2.13 IDEMPOTENT MATRICES 멱등 행렬

A square matrix $\mathbf{A}$ is said to be *idempotent* if $\mathbf{A}^2 = \mathbf{A}$. Most idempotent matrices in this book are symmetric. Many of the sums of squares in regression (Chapters 6–11) and analysis of variance (Chapters 12–15) can be expressed as quadratic forms $\mathbf{y}'\mathbf{A}\mathbf{y}$. The idempotence of $\mathbf{A}$ or of a product involving $\mathbf{A}$ will be used to establish that $\mathbf{y}'\mathbf{A}\mathbf{y}$ (or a multiple of $\mathbf{y}'\mathbf{A}\mathbf{y}$) has a chi-square distribution.

An example of an idempotent matrix is the identity matrix $\mathbf{I}$. $(I^2 = I)$

Theorem 2.13a. The only nonsingular idempotent matrix is the identity matrix $\mathbf{I}$.

PROOF. If $\mathbf{A}$ is idempotent and nonsingular, then $\mathbf{A}^2 = \mathbf{A}$ and the inverse $\mathbf{A}^{-1}$ exists. If we multiply $\mathbf{A}^2 = \mathbf{A}$ by $\mathbf{A}^{-1}$, we obtain $A^{-1}A = I$

$$\mathbf{A}^{-1}\mathbf{A}^2 = \mathbf{A}^{-1}\mathbf{A},$$
$$\mathbf{A} = \mathbf{I}.$$

∎

Many of the matrices of quadratic forms we will encounter in later chapters are singular idempotent matrices. We now give some properties of such matrices.

Theorem 2.13b. If $\mathbf{A}$ is singular, symmetric, and idempotent, then $\mathbf{A}$ is positive semidefinite.

$A' = A$ $AA = A$

PROOF. Since $\mathbf{A} = \mathbf{A}'$ and $\mathbf{A} = \mathbf{A}^2$, we have

$$\mathbf{A} = \mathbf{A}^2 = \mathbf{A}\mathbf{A} = \mathbf{A}'\mathbf{A},$$

which is positive semidefinite by Theorem 2.6d(ii). ∎

nxp matrix

If rank(A) < p, then A'A is positive semidefinite

singular

If a is a real number (실수) such that $a^2 = a$, then a is either 0 or 1. The analogous property (유사한 속성) for matrices is that if $\mathbf{A}^2 = \mathbf{A}$, then the eigenvalues of $\mathbf{A}$ are 0s and 1s.

Theorem 2.13c. If $\mathbf{A}$ is an $n \times n$ symmetric idempotent matrix of rank r, then $\mathbf{A}$ has r eigenvalues equal to 1 and $n - r$ eigenvalues equal to 0.

PROOF. By (2.99), if $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, then $\mathbf{A}^2\mathbf{x} = \lambda^2\mathbf{x}$. Since $\mathbf{A}^2 = \mathbf{A}$, we have $\mathbf{A}^2\mathbf{x} = \mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. Equating the right sides of $\mathbf{A}^2\mathbf{x} = \lambda^2\mathbf{x}$ and $\mathbf{A}^2\mathbf{x} = \lambda\mathbf{x}$, we have

$$\lambda\mathbf{x} = \lambda^2\mathbf{x} \quad \text{or} \quad (\lambda - \lambda^2)\mathbf{x} = \mathbf{0}.$$

But $\mathbf{x} \neq \mathbf{0}$, and therefore $\lambda - \lambda^2 = 0$, from which, λ is either 0 or 1.

By Theorem 2.13b, $\mathbf{A}$ is positive semidefinite, and therefore by Theorem 2.12f(ii), the number of nonzero eigenvalues is equal to rank($\mathbf{A}$). Thus r eigenvalues of $\mathbf{A}$ are equal to 1 and the remaining $n - r$ eigenvalues are equal to 0. ∎

$A_{n\times n}$ 의 고유값 $\lambda_1, \ldots, \lambda_n$ 에 대해
$|A| = \prod_{i=1}^{n} \lambda_i$, $tr(A) = \sum_{i=1}^{n} \lambda_i$

We can use Theorems 2.12e and 2.13c to find the rank of a symmetric idempotent matrix.

A가 $n\times n$ 대칭 멱등 행렬이고 $rank(A) = r$ 일 때,
A의 고유값중 r개는 1이고, 나머지는 0.

Theorem 2.13d. If $\mathbf{A}$ is symmetric and idempotent of rank r, then $\text{rank}(\mathbf{A}) = \text{tr}(\mathbf{A}) = r$.

Proof. By Theorem 2.12e(ii), $\text{tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$, and by Theorem 2.13c, $\sum_{i=1}^{n} \lambda_i = r$. ■

Some additional properties of idempotent matrices are given in the following four theorems.

Theorem 2.13e. If $\mathbf{A}$ ($AA=A$) is an $n \times n$ idempotent matrix, $\mathbf{P}$ ($P^{-1}P=I$) is an $n \times n$ nonsingular matrix, and $\mathbf{C}$ ($C'C=I$) is an $n \times n$ orthogonal matrix, then

(i) $\mathbf{I} - \mathbf{A}$ is idempotent. $(I-A)^2 = I - 2A + \overset{A}{A^2} = I - A$
(ii) $\mathbf{A}(\mathbf{I} - \mathbf{A}) = \mathbf{O}$ and $(\mathbf{I} - \mathbf{A})\mathbf{A} = \mathbf{O}$. $A(I-A) = A - A^2 = A - A = 0$. $(I-A)A$도 마찬가지.
(iii) $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is idempotent. $(P^{-1}AP)^2 = P^{-1}APP^{-1}AP = P^{-1}AP$
(iv) $\mathbf{C}'\mathbf{A}\mathbf{C}$ is idempotent. (If $\mathbf{A}$ is symmetric, $\mathbf{C}'\mathbf{A}\mathbf{C}$ is a symmetric idempotent matrix.) $(C'AC)^2 = C'ACC'AC = C'AC$ ■

Theorem 2.13f. Let $\mathbf{A}$ be $n \times p$ of rank r, let $\mathbf{A}^-$ be any generalized inverse of $\mathbf{A}$, and let $(\mathbf{A}'\mathbf{A})^-$ be any generalized inverse of $\mathbf{A}'\mathbf{A}$. Then $\mathbf{A}^-\mathbf{A}, \mathbf{A}\mathbf{A}^-$, and $\mathbf{A}(\mathbf{A}'\mathbf{A})^-\mathbf{A}'$ are all idempotent. ■

Theorem 2.13g. Suppose that the $n \times n$ symmetric matrix $\mathbf{A}$ can be written as $\mathbf{A} = \sum_{i=1}^{k} \mathbf{A}_i$ for some k, where each $\mathbf{A}_i$ is an $n \times n$ symmetric matrix. Then any two of the following conditions implies the third condition. (이 중 2개가 성립하면, 나머지도 성립.)

$AA = \left(\sum_{i=1}^{k} A_i\right)^2 = \sum_{i=1}^{k} A_i^2 + \underbrace{\sum_{i\neq j} A_iA_j}_{0}$

(i) $\mathbf{A}$ is idempotent.
(ii) Each of $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_k$ is idempotent.
(iii) $\mathbf{A}_i\mathbf{A}_j = \mathbf{O}$ for $i \neq j$. A_i와 A_j는 직교임. ■

Theorem 2.13h. If $\mathbf{I} = \sum_{i=1}^{k} \mathbf{A}_i$, where each $n \times n$ matrix $\mathbf{A}_i$ is symmetric of rank r_i, and if $n = \sum_{i=1}^{k} r_i$, then both of the following are true:

(i) Each of $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_k$ is idempotent. $A_i^2 = A_i$.
(ii) $\mathbf{A}_i\mathbf{A}_j = \mathbf{O}$ for $i \neq j$. ■