COMPACT MODULI OF K3 SURFACES WITH A NONSYMPLECTIC AUTOMORPHISM

VALERY ALEXEEV, PHILIP ENGEL, AND CHANGHO HAN

Abstract. We construct a modular compactification via stable slc pairs for the moduli spaces of K3 surfaces with a nonsymplectic automorphism under the assumption that the fixed locus of the automorphism contains a component of genus $g \geq 2$, and prove that it is semitoroidal.

Contents

1. Introduction

Let X be a smooth K3 surface over the complex numbers. An automorphism σ of X is called *non-symplectic* if it has finite order $n > 1$ and $\sigma^*(\omega_X) = \zeta_n \omega_X$, where $\omega_X \in H^{2,0}(X)$ is a nonzero 2-form and ζ_n is a primitive nth root of identity. By changing the generator of the cyclic group μ_n we can and will assume that $\zeta_n = \exp(2\pi i/n)$. It is well known that a K3 surface admitting such an automorphism is projective. The possibilities for the order are the numbers n whose Euler function satisfies $\varphi(n) \leq 20$ with the single exception $n \neq 60$, see [MO98, Thm. 3].

In this paper we study compactification of moduli spaces of pairs (X, σ). But to begin with, the automorphism group $\mathrm{Aut}(X, \sigma)$, i.e. those automorphisms of X commuting with σ, may be infinite. To fix this, we will usually additionally assume:

$$(\exists g \geq 2) \qquad \text{The fixed locus } \mathrm{Fix}(\sigma) \text{ contains a curve } C_1 \text{ of genus } g \geq 2.$$

By looking at the μ_n-action on the tangent space of any fixed point, it is easy to see that $\mathrm{Fix}(\sigma)$ is a disjoint union of several smooth curves and points. The Hodge index theorem implies at most one of the fixed curves has genus $g \geq 2$. One could instead have one or two fixed curves of genus $g = 1$. All other fixed curves are isomorphic to $\mathbb{P}^1$.

Under the $(\exists g \geq 2)$ assumption, the group $\mathrm{Aut}(X, \sigma)$ is finite. The opposite is almost true. For example let $n = 2$, i.e. σ is an involution. Then σ^* fixes the Neron-Severi lattice $S_X \subset H^2(X, \mathbb{Z})$ and acts as multiplication by (-1) on the lattice $T_X = S_X^{\perp}$ of transcendental cycles. In this case $\mathrm{Aut}(X, \sigma) = \mathrm{Aut}(X)$.

Date: October 26, 2021.

Deformation classes of such K3 surfaces (X, σ) are classified by the primitive 2-elementary hyperbolic sublattices $S \subset L_{K3}$. By Nikulin [Nik79b] there are 75 cases, uniquely determined by certain invariants (g, k, δ). Among them 51 satisfy $(\exists g \geq 2)$. The only case when $|\operatorname{Aut}(X)| < \infty$ but $(\exists g \geq 2)$ is not satisfied is $(g, k, \delta) = (1, 9, 1)$ which is the one-dimensional mirror family to K3 surfaces of degree 2. In the case $(g, k, \delta) = (2, 1, 0)$ one has $|\operatorname{Aut}(X)| = \infty$ but the set $\operatorname{Fix}(\sigma)$ consists of two elliptic curves, so $(\exists g \geq 2)$ does not hold.

Since the moduli stack of smooth quasipolarized K3 surfaces is notoriously non-separated, so is usually the moduli stack of smooth K3s with a nonsymplectic automorphism. For a fixed isometry $\rho \in O(L_{K3})$ of order n, there exists the moduli stack and moduli space of smooth K3 surfaces "of type ρ": those pairs (X, σ) where the action of σ^* on $H^2(X, \mathbb{Z})$ can be modeled by ρ. We construct them in Section 2. The maximal separated quotient of F_ρ is $(\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho$, where $\mathbb{D}_\rho$ is a symmetric Hermitian domain of type IV if $n = 2$ or a complex ball if $n > 2$, Γ_ρ is an arithmetic group, and $\Delta_\rho \subset \mathbb{D}_\rho$ is the discriminant locus.

Under the assumption $(\exists g \geq 2)$, the space $F_\rho^{\mathrm{ade}} := (\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho$ is the coarse moduli space for the K3 surfaces $\overline{X}$ with ADE singularities, obtained from the smooth K3 surfaces X by contracting the (-2)-curves perpendicular to the component C_1 with $g \geq 2$ in $\operatorname{Fix}(\sigma)$. The stack of such ADE K3 surfaces is separated.

The main goal of this paper is to construct a functorial, geometrically meaningful compactification of the moduli space F_ρ^{ade}, under the assumption $(\exists g \geq 2)$. Let $R = C_1$, $\varphi_{|mR|} \colon X \to \overline{X}$ be the contraction as above and $\overline{R}$ be the image of R. Then for any $0 < \epsilon \ll 1$ the pair $(\overline{X}, \epsilon\overline{R})$ is a stable pair with semi log canonical singularities. Then the theory of KSBA moduli spaces (see [Kol21] for the general case or [AET19, ABE20] for the much easier special case needed here) gives a moduli compactification $\overline{F}_\rho^{\mathrm{slc}}$ to a space of stable pairs with automorphism.

Our main Theorem 3.24 says that $\overline{F}_\rho^{\mathrm{slc}}$ is a semitoroidal compactification of $\mathbb{D}_\rho/\Gamma_\rho$. This class of compactifications was introduced by Looijenga [Loo03b] as a common generalization of Baily-Borel and toroidal compactifications. As a corollary, the family of ADE K3 surfaces with an automorphism extends along the inclusion $(\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho \hookrightarrow \mathbb{D}_\rho/\Gamma_\rho$.

The proof applies a modified form of one of the main theorems of [AE21] about so-called *recognizable* divisors. The $g \geq 2$ component of the fixed locus is a canonical choice of a polarizing divisor. We prove that this divisor is recognizable.

The cases $n = 2, 3, 4, 6$ are of the most interest for compactifications. If $n \neq 2, 3, 4, 6$ then the space $\mathbb{D}_\rho/\Gamma_\rho$ is already compact, see [Mat16] or Corollary 3.14.

K3 surfaces with an involution were classified by Nikulin in [Nik79b]. K3s with a non-symplectic automorphism of prime order $p \geq 3$ we classified by Artebani, Sarti, and Taki in [AS08, AST11]. The case $n = 4$ was treated by Artebani-Sarti in [AS15] and the case $n = 6$ by Dillies in [Dil09, Dil12].

We note two cases where our KSBA, semitoroidal compactification $\overline{F}_\rho^{\mathrm{slc}}$ is computed in complete detail: Alexeev-Engel-Thompson [AET19] for the case of K3 surfaces of degree 2, generically double covers of $\mathbb{P}^2$, and a forthcoming work Deopurkar-Han [DH21] which treats a 9-dimensional component in the moduli for $n = 3$.

The paper is organized as follows. In Section 2 we set up the general theory of the moduli of K3 surfaces with a non-symplectic automorphisms. In Section 3 we define the stable pair compactifications and prove the main Theorem 3.24. In Section 4 we relate K3 surfaces with nonsymplectic automorphisms with their quotients $Y = \overline{X}/\mu_n$, and the compactification $\overline{F}_\rho^{\rm slc}$ with the KSBA compactification of the moduli spaces of log del Pezzo pairs $(Y, \frac{n-1+\epsilon}{n}B)$.

Throughout, we work over the field of complex numbers.

Acknowledgements. The first author was partially supported by NSF under DMS-1902157.

2. Moduli of K3s with a nonsymplectic automorphism

2A. **Notations.** A lattice is a free abelian group with an integral-valued symmetric bilinear form. Let $L = H^{\oplus 3} \oplus E_8^{\oplus 2}$ be a fixed copy of the even unimodular lattice of signature $(3, 19)$, where $H = \mathrm{II}_{1,1}$ corresponds to the bilinear form $b(x, y) = xy$ and E_8 is the standard negative definite even lattice of rank 8. For any smooth K3 surface X the cohomology lattice $H^2(X, \mathbb{Z})$ is isometric to L.

Denote by $S = S_X$ the Neron-Severi lattice $\mathrm{Pic}(X) = \mathrm{NS}(X)$. By the Lefschetz $(1, 1)$-theorem, it equals $(H^{2,0}(X))^\perp \cap H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{C})$. We have $H^{2,0}(X) = \mathbb{C}\omega_X$ for some nowhere vanishing holomorphic two-form ω_X. If X is projective, then S_X is nondegenerate of signature $(1, r_X - 1)$. In this case, its orthogonal complement $T_X = (S_X)^\perp \subset H^2(X, \mathbb{Z})$ is the *transcendental lattice*, of signature $(2, 20 - r_X)$. The *Kähler cone* $\mathcal{K}_X \subset H^{1,1}(X, \mathbb{R})$ is the set of classes of Kähler forms on X; it is an open convex cone.

Theorem 2.1 (Torelli Theorem for K3 surfaces, [PSS71])**.** *The isomorphisms $\sigma\colon X' \to X$ are in bijection with the isometries $\sigma^*\colon H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ satisfying the conditions $\sigma^*(H^{2,0}(X)) = H^{2,0}(X')$ and $\sigma^*(\mathcal{K}_X) = \mathcal{K}_{X'}$.*

For any lattice H, a *root* is a vector $\delta \in H$ with $\delta^2 = -2$. The set of all roots is denoted by H_{-2}. The Weyl group $W(H)$ is the group generated by reflections $v \mapsto v + (v, \delta)\delta$ for $\delta \in H_{-2}$. It is a normal subgroup of the isometry group $O(H)$.

2B. **Moduli of marked unpolarized K3s.** The basic reference here is [ast85]. Let X be a K3 surface. A *marking* is an isometry $\phi\colon H^2(X, \mathbb{Z}) \to L$. Let

$$\mathbb{D} = \mathbb{P}\{x \in L_\mathbb{C} \mid x \cdot x = 0,\ x \cdot \bar{x} > 0\}, \quad \dim \mathbb{D} = 20.$$

There exists a fine moduli space $\mathcal{M}$ of *marked K3 surfaces* and a period map $\pi\colon \mathcal{M} \to \mathbb{D}$, $(X, \phi) \mapsto \phi(H^{2,0}(X)) \in \mathbb{P}(L_\mathbb{C})$. $\mathcal{M}$ is a non-Hausdorff 20-dimensional complex manifold with two isomorphic connected components interchanged by negating ϕ. The period map is étale and surjective.

For a period point $x \in \mathbb{D}$, the vector space $(\mathbb{C}x \oplus \mathbb{C}\bar{x}) \cap L_\mathbb{R} \subset L_\mathbb{C}$ is positive definite of rank 2 and its orthogonal complement $x^\perp \cap L_\mathbb{R}$ has signature $(1, 19)$. Let

$$\{v \in x^\perp \cap L_\mathbb{R} \mid v^2 > 0\} = P_x \sqcup (-P_x)$$

be the two connected components of the set of positive square vectors. Then the fiber $\pi^{-1}(x)$ is identified with the set of connected components $\mathcal{C}$ of

$$\big(P_x \sqcup (-P_x)\big) \setminus \cup_\delta \delta^\perp \text{ for } \delta \in (x^\perp \cap L)_{-2}.$$

Namely, an open chamber $\mathcal{C}$ is identified with the Kähler cone $\mathcal{K}_X$ of the corresponding marked K3 surface X via the marking ϕ. The connected components are permuted by the reflections and $\pm\,\mathrm{id}$, and $\pi^{-1}(x)$ is a torsor under the group $\mathbb{Z}_2 \times W_x$, where $W_x = W(x^\perp \cap L)$. Since $x^\perp \cap L_\mathbb{R}$ is hyperbolic, the group and the fiber $\pi^{-1}(x)$ may be infinite. For a general point $x \in \mathbb{D}$, the lattice $x^\perp \cap L$ has no roots and the fiber $\pi^{-1}(x)$ consists of two points, one in each connected component of $\mathcal{M}$.

2C. **Markings of K3 surfaces with automorphism.** Fix $\rho \in O(L)$ an isometry of order $n > 1$, and consider (X, σ) a K3 surface with a non-symplectic automorphism σ of order n. A *ρ-marking* of (X, σ) is an isometry $\phi : H^2(X, \mathbb{Z}) \to L$ such that $\phi \circ \sigma^* = \rho \circ \phi$. We say that (X, σ) is *ρ-markable* if it admits a ρ-marking. It is clear that for any given (X, σ), there exists some such ρ.

A family of smooth K3 surfaces $f\colon (\mathcal{X}, \sigma) \to S$ with automorphism admits a ρ-marking if and only if the local system $R^2 f_* \underline{\mathbb{Z}}$ is constant.

Definition 2.2. Define $\mathbb{D}_\rho \subset \mathbb{D}$ as the set of $x \in \mathbb{D}$ such that $\rho(x) = \zeta_n x$. Define $\Gamma_\rho \subset O(L)$ as the group of changes-of-marking: $\Gamma_\rho := \{\gamma \in O(L) \mid \gamma \circ \rho = \rho \circ \gamma\}$.

Definition 2.3. Let the *generic transcendental lattice* $T_\rho := L_\mathbb{C}^{\mathrm{prim}} \cap L$ be the intersection of L with the sum of all primitive eigenspaces of ρ, and let the *generic Picard lattice* be $S_\rho = (T_\rho)^\perp$. Let $L^\rho \subset S_\rho$ be classes in L fixed by ρ.

Note that the ζ_n-eigenspaces $L_\mathbb{C}^{\zeta_n}$ and $T_{\rho,\mathbb{C}}^{\zeta_n}$ coincide, and that for any K3 surface with a ρ-marking one has $\phi\colon S_X^\sigma = H^2(X, \mathbb{Z})^\sigma \xrightarrow{\sim} L^\rho$.

For there to exist a ρ-markable algebraic K3 surface, the signature of T_ρ must be $(2, \ell)$ for some ℓ, as there is necessarily a vector of positive norm fixed by σ^* (the sum of a σ^*-orbit of an ample class). The converse is also true.

When $n = 2$, we have that $\mathbb{D}_\rho \subset \mathbb{P}(T_{\rho,\mathbb{C}})$ is (two copies of) the Type IV domain associated to the lattice T_ρ. When $n \geq 3$, the condition that $x \cdot x = 0$ is vacuous on $\mathbb{D}_\rho$ because $x \cdot y = 0$ for eigenvectors x, y of ρ with non-conjugate eigenvalue. Thus,

$$\mathbb{D}_\rho = \mathbb{P}\{x \in T_{\rho,\mathbb{C}}^{\zeta_n} \mid x \cdot \bar{x} > 0\}$$

is a complex ball, a Type I domain. The Hermitian form $x \cdot \bar{y}$ on $T_{\rho,\mathbb{C}}^{\zeta_n}$ necessarily has signature $(1, \ell)$ for some ℓ for there to exist a ρ-markable K3 surface.

Definition 2.4. The discriminant locus is $\Delta_\rho := (\cup_\delta\, \delta^\perp) \cap \mathbb{D}_\rho$ ranging over all roots δ in $(L^\rho)^\perp$.

It is clear from the definitions that the moduli space $\mathcal{M}_\rho$ of ρ-marked K3 surfaces admits a period map $\pi_\rho\colon \mathcal{M}_\rho \to \mathbb{D}_\rho$, $(X, \sigma, \phi) \to \phi(H^{2,0}(X))$. There is a natural inclusion $\mathcal{M}_\rho \subset \mathcal{M}$ by forgetting σ, and π_ρ is simply the restriction of π.

Lemma 2.5. *The image of the period map $\pi_\rho\colon \mathcal{M}_\rho \to \mathbb{D}_\rho$ is $\mathbb{D}_\rho \setminus \Delta_\rho$. For a point $x \in \mathbb{D}_\rho \setminus \Delta_\rho$ the fiber $\pi_\rho^{-1}(x)$ is a torsor over $\Gamma_\rho \cap (\mathbb{Z}_2 \times W_x)$.*

Proof. Let $x \in \mathbb{D}_\rho \setminus \Delta_\rho$. Then $L^\rho \not\subset \cup_\delta\, \delta^\perp$ for $\delta \in (x^\perp \cap L)_{-2}$. Thus, there exists a chamber $\mathcal{C}$ in $P_x \setminus \cup_\delta\, \delta^\perp$ such that $\mathcal{C} \cap L^\rho \neq \emptyset$. Let (X, ϕ) be the K3 surface corresponding to this chamber. Consider any $h \in \mathcal{C} \cap L^\rho$ and let $\mathcal{L}_h = \phi^{-1}(h) \in S_X$ be the corresponding ample line bundle on X. The action of ρ fixes $\mathcal{L}_h$, so it fixes the Kähler cone $\mathcal{K}_X$. By the Torelli theorem, the action of ρ on $H^2(X, \mathbb{Z})$ is induced by an automorphism. Thus, $x \in \operatorname{im} \pi_\rho$. Two surfaces (X_1, ϕ_1), (X_2, ϕ_2) in $\pi^{-1}(x)$ are both ρ-markable iff they differ by the action of Γ_ρ.

Now let $x \in \delta^\perp$ for some root $\delta \in (L^\rho)^\perp$ and assume that $x = \pi_\rho((X,\phi))$ for some ρ-markable K3 surface (X,ϕ). Then $\mathcal{L}_\delta = \phi^{-1}(\delta) \in S_X \cap (S_X^\sigma)^\perp$. But the latter can not contain any roots, see [Kon20, Lem. 8.24(3)]. Contradiction. □

Theorem 2.6. *On the level of the coarse moduli spaces, the space $F_\rho = \mathcal{M}_\rho/\Gamma_\rho$ of ρ-markable K3 surfaces with automorphism (X,σ) admits a bijective period map $F_\rho \to (\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho$.*

Proof. The statement is immediate from the definitions and Lemma 2.5, by quotienting the period map π_ρ. The points of $\pi_\rho^{-1}(x)$ are permuted by Γ_ρ, thus they are identified in the Γ_ρ-quotient. □

Remark 2.7. The proof of the surjectivity of the map $\mathcal{M}_\rho \to \mathbb{D}_\rho \setminus \Delta_\rho$ follows that of Dolgachev-Kondo [DK07, Thm. 11.2]. Sections 10 and 11 of [DK07] contain a construction of the moduli space of K3 surfaces with a non-symplectic automorphism that is based on moduli of lattice polarized K3s. But it uses [Dol96, Thm. 3.1] which unfortunately is false, as was noted in [AE21]. For this reason, we decided to give an alternative construction.

Remark 2.8. In fact, the separated quotient $F_\rho^{\rm sep}$ is a stack $[\mathbb{D}_\rho \setminus \Delta_\rho :_W \Gamma_\rho]$ which can be locally constructed near $x \in \mathbb{D}_\rho \setminus \Delta_\rho$ by first taking a coarse quotient by the normal subgroup $\Gamma_\rho \cap (\mathbb{Z}_2 \times W_x) \trianglelefteq \mathrm{Stab}_x(\Gamma_\rho)$ and then taking the stack quotient by $\mathrm{Stab}_x(\Gamma_\rho)/\Gamma_\rho \cap (\mathbb{Z}_2 \times W_x)$. See [AE21, Rem. 2.36].

Proposition 2.9. *Suppose $\sigma \in \mathrm{Aut}(X)$ fixes a curve R of genus at least 2, i.e. the assumption $(\exists g \geq 2)$ holds. Then $\mathrm{Aut}(X,\sigma)$ is finite.*

Proof. Let $h \in \mathrm{Aut}(X,\sigma)$ be an automorphism of X satisfying $h \circ \sigma = \sigma \circ h$. Then h permutes the fixed components of σ. Since there is at most one component R of genus $g \geq 2$, we conclude $h(R) = R$. Hence $h \in \mathrm{Aut}(X, \mathcal{O}(R))$, a finite group. □

By Remark 2.8, the group

$$K_\rho := \ker(\Gamma_\rho \to \mathrm{Aut}(\mathbb{D}_\rho))/\Gamma_\rho \cap (\mathbb{Z}_2 \times W(L^\rho))$$

is the generic stabilizer for either stack $F_\rho^{\rm sep}$ or F_ρ. Note that K_ρ is never the trivial group, as $\rho \in K_\rho$ is a nontrivial element. As this is the automorphism group of a generic element $(X,\sigma) \in F_\rho$, if $(\exists g \geq 2)$ holds then K_ρ is finite by Proposition 2.9.

Example 2.10. Consider the double cover $\pi\colon X \to \mathbb{P}^2$ branched over a smooth sextic B. There is a non-symplectic involution σ switching the two sheets of X, acting on $H^2(X,\mathbb{Z})$ by fixing $h = c_1(\pi^*\mathcal{O}(1))$ and negating $h^\perp$. Choosing a model ρ for the action of σ^* on cohomology, we have that $S_\rho = \langle 2 \rangle$ and $T_\rho = \langle -2 \rangle \oplus H^{\oplus 2} \oplus E_8^{\oplus 2}$ are the $(+1)$- and (-1)-eigenspaces, respectively.

The divisor $\Delta_\rho/\Gamma_\rho \subset \mathbb{D}_\rho/\Gamma_\rho = F_2$ has two irreducible components corresponding to Γ_ρ-orbits of roots $\delta \in (T_\rho)_{-2}$. Such an orbit is uniquely determined by the divisibility (1 or 2) of $\delta \in T_\rho^*$. The case where the divisibility is 2 corresponds to when B acquires a node. Then there is an involution σ on the minimal resolution of the double cover $X \to \overline{X} \to \mathbb{P}^2$, but $\sigma^*(\delta) = \delta$, $\sigma^*(h) = h$ and the $(+1,-1)$-eigenspaces of σ^* have dimensions $(2,20)$. Thus, no ρ-marking can be extended over a family $\mathcal{X} \to C$ with central fiber X and general fiber as above.

When the divisibility of δ is 1, $\mathbb{P}^2$ degenerates to $\mathbb{F}_4^0 = \mathbb{P}(1,1,4)$ and the minimal resolution of the double cover $X \to \overline{X} \to \mathbb{F}_4^0$ is an elliptic K3 surface with σ the elliptic involution. Again the eigenspaces have dimension profile $(2, 20)$ and so (X, σ) is not ρ-markable for the ρ as above.

3. Stable pair compactifications

3A. **Complete moduli of stable slc pairs.** We refer the reader to [ABE20, Sec. 2B] and [AE21, Sec. 7D] for a detailed discussion of stable K3 surface pairs and their compactified moduli. Briefly:

Definition 3.1. In our context, a stable slc surface pair is a pair $(S, \epsilon D)$, where

(1) S is a connected, reduced, projective Gorenstein surface S with $\omega_S \simeq \mathcal{O}_S$ which has semi log canonical singularities.
(2) D is an effective ample Cartier divisor on S that does not contain any log canonical centers of S.

Then for sufficiently small rational number $\epsilon > 0$ the pair $(S, \epsilon D)$ is stable, meaning:

(1) it has semi log canonical singularities, and
(2) the $\mathbb{Q}$-Cartier divisor $K_S + \epsilon D$ is ample.

"Sufficiently small" works in families: for a fixed D^2 there exists ϵ_0 so that if a pair $(S, \epsilon D)$ is stable in the above definition for some ϵ then it is stable for any $0 < \epsilon \le \epsilon_0$.

The main application to K3 surfaces is an observation that for any K3 surface $\overline{X}$ with ADE singularities and an effective ample divisor $\overline{R}$, the pair $(\overline{X}, \epsilon\overline{R})$ is stable. Indeed, $\omega_{\overline{X}} \simeq \mathcal{O}_{\overline{X}}$, the surface $\overline{X}$ has canonical singularities—which is much better than semi log canonical—and there are no log centers.

As usual, let F_{2d} denote the moduli space of polarized K3 surfaces $(\overline{X}, \overline{L})$ with ADE singularities and ample primitive line bundle $\overline{L}$ of degree $\overline{L}^2 = 2d$, and $P_{2d,m} \to F_{2d}$ denote the moduli space of pairs $(\overline{X}, \epsilon\overline{R})$ with an effective divisor $\overline{R} \in |m\overline{L}|$. Then the main result for K3 surfaces is the following:

Theorem 3.2. (1) *For the stable pairs as above there exists an algebraic Deligne-Mumford moduli stack $\mathcal{M}^{\mathrm{slc}}$, with a coarse moduli space M^{slc}.*
(2) *The closure $\overline{P}^{\mathrm{slc}}_{2d,m}$ of $P_{2d,m}$ in M^{slc} is projective and provides a compactification of $P_{2d,m}$ to a moduli space of stable slc pairs.*

To apply this result to a compactification of F^{sep}_ρ one needs to choose, in a canonical manner, a big and nef divisor on the generic $(X, \sigma) \in F_\rho$.

Definition 3.3. A *canonical choice of polarizing divisor* is an algebraically varying big and nef divisor R defined over a Zariski dense subset $U \subset F_\rho$ of the moduli space of ρ-markable K3 surfaces.

3B. **Stable pair compactification of F^{sep}_ρ.** We apply Theorem 3.2 to construct a stable pair compactification in the present context as follows.

Suppose that for each surface $(X, \sigma) \in F_\rho$ assumption $(\exists g \ge 2)$ holds, i.e. the fixed locus $\mathrm{Fix}(\sigma)$ contains a component C_1 of genus $g \ge 2$, as well as possibly several smooth rational curves C_i and some isolated points. In fact, it suffices that a single $(X, \sigma) \in F_\rho$ satisfies assumption $(\exists g \ge 2)$ because the genus of C_1 is constant in a family of smooth K3 surfaces with non-symplectic automorphism. So $R = C_1$ gives a canonical choice of polarizing divisor for all of $U = F_\rho$.

Let $\pi\colon X \to \overline{X}$ be the contraction to an ADE K3 surface such that the divisor $\overline{R} := \pi(C_1)$ is ample; it has degree $\overline{R}^2 = 2g(C_1) - 2 > 0$. It provides us with an ample divisor on $\overline{X}$. If $\mathcal{O}(\overline{R}) = \overline{L}^m$ for a primitive $\overline{L}$ then the pair $(\overline{X}, \mathcal{O}(\overline{R}))$ is a point of $F_{2d,m}$ and the pair $(\overline{X}, \epsilon\overline{R})$ is a point of $P_{2d,m}$.

Definition 3.4. We define the map $\psi\colon F_\rho \to P_{2d,m}$ as follows. Pointwise, it sends (X,σ) to $(\overline{X}, \epsilon\overline{R})$. In every flat family $f\colon \mathcal{X} \to S$ of K3 surfaces with automorphism, the sheaf $\mathcal{O}_{\mathcal{X}}(\mathcal{R})$ is relatively big and nef. Since $R^i\mathcal{L}^d = 0$ for $i > 0$, $d > 0$, it gives a contraction to a flat family $\bar{f}\colon (\overline{\mathcal{X}}, \overline{\mathcal{R}}) \to S$. This induces the map on moduli.

Lemma 3.5. *The map $\psi\colon F_\rho \to P_{2d,m}$ defined above induces an injective map $F_\rho^{\rm sep} \to \operatorname{im}(\psi)$.*

Proof. The map ψ factors through the separated quotient of F_ρ because $P_{2d,m}$ is separated. Now suppose there is an isomorphism of pairs $\overline{f}\colon (\overline{X}_1, \overline{R}_1) \to (\overline{X}_2, \overline{R}_2)$ inducing an isomorphism of the minimal resolutions $f\colon (X_1, R_1) \to (X_2, R_2)$. Consider the morphism $\varphi = \sigma_1^{-1} f^{-1} \sigma_2 f$. Then φ is a *symplectic* automorphism of X_1 fixing the curve R_1 pointwise. Since φ preserves $\mathcal{O}_{X_1}(R_1)$, it has finite order. By [Nik79a] the fixed set of a finite order symplectic K3 automorphism is finite. Thus, $\varphi = \mathrm{id}$ and f preserves the group action. So, (X,σ) is uniquely determined by $(\overline{X}, \overline{R})$. ☐

Remark 3.6. $F_\rho^{\rm sep}$ itself has a moduli interpretation: It is the moduli space $F_\rho^{\rm ade}$ of *ADE* K3 surfaces $(\overline{X}, \overline{\sigma})$ with automorphism, for which $\operatorname{Fix}(\overline{\sigma})$ is ample, and for which the minimal resolution $(X,\sigma) \to (\overline{X}, \overline{\sigma})$ is ρ-markable.

Definition 3.7. Let $Z = \operatorname{im}(\psi)$ and let $\overline{Z}$ be its closure in $\overline{P}_{2d,m}^{\rm slc}$, with reduced scheme structure. The stable pair compactification

$$F_\rho^{\rm sep} = F_\rho^{\rm ade} \hookrightarrow \overline{F}_\rho^{\rm slc}$$

is defined as the normalization of $\overline{Z}$.

In particular, $\overline{F}_\rho^{\rm slc}$ is normal by definition. Points correspond to the pairs $(\overline{X}, \epsilon\overline{R})$, possibly degenerate, with some finite data.

3C. **Kulikov degenerations of K3 surfaces.** A basic tool in the study of degenerations of K3 surfaces is Kulikov models. We use them in the argument below, so we briefly recall the definition.

Let $(C, 0)$ denote the germ of a smooth curve at a point $0 \in C$ and let $C^* = C \setminus 0$. Let $X^* \to C^*$ be a family of algebraic K3 surfaces.

Definition 3.8. A *Kulikov model* $X \to (C, 0)$ is an extension of $X^* \to C^*$ for which X is a smooth algebraic space, $K_X \sim_C 0$, and X_0 has reduced normal crossings. We say the X is *Type I, II, or III*, respectively, depending on whether X_0 is smooth, has double curves but no triple points, or has triple points, respectively. We call the central fiber X_0 of such a family a *Kulikov surface*.

A key result on the degenerations of K3 surfaces is the theorem of Kulikov [Kul77] and Persson-Pinkham [PP81]:

Theorem 3.9. *Let $Y^* \to C^*$ be a family of algebraic K3 surfaces. Then there is a finite base change $(C', 0) \to (C, 0)$ and a sequence of birational modifications of the pull back $Y' \dashrightarrow X$ such that X has smooth total space, $K_X \sim_{C'} 0$, and X_0 has reduced normal crossings.*

We recall some fundamental results about Kulikov models. The primary reference is [FS86]. Let $T : H^2(X_t, \mathbb{Z}) \to H^2(X_t, \mathbb{Z})$ denote the Picard-Lefschetz transformation associated to an oriented simple loop in C^* enclosing 0. Since X_0 is reduced normal crossings, T is unipotent. Let

$$N := \log T = (T - I) - \tfrac{1}{2}(T - I)^2 + \cdots$$

be the logarithm of the monodromy.

Theorem 3.10. [FS86][Fri84] *Let $X \to (C, 0)$ be a Kulikov model. We have that*

if X is Type I, then $N = 0$,
if X is Type II, then $N^2 = 0$ but $N \neq 0$,
if X is Type III, then $N^3 = 0$ but $N^2 \neq 0$.

The logarithm of monodromy is integral, and of the form $Nx = (x \cdot \lambda)\delta - (x \cdot \delta)\lambda$ for $\delta \in H^2(X_t, \mathbb{Z})$ a primitive isotropic vector, and $\lambda \in \delta^\perp/\delta$ satisfying

$$\lambda^2 = \#\{\textit{triple points of } X_0\}.$$

When $\lambda^2 = 0$, its imprimitivity is the number of double curves of X_0.

Thus, the Types I, II, III of Kulikov model are distinguished by the behavior of the *monodromy invariant* λ: either $\lambda = 0$, $\lambda^2 = 0$ but $\lambda \neq 0$, or $\lambda^2 \neq 0$ respectively.

Definition 3.11. Let $J \subset H^2(X_t, \mathbb{Z})$ denote the primitive isotropic lattice $\mathbb{Z}\delta$ in Type III or the saturation of $\mathbb{Z}\delta \oplus \mathbb{Z}\lambda$ in Type II.

3D. **Baily-Borel compactification.** Let N be a lattice of signature $(2, \ell)$, together with an isometry $\rho \in O(N)$ of finite order n, such that all eigenvalues of ρ on $N_{\mathbb{C}}$ are primitive nth roots of unity, and $N_{\mathbb{C}}^{\zeta_n}$ contains a vector x of positive Hermitian norm $x \cdot \bar{x}$. This is the situation which arises for a non-symplectic automorphism of an algebraic K3 surface, with $N = T_\rho$. Then we have a Type IV $(n = 2)$ or I $(n > 2)$ domain

$$\mathbb{D}_\rho = \mathbb{P}\{x \in N_{\mathbb{C}}^{\zeta_n} \mid x \cdot x = 0,\ x \cdot \bar{x} > 0\}$$

admitting the action of the arithmetic group $\widetilde{\Gamma}_\rho := \{\gamma \in O(N) \mid \gamma \circ \rho = \rho \circ \gamma\}$. Fix a finite index subgroup $\Gamma \subset \widetilde{\Gamma}_\rho$.

Recall that $\mathbb{D}_\rho$ embeds into its compact dual $\mathbb{D}_\rho^c$, which is defined by dropping the condition that $x \cdot \bar{x} > 0$. Define $\overline{\mathbb{D}}_\rho \subset \mathbb{D}_\rho^c$ as the topological closure of $\mathbb{D}_\rho \subset \mathbb{D}_\rho^c$.

Definition 3.12. A *rational boundary component* of $\mathbb{D}_\rho$ is an analytic subset $B_J \subset \overline{\mathbb{D}}_\rho$ of the form:

(1) $(\mathbb{P}J_{\mathbb{C}} \setminus \mathbb{P}J_{\mathbb{R}}) \cap \overline{\mathbb{D}}_\rho$ for $\operatorname{rk} J = 2$ a primitive isotropic sublattice of N,
(2) $\mathbb{P}J_{\mathbb{C}} \cap \overline{\mathbb{D}}_\rho$ for $\operatorname{rk} J = 1$ a primitive isotropic sublattice of N.

One defines the *rational closure* of $\mathbb{D}_\rho$ to be $\mathbb{D}_\rho^{\mathrm{bb}} := \mathbb{D}_\rho \cup_J B_J$, topologized via a horoball topology at the boundary. Then the *Baily-Borel compactification* of $\mathbb{D}_\rho/\Gamma$ is (at least topologically) $\overline{\mathbb{D}_\rho/\Gamma}^{\mathrm{bb}} := \mathbb{D}_\rho^{\mathrm{bb}}/\Gamma$. See [Loo03a, Loo03b] for more details.

The space $\overline{\mathbb{D}_\rho/\Gamma}^{\mathrm{bb}}$ was shown to have the structure of a projective variety by Baily-Borel [BB66]. If $\mathbb{D}_\rho$ is a Type IV domain, then the boundary components (1) are isomorphic to $\mathbb{H} \sqcup (-\mathbb{H})$ and the boundary components (2) are points. If $\mathbb{D}_\rho$ is a Type I domain, then boundary components (1) are points, and boundary components (2) cannot exist. If $\operatorname{rk} J = 2$ then a point $x \in B_J$ corresponds to the elliptic curve $E_x = J_{\mathbb{C}}/(J + \mathbb{C}x)$.

Lemma 3.13. *In the case $n > 2$, we necessarily have* $\operatorname{rk} J = 2$ *and* $n \in \{3,4,6\}$. *If* $n = 3$ *or* 6 *then* $j(E_x) = 0$. *If* $n = 4$ *then* $j(E_x) = 1728$.

Proof. Since B_J is a boundary component of $\mathbb{D}_\rho$ and ρ acts trivially on $\mathbb{D}_\rho$, one has $\rho(J) = J$ and $J_{\mathbb{C}} \cap N_{\mathbb{C}}^{\zeta_n} \neq \emptyset$. Since $\zeta_n \notin \mathbb{R}$ and $\operatorname{rk} J = 2$, one has

$$J_{\mathbb{C}} = J_{\mathbb{C}}^{\zeta_n} \oplus J_{\mathbb{C}}^{\bar{\zeta}_n}.$$

Therefore $\rho\big|_J \in \mathrm{GL}(J) \cong \mathrm{GL}_2(\mathbb{Z})$ necessarily has order n. Thus, $n \in \{3,4,6\}$. For a point $x \in B_J$ one has $\mu_n \subset \operatorname{Aut}(E_x)$. This uniquely determines E_x. □

Corollary 3.14. *If* $n \neq 2,3,4,6$ *then the rational closure of* $\mathbb{D}_\rho$ *is simply* $\mathbb{D}_\rho$ *itself. So* $\mathbb{D}_\rho/\Gamma$ *is already compact.*

The following is a well-known consequence of Schmid's nilpotent orbit theorem:

Proposition 3.15. *Let* $X^* \to C^*$ *be a degeneration of a* ρ*-markable K3 surfaces over a punctured analytic disk* C^*. *A lift of the period mapping* $\widetilde{C^*} \cong \mathbb{H} \to \mathbb{D}_\rho$ *approaches the Baily-Borel cusp* B_J *as* $\operatorname{Im}(\tau) \to \infty$, *where* J *is the monodromy lattice in* $H^2(X_t, \mathbb{Z})$, *cf. Definition 3.11. When* $\operatorname{rk}(J) = 2$, *the limiting point* $x \in B_J$ *corresponds to an elliptic curve* E_x *isomorphic to any double curve of the central fiber* X_0 *of a Kulikov model* $X \to C$.

Corollary 3.16. *If* $n \neq 2,3,4,6$, *any degeneration of* $(X,\sigma) \in F_\rho$ *has Type I. If* $n \in \{3,4,6\}$, *any degeneration of* $(X,\sigma) \in F_\rho$ *has Type I or II.*

The last statement was also proved by Matsumoto [Mat16] using different techniques. His proof also holds in some prime characteristics.

3E. **Semitoroidal compactifications.** Semitoroidal compactifications of arithmetic quotients $\mathbb{D}/\Gamma$ for type IV Hermitian symmetric domains $\mathbb{D}$ were defined by Looijenga [Loo03b] (where they were called "semitoric"). They simultaneously generalize toroidal and Baily-Borel compactifications of $\mathbb{D}/\Gamma$. The case of the complex ball $\mathbb{D}$ (a type I symmetric Hermitian domain) is comparatively trivial. The semitoroidal compactifications in this case are implicit in [Loo03a, Loo03b]. We quickly overview the construction in both cases now.

Definition 3.17. A Γ-*admissible semifan* $\mathfrak{F}$ consists of the following data:

When $n = 2$, it is a convex, rational, locally polyhedral decomposition $\mathfrak{F}_J$ of the rational closure $\mathcal{C}^+(J^\perp/J)$ of the positive norm vectors, for all rank 1 primitive isotropic sublattices $J \subset N$, such that:

(1) $\{\mathfrak{F}_J\}_{J \subset N}$ is Γ-invariant. In particular, a fixed $\mathfrak{F}_J$ is invariant under the natural action of $\operatorname{Stab}_J(\Gamma)$ on $\mathcal{C}^+(J^\perp/J)$.
(2) A compatibility condition of the $\{\mathfrak{F}_J\}_{J \subset N}$ along any primitive isotropic lattice $J' \subset N$ of rank 2 holds, see Definition 3.18.

When $n > 2$, the data is much simpler: It consists, for each primitive isotropic sublattice $J \subset N$ satisfying $J_{\mathbb{C}} \cap N_{\mathbb{C}}^{\zeta_n} \neq \emptyset$, of a primitive sublattice $\mathfrak{F}_J \subset J^\perp/J$ such that the collection $\{\mathfrak{F}_J\}$ is Γ-invariant.

Definition 3.18. Let $J' \subset N$ be primitive isotropic of rank 2. We say that the collection $\{\mathfrak{F}_J\}_{J \subset N}$ is *compatible along* J' if, given any primitive sublattice $J \subset J'$ of rank 1, the kernel of the hyperplanes of $\mathfrak{F}_J$ containing J'/J, when intersected with $(J')^\perp/J \subset J^\perp/J$ and then descended to $(J')^\perp/J'$, cut out a fixed sublattice $\mathfrak{F}_{J'} \subset (J')^\perp/J'$ which is independent of J.

In both the $n = 2$ and $n > 2$ cases, we use the same notation $\mathfrak{F} := \{\mathfrak{F}_J\}_{J\subset N}$ even though J ranges over rank 1 isotropic sublattices when $n = 2$ and ranges over rank 2 isotropic sublattices when $n > 2$.

In the Type IV case, Looijenga constructs a compactification $\mathbb{D}/\Gamma \hookrightarrow \overline{\mathbb{D}/\Gamma}^{\mathfrak{F}}$ for any Γ-admissible semifan $\mathfrak{F}$, so consider the Type I case. By Lemma 3.13 we may restrict to $n \in \{3, 4, 6\}$. There is a $\mathbb{Z}[\zeta_n]$-lattice

$$Q := (N \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_n])^{\zeta_n} \subset N_{\mathbb{C}}^{\zeta_n} = Q_{\mathbb{C}}$$

on which Hermitian form $x \cdot \overline{y}$ defines a $\mathbb{Z}[\zeta_n]$-valued Hermitian pairing of signature $(1, \ell)$ for some ℓ. Any element of $\widetilde{\Gamma}_\rho$ (in particular, any element of Γ) preserves Q and the Hermitian form on it. The converse also holds. Thus $\Gamma \subset U(Q)$ is a finite index subgroup of the group of unitary isometries of Q and $\Gamma_{\mathbb{R}} = U(Q_{\mathbb{C}}) = U(1, \ell)$. The boundary components $B_J = \mathbb{P}(J_{\mathbb{C}}^{\zeta_n})$ are then projectivizations of the isotropic $\mathbb{Z}[\zeta_n]$-lines $K \subset Q$. Here $K_{\mathbb{C}} = J_{\mathbb{C}}^{\zeta_n}$.

Choose a generator $k \in K$. Then any $[x] \in \mathbb{D}_\rho \subset \mathbb{P}Q_{\mathbb{C}}$ has a unique representative $x \in Q_{\mathbb{C}}$ for which $k \cdot x = 1$. This realizes $\mathbb{D}_\rho$ as a generalized tube domain in the affine hyperplane $V_k := \{k \cdot x = 1\} \subset Q_{\mathbb{C}}$.

Let $U_K \subset \mathrm{Stab}_K(\Gamma)$ be the unipotent subgroup (i.e. U_K acts on K, $K^\perp/K$, and $Q/K^\perp$ by the identity). Then U_K acts on V_k by translations. Choosing some isotropic $k' \in Q_{\mathbb{C}}$ for which $k' \cdot k = 1$, any element $x \in V_k$ can be written uniquely as $x = k' + x_0 + ck$ for some $x_0 \in \{k, k'\}^\perp$ and $c \in \mathbb{C}$. The image of $\mathbb{D}_\rho$ is exactly those x satisfying $2\mathrm{Re}(c) > -x_0 \cdot \bar{x}_0$.

The fibration $\mathbb{D}_\rho \to K_{\mathbb{C}}^\perp/K_{\mathbb{C}}$ sending $x \mapsto x_0 \bmod K_{\mathbb{C}}$ is a fibration of right half-planes. The action of U_K fibers over the action of a translation subgroup $\overline{U}_K \subset K^\perp/K$ on $K_{\mathbb{C}}^\perp/K_{\mathbb{C}}$ and thus, there is a fibration

$$\mathbb{D}_\rho/U_K \to (K_{\mathbb{C}}^\perp/K_{\mathbb{C}})/\overline{U}_K =: A_K$$

over an abelian variety. The fibers are quotients of the right half-planes with coordinate c by a discrete, purely imaginary, translation group isomorphic to $\mathbb{Z}$. This realizes $\mathbb{D}_\rho/U_K$ is a punctured holomorphic disc bundle over A_K.

Definition 3.19. $\mathbb{D}_\rho/U_K$ is the *first partial quotient* associated to the Baily-Borel cusp K. The extension of this punctured disc bundle to a disc bundle $\overline{\mathbb{D}_\rho/U_K}^{\mathrm{can}} \to A_K$ for a given K is called the *toroidal extension at the cusp* K.

We will identify the divisor at infinity, i.e. the zero section of the disc bundle, with A_K itself.

Construction 3.20. The *toroidal compactification* of $\mathbb{D}_\rho/\Gamma$ is constructed as follows: Let Γ_K be the finite group defined by the exact sequence

$$0 \to U_K \to \mathrm{Stab}_K(\Gamma) \to \Gamma_K \to 0.$$

For each cusp K, quotient the toroidal extension

$$V_K := \overline{\mathbb{D}_\rho/U_K}^{\mathrm{can}}/\Gamma_K \supset \mathbb{D}_\rho/\mathrm{Stab}_K(\Gamma).$$

A well-known theorem states that there exists a horoball neighborhood $\mathbb{P}K_{\mathbb{C}} \in N_K \subset \mathbb{D}_\rho^{\mathrm{bb}}$ such that $(N_K \setminus \mathbb{P}K_{\mathbb{C}})/\mathrm{Stab}_K(\Gamma) \hookrightarrow \mathbb{D}_\rho/\Gamma$ injects. Thus, we can glue a neighborhood of the boundary $A_K/\Gamma_K \subset V_K$ to $\mathbb{D}_\rho/\Gamma$, ranging over all Γ-orbits of cusps K. The result is the toroidal compactification $\overline{\mathbb{D}_\rho/\Gamma}^{\mathrm{tor}}$.

The boundary divisors of $\overline{\mathbb{D}_\rho/\Gamma}^{\rm tor}$ are in bijection with Γ-orbits of isotropic $\mathbb{Z}[\zeta_n]$-lines $K \subset Q$ and the boundary divisor is isomorphic to A_K/Γ_K, where Γ_K acts by a subgroup of the finite group $U(K^\perp/K)$. There is a morphism

$$\overline{\mathbb{D}_\rho/\Gamma}^{\rm tor} \to \overline{\mathbb{D}_\rho/\Gamma}^{\rm bb}$$

which contracts each boundary divisor to a point. As such, the normal bundle of the boundary divisor is anti-ample. Passing to a finite index subgroup $\Gamma_0 \subset \Gamma$, we can assume that Γ_K is trivial for all cusps K and the anti-ampleness still holds. This proves that the normal bundle to $A_K \subset \overline{\mathbb{D}_\rho/U_K}^{\rm can}$ in the first partial quotient is anti-ample.

Using [Gra62] one shows that a divisor in a smooth analytic space, isomorphic to an abelian variety and with anti-ample normal bundle, can be contracted along any abelian subvariety. In particular, for any sub-$\mathbb{Z}[\zeta_n]$-lattice $\mathfrak{F}_K \subset K^\perp/K$, there is a contraction

$$\overline{\mathbb{D}_\rho/U_K}^{\rm can} \to \overline{\mathbb{D}_\rho/U_K}^{\mathfrak{F}_K}$$

which is an isomorphism away from the boundary divisor and contracts exactly the translates of the abelian subvariety $\operatorname{im}(\mathfrak{F}_K)_\mathbb{C} \subset A_K$.

To construct the semitoroidal compactification $\overline{\mathbb{D}_\rho/\Gamma}^{\mathfrak{F}}$, we wish to glue, at each cusp K, a punctured analytic open neighborhood of the boundary of $\overline{\mathbb{D}_\rho/U_K}^{\mathfrak{F}_K}/\Gamma_K$ to $\mathbb{D}_\rho/\Gamma$. This is only possible if the action of Γ_K on $\overline{\mathbb{D}_\rho/U_K}^{\rm can}$ descends along the above contraction. The condition in Definition 3.17 ensures that the collection $\mathfrak{F} = \{\mathfrak{F}_K\}$ is Γ-invariant. So an individual $\mathfrak{F}_K$ is Γ_K-invariant and the Γ_K action descends. Thus, we have constructed the semitoroidal compactification.

Remark 3.21. A feature of the construction is that one can pull back a semifan $\mathfrak{F}$ for a Type IV domain to any Type I subdomain, and there will be a morphism between the corresponding semitoric compactifications.

3F. **Recognizable divisors.** We recall the main new concept "recognizability" introduced in [AE21]. We slightly modify the definition as necessary for moduli spaces of K3 surfaces with ρ-markable automorphism:

Definition 3.22. A canonical choice of polarizing divisor R for $U \subset F_\rho$ is *recognizable* if for every Kulikov surface X_0 of Type I, II, or III which smooths to some ρ-markable K3 surface, there is a divisor $R_0 \subset X_0$ such that on *any* smoothing into ρ-markable K3 surfaces $X \to (C, 0)$ with $C^* \subset U$, the divisor R_0 is, up to the action of $\operatorname{Aut}^0(X_0)$, the flat limit of R_t for $t \neq 0 \in C^*$.

We use the term "smoothing" to mean specifically a Kulikov model $X \to (C, 0)$. Roughly, Definition 3.22 amounts to saying that the canonical choice R can also be made on any Kulikov surface, including smooth K3s.

Theorem 3.23. *If R is recognizable, then $\overline{F}_\rho^{\rm slc}$ is semitoroidal compactification of F_ρ for a unique semifan $\mathfrak{F}_R$.*

Proof. The proof when $n = 2$ is essentially the same as [AE21, Thm. 1.2]. So we restrict our attention to the Type I case $n > 2$, which is ultimately much simpler anyways. First, we show that $\overline{F}_\rho^{\rm slc}$ contains $\mathbb{D}_\rho/\Gamma_\rho$.

Let $\mathcal{M}_\rho^*$ be the closure of the moduli space of ρ-marked K3 surfaces $\mathcal{M}_\rho$ in the space of all marked K3 surfaces $\mathcal{M}$ and let $F_\rho^* = \mathcal{M}_\rho^*/\Gamma_\rho$ be the quotient.

Given any smooth K3 surface $X_0 \in F_\rho^* \setminus U$, the recognizability implies that the universal family $(\mathcal{X}^*, \mathcal{R}^*) \to U$ extends over F_ρ^* by the same argument as [AE21, Prop. 6.3]. Thus, the argument of Lemma 3.5 shows that there is a morphism $(F_\rho^*)^{\rm sep} = \mathbb{D}_\rho/\Gamma_\rho \to P_{2d,m}$ and so we may as well have constructed $\overline{F}_\rho^{\rm slc}$ by taking the normalization of the closure of the image of $\mathbb{D}_\rho/\Gamma_\rho$, which is notably already normal. This completes the proof when $n \neq 3,4,6$.

So let $\mathbb{P}K_\mathbb{C}$ be a Baily-Borel cusp of $\mathbb{D}_\rho$ when $n \in \{3,4,6\}$. We observe that the closure of $\mathbb{D}_\rho/U_K$ in the toroidal extension $\mathbb{D}(J) \subset \mathbb{D}(J)^\lambda$ of the "universal" first partial quotient for unpolarized K3 surfaces, cf. [AE21, Def. 4.18], is simply the first partial quotient $\overline{\mathbb{D}_\rho/U_K}^{\rm can}$. [AE21, Prop. 4.16] shows that $\mathbb{D}(J)$ embeds into a family of affine lines over $J^\perp/J \otimes_\mathbb{Z} \widetilde{\mathcal{E}}$ where $\widetilde{\mathcal{E}}$ is the universal elliptic curve over $\mathbb{H} \sqcup (-\mathbb{H})$ and $\mathbb{D}(J)^\lambda$ is its closure in a projective line bundle. The space $\mathbb{D}_\rho/U_K$ sits inside this affine line bundle as the inverse image of

$$K^{\perp \text{ in } Q}/K \otimes_{\mathbb{Z}[\zeta_n]} E \subset J^\perp/J \otimes_\mathbb{Z} \widetilde{\mathcal{E}}$$

where E is the elliptic curve admitting an action of ζ_n (note that $K = J$ but with the additional structure of a $\mathbb{Z}[\zeta_n]$-lattice).

Thus we may restrict a Type II λ-*family*, cf. [AE21, Def. 5.34], to a family

$$\mathcal{X} \to \overline{\mathbb{D}_\rho/U_K}^{\rm can}$$

of Kulikov surfaces of Types I + II. We call $\mathcal{X}$ a K-*family*. Note that any K-family admits a birational automorphism which is the action of the automorphism σ on the restriction of $\mathcal{X}$ to $(\mathbb{D}_\rho \setminus \Delta_\rho)/U_K$.

The arguments in [AE21, Secs. 6, 8], leading up to the proof of Theorem 1.2 of *loc. cit.* now all apply to K-families $\mathcal{X}$, showing that there is a sandwich of normal compactifications

$$\overline{\mathbb{D}_\rho/\Gamma_\rho}^{\rm tor} \to \overline{F}_\rho^{\rm slc} \to \overline{\mathbb{D}_\rho/\Gamma_\rho}^{\rm bb}.$$

Using that the normal image of an abelian variety is an abelian variety (a similar argument is used in [AE21, Thm. 7.18]), we conclude that there must exist a Γ_ρ-admissible semifan $\mathfrak{F}_R$ for which $\overline{F}_\rho^{\rm slc} = \overline{\mathbb{D}_\rho/\Gamma_\rho}^{\mathfrak{F}_R}$. □

3G. **The main theorem.**

Theorem 3.24. *Under the assumption* $(\exists g \geq 2)$, $R = C_1$ *is recognizable for* F_ρ. *The stable pair compactification* $\overline{F}_\rho^{\rm slc}$ *is a semitoroidal compactification of* $\mathbb{D}_\rho/\Gamma_\rho$.

Proof. By Theorem 3.23, the second statement follows from the first. Let $(X, R) \to (C, 0)$ be a Kulikov model with a flat family of divisors $R \subset X$ for which

1. there is an automorphism σ on $X^* \to C^*$ making $(X_t, \sigma_t) \in F_\rho$ for $t \neq 0$,
2. $R_t \subset \mathrm{Fix}(\sigma_t)$ is the fixed component of genus at least 2 for $t \neq 0$, and
3. $R_0 = \lim_{t\to 0} R_t$.

By [AE21, Prop. 6.12], it suffices to show that if we make a one-parameter deformation the smoothing of X_0 into F_ρ that keeps X_0 constant, the limiting curve R_0 does not deform, up to $\mathrm{Aut}^0(X_0)$.

The automorphism σ on the generic fiber of any smoothing defines a birational automorphism of X. Any two Kulikov models are related by an automorphism followed by a sequence of Atiyah flops of types 0, I, II along curves in X_0 which are either (-2)-curves or (-1)-curves on component(s) of X_0. As such, there are only countably many curves in X_0 along which it is possible to make an Atiyah flop, and this continues to be the case after a flop is made. Thus, up to conjugation by $\mathrm{Aut}^0(X_0)$, there are only countably many possibilities for the birational automorphism $\sigma_0 := \sigma|_{X_0} \colon X_0 \dashrightarrow X_0$.

Hence if $X_0 \hookrightarrow X$ and $X_0 \hookrightarrow \widetilde{X}$ are smoothings into F_ρ as above, we have $\widetilde{\sigma}_0 = \psi \circ \sigma_0 \circ \psi^{-1}$ for some $\psi \in \mathrm{Aut}^0(X_0)$.

Let $\{A_j\}$ be the countable set of curves in X_0 along which σ_0 can be indeterminate. Any such curve A_j is $\mathrm{Aut}^0(X_0)$-invariant. Let $A = \cup_j A_j$ be their union. Clearly, the limit divisor R_0 is contained in the union of $A \cup S$ where S is the closure of the fixed locus of σ_0 in its locus of determinacy. Similarly, $\widetilde{R}_0$ is contained in $A \cup \widetilde{S}$ and $\sigma_0(P) = P$ if and only if $\widetilde{\sigma}_0(\psi(P)) = \psi(P)$. Since the smoothing $\widetilde{X}$ is a deformation of the smoothing X and the limiting divisor of R varies continuously, we conclude that $\widetilde{R}_0 = \psi(R_0)$ and therefore R is recognizable. □

Proposition 3.25. *Any element $(\overline{X}, \epsilon\overline{R}) \in \overline{F}_\rho^{\mathrm{slc}}$ has an automorphism $\overline{\sigma} \in \mathrm{Aut}(\overline{X})$. Furthermore, $\overline{R} = \mathrm{Fix}(\overline{\sigma})$ and $\overline{\sigma}^*$ acts on $H^0(\overline{X}, \omega_{\overline{X}}) \cong \mathbb{C}$ by multiplication by ζ_n.*

Proof. As noted in Remark 3.6, any point in $F_\rho^{\mathrm{sep}} = (\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho$ corresponds to a pair $(\overline{X}, \overline{\sigma})$ of an ADE K3 surface with automorphism, for which $\overline{R} = \mathrm{Fix}(\overline{\sigma})$ is ample and the minimal resolution is ρ-markable. Then any boundary point $(\overline{X}_0, \epsilon\overline{R}_0) \in \overline{F}_\rho^{\mathrm{slc}}$ is a stable limit of such ADE K3 surface pairs $f \colon (\overline{X}, \epsilon\overline{R}) \to C$.

Since $\overline{R}_t$ is $\overline{\sigma}_t$-invariant and the canonical model is unique, $\overline{X}$ admits an automorphism $\overline{\sigma}$ whose fixed locus contains $\overline{R}_0$. In fact, $\mathrm{Fix}(\overline{\sigma}_0) = \overline{R}_0$: $\mathrm{Fix}(\overline{\sigma})$ is a Cartier divisor, and thus forms a flat family of divisors containing $\overline{R}$. But $\mathrm{Fix}(\overline{\sigma}_0)$ already contains the flat limit $\overline{R}_0$. The statement about $\omega_{\overline{X}_0}$ follows from the fact that $f_*\omega_{\overline{X}/C}$ is invertible (by Base Change and Cohomology, since $R^1 f_*\omega_{\overline{X}/C} = 0$) and $\overline{\sigma}_t^*$ acts by ζ_n on the generic fiber of this line bundle. □

4. Moduli of quotient surfaces

We refer the reader to [Kol13] for the notions appearing in the following definitions. The pair (Y, Δ) is called demi-normal if X satisfies Serre's S_2 condition, has double normal crossing singularities in codimension 1, and $\Delta = \sum d_i D_i$ is an effective Weil $\mathbb{Q}$-divisor with $0 < d_i \leq 1$ not containing any components of the double crossing locus of Y.

The following is [Kol13, Prop. 2.50(4)], using our adopted notations.

Proposition 4.1. *Étale locally, there is a one-to-one correspondence between*

(a) *Local demi-normal pairs $(y \in Y, \frac{n-1}{n}B)$ of index n, i.e. such that the divisor $nK_Y + (n-1)B$ is Cartier.*

(b) *Local demi-normal pairs $(\widetilde{y} \in \widetilde{Y})$ such that $K_{\widetilde{Y}}$ is Cartier, with a μ_n-action that is free on a dense open subset, and such that the induced action on $\omega_{\widetilde{Y}} \otimes \mathbb{C}(\widetilde{y})$ is faithful.*

Moreover, the pair $(Y, \frac{n-1}{n}B)$ is slc iff so is $\widetilde{Y}$.

The variety $\widetilde{Y}$ is called the local index-1 cover of the pair $(Y, \frac{n-1}{n}B)$. [Kol13, Sec. 2] also gives a global construction.

Theorem 4.2. *Let $(\overline{X}, \epsilon\overline{R}) \in \overline{F}_\rho^{\mathrm{slc}}$ and let $\pi \colon \overline{X} \to Y = \overline{X}/\mu_n$ be the quotient map with the branch divisor $B = f(\overline{R})$. Then*

(1) *$nK_Y + (n-1)B \sim 0$,*
(2) *B and $-K_Y$ are ample $\mathbb{Q}$-Cartier divisors,*
(3) *the pair $(Y, \frac{n-1+\epsilon}{n}B)$ is stable for any rational $0 < \epsilon \ll 1$, i.e. it has slc singularities and the $\mathbb{Q}$-divisor $K_Y + \frac{n-1+\epsilon}{n}B$ is ample.*

Vice versa, for a pair (Y, B) satisfying the above conditions, its index-1 cover $\overline{X}$ with the ramification divisor $\overline{R}$ satisfies:

(1) *$K_{\overline{X}} \sim 0$ and the μ_n-action on $\overline{X}$ is non-symplectic,*
(2) *$\overline{R}$ is $\mathbb{Q}$-Cartier,*
(3) *the pair $(\overline{X}, \epsilon\overline{R})$ is stable for any rational $0 < \epsilon \ll 1$.*

Proof. Follows from the above Proposition 4.1 and the formulas

$$\pi^*(B) = n\overline{R}, \qquad \pi^*\left(K_Y + \frac{n-1+\epsilon}{n}B\right) = K_{\overline{X}} + \epsilon\overline{R}.$$

□

Corollary 4.3. *The coarse moduli space $\overline{F}_\rho^{\mathrm{slc}}$ coincides with the normalization of the KSBA compactification of the irreducible component in the moduli space of the log canonical pairs $(Y, \frac{n-1+\epsilon}{n}B)$ of log del Pezzo surfaces Y with $(n-1)B \in |-nK_Y|$ in which a generic surface is a quotient of a K3 surface with a non-symplectic automorphism of type ρ. The stack for the former is a μ_n-gerbe over the stack for the latter.*

For the proof, we note that a small deformation of a K3 surface is a K3 surface.

Example 4.4. The KSBA compactification moduli of K3 surfaces of degree 2 for the ramification divisor R constructed in [AET19] is equivalent to the Hacking's compactification [Hac04] of the moduli space of pairs $(\mathbb{P}^2, \frac{1+\epsilon}{2}B_6)$ of plane sextic curves.

References

[ABE20] Valery Alexeev, Adrian Brunyate, and Philip Engel, *Compactifications of moduli of elliptic K3 surfaces: stable pair and toroidal*, Geom. and Topology, to appear (2020), arXiv:2002.07127.

[AE21] Valery Alexeev and Philip Engel, *Compact moduli of K3 surfaces*, Submitted (2021), arXiv:2101.12186.

[AET19] Valery Alexeev, Philip Engel, and Alan Thompson, *Stable pair compactification of moduli of K3 surfaces of degree 2*, arXiv:1903.09742.

[AS08] Michela Artebani and Alessandra Sarti, *Non-symplectic automorphisms of order 3 on K3 surfaces*, Math. Ann. **342** (2008), no. 4, 903–921.

[AS15] ———, *Symmetries of order four on K3 surfaces*, J. Math. Soc. Japan **67** (2015), no. 2, 503–533.

[ast85] *Géométrie des surfaces K3: modules et périodes*, Société Mathématique de France, Paris, 1985, Papers from the seminar held in Palaiseau, October 1981–January 1982, Astérisque No. 126 (1985).

[AST11] Michela Artebani, Alessandra Sarti, and Shingo Taki, *K3 surfaces with non-symplectic automorphisms of prime order*, Math. Z. **268** (2011), no. 1-2, 507–533, With an appendix by Shigeyuki Kondō.

[BB66] W. L. Baily, Jr. and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. (2) **84** (1966), 442–528.

[DH21] Anand Deopurkar and Changho Han, *Stable quadrics, admissible covers, and Kondō's sextic K3 surfaces*, In preparation, 2021.

[Dil09] Jimmy Dillies, *Order 6 non-symplectic automorphisms of K3 surfaces*, arXiv:0912.5228 (2009).

[Dil12] ______, *On some order 6 non-symplectic automorphisms of elliptic K3 surfaces*, Albanian J. Math. **6** (2012), no. 2, 103–114.

[DK07] Igor V. Dolgachev and Shigeyuki Kondō, *Moduli of $K3$ surfaces and complex ball quotients*, Arithmetic and geometry around hypergeometric functions, Progr. Math., vol. 260, Birkhäuser, Basel, 2007, pp. 43–100.

[Dol96] I. V. Dolgachev, *Mirror symmetry for lattice polarized $K3$ surfaces*, J. Math. Sci. **81** (1996), no. 3, 2599–2630, Algebraic geometry, 4.

[Fri84] Robert Friedman, *A new proof of the global Torelli theorem for $K3$ surfaces*, Ann. of Math. (2) **120** (1984), no. 2, 237–269.

[FS86] Robert Friedman and Francesco Scattone, *Type* III *degenerations of $K3$ surfaces*, Invent. Math. **83** (1986), no. 1, 1–39.

[Gra62] Hans Grauert, *Über Modifikationen und exzeptionelle analytische Mengen*, Math. Ann. **146** (1962), 331–368.

[Hac04] P. Hacking, *Compact moduli of plane curves*, Duke Math. J. **124** (2004), no. 2, 213–257.

[Kol13] János Kollár, *Singularities of the minimal model program*, Cambridge Tracts in Mathematics, vol. 200, Cambridge University Press, Cambridge, 2013, With a collaboration of Sándor Kovács.

[Kol21] ______, *Families of varieties of general type*, To appear, 2021, availabe at https://web.math.princeton.edu/∼kollar/.

[Kon20] Shigeyuki Kondō, *K3 surfaces*, vol. 32, EMS Tracts in Mathematics, 2020.

[Kul77] Vik. S. Kulikov, *Degenerations of $K3$ surfaces and Enriques surfaces*, Izv. Akad. Nauk SSSR Ser. Mat. **41** (1977), no. 5, 1008–1042, 1199.

[Loo03a] Eduard Looijenga, *Compactifications defined by arrangements. I. The ball quotient case*, Duke Math. J. **118** (2003), no. 1, 151–187.

[Loo03b] ______, *Compactifications defined by arrangements. II. Locally symmetric varieties of type IV*, Duke Math. J. **119** (2003), no. 3, 527–588.

[Mat16] Yuya Matsumoto, *Degeneration of K3 surfaces with non-symplectic automorphisms*, arXiv:1612.07569 (2016).

[MO98] Natsumi Machida and Keiji Oguiso, *On $K3$ surfaces admitting finite non-symplectic group actions*, J. Math. Sci. Univ. Tokyo **5** (1998), no. 2, 273–297.

[Nik79a] V. V. Nikulin, *Finite groups of automorphisms of Kählerian $K3$ surfaces*, Trudy Moskov. Mat. Obshch. **38** (1979), 75–137.

[Nik79b] ______, *Integer symmetric bilinear forms and some of their geometric applications*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 1, 111–177, 238.

[PP81] Ulf Persson and Henry Pinkham, *Degeneration of surfaces with trivial canonical bundle*, Ann. of Math. (2) **113** (1981), no. 1, 45–66.

[PSS71] I. I. Pjateckiĭ-Shapiro and I. R. Shafarevič, *Torelli's theorem for algebraic surfaces of type* K3, Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 530–572.

Email address: `valery@uga.edu`

Email address: `philip.engel@uga.edu`

Email address: `changho.han@uga.edu`

Department of Mathematics, University of Georgia, Athens GA 30602, USA