figure plots conditional saddle paths for $A = B$ (solid) and $A = G$ (dashed) implied in the canonical RBC model under standard quarterly calibration. The histogram in the background plots the time-series distribution of the aggregate capital stock.

to the other vertically – capital responds from the following period as it is a pre-determined variable one period ahead. In this dynamics, the key force to move the economy is the law of motion for the exogenous state Γ_{exo}. Combined together, both law of motions generate bounded stochastic fluctuations in the economy.

Generalized transition function The conditional saddle path captures the entire transitions across different endogenous (along the saddle) and exogenous states (across the saddle), allowing generalized transition function (GTF) analysis (Lee, 2025), which encompasses generalized impulse response functions (Koop et al., 1996; Andreasen et al., 2017). The economy's response to a sequence of exogenous shocks is not confined to local dynamics around the steady state, and its trajectory is sharply traced in the phase diagram.

Figure 2 illustrates the equilibrium dynamics in consumption and capital stock in phase diagram (panel (a)) and in time domain (panels (b) and (c)) for an arbitrary subsample period (830 – 849, 20 quarters) in the equilibrium path. In period 830, an exogenous negative TFP shock hits leading to a downward jump in consumption across the conditional saddle paths. Then, for four consecutive periods, consumption and capital stock endogenously decline along the saddle. In period 835, an exogenous

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positive TFP shock hits, shifting the economy to the other conditional saddle path, followed by upward endogenous adjustments in consumption and capital. Compared

(a) On the saddles

(b) C - time domain
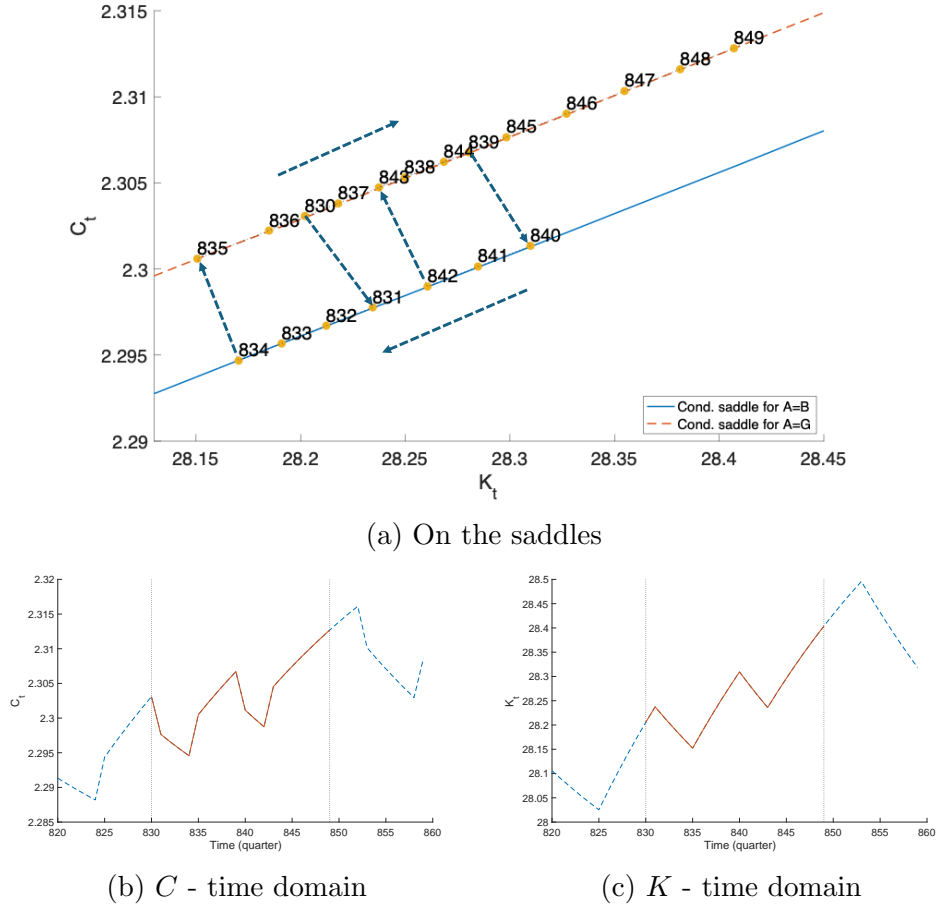

(c) K - time domain

Figure 2: Equilibrium dynamics: saddle vs. time domain

Notes: The figure illustrates the stochastic equilibrium dynamics of consumption and capital stock on the conditional saddle paths (panel (a)) and in the time domain (panels (b) and (c)) implied by a canonical RBC model under standard quarterly calibration. The sample period covers 20 quarters from 830 to 849.

to time-domain figures, the phase diagram more concisely illustrates the economy's equilibrium dynamics in a single figure. Moreover, it enables immediate consideration of counterfactual scenarios by eyeballing the realized exogenous paths.

As the information on all possible global equilibrium dynamics are contained in the conditional saddle paths, generalized impulse response function (GIRF) can be also conveniently represented. For the GIRF illustration I replace the two-state Markov chain with a continuous AR(1) process for A: $A' = (1-\rho)\mu + \rho A + \epsilon$, $\epsilon \sim N(0, \sigma)$. In

this case, there is a continuum of frozen-A conditional saddles indexed by A, and the GIRF path moves both along a given conditional saddle (endogenous propagation) and across the foliation as A evolves.
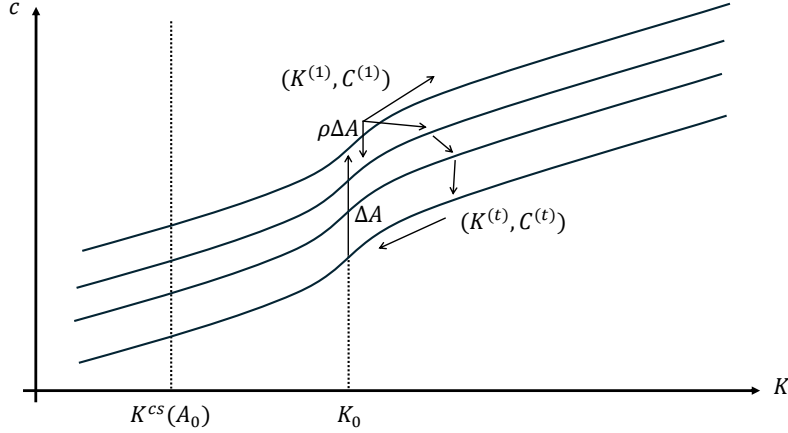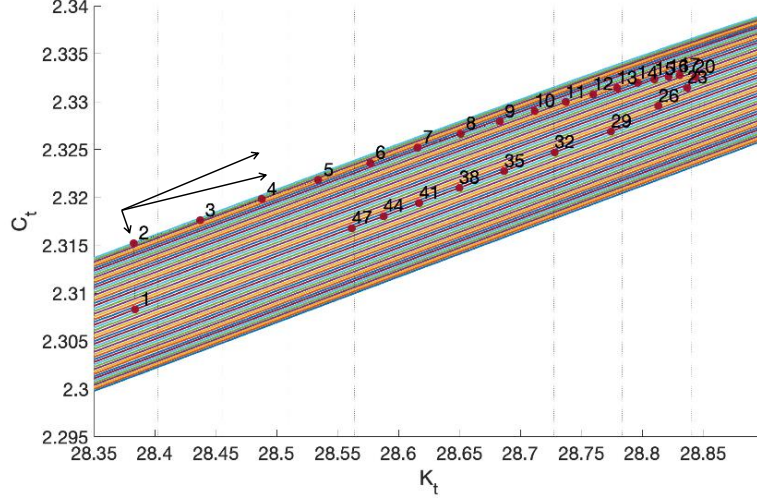

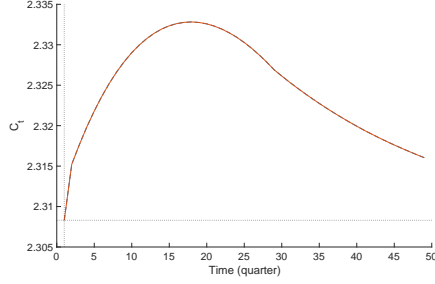Figure 3: Generalized impulse response function (GIRF) at (K_0, A_0)

Notes: The figure illustrates the generalized impulse responses of consumption and capital stock on the conditional saddle paths to a positive TFP shock.

Figure 3 illustrates the post-shock dynamics in the phase diagram following a positive TFP shock, as aggregate productivity gradually reverts to its steady-state level. Upon impact, consumption jumps upward, inducing a vertical shift of the economy across conditional saddle paths. In subsequent periods, forces operating both along and across these saddles jointly shape the post-shock dynamics, generating a right-bending, bow-shaped trajectory. Consequently, the capital stock K exhibits an inverted-U-shaped impulse response over time, while consumption c displays a downward-sloping response in the time domain.

Figure 4 presents the simulated impulse responses of consumption and capital following a positive TFP shock, shown in the phase diagram (panel (a)) and in the time domain (panels (b) and (c)) over periods 1–50. Consistent with the phase-diagram intuition, the bow-shaped capital trajectory translates into a hump-shaped impulse response in the time domain, as shown in panel (c). Consumption also exhibits a hump-shaped response: initially, high-TFP conditional saddle paths push consumption upward, but as TFP gradually decays and crosses a threshold level, the direction of adjustment reverses, pulling consumption back down.

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(a) On the saddles
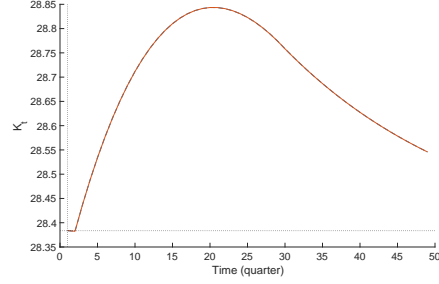
(b) C - time domain

(c) K - time domain

Figure 4: Impulse responses to a positive TFP shock: saddle vs. time domain

Notes: The figure illustrates the impulse responses of consumption and capital stock on the conditional saddle paths (panel (a)) and in the time domain (panels (b) and (c)). The sample period covers 50 quarters from 1 to 50.

Comparison with the perfect-foresight saddles The household's inter-temporal optimality conditions evaluated at the conditional steady states are as follows:

$$\overbrace{\beta\pi_{BB}\left(1+\alpha B(K_B^{cs})^{\alpha-1}-\delta\right)}^{\text{Discounted return for } B \to B}+\overbrace{\beta\pi_{BG}\left(\frac{c(K_B^{cs},G)}{c_B^{cs}}\right)^{-1}\left(1+\alpha G(K_B^{cs})^{\alpha-1}-\delta\right)}^{\text{Discounted return for } B \to G}=1 \quad (15)$$

$$\underbrace{\beta\pi_{GG}\left(1+\alpha G(K_G^{cs})^{\alpha-1}-\delta\right)}_{\text{Discounted return for } G \to G}+\underbrace{\beta\pi_{GB}\left(\frac{c(K_G^{cs},B)}{c_G^{cs}}\right)^{-1}\left(1+\alpha B(K_G^{cs})^{\alpha-1}-\delta\right)}_{\text{Discounted return for } G \to B}=1 \quad (16)$$

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These equations pin down the conditional steady-state conditions for the aggregate capital stocks K_B^{cs} and K_G^{cs}:

$$K_B^{cs} = \left(\frac{\alpha(\pi_{BB}B + \pi_{BG}G)}{1/\beta - (1-\delta)\left(\pi_{BB} + \pi_{BG}\left(g_c^{cs}(B)\right)^{-1}\right)} \right)^{\frac{1}{1-\alpha}} \tag{17}$$

$$K_G^{cs} = \left(\frac{\alpha(\pi_{GG}G + \pi_{GB}B)}{1/\beta - (1-\delta)\left(\pi_{GG} + \pi_{GB}\left(g_c^{cs}(G)\right)^{-1}\right)} \right)^{\frac{1}{1-\alpha}} \tag{18}$$

where $g_c^{cs}(A)$, $A \in \{B, G\}$ are gross consumption growth rates when a regime change happens in each conditional steady state.

In the *perfect-foresight (PF)* economy, agents believe regime switching is impossible and, consistent with this belief, the regime does not switch. In contrast, along the *conditional-saddle (CS)* dynamics, the regime is held fixed ex post, but agents still correctly anticipate that regime switches are possible under Π. Thus PF and CS share the same feasibility accounting and the same K-nullcline under a given productivity level, yet generally differ in their Euler-implied consumption dynamics and hence in their steady states and saddle paths.

From the stationary condition for the capital stock ($\delta K = I$) and the national accounting identity ($Y = C + I$), the conditional nullclines of aggregate capital K for $A \in \{B, G\}$ are as follows:

$$c_B^{Kcnull} = B(K_B^{Kcnull})^\alpha - \delta K_B^{Kcnull} \tag{19}$$

$$c_G^{Kcnull} = G(K_G^{Kcnull})^\alpha - \delta K_G^{Kcnull} \tag{20}$$

Then, I consider a perfect-foresight dynamics where there is no uncertainty in the economy and TFP A is fixed at either B or G. I denote allocations for this perfect-foresight economy with superscript pf. Following a canonical neoclassical growth model analysis, I derive the steady states for the perfect-foresight counterpart with different TFP levels:

$$(K_B^{pf}, c_B^{pf}) = \left(\left(\frac{\alpha B}{1/\beta + \delta - 1}\right)^{\frac{1}{1-\alpha}}, B\left(\frac{\alpha B}{1/\beta + \delta - 1}\right)^{\frac{\alpha}{1-\alpha}} - \delta\left(\frac{\alpha B}{1/\beta + \delta - 1}\right)^{\frac{1}{1-\alpha}} \right) \tag{21}$$

$$(K_G^{pf}, c_G^{pf}) = \left(\left(\frac{\alpha G}{1/\beta + \delta - 1}\right)^{\frac{1}{1-\alpha}}, G\left(\frac{\alpha G}{1/\beta + \delta - 1}\right)^{\frac{\alpha}{1-\alpha}} - \delta\left(\frac{\alpha G}{1/\beta + \delta - 1}\right)^{\frac{1}{1-\alpha}} \right) \tag{22}$$

Notably, two economies share the same capital (conditional) nullclines for each productivity levels, as formalized in Proposition 2.

Proposition 2 (K-nullcline invariance over beliefs).
Conditional K-nullclines are identical between the RBC model with the aggregate uncertainty and the perfect foresight counterpart.

Proof.
From the stationary condition $\delta K = I$, equations (19) and (20) are immediate for both models. Therefore, the conditional nullclines are the same. ∎

This property provides two important insights. First, the differences in the limit behaviors of two economies (steady states) and the corresponding saddle paths are crucially determined by the consumption dynamics – consumption nullclines.[7] In Proposition 3, I compare the rankings between the baseline model and the perfect-foresight model.

Second, both with and without aggregate uncertainty, the saddle-path dynamics require the capital stock K to converge strictly monotonically to the (conditional) steady state. Even when the baseline model is extended to a heterogeneous-household environment, as in Section 4, the K-nullcline remains invariant, preserving the monotonicity of K along conditional saddle paths. This property is crucial for establishing the sufficiency of K in summarizing aggregate fluctuations in the heterogeneous-household setting.

Proposition 3 (Aggregate uncertainty and the conditional steady states).
The following inequalities hold:

$$K_B^{cs} \; < \; K_B^{pf} \; < \; K_G^{pf} \; < \; K_G^{cs}, \qquad c_B^{cs} \; < \; c_B^{pf} \; < \; c_G^{pf} \; < \; c_G^{cs}. \tag{23}$$

Proof.
See Appendix C. ∎

Proposition 3 shows that the (conditional) steady state associated with low TFP features lower capital stock and consumption under the aggregate uncertainty than

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[7]Rigorously speaking, there is no consumption nullcline in the model with aggregate uncertainty. The Euler equations at the conditional steady states play the same role as the consumption nullclines without uncertainty.

under perfect foresight; the one with high TFP features the opposite. Therefore, conditional steady states of capital and consumption with perfect foresight are nested by the counterparts in the steady state with aggregate uncertainty. These are fully driven by rational expectations for future regime shifting under aggregate uncertainty.
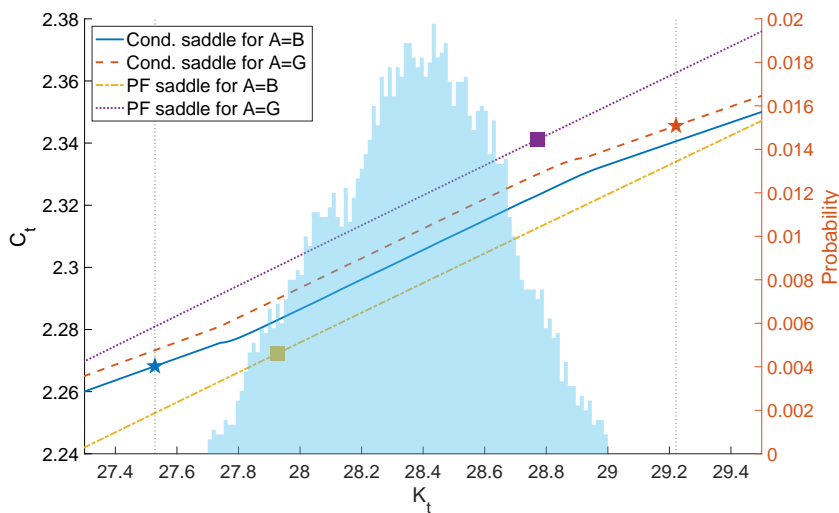

Figure 5: Conditional saddle path comparison: with and without uncertainty

Notes: The figure plots conditional saddle paths for $A = B$ (solid) and $A = G$ (dashed) and perfect-foresight saddle paths for $A = B$ (dash-dotted) and $A = G$ (dotted) implied by a canonical RBC model under standard quarterly calibration. The histogram in the background plots the time-series distribution of the aggregate capital stock.

Figure 5 plots the conditional saddle paths for the model with aggregate uncertainty and the perfect-foresight deterministic saddle paths. As shown in Proposition 3, conditional steady states for both capital stocks and consumption with perfect foresight are nested by the ones with aggregate uncertainty.

Conditional boundary condition   In the canonical deterministic neoclassical growth model, the sequential formulation of competitive equilibrium requires a transversality condition to pin down the economically relevant (non-explosive) equilibrium path.[8] The transversality condition rules out unbounded asset accumulation and ensures equivalence between the sequential and recursive formulations.

In this paper, I work directly with a recursive formulation of stochastic equilib-

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[8]This condition is imposed at the aggregate level and should be distinguished from individual no-Ponzi constraints.

rium. Consistency with an underlying sequential economy therefore implicitly imposes a stochastic analogue of transversality. Given the notion of conditional saddle paths, a natural counterpart is *conditional transversality*:

$$\lim_{t \to \infty} \beta^t u'(c_t) K_t = 0 \quad \text{under a frozen aggregate regime } A \in \{B, G\} \tag{24}$$

$$\text{(along the conditional saddle for each } A \in \{B, G\}).$$

Condition (24) rules out explosive continuations along each conditional saddle and guarantees that the value function remains well defined under the corresponding frozen-regime dynamics. If, instead, some conditional continuations generated explosive paths, then the recursive representation would cease to be internally consistent because continuation values would be ill behaved along those paths.

Under standard assumptions (e.g. concavity and existence of an optimal plan), (24) is equivalent to the usual stochastic transversality conditions (Kamihigashi, 2003, 2005) on the class of equilibria considered here: both select the same bounded recursive competitive equilibrium. The conditional formulation is useful because it makes explicit that the boundary condition operates regime-by-regime on $\mathcal{M}_A$, a perspective that is essential for the dimension-reduction results that follow.

Economies without conditional saddle paths A conditional saddle requires a predetermined endogenous state: without one, the economy jumps each period to the unique bounded allocation implied by the current exogenous state, leaving no nontrivial "along-the-saddle" dynamics.

Remark 1 (Endogenous memorylessness implies degenerate conditional saddle paths)**.** *Consider an RCE with state (Φ_t, A_t). Call the economy endogenously stateless if the endogenous state is memoryless: $\Phi_t = \phi(A_t)$ for some function ϕ, so that all equilibrium allocations depend on A_t only. Then the economy is saddleless: the "conditional steady state" under frozen A is simply $\phi(A)$, and the invariant set is a singleton. Conversely, a non-degenerate conditional saddle requires at least one predetermined endogenous state.*

Example: Gali (2008) three-equation NK model. The textbook New Keynesian model

$$x_t = \mathbb{E}_t x_{t+1} - \tfrac{1}{\sigma}(i_t - \mathbb{E}_t \pi_{t+1} - r_t^n), \tag{25}$$

$$i_t = \varphi_\pi \pi_t + \varphi_x x_t + \nu_t, \tag{26}$$

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + u_t, \tag{27}$$

under determinacy yields $(x_t, \pi_t, i_t) = \Psi s_t$ for $s_t := (r_t^n, u_t, \nu_t)$ and some matrix Ψ. The equilibrium is endogenously stateless: freezing the exogenous state collapses dynamics to an immediate jump to the conditional steady point. Fluctuations occur entirely *across* conditional points, not along a saddle.

Models with predetermined states—capital, habits, interest-rate smoothing, or distributional states in HANK—are endogenously stateful, and conditional saddle paths generically exist.

3 State dependence in a shock response

In this section, I analyze nonlinear shock responsiveness through the lens of conditional saddle paths. In any stochastic dynamic model that admits conditional saddle paths, a response of an aggregate variable to an exogenous shock is represented by shifts across different conditional saddle paths. Then, if and only if all the conditional saddle paths are parallel along the endogenous state, the response becomes state-independent.

Theorem 1 (State-(in)dependence as a geometric condition)**.**
Fix (Φ_0, A_0) and let $\mathcal{M}_{A_0}(\Phi_0, A_0)$ denote the conditional saddle path under the frozen regime A_0. Let $g(\Phi, A)$ be an aggregate equilibrium object (e.g. consumption) defined for $(\Phi, A) \in \mathcal{M}_{A_0}(\Phi_0, A_0) \times \{A_0, A_1\}$. Define the impact gap *between regimes A_1 and A_0 at state Φ by*

$$\Delta_g(\Phi; A_1, A_0) := g(\Phi, A_1) - g(\Phi, A_0). \tag{28}$$

Then the following are equivalent:

(i) (State-independent gap) $\Delta_g(\Phi; A_1, A_0)$ is constant on $\mathcal{M}_{A_0}(\Phi_0, A_0)$, i.e. there

exists $c \in \mathbb{R}$ such that

$$g(\Phi, A_1) - g(\Phi, A_0) = c \qquad \forall \Phi \in \mathcal{M}_{A_0}(\Phi_0, A_0). \tag{29}$$

(ii) (Vertical-translation geometry) Viewed as subsets of $\mathcal{X} \times \mathbb{R}$,

$$\mathcal{G}_{A_j} := \{(\Phi, g(\Phi, A_j)) : \ \Phi \in \mathcal{M}_{A_0}(\Phi_0, A_0)\}, \qquad j \in \{0, 1\}, \tag{30}$$

satisfy $\mathcal{G}_{A_1} = \mathcal{G}_{A_0} + (0, c)$, i.e. $\mathcal{G}_{A_1}$ is a constant vertical translation of $\mathcal{G}_{A_0}$.

Proof. $(i) \Rightarrow (ii)$: if $g(\Phi, A_1) = g(\Phi, A_0) + c$ for all Φ, then $(\Phi, g(\Phi, A_1)) = (\Phi, g(\Phi, A_0) + c)$ for all Φ, hence $\mathcal{G}_{A_1} = \mathcal{G}_{A_0} + (0, c)$. $(ii) \Rightarrow (i)$: if $\mathcal{G}_{A_1} = \mathcal{G}_{A_0} + (0, c)$, then for each Φ we must have $g(\Phi, A_1) = g(\Phi, A_0) + c$, so the gap is constant. ∎

Sharp state independence may be a knife-edge property of an RCE. However, Theorem 1 provides an insight regarding conditions under which state dependence becomes amplified. In particular, when a conditional saddle path is more steeply tilted with respect to the endogenous state, as illustrated in Figure 6, a shock responsiveness becomes state dependent. Such differential slopes may arise from various real (Winberry, 2021; Lee, 2025), financial (Melcangi, 2024), labor market frictions (Petrosky-Nadeau et al., 2018; Pizzinelli et al., 2020), and the scope of the relevant shocks include TFP shocks and fiscal/monetary policy shocks (Tenreyro and Thwaites, 2016; Lee, 2025). The sufficiency and the necessity of tilt in the state dependence implies that models with such nature necessarily implies different slopes of the conditional saddle paths.

An example: asymmetric adjustment cost   As an example where state dependence arises due to the tilt in the conditional saddle, I consider an extended RBC model with asymmetric adjustment cost. Specifically, the representative household's

budget constraint is modified in the following way:

$$c + a' + \mathcal{C}(a', a) = a(1 + r(X)) + w(X) \tag{31}$$

$$\mathcal{C}(a', a) = \frac{\widetilde{\mu}}{2} \left(\frac{a' - a}{a} \right)^2 a \tag{32}$$

$$\widetilde{\mu} = \begin{cases} \mu_+ & \text{if } a' > a \\ \mu_- & \text{if } a' < a \end{cases} \tag{33}$$

where $\mathcal{C}$ is the wealth adjustment cost which indirectly reflects the frictional capital market. The adjustment cost is asymmetric between positive and negative investment, as specified in equation (33). In particular, I consider the case $\mu_+ > \mu_-$.[9]

Figure 6 plots the conditional saddle paths under asymmetric adjustment costs. Relative to the saddle path associated with A=G, the conditional saddle path for A=B is substantially steeper. As a result, a one–standard-deviation TFP shock generates a larger consumption response when the capital stock is low (3.42%) than when it is high (2.42%).

This geometric representation formalizes the intuition that "climbing up is difficult, falling down is easy"—a recurring theme in the literature on asymmetric business cycles and endogenous disasters (Petrosky-Nadeau et al., 2018). In this framework, asymmetric frictions generate differently tilted conditional saddles: the steeper slope under low productivity implies an aggressive contraction across the conditional saddle paths upon negative shocks, while recovery proceeds more slowly along a flatter path. The resulting gap in shock responsiveness (3.42% versus 2.42% in the example above) arises precisely because the geometric distance between conditional saddles varies with the economy's position along them.

This geometric framework suggests that whenever a model in the literature features endogenous state-dependent dynamics, such behavior must fundamentally originate from the differential tilt of the conditional saddle paths. Theorem 1 establishes that state independence is equivalent to a vertical-translation geometry, in which equilibrium conditional saddle paths remain parallel along the endogenous state. It follows that any deviation into state-dependent responsiveness—regardless of whether the underlying friction is financial, labor-related, or based on adjustment costs—necessarily

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[9]The adjustment-cost parameters are not calibrated. For illustrative purposes, I set $\mu_+ = 4$ and $\mu_- = 1$. All other parameters follow a standard quarterly RBC calibration.
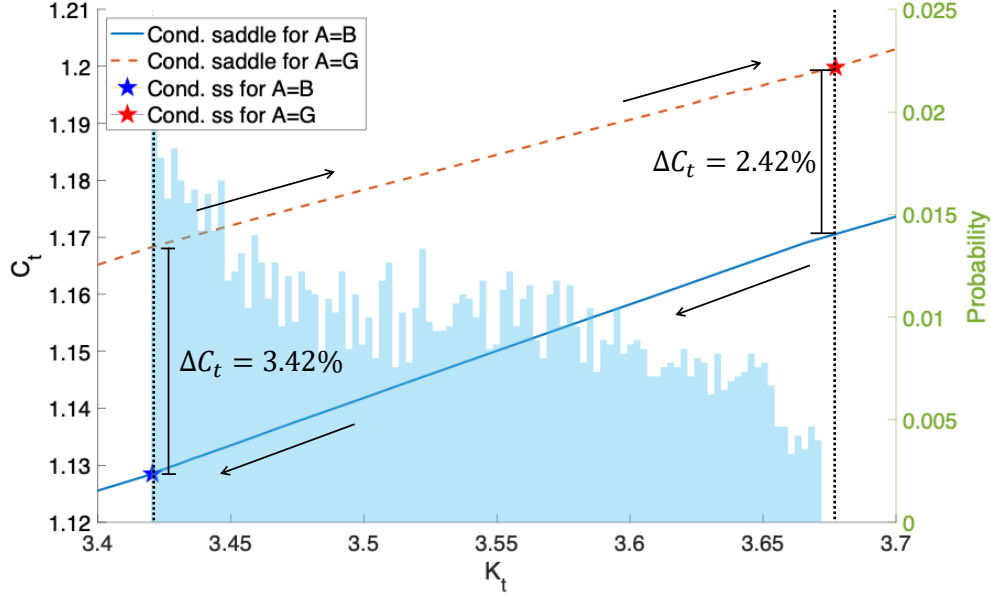
Figure 6: Differently tilted conditional saddle paths – endogenous state dependence

Notes: The figure plots conditional saddle paths for $A = B$ (solid) and $A = G$ (dashed) implied by an RBC model with asymmetric wealth adjustment cost. The conditional saddle path for $A = B$ is steeper than the one for $A = G$. The histogram in the background plots the time-series distribution of the aggregate capital stock.

implies a geometric structure where the local slopes of the conditional saddles diverge across aggregate regimes. In this sense, the "tilt" (the local rate of endogenous accumulation or decumulation) serves as the universal geometric signature of endogenous state dependence.

Furthermore, this framework clarifies the distinction between endogenous and exogenous state dependence. While endogenous state dependence—the sensitivity of shocks to the current endogenous state—necessarily originates from the differential tilt of the conditional saddle paths, exogenous state dependence arises from the interaction and magnitude of the shifts across these paths. Even in economies where saddles are geometrically parallel—resulting in state-independent impact gaps across capital stocks—the equilibrium path may still exhibit nonlinear responses if the exogenous states are "outstretched." In such cases, the vertical distance between saddles reflects a larger exogenous regime shift, creating greater fluctuations that are decomposed into horizontal propagation along the saddles and vertical jumps across them. Consequently, the conditional saddle representation provides a unified visual

tool for assessing both the slope-driven propagation of endogenous frictions and the level-driven impact of exogenous aggregate shocks.

4 Heterogeneous-household economy and dimension reduction

This section introduces conditional saddle path in a canonical heterogeneous-household economy. I consider a continuum of unit measure of ex-ante homogeneous households. The recursive formulation of the households' problem is as follows:

$$v(a, z; \Phi, A) = \max_{c,a'} \quad log(c) + \beta \mathbb{E} v(a', z'; \Phi', A') \tag{34}$$

$$c + a' = a(1 + r(X)) + w(X)z \tag{35}$$

$$a' \geq 0 \tag{36}$$

$$\Phi' = \Gamma_{endo}(\Phi, A), \quad A' \sim \Gamma_{exo}(A'|A). \tag{37}$$

The problem is the same as in the representative-household economy except for 1) uninsurable idiosyncratic labor productivity, which follows a Markov process $z \sim \Gamma_z(z'|z)$; 2) inclusion of distribution of individual states Φ in the aggregate endogenous state; and 3) the corresponding change in the law of motions for the endogenous aggregate state. The RCE is defined as in Krusell and Smith (1998).

The model includes two different stochastic exogenous processes: idiosyncratic productivity and aggregate TFP. Therefore, there are two layers of conditional saddle paths: one is individual conditional saddle path, and the other is aggregate conditional saddle path. The individual saddle path has its own cross-sectional implication which deserves a separate analysis, but it is out of this paper's focus. So, the corresponding analysis is included in Appendix B. I elaborate on the aggregate conditional saddle with stochastic TFP process, where the model closely follows Krusell and Smith (1998).[10]

Krusell and Smith (1998) posits an endogenous law of motion that tracks aggregate capital K rather than the full distribution Φ: $\log K' = \alpha(A) + \beta(A) \log K$, for $A \in$

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[10]In the original model of Krusell and Smith (1998), the exogenous individual labor supply comoves with the exogenous aggregate TFP. All the results stay unaffected after including this feature in the model, but for the expositional brevity, I assume the labor supply is exogenously fixed.

$\{B, G\}$, where α and β are state-dependent coefficients. This formulation embeds two key assumptions: (i) that aggregate capital K is (approximately) sufficient to summarize endogenous aggregate dynamics, and (ii) that the law of motion is log-linear. Using the conditional-saddle framework, this paper establishes that K is an exact sufficient statistic. By contrast, log-linearity follows from the specific model structure in Krusell and Smith (1998) and typically requires numerical verification. The following theorem provides the first step toward establishing the exact sufficiency of K.

Theorem 2 (Monotone aggregate variable as a coordinate).
Fix (Φ_0, A_0) and $A \in \{B, G\}$. Let $\{\Phi_t\}_{t \geq 0}$ be the frozen-regime continuation $\Phi_{t+1} = \Gamma_{\text{endo}}(\Phi_t, A)$ and let $\Phi^{cs} := \lim_{t \to \infty} \Phi_t$. Let $e : \mathcal{M}(A; \Phi_0) \to \mathbb{R}$ be an aggregate equilibrium variable. Suppose that the scalar sequence $\{e(\Phi_t)\}_{t \geq 0}$ is strictly monotone and converges to $e(\Phi^{cs})$. Then e uniquely indexes states on the conditional saddle:

$$e(\Phi) = e(\Phi') \implies \Phi = \Phi' \qquad \forall \Phi, \Phi' \in \mathcal{M}(A; \Phi_0). \tag{38}$$

Proof. Define $\psi_A : \mathbb{N} \cup \{\infty\} \to \mathcal{M}(A; \Phi_0)$ by $\psi_A(t) := \Phi_t$ for $t \in \mathbb{N}$ and $\psi_A(\infty) := \Phi^{cs}$. By definition of $\mathcal{M}(A; \Phi_0)$ and convergence $\Phi_t \to \Phi^{cs}$, we have

$$\mathcal{M}(A; \Phi_0) = \{\Phi_t : t \geq 0\} \cup \{\Phi^{cs}\}, \tag{39}$$

so ψ_A is surjective. Strict monotonicity of $\{e(\Phi_t)\}$ implies that $e(\Phi_t) \neq e(\Phi_\tau)$ for $t \neq \tau$, hence $\Phi_t \neq \Phi_\tau$ for $t \neq \tau$; therefore ψ_A is injective on $\mathbb{N}$. Moreover, since $\{e(\Phi_t)\}$ is strictly monotone and converges to $e(\Phi^{cs})$, we also have $e(\Phi_t) \neq e(\Phi^{cs})$ for all finite t, so ψ_A is injective on $\mathbb{N} \cup \{\infty\}$. Thus ψ_A is a bijection between $\mathbb{N} \cup \{\infty\}$ and $\mathcal{M}(A; \Phi_0)$.

Now let $\Phi, \Phi' \in \mathcal{M}(A; \Phi_0)$. Pick $s, s' \in \mathbb{N} \cup \{\infty\}$ such that $\Phi = \psi_A(s)$ and $\Phi' = \psi_A(s')$. If $e(\Phi) = e(\Phi')$, then $e(\psi_A(s)) = e(\psi_A(s'))$. Since $e \circ \psi_A$ is injective, it follows that $s = s'$, hence $\Phi = \Phi'$. Therefore e is injective on $\mathcal{M}(A; \Phi_0)$, and an inverse map φ_A exists on $e(\mathcal{M}(A; \Phi_0))$. ∎

Consequently, there exists a unique inverse map $\varphi_A : e(\mathcal{M}(A; \Phi_0)) \to \mathcal{M}(A; \Phi_0)$ such that $\varphi_A(e(\Phi)) = \Phi$ on $\mathcal{M}(A; \Phi_0)$, and any equilibrium object restricted to $\mathcal{M}(A; \Phi_0)$ can be written as a (single-valued) function of the scalar coordinate e.

The intuition behind the theoretical result is as follows: Fix a regime $A \in \{B, G\}$

and restrict attention to the set of equilibrium states that lie on the correspond-
ing conditional saddle (the invariant equilibrium path). Starting from any initial
equilibrium state Φ_0 on this set, the equilibrium law of motion determines a unique
subsequent sequence $\{\Phi_t\}_{t \geq 0}$ that remains on the invariant conditional saddle path.
Then, it converges to the unique conditional steady state. The key geometric obser-
vation is that the conditional saddle is effectively a string: it can be viewed as a single
curve of equilibrium states. As a consequence, points on the conditional saddle are
naturally ordered by their position along this curve.

Now consider the aggregate equilibrium variable e. By assumption, along any
equilibrium history on the conditional saddle, the scalar $e_t \equiv e(\Phi_t)$ moves strictly
monotonically over time and converges to its conditional steady-state value. Intu-
itively, e acts as a "progress meter" along the conditional saddle: it always moves in
one direction toward its limiting value and never reverses.

Suppose, toward a contradiction, that e were not injective on the conditional
saddle. Then there exist two distinct equilibrium states $\Phi \neq \Phi'$ on the conditional
saddle such that $e(\Phi) = e(\Phi')$. Consider the two equilibrium histories generated by
these initial states. Since both histories begin with the same value of the progress
meter and e_t must move strictly one-way toward the same limit along the conditional
saddle, the two histories would have to remain synchronized in their progress toward
the conditional steady state. But this is impossible because distinct starting points
on a one-dimensional conditional saddle generate equilibrium histories that cannot
merge. This contradiction implies that no two distinct equilibrium states on the
conditional saddle can share the same value of e.

Therefore e uniquely labels equilibrium states on the conditional saddle: if $e(\Phi) =$
$e(\Phi')$, then $\Phi = \Phi'$. This establishes the injectivity claim in Theorem 2 and explains
why a strictly monotone, convergent aggregate variable can serve as a one-dimensional
indexing variable for equilibrium dynamics on the conditional saddle.

Remark 2 (Injectivity and sufficiency).
*Because e is injective on $\mathcal{M}_A$, any equilibrium object restricted to $\mathcal{M}_A$ can be written
as a function of (e, A)*

Specifically, the following variables can be defined:

$$v^e(\cdot, \cdot; \widetilde{e}, A) \in V(\widetilde{e}, A) := \{v(\cdot, \cdot; \Phi, A) | \forall \Phi \in \mathcal{M}_A(\Phi_0, A_0) \text{ s.t. } e(\Phi) = \widetilde{e}\} \quad (40)$$

$$r^e(\widetilde{e}, A) \in R(\widetilde{e}, A) := \{r(\Phi, A) | \forall \Phi \in \mathcal{M}_A(\Phi_0, A_0) \text{ s.t. } e(\Phi) = \widetilde{e}\} \quad (41)$$

$$w^e(\widetilde{e}, A) \in W(\widetilde{e}, A) := \{w(\Phi, A) | \forall \Phi \in \mathcal{M}_A(\Phi_0, A_0) \text{ s.t. } e(\Phi) = \widetilde{e}\} \quad (42)$$

$$\Gamma^e_{endo}(\widetilde{e}, A) \in G(\widetilde{e}, A) := \{e\left(\Gamma(\Phi, A)\right) | \forall \Phi \in \mathcal{M}_A(\Phi_0, A_0) \text{ s.t. } e(\Phi) = \widetilde{e}\}. \quad (43)$$

V, R, W and G are nonempty by Assumption 1 and singletons by Theorem 2. Therefore, the recursive problem below is equivalent to the original recursive formulation in equilibrium, as they yield the same equilibrium allocations.

$$v^e(a, z; e, A) = \max_{c, a'} \quad log(c) + \beta \mathbb{E} v^e(a', z'; e', A') \quad (44)$$

$$c + a' = a(1 + r^e(e, A)) + w^e(e, A)z \quad (45)$$

$$a' \geq 0 \quad (46)$$

$$e' = \Gamma^e_{endo}(e, A), \quad A' \sim \Gamma_{exo}(A'|A), \quad (47)$$

Now, I show that the conditional K nullclines are the same as in the representative agent model in the following proposition.

Proposition 4 (Conditional K-nullclines of Krusell and Smith (1998)).
The heterogeneous household model's conditional K nullclines are identical to the counterparts of the model with the representative household and invariant over the initial distribution Φ_0:

$$C_B^{Kcnull} = B(K_B^{Kcnull})^\alpha - \delta K_B^{Kcnull} \quad (48)$$

$$C_G^{Kcnull} = G(K_G^{Kcnull})^\alpha - \delta K_G^{Kcnull} \quad (49)$$

where C_A^{Kcnull} is aggregate consumption.

Proof.
As in Proposition 2, the stationary condition $\delta K = I$ immediately implies the form of the conditional nullclines. ∎

Proposition 4 shows that the K-nullcline in the Krusell-Smith model is distribution-invariant. While this provides strong intuition for monotonicity, it does not by itself

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rule out discrete-time pathologies where the saddle path touches the nullcline at an intermediate point $\Phi \neq \Phi_A^{cs}$ (a "turning point") or crosses it repeatedly (a "spiral"). Such behaviors correspond to folds or loops in the (K, C) phase diagram, which would defeat a single-valued global representation indexed by K.

Proposition 5 (K monotonicity and injectivity)**.**
Fix $A \in \{B, G\}$. Under Assumption 1 and Assumption 2, aggregate capital converges to K_A^{cs} strictly monotonically along the frozen-regime continuation on $\mathcal{M}(A; \Phi_0)$. Consequently, K is injective on $\mathcal{M}(A; \Phi_0)$.

Proof.
Fix A and let $\{\Phi_t\}_{t \geq 0}$ be the frozen-regime continuation, with $K_t := K(\Phi_t)$ and $C_t := C(\Phi_t, A)$. By Proposition 4, the conditional K-nullcline is

$$C_A^{Kcnull}(K) = AK^\alpha - \delta K, \tag{50}$$

and along the frozen-regime law of motion,

$$\Delta K_t := K_{t+1} - K_t = C_A^{Kcnull}(K_t) - C_t = -H_t, \qquad H_t := C_t - C_A^{Kcnull}(K_t). \tag{51}$$

Therefore, Assumption 2 implies $H_t H_{t+1} > 0$ for all t such that $\Phi_t \neq \Phi^{cs}(A; \Phi_0)$, hence $\{H_t\}$ has a constant sign until convergence. Therefore $\Delta K_t = -H_t$ has a constant nonzero sign until convergence, so $\{K_t\}$ is strictly monotone. By Assumption 1 (i) and continuity of K, $K_t \to K_A^{cs} := K(\Phi^{cs}(A; \Phi_0))$.

Finally, strict monotonicity implies $K(\Phi_t) \neq K(\Phi_\tau)$ for $t \neq \tau$. Since $\Phi_t \to \Phi^{cs}(A; \Phi_0)$,

$$\mathcal{M}(A; \Phi_0) = \overline{\{\Phi_t : t \geq 0\}} = \{\Phi_t : t \geq 0\} \cup \{\Phi^{cs}(A; \Phi_0)\}, \tag{52}$$

and a strictly monotone convergent sequence cannot attain its limit at finite time. Hence K takes distinct values at distinct points of $\mathcal{M}(A; \Phi_0)$, i.e. K is injective on $\mathcal{M}(A; \Phi_0)$. ∎

Proposition 4 plays a key role in the proof above because it makes the K-nullcline *sign-determining*: the graph $C = C_A^{Kcnull}(K)$ partitions the (K, C) plane into $\Delta K > 0$ and $\Delta K < 0$ regions via $\Delta K = C_A^{Kcnull}(K) - C$. Assumption 2 rules out pathological discrete-time *crossings* of this partition between successive dates by requiring that

the line segment connecting (K_t, C_t) and (K_{t+1}, C_{t+1}) does not intersect the nullcline. Together, these two ingredients imply that net investment cannot reverse sign along the conditional saddle, yielding strictly monotone capital dynamics and hence an exact one-dimensional representation indexed by K.

Proposition 5 establishes that aggregate capital K is an exact sufficient statistic for endogenous state dynamics in a canonical heterogeneous-household business-cycle model. To complement this theoretical result, Figure 7 plots the computed conditional saddle paths in the (K, C) phase diagram. The model is solved globally using the repeated transition method, and the dynamics under each frozen aggregate state are simulated for 2,000 periods. Although the true endogenous state of the model is the full distribution Φ rather than K alone, the figure shows that conditional saddle paths are strictly and monotonically ordered in K, providing clear computational support for the sufficiency result.
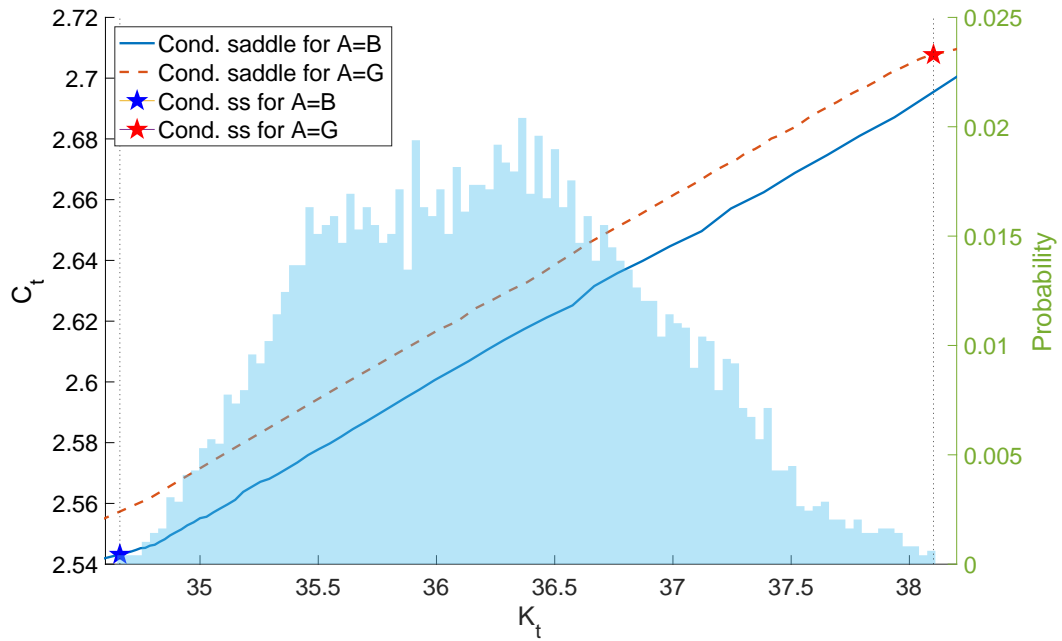

Figure 7: Conditional saddle paths in Krusell and Smith (1998)

Notes: The figure plots conditional saddle paths for $A = B$ (solid) and $A = G$ (dashed) implied by a canonical heterogeneous household business cycle model (Krusell and Smith, 1998). The histogram in the background plots the time-series distribution of the aggregate capital stock.

From near rationality to complete rationality Krusell and Smith (1998) as-

sumes the specific parametric law of motion to compute the heterogeneous-household model. Then, they confirm the accuracy of the law of motion through the accurate consistency between the realized and assumed dynamics. After the celebrated contribution, the approach is often labeled as near or bounded rational approach. The conditional-saddle framework shows that, restricted to the conditional saddle, conditioning on K can be exact (a sufficient statistic) for equilibrium objects. The additional log-linear functional form used by Krusell and Smith (1998) remains a quantitative approximation.

Parametric form of the conditional saddle paths   Despite the indexing function K, specific conditional saddle paths and the form of the law of motion remains undetermined. This problem exists even for representative agent business cycle models. Lee (2025) develops repeated transition method that utilizes the recurrence of the equilibrium allocations along the conditional saddle path. For heterogeneous-agent models, the existence of the indexing variable (sufficient statistic) starkly eases the implementation of the RTM. Theorem 2 and Proposition 5 theoretically supports the implementation feasibility of the RTM using the sufficient statistic.

4.1   Extensions

Theorem 2 provides verifiable conditions for dimension reduction: identify the K-nullcline and verify monotonicity. This section applies the diagnostic to several extensions, illustrating both the framework's scope and its limits.

Economies with multiple endogenous states   Theorem 2 applies to an economy with the multivariate (distributional) endogenous state. As long as there is an aggregate equilibrium variable e that strictly monotonically converges to the conditional steady-state level, then e is a sufficient statistic.

For example, consider the following model that extends the heterogeneous-agent model above by adding endogenous bond holding. The following is the corresponding budget constraint:

$$c + a' + q(\Phi, A)b' = a(1 + r(\Phi, A)) + b + w(\Phi, A)z \tag{53}$$

where b is bond holding and q is the bond price competitively determined by

$$\int b'(a, z; \Phi, A)d\Phi = 0. \tag{54}$$

As long as the production function and the labor supply is assumed as in the canonical setup as above, the K nullcline stays invariant from the representative counterpart, enabling K sufficiency. Lee (2025) showed that the equilibrium allocations are strictly monotonically sorted along K through the globally solved computational outcome. This is because the conditional saddle's strict monotone property along K implied by its conditional nullcline is unaffected by the inclusion of bond dimension. It confirms the prediction of Theorem 2.

Models with endogenous labor supply A heterogeneous-household model with endogenous labor supply and CRRA–GHH preferences is a natural extension of the exogenous-labor-supply framework studied in this paper. GHH utility is particularly useful in this context because it eliminates wealth effects in labor-supply decisions, which could otherwise disrupt the monotonicity properties through distributional channels. Consider preferences given by

$$u(c, l_H) = \frac{1}{1 - \sigma} \left(c - \frac{\eta}{1 + \frac{1}{\chi}} l_H^{1 + \frac{1}{\chi}} \right)^{1 - \sigma}, \tag{55}$$

and the budget constraint

$$c + a' = a(1 + r(\Phi, A)) + w(\Phi, A)z l_H. \tag{56}$$

where z denotes idiosyncratic labor productivity. The individual labor-supply optimality condition then implies the following aggregate labor supply $L(\Phi, A)$:

$$l_H(a, z; \Phi, A) = \left(\frac{z}{\eta} \right)^{\chi} w(\Phi, A)^{\chi} \tag{57}$$

$$L(\Phi, A) = \int z l_H(a, z; \Phi, A)d\Phi = w(\Phi, A)^{\chi} \int \frac{z^{\chi+1}}{\eta^{\chi}} d\Phi_z \tag{58}$$

where $M = \int \frac{z^{\chi+1}}{\eta^{\chi}} d\Phi_z$ and Φ_z is the cumulative distribution function of the stationary productivity distribution. Then, from the optimality condition in the production

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sector with respect to the labor demand,

$$(1 - \alpha)AK^\alpha L(\Phi, A)^{-\alpha} = w(\Phi, A) = \left(\frac{L(\Phi, A)}{M}\right)^{\frac{1}{\chi}}. \tag{59}$$

Combining these two conditions, the conditional aggregate capital nullcline is as follows:

$$c_A^{Kcnull} = A(M^{\frac{1}{1+\alpha\chi}}((1-\alpha)A)^{\frac{\chi}{1+\alpha\chi}})^{1-\alpha}(K_A^{Kcnull})^{\alpha+(1-\alpha)\frac{\alpha\chi}{1+\alpha\chi}} - \delta K_A^{Kcnull}, \tag{60}$$

which satisfies the same monotone convergence to the conditional steady state.

Distributional disturbance Now consider the following CRRA-utility setup:

$$u(c, l_H) = \frac{1}{1-\sigma}c^{1-\sigma} - \frac{\eta}{1+\frac{1}{\chi}}l_H^{1+\frac{1}{\chi}}. \tag{61}$$

The corresponding individual labor supply is as follows:

$$l_H(a, z; \Phi, A) = \left(\frac{z}{\eta c(a, z; \Phi, A)^\sigma}\right)^\chi w(\Phi, A)^\chi \tag{62}$$

Following the same step as in the GHH case above, I obtain the following conditional K nullcline condition:

$$c_A^{Kcnull} = A(M(\Phi^{Kcnull})^{\frac{1}{1+\alpha\chi}}((1-\alpha)A)^{\frac{\chi}{1+\alpha\chi}})^{1-\alpha}(K_A^{Kcnull})^{\alpha+(1-\alpha)\frac{\alpha\chi}{1+\alpha\chi}} - \delta K_A^{Kcnull}, \tag{63}$$

where $M(\Phi^{Kcnull}) = \int \frac{z^{\chi+1}}{\eta^\chi c^{\sigma\chi}}d\Phi^{Kcnull}$. Because of heterogeneous wealth effects, the conditional capital nullcline depends on the distribution of individual states Φ^{Kcnull} associated with the level K^{cnull}. As a result, strict monotonicity of the capital dynamics cannot be verified analytically. Nevertheless, Lee (2025) provides computational evidence that the recursive competitive equilibrium of the model with this utility specification exhibits strictly monotone convergence of aggregate capital K under the standard calibration.

5   Concluding remarks

This paper develops a geometric framework for stochastic equilibrium dynamics by introducing *conditional saddle paths*: invariant equilibrium paths defined under frozen exogenous states. This object extends the familiar saddle-path intuition from deterministic models to environments under aggregate uncertainty. In the resulting phase-diagram representation, business-cycle fluctuations decompose into movements *along* a conditional saddle (endogenous propagation within a regime) and *across* conditional saddles (transitions in exogenous states). The framework clarifies why impulse responses can be state-dependent: such dependence is a geometric property of the stable equilibrium branch and arises precisely when conditional saddles differ in slope rather than by mere vertical translation.

When an aggregate equilibrium variable varies strictly monotonically along a conditional saddle, it provides a global coordinate: it uniquely indexes equilibrium states and therefore summarizes all equilibrium allocations and prices on the relevant invariant set. Applying this logic, I provide a theoretical proof of the sufficiency of aggregate capital in a canonical heterogeneous-household model. Beyond the Krusell–Smith benchmark, the same reasoning applies in multi-asset and richer heterogeneous-agent environments whenever a monotone-convergent aggregate coordinate exists.

More broadly, conditional saddles offer a complementary lens on stochastic models: they provide a language for interpreting nonlinear dynamics and for assessing when scalar state approximations are exact rather than merely accurate. They also provide a natural geometry for *state-contingent* policy analysis: by making state dependence explicit in the phase diagram, the framework clarifies when the same intervention should be expected to have different quantitative effects across regimes and over the cycle. A promising direction for future work is to use these geometric objects to sharpen empirical restrictions on state dependence and to discipline the design of state-contingent stabilization policies in heterogeneous-agent economies under aggregate uncertainty.

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