Fixed point representations

- If we have values with fractional parts, one way to represent them is to consider using a **fixed point representation**.
- In this representation, we allow a certain number of digits prior to the fixed point (the unsigned integer part of the value), and a certain number of digits after the fixed point (the fractional part of the value).
- The unsigned integer part of the number will be represented the same as with positional number notation.
- We only need to consider the fractional part...

Fixed point representation

- Consider a positive value V which has only a fractional part.
- In base-r using k digits after the fixed point (not called a decimal point if not base-10), V is represented by $D = (0.d_1 d_2 \cdots d_{-k})_r$.
- Any positive value V which has both an whole part (using positional notation) and a fractional part can be represented in base-r as

$$D = (\underbrace{d_{n-1} d_{n-2} \cdots d_1 d_0}_{\text{whole part}} \underbrace{.}_{\text{fixed point}} \underbrace{d_1 d_2 \cdots d_{-k}}_{\text{fractional part}})_r$$

- This is called fixed point notation (fixed because we define the number of digits n in front of the fixed point and the number of digits k after the fixed point).

Fixed point representation

- Given a representation D of some value V in base r using fixed point notation, we can compute the value of V using

$$\begin{aligned} V \quad = \quad & d_{n-1} \times r^{n-1} + d_{n-2} \times r^{n-2} + \cdots + d_1 \times r^1 + d_0 \times r^0 \\ & + d_{-1} \times r^{-1} + d_{-2} \times r^{-2} + \cdots + d_{-k} \times r^{-k} \end{aligned}$$

- Examples...
  - A number represented in base-10:

$$\begin{aligned} (213.78)_{10} &= 2 \times 10^2 + 1 \times 10^1 + 3 \times 10^0 + 7 \times 10^{-1} + 8 \times 10 \\ &= 200 + 10 + 3 + 7/10 + 8/100 \\ &= 200 + 10 + 3 + 0.70 + 0.08 \\ &= 213.78 \end{aligned}$$

  - A number represented in base-4:

$$\begin{aligned} (1321.312)_4 &= 1 \times 4^3 + 3 \times 4^2 + 2 \times 4^1 + 1 \times 4^0 \\ &\qquad + 3 \times 4^{-1} + 1 \times 4^{-2} + 2 \times 4^{-3} \\ &= 64 + 48 + 8 + 4 + 0.75 + 0.0625 + 0.031250 \\ &= 121.84375 \end{aligned}$$

Fixed point representation

- A value's fractional representation in base r can be computed using successive multiplication by r:

$$V = 0.d_{-1} \times r^{-1} + d_{-2} \times r^{-2} + \cdots d_{-k} \times r^{-k}$$

- Multiplication by r gives

$$\frac{V}{r} = \underbrace{d_{-1}}_{\text{one of the digits}} . \underbrace{d_{-2} \times r^{-1} + \cdots + d_{-k} \times r^{-k+1}}_{\text{remaining fractional part}}$$

  - One of the digits "pops out" in front of the fixed point.
  - Repeat the multiplication with the remaining fractional part.
  - Stop once everything is 0... (All subsequent digits will be zero).

- The value of the integer part is computed as with positional number notation.

Fixed point representation

- Example... Represent the value 0.625 in base-2.

$$\begin{aligned} 0.625 \times 2 &= 1.250 &\rightarrow d_{-1} = 1 \\ 0.250 \times 2 &= 0.500 &\rightarrow d_{-2} = 0 \\ 0.500 \times 2 &= 1.000 &\rightarrow d_{-3} = 1 \end{aligned}$$

  - Stop here since further multipcation would simply indicate that $d_{-4} = 0$, $d_{-5} = 0$, etc.
  - Therefore 0.625 is represented in base-2 as 0.101000_2.

Fixed point representation

- To determine the representation for a value V that has both a whole part and a fractional part, we convert the whole part and the fractional part separately.
- The whole part is converted just like with positional number representation (i.e., through repeated division).
- The fractional part is converted as just discussed (i.e., through repeated multiplication).
- Converting fixed point representations from base-a to base-b is the same as before: First determine the value of the representation in base-a and then convert the value to base-b.
- Note that fast conversion between base-2, base-8 and base-16 can still be done by grouping bits.

Fixed point representation

- For fixed point notation, it we need to represent a negative number, then we will use a *sign bit* and deal with the consequences of doing it this way.