ECE 124 digital circuits and systems
Assignment #3

Q1: Find minimum SOP and POS expressions for the following functions using 3-variable Kmaps.

(a) $f(a, b, c) = \sum(0, 2, 6, 7)$

(b) $f(a, b, c) = \Pi(1, 2, 4)$

Solution:

(a) $f(a, b, c) = \sum(0, 2, 6, 7) = a'c' + ab = (a + c')(a' + b)$

a	b	c	f
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

a \ bc	00	01	11	10
0	1	0	0	1
1	0	0	1	1

a \ bc	00	01	11	10
0	1	0	0	1
1	0	0	1	1

(b) $f(a,b,c) = \Pi(1,2,4) = a'b'c' + bc + ac + ab = (a+b+c')(a+b'+c)(a'+b+c)$

a	b	c	f
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

a \ bc	00	01	11	10
0	1	0	1	0
1	0	1	1	1

a \ bc	00	01	11	10
0	1	0	1	0
1	0	1	1	1

Q2: Find minimum SOP and POS expressions for the following functions using 3-variable Kmaps.

(a) $f = xy + x'y'z' + x'yz'$

(b) $f = A'B + BC' + B'C'$

(c) $f = (x' + y + z)(x' + y + z')(x' + y' + z)$

Solution:

(a) $f = xy + x'y'z' + x'yz' = x'z' + xy = (x + z')(x' + y)$

x \ yz	00	01	11	10
0	1	0	0	1
1	0	0	1	1

x \ yz	00	01	11	10
0	1	0	0	1
1	0	0	1	1

(b) $f = A'B + BC' + B'C' = C' + A'B = (B + C')(A' + C')$

A \ BC	00	01	11	10
0	1	0	1	1
1	1	0	0	1

A \ BC	00	01	11	10
0	1	0	1	1
1	1	0	0	1

(c) $f = (x' + y + z)(x' + y + z')(x' + y' + z) = x' + yz = (x' + y)(x' + z)$

x \ yz	00	01	11	10
0	1	1	1	1
1	0	0	1	0

x \ yz	00	01	11	10
0	1	1	1	1
1	0	0	1	0

Q3: Find minimum SOP and POS expressions for the following functions using 4-variable Kmaps.

(a) $f = w'z + xz + x'y + wx'z$.

(b) $f = B'D + A'BC' + AB'C + ABC'$

(c) $f = (A' + B' + D')(A + B' + C')(A' + B' + D')(B + C' + D')$.

Solution:

(a)

$$f = w'z + xz + x'y + wx'z = z + x'y = (y + z)(x' + z) \ .$$

wx \ yz	00	01	11	10
00	0	1	1	1
01	0	1	1	0
11	0	1	1	0
10	0	1	1	1

wx \ yz	00	01	11	10
00	0	1	1	1
01	0	1	1	0
11	0	1	1	0
10	0	1	1	1

(b)

$$f = B'D + A'BC' + AB'C + ABC' = BC' + B'D + AB'C = (B' + C')(B + C + D)(A + C' + D)$$

AB \ CD	00	01	11	10
00	0	1	1	0
01	1	1	0	0
11	1	1	0	0
10	0	1	1	1

AB \ CD	00	01	11	10
00	0	1	1	0
01	1	1	0	0
11	1	1	0	0
10	0	1	1	1

(c)

$f = (A' + B' + D')(A + B' + C')(A' + B + D')(B + C' + D') = A'C' + C'D' + AD' + B'D' = (C' + D')(A' + D')(A + B' + C')$.

AB \ CD	00	01	11	10
00	1	1	0	1
01	1	1	0	0
11	1	0	0	1
10	1	0	0	1

AB \ CD	00	01	11	10
00	1	1	0	1
01	1	1	0	0
11	1	0	0	1
10	1	0	0	1

Q4: Find minimum SOP and POS expressions for the following functions using 4-variable Kmaps.

(a) $f(w,x,y,z) = \sum(0,1,2,3,4,12,13,14,15)$

(b) $f(w,x,y,z) = \Pi(0,1,2,4,5,7,8,9,10,12,14,15)$

Solution:

(a)

$$f(w,x,y,z) = \sum(0,1,2,3,4,12,13,14,15) = w'x' + wx + w'y'z' = (w'+x)(w+x'+z')(w+x'+y')$$

wx \ yz	00	01	11	10
00	1	1	1	1
01	1	0	0	0
11	1	1	1	1
10	0	0	0	0

wx \ yz	00	01	11	10
00	1	1	1	1
01	1	0	0	0
11	1	1	1	1
10	0	0	0	0

(b)

$$f(w,x,y,z) = \Pi(0,1,2,4,5,7,8,9,10,12,14,15) = x'yz + w'xyz' + wxy'z = (y+z)(x+y)(x+z)(w+x'+z')(w'+x'+y')$$

wx \ yz	00	01	11	10
00	0	0	1	0
01	0	0	0	1
11	0	1	0	0
10	0	0	1	0

wx \ yz	00	01	11	10
00	0	0	1	0
01	0	0	0	1
11	0	1	0	0
10	0	0	1	0

Q5: Find minimum SOP and POS expressions for the following functions using 5-variable Kmaps.

(a) $f(x_1, x_2, x_3, x_4, x_5) = \sum(0, 1, 4, 5, 16, 17, 21, 25, 29)$

(b) $f = x_1'x_2'x_3x_4' + x_1'x_2'x_3'x_4' + x_2'x_4'x_5' + x_2'x_3x_4' + x_3x_4x_5' + x_2x_4x_5'$.

(c) $f(x_1, x_2, x_3, x_4, x_5) = \Pi(1, 4, 6, 7, 9, 12, 15, 17, 20, 21, 22, 23, 28, 31)$

Solution:

(a) $f(x_1, x_2, x_3, x_4, x_5) = \sum(0, 1, 4, 5, 16, 17, 21, 25, 29) = x_1x_4'x_5 + x_1'x_2'x_4' + x_2'x_3'x_4'$

x_4x_5 \ $x_1x_2x_3$	000	001	011	010		100	101	111	110
00	1	1				1			
01	1	1				1	1	1	1
11									
10									

The POS can be found to be $f = (x_4')(x_1 + x_2')(x_2' + x_5)(x_1' + x_3' + x_5)$.

(b) $f = x_1'x_2'x_3x_4' + x_1'x_2'x_3'x_4' + x_2'x_4'x_5' + x_2'x_3x_4' + x_3x_4x_5' + x_2x_4x_5'$.

x_4x_5 \ $x_1x_2x_3$	000	001	011	010		100	101	111	110
00	1	1	1			1	1		
01	1	1	1				1		
11			1	1					
10			1	1					

The SOP is $f = x_5'x_4x_2 + x_5'x_4x_3 + x_5'x_4'x_2' + x_4'x_3x_2' + x_4'x_2'x_1'$. The POS is $f = (x_4 + x_2')(x_5' + x_4')(x_4' + x_3 + x_2)(x_5' + x_3 + x_1')$.

(c) $f(x_1, x_2, x_3, x_4, x_5) = \Pi(1, 4, 6, 7, 9, 12, 15, 17, 20, 21, 22, 23, 28, 31)$

$x_4x_5 \backslash x_1x_2x_3$	000	001	011	010		100	101	111	110
00		0	0				0	0	
01	0			0		0	0		
11		0	0				0	0	
10		0					0		

The minimum POS is $f = (x_2+x_3'+x_4')(x_3'+x_4'+x_5')(x_3'+x_4+x_5)(x_1'+x_2+x_4+x_5')(x_1+x_3+x_4+x_5')$.
The minimum SOP can be found to be $f = x_3'x_4 + x_3'x_5' + x_2x_4x_5' + x_1'x_3x_4'x_5 + x_1x_2x_4'x_5$.

Q6: Find minimum SOP and POS expressions for the following functions together with the don't care conditions D.

(a) $f(w,x,y,z) = \sum(0,2,8,9,10,15) + D(1,3,6,7)$

(b) $f(w,x,y,z) = \Pi(1,3,5,7,13) + D(0,2,15)$

Solution:

(a)

$$f(w,x,y,z) = \sum(0,2,8,9,10,15) + D(1,3,6,7) = x'z' + x'y' + xyz = (x'+y)(x'+z)(x+y'+z')$$

wx \ yz	00	01	11	10
00	1	X	X	1
01	0	0	X	X
11	0	0	1	0
10	1	1	0	1

wx \ yz	00	01	11	10
00	1	X	X	1
01	0	0	X	X
11	0	0	1	0
10	1	1	0	1

(b)

$$f(w,x,y,z) = \Pi(1,3,5,7,13) + D(0,2,15) = z' + wx' = (w+x)(x'+z')$$

wx \ yz	00	01	11	10
00	X	0	0	X
01	1	0	0	1
11	1	0	X	1
10	1	1	1	1

wx \ yz	00	01	11	10
00	X	0	0	X
01	1	0	0	1
11	1	0	X	1
10	1	1	1	1

Q7: Find the prime implicants for the following functions. Determine which prime implicants are essential.

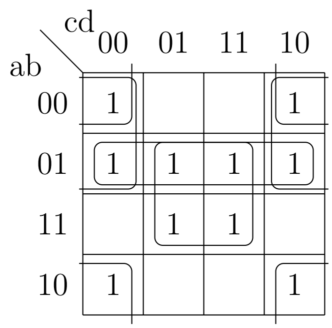
(a) $f(a, b, c, d) = \sum(0, 2, 4, 5, 6, 7, 8, 10, 13, 15)$

(b) $f(a, b, c, d) = \sum(1, 3, 4, 5, 10, 11, 12, 13, 14, 15)$

Solution:

(a)

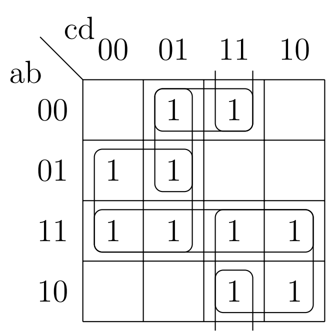
$$f(a, b, c, d) = \sum(0, 2, 4, 5, 6, 7, 8, 10, 13, 15)$$



The prime implicants are: bd, $a'b$, $a'd'$, and $b'd'$. The essential prime implicants are: bd, $b'd'$.

(b)

$$f(a, b, c, d) = \sum(1, 3, 4, 5, 10, 11, 12, 13, 14, 15)$$



The prime implicants are: bc', ab, ac, $b'cd$, $a'b'd$, and $a'c'd$. The essential prime implicants are: bc', ac.

Q8: A four-variable logic function that equals 1 if any three or all four input variables are equal to 1 is called a *majority* function. Design a minimum cost SOP circuit that implements the majority function.

Solution:

a	b	c	d	f
0	0	0	0	0
0	0	0	1	0
0	0	1	0	0
0	0	1	1	0
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	1
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	1
1	1	0	0	0
1	1	0	1	1
1	1	1	0	1
1	1	1	1	1

ab \ cd	00	01	11	10
00	0	0	0	0
01	0	0	1	0
11	0	1	1	1
10	0	0	1	0

$$f = abd + bcd + acd + abc.$$

Q9: Derive a minimum cost 2-level circuit for a four variable function that is equal to 1 if exactly two or exactly three of its input variables are equal to 1; otherwise it is equal to 0.

Solution:

a	b	c	d	f
0	0	0	0	0
0	0	0	1	0
0	0	1	0	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	1
1	0	1	0	1
1	0	1	1	1
1	1	0	0	1
1	1	0	1	1
1	1	1	0	1
1	1	1	1	0

ab \ cd	00	01	11	10
00	0	0	1	0
01	0	1	1	1
11	1	1	0	1
10	0	1	1	1

$$f = (a' + b' + c' + d')(a + b + c)(a + c + d)(a + b + d)(b + c + d)$$

The POS has cost of 27. You can find the minimum SOP too, but it turns out the POS is cheaper.

Q10: A circuit with two outputs is required to implement the two logic functions f and g given by

$$f(a,b,c,d) = \sum(0,2,4,6,7,9) + D(10,11)$$

and

$$g(a,b,c,d) = \sum(2,4,9,10,15) + D(0,13,14)$$

where D denotes the don't cares for each function.

(a) Design a minimum SOP for both f and g separately and compute cost of each function assuming each gate costs 1 and each gate input costs 1. You may assume circuit inputs are available in both complemented and uncomplemented forms.

(b) Design a minimum cost circuit that implements both f and g as SOPs. Determine the cost of the circuit.

Solution:

(a)

Consider f and g separately.

$f = a'd' + a'bc + ab'd.$

ab \ cd	00	01	11	10
00	1	0	0	1
01	1	0	1	1
11	0	0	0	0
10	0	1	X	X

The cost of f is AND (2 input) plus AND (3 input) plus AND (3 input) plus OR (3 input) = 15.

$g = a'c'd' + ac'd + b'cd' + abc.$

ab \ cd	00	01	11	10
00	X	0	0	1
01	1	0	0	0
11	0	X	1	X
10	0	1	0	1

The cost of g is AND (3 input) plus AND (3 input) plus AND (3 input) plus AND (3 input) plus OR (4 input) = 21.

If you look at the Karnaugh maps for f and g, there are no product terms which are the same. Therefore, the total cost of implementing f and g individually is 36.

(b)

ab \ cd	00	01	11	10
00	1	0	0	1
01	1	0	1	1
11	0	0	0	0
10	0	1	X	X

ab \ cd	00	01	11	10
00	X	0	0	1
01	1	0	0	0
11	0	X	1	X
10	0	1	0	1

$f = a'c'd' + b'c'd' + ab'c'd + a'bc \quad g = a'c'd' + b'c'd' + ab'c'd + abc.$

If f and g are considered together, there are several product terms which can be shared. The total cost of f and g is:

- **shared terms**: AND (3 input) plus AND (3 input) plus AND (4 input);
- **unique to** f: AND (3 input) plus OR (4 input);
- **unique to** g: AND (3 input) plus OR (4 input).

Therefore, the total cost is 31 which is cheaper that implementing f and g individually. Note that here, neither function is minimized.

Q11: Repeat problem 10 for the following functions.

$$f(x_1, x_2, x_3, x_4, x_5) = \sum(1, 4, 5, 11, 27, 28) + D(10, 12, 14, 15, 20, 31)$$

and

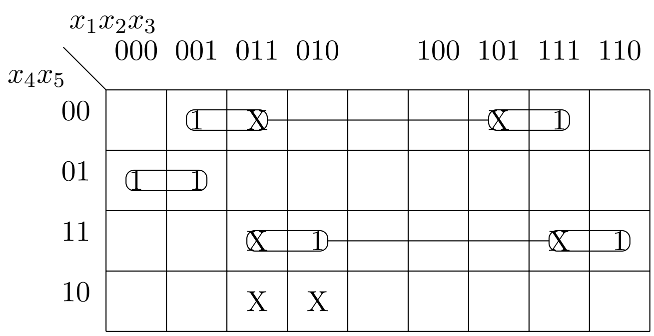
$$g(x_1, x_2, x_3, x_4, x_5) = \sum(0, 1, 2, 4, 5, 8, 14, 15, 16, 18, 20, 24, 26, 28, 31) + D(10, 11, 12, 27)$$

Solution:

(a)

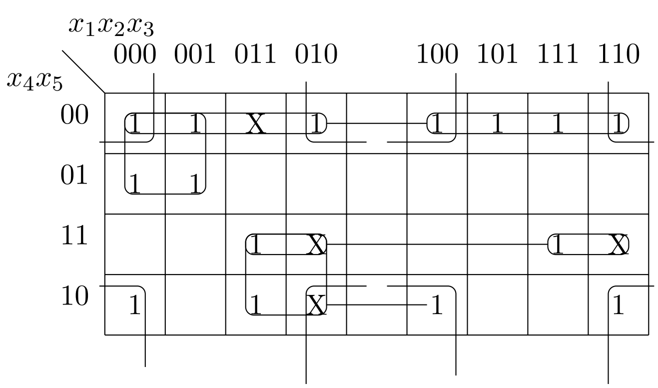
Consider f and g separately.

$$f = x_3x_4'x_5' + x_2x_4x_5 + x_1'x_2'x_4'x_5.$$



The cost of f is AND (3 input) plus AND (3 input) plus AND (4 input) plus OR (3 input) = 17.

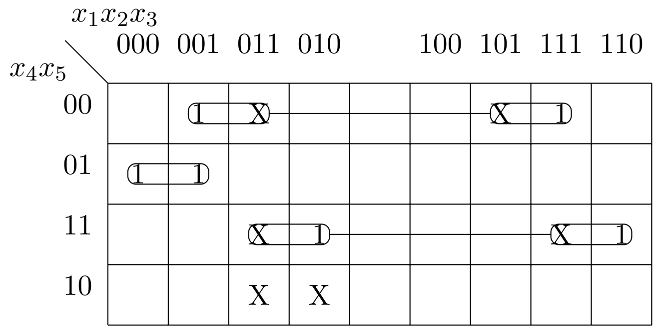
$$g = x_4'x_5' + x_3'x_5' + x_1'x_2'x_4' + x_1'x_2x_4 + x_2x_4x_5.$$

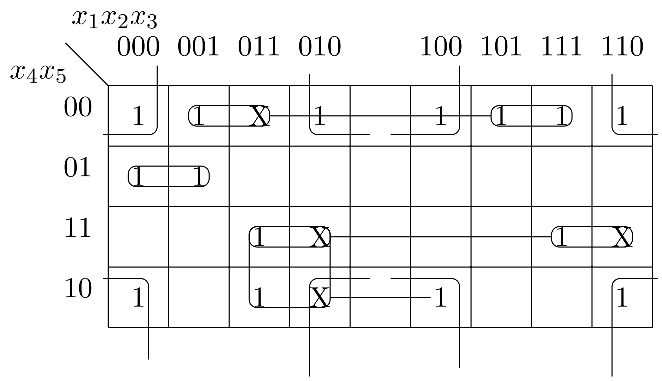


The cost of g is AND (2 input) plus AND (2 input) plus AND (3 input) plus AND (3 input) plus AND (3 input) plus OR (5 input) = 24.

If you look at the Karnaugh maps for f and g, there are some product terms that are already the same. Specifcially, $x_2x_4x_5$ is used in both functions. Therefore, implementing f and g separately would cost $24 + 17 = 41$. If the single product term is shared, then the cost would be only 37.

(b)





If f and g are considered together, we can see that some additional product terms can be shared. We can see that $f = x_3x_4'x_5' + x_2x_4x_5 + x_1'x_2'x_4'x_5$. We can see that $g = x_3x_4'x_5' + x_2x_4x_5 + x_1'x_2'x_4'x_5 + x_3'x_5' + x_1'x_2x_4$. The total cost of f and g is:

- **shared terms**: AND (3 input) plus AND (3 input) plus AND (4 input);
- **unique to** f: OR (3 input);
- **unique to** g: AND (2 input) plus AND (3 input) plus OR (5 input).

Therefore, the total cost is 30.

If we are not limited to SOP implementations, we can also do even better. We can observe that $g = f + x_3'x_5' + x_1'x_2x_4$. This would further reduce the cost of g by reducing the size of the OR gate from 5-input down to 3-input. However, in this case g is not strictly and SOP since is a function of f.

Q12: Derive a minimum cost circuit that implements the function $f(a, b, c, d) = \sum(4, 7, 8, 11) + D(12, 15)$. You are not restricted to 2-level circuits and may use any sort of logic gate.

Solution:

ab \ cd	00	01	11	10
00	0	0	0	0
01	1	0	1	0
11	X	0	X	0
10	1	0	1	0

We find that $f = bc'd' + ac'd' + bcd + acd$. The minimum SOP has cost of 21.

Since we are told we can use any sort of gate (and assuming a gate costs 1), we can write $f = bc'd' + ac'd' + bcd + acd = a(c'd' + cd) + b(c'd' + cd) = (a + b)(c'd' + cd) = (a + b)(\overline{c \oplus d})$. This requires one 2-input OR, one 2-input NXOR, and one 2-input AND. This has cost 9.

Q13: Find a minimum cost circuit that implements the function $f(a,b,c,d) = \sum(0,4,8,13,14,15)$. The input variables are available in uncomplemented form only.

Solution:

ab \ cd	00	01	11	10
00	1	0	0	0
01	1	0	0	0
11	0	1	1	1
10	1	0	0	0

We find that $f = a'c'd' + b'c'd' + abc + abd$. Since the input variables are only available in uncomplemented form, we need to count the inverters when computing the cost. This circuit therefore costs 21 (for the SOP) plus another 8 since we need to invert every input. The total cost is 29.

Since we are asked for the minimum cost circuit, we can consider other sorts of gates. We can write $f = a'c'd' + b'c'd' + abc + abd = (c'd')(a'+b') + (c+d)(ab)$. Let $A = c + d$ and let $B = ab$. Then we can see that $f = A'B' + AB = \overline{A \oplus B} = \overline{(c+d) \oplus (ab)}$. This implementation requires one OR (2 input), one AND (2 input) and one NXOR (2 input) and no input inversions. This circuits costs 9.

Q14: Find the simplest realization of the function $f = \sum(0, 3, 4, 7, 9, 10, 13, 14)$ assuming you can only use logic gates with a maximum of 2-inputs.

Solution:

ab \ cd	00	01	11	10
00	1	0	1	0
01	1	0	1	0
11	0	1	0	1
10	0	1	0	1

We find that $f = a'c'd' + a'cd + ac'd + acd'$. The SOP (and the POS, if you find it) require gates with more than 2 inputs. Of course, you can implement AND/OR gates of larger inputs from smaller AND/OR gates (due to associative property), but perhaps we can do better.

We can write $f = a'c'd' + a'cd + ac'd + acd' = d'(a'c' + ac) + d(a'c + ac') = d'(a \oplus c)' + d(a \oplus c) = (d \oplus (a \oplus c))'$. This expression requires a 2-input XOR gate to form $a \oplus c$ and then a 2-input NXOR gate to form $(d \oplus (a \oplus c))'$. This is shown below.