Representations of Belief: Ranking Theory

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***Mind*: Degrees of Belief under Uncertainty**

It is unclear how humans represent environmental uncertainty. When engaging with the world, taking uncertainty into account in perceptual processing may be crucial. For instance, a hiker descending a mountain must determine whether or not to take a certain path based on their perception of the slope's steepness. It would be useful to account for the uncertainty about the slope. The hiker can do so by getting closer to the slope to gather more information before deciding whether to take an alternative path. The observer's degree of belief in the ideal path to descend will alter as new information about the steepness becomes available. Uncertainty can be characterized as the observer's lack of knowledge about the state of the world, as humans cannot represent all conceivable information in the environment (Walker et al., 2022).

When faced with uncertainty, humans struggle to know what to do or what to believe in. The common stance in the literature suggests that people are Bayesian and represent uncertainty by assigning a probability to each possible state of the world, using probability laws to calculate the best action (Sanborn et al., 2016). In this view, degrees of belief simply follow the laws of probability and are assumed to be real numbers in the range [0,1]. The greater an agent's degree of belief in a proposition, the greater their confidence in the truth of that proposition (Huber, 2009). There is a possibility that humans use probability distributions to develop beliefs, but probability alone may not be sufficient to fully explain degrees of belief; the problem is that belief is not a probability (Spohn, 2013). But rather, the probability function can be thought of as simply the additive measure required to compute the expected utility for decision making (Smets, 2002).

The gambling and lottery paradox reveals a common tendency to make incorrect probability judgments, revealing the difficulties that people have in conceiving about the nature of probability. Consider a fair 1000-ticket lottery with only one winning ticket. Suppose that an event is very likely if its probability is greater than 0.99. On these grounds, it is presumed rational to accept that lottery ticket 1 will not win (i.e., probability of 0.001). Because the lottery is fair, it is reasonable to assume that ticket 2 will not win either, and so on until accepting that ticket 1000 will not win entails accepting that no ticket will win, which contradicts the proposition that one ticket will win (Wheeler, 2007). It is possible to argue that we have beliefs and probabilities, and that neither is reducible to the other (Spohn, 2013). Thus, our exploration is motivated by the notion that probabilities alone are not a good model for uncertain reasoning in humans, and that a better account for uncertain reasoning in humans must be developed.

Ranking theory is a formal epistemology that seeks to provide a normative account of belief dynamics as an alternative to current probabilistic approaches (Spohn, 2013; Skovgaard-Olsen, 2015). Ranking theory eliminates the need to assign probabilities to events, as a result, it would be very useful to know whether humans follow the laws of ranking theory when estimating their beliefs about uncertainty. One crucial notion is that ranks and probabilities express two distinct forms of uncertainty. Ranks should not be regarded as a replacement for probabilities. Rather, ranks are better characterized in terms of degrees of surprise/disbelief, and the problem is that differences in probability often do not warrant differences in degrees of surprise/disbelief.

Theory: **A Ranking-Theoretic Approach**

In ranking theory, beliefs are expressed in terms of ranking functions (e.g., a two-sided ranking function expresses belief and disbelief at once). The negative ranking function, κ, can be thought of as varying degrees of disbelief or surprise an observer has about a proposition. We assume a non-empty set W (“the world”) of mutually exclusive and jointly exhaustive possible worlds which is the set of all possibilities or propositions. Every subset of W is a proposition, and the set of all propositions is W. The degree of disbelief in propositions is formally quantified by a *negative ranking function*, denoted by κ. More precisely, a negative ranking function κ is a function from **A** into the set of non-negative reals plus infinity, such that $\kappa(W) = 0$, $\kappa(\emptyset) = 0$, and for all propositions $A, B \in$ **A,** $\kappa(A \cup B) = min\{\kappa(A), \kappa(B)\}$. In ranking theory, we are taking A as an input and returning either a non-negative real number or infinity [Eq 1]. If $\kappa(A) = 0$, A is not disbelieved at all (i.e. “fully believed”), Conversely, if $\kappa(A) > 0$, A is disbelieved to some positive degree [Eq 2]. Belief in A is the same as disbelief in $\bar{A}$ [Eq 2]. It then follows that for each $A \in \mathbf{A}$, either $\kappa(A)$ or $\kappa(\bar{A})$ or both are equal to 0 [Eq 2]. Finally, the law of disjunction states that for a set containing propositions A and B, the negative ranking of that set is just the minimum of the two [Eq 3].

Ranking theory's conception of uncertainty and belief accounts for certain intuitive uses of certainty and uncertainty in language. Consider the following question: "Are we certain that the sun will rise tomorrow?" While it is apparent that no one will (in earnest) disaffirm their belief in this proposition, Hume's Problem of Induction has demonstrated that we cannot know this proposition with certainty. Hume illustrates that we lack a certain or *a priori* foundation for our inductive beliefs because they are justified by prior inductive inferences and an assumption of nature's uniformity (Hume, 1739; 1748). In other words, our belief X (e.g., that the sun will rise tomorrow) is based on our inference from our past observance Y (e.g., that the sun has always risen). An assumption in this chain of reasoning that the present will resemble the past – that our past observance Y will be true in the present (known as the Uniformity Principle). However, Hume points out that our belief that our past observance Y will remain true (our belief in the Uniformity Principle) is *itself* believed, based on induction and the observance that this has always been true over time. Consequently, a circularity emerges whereby the Principle of Uniformity (our belief that the future will resemble the past) is justified by itself. This argument culminates in the recognition that we do not possess *a priori* certainty that the sun will rise tomorrow.

Nonetheless, it appears unusual to assert that there is a 99.9999% (or to an exponent one is inclined to stop at) chance that the sun will rise tomorrow. Moreover, the statement, "we are 99.9999% certain that the sun will rise tomorrow" does not appear to capture people's intuitions and degree of certainty concerning the likelihood of the sun rising. In other words, probabilistic models of people's belief that the sun will rise tomorrow do not seem to fully explain the intuitive sense with which these beliefs are maintained. In contrast, ranking theory's emphasis on degrees of belief and disbelief can better account for this intuitive sense because, rather than assigning a probability, it explains that people have an extremely strong degree of belief in the proposition 'the sun will rise tomorrow' and nigh-infinite disbelief in the possibility that this is not the case. Consequently, despite the lack of *a priori* foundations for the inductive inference, people have a complete or overwhelming belief in the proposition's truth and a complete or overwhelming disbelief in its negation. In ranking theory, the contrast between people's belief/disbelief in the proposition ‘the sun will rise tomorrow’ better explains the strength of people’s beliefs in the proposition than assigning a 99.9999% probability does.

Degrees of belief may be more easily expressed in a ranking function than as subjective probability, but there are also limitations to this approach. Suppose that in a set W of all possible propositions, the negative ranks of three possible propositions are $\kappa(A) = 0$, $\kappa(B) = 30$, and $\kappa(C) = 10$. Given these ranks, the B proposition has the greatest disbelief, followed by the C proposition, and then the A proposition is not disbelieved or believed. However, simply stating that an agent has a $\kappa(B) = 30$ makes it difficult to interpret what this value means without knowing the agent's ranks to other sets of propositions. We have a clear definition of what that $\kappa(W) = 0$ and $\kappa(\emptyset)= \infty$ means [Eq 1]. Yet, stating that someone has a disbelief rank of 30 without reference to other possible propositions may not be informative. Subjective probability, on the other hand, sums to 1, allowing for more intuition into what an agent means by a probability of 0.5, but still lacks our understanding of their degree of belief. Thus, our first goal was to determine if ranks have a consistent mapping to subjective probabilities. Humans may set their degrees of belief based on some subjective probability value(s) of a proposition. Spohn (2013) suggests that negative ranks are simply the logarithm of probabilities concerning some base <1 [Eq 4]. However, it is unclear what this logarithm base is for each agent. We would see a consistent logarithm base across the three questionnaire sets (Table 1) if an agent were to use subjective probabilities to assign ranks, and vice versa. It is important to note that our mapping between negative ranking functions and subjective probabilities does not indicate a causal relation, rather just explores whether there is a correlation between negative ranks and subjective probabilities. Even if we were to observe a one-to-one correspondence between subjective probabilities and rankings, it remains unclear whether there are any other unknown factors (e.g., higher-order probability distributions).

These limitations also give rise to potential theoretical constraints on ranking theory, such as the normativity of the account and how generalizable it is across various people(s). This concern may be summarized in the question: “Are we certain that different observers form their beliefs about given propositions according to a similar or comparable schema?” Some observers may be naturally more skeptical, ranking/perceiving the world primarily through their disbelief, whereas others, more credulous, rank/perceive the world through their beliefs. Thus, it is possible that different observers may be more or less conservative in their rankings, resulting in a qualitative difference in the level of credulity between various individuals. This may pose the following consideration for ranking theory: if there is a qualitative difference in the schema whereby various individuals form, conceive, or rank their beliefs about a given proposition, then the rankings of individuals operating under a qualitatively different schema may not generalize and offer a singular normative account.

This raises the theoretical question of ranking theory: do all individuals operate within a similar and comparable schema of ranking possible outcomes (such that all rankings are normative and highly generalizable), or do some individuals operate under different schemas for ranking possible outcomes? To examine how comparable two observers' ranking or probability responses are toward the same set of propositions, we utilized the Jaccard Index and Kendall Tau Distance for each pair of ranking responses [Eq 6, Eq 7].

***Math*: The Laws of Ranking Theory**

Here we outline the mathematics used to conceptualize and implement ranking theory and similarity/dissimilarity measures in Python code.

Eq 1. Let A be an algebra over W. Then κ is a *negative ranking function* for A iff κ is a function from A into $\mathbb{R}* = \mathbb{R}+ \cup \{\infty\}$ (the set of non-negative reals plus infinity) such that for all $A, B \in$ ***A*****.** In ranking theory, we are taking A as an input and returning either a non-negative real number or infinity. Since W represents all the "worlds", $\kappa(W) = 0$, because W, the set of all propositions, contains your beliefs. Conversely, the empty set represents a contradiction to the "worlds" and therefore kappa of the empty set is equal to infinity which is the maximal value, essentially the empty set contains no propositions and therefore will never contain your beliefs

$$\kappa(W) = 0 \text{ and } \kappa(\varnothing) = \infty.$$

Eq 2. Law of negation. For each proposition A, either the negative rank of A is 0 or the negative rank of non-A is 0. Essentially, if A is a proposition in W, then the non-A is all the other propositions that are not A in W. Therefore, your belief can be contained in A or in non-A. Via the law of negation A is believed in iff $\kappa(\bar{A}) > 0$, which means that there is some degree of disbelief in the non-A. Finally when $\kappa(\bar{A}) = 0$, κ is neutral or un-opinionated concerning A.

$\kappa(A) = 0$ or $\kappa(\bar{A}) = 0$ or both

A is believed in κ iff $\kappa(\bar{A}) > 0$

$\kappa(\bar{A}) = 0$; κ is neutral or un-opinionated concerning A

Eq 3. Law of disjunction. $\kappa(A)$ is often referred to as the negative rank of A and the law of disjunction says that for a set containing propositions A and B, the negative ranking of that set is just the minimum of the two.

$$\kappa(A \cup B) = \min\{\kappa(A), \kappa(B)\}.$$

Def. 1 (Positive ranking function): The positive ranking function β of A is the negative ranking of the non-A

$$\beta(A) := \kappa(\bar{A}).$$

Def. 2 (Two-sided ranking function): The two-sided ranking function τ is the negative ranking function subtracted from the positive ranking function

$$\tau(A) := \beta(A) - \kappa(A).$$

Def. 3 (Conditional negative ranking function): Let κ be a negative ranking function over **A** and A $\in$ **A** such that $\kappa(A) < \infty$. The conditional negative ranking function given A is defined as

$$\kappa(B|A) := \kappa(B \cap A) - \kappa(A),$$

for any B $\in$ **A.**

Eq. 4 (Probability to rankings): Let Ω be a set of possible worlds, and $P(w)$, $w \in \Omega$ a probability distribution function, i.e. $P(w) \geq 0$ and

$$\sum_{w \in \Omega} P(w) = 1.$$

The probabilities can then be mapped to rankings by the transformation

$$P(w) \mapsto \kappa_P^a(w) := log_a\left(\frac{P(w)}{p}\right),$$

where $a \in (0, 1)$ and $p := max_{\nu \in \Omega} P(\nu)$ is the maximum of the distribution function.

Eq. 5 (Rankings to probability): Let κ be a negative ranking function and $w \in \Omega$ and event in the world space. The rankings can then be mapped to probabilities by the transformation

$$\kappa(w) \mapsto P_\kappa^a(w) := \frac{a^{\kappa(w)}}{Z(a, \kappa)}.$$

where $a \in (0, 1)$ and

$$Z(a, \kappa) := \sum_{\nu \in \Omega} a^{\kappa(\nu)}$$

is the normalizing factor for the probability distribution.

Similarity/Dissimilarity Distances

Eq. 6 (Jaccard index): The Jaccard index is used to compare the similarity or distinctness between two sets by measuring the overlap they share with their attributes. Each attribute of A and B can either be 0 or 1, with 0 meaning not similar and 1 meaning similar.

$$J(A,B) = \frac{|A \cap B|}{|A \uplus B|} = \frac{|A \cap B|}{|A| + |B|}$$

Eq. 7 (Kendall Tau rank distance): Kendall Tau distance is calculated using pairwise comparison, or more specifically, disagreements between two ranking lists. The Kendall Tau distance for two ranking lists τ_1 and τ_2 is defined as

$$K_d(\tau_1, \tau_2) = |\{(i,j) : i<j, [\tau_1(i) < \tau_1(j) \wedge \tau_2(i) > \tau_2(j)] \vee [\tau_1(i) > \tau_1(j) \wedge \tau_2(i) < \tau_2(j)]\}|,$$

where $\tau_1(i)$ and $\tau_2(i)$ are the rankings of the element i in τ_1 and τ_2, respectively. The greater the distance between the two lists τ_1 and τ_2, the more dissimilar they are.

***Code*: Investigating Ranking Theory**

Participants 1 and 2 completed ranking and probability ratings for the same three scenarios (see Table 1). First, participants were asked to rank their degree of disbelief for three fictional scenarios involving 1) language demographics of Quebec, 2) September weather in Boston, and 3) music genre preference. Participants then provided probability ratings for the same scenarios. Curve fitting was used to compute the optimal logarithm base values for the mapping between ranks and probability estimates. Ranks and probability estimates were then compared using a similarity and dissimilarity distance measure, Jaccard Index [Eq 6] and Kendall's Tau distance [Eq 7] to determine how comparable two participants' ranking or probability responses are toward the same set of propositions. All analyses were conducted using Python 3.10.4 (Python Software Foundation, https://www.python.org/). While Python supports multiple programming paradigms, it specifically supports functional programming. A benefit of functional programming is that it allows for modularity and high flexibility. Essentially, it made it effortless to implement various distance measures to compare ranking and probability ratings.

Table 1

Rank and probability ratings from participant 1 and participant 2

	Hanbin	Sophia

Suppose you meet a person who has only lived in Quebec since birth and is monolingual. Suppose you meet a person who has only lived in Quebec since birth and is monolingual. Would you be surprised to learn that they only speak?

	Rank	*Probability*	*Rank*	*Probability*
French	0	0.880	0	0.970
Spanish	3	0.030	75	0.003
English	0	0.100	5	0.010
Greek	12	0.002	300	0.002
Polish	12	0.002	90	0.003
Korean	12	0.002	250	0.001
Urdu	10	0.003	80	0.004
Japanese	12	0.002	200	0.003
Spanish or Urdu	3	0.030	50	0.004

Suppose you are planning a day trip to Boston in September. Suppose you are planning a day trip to Boston in September. Would you be surprised to learn that the daytime high temperatures will range between ________ degrees Celsius? Would you be surprised to learn that the daytime high temperatures will range between degrees Celsius ($\circ$C)?

	Rank	*Probability*	*Rank*	*Probability*
-29 to -20	40	0.0001	50	0.005

-19 to -10	20	0.0020	20	0.020
-9 to 0	5	0.0030	15	0.040
1 to 10	1	0.0500	10	0.080
11 to 20	0	0.4000	0	0.700
21 to 30	0	0.5000	10	0.040
31 to 40	4	0.0200	40	0.020
41 to 50	10	0.0070	50	0.005
-9 to 0 or 1 to 10	5	0.0500	5	0.090

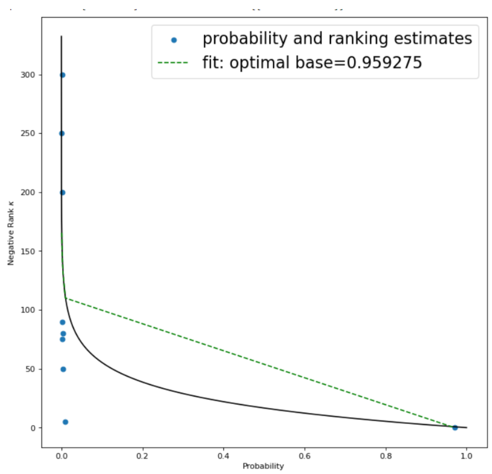
Suppose you are an employee at an automotive service centre. Suppose you are an employee at an automotive service centre. You have been invited to a work karaoke party, and you know your co-worker Tom will be there. You have been invited to a work karaoke party, and you know your co-worker Tom will be there. Tom is a Caucasian male in his 20s who grew up in a rural area and loves to wear his cowboy hat. Tom is a Caucasian male in his 20s who grew up in a rural area and loves to wear his cowboy hat. Would it surprise you to learn that Tom enjoys singing along to ________ music?

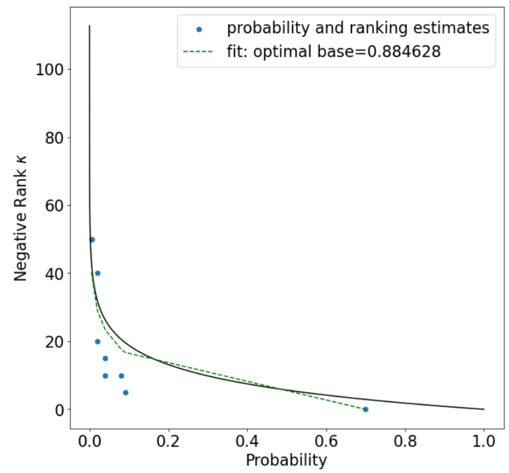
	Rank	*Probability*	*Rank*	*Probability*
Pop	2	0.050	20	0.015
Rock	1	0.200	5	0.020
Hip Hop	7	0.010	25	0.005

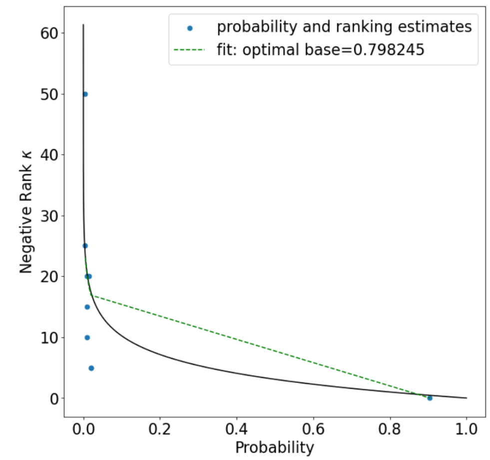
Heavy Metal	5	0.020	10	0.010
Country	0	0.500	0	0.900
Jazz	10	0.010	20	0.10
Opera	50	0.005	50	0.005
Folk	1	0.200	5	0.020
Jazz or Opera	10	0.010	15	0.010

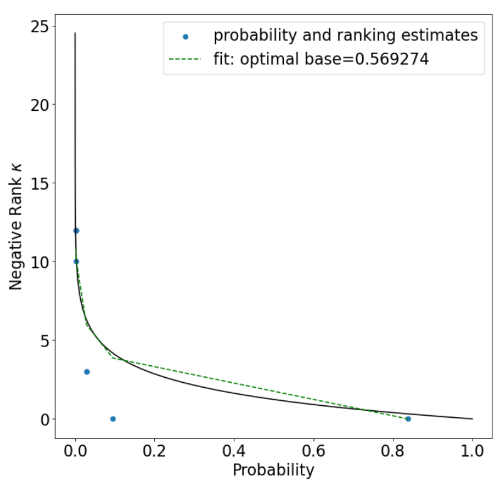
Table 2

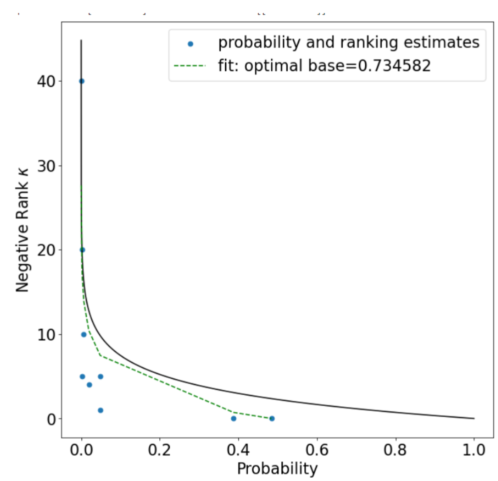
Optimal base and estimated variances for participant 1 and 2

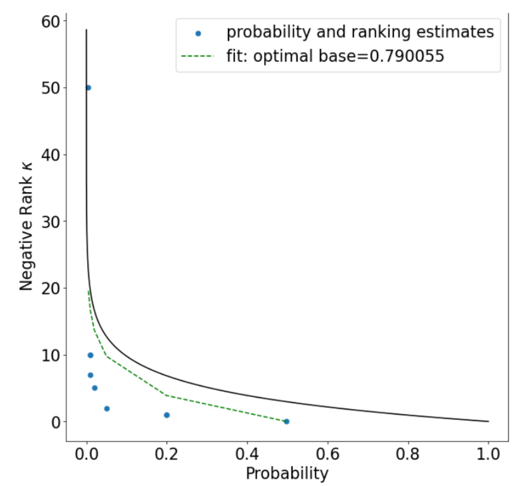
<table>
<tr><th></th><th>Language Demographics of Quebec</th><th>September weather in Boston</th><th>Music Genre Preference</th></tr>
<tr><td>Participant 1</td>
<td>

Optimal base: 0.9592746
Estimated covariance: 7.82011479e-05</td>
<td>

Optimal base: 0.88462776
Estimated covariance: 0.00018543</td>
<td>

Optimal base: 0.79824458
Estimated covariance: 0.0014603</td></tr>
<tr><td>Participant 2</td>
<td>

Optimal base: 0.56927351
Estimated covariance: 0.00075721</td>
<td>

Optimal base: 0.73458166
Estimated covariance 0.00018543</td>
<td>

Optimal base: 0.79005484
Estimated covariance 0.00358675</td></tr>
</table>

SciPy's *optimize* function was used to compute the optimal logarithmic base and covariance for the mapping between ranks and probability estimates. This function minimizes or maximizes objective functions that are subject to constraints. The optimal base values varied between and within participants across questionnaires, as shown in Table 2, indicating inconsistency in ranking responses in relation to probability estimates (Table 2). We did not expect to see a consistent mapping between subjective probability and negative ranks because subjective probability does not account for degrees of belief. However, it is reasonable to think that the inconsistency observed in the mapping between ranks and probability is due to participant's poor ability to assign ranks and/or possibly subjective probability estimates to each proposition.

To examine how similar two ranking and probability ratings are, Jaccard index was used [Eq 6]. Given two sets with *n* binary attributes, each attribute can either be 0 or 1 (Chung et al., 2019). The closer to 1, the more similar the two sets are (Chung et al., 2019). The index is then calculated by taking the number of observations in both sets and dividing by the number in either set (a ratio of the intersection to the union; Chung et al., 2019). Using the Jaccard index, we wanted to see how similar the rank and probability measures for the same questionnaires were for each participant (Chung et al., 2019). For example, say we have two sets: **A = 0, 3, 6, 7, 8, 9; B= 0, 1, 2, 3, 4, 5.** The number of observations in both (0, 3 = 2) divided by the number of observations in either (0, 1, 2, 3, 4, 5, 6, 7, 8, 9 = 10) would result in a Jaccard index of 0.2 (2/10).

Table 3

Participants' rank and probability distance using the Jaccard Index

	Ranks	Probability
Language Demographics of Quebec	0.08333333333333333	0.0
September Weather in Boston	0.5555555555555556	0.0
Music Genre Preference	0.4	0.0

The Jaccard Index can be useful for comparing rankings, as it compares members from two sets to see which members are shared and which are distinct, but is not useful for comparing probability estimates, since two lists are unlikely to contain the 'exact' probability values (e.g., unlikely to observe an intersection), unless probability estimates were rounded to the nearest tenth. However, it is unable to compare the magnitude of the differences between the two members (see Table 3).

Next, we used the Kendall tau rank distance which is a distance function that counts the number of pairwise disagreements (inversions) between lists. The greater the distance between the lists, the more dissimilar they are (Kendall, 1938). Kendall tau distance was normalized, thus if the K distance for the lists is 0 then the rankings completely agree. Conversely, a K distance of 1 indicates maximum disagreement between the lists (Kendall, 1938).

Table 4

Participants' rank and probability distance using Kendall's Tau

	Ranks	Probability
Language Demographics of Quebec	0.5277777777777778	0.3055555555555556
September Weather in Boston	0.4166666666666667	0.4444444444444444
Music Genre Preference	0.3611111111111111	0.3611111111111111

In contrast to the Jaccard index, the Kendall tau distance function allowed us to compute the dissimilarity between our rank and probability sets. As seen in Table 4, the greatest disparity was discovered in our two ranking sets of the Language Demographics of Quebec questionnaire, and the greatest disparity in the probability sets of September Weather in Boston. The disadvantage of this calculation is that it does not take into account the actual values of the ranks.

Discussion

This project was approached from the theoretical perspective that ranking theory provides an alternative normative account of belief systems to current probabilistic approaches. Probability theory proposes that humans represent uncertainty by assigning probability to each possible state of the world, but it should not be regarded as a metric of beliefs. This paper introduced ranking theory, which is based on belief metrics that quantify a grading of disbelief expressed by negative ranking functions. We explored the relationship between negative ranks and probabilities for three sets of questionnaires by computing the optimal logarithmic base for rank-to-probability translation. A difference in the logarithmic base was observed between the two participants for each set of questionnaires. The logarithmic base was also inconsistent across the three different questionnaires within subjects. We would have expected participants' mapping to be consistent across the three questionnaires if they solely relied on subjective probability to rank their degrees of disbelief. We also expect individual variability in the relationship between subjective probability and degrees of belief (e.g., different logarithmic base for each participant), as ‘0.001’ probability may not imply the same degree of disbelief for everyone.

The degree to which different observers’ beliefs/disbeliefs are comparable should be investigated and discussed further. The question here results: is the inner variation between given observers so great as to render their rankings difficult or problematic to compare? Although some variation between observers is to be expected, a better understanding of how normative or generalizable the rankings of a given individual are to others may serve to explain the nature of ranks. If it should be discovered that the process by which different observers rank their beliefs/disbeliefs is fundamentally distinct (such that their ranks mean different things) then this would suggest that the nature of these ranks is in some sense personally housed or defined by the given observer. In contrast, if it should be discovered that the process of ranking beliefs/disbeliefs is highly normative and generalizable, then this would suggest that the presence of an innate or inherited cognitive ranking ability is similar/shared across individuals. Finally, it is possible that the process of ranking beliefs/disbeliefs is in part innate and in part personally developed over one’s life.

These possibilities may be drawn out by the following example: there is a general cognitive (sensory-perceptual) ability to discern color in the human organism which is stable across individuals. However, it has also been demonstrated that there is a variance and personally acquired/learned perception of color across various individuals (Emery, 2019). This example falls under the third possibility stated above, whereby there is an inherited and generally stable ability that is also subject to a degree of variance across different observers. With respect to ranking theory, discerning to what extent the process/reasoning whereby individuals rank their degrees of belief/disbelief is innately shaped as opposed to personally learned is a non-trivial matter. A greater clarity in this capacity would serve to help elucidate whether the phenomena of individuals assigning ranks is a cognitively acquired or learned activity.

A final point for further consideration is the limitations of using the Jaccard Index and Kendall Tau's distance. As noted previously, neither of these distance measures account for both the order and magnitude of difference. To address this limitation, further comparisons of participants' rankings and probability ratings can be conducted by using a distance measure. For example, the distance induced by a p-norm can account for both order and magnitude of difference in two lists. For any $p \geq 1$, the distance between two ranked lists **X** and **Y**, represented as n-dimensional vectors is given by:

$$d_p(X, Y) := \left(\sum_{i=1}^{n} |X_i - Y_i|^p \right)^{1/p} .$$

In the case where $p = 1$, we have

$$d_1(X, Y) := \sum_{i=1}^{n} |X_i - Y_i| ,$$

which is commonly called the "Manhattan" or "Taxicab" (L^1 $norm$) distance. As an example, for two lists: A: 1, 2, 3; B: 0, 5, 2. The Manhattan distance would then be $|1 - 0| + |2 - 5| + |3 - 2|$ = 5. In the case where $p = 2$, we have

$$d_2(X, Y) := \sqrt{\sum_{i=1}^{n} |X_i - Y_i|^2},$$

which is the usual Euclidean distance (L^2 $norm$) between two vectors. As an example, for two lists: A: 1, 2, 3; B: 0, 5, 2. The Euclidean distance would then be $\sqrt{|1 - 0|^2 + |2 - 5|^2 + |3 - 2|^2}$ = $\sqrt{11}$. The use of distance measures that account for both order and magnitude will hopefully provide a more comprehensive analysis of participants' rankings and probability ratings.

While the question of how humans represent environmental uncertainty remains unclear, a further examination into whether individuals operate with comparable and generalizable ranking *schemata* will hopefully provide more clarity on the cognitive representation of uncertainty.

References

Chung, N. C., Miasojedow, B., Startek, M., & Gambin, A. (2019). Jaccard/Tanimoto similarity test and estimation methods for biological presence-absence data. *BMC Bioinformatics*, *20*(644), 1-11. doi: 10.1186/s12859-019-3118-5

Emery, K. J., & Webster, M. A. (2019). Individual differences and their implications for color perception. Current opinion in behavioral sciences, 30, 28-33.

Huber, F. (2009). Belief and degrees of belief. In *Degrees of belief* (pp. 1-33). Springer, Dordrecht.

Hume, D. 1739, *A Treatise of Human Nature*. Oxford, Oxford University Press.

Hume, D. 1748, *An Enquiry Concerning Human Understanding.* Oxford, Oxford University Press.

Kendall, M. G., A new measure of rank correlation, *Biometrika*, *30*,(1-2), 81–93. doi:10.1093/biomet/30.1-2.81

Python Software Foundation. [Python 3.10.4]. (2022). Retrieved from https://www.python.org/.

Sanborn, A. N., & Chater, N. (2016). Bayesian brains without probabilities. *Trends in cognitive sciences*, *20*(12), 883-893.

Spohn, W. (2013). A ranking‐theoretic approach to conditionals. *Cognitive Science*, *37*(6), 1074-1106.

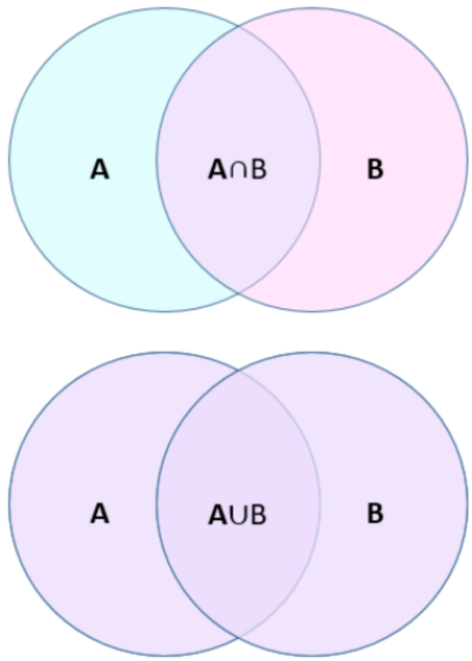
Smets, P. (2002). Decision making in a context where uncertainty is represented by belief functions. In *Belief functions in business decisions* (pp. 17-61). Physica, Heidelberg.

Walker, E. Y., Pohl, S., Denison, R. N., Barack, D. L., Lee, J., Block, N., Lee, J., Block, N., Ma, W. J., & Meyniel, F. (2022). Studying the neural representations of uncertainty. *arXiv preprint arXiv:2202.04324*.

Wheeler, G. (2007). A review of the lottery paradox. *Probability and inference: Essays in honour of Henry E. Kyburg, Jr*, 1-31.

Appendix A

Visualization of Jaccard Index



Intersection and union of two sets A and B

Appendix B

Visualization of Kendall Tau Rank Distance

Person	A	B	C	D	E	F	G	H	I	J	ranking
Rank by Test 1	1	2	3	4	5	6	7	8	9	10	A>B>C>D>E>F>G>H>I>J
Rank by Test 2	3	4	1	2	5	8	9	6	7	10	C>D>A>B>E>H>I>F>G>J

Pair	Test1	Test 2	Count	Pair	Test1	Test 2	Count	Pair	Test1	Test 2	Count
(A,B)	1 < 2	3 < 4		(A,C)	1 < 3	3 > 1	x	(A,D)	1 < 4	3 > 2	x
(A,E)	1 < 5	3 < 5		(A,F)	1 < 6	1 < 8		(A,G)	1 < 7	1 < 9	
(A,H)	1 < 8	1 < 6		(A,I)	1 < 9	1 < 7		(A,J)	1 < 10	1 < 10	
(B,C)	2 < 3	4 > 1	x	(B,D)	2 < 4	4 > 2	x	(B,E)	2 < 5	4 < 5	
(B,F)	2 < 6	4 < 8		(B,G)	2 < 7	4 < 9		(B,H)	2 < 8	4 < 6	
(B,I)	2 < 9	4 < 7		(B,J)	2 < 10	4 < 10					
(C,D)	3 < 4	1 < 2		(C,E)	3 < 5	1 < 5		(C,F)	3 < 6	1 < 8	
(C,G)	3 < 7	1 < 9		(C,H)	3 < 8	1 < 6		(C,I)	3 < 9	1 < 7	
(C,J)	3 < 10	1 < 10									
(D,E)	4 < 5	2 < 5		(D,F)	4 < 6	2 < 8		(D,G)	4 < 7	2 < 9	
(D,H)	4 < 8	2 < 6		(D,I)	4 < 9	2 < 7		(D,J)	4 < 10	2 < 10	
(E,F)	5 < 6	5 < 8		(E,G)	5 < 7	5 < 9		(E,H)	5 < 8	5 < 6	
(E,I)	5 < 9	5 < 7		(E,J)	5 < 10	5 < 10					
(F,G)	6 < 7	8 < 9		(F,H)	6 < 8	8 > 6	X	(F,I)	6 < 9	8 > 7	X
(F,J)	6 < 10	8 < 10									
(G,H)	7 < 8	9 > 6	X	(G,I)	7 < 9	9 > 7	X	(G,J)	7 < 10	9 < 10	
(H,I)	8 < 9	6 < 7		(H,J)	8 < 10	6 < 10					
(I,J)	9 < 10	7 < 10									

Since there are eight pairs whose values are in opposite order, the Kendall tau distance is eight.