A square matrix of 1s is denoted by $\mathbf{J}$; for example

$$\mathbf{J} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{2.7}$$

We denote a vector of zeros by $\mathbf{0}$ and a matrix of zeros by $\mathbf{O}$; for example

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{O} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.8}$$

2.2 OPERATIONS

합 곱 성질

We now define sums and products of matrices and vectors and consider some properties of these sums and products.

2.2.1 Sum of Two Matrices or Two Vectors

If two matrices or two vectors are the same size, they are said to be *conformal for addition*. Their *sum* is found by adding corresponding elements. Thus, if $\mathbf{A}$ is $n \times p$ and $\mathbf{B}$ is $n \times p$, then $\mathbf{C} = \mathbf{A} + \mathbf{B}$ is also $n \times p$ and is found as $\mathbf{C} = (c_{ij}) = (a_{ij} + b_{ij})$; for example

$$\begin{pmatrix} 7 & -3 & 4 \\ 2 & 8 & -5 \end{pmatrix} + \begin{pmatrix} 11 & 5 & -6 \\ 3 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 18 & 2 & -2 \\ 5 & 12 & -3 \end{pmatrix}.$$

The *difference* $\mathbf{D} = \mathbf{A} - \mathbf{B}$ between two conformal matrices $\mathbf{A}$ and $\mathbf{B}$ is defined similarly: $\mathbf{D} = (d_{ij}) = (a_{ij} - b_{ij})$.

Two properties of matrix addition are given in the following theorem.

Theorem 2.2a. If $\mathbf{A}$ and $\mathbf{B}$ are both $n \times m$, then

(i) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$. (2.9)

(ii) $(\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'$. (2.10)

$(a_{ij} + b_{ij})' = (c_{ij})'$
$= c_{ji}$

$(a_{ij})' + (b_{ij})' = a_{ji} + b_{ji}$
$= c_{ji}$

$\square$

2.2.2 Product of a Scalar and a Matrix

Any scalar can be multiplied by any matrix. The product of a scalar and a matrix is defined as the product of each element of the matrix and the scalar:

$$c\mathbf{A} = (ca_{ij}) = \begin{pmatrix} ca_{11} & ca_{12} & \cdots & ca_{1m} \\ ca_{21} & ca_{22} & \cdots & ca_{2m} \\ \vdots & \vdots & & \vdots \\ ca_{n1} & ca_{n2} & \cdots & ca_{nm} \end{pmatrix}. \tag{2.11}$$

스칼라와 행렬의 곱은 교환법칙이 성립한다.

Since $ca_{ij} = a_{ij}c$, the product of a scalar and a matrix is commutative:

$$c\mathbf{A} = \mathbf{A}c. \tag{2.12}$$

2.2.3 Product of Two Matrices or Two Vectors

In order for the product **AB** to be defined, the number of columns in **A** must equal the number of rows in **B**, in which case **A** ($n \times m$) and **B** ($m \times p$) are said to be *conformal for multiplication*. Then the (ij)th element of the product $\mathbf{C} = \mathbf{AB}$ is defined as

$$c_{ij} = \sum_k a_{ik} b_{kj}, \tag{2.13}$$

which is the sum of products of the elements in the ith row of **A** and the elements in the jth column of **B**. Thus we multiply every row of **A** by every column of **B**. If **A** is $n \times m$ and **B** is $m \times p$, then $\mathbf{C} = \mathbf{AB}$ is $n \times p$. We illustrate matrix multiplication in the following example.

Example 2.2.3. Let

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 6 & 5 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 4 \\ 2 & 6 \\ 3 & 8 \end{pmatrix}.$$

Then

$$\mathbf{AB} = \begin{pmatrix} 2 \cdot 1 + 1 \cdot 2 + 3 \cdot 3 & 2 \cdot 4 + 1 \cdot 6 + 3 \cdot 8 \\ 4 \cdot 1 + 6 \cdot 2 + 5 \cdot 3 & 4 \cdot 4 + 6 \cdot 6 + 5 \cdot 8 \end{pmatrix} = \begin{pmatrix} 13 & 38 \\ 31 & 92 \end{pmatrix},$$

2+16

$$\mathbf{BA} = \begin{pmatrix} 18 & 25 & 23 \\ 28 & 38 & 36 \\ 38 & 51 & 49 \end{pmatrix}.$$

1x1 크기의 행렬과 스칼라의 차이

Note that a 1×1 *matrix* **A** can only be multiplied on the right by a $1 \times n$ matrix **B** or on the left by an $n \times 1$ matrix **C**, whereas a *scalar* can be multiplied on the right or left by a matrix of any size.

If $\mathbf{A}$ is $n \times m$ and $\mathbf{B}$ is $m \times p$, where $n \neq p$, then $\mathbf{AB}$ is defined, but $\mathbf{BA}$ is not defined. If $\mathbf{A}$ is $n \times p$ and $\mathbf{B}$ is $p \times n$, then $\mathbf{AB}$ is $n \times n$ and $\mathbf{BA}$ is $p \times p$. In this case, of course, $\mathbf{AB} \neq \mathbf{BA}$, as illustrated in Example 2.2.3. If $\mathbf{A}$ and $\mathbf{B}$ are both $n \times n$, then $\mathbf{AB}$ and $\mathbf{BA}$ are the same size, but, in general

$$\mathbf{AB} \neq \mathbf{BA}. \tag{2.14}$$

예외존재 두 대각 행렬, 정사각행렬과 항등행렬.

[There are a few exceptions to (2.14), for example, two diagonal matrices or a square matrix and an identity.] Thus matrix multiplication is not commutative, and certain
행렬 곱은 교환 법칙 x.
familiar manipulations with real numbers cannot be done with matrices. However, matrix multiplication is distributive over addition or subtraction:

덧셈, 뺄셈에 대해 분배 법칙은 성립.

$$\mathbf{A}(\mathbf{B} \pm \mathbf{C}) = \mathbf{AB} \pm \mathbf{AC}, \tag{2.15}$$

$$(\mathbf{A} \pm \mathbf{B})\mathbf{C} = \mathbf{AC} \pm \mathbf{BC}. \tag{2.16}$$

Using (2.15) and (2.16), we can expand products such as $(\mathbf{A} - \mathbf{B})(\mathbf{C} - \mathbf{D})$:

$$\begin{aligned} (\mathbf{A} - \mathbf{B})(\mathbf{C} - \mathbf{D}) &= (\mathbf{A} - \mathbf{B})\mathbf{C} - (\mathbf{A} - \mathbf{B})\mathbf{D} \qquad [\text{by } (2.15)] \\ &= \mathbf{AC} - \mathbf{BC} - \mathbf{AD} + \mathbf{BD} \qquad [\text{by } (2.16)]. \end{aligned} \tag{2.17}$$

Multiplication involving vectors follows the same rules as for matrices. Suppose that $\mathbf{A}$ is $n \times p$, $\mathbf{b}$ is $p \times 1$, $\mathbf{c}$ is $p \times 1$, and $\mathbf{d}$ is $n \times 1$. Then $\mathbf{Ab}$ is a column vector of size $n \times 1$, $\mathbf{d}'\mathbf{A}$ is a row vector of size $1 \times p$, $\mathbf{b}'\mathbf{c}$ is a sum of products (1×1), $\mathbf{bc}'$ is a $p \times p$ matrix, and $\mathbf{cd}'$ is a $p \times n$ matrix. Since $\mathbf{b}'\mathbf{c}$ is a 1×1 sum of products, it is equal to $\mathbf{c}'\mathbf{b}$:

$A_{n\times p}$ $b_{p\times 1}$ $c_{p\times 1}$ $d_{n\times 1}$

$$\begin{aligned} \mathbf{b}'\mathbf{c} &= b_1c_1 + b_2c_2 + \cdots + b_pc_p, \\ \mathbf{c}'\mathbf{b} &= c_1b_1 + c_2b_2 + \cdots + c_pb_p, \\ \mathbf{b}'\mathbf{c} &= \mathbf{c}'\mathbf{b}. \end{aligned} \tag{2.18}$$

The matrix $\mathbf{cd}'$ is given by

$$\underset{p\times 1\ 1\times n}{\mathbf{cd}'} = \begin{pmatrix} c_1d_1 & c_1d_2 & \cdots & c_1d_n \\ c_2d_1 & c_2d_2 & \cdots & c_2d_n \\ \vdots & \vdots & & \vdots \\ c_pd_1 & c_pd_2 & \cdots & c_pd_n \end{pmatrix}_{p\times n}. \tag{2.19}$$

Similarly

$$\underset{1\times p}{\mathbf{b}'}\underset{p\times 1}{\mathbf{b}} = b_1^2 + b_2^2 + \cdots + b_p^2, \tag{2.20}$$

$$\underset{p\times 1}{\mathbf{b}}\underset{1\times p}{\mathbf{b}'} = \begin{pmatrix} b_1^2 & b_1b_2 & \cdots & b_1b_p \\ b_2b_1 & b_2^2 & \cdots & b_2b_p \\ \vdots & \vdots & & \vdots \\ b_pb_1 & b_pb_2 & \cdots & b_p^2 \end{pmatrix}_{p\times p}. \tag{2.21}$$

제곱의 합 (대칭) 정사각행렬

Thus, $\mathbf{b}'\mathbf{b}$ is a sum of squares and $\mathbf{bb}'$ is a (symmetric) square matrix.

The square root of the sum of squares of the elements of a $p \times 1$ vector $\mathbf{b}$ is the distance from the origin to the point $\mathbf{b}$ and is also referred to as the *length* of $\mathbf{b}$:

원점에서 점 b까지 거리.

$$\text{Length of } \mathbf{b} = \sqrt{\mathbf{b}'\mathbf{b}} = \sqrt{\sum_{i=1}^{p} b_i^2}. \tag{2.22}$$

If $\mathbf{j}$ is an $n \times 1$ vector of 1s as defined in (2.6), then by (2.20) and (2.21), we have

$$\underset{1\times n}{\mathbf{j}'}\underset{n\times 1}{\mathbf{j}} = n, \quad \underset{n\times 1}{\mathbf{j}}\underset{1\times n}{\mathbf{j}'} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}_{n\times n} = \mathbf{J}, \tag{2.23}$$

where $\mathbf{J}$ is an $n \times n$ square matrix of 1s as illustrated in (2.7). If $\mathbf{a}$ is $n \times 1$ and $\mathbf{A}$ is $n \times p$, then

$$\underset{1\times n}{\mathbf{a}'}\underset{n\times 1}{\mathbf{j}} = \underset{1\times n}{\mathbf{j}'}\underset{n\times 1}{\mathbf{a}} = \sum_{i=1}^{n} a_i, \tag{2.24}$$

$$\underset{1\times n}{\mathbf{j}'}\underset{n\times p}{\mathbf{A}} = \left(\sum_i a_{i1}, \sum_i a_{i2}, \ldots, \sum_i a_{ip} \right), \qquad \underset{n\times p}{\mathbf{A}}\underset{p\times 1}{\mathbf{j}} = \begin{pmatrix} \sum_j a_{1j} \\ \sum_j a_{2j} \\ \vdots \\ \sum_j a_{nj} \end{pmatrix}. \tag{2.25}$$

벡터 a의 원소들의 합 각 열의 합. 각 행의 합

Thus $\mathbf{a}'\mathbf{j}$ is the sum of the elements in $\mathbf{a}$, $\mathbf{j}'\mathbf{A}$ contains the column sums of $\mathbf{A}$, and $\mathbf{Aj}$ contains the row sums of $\mathbf{A}$. Note that in $\mathbf{a}'\mathbf{j}$, the vector $\mathbf{j}$ is $n \times 1$; in $\mathbf{j}'\mathbf{A}$, the vector $\mathbf{j}$ is $n \times 1$; and in $\mathbf{Aj}$, the vector $\mathbf{j}$ is $p \times 1$.

전치

The transpose of the product of two matrices is the product of the transposes in reverse order.

역순

Theorem 2.2b. If $\mathbf{A}$ is $n \times p$ and $\mathbf{B}$ is $p \times m$, then

$$(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'. \tag{2.26}$$

Proof. Let $\mathbf{C} = \mathbf{AB}$. Then by (2.13)

$$\mathbf{C} = (c_{ij}) = \left(\sum_{k=1}^{p} a_{ik} b_{kj} \right).$$

By (2.3), the transpose of $\mathbf{C} = \mathbf{AB}$ becomes

$$\begin{aligned}
(\mathbf{AB})' &= \mathbf{C}' = (c_{ij})' = (c_{ji}) \\
&= \left(\sum_{k=1}^{p} a_{jk} b_{ki} \right) = \left(\sum_{k=1}^{p} b_{ki} a_{jk} \right) = \mathbf{B}'\mathbf{A}'.
\end{aligned}$$

i는 행, j는 열을 나타내므로 위치가 바뀌어야 함.

We illustrate the steps in the proof of Theorem 2.2b using a 2×3 matrix $\mathbf{A}$ and a 3×2 matrix $\mathbf{B}$:

$$\begin{aligned}
\mathbf{AB} &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} \\
&= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{pmatrix}, \\
(\mathbf{AB})' &= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} \\ a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{pmatrix} \\
&= \begin{pmatrix} b_{11}a_{11} + b_{21}a_{12} + b_{31}a_{13} & b_{11}a_{21} + b_{21}a_{22} + b_{31}a_{23} \\ b_{12}a_{11} + b_{22}a_{12} + b_{32}a_{13} & b_{12}a_{21} + b_{22}a_{22} + b_{32}a_{23} \end{pmatrix} \\
&= \begin{pmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix} \\
&= \mathbf{B}'\mathbf{A}'.
\end{aligned}$$

The following corollary to Theorem 2.2b gives the transpose of the product of three matrices.

Corollary 1. If $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ are conformal so that $\mathbf{ABC}$ is defined, then $(\mathbf{ABC})' = \mathbf{C}'\mathbf{B}'\mathbf{A}'$.

D $\searrow$ $(DC)' = C'D' = C'(AB)' = C'B'A'$

Suppose that $\mathbf{A}$ is $n \times m$ and $\mathbf{B}$ is $m \times p$. Let $\mathbf{a}_i'$ be the ith *row* of $\mathbf{A}$ and $\mathbf{b}_j$ be the jth *column* of $\mathbf{B}$, so that

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_1' \\ \mathbf{a}_2' \\ \vdots \\ \mathbf{a}_n' \end{pmatrix}, \qquad \mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p).$$

Then, by definition, the (ij)th element of $\mathbf{AB}$ is $\mathbf{a}_i'\mathbf{b}_j$:

$$\mathbf{AB} = \begin{pmatrix} \mathbf{a}_1'\mathbf{b}_1 & \mathbf{a}_1'\mathbf{b}_2 & \cdots & \mathbf{a}_1'\mathbf{b}_p \\ \mathbf{a}_2'\mathbf{b}_1 & \mathbf{a}_2'\mathbf{b}_2 & \cdots & \mathbf{a}_2'\mathbf{b}_p \\ \vdots & \vdots & & \vdots \\ \mathbf{a}_n'\mathbf{b}_1 & \mathbf{a}_n'\mathbf{b}_2 & \cdots & \mathbf{a}_n'\mathbf{b}_p \end{pmatrix}.$$

A의 행 기준으로

This product can be written in terms of the rows of $\mathbf{A}$:

$$\mathbf{AB} = \begin{pmatrix} \mathbf{a}_1'(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p) \\ \mathbf{a}_2'(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p) \\ \vdots \\ \mathbf{a}_n'(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p) \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1'\mathbf{B} \\ \mathbf{a}_2'\mathbf{B} \\ \vdots \\ \mathbf{a}_n'\mathbf{B} \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1' \\ \mathbf{a}_2' \\ \vdots \\ \mathbf{a}_n' \end{pmatrix} \mathbf{B}. \tag{2.27}$$

The first column of **AB** can be expressed in terms of $\mathbf{A}$ as

첫번째 열

$$\begin{pmatrix} \mathbf{a}_1'\mathbf{b}_1 \\ \mathbf{a}_2'\mathbf{b}_1 \\ \vdots \\ \mathbf{a}_n'\mathbf{b}_1 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1' \\ \mathbf{a}_2' \\ \vdots \\ \mathbf{a}_n' \end{pmatrix} \mathbf{b}_1 = \mathbf{A}\mathbf{b}_1.$$

두 번째 열 $\rightarrow$

Likewise, the second column is $\mathbf{A}\mathbf{b}_2$, and so on. Thus $\mathbf{AB}$ can be written in terms of the columns of $\mathbf{B}$:

$$\mathbf{AB} = \mathbf{A}(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p) = (\mathbf{A}\mathbf{b}_1, \mathbf{A}\mathbf{b}_2, \ldots, \mathbf{A}\mathbf{b}_p). \tag{2.28}$$

Any matrix $\mathbf{A}$ can be multiplied by its transpose to form $\mathbf{A}'\mathbf{A}$ or $\mathbf{A}\mathbf{A}'$. Some properties of these two products are given in the following theorem.

Theorem 2.2c. Let $\mathbf{A}$ be any $n \times p$ matrix. Then $\mathbf{A}'\mathbf{A}$ and $\mathbf{A}\mathbf{A}'$ have the following properties.

A의 열의 곱.

(i) $\mathbf{A}'\mathbf{A}$ is $p \times p$ and its elements are products of the *columns* of $\mathbf{A}$.
(ii) $\mathbf{A}\mathbf{A}'$ is $n \times n$ and its elements are products of the *rows* of $\mathbf{A}$.

A의 행의 곱.

(iii) Both $\mathbf{A}'\mathbf{A}$ and $\mathbf{A}\mathbf{A}'$ are symmetric.
(iv) If $\mathbf{A}'\mathbf{A} = \mathbf{O}$, then $\mathbf{A} = \mathbf{O}$.

Let $\mathbf{A}$ be an $n \times n$ matrix and let $\mathbf{D} = \text{diag}(d_1, d_2, \ldots, d_n)$. In the product $\mathbf{DA}$, the ith row of $\mathbf{A}$ is multiplied by d_i, and in $\mathbf{AD}$, the jth column of $\mathbf{A}$ is multiplied by d_j. For example, if $n = 3$, we have

$$\mathbf{DA} = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$= \begin{pmatrix} d_1 a_{11} & d_1 a_{12} & d_1 a_{13} \\ d_2 a_{21} & d_2 a_{22} & d_2 a_{23} \\ d_3 a_{31} & d_3 a_{32} & d_3 a_{33} \end{pmatrix}, \tag{2.29}$$

$$\mathbf{AD} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$$

$$= \begin{pmatrix} d_1 a_{11} & d_2 a_{12} & d_3 a_{13} \\ d_1 a_{21} & d_2 a_{22} & d_3 a_{23} \\ d_1 a_{31} & d_2 a_{32} & d_3 a_{33} \end{pmatrix}, \tag{2.30}$$

$$\mathbf{DAD} = \begin{pmatrix} d_1^2 a_{11} & d_1 d_2 a_{12} & d_1 d_3 a_{13} \\ d_2 d_1 a_{21} & d_2^2 a_{22} & d_2 d_3 a_{23} \\ d_3 d_1 a_{31} & d_3 d_2 a_{32} & d_3^2 a_{33} \end{pmatrix}. \tag{2.31}$$

Note that $\mathbf{DA} \neq \mathbf{AD}$. However, in the special case where the diagonal matrix is the identity, (2.29) and (2.30) become

$$\mathbf{IA} = \mathbf{AI} = \mathbf{A}. \tag{2.32}$$

A크기가 n×n이 아니어도. 성립함. 근데 두 항등 행렬의 크기는 다름.

If $\mathbf{A}$ is rectangular, (2.32) still holds, but the two identities are of different sizes.

If $\mathbf{A}$ is a symmetric matrix and $\mathbf{y}$ is a vector, the product

$$\mathbf{y}'\mathbf{A}\mathbf{y} = \sum_i a_{ii}y_i^2 + \sum_{i \neq j} a_{ij}y_iy_j \tag{2.33}$$

$$(y_1\ y_2)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = a_{11}y_1^2 + a_{22}y_2^2 + (a_{12}+a_{22})y_1y_2$$

이차 형식.

is called a *quadratic form*. If $\mathbf{x}$ is $n \times 1$, $\mathbf{y}$ is $p \times 1$, and $\mathbf{A}$ is $n \times p$, the product

$$\mathbf{x}'\mathbf{A}\mathbf{y} = \sum_{ij} a_{ij}x_iy_j \tag{2.34}$$

쌍선형 형식

is called a *bilinear form*.

2.2.4 Hadamard Product of Two Matrices or Two Vectors

Sometimes a third type of product, called the *elementwise* or *Hadamard product*, is useful. If two matrices or two vectors are of the same size (conformal for addition), the Hadamard product is found by simply multiplying corresponding elements:

$$(a_{ij}b_{ij}) = \begin{pmatrix} a_{11}b_{11} & a_{12}b_{12} & \cdots & a_{1p}b_{1p} \\ a_{21}b_{21} & a_{22}b_{22} & \cdots & a_{2p}b_{2p} \\ \vdots & \vdots & & \vdots \\ a_{n1}b_{n1} & a_{n2}b_{n2} & \cdots & a_{np}b_{np} \end{pmatrix}.$$

2.3 PARTITIONED MATRICES

It is sometimes convenient to partition a matrix into submatrices. For example, a partitioning of a matrix $\mathbf{A}$ into four (square or rectangular) submatrices of appropriate sizes can be indicated symbolically as follows:

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}.$$