where $b = a_{22} - \mathbf{a}'_{12}\mathbf{A}_{11}^{-1}\mathbf{a}_{12}$. As another special case of (2.50), we have

$$\begin{pmatrix} \mathbf{A}_{11} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}_{22} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}_{11}^{-1} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}_{22}^{-1} \end{pmatrix}. \tag{2.52}$$

If a square matrix of the form $\mathbf{B} + \mathbf{cc}'$ is nonsingular, where $\mathbf{c}$ is a vector and $\mathbf{B}$ is a nonsingular matrix, then

$$(\mathbf{B} + \mathbf{cc}')^{-1} = \mathbf{B}^{-1} - \frac{\mathbf{B}^{-1}\mathbf{cc}'\mathbf{B}^{-1}}{1 + \mathbf{c}'\mathbf{B}^{-1}\mathbf{c}}. \tag{2.53}$$

In more generality, if $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{A} + \mathbf{PBQ}$ are nonsingular, then

$$(\mathbf{A} + \mathbf{PBQ})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{PB}(\mathbf{B} + \mathbf{BQA}^{-1}\mathbf{PB})^{-1}\mathbf{BQA}^{-1}. \tag{2.54}$$

Both (2.53) and (2.54) can be easily verified (Problems 2.33 and 2.34).

2.6 POSITIVE DEFINITE MATRICES 양의 정부호 행렬

벡터 y, 대칭행렬 A에 대하여 $y'Ay = \sum_i a_{ii}y_i^2 + \sum_{i\neq j} a_{ij}y_iy_j$

Quadratic forms were introduced in (2.33). For example, the quadratic form $3y_1^2 + y_2^2 + 2y_3^2 + 4y_1y_2 + 5y_1y_3 - 6y_2y_3$ can be expressed as

$$3y_1^2 + y_2^2 + 2y_3^2 + 4y_1y_2 + 5y_1y_3 - 6y_2y_3 = \mathbf{y}'\mathbf{A}\mathbf{y},$$

where

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} 3 & 4 & 5 \\ 0 & 1 & -6 \\ 0 & 0 & 2 \end{pmatrix}.$$

However, the same quadratic form can also be expressed in terms of the symmetric matrix

$$\frac{1}{2}(\mathbf{A} + \mathbf{A}') = \begin{pmatrix} 3 & 2 & \frac{5}{2} \\ 2 & 1 & -3 \\ \frac{5}{2} & -3 & 2 \end{pmatrix}.$$

In general, any quadratic form $\mathbf{y}'\mathbf{A}\mathbf{y}$ can be expressed as

$$\mathbf{y}'\mathbf{A}\mathbf{y} = \mathbf{y}'\left(\frac{\mathbf{A}+\mathbf{A}'}{2}\right)\mathbf{y}, \tag{2.55}$$

and thus the matrix of a quadratic form can always be chosen to be symmetric (and thereby unique).

The sums of squares we will encounter in regression (Chapters 6–11) and analysis–of–variance (Chapters 12–15) can be expressed in the form $\mathbf{y}'\mathbf{A}\mathbf{y}$, where $\mathbf{y}$ is an observation vector. Such quadratic forms remain positive (or at least nonnegative) for all possible values of $\mathbf{y}$. We now consider quadratic forms of this type.

If the symmetric matrix $\mathbf{A}$ has the property $\mathbf{y}'\mathbf{A}\mathbf{y} > 0$ for all possible $\mathbf{y}$ except $\mathbf{y} = \mathbf{0}$, then the quadratic form $\mathbf{y}'\mathbf{A}\mathbf{y}$ is said to be *positive definite*, and $\mathbf{A}$ is said to be a *positive definite* matrix. Similarly, if $\mathbf{y}'\mathbf{A}\mathbf{y} \geq 0$ for all $\mathbf{y}$ and there is at least one $\mathbf{y} \neq \mathbf{0}$ such that $\mathbf{y}'\mathbf{A}\mathbf{y} = 0$, then $\mathbf{y}'\mathbf{A}\mathbf{y}$ and $\mathbf{A}$ are said to be *positive semidefinite*. Both types of matrices are illustrated in the following example.

양의 정부호행렬

y'Ay=0을 만족하는 y≠0인 y가 적어도 하나가 존재한다면 → 양의 준정부호

Example 2.6. To illustrate a positive definite matrix, consider

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$$

and the associated quadratic form

$(y_1 \ y_2)\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (2y_1 - y_2, -y_1 + 3y_2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 2y_1^2 - y_1y_2 - y_1y_2 + 3y_2^2$
$= 2y_1^2 - 2y_1y_2 + 3y_2^2$

$$\mathbf{y}'\mathbf{A}\mathbf{y} = 2y_1^2 - 2y_1y_2 + 3y_2^2 = 2(y_1 - \tfrac{1}{2}y_2)^2 + \tfrac{5}{2}y_2^2,$$

which is clearly positive as long as y_1 ($\neq 0$) and y_2 ($\neq 0$) are not both zero.

To illustrate a positive semidefinite matrix, consider

$$(2y_1 - y_2)^2 + (3y_1 - y_3)^2 + (3y_2 - 2y_3)^2,$$

which can be expressed as $\mathbf{y}'\mathbf{A}\mathbf{y}$, with

$$\mathbf{A} = \begin{pmatrix} 13 & -2 & -3 \\ -2 & 10 & -6 \\ -3 & -6 & 5 \end{pmatrix}.$$

If $2y_1 = y_2, 3y_1 = y_3$, and $3y_2 = 2y_3$, then $(2y_1 - y_2)^2 + (3y_1 - y_3)^2 + (3y_2 - 2y_3)^2 = 0$. Thus $\mathbf{y}'\mathbf{A}\mathbf{y} = 0$ for any multiple of $\mathbf{y} = (1, 2, 3)'$. Otherwise $\mathbf{y}'\mathbf{A}\mathbf{y} > 0$ (except for $\mathbf{y} = \mathbf{0}$). ■

In the matrices in Example 2.6, the diagonal elements are positive. For positive definite matrices, this is true in general.

Theorem 2.6a 모든 벡터 y에 대해 (y≠0) y'Ay > 0.

(i) If **A** is positive definite, then all its diagonal elements a_{ii} are positive.
(ii) If **A** is positive semidefinite, then all $a_{ii} \geq 0$.

PROOF

양의 정부호이므로 y'Ay > 0.

(i) Let $\mathbf{y}' = (0, \ldots, 0, 1, 0, \ldots, 0)$ with a 1 in the ith position and 0's elsewhere. Then $\mathbf{y}'\mathbf{A}\mathbf{y} = a_{ii} > 0$.
(ii) Let $\mathbf{y}' = (0, \ldots, 0, 1, 0, \ldots, 0)$ with a 1 in the ith position and 0's elsewhere. Then $\mathbf{y}'\mathbf{A}\mathbf{y} = a_{ii} \geq 0$. □

Some additional properties of positive definite and positive semidefinite matrices are given in the following theorems.

Theorem 2.6b. Let **P** be a nonsingular matrix. 역행렬 존재, $P^{-1}P = I$

모든 0이 아닌 벡터 y에 대해 y'Ay > 0. → y'P'APy > 0.

(i) If **A** is positive definite, then $\mathbf{P}'\mathbf{A}\mathbf{P}$ is positive definite.
(ii) If **A** is positive semidefinite, then $\mathbf{P}'\mathbf{A}\mathbf{P}$ is positive semidefinite.

y'Ay ≥ 0

PROOF

z=Py 라고 하면, (Py)'A(Py) = z'Az. 모든 z≠0에 대해 z'Az > 0 임을 보이면 됨!

(Py)'APy

(i) To show that $\mathbf{y}'\mathbf{P}'\mathbf{A}\mathbf{P}\mathbf{y} > 0$ for $\mathbf{y} \neq \mathbf{0}$, note that $\mathbf{y}'(\mathbf{P}'\mathbf{A}\mathbf{P})\mathbf{y} = (\mathbf{P}\mathbf{y})'\mathbf{A}(\mathbf{P}\mathbf{y})$. Since **A** is positive definite, $(\mathbf{P}\mathbf{y})'\mathbf{A}(\mathbf{P}\mathbf{y}) > 0$ provided that $\mathbf{P}\mathbf{y} \neq \mathbf{0}$. By (2.47), $\mathbf{P}\mathbf{y} = \mathbf{0}$ only if $\mathbf{y} = \mathbf{0}$, since $\mathbf{P}^{-1}\mathbf{P}\mathbf{y} = \mathbf{P}^{-1}\mathbf{0} = \mathbf{0}$. Thus $\mathbf{y}'\mathbf{P}'\mathbf{A}\mathbf{P}\mathbf{y} > 0$ if $\mathbf{y} \neq \mathbf{0}$.

P가 nonsingular 라서 P≠0 임은 보장되어 있음. 이 식에 따라 Py=0 이면 y=0 이어야 하는데, y≠0 라는 조건이 있으므로 Py≠0 이다.

(ii) See problem 2.36. □

Corollary 1. Let **A** be a $p \times p$ positive definite matrix and let **B** be a $k \times p$ matrix of rank $k \leq p$. Then $\mathbf{B}\mathbf{A}\mathbf{B}'$ is positive definite. □

k×k z∈ℝᵏ, z≠0에 대해 z'(BAB')z = (B'z)'A(B'z). B의 rank는 k고 z≠0 이므로 B'z ≠0.

Corollary 2. Let **A** be a $p \times p$ positive definite matrix and let **B** be a $k \times p$ matrix. If $k > p$ or if $\text{rank}(\mathbf{B}) = r$, where $r < k$ and $r < p$, then $\mathbf{B}\mathbf{A}\mathbf{B}'$ is positive semidefinite. □

대칭행렬 A가 양의 정부호라면, 즉, y'Ay>0라면, A=P'P를 만족하는 비특이행렬 P가 존재.

Theorem 2.6c. A symmetric matrix **A** is positive definite if ~~and~~ only if there exists a nonsingular matrix **P** such that $\mathbf{A} = \mathbf{P}'\mathbf{P}$.

PROOF. We prove the "if" part only. Suppose $\mathbf{A} = \mathbf{P}'\mathbf{P}$ for nonsingular $\mathbf{P}$. Then

$\Leftarrow$

A=PP'를 만족하는 비특이행렬 P가 존재한다면,
A는 대칭행렬이며 양의 정부호이다.

$$\mathbf{y}'\mathbf{A}\mathbf{y} = \mathbf{y}'\mathbf{P}'\mathbf{P}\mathbf{y} = (\mathbf{P}\mathbf{y})'(\mathbf{P}\mathbf{y}).$$

This is a sum of squares [see (2.20)] and is positive unless $\mathbf{P}\mathbf{y} = \mathbf{0}$. By (2.47), $\mathbf{P}\mathbf{y} = \mathbf{0}$ only if $\mathbf{y} = \mathbf{0}$.

$(Py)'(Py) > 0$

↳ $y \neq 0$ 라면 $Py \neq 0$

Corollary 1. A positive definite matrix is nonsingular.

↳ $x'Ax > 0$. $x \neq 0$.

GPT : A가 특이 행렬이면, rank(A) < n 이므로
$x \neq 0$ 인 벡터 x에 대해 $Ax = 0$가 존재.
그럼 $x'Ax = 0$ 이라 $x'Ax > 0$ 가 아님....

One method of factoring a positive definite matrix $\mathbf{A}$ into a product $\mathbf{P}'\mathbf{P}$ as in Theorem 2.6c is provided by the Cholesky decomposition (Seber and Lee 2003, pp. 335–337), by which $\mathbf{A}$ can be factored uniquely into $\mathbf{A} = \mathbf{T}'\mathbf{T}$, where $\mathbf{T}$ is a nonsingular upper triangular matrix.

For any square or rectangular matrix $\mathbf{B}$, the matrix $\mathbf{B}'\mathbf{B}$ is positive definite or positive semidefinite.

Theorem 2.6d. Let $\mathbf{B}$ be an $n \times p$ matrix.

(i) If $\text{rank}(\mathbf{B}) = p$, then $\mathbf{B}'\mathbf{B}$ is positive definite.
(ii) If $\text{rank}(\mathbf{B}) < p$, then $\mathbf{B}'\mathbf{B}$ is positive semidefinite.

PROOF

(i) To show that $\mathbf{y}'\mathbf{B}'\mathbf{B}\mathbf{y} > 0$ for $\mathbf{y} \neq \mathbf{0}$, we note that

$$\mathbf{y}'\mathbf{B}'\mathbf{B}\mathbf{y} = (\mathbf{B}\mathbf{y})'(\mathbf{B}\mathbf{y}),$$

which is a sum of squares and is thereby positive unless $\mathbf{B}\mathbf{y} = \mathbf{0}$. By (2.37), we can express $\mathbf{B}\mathbf{y}$ in the form

$$\mathbf{B}\mathbf{y} = y_1\mathbf{b}_1 + y_2\mathbf{b}_2 + \cdots + y_p\mathbf{b}_p.$$

$= 0$ 을 만족하려면
$y = 0$ 이여야만 한다! rank(B) = p 니까

This linear combination is not $\mathbf{0}$ (for any $\mathbf{y} \neq \mathbf{0}$) because $\text{rank}(\mathbf{B}) = p$, and the columns of $\mathbf{B}$ are therefore linearly independent [see (2.40)].

(ii) If $\text{rank}(\mathbf{B}) < p$, then we can find $\mathbf{y} \neq \mathbf{0}$ such that

$$\mathbf{B}\mathbf{y} = y_1\mathbf{b}_1 + y_2\mathbf{b}_2 + \cdots + y_p\mathbf{b}_p = \mathbf{0}$$

since the columns of $\mathbf{B}$ are linearly dependent [see (2.40)]. Hence $\mathbf{y}'\mathbf{B}'\mathbf{B}\mathbf{y} \geq 0$.

Note that if **B** is a square matrix, the matrix $\mathbf{BB} = \mathbf{B}^2$ is not necessarily positive semidefinite. For example, let

이건 무조건 양의 준정부호는 아니다.

$$\mathbf{B} = \begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}.$$

Then

$$\begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 1-2 & -2+4 \\ 1-2 & -2+4 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}$$

$$\mathbf{B}^2 = \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}, \quad \mathbf{B}'\mathbf{B} = \begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix}.$$

In this case, $\mathbf{B}^2$ is not positive semidefinite, but $\mathbf{B}'\mathbf{B}$ is positive semidefinite, since $\mathbf{y}'\mathbf{B}'\mathbf{B}\mathbf{y} = 2(y_1 - 2y_2)^2$. ≥ 0

Two additional properties of positive definite matrices are given in the following theorems.

Theorem 2.6e. If **A** is positive definite, then $\mathbf{A}^{-1}$ is positive definite.

PROOF. By Theorem 2.6c, $\mathbf{A} = \mathbf{P}'\mathbf{P}$, where **P** is nonsingular. By Theorems 2.5a and 2.5b, $\mathbf{A}^{-1} = (\mathbf{P}'\mathbf{P})^{-1} = \mathbf{P}^{-1}(\mathbf{P}')^{-1} = \mathbf{P}^{-1}(\mathbf{P}^{-1})'$, which is positive definite by Theorem 2.6c. ∎

nonsingular matrix

Theorem 2.6f. If **A** is positive definite and is partitioned in the form

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix},$$

where $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ are square, then $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ are positive definite.

PROOF. We can write $\mathbf{A}_{11}$, for example, as $\underset{n\times n}{\mathbf{A}_{11}} = \underbrace{(\mathbf{I}, \mathbf{O})\mathbf{A}\begin{pmatrix} \mathbf{I} \\ \mathbf{O} \end{pmatrix}}_{P'AP}$, where **I** is the same size as $\mathbf{A}_{11}$. Then by Corollary 1 to Theorem 2.6b, $\mathbf{A}_{11}$ is positive definite. ∎

2.7 SYSTEMS OF EQUATIONS

The system of n (linear) equations in p unknowns

$$n개. \left\{ \begin{aligned} &\overbrace{a_{11}x_1 + a_{12}x_2 + \cdots + a_{1p}x_p}^{x_1 \sim x_p} = c_1 \\ &a_{21}x_1 + a_{22}x_2 + \cdots + a_{2p}x_p = c_2 \\ &\qquad\qquad\vdots \\ &a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{np}x_p = c_n \end{aligned} \right. \qquad (2.56)$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \Rightarrow Ax = C$$