Date. No.

* Relation between zero covariance and independence.

$X \& Y$: indep $\Rightarrow Cov(X,Y) = O_{p\times q}$

$\Leftarrow$ (*)

성립조건 : 정규분포! $\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_{p\times q}\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{pmatrix} \right)$

* 독립인지 확인하는 방법! X, Y : independent $\Leftrightarrow Cov(X,Y) = \Sigma_{XY} = 0$

$\Leftarrow$ 인지 확인해보자.

① $f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$ ✓

② $mgf_{XY}(s,t) = mgf_X(s)\, mgf_Y(t)$

• $$f_{XY}(\underbrace{x}_{p차원}, \underbrace{y}_{q차원}) = \frac{1}{|2\pi\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2} \begin{pmatrix} x-\mu_X \\ y-\mu_Y \end{pmatrix}^T \begin{pmatrix} \Sigma_{XX}^{-1} & 0 \\ 0 & \Sigma_{YY}^{-1} \end{pmatrix} \begin{pmatrix} x-\mu_X \\ y-\mu_Y \end{pmatrix} \right)$$

p차원 q차원 벡터

$$= \frac{1}{|2\pi\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(x-\mu_X)^T \Sigma_{XX}^{-1}(x-\mu_X) - \frac{1}{2}(y-\mu_Y)^T \Sigma_{YY}^{-1}(y-\mu_Y) \right)$$ X와 Y로 분해.

$$\underbrace{\left((x-\mu_X)^T, (y-\mu_Y)^T\right)}_{1\times 2} \underbrace{\begin{pmatrix} \Sigma_{XX}^{-1} & 0 \\ 0 & \Sigma_{YY}^{-1} \end{pmatrix}}_{2\times 2} \underbrace{\begin{pmatrix} x-\mu_X \\ y-\mu_Y \end{pmatrix}}_{2\times 1} = \left((x-\mu_X)^T\Sigma_{XX}^{-1}, (y-\mu_Y)^T\Sigma_{YY}^{-1}\right) \begin{pmatrix} x-\mu_X \\ y-\mu_Y \end{pmatrix}$$

$$= (x-\mu_X)^T\Sigma_{XX}^{-1}(x-\mu_X) + (y-\mu_Y)^T\Sigma_{YY}^{-1}(y-\mu_Y)$$

* The conditional distributions of the components of X :

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \right)$$

Then, the conditional distribution of X_1 given $X_2 = x_2$ is $X_1 | X_2 = x_2$.

$$\begin{pmatrix} I_p & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I_q \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} X_1 - \Sigma_{12}\Sigma_{22}^{-1}X_2 \\ X_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 - \Sigma_{12}\Sigma_{22}^{-1}\mu_2 \\ \mu_2 \end{pmatrix}, \underset{(*)}{0} \right)$$

ind.

transpose

(*) $$\begin{pmatrix} I_p & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I_q \end{pmatrix} \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ -\Sigma_{22}^{-1}\Sigma_{21} & I \end{pmatrix}$$

여기를 0으로 만드는 녀석을 찾아라. (분산이 0임을 보여서 $X_1 - \Sigma_{12}\Sigma_{22}^{-1}X_2$ 와 X_2가 독립임을 보이자.)

$$= \begin{pmatrix} \underbrace{\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}_{= \Sigma_{11.2}} & 0 \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ -\Sigma_{22}^{-1}\Sigma_{21} & I \end{pmatrix} = \begin{pmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & 0 \\ 0 & \Sigma_{22} \end{pmatrix}$$

$X_1 - \Sigma_{12}\Sigma_{22}^{-1}X_2,\ X_2$: ind $\Rightarrow X_1 - \Sigma_{12}\Sigma_{22}^{-1}X_2 \mid X_2 = x_2 = X_1 - \Sigma_{12}\Sigma_{22}^{-1}X_2 \sim N(\mu_1 - \Sigma_{12}\Sigma_{22}^{-1}\mu_2, \Sigma_{11.2})$

$\Rightarrow X_1 - \Sigma_{12}\Sigma_{22}^{-1}X_2 + \Sigma_{12}\Sigma_{22}^{-1}X_2 \mid X_2 = x_2 \sim N(\mu_1 - \Sigma_{12}\Sigma_{22}^{-1}\mu_2 + \Sigma_{12}\Sigma_{22}^{-1}x_2, \Sigma_{11.2})$

여기서 x_2로 주어지니, 상수 취급.

$\Rightarrow X_1 | X_2 = x_2 \sim N(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2), \Sigma_{11.2})$