COMPACT MODULI OF K3 SURFACES WITH A NONSYMPLECTIC AUTOMORPHISM

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Abstract. We construct a modular compactification via stable slc pairs for the moduli spaces of K3 surfaces with a nonsymplectic group of automorphisms under the assumption that some combination of the fixed loci of automorphisms defines an effective big divisor, and prove that it is semitoroidal.

Contents

1. Introduction

Let X be a smooth K3 surface over the complex numbers. An automorphism σ of X is called *non-symplectic* if it has finite order $n > 1$ and $\sigma^*(\omega_X) = \zeta_n \omega_X$, where $\omega_X \in H^{2,0}(X)$ is a nonzero 2-form and ζ_n is a primitive nth root of identity. By changing the generator of the cyclic group μ_n we can and will assume that $\zeta_n = \exp(2\pi i/n)$. It is well known that a K3 surface admitting such an automorphism is projective. The possibilities for the order are the numbers n whose Euler function satisfies $\varphi(n) \le 20$ with the single exception $n \ne 60$, see [MO98, Thm. 3].

In this paper we study compactification of moduli spaces of pairs (X, σ). But to begin with, the automorphism group $\mathrm{Aut}(X, \sigma)$, i.e. those automorphisms of X commuting with σ, may be infinite. To fix this, we will usually additionally assume:

$(\exists g \ge 2)$ The fixed locus $\mathrm{Fix}(\sigma)$ contains a curve C_1 of genus $g \ge 2$.

By looking at the μ_n-action on the tangent space of any fixed point, it is easy to see that $\mathrm{Fix}(\sigma)$ is a disjoint union of several smooth curves and points. The Hodge index theorem implies at most one of the fixed curves has genus $g \ge 2$. One could instead have one or two fixed curves of genus $g = 1$. All other fixed curves are isomorphic to $\mathbb{P}^1$.

Under the $(\exists g \ge 2)$ assumption, the group $\mathrm{Aut}(X, \sigma)$ is finite. The opposite is almost true. For example let $n = 2$, i.e. σ is an involution. Then σ^* fixes the

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Neron-Severi lattice $S_X \subset H^2(X,\mathbb{Z})$ and acts as multiplication by (-1) on the lattice $T_X = S_X^\perp$ of transcendental cycles. In this case $\mathrm{Aut}(X,\sigma) = \mathrm{Aut}(X)$.

Deformation classes of such K3 surfaces (X,σ) are classified by the primitive 2-elementary hyperbolic sublattices $S \subset L_{K3}$. By Nikulin [Nik79b] there are 75 cases, uniquely determined by certain invariants (g,k,δ). Among them 51 satisfy $(\exists g \geq 2)$. The only case when $|\mathrm{Aut}(X)| < \infty$ but $(\exists g \geq 2)$ is not satisfied is $(g,k,\delta) = (1,9,1)$ which is the one-dimensional mirror family to K3 surfaces of degree 2. In the case $(g,k,\delta) = (2,1,0)$ one has $|\mathrm{Aut}(X)| = \infty$ but the set $\mathrm{Fix}(\sigma)$ consists of two elliptic curves, so $(\exists g \geq 2)$ does not hold.

Since the moduli stack of smooth quasipolarized K3 surfaces is notoriously non-separated, so is usually the moduli stack of smooth K3s with a nonsymplectic automorphism. For a fixed isometry $\rho \in O(L_{K3})$ of order n, there exists the moduli stack and moduli space of smooth K3 surfaces "of type ρ": those pairs (X,σ) where the action of σ^* on $H^2(X,\mathbb{Z})$ can be modeled by ρ. We construct them in Section 2. The maximal separated quotient of F_ρ is $(\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho$, where $\mathbb{D}_\rho$ is a symmetric Hermitian domain of type IV if $n=2$ or a complex ball if $n>2$, Γ_ρ is an arithmetic group, and $\Delta_\rho \subset \mathbb{D}_\rho$ is the discriminant locus.

Under the assumption $(\exists g \geq 2)$, the space $F_\rho^{\mathrm{ade}} := (\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho$ is the coarse moduli space for the K3 surfaces $\overline{X}$ with ADE singularities, obtained from the smooth K3 surfaces X by contracting the (-2)-curves perpendicular to the component C_1 with $g \geq 2$ in $\mathrm{Fix}(\sigma)$. The stack of such ADE K3 surfaces is separated.

The main goal of this paper is to construct a functorial, geometrically meaningful compactification of the moduli space F_ρ^{ade}, under the assumption $(\exists g \geq 2)$. Let $R = C_1$, $\varphi_{|mR|}\colon X \to \overline{X}$ be the contraction as above and $\overline{R}$ be the image of R. Then for any $0 < \epsilon \ll 1$ the pair $(\overline{X}, \epsilon\overline{R})$ is a stable pair with semi log canonical singularities. Then the theory of KSBA moduli spaces (see [Kol21] for the general case or [AET19, ABE20] for the much easier special case needed here) gives a moduli compactification $\overline{F}_\rho^{\mathrm{slc}}$ to a space of stable pairs with automorphism.

Our main Theorem 3.24 says that $\overline{F}_\rho^{\mathrm{slc}}$ is a semitoroidal compactification of $\mathbb{D}_\rho/\Gamma_\rho$. This class of compactifications was introduced by Looijenga [Loo03b] as a common generalization of Baily-Borel and toroidal compactifications. As a corollary, the family of ADE K3 surfaces with an automorphism extends along the inclusion $(\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho \hookrightarrow \mathbb{D}_\rho/\Gamma_\rho$.

The proof applies a modified form of one of the main theorems of [AE21] about so-called *recognizable* divisors. The $g \geq 2$ component of the fixed locus is a canonical choice of a polarizing divisor. We prove that this divisor is recognizable.

As we point out in Section 5, the results also extend to the more general situation of a symmetry group $G \subset \mathrm{Aut}\, X$ which is not purely symplectic.

The cases $n = 2,3,4,6$ are of the most interest for compactifications. If $n \neq 2,3,4,6$ then the space $\mathbb{D}_\rho/\Gamma_\rho$ is already compact, see [Mat16] or Corollary 3.14.

K3 surfaces with an involution were classified by Nikulin in [Nik79b]. K3s with a non-symplectic automorphism of prime order $p \geq 3$ we classified by Artebani, Sarti, and Taki in [AS08, AST11]. The case $n=4$ was treated by Artebani-Sarti in [AS15] and the case $n=6$ by Dillies in [Dil09, Dil12].

We note two cases where our KSBA, semitoroidal compactification $\overline{F}_\rho^{\mathrm{slc}}$ is computed in complete detail: Alexeev-Engel-Thompson [AET19] for the case of K3 surfaces of degree 2, generically double covers of $\mathbb{P}^2$, and a forthcoming work Deopurkar-Han [DH22] which treats a 9-dimensional component in the moduli for $n = 3$.

The paper is organized as follows. In Section 2 we set up the general theory of the moduli of K3 surfaces with a non-symplectic automorphisms. In Section 3 we define the stable pair compactifications and prove the main Theorem 3.24. In Section 4 we relate K3 surfaces with nonsymplectic automorphisms with their quotients $Y = \overline{X}/\mu_n$, and the compactification $\overline{F}_\rho^{\rm slc}$ with the KSBA compactification of the moduli spaces of log del Pezzo pairs $(Y, \frac{n-1+\epsilon}{n}B)$.

In Section 5 we extend the results in two different ways: to K3 surfaces with a finite group of symmetries $G \subset \operatorname{Aut} X$ which is not purely symplectic, and to more general polarizing divisors associated with such a group action.

Throughout, we work over the field of complex numbers.

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2. Moduli of K3s with a nonsymplectic automorphism

2A. **Notations.** A lattice is a free abelian group with an integral-valued symmetric bilinear form. Let $L = H^{\oplus 3} \oplus E_8^{\oplus 2}$ be a fixed copy of the even unimodular lattice of signature $(3, 19)$, where $H = \mathrm{II}_{1,1}$ corresponds to the bilinear form $b(x, y) = xy$ and E_8 is the standard negative definite even lattice of rank 8. For any smooth K3 surface X the cohomology lattice $H^2(X, \mathbb{Z})$ is isometric to L.

Denote by $S = S_X$ the Neron-Severi lattice $\operatorname{Pic}(X) = \operatorname{NS}(X)$. By the Lefschetz $(1,1)$-theorem, it equals $(H^{2,0}(X))^\perp \cap H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{C})$. We have $H^{2,0}(X) = \mathbb{C}\omega_X$ for some nowhere vanishing holomorphic two-form ω_X. If X is projective, then S_X is nondegenerate of signature $(1, r_X - 1)$. In this case, its orthogonal complement $T_X = (S_X)^\perp \subset H^2(X, \mathbb{Z})$ is the *transcendental lattice*, of signature $(2, 20 - r_X)$. The *Kähler cone* $\mathcal{K}_X \subset H^{1,1}(X, \mathbb{R})$ is the set of classes of Kähler forms on X; it is an open convex cone.

Theorem 2.1 (Torelli Theorem for K3 surfaces, [PSS71])**.** *The isomorphisms $\sigma\colon X' \to X$ are in bijection with the isometries $\sigma^*\colon H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ satisfying the conditions $\sigma^*(H^{2,0}(X)) = H^{2,0}(X')$ and $\sigma^*(\mathcal{K}_X) = \mathcal{K}_{X'}$.*

For any lattice H, a *root* is a vector $\delta \in H$ with $\delta^2 = -2$. The set of all roots is denoted by H_{-2}. The Weyl group $W(H)$ is the group generated by reflections $v \mapsto v + (v, \delta)\delta$ for $\delta \in H_{-2}$. It is a normal subgroup of the isometry group $O(H)$.

2B. **Moduli of marked unpolarized K3s.** The basic reference here is [ast85]. Let X be a K3 surface. A *marking* is an isometry $\phi\colon H^2(X, \mathbb{Z}) \to L$. Let

$$\mathbb{D} = \mathbb{P}\{x \in L_\mathbb{C} \mid x \cdot x = 0,\ x \cdot \bar{x} > 0\}, \quad \dim \mathbb{D} = 20.$$

There exists a fine moduli space $\mathcal{M}$ of *marked K3 surfaces* and a period map $\pi\colon \mathcal{M} \to \mathbb{D}$, $(X, \phi) \mapsto \phi(H^{2,0}(X)) \in \mathbb{P}(L_\mathbb{C})$. $\mathcal{M}$ is a non-Hausdorff 20-dimensional complex manifold with two isomorphic connected components interchanged by negating ϕ. The period map is étale and surjective.

For a period point $x \in \mathbb{D}$, the vector space $(\mathbb{C}x \oplus \mathbb{C}\bar{x}) \cap L_\mathbb{R} \subset L_\mathbb{C}$ is positive definite of rank 2 and its orthogonal complement $x^\perp \cap L_\mathbb{R}$ has signature $(1, 19)$. Let

$$\{v \in x^\perp \cap L_\mathbb{R} \mid v^2 > 0\} = P_x \sqcup (-P_x)$$

be the two connected components of the set of positive square vectors. Then the fiber $\pi^{-1}(x)$ is identified with the set of connected components $\mathcal{C}$ of

$$\big(P_x \sqcup (-P_x)\big) \setminus \cup_\delta \, \delta^\perp \text{ for } \delta \in (x^\perp \cap L)_{-2}. \tag{1}$$

Namely, an open chamber $\mathcal{C}$ is identified with the Kähler cone $\mathcal{K}_X$ of the corresponding marked K3 surface X via the marking ϕ. The connected components are permuted by the reflections and $\pm\,\mathrm{id}$, and $\pi^{-1}(x)$ is a torsor under the group $\mathbb{Z}_2 \times W_x$, where $W_x = W(x^\perp \cap L)$. Since $x^\perp \cap L_\mathbb{R}$ is hyperbolic, the group and the fiber $\pi^{-1}(x)$ may be infinite. For a general point $x \in \mathbb{D}$, the lattice $x^\perp \cap L$ has no roots and the fiber $\pi^{-1}(x)$ consists of two points, one in each connected component of $\mathcal{M}$.

2C. **Moduli of ρ-marked and ρ-markable K3 surfaces with automorphisms.** Fix $\rho \in O(L)$ an isometry of order $n > 1$ and consider a K3 surface X with a non-symplectic automorphism σ of order n.

Definition 2.2. A *ρ-marking* of (X, σ) is an isometry $\phi : H^2(X, \mathbb{Z}) \to L$ such that $\sigma^* = \phi^{-1} \circ \rho \circ \phi$. We say that (X, σ) is *ρ-markable* if it admits a ρ-marking.

A family of ρ-marked surfaces is a smooth morphism $f\colon (\mathcal{X}, \sigma_B) \to B$ with an automorphism $\sigma_B\colon \mathcal{X} \to \mathcal{X}$ over B, together with an isomorphism of local systems $\phi_S\colon R^2 f_* \underline{\mathbb{Z}} \to L \otimes \underline{\mathbb{Z}}_B$ such that every fiber is a K3 surface with a ρ-marking. A family $f\colon (\mathcal{X}, \sigma_B) \to B$ is ρ-markable if such an isomorphism exists locally in complex-analytic topology on B.

We define the moduli stacks $\mathcal{M}_\rho$ of ρ-marked, resp. F_ρ of ρ-markable K3 by taking $\mathcal{M}_\rho(B)$, resp. $F_\rho(B)$ to be the groupoids of such families over base B.

Definition 2.3. Define $L_\mathbb{C}^{\zeta_n}$ to be the eigenspace $x \in L_\mathbb{C}$ such that $\rho(x) = \zeta_n x$, and the subdomain $\mathbb{D}_\rho = \mathbb{P}(L_\mathbb{C}^{\zeta_n}) \cap \mathbb{D} \subset \mathbb{D}$. Define $\Gamma_\rho \subset O(L)$ as the group of changes-of-marking: $\Gamma_\rho := \{\gamma \in O(L) \mid \gamma \circ \rho = \rho \circ \gamma\}$.

Definition 2.4. Let the *generic transcendental lattice* $T_\rho := L_\mathbb{C}^{\mathrm{prim}} \cap L$ be the intersection of L with the sum of all primitive eigenspaces of ρ, and let the *generic Picard lattice* be $S_\rho = (T_\rho)^\perp$. Let $L^G = \mathrm{Fix}(\rho) \subset S_\rho$ be classes in L fixed by ρ. (Here, we use $G = \langle \rho \rangle \simeq \mathbb{Z}_n$ to avoid confusing notation, as L^G would be.)

Note that the ζ_n-eigenspaces $L_\mathbb{C}^{\zeta_n}$ and $T_{\rho,\mathbb{C}}^{\zeta_n}$ coincide, and that for any K3 surface with a ρ-marking the two fixed sublattices $\phi\colon S_X^G = H^2(X, \mathbb{Z})^G \xrightarrow{\sim} L^G$ are identified.

For there to exist a ρ-markable algebraic K3 surface, the signature of T_ρ must be $(2, \ell)$ for some ℓ, as there is necessarily a vector of positive norm fixed by σ^* (the sum of a σ^*-orbit of an ample class). The converse is also true.

When $n = 2$, we have that $\mathbb{D}_\rho \subset \mathbb{P}(T_{\rho,\mathbb{C}})$ is (two copies of) the Type IV domain associated to the lattice T_ρ. When $n \geq 3$, the condition that $x \cdot x = 0$ is vacuous on $\mathbb{D}_\rho$ because $x \cdot y = 0$ for eigenvectors x, y of ρ with non-conjugate eigenvalue. Thus,

$$\mathbb{D}_\rho = \mathbb{P}\{x \in T_{\rho,\mathbb{C}}^{\zeta_n} \mid x \cdot \bar{x} > 0\}$$

is a complex ball, a Type I domain. The Hermitian form $x \cdot \bar{y}$ on $T_{\rho,\mathbb{C}}^{\zeta_n}$ necessarily has signature $(1, \ell)$ for some ℓ for there to exist a ρ-markable K3 surface.

Definition 2.5. The discriminant locus is $\Delta_\rho := (\cup_\delta \, \delta^\perp) \cap \mathbb{D}_\rho$ ranging over all roots δ in $(L^G)^\perp$.

Lemma 2.6. *Let $\rho \in O(L)$ be an isometry of order $n > 1$. Then*

(1) *A marking $\phi\colon H^2(X,\mathbb{Z}) \to L$ defines a ρ-marking, i.e. defines an automorphism σ such that $\sigma^* = \phi^{-1}\circ\rho\circ\phi$ iff the period $x = \pi((X,\phi))$ lies in $\mathbb{D}_\rho \setminus \Delta_\rho$ and there exists an ample line bundle $\mathcal{L}_h$ on X with $h = \phi(\mathcal{L}_h) \in L^G$.*
(2) *For a point $x \in \mathbb{D}_\rho \setminus \Delta_\rho$ the set of ρ-marked K3s with this period is a torsor over the group $\Gamma_\rho \cap (\mathbb{Z}_2 \times W_x)$.*

Proof. Because the action is nonsymplectic, $\rho(x) = \zeta_n x \neq x$. For any $h \in L^G$ one has $\rho(h) = h$, which implies that $hx = 0$. Thus, $L^G \perp x$ and $S_X^G \simeq L^G$.

Clearly, one must have $x \in \mathbb{D}_\rho$. By the Torelli theorem, automorphism $a = \phi^{-1}\circ\rho\circ\phi$ of $H^2(X,\mathbb{Z})$ is induced by an automorphism σ of X iff it sends the Kähler cone $\mathcal{K}_X$ to itself. By averaging, this is equivalent to having an a-invariant Kähler class $\mathcal{L}_h \in \mathcal{K}_X \cap H^2(X,\mathbb{Z})$. And since $L^G \perp x$, one has $\mathcal{L}_h \perp \omega_X$, so $\mathcal{L}_h \in S_X$ and $\mathcal{L}_h$ is an ample line bundle. This proves (1).

If $x \perp \delta$ for some root $\delta \in (L^G)^\perp$ then $\mathcal{L}_\delta = \phi^{-1}(\delta) \in \operatorname{Pic}(X)$ and either $\mathcal{L}_\delta$ or $\mathcal{L}_\delta^{-1}$ is effective. Then for $\mathcal{L}_h$ as in part (1) one has both $\mathcal{L}_h \cdot \mathcal{L}_\delta = 0$ because $h \perp \delta$ and $\mathcal{L}_h \cdot \mathcal{L}_\delta \neq 0$ because $\mathcal{L}_h$ is ample. Contradiction.

On the other hand, let $x \in \mathbb{D}_\rho \setminus \Delta_\rho$. Then $L^G \not\subset \cup_\delta\, \delta^\perp$ for $\delta \in (x^\perp \cap L)_{-2}$. Thus, there exists a chamber $\mathcal{C}$ in $P_x \setminus \cup_\delta\, \delta^\perp$ such that $\mathcal{C} \cap L^G \neq \emptyset$. Let (X,ϕ) be the K3 surface corresponding to this chamber. Then there exists $h \in \mathcal{C} \cap L^G$ and by part (1) the marking ϕ is a ρ-marking.

Any surface with the same period x is isomorphic to X, but with a marking $\phi' = g \circ \phi$ for some $g \in \mathbb{Z}_2 \times W_x$. Then one has both $\sigma^* = \phi^{-1}\circ\rho\circ\phi$ and $\sigma^* = (\phi')^{-1}\circ\rho\circ\phi'$ iff $g \in \Gamma_\rho$. This proves (2). ☐

Lemma 2.7. *There exists a fine moduli space $\mathcal{M}_\rho$ of ρ-marked K3 surfaces with a non-symplectic automorphism. $\mathcal{M}_\rho$ an open subset of $\pi^{-1}(\mathbb{D}_\rho \setminus \Delta_\rho)$.*

Proof. The points of $\mathcal{M}$ are chambers $\mathcal{C}$ in Equation (1) over $x \in \mathbb{D}_\rho \setminus \Delta_\rho$. As in the proof of Lemma 2.6, one has $\mathcal{C} \in \mathcal{M}_\rho$ iff $\mathcal{C} \cap L^G \neq \emptyset$. This is an open condition. ☐

The restriction of $\pi\colon \mathcal{M} \to \mathbb{D}$ gives the period map $\pi_\rho\colon \mathcal{M}_\rho \to \mathbb{D}_\rho \setminus \Delta_\rho$. The general fiber of π_ρ is a torsor over $\Gamma_\rho \cap (\mathbb{Z}_2 \times W(S_\rho))$. Thus, $\mathcal{M}_\rho$ is not separated iff there exists $x \in \mathbb{D}_\rho \setminus \Delta_\rho$ such that $\Gamma_\rho \cap W_x \supsetneq \Gamma_\rho \cap W(S_\rho)$. This indeed happens:

Example 2.8. Consider the 9-dimensional family of μ_3-covers of $\mathbb{P}^1 \times \mathbb{P}^1$ branched in a curve B of bidegree $(3,3)$, studied by Kondō [Kon02]. In this case,

$$S_\rho = L^G = \big(\operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)\big)(3) = H(3) \quad \text{and} \quad T_\rho = (L^G)^\perp = H \oplus H(3) \oplus E_8^2.$$

Let $\overline{Y}$ be a degeneration of the quadric $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ to a quadratic cone and $\overline{X} \to \overline{Y}$ be the μ_3-cover branched in a curve $\overline{B} \in |\mathcal{O}_{\overline{Y}}(3)|$ not passing through the apex. Let $Y = \mathbb{F}_2$ and X be the minimal resolutions of $\overline{Y}$ and $\overline{X}$. The $\mathbb{P}^1$-fibration on Y gives an elliptic fibration on X, and the preimage of the (-2)-section of Y is a union of three disjoint (-2)-sections e, σe, $\sigma^2 e$ on X, interchanged by the automorphism σ. The invariant sublattice $S_X^\sigma = \big(\operatorname{Pic}(\mathbb{F}_2)\big)(3) = H(3)$ is generated by f and $f' = f + \sum_{i=0}^2 \sigma^i e$.

The only (-2)-curves on X are $\sigma^i e$ and they do not lie in $(S_X^\sigma)^\perp$. Thus, once we fix a marking ϕ, the period x of X will be in $\mathbb{D}_\rho \setminus \Delta_\rho$. The reflections w_i in the roots $\rho^i\phi(e)$ commute. Their product $w = w_0 w_1 w_2$ is non-trivial: on L^G it acts as the reflection that interchanges $\phi(f)$ and $\phi(f')$. It is easy to check that $w \in \Gamma_\rho$. So $\Gamma_\rho \cap W_x \neq 1$ and $W(L^G) = 1$.

Thus, the map $\mathcal{M}_\rho \to \mathbb{D}_\rho \setminus \Delta_\rho$ is not separated in this case. Locally it looks like the "double-headed snake" $\mathbb{A}^1 \cup_{\mathbb{A}^1\setminus 0} \mathbb{A}^1 \to \mathbb{A}^1$ times $\mathbb{A}^8$. Here is another way to see the same. The positive cone P in $H(3)_\mathbb{R}$ is the unique Weyl chamber for the Weyl group $W\big(H(3)\big) = 1$; its rays are $\phi(f)$ and $\phi(f')$. The hyperplane $\phi(e)^\perp$ cuts it in half. The intersections of the Weyl chambers $\mathcal{C} \subset P_x \setminus \cup\delta^\perp$ of Equation 1 with P are either halves of P.

Theorem 2.9. *The moduli stack F_ρ of ρ-markable K3 surfaces with a non-symplectic automorphism is the quotient $F_\rho = \mathcal{M}_\rho/\Gamma_\rho$. Its coarse moduli space admits a bijective period map to $(\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho$, and the coarse moduli space of the separated quotient $F_\rho^{\rm sep}$ is $(\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho$. The generic stabilizer is the group*

$$K_\rho := \ker(\Gamma_\rho \to \mathrm{Aut}(\mathbb{D}_\rho))/\Gamma_\rho \cap (\mathbb{Z}_2 \times W(S_\rho))$$

Proof. The statement is immediate from the definitions and the above two Lemmas by quotienting the period map π_ρ. The points of $\pi_\rho^{-1}(x)$ are permuted by Γ_ρ, thus they are identified in the Γ_ρ-quotient. They are also identified in the separated quotient.

For ρ to correspond to any K3 surface with a nonsymplectic automorphism, S_ρ must have signature $(1, r-1)$ for some r, and for T_ρ to have signature $(2, 20-r)$. The action of Γ_ρ on the Type IV domain $\mathbb{D}(T_\rho)$ factors through $O(T_\rho)$ and is therefore properly discontinuous. Thus, the action of Γ_ρ on $\mathbb{D}_\rho$ is properly discontinuous, and so $(\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho$ is makes sense as a complex-analytic space. (It is also quasiprojective by Baily-Borel.)

The last statement follows from Lemma 2.6(2) by noting that for a generic $x \in \mathbb{D}_\rho \setminus \Delta_\rho$ one has $x^\perp \cap L = S_\rho$. $\square$

Remark 2.10. The proof of part (1) of Lemma 2.6 and of Theorem 2.9 follow the arguments of Dolgachev-Kondo [DK07, Thms. 11.2, 11.3]. Sections 10 and 11 of [DK07] contain a construction of the moduli space of K3 surfaces with a non-symplectic automorphism that is based on moduli of lattice polarized K3s. But it uses [Dol96, Thm. 3.1] which unfortunately is false, as was noted in [AE21] and as Example 2.8 also shows. For this reason, we decided to give an alternative construction.

Remark 2.11. Even though the map to $(\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho$ in Theorem 2.9 is bijective, the coarse moduli space of F_ρ is a non-separated algebraic space when $\mathcal{M}_\rho$ is not separated. This is very similar to the algebraic space obtained by dividing a "two-headed snake" $\mathbb{A}^1 \cup_{\mathbb{A}^1\setminus 0} \mathbb{A}^1$ by the involution $z \to -z$ exchanging the heads. The quotient is a non-separated algebraic space with a bijection to $\mathbb{A}^1 = \mathbb{A}^1/\pm$.

We note that the separated quotient $F_\rho^{\rm sep}$ is a stack $[\mathbb{D}_\rho \setminus \Delta_\rho :_W \Gamma_\rho]$ which can be locally constructed near $x \in \mathbb{D}_\rho \setminus \Delta_\rho$ by first taking a coarse quotient by the normal subgroup $\Gamma_\rho \cap (\mathbb{Z}_2 \times W_x) \trianglelefteq \mathrm{Stab}_x(\Gamma_\rho)$ and then taking the stack quotient by $\mathrm{Stab}_x(\Gamma_\rho)/\Gamma_\rho \cap (\mathbb{Z}_2 \times W_x)$. See [AE21, Rem. 2.36].

Proposition 2.12. *Suppose $\sigma \in \mathrm{Aut}(X)$ fixes a curve R of genus at least 2, i.e. the assumption $(\exists g \geq 2)$ holds. Then $\mathrm{Aut}(X, \sigma)$ is finite.*

Proof. Let $h \in \mathrm{Aut}(X, \sigma)$ be an automorphism of X satisfying $h \circ \sigma = \sigma \circ h$. Then h permutes the fixed components of σ. Since there is at most one component R of genus $g \geq 2$, we conclude $h(R) = R$. Hence $h \in \mathrm{Aut}(X, \mathcal{O}(R))$, a finite group. $\square$

Note that generic stabilizer K_ρ from Theorem 2.9 is never the trivial group, as $\rho \in K_\rho$ is a nontrivial element. As this is the automorphism group of a generic element $(X,\sigma) \in F_\rho$, if $(\exists g \geq 2)$ holds then K_ρ is finite by Proposition 2.12.

Example 2.13. Consider the double cover $\pi\colon X \to \mathbb{P}^2$ branched over a smooth sextic B. There is a non-symplectic involution σ switching the two sheets of X, acting on $H^2(X,\mathbb{Z})$ by fixing $h = c_1(\pi^*\mathcal{O}(1))$ and negating $h^\perp$. Choosing a model ρ for the action of σ^* on cohomology, we have that $S_\rho = \langle 2\rangle$ and $T_\rho = \langle -2\rangle \oplus H^{\oplus 2} \oplus E_8^{\oplus 2}$ are the $(+1)$- and (-1)-eigenspaces, respectively.

The divisor $\Delta_\rho/\Gamma_\rho \subset \mathbb{D}_\rho/\Gamma_\rho = F_2$ has two irreducible components corresponding to Γ_ρ-orbits of roots $\delta \in (T_\rho)_{-2}$. Such an orbit is uniquely determined by the divisibility (1 or 2) of $\delta \in T_\rho^*$. The case where the divisibility is 2 corresponds to when B acquires a node. Then there is an involution σ on the minimal resolution of the double cover $X \to \overline{X} \to \mathbb{P}^2$, but $\sigma^*(\delta) = \delta$, $\sigma^*(h) = h$ and the $(+1,-1)$-eigenspaces of σ^* have dimensions $(2,20)$. Thus, no ρ-marking can be extended over a family $\mathcal{X} \to C$ with central fiber X and general fiber as above.

When the divisibility of δ is 1, $\mathbb{P}^2$ degenerates to $\mathbb{F}_4^0 = \mathbb{P}(1,1,4)$ and the minimal resolution of the double cover $X \to \overline{X} \to \mathbb{F}_4^0$ is an elliptic K3 surface with σ the elliptic involution. Again the eigenspaces have dimension profile $(2,20)$ and so (X,σ) is not ρ-markable for the ρ as above.

3. Stable pair compactifications

3A. **Complete moduli of stable slc pairs.** We refer the reader to [ABE20, Sec. 2B] and [AE21, Sec. 7D] for a detailed discussion of stable K3 surface pairs and their compactified moduli. Briefly:

Definition 3.1. In our context, a stable slc surface pair is a pair $(S,\epsilon D)$, where

(1) S is a connected, reduced, projective Gorenstein surface S with $\omega_S \simeq \mathcal{O}_S$ which has semi log canonical singularities.
(2) D is an effective ample Cartier divisor on S that does not contain any log canonical centers of S.

Then for sufficiently small rational number $\epsilon > 0$ the pair $(S,\epsilon D)$ is stable, meaning:

(1) it has semi log canonical singularities, and
(2) the $\mathbb{Q}$-Cartier divisor $K_S + \epsilon D$ is ample.

"Sufficiently small" works in families: for a fixed D^2 there exists ϵ_0 so that if a pair $(S,\epsilon D)$ is stable in the above definition for some ϵ then it is stable for any $0 < \epsilon \leq \epsilon_0$.

The main application to K3 surfaces is an observation that for any K3 surface $\overline{X}$ with ADE singularities and an effective ample divisor $\overline{R}$, the pair $(\overline{X},\epsilon\overline{R})$ is stable. Indeed, $\omega_{\overline{X}} \simeq \mathcal{O}_{\overline{X}}$, the surface $\overline{X}$ has canonical singularities—which is much better than semi log canonical—and there are no log centers.

As usual, let F_{2d} denote the moduli space of polarized K3 surfaces $(\overline{X},\overline{L})$ with ADE singularities and ample primitive line bundle $\overline{L}$ of degree $\overline{L}^2 = 2d$, and $P_{2d,m} \to F_{2d}$ denote the moduli space of pairs $(\overline{X},\epsilon\overline{R})$ with an effective divisor $\overline{R} \in |m\overline{L}|$. Then the main result for K3 surfaces is the following:

Theorem 3.2. (1) *For the stable pairs as above there exists an algebraic Deligne-Mumford moduli stack $\mathcal{M}^{\mathrm{slc}}$, with a coarse moduli space M^{slc}.*

(2) *The closure* $\overline{P}^{\mathrm{slc}}_{2d,m}$ *of* $P_{2d,m}$ *in* M^{slc} *is projective and provides a compactification of* $P_{2d,m}$ *to a moduli space of stable slc pairs.*

To apply this result to a compactification of F^{sep}_ρ one needs to choose, in a canonical manner, a big and nef divisor on the generic $(X,\sigma)\in F_\rho$.

Definition 3.3. A *canonical choice of polarizing divisor* is an algebraically varying big and nef divisor R defined over a Zariski dense subset $U\subset F_\rho$ of the moduli space of ρ-markable K3 surfaces.

3B. **Stable pair compactification of F^{sep}_ρ.** We apply Theorem 3.2 to construct a stable pair compactification in the present context as follows.

Suppose that for each surface $(X,\sigma)\in F_\rho$ assumption $(\exists g\ge 2)$ holds, i.e. the fixed locus $\mathrm{Fix}(\sigma)$ contains a component C_1 of genus $g\ge 2$, as well as possibly several smooth rational curves C_i and some isolated points. In fact, it suffices that a single $(X,\sigma)\in F_\rho$ satisfies assumption $(\exists g\ge 2)$ because the genus of C_1 is constant in a family of smooth K3 surfaces with non-symplectic automorphism. So $R=C_1$ gives a canonical choice of polarizing divisor for all of $U=F_\rho$.

Let $\pi\colon X\to\overline{X}$ be the contraction to an ADE K3 surface such that the divisor $\overline{R}:=\pi(C_1)$ is ample; it has degree $\overline{R}^2=2g(C_1)-2>0$. It provides us with an ample divisor on $\overline{X}$. If $\mathcal{O}(\overline{R})=\overline{L}^m$ for a primitive $\overline{L}$ then the pair $(\overline{X},\mathcal{O}(\overline{R}))$ is a point of $F_{2d,m}$ and the pair $(\overline{X},\epsilon\overline{R})$ is a point of $P_{2d,m}$.

Definition 3.4. We define the map $\psi\colon F_\rho\to P_{2d,m}$ as follows. Pointwise, it sends (X,σ) to $(\overline{X},\epsilon\overline{R})$. In every flat family $f\colon\mathcal{X}\to S$ of K3 surfaces with automorphism, the sheaf $\mathcal{O}_{\mathcal{X}}(\mathcal{R})$ is relatively big and nef. Since $R^i\mathcal{L}^d=0$ for $i>0$, $d>0$, it gives a contraction to a flat family $\overline{f}\colon(\overline{\mathcal{X}},\overline{\mathcal{R}})\to S$. This induces the map on moduli.

Lemma 3.5. *The map* $\psi\colon F_\rho\to P_{2d,m}$ *defined above induces an injective map* $F^{\mathrm{sep}}_\rho\to\mathrm{im}(\psi)$.

Proof. The map ψ factors through the separated quotient of F_ρ because $P_{2d,m}$ is separated. Now suppose there is an isomorphism of pairs $\overline{f}\colon(\overline{X}_1,\overline{R}_1)\to(\overline{X}_2,\overline{R}_2)$ inducing an isomorphism of the minimal resolutions $f\colon(X_1,R_1)\to(X_2,R_2)$. Consider the morphism $\varphi=\sigma_1^{-1}f^{-1}\sigma_2 f$. Then φ is a *symplectic* automorphism of X_1 fixing the curve R_1 pointwise. Since φ preserves $\mathcal{O}_{X_1}(R_1)$, it has finite order. By [Nik79a] the fixed set of a finite order symplectic K3 automorphism is finite. Thus, $\varphi=\mathrm{id}$ and f preserves the group action. So, (X,σ) is uniquely determined by $(\overline{X},\overline{R})$. □

Remark 3.6. F^{sep}_ρ itself has a moduli interpretation: It is the moduli space F^{ade}_ρ of *ADE* K3 surfaces $(\overline{X},\overline{\sigma})$ with automorphism, for which $\mathrm{Fix}(\overline{\sigma})$ is ample, and for which the minimal resolution $(X,\sigma)\to(\overline{X},\overline{\sigma})$ is ρ-markable.

Definition 3.7. Let $Z=\mathrm{im}(\psi)$ and let $\overline{Z}$ be its closure in $\overline{P}^{\mathrm{slc}}_{2d,m}$, with reduced scheme structure. The stable pair compactification

$$F^{\mathrm{sep}}_\rho=F^{\mathrm{ade}}_\rho\hookrightarrow\overline{F}^{\mathrm{slc}}_\rho$$

is defined as the normalization of $\overline{Z}$.

In particular, $\overline{F}^{\mathrm{slc}}_\rho$ is normal by definition. Points correspond to the pairs $(\overline{X},\epsilon\overline{R})$, possibly degenerate, with some finite data.

3C. **Kulikov degenerations of K3 surfaces.** A basic tool in the study of degenerations of K3 surfaces is Kulikov models. We use them in the argument below, so we briefly recall the definition.

Let $(C, 0)$ denote the germ of a smooth curve at a point $0 \in C$ and let $C^* = C \setminus 0$. Let $X^* \to C^*$ be a family of algebraic K3 surfaces.

Definition 3.8. A *Kulikov model* $X \to (C, 0)$ is an extension of $X^* \to C^*$ for which X is a smooth algebraic space, $K_X \sim_C 0$, and X_0 has reduced normal crossings. We say the X is *Type I, II, or III*, respectively, depending on whether X_0 is smooth, has double curves but no triple points, or has triple points, respectively. We call the central fiber X_0 of such a family a *Kulikov surface*.

A key result on the degenerations of K3 surfaces is the theorem of Kulikov [Kul77] and Persson-Pinkham [PP81]:

Theorem 3.9. *Let $Y^* \to C^*$ be a family of algebraic K3 surfaces. Then there is a finite base change $(C', 0) \to (C, 0)$ and a sequence of birational modifications of the pull back $Y' \dashrightarrow X$ such that X has smooth total space, $K_X \sim_{C'} 0$, and X_0 has reduced normal crossings.*

We recall some fundamental results about Kulikov models. The primary reference is [FS86]. Let $T : H^2(X_t, \mathbb{Z}) \to H^2(X_t, \mathbb{Z})$ denote the Picard-Lefschetz transformation associated to an oriented simple loop in C^* enclosing 0. Since X_0 is reduced normal crossings, T is unipotent. Let

$$N := \log T = (T - I) - \tfrac{1}{2}(T - I)^2 + \cdots$$

be the logarithm of the monodromy.

Theorem 3.10. [FS86][Fri84] *Let $X \to (C, 0)$ be a Kulikov model. We have that*

if X is Type I, then $N = 0$,
if X is Type II, then $N^2 = 0$ but $N \neq 0$,
if X is Type III, then $N^3 = 0$ but $N^2 \neq 0$.

The logarithm of monodromy is integral, and of the form $Nx = (x \cdot \lambda)\delta - (x \cdot \delta)\lambda$ for $\delta \in H^2(X_t, \mathbb{Z})$ a primitive isotropic vector, and $\lambda \in \delta^\perp/\delta$ satisfying

$$\lambda^2 = \#\{\textit{triple points of } X_0\}.$$

When $\lambda^2 = 0$, its imprimitivity is the number of double curves of X_0.

Thus, the Types I, II, III of Kulikov model are distinguished by the behavior of the *monodromy invariant* λ: either $\lambda = 0$, $\lambda^2 = 0$ but $\lambda \neq 0$, or $\lambda^2 \neq 0$ respectively.

Definition 3.11. Let $J \subset H^2(X_t, \mathbb{Z})$ denote the primitive isotropic lattice $\mathbb{Z}\delta$ in Type III or the saturation of $\mathbb{Z}\delta \oplus \mathbb{Z}\lambda$ in Type II.

3D. **Baily-Borel compactification.** Let N be a lattice of signature $(2, \ell)$, together with an isometry $\rho \in O(N)$ of finite order n, such that all eigenvalues of ρ on $N_{\mathbb{C}}$ are primitive nth roots of unity, and $N_{\mathbb{C}}^{\zeta_n}$ contains a vector x of positive Hermitian norm $x \cdot \bar{x}$. This is the situation which arises for a non-symplectic automorphism of an algebraic K3 surface, with $N = T_\rho$. Then we have a Type IV domain

$$\mathbb{D}_N = \mathbb{P}\{x \in N_{\mathbb{C}} \mid x \cdot x = 0,\ x \cdot \bar{x} > 0\}$$

For $n = 2$ one has $\mathbb{D}_\rho = \mathbb{D}_N$. For $n > 2$ one has a Type I subdomain of $\mathbb{D}_N$

$$\mathbb{D}_\rho = \mathbb{P}\{x \in N_{\mathbb{C}}^{\zeta_n} \mid x \cdot \bar{x} > 0\}$$

$\mathbb{D}_\rho$ admits the action of the arithmetic group $\widetilde{\Gamma}_\rho := \{\gamma \in O(N) \mid \gamma\circ\rho = \rho\circ\gamma\}$. Fix a finite index subgroup $\Gamma \subset \widetilde{\Gamma}_\rho$.

Recall that $\mathbb{D}_N$ and $\mathbb{D}_\rho$ embed into their compact duals $\mathbb{D}_N^c$, $\mathbb{D}_\rho^c$, which are defined by dropping the condition that $x\cdot\bar{x} > 0$. Define $\overline{\mathbb{D}}_N \subset \mathbb{D}_N^c$, $\overline{\mathbb{D}}_\rho \subset \mathbb{D}_\rho^c$ as their topological closures. One has a well known description of the rational boundary components of $\mathbb{D}_N$, see e.g. see [Loo03b].

Definition 3.12. A *rational boundary component* of $\mathbb{D}_N$ is an analytic subset $B_J \subset \overline{\mathbb{D}}_N$ of the form:

(1) $(\mathbb{P}J_{\mathbb{C}} \setminus \mathbb{P}J_{\mathbb{R}}) \cap \overline{\mathbb{D}}_N$ for $\operatorname{rk} J = 2$ a primitive isotropic sublattice of N,
(2) $\mathbb{P}J_{\mathbb{C}} \cap \overline{\mathbb{D}}_N$ for $\operatorname{rk} J = 1$ a primitive isotropic sublattice of N.

The rational boundary components of $\mathbb{D}_\rho$ are intersections of $B'_J = B_J \cap \overline{\mathbb{D}}_\rho$.

One defines the *rational closure* of $\mathbb{D}_\rho$ to be $\overline{\mathbb{D}}_\rho^{\,\mathrm{bb}} := \mathbb{D}_\rho \cup_J B'_J,$, topologized via a horoball topology at the boundary. Then the *Baily-Borel compactification* of $\mathbb{D}_\rho/\Gamma$ is (at least topologically) $\overline{\mathbb{D}_\rho/\Gamma}^{\mathrm{bb}} := \overline{\mathbb{D}}_\rho^{\,\mathrm{bb}}/\Gamma$.

The space $\overline{\mathbb{D}_\rho/\Gamma}^{\mathrm{bb}}$ was shown to have the structure of a projective variety by Baily-Borel [BB66]. For Type IV domains $\mathbb{D}_N$ and $\mathbb{D}_\rho$ if $n = 2$, the boundary components (1) are isomorphic to $\mathbb{H} \sqcup (-\mathbb{H})$ and the boundary components (2) are points. For $n > 2$, the boundary components of the Type I domain $\mathbb{D}_\rho$ are points. If $\operatorname{rk} J = 2$ then a point $[x] \in B_J$ corresponds to the elliptic curve $E_x = J_{\mathbb{C}}/(J + \mathbb{C}x)$.

Lemma 3.13. *In the case $n > 2$, for each boundary component B'_J we necessarily have $\operatorname{rk} J = 2$ and $n \in \{3,4,6\}$, and $x \in B'_J$ corresponds to the elliptic curve with $j(E_x) = 0$ if $n = 3$ or 6, and with $j(E_x) = 1728$ if $n = 4$.*

Proof. If B'_J is boundary component of $\mathbb{D}_\rho$ then $N_{\mathbb{C}}^{\zeta_n} \cap J_{\mathbb{C}} \neq 0$. Since J is defined over $\mathbb{Z}$ and $\zeta_n \notin \mathbb{R}$, then $N_{\mathbb{C}}^{\bar{\zeta}_n} \cap J_{\mathbb{C}} \neq 0$ as well. This implies that $\operatorname{rk} J = 2$ and

$$J_{\mathbb{C}} = J_{\mathbb{C}}^{\zeta_n} \oplus J_{\mathbb{C}}^{\bar{\zeta}_n}.$$

Thus, $\rho(J_{\mathbb{C}}) = J_{\mathbb{C}}$, implying that $\rho(J) = J$. Therefore $\rho\big|_J \in \mathrm{GL}(J) \cong \mathrm{GL}_2(\mathbb{Z})$ necessarily has order n. Thus, $n \in \{3,4,6\}$. For a point $[x] \in B'_J$ one has $x \in N_{\mathbb{C}}^{\zeta_n}$ and $\mu_n \subset \operatorname{Aut}(E_x)$. This determines E_x. $\square$

Corollary 3.14. *If $n \neq 2,3,4,6$ then the rational closure of $\mathbb{D}_\rho$ is simply $\mathbb{D}_\rho$ itself. So $\mathbb{D}_\rho/\Gamma$ is already compact.*

The following is a well-known consequence of Schmid's nilpotent orbit theorem:

Proposition 3.15. *Let $X^* \to C^*$ be a degeneration of a ρ-markable K3 surfaces over a punctured analytic disk C^*. A lift of the period mapping $\widetilde{C^*} \cong \mathbb{H} \to \mathbb{D}_\rho$ approaches the Baily-Borel cusp B_J as $\operatorname{Im}(\tau) \to \infty$, where J is the monodromy lattice in $H^2(X_t, \mathbb{Z})$, cf. Definition 3.11. When $\operatorname{rk}(J) = 2$, the limiting point $x \in B_J$ corresponds to an elliptic curve E_x isomorphic to any double curve of the central fiber X_0 of a Kulikov model $X \to C$.*

Corollary 3.16. *If $n \neq 2,3,4,6$, any degeneration of $(X,\sigma) \in F_\rho$ has Type I. If $n \in \{3,4,6\}$, any degeneration of $(X,\sigma) \in F_\rho$ has Type I or II.*

The last statement was also proved by Matsumoto [Mat16] using different techniques. His proof also holds in some prime characteristics.

3E. **Semitoroidal compactifications.** Semitoroidal compactifications of arithmetic quotients $\mathbb{D}/\Gamma$ for type IV Hermitian symmetric domains $\mathbb{D}$ were defined by Looijenga [Loo03b] (where they were called "semitoric"). They simultaneously generalize toroidal and Baily-Borel compactifications of $\mathbb{D}/\Gamma$. The case of the complex ball $\mathbb{D}$ (a type I symmetric Hermitian domain) is comparatively trivial. The semitoroidal compactifications in this case are implicit in [Loo03a, Loo03b]. We quickly overview the construction in both cases now.

Definition 3.17. A Γ-*admissible semifan* $\mathfrak{F}$ consists of the following data:

When $n = 2$, it is a convex, rational, locally polyhedral decomposition $\mathfrak{F}_J$ of the rational closure $\mathcal{C}^+(J^\perp/J)$ of the positive norm vectors, for all rank 1 primitive isotropic sublattices $J \subset N$, such that:

(1) $\{\mathfrak{F}_J\}_{J\subset N}$ is Γ-invariant. In particular, a fixed $\mathfrak{F}_J$ is invariant under the natural action of $\mathrm{Stab}_J(\Gamma)$ on $\mathcal{C}^+(J^\perp/J)$.
(2) A compatibility condition of the $\{\mathfrak{F}_J\}_{J\subset N}$ along any primitive isotropic lattice $J' \subset N$ of rank 2 holds, see Definition 3.18.

When $n > 2$, the data is much simpler: It consists, for each primitive isotropic sublattice $J \subset N$ satisfying $J_{\mathbb{C}} \cap N_{\mathbb{C}}^{\zeta_n} \neq \emptyset$, of a primitive sublattice $\mathfrak{F}_J \subset J^\perp/J$ such that the collection $\{\mathfrak{F}_J\}$ is Γ-invariant.

Definition 3.18. Let $J' \subset N$ be primitive isotropic of rank 2. We say that the collection $\{\mathfrak{F}_J\}_{J\subset N}$ is *compatible along* J' if, given any primitive sublattice $J \subset J'$ of rank 1, the kernel of the hyperplanes of $\mathfrak{F}_J$ containing J'/J, when intersected with $(J')^\perp/J \subset J^\perp/J$ and then descended to $(J')^\perp/J'$, cut out a fixed sublattice $\mathfrak{F}_{J'} \subset (J')^\perp/J'$ which is independent of J.

In both the $n = 2$ and $n > 2$ cases, we use the same notation $\mathfrak{F} := \{\mathfrak{F}_J\}_{J\subset N}$ even though J ranges over rank 1 isotropic sublattices when $n = 2$ and ranges over rank 2 isotropic sublattices when $n > 2$.

In the Type IV case, Looijenga constructs a compactification $\mathbb{D}/\Gamma \hookrightarrow \overline{\mathbb{D}/\Gamma}^{\mathfrak{F}}$ for any Γ-admissible semifan $\mathfrak{F}$, so consider the Type I case. By Lemma 3.13 we may restrict to $n \in \{3, 4, 6\}$. There is a $\mathbb{Z}[\zeta_n]$-lattice

$$Q := (N \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_n])^{\zeta_n} \subset N_{\mathbb{C}}^{\zeta_n} = Q_{\mathbb{C}}$$

on which Hermitian form $x \cdot \overline{y}$ defines a $\mathbb{Z}[\zeta_n]$-valued Hermitian pairing of signature $(1, \ell)$ for some ℓ. Any element of $\widetilde{\Gamma}_\rho$ (in particular, any element of Γ) preserves Q and the Hermitian form on it. The converse also holds. Thus $\Gamma \subset U(Q)$ is a finite index subgroup of the group of unitary isometries of Q and $\Gamma_{\mathbb{R}} = U(Q_{\mathbb{C}}) = U(1, \ell)$. The boundary components $B_J = \mathbb{P}(J_{\mathbb{C}}^{\zeta_n})$ are then projectivizations of the isotropic $\mathbb{Z}[\zeta_n]$-lines $K \subset Q$. Here $K_{\mathbb{C}} = J_{\mathbb{C}}^{\zeta_n}$.

Choose a generator $k \in K$. Then any $[x] \in \mathbb{D}_\rho \subset \mathbb{P}Q_{\mathbb{C}}$ has a unique representative $x \in Q_{\mathbb{C}}$ for which $k \cdot x = 1$. This realizes $\mathbb{D}_\rho$ as a generalized tube domain in the affine hyperplane $V_k := \{k \cdot x = 1\} \subset Q_{\mathbb{C}}$.

Let $U_K \subset \mathrm{Stab}_K(\Gamma)$ be the unipotent subgroup (i.e. U_K acts on K, $K^\perp/K$, and $Q/K^\perp$ by the identity). Then U_K acts on V_k by translations. Choosing some isotropic $k' \in Q_{\mathbb{C}}$ for which $k' \cdot k = 1$, any element $x \in V_k$ can be written uniquely as $x = k' + x_0 + ck$ for some $x_0 \in \{k, k'\}^\perp$ and $c \in \mathbb{C}$. The image of $\mathbb{D}_\rho$ is exactly those x satisfying $2\mathrm{Re}(c) > -x_0 \cdot \bar{x}_0$.

The fibration $\mathbb{D}_\rho \to K_{\mathbb{C}}^\perp/K_{\mathbb{C}}$ sending $x \mapsto x_0 \bmod K_{\mathbb{C}}$ is a fibration of right half-planes. The action of U_K fibers over the action of a translation subgroup $\overline{U}_K \subset K^\perp/K$ on $K_{\mathbb{C}}^\perp/K_{\mathbb{C}}$ and thus, there is a fibration

$$\mathbb{D}_\rho/U_K \to (K_{\mathbb{C}}^\perp/K_{\mathbb{C}})/\overline{U}_K =: A_K$$

over an abelian variety. The fibers are quotients of the right half-planes with coordinate c by a discrete, purely imaginary, translation group isomorphic to $\mathbb{Z}$. This realizes $\mathbb{D}_\rho/U_K$ is a punctured holomorphic disc bundle over A_K.

Definition 3.19. $\mathbb{D}_\rho/U_K$ is the *first partial quotient* associated to the Baily-Borel cusp K. The extension of this punctured disc bundle to a disc bundle $\overline{\mathbb{D}_\rho/U_K}^{\mathrm{can}} \to A_K$ for a given K is called the *toroidal extension at the cusp* K.

We will identify the divisor at infinity, i.e. the zero section of the disc bundle, with A_K itself.

Construction 3.20. The *toroidal compactification* of $\mathbb{D}_\rho/\Gamma$ is constructed as follows: Let Γ_K be the finite group defined by the exact sequence

$$0 \to U_K \to \mathrm{Stab}_K(\Gamma) \to \Gamma_K \to 0.$$

For each cusp K, quotient the toroidal extension

$$V_K := \overline{\mathbb{D}_\rho/U_K}^{\mathrm{can}}/\Gamma_K \supset \mathbb{D}_\rho/\mathrm{Stab}_K(\Gamma).$$

A well-known theorem states that there exists a horoball neighborhood $\mathbb{P}K_{\mathbb{C}} \in N_K \subset \mathbb{D}_\rho^{\mathrm{bb}}$ such that $(N_K \setminus \mathbb{P}K_{\mathbb{C}})/\mathrm{Stab}_K(\Gamma) \hookrightarrow \mathbb{D}_\rho/\Gamma$ injects. Thus, we can glue a neighborhood of the boundary $A_K/\Gamma_K \subset V_K$ to $\mathbb{D}_\rho/\Gamma$, ranging over all Γ-orbits of cusps K. The result is the toroidal compactification $\overline{\mathbb{D}_\rho/\Gamma}^{\mathrm{tor}}$.

The boundary divisors of $\overline{\mathbb{D}_\rho/\Gamma}^{\mathrm{tor}}$ are in bijection with Γ-orbits of isotropic $\mathbb{Z}[\zeta_n]$-lines $K \subset Q$ and the boundary divisor is isomorphic to A_K/Γ_K, where Γ_K acts by a subgroup of the finite group $U(K^\perp/K)$. There is a morphism

$$\overline{\mathbb{D}_\rho/\Gamma}^{\mathrm{tor}} \to \overline{\mathbb{D}_\rho/\Gamma}^{\mathrm{bb}}$$

which contracts each boundary divisor to a point. As such, the normal bundle of the boundary divisor is anti-ample. Passing to a finite index subgroup $\Gamma_0 \subset \Gamma$, we can assume that Γ_K is trivial for all cusps K and the anti-ampleness still holds. This proves that the normal bundle to $A_K \subset \overline{\mathbb{D}_\rho/U_K}^{\mathrm{can}}$ in the first partial quotient is anti-ample.

Using [Gra62] one shows that a divisor in a smooth analytic space, isomorphic to an abelian variety and with anti-ample normal bundle, can be contracted along any abelian subvariety. In particular, for any sub-$\mathbb{Z}[\zeta_n]$-lattice $\mathfrak{F}_K \subset K^\perp/K$, there is a contraction

$$\overline{\mathbb{D}_\rho/U_K}^{\mathrm{can}} \to \overline{\mathbb{D}_\rho/U_K}^{\mathfrak{F}_K}$$

which is an isomorphism away from the boundary divisor and contracts exactly the translates of the abelian subvariety $\mathrm{im}(\mathfrak{F}_K)_{\mathbb{C}} \subset A_K$.

To construct the semitoroidal compactification $\overline{\mathbb{D}_\rho/\Gamma}^{\mathfrak{F}}$, we wish to glue, at each cusp K, a punctured analytic open neighborhood of the boundary of $\overline{\mathbb{D}_\rho/U_K}^{\mathfrak{F}_K}/\Gamma_K$ to $\mathbb{D}_\rho/\Gamma$. This is only possible if the action of Γ_K on $\overline{\mathbb{D}_\rho/U_K}^{\mathrm{can}}$ descends along the above contraction. The condition in Definition 3.17 ensures that the collection

$\mathfrak{F} = \{\mathfrak{F}_K\}$ is Γ-invariant. So an individual $\mathfrak{F}_K$ is Γ_K-invariant and the Γ_K action descends. Thus, we have constructed the semitoroidal compactification.

Remark 3.21. A feature of the construction is that one can pull back a semifan $\mathfrak{F}$ for a Type IV domain to any Type I subdomain, and there will be a morphism between the corresponding semitoric compactifications.

3F. **Recognizable divisors.** We recall the main new concept "recognizability" introduced in [AE21]. We slightly modify the definition as necessary for moduli spaces of K3 surfaces with ρ-markable automorphism:

Definition 3.22. A canonical choice of polarizing divisor R for $U \subset F_\rho$ is *recognizable* if for every Kulikov surface X_0 of Type I, II, or III which smooths to some ρ-markable K3 surface, there is a divisor $R_0 \subset X_0$ such that on *any* smoothing into ρ-markable K3 surfaces $X \to (C, 0)$ with $C^* \subset U$, the divisor R_0 is, up to the action of $\mathrm{Aut}^0(X_0)$, the flat limit of R_t for $t \neq 0 \in C^*$.

We use the term "smoothing" to mean specifically a Kulikov model $X \to (C, 0)$. Roughly, Definition 3.22 amounts to saying that the canonical choice R can also be made on any Kulikov surface, including smooth K3s.

Theorem 3.23. *If R is recognizable, then $\overline{F}_\rho^{\mathrm{slc}}$ is semitoroidal compactification of F_ρ for a unique semifan $\mathfrak{F}_R$.*

Proof. The proof when $n = 2$ is essentially the same as [AE21, Thm. 1.2]. So we restrict our attention to the Type I case $n > 2$, which is ultimately much simpler anyways. First, we show that $\overline{F}_\rho^{\mathrm{slc}}$ contains $\mathbb{D}_\rho/\Gamma_\rho$.

Let $\mathcal{M}_\rho^*$ be the closure of the moduli space of ρ-marked K3 surfaces $\mathcal{M}_\rho$ in the space of all marked K3 surfaces $\mathcal{M}$ and let $F_\rho^* = \mathcal{M}_\rho^*/\Gamma_\rho$ be the quotient. Given any smooth K3 surface $X_0 \in F_\rho^* \setminus U$, the recognizability implies that the universal family $(\mathcal{X}^*, \mathcal{R}^*) \to U$ extends over F_ρ^* by the same argument as [AE21, Prop. 6.3]. Thus, the argument of Lemma 3.5 shows that there is a morphism $(F_\rho^*)^{\mathrm{sep}} = \mathbb{D}_\rho/\Gamma_\rho \to P_{2d,m}$ and so we may as well have constructed $\overline{F}_\rho^{\mathrm{slc}}$ by taking the normalization of the closure of the image of $\mathbb{D}_\rho/\Gamma_\rho$, which is notably already normal. This completes the proof when $n \neq 3, 4, 6$.

So let $\mathbb{P}K_\mathbb{C}$ be a Baily-Borel cusp of $\mathbb{D}_\rho$ when $n \in \{3, 4, 6\}$. We observe that the closure of $\mathbb{D}_\rho/U_K$ in the toroidal extension $\mathbb{D}(J) \subset \mathbb{D}(J)^\lambda$ of the "universal" first partial quotient for unpolarized K3 surfaces, cf. [AE21, Def. 4.18], is simply the first partial quotient $\overline{\mathbb{D}_\rho/U_K}^{\mathrm{can}}$. [AE21, Prop. 4.16] shows that $\mathbb{D}(J)$ embeds into a family of affine lines over $J^\perp/J \otimes_\mathbb{Z} \widetilde{\mathcal{E}}$ where $\widetilde{\mathcal{E}}$ is the universal elliptic curve over $\mathbb{H} \sqcup (-\mathbb{H})$ and $\mathbb{D}(J)^\lambda$ is its closure in a projective line bundle. The space $\mathbb{D}_\rho/U_K$ sits inside this affine line bundle as the inverse image of

$$K^{\perp \text{ in } Q}/K \otimes_{\mathbb{Z}[\zeta_n]} E \subset J^\perp/J \otimes_\mathbb{Z} \widetilde{\mathcal{E}}$$

where E is the elliptic curve admitting an action of ζ_n (note that $K = J$ but with the additional structure of a $\mathbb{Z}[\zeta_n]$-lattice).

Thus we may restrict a Type II λ-*family*, cf. [AE21, Def. 5.34], to a family

$$\mathcal{X} \to \overline{\mathbb{D}_\rho/U_K}^{\mathrm{can}}$$

of Kulikov surfaces of Types I + II. We call $\mathcal{X}$ a K-*family*. Note that any K-family admits a birational automorphism which is the action of the automorphism σ on the restriction of $\mathcal{X}$ to $(\mathbb{D}_\rho \setminus \Delta_\rho)/U_K$.

The arguments in [AE21, Secs. 6, 8], leading up to the proof of Theorem 1.2 of *loc. cit.* now all apply to K-families $\mathcal{X}$, showing that there is a sandwich of normal compactifications

$$\overline{\mathbb{D}_\rho/\Gamma_\rho}^{\rm tor} \to \overline{F}_\rho^{\rm slc} \to \overline{\mathbb{D}_\rho/\Gamma_\rho}^{\rm bb}.$$

Using that the normal image of an abelian variety is an abelian variety (a similar argument is used in [AE21, Thm. 7.18]), we conclude that there must exist a Γ_ρ-admissible semifan $\mathfrak{F}_R$ for which $\overline{F}_\rho^{\rm slc} = \overline{\mathbb{D}_\rho/\Gamma_\rho}^{\mathfrak{F}_{\rm R}}$. ☐

3G. **The main theorem.**

Theorem 3.24. *Under the assumption* ($\exists g \geq 2$)*,* $R = C_1$ *is recognizable for* F_ρ*. The stable pair compactification* $\overline{F}_\rho^{\rm slc}$ *is a semitoroidal compactification of* $\mathbb{D}_\rho/\Gamma_\rho$*.*

Proof. By Theorem 3.23, the second statement follows from the first. Let $(X, R) \to (C, 0)$ be a Kulikov model with a flat family of divisors $R \subset X$ for which

1. there is an automorphism σ on $X^* \to C^*$ making $(X_t, \sigma_t) \in F_\rho$ for $t \neq 0$,
2. $R_t \subset \mathrm{Fix}(\sigma_t)$ is the fixed component of genus at least 2 for $t \neq 0$, and
3. $R_0 = \lim_{t\to 0} R_t$.

By [AE21, Prop. 6.12], it suffices to show that if we make a one-parameter deformation the smoothing of X_0 into F_ρ that keeps X_0 constant, the limiting curve R_0 does not deform, up to $\mathrm{Aut}^0(X_0)$.

The automorphism σ on the generic fiber of any smoothing defines a birational automorphism of X. Any two Kulikov models are related by an automorphism followed by a sequence of Atiyah flops of types 0, I, II along curves in X_0 which are either (-2)-curves or (-1)-curves on component(s) of X_0. As such, there are only countably many curves in X_0 along which it is possible to make an Atiyah flop, and this continues to be the case after a flop is made. Thus, up to conjugation by $\mathrm{Aut}^0(X_0)$, there are only countably many possibilities for the birational automorphism $\sigma_0 := \sigma|_{X_0} \colon X_0 \dashrightarrow X_0$.

Hence if $X_0 \hookrightarrow X$ and $X_0 \hookrightarrow \widetilde{X}$ are smoothings into F_ρ as above, we have $\widetilde{\sigma}_0 = \psi \circ \sigma_0 \circ \psi^{-1}$ for some $\psi \in \mathrm{Aut}^0(X_0)$.

Let $\{A_j\}$ be the countable set of curves in X_0 along which σ_0 can be indeterminate. Any such curve A_j is $\mathrm{Aut}^0(X_0)$-invariant. Let $A = \cup_j A_j$ be their union. Clearly, the limit divisor R_0 is contained in the union of $A \cup S$ where S is the closure of the fixed locus of σ_0 in its locus of determinacy. Similarly, $\widetilde{R}_0$ is contained in $A \cup \widetilde{S}$ and $\sigma_0(P) = P$ if and only if $\widetilde{\sigma}_0(\psi(P)) = \psi(P)$. Since the smoothing $\widetilde{X}$ is a deformation of the smoothing X and the limiting divisor of R varies continuously, we conclude that $\widetilde{R}_0 = \psi(R_0)$ and therefore R is recognizable. ☐

Proposition 3.25. *Any element* $(\overline{X}, \epsilon\overline{R}) \in \overline{F}_\rho^{\rm slc}$ *has an automorphism* $\overline{\sigma} \in \mathrm{Aut}(\overline{X})$*. Furthermore,* $\overline{R} = \mathrm{Fix}(\overline{\sigma})$ *and* $\overline{\sigma}^*$ *acts on* $H^0(\overline{X}, \omega_{\overline{X}}) \cong \mathbb{C}$ *by multiplication by* ζ_n*.*

Proof. As noted in Remark 3.6, any point in $F_\rho^{\rm sep} = (\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho$ corresponds to a pair $(\overline{X}, \overline{\sigma})$ of an ADE K3 surface with automorphism, for which $\overline{R} = \mathrm{Fix}(\overline{\sigma})$ is ample and the minimal resolution is ρ-markable. Then any boundary point $(\overline{X}_0, \epsilon\overline{R}_0) \in \overline{F}_\rho^{\rm slc}$ is a stable limit of such ADE K3 surface pairs $f\colon (\overline{X}, \epsilon\overline{R}) \to C$.

Since $\overline{R}_t$ is $\overline{\sigma}_t$-invariant and the canonical model is unique, $\overline{X}$ admits an automorphism $\overline{\sigma}$ whose fixed locus contains $\overline{R}_0$. In fact, $\mathrm{Fix}(\overline{\sigma}_0) = \overline{R}_0$: $\mathrm{Fix}(\overline{\sigma})$ is a

Cartier divisor, and thus forms a flat family of divisors containing $\overline{R}$. But $\mathrm{Fix}(\overline{\sigma}_0)$ already contains the flat limit $\overline{R}_0$. The statement about $\omega_{\overline{X}_0}$ follows from the fact that $f_*\omega_{\overline{X}/C}$ is invertible (by Base Change and Cohomology, since $R^1 f_*\omega_{\overline{X}/C} = 0$) and $\overline{\sigma}_t^*$ acts by ζ_n on the generic fiber of this line bundle. □

4. Moduli of quotient surfaces

We refer the reader to [Kol13] for the notions appearing in the following definitions. The pair (Y, Δ) is called demi-normal if X satisfies Serre's S_2 condition, has double normal crossing singularities in codimension 1, and $\Delta = \sum d_i D_i$ is an effective Weil $\mathbb{Q}$-divisor with $0 < d_i \le 1$ not containing any components of the double crossing locus of Y.

The following is [Kol13, Prop. 2.50(4)], using our adopted notations.

Proposition 4.1. *Étale locally, there is a one-to-one correspondence between*

(a) *Local demi-normal pairs* $(y \in Y, \frac{n-1}{n}B)$ *of index* n*, i.e. such that the divisor* $nK_Y + (n-1)B$ *is Cartier.*

(b) *Local demi-normal pairs* $(\widetilde{y} \in \widetilde{Y})$ *such that* $K_{\widetilde{Y}}$ *is Cartier, with a* μ_n*-action that is free on a dense open subset, and such that the induced action on* $\omega_{\widetilde{Y}} \otimes \mathbb{C}(\widetilde{y})$ *is faithful.*

Moreover, the pair $(Y, \frac{n-1}{n}B)$ *is slc iff so is* $\widetilde{Y}$.

The variety $\widetilde{Y}$ is called the local index-1 cover of the pair $(Y, \frac{n-1}{n}B)$. [Kol13, Sec. 2] also gives a global construction.

Theorem 4.2. *Let* $(\overline{X}, \epsilon\overline{R}) \in \overline{F}_\rho^{\mathrm{slc}}$ *and let* $\pi\colon \overline{X} \to Y = \overline{X}/\mu_n$ *be the quotient map with the branch divisor* $B = f(\overline{R})$*. Then*

(1) $nK_Y + (n-1)B \sim 0$,

(2) B *and* $-K_Y$ *are ample* $\mathbb{Q}$*-Cartier divisors,*

(3) *the pair* $(Y, \frac{n-1+\epsilon}{n}B)$ *is stable for any rational* $0 < \epsilon \ll 1$*, i.e. it has slc singularities and the* $\mathbb{Q}$*-divisor* $K_Y + \frac{n-1+\epsilon}{n}B$ *is ample.*

Vice versa, for a pair (Y, B) *satisfying the above conditions, its index-1 cover* $\overline{X}$ *with the ramification divisor* $\overline{R}$ *satisfies:*

(1) $K_{\overline{X}} \sim 0$ *and the* μ_n*-action on* $\overline{X}$ *is non-symplectic,*

(2) $\overline{R}$ *is* $\mathbb{Q}$*-Cartier,*

(3) *the pair* $(\overline{X}, \epsilon\overline{R})$ *is stable for any rational* $0 < \epsilon \ll 1$.

Proof. Follows from the above Proposition 4.1 and the formulas

$$\pi^*(B) = n\overline{R}, \qquad \pi^*\left(K_Y + \frac{n-1+\epsilon}{n}B\right) = K_{\overline{X}} + \epsilon\overline{R}.$$

□

Corollary 4.3. *The coarse moduli space* $\overline{F}_\rho^{\mathrm{slc}}$ *coincides with the normalization of the KSBA compactification of the irreducible component in the moduli space of the log canonical pairs* $(Y, \frac{n-1+\epsilon}{n}B)$ *of log del Pezzo surfaces* Y *with* $(n-1)B \in |-nK_Y|$ *in which a generic surface is a quotient of a K3 surface with a non-symplectic automorphism of type* ρ*. The stack for the former is a* μ_n*-gerbe over the stack for the latter.*

For the proof, we note that a small deformation of a K3 surface is a K3 surface.

Example 4.4. The KSBA compactification moduli of K3 surfaces of degree 2 for the ramification divisor R constructed in [AET19] is equivalent to the Hacking's compactification [Hac04] of the moduli space of pairs $(\mathbb{P}^2, \frac{1+\epsilon}{2}B_6)$ of plane sextic curves.

5. Extensions

The results of this paper are easily extended to the case of a nonsymplectic action by an arbitrary finite group G and to more general divisors defined by group actions. Most of the changes amount to introducing more cumbersome notations.

5A. **A general nonsymplectic group of automorphisms.**

Definition 5.1. Let X be a smooth K3 surface and $\sigma\colon G \subset \operatorname{Aut} X$ be a finite symmetry group. The action of G on $H^{2,0}(X) = \mathbb{C}\omega_X$ gives the exact sequence

$$0 \to G_0 \to G \xrightarrow{\alpha} \mu_n \to 1, \qquad \mu_n \subset \mathbb{C}^*.$$

One says that G is nonsymplectic (or not purely symplectic) if $G \neq G_0$, i.e. $\alpha \neq 1$.

We now extend the results of Section 2 directly to this more general setting.

Definition 5.2. Fix a finite subgroup $\rho\colon G \to O(L)$ and a nontrivial character $\chi\colon G \to \mathbb{C}^*$. Let $(X, \sigma\colon G \to \operatorname{Aut} X)$ be a K3 surface with a non-symplectic automorphism group.

A (ρ,χ)-*marking* of (X,σ) is an isometry $\phi : H^2(X,\mathbb{Z}) \to L$ such that for any $g \in G$ one has $\phi \circ \sigma(g)^* = \rho(g) \circ \phi$ and such that the character $\alpha\colon G \to \mathbb{C}^*$ induced by σ coincides with χ. We say that (X,σ) is ρ-*markable* if it admits a ρ-marking.

A family of (ρ,χ)-marked K3 surfaces is a smooth family $f\colon (\mathcal{X}, \sigma_B, \phi_B) \to B$ with a group of automorphisms $\sigma_B\colon G \to \operatorname{Aut}(\mathcal{X}/B)$ and with a marking $\phi_B\colon R^2 f_*\mathbb{Z} \to L \otimes \underline{\mathbb{Z}}_B$ such that every fiber is a (ρ,χ)-marked K3 surface.

A family of smooth ρ-markable K3 surfaces is a family $f\colon (\mathcal{X}, \sigma_B) \to B$ of K3 surfaces with a group of automorphisms over base B which admits a ρ-marking locally on B.

We define the moduli stacks $\mathcal{M}_{\rho,\chi}$ of (ρ,χ)-marked, resp. $F_{\rho,\chi}$ of (ρ,χ)-markable K3 by taking $\mathcal{M}_{\rho,\chi}(B)$, resp. $F_{\rho,\chi}(B)$ to be the groupoids of such families over B.

Definition 5.3. Define the vector space

$$L_{\mathbb{C}}^{\rho,\chi} = \{x \in L_{\mathbb{C}} \mid \rho(g)(x) = \chi(g)x\}$$

to be the intersection of the eigenspaces for the individual $g \in G$, and the period domain as

$$\mathbb{D}_{\rho,\chi} = \mathbb{P}\{x \in L_{\mathbb{C}}^{\rho,\chi} \mid x \cdot \bar{x} > 0\}$$

The second condition is redundant if there exists $g \in G$ with $\chi(g) > 2$. Thus, $\mathbb{D}_\rho$ is a type IV domain if $|\chi(G)| = 2$ and a complex ball, a type I domain if $|\chi(G)| > 2$.

The discriminant locus is $\Delta_\rho := \cup_\delta \delta^\perp \cap \Delta_\rho$ ranging over all roots δ in $(L^G)^\perp$, where $L^G = \{a \in L \mid \rho(g)(a) = a\}$ is the sublattice of L fixed by G.

Definition 5.4. The group of changes-of-marking is

$$\Gamma_\rho := \{\gamma \in O(L) \mid \gamma \circ \rho = \rho \circ \gamma\}.$$

Then the direct analogue of Lemma 2.6 and Theorem 2.9 is

Theorem 5.5. *For a fixed finite group $\rho\colon G \to O(L)$ with a nontrivial character $\chi\colon G \to \mathbb{C}^*$:*

(1) *There exists a fine moduli space* $\mathcal{M}_{\rho,\chi}$ *of* (ρ,χ)*-marked K3 surfaces* (X,σ,ϕ). *It admits an étale period map* $\pi_\rho\colon \mathcal{M}_{\rho,\chi} \to \mathbb{D}_{\rho,\chi} \setminus \Delta_\rho$. *The fiber* $\pi_\rho^{-1}(x)$ *over a point* $x \in \mathbb{D}_{\rho,\chi} \setminus \Delta_\rho$ *is a torsor over* $\Gamma_\rho \cap (\mathbb{Z}_2 \cap W_x)$.

(2) *The moduli stack of* ρ*-markable K3 surfaces* (X,σ) *is obtained as a quotient of* $F_{\rho,\chi}$ *by* Γ_ρ. *On the level of coarse moduli spaces it admits a bijective map to* $(\mathbb{D}_{\rho,\chi} \setminus \Delta_\rho)/\Gamma_\rho$.

Proof. If the group G does not act purely symplectically, i.e. there exists $g \in G$ with $\rho(g)(x) \neq x$ then $L^G \perp x$ and $S_X^G \simeq L^G$. The rest of the proof of Lemma 2.6 works the same for any finite group. And the proof of Theorem 2.9 goes through verbatim. □

5B. **More general polarizing divisors.** With a more general group action, there are more choices for the polarizing divisors. For a generic K3 surface X with a period $x \in \mathbb{D}_{\rho,\chi} \setminus \Delta_\rho$ we can consider any combination $\sum b_i B_i$ of curves B_i which are either fixed by some element $g \in G$ or are some of the (-2)-curves corresponding to the roots in the generic Picard lattice $(L_{\mathbb{C}}^{\rho,\chi})^\perp \cap L$ that generically gives a big and nef divisor on X. Theorem 3.24 extends immediately to this situation with the same proof.

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