ARITHMETIC INFLECTION FORMULAE FOR LINEAR SERIES ON HYPERELLIPTIC CURVES

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Abstract. Over the complex numbers, *Plücker's formula* computes the number of inflection points of a linear series of fixed degree and projective dimension on an algebraic curve of fixed genus. Here we explore the geometric meaning of a natural analogue of Plücker's formula and its constituent local indices in $\mathbb{A}^1$-homotopy theory for certain linear series on hyperelliptic curves defined over an arbitrary field.

1. Introduction

1.1. **Inflection of linear series on algebraic curves.** The *inflectionary index* of a linear series of rank r and degree d in a point p of an algebraic curve, as measured by the total deviation of the vanishing orders of global sections in p from the generic sequence $(0, 1, \ldots, r)$, is a fundamental local invariant. Over the complex numbers, a classical formula of Plücker expresses the sum of inflectionary indices as p varies, as an explicit polynomial in r, d, and the genus g of the underlying curve. Plücker's inflection formula plays a foundational role in the Eisenbud and Harris' reworking of the *Brill–Noether theorem* for very general curves of genus g; see [10, Prop. 1.1]. Plücker's formula notwithstanding, the behavior of the individual inflectionary indices that contribute to the inflection of a given series are quite mysterious, even over $\mathbb{C}$; indeed, the stratification of curves in moduli according to the local inflectionary behavior of their canonical series is ongoing [11, 22]. If we furthermore replace $\mathbb{C}$ by a non-algebraically closed field, it is natural to wonder whether an analogue of Plücker's formula is still operative, and how we may interpret its constituent inflectionary indices. The answer to the first question is often "yes", and $\mathbb{A}^1$-homotopy theory provides the tools both to produce a conserved global quantity à la Plücker and to interpret its local contributions.

1.2. **The $\mathbb{A}^1$-homotopy machine.** $\mathbb{A}^1$-homotopy theory was originally developed by Morel, Voevodsky and others as a purely algebraic approach to homotopy theory modeled on Grothendieck's approach to algebraic geometry. Recently, J. Kass and K. Wickelgren, M. Levine, and M. Wendt have applied $\mathbb{A}^1$-homotopy theory to spectacular effect to investigate enumerative algebraic geometry over fields F other than $\mathbb{C}$. In this paper, we will apply the Kass–Wickelgren program, as developed in [15, 16], to investigate the inflection of linear series on hyperelliptic curves. To this end, we will exhibit inflection as the Euler class of a vector bundle $\mathcal{E}$, and then show that $\mathcal{E}$ is (relatively) orientable. These conditions imply that a well-defined $\mathbb{A}^1$-*inflection class* exists in the Grothendieck–Witt group of F, which we review below.

1.3. **Inflection as an Euler class.** Let X denote a smooth projective curve of genus g. A g^r_d on X is a linear series comprised of a degree-d line bundle L, together with an $(r+1)$-dimensional subspace of holomorphic sections $V \subset H^0(X, L)$. By the *inflection divisor* of (L, V) on X we mean the nonsurjective locus of the natural evaluation morphism

$$V \otimes \mathcal{O} \to J^{r+1}(L) \tag{1}$$

where $J^{r+1}(L)$ is the principal parts bundle with fibers $H^0(L/L(-(r+1)p))$, $p \in X$. Taking top exterior powers in (1), the inflection divisor is precisely the zero locus of a nonvanishing section $s \in H^0(\mathcal{E})$, where $\mathcal{E} := \det J^{r+1}(L)$. Thus the class of this divisor is $c_1(\mathcal{E})$, i.e. the Euler class of $\mathcal{E}$. A useful standard fact is that $J^{r+1}(L)$ and $J^r(L)$ are related by an exact sequence, namely

$$0 \to L \otimes K_X^{\otimes r} \to J^{r+1}(L) \to J^r(L) \to 0$$

where K_X is the canonical bundle of X; it follows that $\mathcal{E} = L^{\otimes(r+1)} \otimes K_X^{\otimes\binom{r+1}{2}}$.

1.4. **Relative orientability.** Recall from [16] that a vector bundle $\mathcal{F}$ on an algebraic variety Y is *relatively orientable* when $\mathrm{Hom}(\det T_Y, \det \mathcal{F})$ is a tensor square. When $Y = X$ is a curve and $\mathcal{F} = \mathcal{E}$ is as above, the corresponding hom-bundle is isomorphic to $L^{\otimes(r+1)} \otimes K_X^{\otimes(\binom{r+1}{2}+1)}$. Thus for arbitrary choices of curve X and linear series (L, V), the hom-bundle is not relatively orientable. However, relative orientability is guaranteed in many situations of interest.

1.5. **Inflection of linear series on hyperelliptic curves.** In this paper, the primary situation of interest will be that in which X is *hyperelliptic* and L is the line bundle obtained from any multiple of the g^1_2 on X. Relative orientability is then trivial, as the the hyperelliptic projection $\pi : X \to \mathbb{P}^1$ is of degree 2 and both L and K_X are pullbacks of line bundles on $\mathbb{P}^1$. Note that when $X = E$ is of genus *one*, inflection points of any complete linear series of degree N are in bijection with N-torsion points on E; so the study of inflection on X generalizes the study of torsion on E. The analogy between inflection and torsion is itself not new (see, e.g., [25]), though to our knowledge it hasn't been examined systematically through the lens of $\mathbb{A}^1$-homotopy theory.

Our strategy for calculating inflection on a given hyperelliptic curve is a two-step process, which involves separately computing contributions arising from the hyperelliptic ramification locus and its complement. Roughly speaking, inflection that arises from the ramification locus is constant in moduli, while inflection supported away from the ramification locus varies nontrivially when the underlying (marked) curve moves. This in turn leads naturally to what we call *inflectionary varieties* associated to any given flat family of linear series on marked hyperelliptic curves, whose points parameterize those inflection points supported away from the ramification locus. They are analogues of modular curves $X_1(N)$ that parameterize pairs $((E, 0), p)$ of elliptic curves $(E, 0)$ together with N-torsion points $p \in E[N]$. The study of rational points on $X_1(N)$ is a classical and enduring area of inquiry; here we initiate an experimental study of rational points on inflectionary curves derived from Legendre and Weierstrass pencils of elliptic curves.

1.6. **The Grothendieck–Witt group.** We will be interested in the class of $e(\mathcal{E})$ in the Grothendieck–Witt group $\mathrm{GW}(F)$ of an arbitrary field F. Here $\mathrm{GW}(F)$ is the (additive) groupification of the monoid of symmetric nondegenerate bilinear forms. It is worth recalling here that $\mathrm{GW}(F)$ has an explicit presentation in terms of generators and relations; see [18]. We use $\langle a \rangle$ to denote the class of $a \in F$. The group $\mathrm{GW}(F)$ contains a distinguished *hyperbolic form* $\mathbb{H} := \langle 1 \rangle + \langle -1 \rangle$.

In the classical situation, we have $\mathrm{GW}(\mathbb{C}) \cong \mathbb{Z}$, which reflects the fact that any quadratic form over the complex numbers is determined up to isomorphism by its rank. The nonclassical situations of primary interest to us in the sequel will be $F = \mathbb{R}$ and $F = \mathbb{F}_q$, in which $q = p^n$ is a finite prime power. We note that $\mathrm{GW}(\mathbb{R}) \cong \mathbb{Z}^2$ (resp., $\mathrm{GW}(\mathbb{F}_q) \cong \mathbb{Z} \times \mathbb{F}_q^*/(\mathbb{F}_q^*)^2$); indeed, quadratic forms over the real numbers (resp., over a finite field) are characterized by rank and signature (resp., rank and discriminant, modulo squares).

1.7. **Enriched Euler classes and local indices.** According to [6, Thm 1.1], the class in $\mathrm{GW}(F)$ of the Euler class $e(\mathcal{E})$ may be recovered as a sum

$$e(\mathcal{E}) = \sum_{p : s(p) = 0} \mathrm{ind}_p(s)$$

of local Euler indices $\mathrm{ind}_p(s)$ over (all points of) the vanishing locus of the section $s \in H^0(\mathcal{E})$ described above, provided s has isolated zeroes. In particular, $e(\mathcal{E})$ is independent of the particular section s chosen. It will turn out that our global Euler classes are themselves uninteresting inasmuch as they are uniform, whereas the local indices reflect the features of our particular choice of base field F.

1.8. **Calculating local Euler indices via local charts and trivializations.** Computing the local Euler indices $\mathrm{ind}_p(s)$ that contribute to $e(\mathcal{E})$, in turn, is a three-step procedure. We begin by calculating a *local Wronskian* expression for the determinant of (1) in an étale chart of the inflection point p in question. For $\mathbb{A}^1$-homotopy theory, we need to work in the more refined Nisnevich topology, however, so in a second step we rewrite our local Wronskian in terms of a Nisnevich uniformizer, using standard facts about how Wronskians transform under changes of coordinates. In practice, the étale charts arise from projections to the coordinate axes, while the Nisnevich charts are associated with generic projections. Finally, we apply a linear algebraic result originally due to Scheja and Storch (we follow [17]) to extract $\mathrm{ind}_p(s)$ from our Nisnevich local Wronskian. The output of this procedure is a *trace* of a certain class in $\mathrm{GW}(k(p))$, where $k(p)$ is the splitting field of p and the trace is canonically induced by the usual field trace of $k(p)$ over F.

1.9. **Outline.** The paper following this introduction is organized as follows. We work over an arbitrary field of characteristic not equal to 2. In section 2, we carry out an enriched count of the ramification points of the g^1_2 on a hyperelliptic curve X, which corresponds to the special case when $r = 1$. A particular feature of this case is that producing compatible Nisnevich charts is easy. We then proceed to the general case and in section 3 we carry out an enriched count of inflection for arbitrary multiples of the g^1_2 on X. Theorem 3.1 is a Plücker-type formula for the class of the inflection locus of an arbitrary multiple ℓg^1_2 of the g^1_2 on a hyperelliptic curve. The remainder of section 3 is devoted to the calculation of the local Euler indices that contribute to the global inflection class. Lemma 3.2 is a now-standard general result that characterizes the local Euler index at an isolated zero of arbitrary multiplicity.

Section 3.5 is the technical core of this paper, in which we compute local Euler multiplicities at points of the ramification locus R_π of the hyperelliptic projection π. We use *Hasse* derivatives, as these behave better than usual derivatives in positive characteristic, to compute local Wronskians. Theorems 3.7 and 3.12 (resp., Theorems 3.9 and 3.13) give explicit formulas for the Hasse Wronskian in terms of a Nisnevich (resp., étale) uniformizer at a point of R_π whenever $\ell \leq g$ (resp., $\ell > g$), which we subsequently promote in Theorems 3.8 and 3.14 (resp., Theorems 3.11 and 3.15) to explicit expressions for the corresponding local Euler indices; when $\ell > g$, "explicit" means a Gessel–Viennot count of nonintersecting lattice paths in the plane. In subsection 3.6, we generalize (and redefine) the *inflection polynomials* studied in [7] in terms of Hasse derivatives; their roots parameterize the x-coordinates of $\overline{F}$-inflection points away from R_π whenever $\ell > g$, and from them it is straightforward to read off the corresponding local Euler indices. The case $\ell = g + 1$ is distinguished in that the corresponding *atomic* inflection polynomials are prescribed recursively (Proposition 3.17), while general inflection polynomials are determinants whose entries are atomic inflection polynomials (Proposition 3.16).

Finally, in section 4, we give concrete interpretations of our formulas for local Euler indices over $\mathbb{R}$, $\mathbb{F}_q$, and $\mathbb{C}((t))$; and we study inflectionary curves (defined by inflectionary polynomials) derived from Lefschetz and Weierstrass pencils of elliptic curves. In Conjecture 4.1 (resp., Proposition 4.2) we give some explicit speculations regarding the F-rationality loci of these curves when $F = \mathbb{R}$ (resp., $F = \mathbb{F}_q$).

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2. Inflection of the hyperelliptic projection

Let X denote a smooth hyperelliptic curve of genus g. By definition, X comes equipped with a degree-2 morphism $\pi : X \to \mathbb{P}^1$. It is convenient to assume the projection π is ramified over $\infty \in \mathbb{P}^1$;

away from ∞, then, X is given by an affine equation

$$y^2 = f(x) \tag{2}$$

in which $\deg_x(f) = 2g+1$ and π becomes the projection $(x, y) \mapsto x$. As a matter of notation, we also set $\infty_X := \pi^{-1}(\infty)$. The line bundle L from Section 1 is then $L = \mathcal{O}(2\infty_X)$, and the space of sections that defines π is $V = H^0(X, L)$. So the inflectionary locus of the pair (L, V) is exactly the ramification locus of π.

In [19, §12] and [3], $\mathrm{ind}_p(s)$ and $e(\mathcal{E})$ were deduced by comparing the canonical bundle of X against that of $\mathbb{P}^1$. Here, we instead use an explicit calculation using local coordinate charts. An advantage of our method is that it generalizes readily to calculate local inflectionary indices associated with linear series of higher rank.

2.1. **Local Euler indices away from ∞.** To calculate local inflectionary indices, we carry out calculations in *Nisnevich* local charts. These are étale charts $\varphi : U \to \mathbb{A}^1$ in which $U \subset X$ is open, and over which there is an isomorphism between the residue fields of (each point in the) fiber and target. We will refer to the latter requirement as the *Nisnevich condition*.

We begin by calculating local indices away from the ramification locus R_π of π. This means precisely that our affine coordinate y is nonvanishing. Accordingly, we choose our chart φ to be the (restriction of) the hyperelliptic projection $U_y \to \mathbb{A}^1_x$, where $U_y := (y \neq 0) \subset X \setminus \{\infty_X\}$. Since local sections of $\mathcal{O}(2\infty_X)$ are holomorphic away from ∞, the map $\mathcal{O}_{\mathbb{A}^1} \to \mathcal{O}_{U_y}$ induced by φ is simply the constant map induced by the assignment $1 \mapsto 1$, so $\mathcal{O}_X(2\infty_X)$ trivializes in this chart as $\langle 1 \rangle$. Similarly, we have $K_{U_y} = \langle dx \rangle$ and $V \otimes \mathcal{O}_{U_y} = \langle 1 \rangle \oplus \langle x \rangle$. It follows that the evaluation morphism (1) trivializes as the matrix

$$\begin{array}{cc} \mathrm{ev} & \begin{array}{cc} 1 & x \end{array} \\ \begin{array}{c} 1 \\ 1 \cdot dx \end{array} & \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \end{array}$$

which has determinant 1. Consequently, the local inflectionary indices along U_y are all zero.

We now turn to the local index calculation along R_π. This time, to ensure that the Nisnevich condition is met, we use a generic projection. Specifically, let φ be the restriction to an open neighborhood U in X of the projection from the point $(1 : b : 0)$ in the $\mathbb{P}^2$ obtained by compactifying X by homogenizing (2) with respect to z, where $b \in F$. Fix a choice $(\gamma, 0) \in R_\pi$ of ramification point. Now set $u_b := y - bx$. Our local chart $U \to \mathbb{A}^1$ is then given by $(x, y) \mapsto u_b$, and accordingly our affine hyperelliptic equation (2) becomes

$$(u_b + bx)^2 - f(x) = 0. \tag{3}$$

We will in fact trivialize the evaluation morphism (1) over the completed local rings of the source and target. To this end, note that $\mathcal{O}_X(2\infty_X)$ trivializes in this chart as $\langle 1 \rangle$, while K_X trivializes as $\langle du_b \rangle$. Accordingly, the evaluation morphism in this chart is of the form

$$\begin{array}{cc} \mathrm{ev} & \begin{array}{cc} 1 & x \end{array} \\ \begin{array}{c} 1 \\ 1 \cdot du_b \end{array} & \begin{pmatrix} 1 & x \\ 0 & \frac{dx}{du_b} \end{pmatrix} \end{array} \tag{4}$$

Next, we differentiate (3) with respect to u_b. The result is

$$\frac{dx}{u_b} = \frac{2(u_b + bx)}{f'(x) - 2b(u_b + bx)} \tag{5}$$

where $f' = \frac{df}{dx}$. In particular, differentiating (5) a second time with respect to u_b and evaluating in $x = \gamma, y = 0$ yields precisely $\frac{2}{f'(\gamma)}$. It follows that the *Jacobian* determinant of (4) evaluated in $x = \gamma, y = 0$ is precisely $\frac{2}{f'(\gamma)}$, and applying the main result of [5] we deduce that the corresponding local inflectionary index is $\mathrm{Tr}_{k(\gamma)/F}\langle \frac{2}{f'(\gamma)} \rangle$, which yields $\mathrm{Tr}_{k(\gamma)/F}\langle 1 \rangle$ whenever $\frac{2}{f'(\gamma)}$ is a square in $k(\gamma)$.

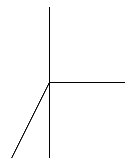

FIGURE 1. The fan of $\mathbb{F}_d$, with rays $u_1 = e_1$, $u_2 = e_2$, $u_3 = -e_1 - de_2$, $u_4 = -e_2$. Here $\tau_1 = \mathrm{Span}\{u_1, u_2\}$, $\tau_2 = \mathrm{Span}\{u_2, u_3\}$, and $d = 2$.

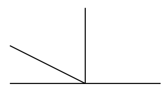

FIGURE 2. The dual fan of $\mathbb{F}_d$, with rays $x = e_1$, $y = e_2$, $z = -e_1$, $w = -de_1 + e_2$. Here $\tau_1^\vee = \mathrm{Span}\{x, y\}$ and $\tau_2^\vee = \mathrm{Span}\{z, w\}$.

2.2. **Local Euler indices around ∞.** To compute the local inflectionary index at the point ∞_X, we will use an alternative point of view that is interesting in its own right. Any fact on toric varieties in this subsection is standard; we follow [9]. Our point of departure is that X naturally embeds in a Hirzebruch surface, namely $\mathbb{F}_d = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(D))$, where $2D = B$ is the branch divisor of the hyperelliptic structure map π and $d = \deg D = g + 1$. The cone of curves on $\mathbb{F}_d$ is generated by (the classes of) a fiber f and by the unique section σ with self-intersection $-d$. The adjunction formula implies that $X \sim 2(\sigma + df)$, so in particular X and σ do not intersect. Accordingly, it suffices to work locally over those two toric local coordinate charts U_{τ_1}, U_{τ_2} of $\mathbb{F}_d$ corresponding to the cones τ_1 and τ_2, as in Figure 1.

Passing to the dual fan as in Figure 2, we see that $U_{\tau_1} \cong \mathbb{A}^2_{x,y}$ and $U_{\tau_2} \cong \mathbb{A}^2_{z,w}$. Now let D_{u_i}, $i = 1, 2$ denote the divisors of $\mathbb{F}_d$ determined by the rays u_i of the fan. Since $z = x^{-1}$ and $w = x^{-d}y$, the transition map $\varphi : U_{\tau_1} \backslash D_{u_1} \to U_{\tau_2} \backslash D_{u_1}$ is given by $\varphi(x, y) = (x^{-1}, yx^{-d})$.

Now let $\theta := \sigma + df$. Since X is given by a vanishing locus of a section of $\mathcal{O}(2\theta)$, we also need to trivialize the bundle $\mathcal{O}(\theta)$ over U_{τ_2}. Since $D_{u_2} \sim \theta$, it is easy to see that the transition map of $\mathcal{O}(\theta)$ corresponding to φ is given by multiplication by z^d. Similarly, the transition map for $\mathcal{O}(f)$ (which restricts to $\mathcal{O}_X(2\infty)$) is given by multiplication by z.

It follows from the preceding discussion that X is given in $U_{\tau_2} \cong \mathbb{A}^2_{z,w}$ by

$$w^2 - z(z^{2g+1} f(z^{-1})) = 0 \tag{6}$$

as $d = g + 1$. Now let $h(z) := z(z^{2g+1} f(z^{-1}))$. To calculate the local index $\mathrm{ind}_\infty(s)$ at ∞, choose a local coordinate chart $(X \cap U_{\tau_2}) \backslash V(dw) \to \mathbb{A}^1_w$ obtained by restriction from the projection $\mathbb{A}^2_{z,w} \to \mathbb{A}^2_w$. Note that $\mathcal{O}_X(2\infty_X)$ trivializes in this chart as $\langle z \rangle$, while K_X trivializes as $\langle dw \rangle$. The evaluation morphism in this chart is then of the form

$$\begin{array}{cc} \mathrm{ev} & \begin{array}{cc} z & 1 \end{array} \\ \begin{array}{c} 1 \\ 1 \cdot dw \end{array} & \begin{pmatrix} z & 1 \\ \frac{dz}{dw} & 0 \end{pmatrix}. \end{array} \tag{7}$$

Differentiating (6) with respect to dw then dividing by $h'(z)$, we conclude that

$$\frac{dz}{dw} = \frac{2w}{h'(z)}.$$

Since the section s is the Wronskian of the evaluation map, s is given in local coordinates by the determinant of (7), which is $-\frac{2w}{h'(z)}$. Consequently, the local Euler index at ∞ is given by

$$\mathrm{ind}_\infty(s) = \left\langle \mathrm{Jac}_{w=0}\left(-\frac{2w}{h'(z)}\right)\right\rangle = \left\langle -\frac{2}{h'(0)} \right\rangle.$$

Note that whenever $f(z) = \sum a_i z^i$, we have $h(z) = a_{2g+1}z + a_{2g}z^2 + \cdots + a_0 z^{2g+2}$, so $h'(0) = a_{2g+1}$ is the leading coefficient of f.

2.3. **Compatibility of Nisnevich charts.** Finally, we need to check that the local Euler indices obtained by the choices of Nisnevich local coordinates and local trivializations of $\mathcal{E} = \mathcal{L}^{\otimes 2} \otimes K_X$ as above are consistent with the relative orientation described in §1.4. To explain how this works, assume that X is a smooth projective variety X of dimension $d > 0$, that $\mathcal{E}$ is a vector bundle of rank d on X, and that $(X, \mathcal{E})$ is relatively orientable, i.e., equipped with a (fixed) isomorphism $\mathrm{Hom}(\det T_X, \det \mathcal{E}) \to \mathcal{L}^{\otimes 2}$. Given a local Nisnevich chart $X \supset U \to \mathbb{A}^d$, let $\psi : \mathcal{E}|_U \to \mathcal{O}_U^d$ denote the corresponding trivialization of $\mathcal{E}$. Our Nisnevich chart and the associated trivialization ψ of $\mathcal{E}$ determine *distinguished bases* on $T_X|_U$ (respectively, on $\mathcal{E}|_U$) induced by the standard coordinate bases on $T_{\mathbb{A}^d}$ (respectively, on $\mathcal{O}_U^d$). The following *Nisnevich compatibility* condition was introduced in [16, Def. 19].

Definition 2.1. *Given a Nisnevich local coordinate ϕ and a local trivialization ψ as above, ψ is compatible with ϕ if the element of Hom$(\det T_X, \det \mathcal{E})$ sending the distinguished basis of $\det T_X$ to that of $\det \mathcal{E}$ is a square, under the identification Hom$(\det T_X, \det \mathcal{E}) \to \mathcal{L}^2$ prescribed by the relative orientation.*

In practice, it is equivalent to verify the compatibility conditions of various local trivializations by checking that the transition maps of $\mathrm{Hom}(\det T_X, \det \mathcal{E})$ between pairs of Nisnevich local coordinates and local trivializations are squares; see, e.g., [16, Ex. 31]. This is because the collection of such trivializations determines the identification $\mathrm{Hom}(\det T_X, \det \mathcal{E}) \to \mathcal{L}^2$ up to squares of scalars.

We now claim that in the problem at hand, namely when $L = \mathcal{O}(2\infty_X)$ and $V = H^0(X, L)$, there are natural local trivializations associated to the local Nisnevich coordinates prescribed by generic projections, and the transition maps between them are indeed squares. Here the trivialization of $\mathcal{E} \cong \det(J^2(X)) \cong L^{\otimes 2} \otimes K_X$ is naturally induced from those of L and K_X, in which the trivialization of K_X is the dual of the standard trivialization of T_X introduced previously.

Indeed, let's begin by comparing local trivializations away from ∞. According to §2.1, given a local chart $X \supset U \to \mathbb{A}_t^1$ induced by a linear projection from $\mathbb{A}_{x,y}^2$, $\mathrm{Hom}(\det T_X, \det \mathcal{E})$ trivializes as $\langle dt \times 1^2 \times dt \rangle = \langle dt^{\otimes 2} \rangle$. So clearly, the transition map between any two local charts induced by linear projections is a square of the change-of-basis multiplier that relates differentials in each respective set of local coordinates. More precisely, to pass from the a chart with local coordinate t to a chart with local coordinate $\tilde{t}$, we multiply by $\left(\frac{dt}{d\tilde{t}}\right)^2$.

Finally, it remains to show that the transition map between a local trivialization induced by a linear projection as above and the local trivialization at ∞ as described in §2.2 is a square. To this end, we compare $U_y \to \mathbb{A}_y^1$ against $(X \cap U_{\tau_2}) \backslash V(dw) \to \mathbb{A}_w^1$. In this case, $\mathrm{Hom}(\det T_X, \det \mathcal{E})$ trivializes as $\langle dx^{\otimes 2} \rangle$ on U_y and $\langle z^2 dw^{\otimes 2} \rangle$ on $(X \cap U_{\tau_2}) \backslash V(dw)$. An easy adaptation of the argument used in the preceding paragraph shows that the corresponding transition map is a square.

3. Arithmetic inflection formulae

3.1. A global arithmetic Euler class.

Theorem 3.1. *Let F be a field with $\mathrm{char}(F) \neq 2$, let $\ell \geq 1$ be a positive integer, and let $L = \mathcal{O}(2\ell\infty_X)$, where $\pi : X \to \mathbb{P}^1$ is a hyperelliptic curve of genus g defined over F and ramified in $\infty_X = \pi^{-1}(\infty)$. Associated to the complete linear series $|L|$ on X there is a well-defined arithmetic F-inflection class in the Grothendieck–Witt group $GW(F)$ given by*

$$[\mathrm{Inf}_F(|L|)]_{\mathbb{A}^1} = \frac{\gamma_{\mathbb{C}}}{2} \cdot \mathbb{H}$$

where $\gamma_{\mathbb{C}} := g(2\ell - g + 1)^2$ is the $\mathbb{C}$-inflectionary degree computed by the usual Plücker formula.

Proof. The result follows from the fact that ours is the Euler class of a vector bundle of *odd* rank; see [26, proof of Prop. 19], and formally treat every instance of the first bundle summand as 0. $\square$

3.2. **Local Euler indices in the general case.** In subsection 1.7, we saw that the global Euler class is a sum of local Euler indices. Often, local Euler indices encode subtle geometric data. Here, we explain how to compute local Euler indices using Nisnevich charts.

Suppose, then, that $\mathcal{E}$ is a line bundle on a smooth curve X, equipped with a local Nisnevich chart near an isolated zero p of a section σ of $\mathcal{E}$, say with multiplicity m. When the residue field of p is F, the following proposition characterizes the corresponding local Euler index.

Lemma 3.2. *Let F be a field of characteristic $\neq 2$. Suppose that $\sigma \in H^0(\mathcal{O}_{\mathbb{A}^1_F})$ is given by $x^m(a + xg(x))$, for some $a \in F^*$, $m \in \mathbb{N}$ and $g(x) \in F[x]$; then*

$$\mathrm{ind}_0\sigma = \begin{cases} \frac{m}{2} \cdot \mathbb{H} & \text{if } m \text{ is even} \\ \frac{m-1}{2} \cdot \mathbb{H} + \langle a \rangle & \text{otherwise.} \end{cases}$$

Proof. Applying [17, Cor. 8], it suffices to find the class in $\mathrm{GW}(F)$ of a bilinear form associated to the local Newton matrix $\mathrm{New}(x^m(a+xg(x)), 0)$ from [17, Def. 7]; since $x^m(a + xg(x))$ is equal to a unit times ax^m in the local ring $F[x]_{(x)}$, [17, Def. 7] implies that $\mathrm{New}(x^m(a+xg(x)), 0) = \mathrm{New}(ax^m, 0)$ is the following m by m matrix:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & a^{-1} \\ 0 & 0 & \cdots & a^{-1} & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots \\ 0 & a^{-1} & \cdots & 0 & 0 \\ a^{-1} & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

By [17, Lemma 6], the class in $\mathrm{GW}(F)$ of $\mathrm{New}(ax^m, 0)$ matches with the statement of Lemma 3.2 except that a is replaced with a^{-1}; we conclude using $\langle a^{-1} \rangle = \langle a^2 \cdot a^{-1} \rangle = \langle a \rangle$. $\square$

Remark 3.3. *When $F = \mathbb{R}$, we have $\langle a \rangle = \langle 1 \rangle$ if $a > 0$, and $\langle a \rangle = \langle -1 \rangle$ otherwise. In this case, the local index recovers Milnor's real oriented index. Indeed, suppose that we have a smooth function $f : \mathbb{R}^n \to \mathbb{R}^n$ and $q \in \mathbb{R}^n$ is a regular value of f (Sard's theorem implies that such a regular value exists). The inverse function theorem guarantees that for any point $p \in f^{-1}(\{q\})$, the map f is an diffeomorphism between a neighborhood of p and q. The Milnor index at the point p will be 1 if the restriction of f to that neighborhood, is orientation-preserving and -1 if it is orientation-reversing. In particular, when $m = 1$, we have a map $\sigma : \mathbb{R} \to \mathbb{R}$, $\sigma(x) = ax + g(x)x^2$ that has a regular value at 0, and its derivative at the origin is $\sigma'(0) = a$; the sign of a determines whether or not σ is orientation-preserving.*

Now say $F = \mathbb{F}_q$, where $2 \not| q$. Recall that $\mathrm{GW}(\mathbb{F}_q) \cong \mathbb{Z} \times \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2 \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Thus $\langle a \rangle = \langle 1 \rangle$ if and only if a is a square in $\mathbb{F}_q$. In particular, $\langle -1 \rangle = \langle 1 \rangle$ if and only if -1 is a square in $\mathbb{F}_q$.

A significant and interesting case arises when $q = p$ is an odd prime. Then $\langle -1 \rangle = \langle 1 \rangle$ if and only if $p \equiv 1$ modulo 4, while $\langle a \rangle = \langle 1 \rangle$ if and only if $\left(\frac{a}{p}\right) = 1$ where $\left(\frac{a}{p}\right)$ is the Legendre symbol. For example, when $p \equiv 1$ modulo 4, quadratic reciprocity reduces Lemma 3.2 to the statement that

$$\mathrm{ind}_0\sigma = \begin{cases} m \langle 1 \rangle & \text{if } m \text{ is even} \\ (m-1) \langle 1 \rangle + \langle a \rangle & \text{otherwise.} \end{cases}$$

In general, an isolated zero p of a section σ may not be defined over the base field F. Nonetheless, under a mild hypothesis on the residue field of p, the local Euler index may be computed using [5, Theorem 1.3], as follows.

Lemma 3.4. *Let F be a field. If $\sigma \in H^0(\mathcal{O}_{\mathbb{A}^1_F})$ has an isolated zero at $p \in \mathbb{A}^1_F$ and $k(p)/F$ is separable, then*

$$\mathrm{ind}_p\sigma = \mathrm{Tr}_{k(p)/F}\mathrm{ind}_{\overline{p}}(\sigma \otimes_F k(p))$$

where $\overline{p}$ is the canonical point above p in $\mathbb{A}^1_{k(p)} \cong \mathbb{A}^1_F \otimes_F k(p)$, and $\mathrm{Tr}_{k(p)/F} : \mathrm{GW}(k(p)) \to \mathrm{GW}(F)$ denotes the trace on bilinear forms induced via post-composition by the field trace $\mathrm{tr}_{k(p)/F} : k(p) \to F$.

3.3. **Inflection via Hasse–Witt derivatives.** Given a polynomial $f = a_nx^n + \ldots + a_1x + a_0 \in F[x]$, its k-*th Hasse derivative* is

$$\frac{D^kf}{dx} := \sum_{i=k}^{n} \binom{i}{k} a_i x^{i-k} \in F[x]. \tag{8}$$

Note that $k!D^kf$ is simply the usual k-th derivative of f; (8) gives an alternative way to differentiate that is well-suited to positive characteristic. There is also a version of Taylor's formula for Hasse derivatives; namely,

$$f = \sum_{i=0}^{n} \frac{D^if}{dx}(a)(x-a)^i.$$

Here we will use the Hasse derivative to define inflection in arbitrary characteristic. Most of the content in this section is contained, at least implicitly, in [27], but our presentation will be slightly different.

Accordingly, let (V, L) be a g^r_d on a curve C. Let $\{U_\alpha\}$ be an open covering of C together with local coordinates $z_\alpha : U_\alpha \to \mathbb{A}^1$, and let $\ell_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{G}_m$ be transition functions of a line bundle L with respect to $\{U_\alpha\}$. Given a basis $\lambda = (\lambda_0, \ldots, \lambda_r)$ of $V \in \mathrm{Gr}(r+1, H^0(L))$, let $\lambda_{i,\alpha}$ denote the restriction of λ_i to U_α. We define the *Hasse Wronskian* of λ to be the section $w(\lambda)$ of $L^{\otimes r+1} \otimes K^{\otimes\binom{r+1}{2}}$ given locally by

$$w_\alpha(\lambda) = \det\begin{pmatrix} \lambda_{0,\alpha}(z_\alpha) & \cdots & \lambda_{r,\alpha}(z_\alpha) \\ \dfrac{D^1\lambda_{0,\alpha}(z_\alpha)}{dz_\alpha} & \cdots & \dfrac{D^1\lambda_{r,\alpha}(z_\alpha)}{dz_\alpha} \\ \vdots & \ddots & \vdots \\ \dfrac{D^r\lambda_{0,\alpha}(z_\alpha)}{dz_\alpha} & \cdots & \dfrac{D^r\lambda_{r,\alpha}(z_\alpha)}{dz_\alpha} \end{pmatrix}. \tag{9}$$

Hereafter, we will let $W(\lambda)$ and its local version $W_\alpha(\lambda)$ denote Wronskian matrices, whose determinants are $w(\lambda)$ and $w_\alpha(\lambda)$, respectively. We need to verify that (9) actually defines a section of $L^{\otimes r+1} \otimes K^{\otimes\binom{r+1}{2}}$, which means that the transition functions are of the form $\ell_{\alpha\beta}(z_\beta)^{r+1}\kappa_{\alpha\beta}(z_\beta)^{\binom{r+1}{2}}$ where $\kappa_{\alpha\beta}$ are the transition functions of the canonical bundle. To this end, we compute the k-th Hasse derivative with respect to z_α of the equation

$$\lambda_{i,\alpha}(z_\alpha) = \ell_{\alpha\beta}(z_\beta)\lambda_{i,\beta}(z_\beta).$$

In the Taylor expansion of $\lambda_{i,\beta}$, the only relevant term is that corresponding to $\dfrac{D^k\lambda_{i,\beta}}{dz_\beta}$. Indeed, indices strictly less than k will not will not appear the determinant because of row operations; while indices strictly larger than k will vanish when evaluating the Taylor expansion. A straightforward computation now shows that the term corresponding to $\dfrac{D^k\lambda_{i,\beta}}{dz_\beta}$ is $\ell_{\alpha\beta}(z_\beta)\kappa_{\alpha\beta}(z_\beta)^k$. So the Hasse Wronskian indeed defines a section of $L^{\otimes r+1} \otimes K^{\binom{r+1}{2}}$, whose vanishing defines the *Hasse inflection locus*. In particular, Hasse inflection agrees with the usual notion of inflection in characteristic 0.

3.4. **Compatible Nisnevich coordinates via general linear projections.** In order to compute local Euler indices, we first need to specify local Nisnevich coordinates with compatible local trivializations; see §1.8 and Definition 2.1. The key to producing compatible Nisnevich trivializations, on the other hand, is to generalize the $\ell = 1$ case treated in §2.3. To do so, we assume in this subsection that F is a perfect field. Then, away from ∞, we use a general linear projection to define our chart $\mathbb{A}^2_{x,y} \to \mathbb{A}^1_t$, where t is given by a nontrivial F-linear combination of x and y. At ∞, as in §2.2, we use the linear projection $\mathbb{A}^2_{z,w} \to \mathbb{A}^1_w$. For the local trivializations of $\mathcal{E} = \det J^{r+1}(L)$ in either case, we first need to fix a local trivialization of L, so that we can then use the associated local trivializations on $\mathcal{E}$ induced by those of L and K_X.

Remark 3.5. *Perfectness of the base field F is necessary in order to ensure that we can cover a given hyperelliptic curve X by local Nisnevich charts. Indeed, given $(x, y) \in \mathbb{A}^2$, we have $k(x, y) = k(ax+by)$ for some $a, b \in F$ whenever $k(x, y)/F$ is a separable extension; and separability is guaranteed whenever F is perfect. If F is not perfect, our basic strategy for producing local Nisnevich charts via linear projection only works provided we know that all residue fields of inflection points on X are separable extensions of F; checking separability explicitly when $\ell > g$, however, is difficult; see §3.6 for further details.*

In order to understand local trivializations of L, consider the toric fan associated to $\mathbb{F}_{g+1} \supset X$ as in §2; our goal is to trivialize L by trivializing a line bundle on $\mathbb{F}_{g+1}$ that restricts to L. For this purpose, note that $L \cong \mathcal{O}_X(a\sigma + \ell f)$ for any a, as $\mathcal{O}_X(\sigma) \cong \mathcal{O}_X$ and $\mathcal{O}_X(f) \cong \mathcal{O}_X(2\infty)$. It therefore suffices to produce an integer $a \in \mathbb{Z}$ so that $|L|$ is induced by a linear series in $|\mathcal{O}_{\mathbb{F}_{g+1}}(a\sigma + \ell f)|$. The appropriate choice, for any ℓ, is $a = 1$ (note that $a = 0$ also works when $\ell \leq g$). Indeed, whenever $\ell \leq g$, any effective divisor on $\mathbb{F}_{g+1}$ linearly equivalent to $\sigma + \ell f$ must be a union of σ together with ℓ fiber classes up to multiplicity, while any section of L arises from $H^0(\mathcal{O}_{\mathbb{P}^1}(\ell))$ via pullback under the hyperelliptic structure map $X \to \mathbb{P}^1$. If instead $\ell > g$, notice that $|L|$ is generated by $n(D_{u_1} \cap X) + (\ell - n)(2\infty)$ and $R_\pi + m(D_{u_1} \cap X) + (\ell - m - g - 1)(2\infty)$ for all $0 \leq m \leq \ell - g - 1$ and $0 \leq n \leq \ell$ (here D_{u_1} compactifies $V(x) \subset \mathbb{A}^2_{x,y}$, while $2\infty = D_{u_3} \cap X$). Similarly, D_{u_2} compactifies $V(y) \subset \mathbb{A}^2_{x,y}$, and $R_\pi = X \cap D_{u_2}$; the fact that $|\sigma + \ell f|$ restricts to $|L|$ on X follows. Therefore, as in §2, L trivializes over $\mathbb{A}^2_{x,y}$ (resp., over $\mathbb{A}^2_{z,w}$) as $\langle 1 \rangle$ (resp., as $\langle z^\ell \rangle$).

We next compute transition maps between local trivializations of $\mathrm{Hom}(\det T_X, \det \mathcal{E})$ in terms of the local coordinates given by linear projections. Along the chart $\mathbb{A}^2_{x,y} \to \mathbb{A}^1_t$ away from ∞, $\mathrm{Hom}(\det T_X, \det \mathcal{E})$ trivializes as $\langle dt^{\otimes \binom{\ell+1}{2}+1} \rangle$. On the other hand, along the chart $\mathbb{A}^2_{z,w} \to \mathbb{A}^1_w$ near ∞, $\mathrm{Hom}(\det T_X, \det \mathcal{E})$ trivializes as $\langle z^{\ell+1} dw^{\otimes \binom{\ell+1}{2}+1} \rangle$. Arguments analogous to those used in §2.3 show that all of these transition maps are squares if $\ell \equiv 1 \bmod 4$. Accordingly, in the sequel we will assume that $\ell \equiv 1 \bmod 4$; see Remark 3.6 below for further discussion.

Sometimes it is useful to remember the transition maps between local trivializations of $\mathcal{E}$, as it is often easier to represent the Wronskian $w(\lambda)$ under a desired local trivialization by using transition maps and the representation of s under a special local trivialization. Since the local Euler index of $w(\lambda)$ only depends on the power series representation, we can apply this method to find the power series representation of $w(\lambda)$ under a desired Nisnevich local trivialization from a chosen étale local trivialization (meaning a choice of étale local coordinate together with a compatible local trivialization). The transition map that relates a local trivialization with uniformizer t to another local trivialization with uniformizer $\tilde{t}$ is multiplication by $\left(\frac{Dt}{d\tilde{t}}\right)^{\binom{l+1}{2}}$. Hereafter, a local Nisnevich/étale coordinate t means a local Nisnevich/étale coordinate with a uniformizer t that is equipped with a compatible local trivialization.

Remark 3.6. *Bachmann and Wickelgren explain in* [6, p.11] *that any given local trivialization may be modified to a compatible local trivialization. Doing so involves rescaling the local trivialization by an appropriate scalar function; without precisely identifying this scalar function, however, it is difficult to compute the transition maps required for $w(\lambda)$ explicitly. Consequently, we always assume in this* *section that $\ell \equiv 1$ mod 4, though it would be nice to know whether there is a natural way to overcome this obstacle.*

3.5. **Local arithmetic Euler indices along R_π.** We will argue separately according to whether the underlying ramification point is ∞_X; and whether $\ell \leq g$ or $\ell > g$.

3.5.1. *Local arithmetic Euler indices along $R_\pi \setminus \{\infty_X\}$.* We begin by proving the Hasse-inflectionary analogue of [4, Theorem 3.3].

Theorem 3.7. *Assume that $\ell \leq g$, in which case the complete linear series defined by $\mathcal{O}(2\ell\infty_X)$ has basis $\lambda = (1, x, x^2, \ldots, x^\ell)$. The zero locus of the Hasse Wronskian $w(\lambda)$ is $\binom{\ell+1}{2}$ times the ramification divisor R_π. Moreover, near any inflection point $p \neq \infty_X$ in R_π, we have*

$$w(\lambda) = \left(\frac{Dx}{dz}\right)^{\binom{\ell+1}{2}}$$

where z is a local Nisnevich uniformizer in p.

Proof. We proceed as in [4, Theorem 3.3], except that we replace the usual Leibniz rule by the Hasse-Leibniz rule

$$\frac{D^k(fg)}{dt} = \sum_{i=0}^{k} \frac{D^i f}{dt} \cdot \frac{D^{k-i} f}{dt} \tag{10}$$

and its long-form generalization

$$\frac{D^k}{dt} \prod_{j=1}^{e} f_j = \sum_{\substack{i_1 + \cdots i_e = k \\ i_m \geq 0 \text{ for all } i_m}} \prod_{j=1}^{e} \frac{D^{i_j} f_j}{dt} \,. \tag{11}$$

More precisely, we will show that the square matrix $W(\lambda)$ may be column–reduced to a lower triangular matrix for which the diagonal entry of the column indexed by each $\lambda_i(t)$ is $\left(\frac{Dx}{dt}\right)^i$; the theorem follows from the fact that away from ∞, $\frac{Dx}{dt}$ vanishes precisely to order 1 along the ramification divisor R_π.

To this end, note that the first column of $W(\lambda)$, corresponding to $\lambda_0 = 1$, is zero in every entry except the first one, which is 1. Similarly, the second column, corresponding to $\lambda_1 = t$, has entries $\frac{D^k x}{dt}$, $k \geq 0$. On the other hand, whenever $e \geq 2$, we may rewrite the entries of each column corresponding to λ_e using equation (11), obtaining

$$\frac{D^k(x^e)}{dt} = \sum_{\gamma=1}^{e} \binom{e}{\gamma} x^{e-\gamma} \sum_{\substack{i_1 + \cdots i_\gamma = k \\ i_m \geq 1 \text{ for all } i_m}} \prod_{j=1}^{\gamma} \frac{D^{i_j} x}{dt} \tag{12}$$

for every $k \geq 1$. Now define C_0 to be a column vector all of whose entries are zero except for the first entry $C_{0,0} = 1$, and for every $\gamma \geq 1$, let C_γ denote a column vector with entries

$$C_{k,\gamma} = \sum_{\substack{i_1 + \cdots i_\gamma = k \\ i_m \geq 1 \text{ for all } i_m}} \prod_{j=1}^{\gamma} \frac{D^{i_j} x}{dt}$$

for $k = 0, \ldots, \ell$. Note that each column $C_{(e)}$ corresponding to $\lambda_e = x^e$ for $e \geq 1$ is spanned by linearly independent vectors C_γ for $\gamma = 0, \ldots, e$. More precisely, we have

$$C_{(e)} = \sum_{\gamma=0}^{e} \binom{e}{\gamma} x^{e-\gamma} C_\gamma \,.$$

Since the coefficient of $C_{(e)}$ is 1 in the above sum, we conclude that the matrix corresponding to $W(\lambda)$ column-reduces to a matrix with columns C_γ, $0 \leq \gamma \leq \ell$.

On the other hand, note that $C_{k,\gamma} = 0$ when $1 \leq k < \gamma$, as not all i_m, $m = 1, \ldots, \gamma$, satisfy $i_m \geq 1$. Since

$$C_{\gamma,\gamma} = \sum_{\substack{i_1+\cdots i_\gamma=\gamma \\ i_m \geq 1 \text{ for all } i_m}} \prod_{j=1}^{\gamma} \frac{D^{i_j} x}{dt} = \prod_{j=1}^{\gamma} \frac{Dx}{dt}$$

it follows that the diagonal entries of our column-reduced matrix are $(\frac{Dx}{dt})^m$, $0 \leq m \leq \ell$. Multiplying them together yields the desired local description of $w(\lambda)$. □

Theorem 3.8. *Let X denote a hyperelliptic curve defined over a field F of characteristic $p \neq 2$ as above. Whenever $\ell \leq g$, the local Euler index of the complete linear series $|2\ell\infty_X|$ in $\mathrm{GW}(F)$ associated to a ramification point $(\gamma, 0) \in R_\pi \setminus \{\infty_X\}$ of the hyperelliptic projection $\pi : X \to \mathbb{P}^1$ is given by*

$$\operatorname{ind}_{(\gamma,0)} w(\lambda) = \operatorname{Tr}_{k(\gamma)/F} \left(\frac{\binom{\ell+1}{2} - 1}{2} \cdot \mathbb{H} + \left\langle \frac{(D^1 f)(\gamma)}{2} \right\rangle \right). \tag{13}$$

Proof. To compute the local Euler index at a ramification point $(\gamma, 0) \in R_\pi$ using Theorem 3.7, we first need to understand the local description of $w(\lambda)$ when $t = u_b = y - bx$. In this case, the affine equation $y^2 = f(x)$ for X becomes $(u_b + bx)^2 = f(x)$, and differentiating both sides of the latter equation with respect to u_b yields

$$2(u_b + bx) \left(1 + b \frac{D^1 x}{du_b} \right) = (D^1 f)(x) \cdot \frac{D^1 x}{du_b}.$$

After collecting terms involving $\frac{D^1 x}{du_b}$, we obtain

$$\frac{D^1 x}{du_b} = \frac{2(u_b + bx)}{(D^1 f)(x) - 2(u_b + bx)b}. \tag{14}$$

Here $x(u_b) = \gamma$ when $u_b = -b\gamma$. Note that $\frac{D^1 x}{du_b}$ vanishes at $u_b = -b\gamma$ to order 1, in light of the facts that $f(x) = 0$ and $D^1 f(\gamma) \neq 0$, and that the power series expansion of $\frac{D^1 x}{du_b}$ is of the form

$$\frac{D^1 x}{du_b} = \frac{2(u_b + b\gamma)}{(D^1 f)(\gamma)} + h(u_b)(u_b + b\gamma)^2 \tag{15}$$

where $h \in F[\![u_b]\!]$. Applying Lemma 3.2 and 3.4 in tandem with Theorem 3.7, (13) follows. □

Now suppose that $\ell > g$. It is difficult to directly compute $w(\lambda)$ with respect to a local Nisnevich coordinate. To circumvent this difficulty, we first prove the following result with respect to the local étale coordinate y, which refines [4, Theorem 5.7].

Theorem 3.9. *Assume that $\ell > g$, and let $\lambda := (1, y, \ldots, x^{\ell-g-1}, x^{\ell-g-1}y; x^{\ell-g}, x^{\ell-g+1}, \ldots, x^\ell)$ denote the corresponding monomial basis of the complete linear series $|\mathcal{O}(2\ell\infty_X)|$. With respect to the local étale coordinate y, the lowest-order term of the Hasse Wronskian $w(\lambda)$ in a ramification point $(\gamma, 0) \in R_\pi \setminus \{\infty_X\}$ is given by that of*

$$\det M(\ell, g) \cdot \left(D_y^1 x\right)^{\binom{g+1}{2}} (D_y^2 x)^{\ell(\ell-g)}$$

whenever $\det M(\ell, g)$ is nonzero in F, in which $D_y^i = \frac{D^i x}{dy^i}$ and $M(\ell, g)$ denotes the $(g+1) \times (g+1)$ matrix with entries $M_{ij} = \binom{\ell-g+j}{2j-i}$, $0 \leq i, j \leq g$.

Remark 3.10. *Let ℓ_1 and ℓ_2 denote the lines $x+y = 0$ and $2y+x = 2\ell-2g$ in the xy-plane, respectively. Let a_i (resp., b_i) denote the point of ℓ_1 (resp., ℓ_2) with coordinates $(i, -i)$ (resp., $(2i, \ell - g - i)$), $i = 0, \ldots, g$. The Gessel–Viennot lemma [12] implies that (the integer underlying) $\det M(\ell, g)$ is equal* *to the number of non-intersecting lattice paths connecting the $(g+1)$-tuple of points a_i, $i = 0, \dots, g$ with the $(g+1)$-tuple of points b_i, $i = 0, \dots, g$.*

Proof. In each ramification point $(\gamma, 0)$, note that

$$\lambda^\gamma := (1, y, (x-\gamma), (x-\gamma)y, \dots, (x-\gamma)^{\ell-g-1}, (x-\gamma)^{\ell-g-1}y; (x-\gamma)^{\ell-g}, (x-\gamma)^{\ell-g+1}, \dots, (x-\gamma)^{\ell}) \tag{16}$$

determines an inflectionary basis for $\mathcal{O}(2\ell\infty_X)$, i.e., the elements of λ^γ are ordered according to their orders-of-vanishing in the uniformizer y, which are strictly increasing. Note that for each i, the i^{th} element of λ^γ is a linear combination of the first i elements of λ in which the i^{th} element of λ appears with coefficient one; as a result, the Hasse Wronskians of λ and λ^γ are equal.

As a result, we may assume without loss of generality that $\gamma = 0$. Note first that the Hasse Leibniz rule (10) implies

$$D_y^k(x^j y) = D_y^{k-1} x^j + y D_y^k x^j$$

for all positive integers j and k; it follows easily that the column of $W(\lambda)$ indexed by $x^j y$ may be replaced by a column whose kth entry (when counting starting from 0 at the top of the column) is $D_y^{k-1} x^j$, for every $j = 0, \dots, \ell - g - 1$. We next apply Faà di Bruno's chain rule for Hasse derivatives, which states in general that

$$D^k(f \circ g) = \sum_{\substack{\sum_{i=1}^k i c_i = k \\ c_i \geq 0 \text{ for all } i}} \binom{c_1 + \cdots + c_k}{c_1, \dots, c_k} (D^{c_1 + \cdots + c_k}(f)) \circ g \cdot \prod_{j=1}^k (D^j g)^{c_j}. \tag{17}$$

Computing the Hasse derivative of powers of x with respect to y via (17), we find that

$$D_y^k x^j = \sum_{\substack{\sum_{i=1}^k i c_i = k \\ c_i \geq 0 \text{ for all } i}} \binom{c_1 + \cdots + c_k}{c_1, \dots, c_k} \binom{j}{c_1 + \cdots + c_k} x^{j - (c_1 + \cdots + c_k)} \cdot \prod_{i=1}^k (D_y^i x)^{c_i}. \tag{18}$$

We claim that via further column reduction, all entries of the form $D_y^k x^j$ may be systematically replaced by

$$\sum_{\substack{\sum_{i=1}^k i c_i = k \\ \sum_{i=1}^k c_i = j}} \binom{j}{c_1, \dots, c_k} \prod_{i=1}^k (D_y^i x)^{c_i} \tag{19}$$

or equivalently, that all terms in (18) indexed by k-tuples of non-negative integers $(c_1, \dots, c_k)$ with $\sum_{i=1}^k c_i < j$ may be ignored. Notice that this is vacuously true when $j = 0, 1$. To see this when $j > 1$, fix a strictly positive integer $j' < j$, and let C_j (resp., C'_j) denote any column whose kth entry is $D_y^k x^j$ (resp., $D_y^{k-1} x^j$); note that with the exception of the "trivial" columns indexed by 1 and y, every column of $W(\lambda)$ is either of the form C_j or C'_j for some j.

We now iteratively define a sequence of column reductions, as follows. Let $C_{j;1} := C_j - j x^{j-1} C_1$ and $C'_{j;1} := C'_j - j x^{j-1} C'_1$ for every $j = 2, \dots, \ell$. Replace column C_j (resp., C'_j) by $C_{j;1}$ (resp., $C'_{j;1}$) for every $j = 2, \dots, \ell$; doing so eliminates all terms involving x^{j-1}. For convenience, continue labeling the columns of $W(\lambda)$ by C_j and C'_j, respectively. Now replace C_j (resp., C'_j) by $C_{j;2} := C_j - \binom{j}{2} x^{j-2} C_2$ (resp., $C'_{j;2} := C'_j - \binom{j}{2} x^{j-2} C'_2$) for every $j = 3, \dots, \ell$; doing so eliminates all terms involving x^{j-2}. Iterating this procedure, at the m^{th} step replace C_j (resp., C'_j) by $C_{j;m} := C_j - \binom{j}{m} x^{j-m} C_m$ (resp., $C'_{j;m} := C'_j - \binom{j}{m} x^{j-m} C'_m$). After iterating ℓ times, the columns of $W(\lambda)$ are clearly as in (19).

The preceding analysis shows, in particular, that the kth entry of the column indexed by $x^{\ell-g+j}$, $j = 0, \dots, g$ of (the column-reduced version of) $W(\lambda)$ is a sum of monomials $\prod_{i=1}^{2\ell-g} (D_y^i x)^{c_i}$ whose associated (appropriately reordered) exponent vector $(c_1, \dots, c_{2\ell-g})$ determines a partition $(i^{c_i})_i$ of weight k with $\ell - g + j$ parts. Here we may assume $\ell - g + j \leq k \leq \ell$ without loss of generality; indeed, whenever $k < \ell - g + j$, there are no such partitions, and the corresponding entry of $W(\lambda)$ is zero. Likewise, it is easy to see that none of the leftmost $(2\ell - 2g)$ columns of $W(\lambda)$ are divisible by $D^1_y x$, but that their reductions modulo $D^1_y x$ are zero above the diagonal. Indeed, this follows from the fact that the only weight-k partitions with j parts for which $j \leq k \leq 2j - 1$ have at least one singleton part.

Note that modulo $D^1_y x$, the diagonal entries that appear among the leftmost $(2\ell - 2g)$ columns of $W(\lambda)$ are precisely $1, 1, (D^2_y x), (D^2_y x), \ldots, (D^2_y x)^{\ell-g-1}, (D^2_y x)^{\ell-g-1}$; their product is $(D^2_y x)^{(\ell-g)(\ell-g-1)}$. In order to conclude, we will more closely examine the rightmost $(g+1)$ columns of $W(\lambda)$, namely those indexed by $x^{\ell-g+i}$, $i = 0, \ldots, g$. Accordingly, let M denote the $(g+1) \times (g+1)$ matrix obtained from the submatrix of $W(\lambda)$ determined by the bottom-most $(g+1)$ rows of the rightmost $(g+1)$ columns, recording the multiplicity with which $D^1_y x$ divides the corresponding entry of $W(\lambda)$; as a matter of convention, we take this multiplicity to be ∞ whenever the corresponding entry of $W(\lambda)$ is zero. Let M_i, $i = 0, \ldots, g$ denote the ith column of M. Whenever it is finite, the kth entry of M_i is precisely the minimal number of singleton parts of weight-$(2\ell - 2g + k)$ partitions with $\ell - g + i$ parts, $k = 0, \ldots, g$. This minimal number, in turn, depends on how large the number of parts is relative to the weight. More precisely, whenever $2\ell - 2g + k > 2(\ell - g + i)$, i.e., when $k > 2i$, the minimal number is clearly zero. On the other hand, whenever $k \leq 2i$, the minimal number is realized by the partition $(2^{\lambda_1}, 1^{\lambda_2})$ for which $\lambda_1 + \lambda_2 = \ell - g + i$ and $2\lambda_1 + \lambda_2 = 2\ell - 2g + k$; solving, we find $\lambda_1 = \ell - g + k - i$ and $\lambda_2 = 2i - k$. Since we require λ_1 to be non-negative, it follows that whenever $k \leq 2i$, the kth entry of M_i is either $2i - k$, with $i \leq \ell - g + k$; otherwise, it is ∞.

To compute the multiplicity with which $D^1_y x$ divides $w(\lambda)$, the key point is that the tropical *permanent* $\mathrm{perm}(M)$ of M with respect to addition and taking minima is precisely $\binom{g+1}{2}$. Indeed, the tropical permanent of M is bounded below by the tropical permanent of the matrix M' whose $(k,i)^{\text{th}}$ entry is $2i - k$, for every $0 \leq i, k \leq g$. On the other hand, every $(g+1)$-tuple tropical product that contributes to $\mathrm{perm}(M')$ is the same, namely $\binom{g+1}{2}$, which also gives the contribution of the diagonal to $\mathrm{perm}(M)$. Since $(k,i)^{\text{th}}$ entry of M' records λ_2 of the corresponding partition $(2^{\lambda_1}, 1^{\lambda_2})$ (where λ_1, λ_2 are allowed to be negative) of $2\ell - 2g + k$ by $\ell - g + i$ parts, the determinant of the bottom-right $(g+1) \times (g+1)$ part of $W(\lambda)$ must be a constant times $(D^1_y x)^{\binom{g+1}{2}} (D^2_y x)^{(g+1)(\ell-g)}$. So we deduce that, up to a scalar multiple, the lowest-order term of $w(\lambda)$ is equal to $(D^1_y x)^{\binom{g+1}{2}} (D^2_y x)^{\ell(\ell-g)}$.

More precisely, the lowest-order term of $w(\lambda)$ is equal to $(D^1_y x)^{\binom{g+1}{2}} (D^2_y x)^{\ell(\ell-g)}$ times a $(g+1) \times (g+1)$ determinant of Faà di Bruno coefficients associated with monomials $(D^1_y x)^{c_1} (D^2_y x)^{c_2}$ in the (entries of) the lower right-hand $(g+1) \times (g+1)$ submatrix of $W_\alpha(\lambda)$ that contribute to the lowest-order term of $w(\lambda)$. According to the preceding two paragraphs, those monomials $(D^1_y x)^{c_1} (D^2_y x)^{c_2}$ in the (k,i) entry of the lower right-hand $(g+1) \times (g+1)$ submatrix of $W(\lambda)$ that contribute are those for which $c_2 = \ell - g + k - i$ and $c_1 = 2i - k$, respectively, whenever these quantities are nonnegative (if either $\ell - g + k - i$ or $2i - k$ is negative, there is no contributing monomial, and the associated coefficient is zero). According to (19), this is precisely the (k,i)-entry of the matrix $M(\ell, g)$; the desired result follows. $\square$

With Theorem 3.9 in hand, we can compute the local Euler index of $|\mathcal{O}(2\ell \infty_X)|$ in a hyperelliptic ramification point whenever $\ell > g$.

Theorem 3.11. *Let X denote a hyperelliptic curve defined over a perfect field F of characteristic not equal to 2 as above. Whenever $\ell > g$ and $\det M(\ell, g)$ is nonzero in F, the local Euler index of the complete linear series $|2\ell \infty_X|$ in $\mathrm{GW}(F)$ associated to a ramification point $(\gamma, 0) \in R_\pi \setminus \{\infty_X\}$ of the* *hyperelliptic projection $\pi : X \to \mathbb{P}^1$ is given by*

$$\begin{aligned}&\mathrm{ind}_{(\gamma,0)} w(\lambda)\\ &= \begin{cases}\mathrm{Tr}_{k(\gamma)/F}\left(\frac{1}{2}\binom{g+1}{2}\cdot \mathbb{H}\right) & \text{if } \binom{g+1}{2} \text{ is even}\\ \mathrm{Tr}_{k(\gamma)/F}\left(\frac{\binom{g+1}{2}-1}{2}\cdot\mathbb{H} + \left\langle (\det M(\ell,g)) 2^{\binom{g+1}{2}} (D^1 f)(\gamma)^{\binom{g+1}{2}+\ell(\ell-g)}\right\rangle\right) & \text{otherwise.}\end{cases}\end{aligned} \tag{20}$$

Proof. To compute the local Euler index at a ramification point $(\gamma, 0) \in R_\pi$ using Theorem 3.9, we first need to rewrite $w(\lambda)$ in terms of the local Nisnevich coordinate $u_b := y - bx$. Note that equation (14) describes $D^1_{u_b}x$, and Faà di Bruno's chain rule (17) implies that $D^1_y x = (D^1_{u_b}x)(D^1_{u_b}y)^{-1}$ (where we interpret u_b as a function of y). Since $y = u_b + bx$, we have $(D^1_{u_b}y)(-b\gamma) = 1$ as $(D^1_{u_b}x)(-b\gamma) = 0$ by (15). Thus, $D^1_y x$ vanishes once at $-b\gamma$, as a function of u_b.

Similarly, in order to rewrite $D^2_y x$ as a function of u_b, we first apply Faà di Bruno, which yields

$$D^2_y x = (D^1_{u_b}x)(D^2_y u_b) + (D^2_{u_b}x)(D^1_y u_b)^2.$$

To understand the implications of this, we first rewrite $D^2_{u_b}x$. Applying Faà di Bruno to compute the second Hasse derivative of $(u_b + bx)^2 = f(x)$, we obtain

$$\left(1 + bD^1_{u_b}x\right)^2 + 2(u_b + bx)bD^2_{u_b}x = (D^2 f)(x)\left(D^1_{u_b}x\right)^2 + (D^1 f)(x)D^2_{u_b}x.$$

After collecting terms involving $D^1_{u_b}x$, we obtain

$$D^2_{u_b}x = \frac{(b^2 - (D^2 f)(x))\left(D^1_{u_b}x\right)^2 + 2bD^1_{u_b}x + 1}{(D^1 f)(x) - 2(u_b + bx)b}.$$

Note that when $u_b = -b\gamma$, we have $x(-b\gamma) = \gamma$ and $D^1 f(\gamma) \neq 0$. Moreover, the power series expansion of $D^1_{u_b}x$ of equation (15) implies that the power series expansion of $D^2_{u_b}x$ is of the form

$$D^2_{u_b}x = \frac{1}{(D^1 f)(\gamma)} + q(u_b)(u_b + b\gamma)$$

where $q \in F[\![u_b]\!]$. According to Lemma 3.2, to compute the local Euler index we need only the lowest order term of the power series expansion of $D^2_y x$ at $u_b = -b\gamma$; so we may ignore $D^2_y u_b$, as $D^2_{u_b}x$ and $D^1_y u_b$ do not vanish at $u_b = -b\gamma$ but $D^1_{u_b}x$ does.

Putting this all together, we see that as a function of u_b, the lowest order term of $w(\lambda)$ with respect to y is

$$(D^1_{u_b}y)^{\binom{l+1}{2}} \det M(\ell, g) \cdot (D^1_{u_b}x)^{\binom{g+1}{2}}(D^1_{u_b}y)^{-\binom{g+1}{2}}((D^1_{u_b}x)(D^2_y u_b) + (D^2_{u_b}x)(D^1_y u_b)^2)^{\ell(\ell-g)},$$

where $(D^1_{u_b}y)^{\binom{l+1}{2}}$ comes from the transition map described in §3.4. Simplifying and ignoring terms with $D^2_y u_b$, our local Euler index reduces to that of

$$\det M(\ell, g) \cdot (D^1_{u_b}y)^{\binom{\ell+1}{2}-\binom{g+1}{2}-2\ell(\ell-g)}(D^1_{u_b}x)^{\binom{g+1}{2}}(D^2_{u_b}x)^{\ell(\ell-g)}.$$

Applying Lemma 3.2 and 3.4 in tandem with Theorem 3.9, equation (20) follows. □

3.5.2. *Local arithmetic Euler indices at ∞_X.* Recall from Section 2.2 that along $\mathbb{A}^2_{z,w}$, X is defined by $w^2 = h(z)$, where $h(z) = z^{2g+2}f(z^{-1})$. According to Section 3.4, the transition map for L that results from exchanging any local coordinate away from ∞ for the local Nisnevich coordinate w at ∞ is multiplication by z^ℓ.

Whenever $\ell \leq g$, the basis $\lambda = (1, x, x^2, \ldots, x^\ell)$ for $|2\infty_X|$, when rewritten in terms of z and w, becomes $(z^\ell, z^{\ell-1}, z^{\ell-2}, \ldots, 1)$. The latter is a permutation of $(1, z, z^2, \ldots, z^\ell)$ with sign equal to

$(-1)^{\lfloor \frac{\ell+1}{2} \rfloor} = -1$, since we assume $\ell \equiv 1 \bmod 4$. The induced action of the permutation on Wronskian matrices yields $w(\lambda) = (-1)^{\binom{\ell+1}{2}} \cdot w(1, z, z^2, \ldots, z^\ell)$. As a result, we easily obtain analogues of Theorems 3.7 and 3.8 by replacing f by h, y and t and $u_b = y - bx$ by w, γ by 0, and x by z.

Theorem 3.12. *Assume that $\ell \leq g$, in which case the complete linear series defined by $\mathcal{O}(2\ell\infty_X)$ has basis $\lambda = (z^\ell, z^{\ell-1}, \ldots, z, 1)$. In terms of the local Nisnevich coordinate w at ∞_X, we have*

$$w(\lambda) = -\left(\frac{Dz}{dw}\right)^{\binom{\ell+1}{2}}.$$

Theorem 3.13. *Let X denote a hyperelliptic curve defined over a field F of characteristic $p \neq 2$ as above. Whenever $\ell \leq g$, the local Euler index of the complete linear series $|2\ell\infty_X|$ in $\mathrm{GW}(F)$ associated to ∞_X is given by*

$$\mathrm{ind}_{\infty_X} w(\lambda) = \left(\frac{\binom{\ell+1}{2} - 1}{2} \cdot \mathbb{H} + \left\langle -\frac{(D^1 h)(0)}{2} \right\rangle\right). \tag{21}$$

Now suppose that $\ell > g$. Rewritten in terms of z and w, the basis

$$\lambda = (1, y, \ldots, x^{\ell-g-1}, x^{\ell-g-1}y; x^{\ell-g}, x^{\ell-g+1}, \ldots, x^\ell)$$

becomes

$$(z^\ell, z^{\ell-g-1}w, z^{\ell-1}, z^{\ell-g-2}w, \ldots, z^{g+1}, w; z^g, z^{g-1}, \ldots, 1).$$

The latter is a permutation of $(1, w, z, zw, \ldots, z^{\ell-g-1}, z^{\ell-g-1}w; z^{\ell-g}, z^{\ell-g+1}, \ldots, z^\ell)$ with sign

$$(-1)^{\lfloor \frac{\ell+1}{2} \rfloor} \cdot (-1)^{\lfloor \frac{\ell-g}{2} \rfloor} = (-1)^{1+\binom{\ell-g}{2}}$$

as $(-1)^{\lfloor \frac{n}{2} \rfloor} = (-1)^{\binom{n}{2}}$ for every $n \in \mathbb{N}$. Essentially the same argument as that used when $\ell \leq g$ yields the following analogues of Theorems 3.9 and 3.11.

Theorem 3.14. *Assume that $\ell > g$, and let $\lambda := (z^\ell, z^{\ell-g-1}w, z^{\ell-1}, z^{\ell-g-2}w, \ldots, z^{g+1}, w; z^g, z^{g-1}, \ldots, 1)$ denote the corresponding monomial basis of the complete linear series $|\mathcal{O}(2\ell\infty_X)|$. With respect to the local étale coordinate w, the lowest-order term of the Hasse Wronskian $w(\lambda)$ at ∞_X is equal to that of*

$$(-1)^{1+\binom{\ell-g}{2}} \cdot \det M(\ell, g) \cdot \left(D^1_w z\right)^{\binom{g+1}{2}} (D^2_w z)^{\ell(\ell-g)}$$

whenever $\det M(\ell, g)$ is nonzero in F, in which $D^i_w = \frac{D^i z}{dw^i}$ and $M(\ell, g)$ denotes the $(g+1) \times (g+1)$ matrix with entries $M_{ij} = \binom{\ell-g+j}{2j-i}$, $0 \leq i, j \leq g$.

Theorem 3.15. *Let X denote a hyperelliptic curve defined over a field F of characteristic $p \neq 2$ as above. Whenever $\ell > g$ and $\det M(\ell, g)$ is nonzero in F, the local Euler index of the complete linear series $|2\ell\infty_X|$ in $\mathrm{GW}(F)$ associated at ∞_X is given by*

$$\begin{aligned}
&\mathrm{ind}_{\infty_X} w(\lambda) \\
&= \begin{cases} \dfrac{1}{2}\binom{g+1}{2} \cdot \mathbb{H} & \text{if } \binom{g+1}{2} \text{ is even} \\[2ex] \dfrac{\binom{g+1}{2} - 1}{2} \cdot \mathbb{H} + \left\langle (-1)^{1+\binom{\ell-g}{2}} (\det M(\ell, g)) 2^{\binom{g+1}{2}} (D^1 h)(0)^{\binom{g+1}{2} + \ell(\ell-g)} \right\rangle & \text{otherwise.} \end{cases}
\end{aligned} \tag{22}$$

3.6. **Local Euler indices away from R_π, and Hasse inflection polynomials.** In this subsection, we will generalize the description of local inflection indices given in [7] and [8] to arbitrary characteristic. Accordingly, given positive integers $\ell > g$, we define the *(g, ℓ)th Hasse inflection polynomial* $P_{g,\ell}(x) \in F[x]$ according to

$$\det(D^j x^i y)_{0 \leq i \leq \ell-g-1; \ell+1 \leq j \leq 2\ell-g} = (f^{-(\ell+1)} y)^{\ell-g} P_{g,\ell}(x) \tag{23}$$

where $D^j = D^j_x$. The characteristic property of $P_{g,\ell}$ is that its roots parameterize precisely the x-coordinates of $\overline{F}$-rational Hasse inflection points of the complete linear series $|2\ell\infty_X|$ on X *supported* *on the complement of R_π.* It is worth noting that when $\ell = g+1$, the equation (23) reduces to the statement that

$$D^{g+2}y = f^{-(g+2)}y \cdot P_{g,g+1}(x). \tag{24}$$

In general, we can always realize Hasse inflection polynomials as determinants in the "atomic" polynomials $P_{g,g+1}(x)$, according to the following Hasse analogue of [8, Lem. 2.1].

Proposition 3.16. *Given positive integers $\ell > g$, let $\mu := \ell - g$. There exists a homogeneous polynomial $Q_{\mu,\ell+1} \in \mathbb{Z}[t_{-\mu}, \ldots, t_0, \ldots, t_{\mu-1}]$ of degree μ for which $P_{g,\ell} = Q_{\mu,\ell+1}|_{t_i = P_{\ell+1+i,\ell+2+i}}$ where*

$$Q_{\mu,\ell+1} = \det(t_{j-i})_{0 \le i,j \le \mu-1}.$$

Proof. The desired result follows immediately from the (first part of the) Hasse analogue of [4, Rmk 3.5]; namely, that away from the ramification locus R_π, the Hasse Wronskian is locally given by

$$w(\lambda) = \det(D^{\ell+1+j-i}y)_{1 \le i,j \le \ell-g}.$$

The latter equality results from an easy, if slightly tedious, row reduction that we leave to the reader. □

The atomic inflection polynomials $P_{g,g+1}(x)$, in turn, may be calculated recursively. Namely, given a positive integer g, let $n = g+2$; for simplicity (and consistently with [7]) we write P_n in place of $P_{g,g+1}$. The following result is a Hasse analogue of [7, Sect. 2, eq. (3)].

Proposition 3.17. *Suppose that* $\mathrm{char}(F) \neq 2$. *The atomic inflection polynomials of the hyperelliptic curve defined by the affine equation $y^2 = f(x)$ may then be obtained by applying the recursive relation*

$$P_{n+1} = \frac{1}{n+1}\left((D^1 P_n) \cdot f + \left(-n + \frac{1}{2}\right) P_n \cdot (D^1 f)\right) \tag{25}$$

for every $n \ge 1$, subject to the seed datum $P_1 = \frac{1}{2}D^1 f$.

Proof. Assume first that $\mathrm{char}(F) = 0$. Note that differentiating $y^2 = f(x)$ yields $2yD^1y = D^1 f$. Similarly, differentiating both sides of the defining equation (24) for atomic inflection polynomials yields

$$\begin{aligned} D^1 D^n y &= (D^1 P_n) f^{-n} y + P_n \cdot (-n f^{-(n+1)}(D^1 f) \cdot y + f^{-n} D^1 y) \\ &= (D^1 P_n) f^{-n} y + P_n \cdot \left(-n f^{-(n+1)}(D^1 f) \cdot y + f^{-n} \cdot \frac{1}{2} f^{-1}(D^1 f) \cdot y\right) \\ &= f^{-(n+1)} y \left((D^1 P_n) \cdot f + P_n \cdot (D^1 f) \cdot \left(-n + \frac{1}{2}\right)\right). \end{aligned}$$

The desired recursion now follows from the fact that $D^1 D^n = (n+1)D^{n+1}$; see §3.3.

Assume now that the base field F has characteristic $p > 2$. The recursion (25) makes sense provided $p \nmid (n+1)$, as does the argument used in proving it. More generally, as we now explain, P_{n+1} may be obtained by lifting to characteristic zero and then specializing to the base field F. The upshot is that there is a well-defined way to "divide" by $n+1$.

Specifically, we claim that P_{n+1} over F is the specialization of an inflection polynomial defined over a discrete valuation ring R whose field of fractions has characteristic zero, associated to a "spreading out" of the underlying F-curve $y^2 = f(x)$ over R. Clearly such an R exists; moreover, the argument we used to prove the recursion (25) in the characteristic zero case extends to R and yields

$$(n+1)P_{n+1} = (D^1 P_n) \cdot f + \left(-n + \frac{1}{2}\right) P_n \cdot (D^1 f). \tag{26}$$

Since Hasse differentiation commutes with base change (see [28, Prop. 5.6]), equation (26) descends to F. In particular, the p-adic multiplicity of the right side of equation (26) is at least that of $n+1$; this means in turn that the right side of equation (25) is well-defined over F whenever $p > 2$, provided we divide both sides of (26) by suitable powers of p (here p is a power of the uniformizer of R up to multiplication by a unit). Finally, it is straightforward to check that the recursion we obtain is independent of our choice of mixed-characteristic extension. □

By construction, the Hasse Wronskian of the complete linear series $|\mathcal{O}_X(2\ell\infty_X)|$ on the hyperelliptic curve X of genus g is $w(\lambda) = (-1)^{\binom{\ell-g+1}{2}}(f^{-(\ell+1)}y)^{\ell-g}P_{g,\ell}(x)$ with respect to the local étale coordinate x, where $\lambda = (1, y, \ldots, x^{\ell-g-1}, x^{\ell-g-1}y; x^{\ell-g}, x^{\ell-g+1}, \ldots, x^{\ell})$, and $(-1)^{\binom{\ell-g+1}{2}}$ is the sign of the permutation that reorders λ as $(1, x, x^2, \ldots, x^{\ell}; y, xy, \ldots, x^{\ell-g-1}y)$. In particular, every root $x = \gamma$ of an inflection polynomial $P_{g,\ell}(x)$ lifts to two inflection points $(\gamma, \pm\sqrt{f(\gamma)})$ on X. As detailed in §3.4, computing the local Euler index of $w(\lambda)$ at $(\gamma, \pm\sqrt{f(\gamma)})$ involves identifying the leading term of the power series expansion of $w(\lambda)$ with respect to a local Nisnevich coordinate. In full generality, this is somewhat delicate; however, if we further assume that the residue field of $(\gamma, \pm\sqrt{f(\gamma)})$ is $k(\gamma)$, then x is already a local Nisnevich coordinate, and we obtain the following result.

Proposition 3.18. *Choose a square root $\sqrt{f(\gamma)}$ of $f(\gamma)$, where γ is an x-coordinate of an inflection point of a complete linear series $|\mathcal{O}_X(2\ell\infty_X)|$ on a hyperelliptic curve X of genus g. If $\sqrt{f(\gamma)} \in k(\gamma)$, then*

$$\mathrm{ind}_{(\gamma,\sqrt{f(\gamma)})} w(\lambda) = \mathrm{Tr}_{k(\gamma)/F}\left((-1)^{\binom{\ell-g+1}{2}}(f^{-(\ell+1)}(\gamma)\sqrt{f(\gamma)})^{\ell-g} \cdot \mathrm{ind}^{k(\gamma)}_{(\gamma,0)} P_{g,\ell}(x)\right)$$

and $\mathrm{ind}_{(\gamma,-\sqrt{f(\gamma)})} w(\lambda) = \mathrm{Tr}_{k(\gamma)/F}\left((-1)^{\ell-g} \cdot \mathrm{ind}^{k(\gamma)}_{(\gamma,\sqrt{f(\gamma)})} w(\lambda)\right)$, *where* $\mathrm{ind}^{k(\gamma)}$ *is the local Euler index over the field $k(\gamma)$.*

Proof. The condition $\sqrt{f(\gamma)} \in k(\gamma)$ implies that the residue field of $(\gamma, \pm\sqrt{f(\gamma)})$ is $k(\gamma)$, hence x is itself a local Nisnevich coordinate near those inflection points. The proposition now follows from our discussion (and proof of) Proposition 3.16. □

When $\ell = g+1$, the inflection polynomial is of the form $P_n(x)$ for some $n \geq 3$, and may be computed recursively as in Proposition 3.17; the general inflection polynomial $P_{g,\ell}(x)$ may then be computed as a determinant in the atomic inflection polynomials $P_n(x)$ as in Proposition 3.16. The defining equation (23) for inflection polynomials reduces the calculation of the local Euler index in one of the two $\overline{F}$-preimages $(\gamma, \pm\sqrt{f(\gamma)})$ of $x = \gamma$ to a calculation purely in terms of x. So whenever we understand the power series expansion of all $P_n(x)$ at $x = \gamma$, we may deduce the local Euler index of $w(\lambda)$ at inflection points $(\gamma, \pm\sqrt{f(\gamma)})$.

Remark 3.19. *Note that the condition $\sqrt{f(\gamma)} \in k(\gamma)$ is necessary in Proposition 3.18 in order to obtain a local Euler index formula that depends only on γ, f, and ℓ. To see the difficulty in computing the local Euler index when $\sqrt{f(\gamma)} \notin k(\gamma)$, first assume that $k(\gamma, \sqrt{f(\gamma)})/F$ is a separable extension as in Remark 3.5. In this case, x is no longer a local Nisnevich coordinate; rather, we need to replace it by a general F-linear combination $ax+by$ corresponding to a general linear projection. The transition map that sends the local étale coordinate x to the local Nisnevich coordinate $ax+by$ is $(D^1_{ax+by}x)^{\binom{\ell+1}{2}}$, which is the reciprocal of*

$$(D^1_x(ax+by))^{\binom{\ell+1}{2}} = (a + bD^1_y x)^{\binom{\ell+1}{2}}.$$

Note that near $(\gamma, \sqrt{f(\gamma)})$, we have $D^1_y x = (D^1_x y)^{-1} = (D^1_x\sqrt{f(x)})^{-1} = \frac{2\sqrt{f(x)}}{f'(x)}$. Nonetheless, the evaluation of $(D^1_{ax+by}x)^{\binom{\ell+1}{2}}$ at $(\gamma, \sqrt{f(\gamma)})$ depends on a and b, which makes it difficult to extract a local Euler index formula independent of our choice of (a,b) (even if the local Euler index does not itself depend on the choice of (a,b)). This suggests that it only makes sense to work with such a formula provided we assume F to be a specifically-chosen field, as in §4.

4. Geometric interpretations of Euler indices

The objective of this section is twofold. We first give concrete interpretations for several interesting base fields (not of characteristic 2) of the global Euler index computed by Theorem 3.1, as well as of the local Euler indices for points that belong to the ramification locus R_π of the hyperelliptic projection calculated in Theorems 3.8 and 3.11. We then give analogous concrete interpretations and speculations for points in the complement of R_π when the underlying curve is of genus one.

4.1. **Euler indices over $\mathbb{R}$.** As explained in the introduction, we have $\mathrm{GW}(\mathbb{R}) \cong \mathbb{Z}^2$, in which the two $\mathbb{Z}$ factors correspond, respectively, to the rank and signature of quadratic forms. Computing the rank of the classes in Theorem 3.1 (resp., in Theorems 3.8 and 3.11), we clearly recover the total $\mathbb{C}$-inflection of $|\ell\infty_X|$ (resp., the $\mathbb{C}$-inflection multiplicity of $|\ell\infty_X|$ over R_π). On the other hand, the fact that $\mathrm{sign}(\mathbb{H}) = 0$ implies that the signature of the global Euler class is zero. In light of Remark 3.3, this means precisely that the sum of oriented Milnor indices over all inflection points of $|\ell\infty_X|$ is zero.

The signatures of the local Euler indices at ramification points away from ∞_X in Theorems 3.8 and 3.11 depend on the field of definition of the x-coordinates γ of the underlying ramification points $(\gamma, 0)$. If $\gamma \in \mathbb{C} \setminus \mathbb{R}$, then exactly as in [20, Rmk 6.2], the signature of the local Euler index is zero. On the other hand, if $\gamma \in \mathbb{R}$ and $\ell \leq g$, the signature of the local Euler index is equal to the sign of $f'(\gamma)$. An analogous and only slightly more elaborate statement holds when $\gamma \in \mathbb{R}$ and $\ell > g$, in which case the local Euler index is zero when $\binom{g+1}{2}$ is even, and equal to the sign of $(f'(\gamma))^{\ell(\ell-g)+1}$ when $\binom{g+1}{2}$ is odd.

On the other hand, according to Theorems 3.13 and 3.15, the signature of the local Euler index at ∞_X depends on the sign of $h'(0)$, where $h(z) = z^{2g+2} f(z^{-1})$; moreover, $h'(0)$ is exactly the leading coefficient of f. If $\ell \leq g$, then the signature of the local Euler index is equal to the sign of $-h'(0)$. Analogous arguments show that when $\ell > g$, the local Euler index is zero when $\binom{g+1}{2}$ is even, and otherwise it is equal to the sign of $(-1)^{1+\binom{\ell-g}{2}}(h'(0))^{1+\ell(\ell-g)}$.

This means, in particular, that whenever f is *monic* and $\ell \leq g$, the difference between the number of ramification points $\gamma \in \mathbb{R}$ for which $f'(\gamma)$ is negative and the number of ramification points for which $f'(\gamma)$ is positive is precisely one.

4.2. **Euler indices over $\mathbb{F}_q$.** We have $\mathrm{GW}(\mathbb{F}_q) \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, in which the factors correspond, respectively, to the rank and the discriminant. More precisely, elements of $\mathrm{GW}(\mathbb{F}_q)$ of given rank are classified according to whether or not their discriminants are squares; note that for explicit calculations we think of the second factor as $\mathbb{F}_q^*/(\mathbb{F}_q^*)^2$ with its multiplicative group law. The discriminant of the global Euler class computed by Theorem 3.1 is $(-1)^{\frac{\gamma_{\mathbb{C}}}{2}} = (-1)^{\frac{g(2\ell-g+1)^2}{2}}$, which is a square exactly when $g(2\ell - g + 1)^2$ or $(q-1)$ is divisible by 4, i.e., if and only if g is odd or g is divisible by 4 or $(q-1)$ is divisible by 4. To compute the discriminants of the local Euler indices at ramification points $\gamma \in \mathbb{A}^1_{\mathbb{F}_q}$ in Theorems 3.8 and 3.11, we proceed as in [20, §6.3]. Namely, we apply the facts that $\mathrm{Tr}_{k(\gamma)/\mathbb{F}_q}\langle\alpha\rangle = \langle\mathrm{norm}(\alpha)\rangle \cdot \mathrm{Tr}_{k(\gamma)/\mathbb{F}_q}\langle 1\rangle$ for every $\alpha \in k(\gamma)$; that $\mathrm{norm}(\alpha)$ is a square in $\mathbb{F}_q$ if and only if α is a square in $k(\gamma)$; and that $\mathrm{Tr}_{k(\gamma)/\mathbb{F}_q}\langle 1\rangle$ is a square if and only if $k(\gamma) = \mathbb{F}_{q^n}$ with n odd. More generally, given $\beta = \sum\langle\alpha_i\rangle \in \mathrm{GW}(\mathrm{k}(\gamma))$, the corresponding trace $\mathrm{Tr}_{k(\gamma)/\mathbb{F}_q}\beta = \sum\langle\mathrm{norm}(\alpha_i)\rangle \cdot \mathrm{Tr}_{k(\gamma)/\mathbb{F}_q}\langle 1\rangle$ has discriminant $\mathrm{disc}(\mathrm{norm}(\prod \alpha_i)) \cdot \mathrm{disc}(\mathrm{Tr}_{k(\gamma)/\mathbb{F}_q}\langle 1\rangle)^{\mathrm{rank}(\beta)}$.

When $\ell \leq g$, the local index $\beta = \sum\langle\alpha_i\rangle$ computed by Theorem 3.8 satisfies $\prod \alpha_i = (-1)^{\frac{\binom{\ell+1}{2}-1}{2}} \frac{f'(\gamma)}{2}$. The arguments of the preceding paragraph now show that the local index depends only on whether $\frac{f'(\gamma)}{2}$ is a square in $k(\gamma)$, together with certain parity conditions on ℓ and the degree of $k(\gamma)/\mathbb{F}_q$. When $\ell > g$, we further assume that $\det M(\ell, g)$ is nonzero in $\mathbb{F}_q$ so that Theorem 3.11 applies; the local index at γ then depends on whether 2 and $f'(\gamma)$ is a square in $k(\gamma)$, along with parity conditions on ℓ and g and the degree of $k(\gamma)/\mathbb{F}_q$.

Similarly, when $\ell \leq g$, the local index at ∞_X depends only on whether $\frac{h'(0)}{2}$ is a square in $\mathbb{F}_q$, together with certain parity conditions on ℓ. When $\ell > g$ and $\det M(\ell, g)$ is nonzero in $\mathbb{F}_q$, the local index at ∞_X depends on whether or not 2 and $h'(0)$ are squares in $\mathbb{F}_q$, along with parity conditions on ℓ and g.

Summing local Euler indices and assuming that $p \equiv 1 \pmod 8$ (so that -1 and 2 are squares in $\mathbb{F}_q$), we see, for example, that when $\ell \leq g$ and f is monic, the difference between the number of ramification points $\gamma \in \overline{F}_q$ for which the discriminant is not a square (either $f'(\gamma)$ is a not a square in $k(\gamma)$ and the degree of $k(\gamma)/\mathbb{F}_q$ is odd, or $f'(\gamma)$ is a square and the degree of $k(\gamma)/\mathbb{F}_q$ is even) and the number of ramification points $\gamma \in \overline{F}$ for which the discriminant is a square is 1 modulo 2.

4.3. **Euler indices over $\mathbb{C}((t))$.** We have $\mathrm{GW}(\mathbb{C}((t))) \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, in which the factors correspond, respectively, to the rank and the discriminant; see, e.g., [20, Prop. 6.9]. Let $\nu = \nu_t$ denote the standard t-adic valuation on $\mathbb{C}((t))$ given by the least non-vanishing degree in t. If $g = ut^{\nu(g)}$ for some unit u, then $\langle g \rangle$ is $\langle 1 \rangle$ (resp., $\langle t \rangle$) when $\nu(g)$ is even (resp., odd). Since $\mathbb{H} = 2\langle 1 \rangle = 2\langle t \rangle$, it follows that the discriminant of the global Euler class computed by Theorem 3.1 is 0.

An explicit description of the discriminants of the local Euler indices in Theorems 3.8 and 3.11 is as follows. When $\ell \leq g$ and $(\gamma, 0)$ is a ramification point defined over $\mathbb{C}((t^{1/m}))$, the corresponding local index depends on the parity of the $t^{1/m}$-adic valuation of $f'(\gamma)$. More explicitly, $\mathrm{disc}(\mathrm{ind}_{(\gamma,0)})$ is given by $\mathrm{Tr}_{\mathbb{C}((t^{1/m}))/\mathbb{C}((t))}\langle 1 \rangle$ (resp., $\mathrm{Tr}_{\mathbb{C}((t^{1/m}))/\mathbb{C}((t))}\langle t^{1/m} \rangle$) when the valuation is even (resp., odd). Similarly, when $\ell > g$, $\binom{g+1}{2}$ is odd, and $(\gamma, 0)$ is a ramification point defined over $\mathbb{C}((t^{1/m}))$, we see that $\mathrm{disc}(\mathrm{ind}_{(\gamma,0)})$ is given by $\mathrm{Tr}_{\mathbb{C}((t^{1/m}))/\mathbb{C}((t))}\langle 1 \rangle$ (resp., $\mathrm{Tr}_{\mathbb{C}((t^{1/m}))/\mathbb{C}((t))}\langle t^{1/m} \rangle$) if the valuation of $f'(\gamma)^{1+\ell(\ell-g)}$ is even (resp., odd); and that $\mathrm{disc}(\mathrm{ind}_{(\gamma,0)})$ always vanishes when $\binom{g+1}{2}$ is even. The above analysis goes through verbatim if we replace $(\gamma, 0)$ by ∞, m by 1, and $f'(\gamma)$ by $h'(0)$.

The value of these traces, in turn, only depends on the parity of m. Indeed, proceeding as in the $\mathbb{F}_q$ case, we see that $\mathrm{Tr}_{\mathbb{C}((t^{1/m}))/\mathbb{C}((t))}\langle 1 \rangle \equiv m - 1 \pmod 2$ and $\mathrm{Tr}_{\mathbb{C}((t^{1/m}))/\mathbb{C}((t))}\langle t^{1/m} \rangle \equiv m \pmod 2$.

Summing local Euler indices, we deduce that when $\ell \leq g$ and f is monic, the difference between the number of ramification points $(\gamma, 0)$ with odd discriminant (either the degree m of the field of definition $\mathbb{C}((t^{1/m}))$ is odd and $f'(\gamma)$ has odd valuation, or m is even and $f'(\gamma)$ has even valuation) and the number of ramification points $\gamma \in \overline{\mathbb{C}((t))}$ with even discriminant is 1 modulo 2.

4.4. **Inflection polynomials for elliptic curves.** Over the complex numbers, the *Legendre* family of elliptic curves parameterized by

$$y^2 = x(x-1)(x-\kappa) \tag{27}$$

leads to a nice presentation of the moduli stack of (marked) elliptic curves. In [7, 8] the authors studied the variation of inflection polynomials in the Legendre parameter τ.[1] Their roots, as κ varies, trace out *inflectionary* curves in the (x, κ)-plane. According to the analogy between torsion and inflection introduced in §1.5, we may think of these inflectionary curves as generalizations of modular curves. Whenever κ is a *real* parameter, so that the corresponding elliptic curve E_κ has a real locus $E_\kappa(\mathbb{R})$ with two connected components, [8, Conj. 3.1] predicts that the corresponding inflection polynomial is separable, i.e., has only simple roots.

In [13], meanwhile, J. Huisman studies the moduli of elliptic curves over base fields F of characteristic $\mathrm{char}(F) \neq 2, 3$ via their *Weierstrass* presentations $E_{(a,b)} : y^2 = x^3 + ax + b$. He shows that isomorphism classes of $\mathbb{R}$-elliptic curves are parameterized by a real projective j-line that is an $\mathbb{R}^*$-quotient of the punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$ with coordinates (a, b). The j-line, in turn, is stratified according to the sign of the discriminant $\Delta(a, b) = -16(4a^3 + 27b^2)$; elliptic curves $E_{(a,b)}$ of strictly positive (resp., negative) discriminant are those whose real loci split as two connected components (resp., comprise a single connected component). Note that when $\Delta = 0$, the corresponding fiber $E_{(a,b)}$ is a nodal rational curve. We now turn to the behavior of inflection polynomials along the open sublocus of the j-line where b is nonzero. As a convenient normalization, we set $b = 2$; then a becomes a local parameter for our punctured j-line, and the ray $(a < -3)$ (resp., $(a > -3)$ parameterizes those isomorphism classes of elliptic curves with strictly positive (resp., negative) discriminants. The behavior of the

[1] In this situation, the linear series on the underlying curves are no longer generally complete; rather, they are distinguished codimension-$(g-1)$ subseries of complete series of degree 2ℓ generated by the monomial bases λ in the toric coordinates x and y introduced previously.

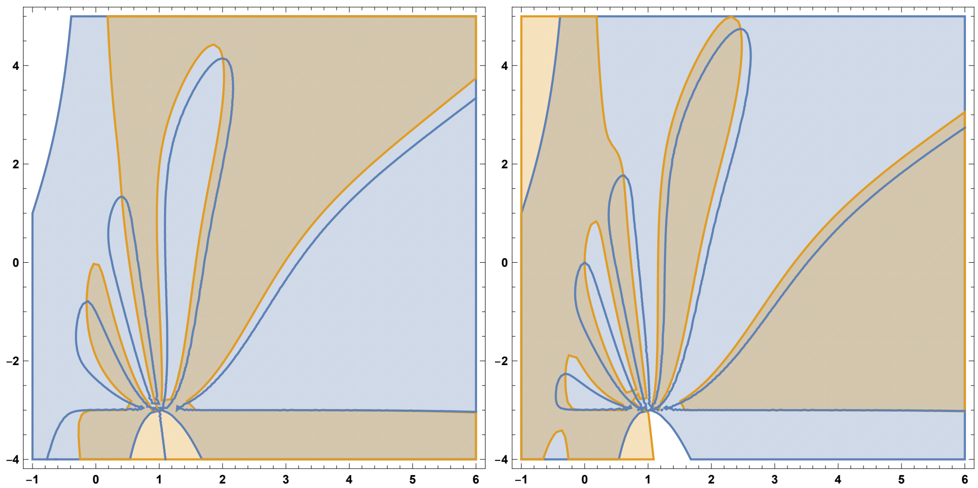

FIGURE 3. Dark blue curves trace out the real loci of $(P_n = 0)$ for $n = 9, 10$ in the (x, a)-plane. Here a parameterizes the punctured j-line, and the fiber over a is the elliptic curve $E_{(a,2)} : z^2 = x^3 + ax + 2$ in the (x, z)-plane. Grey (resp., orange) shading indicates that the Weierstrass cubic $f(x) = x^3 + ax + 2$ (resp., $\frac{dP_n}{dx}$) is strictly positive.

negative-discriminant regime $(a > -3)$ displays some distinctive features relative to that of the positive-discriminant case explored in [8].

First of all, the number $\omega_{\mathbb{R}}$ of real inflection points is no longer uniform in the modular parameter a; rather, there are several critical intervals in a along which $\omega_{\mathbb{R}}$ is constant, corresponding to the petals in Figure 3. Computer experiments indicate that the number of petals, as a function of n, is precisely $\frac{n}{2} - 1$ (resp., $\frac{n-1}{2} - 1$) when n is even (resp., odd). The *peak* of each petal, on the other hand, belongs to the discrete set of points in the (x, a)-plane for which $P_n = \frac{dP_n}{dx} = 0$; indeed, a peak is tautologically a point where $\frac{da}{dx} = 0$, and $\frac{dP_n}{dx} = 0$ follows by the chain rule. Similarly, those a-values at which (the corresponding specialization of) P_n fails to be separable are roots of the x-discriminant of P_n. In particular, the non-separable set of a-values includes the a-coordinates of the petal peaks mentioned above. Our empirical (computer-based) evidence suggests the following is true.

Conjecture 4.1. *Let $a \in \mathbb{R}$, and let $P_n(x)$, $n \geq 2$ denote the nth atomic inflection polynomial associated to the real Weierstrass elliptic curve $E_{(a,2)} : y^2 = x^3 + ax + 2$ as above. The possible numbers of real zeroes of $P_n(x)$, as a set-valued function of the modular parameter a, are as follows.*

Value of a	*n odd*	*n even*
$a < -3$	*4, of which 2 satisfy $f > 0$*	*2, of which 1 satisfies $f > 0$*
$a > -3$	$2i, i = 1, \ldots, \frac{n-1}{2}$, *of which $(2i - 1)$ satisfy $f > 0$*	$2i, i = 1, \ldots, \frac{n}{2}$, *of which $(2i - 1)$ satisfy $f > 0$*

The number of real roots (in x) of P_n is nonincreasing as a function of a, as a increases from -3 to ∞. Moreover, when n is even (resp., odd) there are precisely $\frac{n}{2} - 1$ (resp., $\frac{n-1}{2} - 1$) values of a at which P_n fails to be separable, and these are the a-coordinates of the petal peaks described above. At each of these a-values, P_n has a double root and all other roots are simple.

Note, in particular, that the conjecture predicts that for every fixed value of a, P_n has an *even* number of real roots; and that whenever n is *even*, there are values of a for which P_n has *only* real roots. Another counter-intuitive upshot of Conjecture 4.1 is that for fixed values of n, those real elliptic curves whose linear series are maximally $\mathbb{R}$-inflected are not necessarily those with the maximal number (two) of real components. On the other hand, the precise distribution of *signatures* of the roots of P_n appears to be somewhat intricate, as evidenced in Figure 3 by the pattern in which the petals intersect the domains of positivity for $\frac{dP_n}{dx}$.

It is natural to ask for analogues of Conjecture 4.1 over $\mathbb{Q}$ or $\mathbb{F}_q$. Faltings' theorem implies that the number of $\mathbb{Q}$-rational points of the inflectionary curve $\mathcal{C}_n := (P_n = 0)$ is finite; and in principle the Chabauty–Coleman method of p-adic integration [14, 21] may be used to compute $\mathcal{C}_n(\mathbb{Q})$. Implementation, however, is beyond the scope of the current paper.

Over $\mathbb{F}_q$, on the other hand, Hasse–Weil theory [24] applies, and establishes that

$$\#\mathcal{C}_n(\mathbb{F}_q) = q + 1 + e_{n,q},$$

where $e_{n,q}$ is a bounded error term. More precisely, Deligne showed that $|e_{n,q}| \leq 2g\sqrt{q}$ for *smooth* curves of genus g. In our case, $\mathcal{C}_n$ is singular for every $n \geq 2$, but Aubry and Perret [1] showed that the same basic inequality holds, provided we interpret g as the arithmetic genus. Note that $\mathcal{C}_n$ is always a plane curve of degree $2n$. Accordingly, it is instructive to examine the distribution of the renormalized error terms $\widetilde{e}_{n,p} := \frac{e_{n,p}}{(2n-1)(2n-2)\sqrt{p}}$ associated with the inflectionary curves $\mathcal{C}_n$ derived from either the Legendre or Weierstrass presentations as p varies over all odd primes p, in the spirit of the Sato–Tate conjecture for elliptic curves proved in [2].[2]

Somewhat counter-intuitively, (the $\mathbb{F}_p$-rational behavior of) the inflectionary curves $\mathcal{C}_n$ depends strongly upon whether the underlying family of elliptic curves is of Legendre or Weierstrass type. Most strikingly, the delta-invariants of the singularities and the geometric genera of Legendre and Weierstrass inflectionary curves $\mathcal{C}_n$ are generally distinct. The singularities and geometric genera of Legendre inflectionary curves $\mathcal{C}_n$ were addressed in detail in [8]; the most salient points (which are conjectural in general, but hold when n is small) are that a) $\mathcal{C}_n$ has precisely three singularities, each of which is defined over $\mathbb{Q}$, and which are permuted by automorphisms of $\mathcal{C}_n$; and b) each singularity of $\mathcal{C}_n$ has delta-invariant $\lfloor\frac{(n-1)^2}{2}\rfloor + n - 1$. In particular, this implies that the Legendre inflectionary curve $\mathcal{C}_n$ is of geometric genus *zero* whenever $2 \leq n \leq 5$, provided $\mathcal{C}_n$ is irreducible.[3] Moreover, we expect that $\mathcal{C}_n$ is almost always irreducible; the unique exception to this rule of which we are aware is provided by P_3, which factors over $\mathbb{Q}$ as

$$P_3 = \frac{1}{16}(\kappa - x^2)(\kappa - 2x + x^2)(\kappa - 2x\kappa + x^2).$$

In particular, the normalization of $\mathcal{C}_3$ is the disjoint union of three conics.

By contrast, the Weierstrass inflectionary curve $\mathcal{C}_2$ has geometric genus *one*. Indeed, $\mathcal{C}_2$ is an irreducible curve of arithmetic genus 3 cut out by $P_2 = \frac{1}{8}(3x^4 + 6x^2a + 24x - a^2)$; its projective completion to a curve in $\mathbb{P}^2_{x,a,z}$ (obtained by homogenizing P_2 with respect to z) has a unique singularity in $(0,1,0)$, namely the tacnode (of delta-invariant 2) with affine equation $3x^4 + 6x^2z - z^2 = 0$.

Proposition 4.2. *The values of the renormalized errors $\widetilde{e}_{2,p}$ are equidistributed with respect to the Sato–Tate measure on an elliptic curve with complex multiplication.*

See Figure 4 for a graphical representation of the renormalized errors $\widetilde{e}_{2,p}$ as p varies in the range $p \leq 10000$ and p splits in $\mathbb{Q}(\sqrt{-3})$.

Proof. To verify Proposition 4.2, we will show that the normalization $\widetilde{\mathcal{C}_2}$ of $\mathcal{C}_2$ is an elliptic curve with complex multiplication over $\mathbb{Q}(\sqrt{-3})$. Indeed, we start in $\mathbb{A}^2_{x,z}$, where $\mathcal{C}_2$ is given by the equation $\widetilde{P}_2 := 3x^4 + 6x^2z + 24xz^3 - z^2 = 0$, and the tacnode is supported in the origin. Introducing an auxiliary (weighted) blow-up variable $t = \frac{x^2}{z}$, we can rewrite the pullback of $\widetilde{P}_2$ to $\mathbb{Q}[x, z, \frac{1}{z}, t]$ as the system of

[2] If we further assume that $p \equiv 2 \pmod 3$, then every element of $\mathbb{F}_p$ is a cube, and it then follows from Cardano's cubic formula that an elliptic curve $E_{(a,2)}$ admits a Legendre presentation if and only if its discriminant $\Delta = \Delta(a, 2)$ is a square in $\mathbb{F}_p$.

[3] This might seem to suggest that $\mathcal{C}_3$ is of geometric genus $10 - 3(4) = -2$, an absurd conclusion; the explanation, however, is that $\mathcal{C}_3$ in fact has 3 irreducible components, so the geometric genus is two more than we expected.

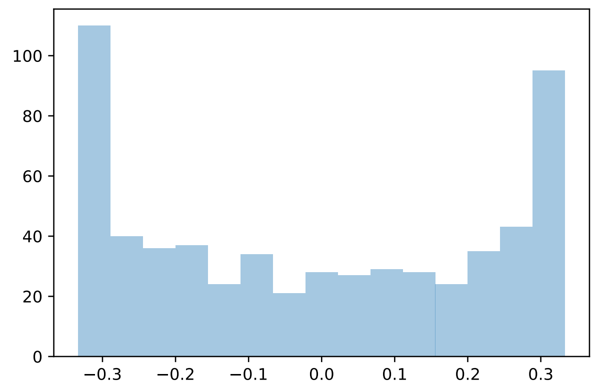

FIGURE 4. Distribution of renormalized errors for the Weierstrass inflectionary curve $\mathcal{C}_2$.

equations $Q_1 := tz - x^2 = 0, Q_2 := 3t^2 + 6t + 24xz - 1 = 0$, whose homogenization with respect to a gives an affine presentation for $\widetilde{\mathcal{C}}_2$ as the intersection of quadrics in $\mathbb{P}^3_{x,a,z,t}$. There is a well-known recipe for converting a space cubic of the latter type to an isomorphic plane cubic; see, e.g., [23, §1.4]. Namely, identifying Q_1 and Q_2 with the 4×4 bilinear forms to which they correspond, the plane cubic presentation is $y^2 = f(x)$, where $f(x) := \det(Q_1 x + Q_2)$. In our case, we find that $f(x) = -\frac{1}{4}x^3 + 1728$. The corresponding elliptic curve E has complex multiplication over $\mathbb{Q}(\sqrt{-3})$.

Now the normalization map $\widetilde{\mathcal{C}}_2 \to \mathcal{C}_2$ is an isomorphism over $\mathbb{Q}$ when restricted to the complement of the point $(0:1:0) \in \mathcal{C}_2$. The fiber above the latter point consists of those points $(0:1:0:t) \in \widetilde{\mathcal{C}}_2$ such that $3t^2 + 6t - 1 = 0$. It follows that for every primes $p > 3$, we have

$$\#\mathcal{C}_2(\mathbb{F}_p) = \#\widetilde{\mathcal{C}}_2(\mathbb{F}_p) - \left(\frac{3}{p}\right) = \#E(\mathbb{F}_p) - \left(\frac{3}{p}\right).$$

Since the Legendre symbol only takes the values ± 1, it becomes negligible when renormalizing the error terms $\widetilde{e}_{2,p}$. Therefore the error terms of $\mathcal{C}_2$ obey the same distribution as those of E. □

This is somewhat mysterious, as $\mathcal{C}_2$ does not itself parameterize (x-coordinates of) inflection points of linear series on elliptic curves[4]. It would be interesting to have a modular interpretation of $\mathcal{C}_2$ and its normalization.

REFERENCES

[1] M. Aubry and M. Perret, *A Weil theorem for singular curves*, in "Arithmetic, geometry, and coding theory", Pellikaan, Perret, Vlăduţ, eds., de Gruyter, New York, 1996.
[2] T. Barnet-Lamb, D. Geraghty, M. Harris, and R. Taylor, *A family of Calabi–Yau varieties and potential automorphy*, Publ. RIMS **47** (2011), no. 1, 29–98.
[3] C. Bethea, J. Kass, and K. Wickelgren, *An example of wild ramification in an enriched Riemann-Hurwitz formula*, in "Motivic homotopy theory and refined enumerative geometry", Contemp. Math. **745** (2020), 69–82.
[4] I. Biswas, E. Cotterill, and C. Garay López, *Real inflection points of real hyperelliptic curves*, arXiv:1708.08400, Trans. AMS **372** (2019), no. 7, 4805–4827.
[5] T. Brazelton, R. Burklund, S. McKean, M. Montoro, and M. Opie, *The trace of the local* $\mathbb{A}^1$*–degree*, Homology Homotopy Appl. **23.1** (2021), 243–255.
[6] T. Bachmann and K. Wickelgren, $\mathbb{A}^1$*-Euler classes: Six functors formalisms, dualities, integrality and linear subspaces of complete intersections*, arXiv:2002.01848.
[7] E. Cotterill and C. Garay López, *Real inflection points of real linear series on an elliptic curve*, arXiv:1804.06524, Experimental Math (2019), doi:10.1080/10586458.2019.1655815.

[4]Indeed, because $n = g + 2$, those atomic inflectionary curves $\mathcal{C}_n$ that parameterize inflection points of linear series on elliptic curves satisfy $n \geq 3$.

[8] E. Cotterill and C. Garay López, *Inflection divisors of linear series on an elliptic curve*, `arXiv:1903.03222`, 2018 ICM satellite conference on moduli proceedings, Mat. Contemp. **47** (2020), 73–82.
[9] D. Cox, J. Little, and H. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, 2011.
[10] D. Eisenbud and J. Harris, *Divisors on general curves and cuspidal rational curves*, Invent. Math. **74** (1983), 371–418.
[11] D. Eisenbud and J. Harris, *Existence, decomposition, and limits of certain Weierstrass points*, Invent. Math. **87** (1987), 495–515.
[12] I. Gessel and G. Viennot, *Binomial determinants, paths, and hook length formulae*, Adv. Math. **58** (1985), no. 3, 300–321.
[13] J. Huisman, *Algebraic moduli of real elliptic curves*, Comm. Alg. **29** (2001), no. 8, 3459–3476.
[14] E. Katz, J. Rabinoff, and D. Zureick–Brown, *Diophantine and tropical geometry, and uniformity of rational points on curves*, in "Algebraic Geometry: Salt Lake City 2015", AMS Proc. Sympos. Pure Math. (2018), 231-279.
[15] J. Kass and K. Wickelgren, *The class of Eisenbud–Khimshiashvili–Levine is the local $\mathbb{A}^1$-Brouwer degree*, Duke Math J. **168** (2019), no. 3, 429–469.
[16] J. Kass and K. Wickelgren, *An arithmetic count of the lines on a smooth cubic surface*, `arXiv:1708.01175`, to appear in Compositio Math.
[17] J. Kass and K. Wickelgren, *A classical proof that the algebraic homotopy class of a rational function is the residue pairing*, Linear Algebra Appl. **595** (2020), 157–181.
[18] T. Lam, *Introduction to quadratic forms over fields*, Graduate Studies in Mathematics, vol. 67, American Mathematical Society, Providence, 2005.
[19] M. Levine, *Toward an enumerative geometry with quadratic forms*, `arXiv:1703.03049`.
[20] S. McKean, *An arithmetic enrichment of Bézout's theorem*, Math. Ann. **379** (2021), no. 1, 633–660.
[21] W. McCallum and B. Poonen, *The method of Chabauty and Coleman*, in "Explicit methods in number theory; rational points and Diophantine equations", Panoramas et synthèses **36** (2012), Soc. Math. de France.
[22] N. Pflueger, *On non-primitive Weierstrass points*, Alg. Number Th. **12** (2018), 1923–1947.
[23] E. Previato, *Poncelet's theorem in space*, Proc. Amer. Math. Soc. **127** (1999), no. 9, 2547–2556.
[24] J.-P. Serre, *Lectures on $N_X(p)$*, Research Notes in Mathematics, CRC Press, 2011.
[25] J. Silverman, *Some arithmetic properties of Weierstrass points: hyperelliptic curves*, Bull. Braz. Math. Soc. **21** (1990), no. 1, 11–50.
[26] P. Srinivasan and K. Wickelgren, *An arithmetic count of the lines meeting four lines in $\mathbb{P}^3$*, with an appendix by B. Kadets, P. Srinivasan, A.A. Swaminathan, L. Taylor, and D. Tseng, to appear in Trans. AMS.
[27] K.-O. Stöhr and J. F. Voloch, *Weierstrass points and curves over finite fields*. Proc. London Math. Soc. **52** (1986), no. 3, 1–19.
[28] P. Vojta, *Jets via Hasse-Schmidt derivatives*, Diophantine geometry, CRM Series **4** (2007), 335–361.

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