RESEARCH STATEMENT

CHANGHO HAN

1. Introduction

Algebraic geometry originated in the study of shapes defined by zero sets of polynomials, i.e., algebraic varieties. Since functions can be approximated by polynomials, problems in other subjects such as differential geometry, physics, statistics, and geometric optimization, can be modeled by problems in algebraic geometry. I am interested in finding geometric properties of such shapes and how they change as coefficients of the underlying polynomials vary. By changing the fields of definition for such polynomials, I also consider problems in arithmetic geometry, which has direct applications in number theory.

For example, an important question in algebraic geometry is the classification of projective varieties. It is remarkable that certain collections of projective varieties are themselves parameterized by points of an algebraic variety, called a *moduli space*. A well-known example is the moduli space $\mathcal{M}_g$ of smooth projective curves (i.e., compact Riemann surfaces) of a fixed genus g. In general, a point of a moduli space $\mathcal{M}$ corresponds to an object in the classification, and a map from a space V to $\mathcal{M}$ corresponds to a family of objects parameterized by V. In this way, $\mathcal{M}$ not only gives ways to understand individual objects, but also naturally gives ways to understand families of objects. Typically, points in $\mathcal{M}$ are distinguished by geometric and/or arithmetic properties, called invariants. For example, both the genus of a curve and the number of rational points are invariants.

My work focuses on the geometry and arithmetic of projective varieties and moduli spaces, particularly moduli spaces of curves and surfaces. Some of my projects involve producing explicit geometric and arithmetic descriptions of moduli spaces. The other projects instead focus on understanding invariants of curves and how they vary in special families. See § 1.1, § 1.2, and § 1.3 below for snapshots of my research projects.

1.1. **Birational geometry of moduli spaces.** (See Section 2) A crucial observation in the study of moduli spaces is that a space $\mathcal{M}$ may admit more than one modular interpretation. For example, Kondō considered in [27] a space over $\mathbb{C}$, which is both 1) an open subset of the moduli $\mathcal{M}_4$ of smooth curves of genus 4, and 2) a moduli space parameterizing certain special K3 surfaces. Typically, such spaces are not compact, which in turn poses difficulties in understanding their geometric and topological properties. Instead, I consider different choices of *compactifications* $\overline{\mathcal{M}}$ of a given moduli space $\mathcal{M}$.

On one hand, one can carefully enlarge the moduli space $\mathcal{M}$ by allowing the varieties in question to acquire certain singularities. The resulting moduli space $\overline{\mathcal{M}}$ is called a *modular compactification* of $\mathcal{M}$. For example, enlarging the class of smooth projective curves by adding stable curves, Deligne and Mumford in [11] constructed a compact moduli space $\overline{\mathcal{M}}_g$ of stable curves as a modular compactification of $\mathcal{M}_g$; this led to significant advances not only in algebraic geometry, but also in number theory and mathematical physics (for example, see [23, 18, 11]). On the other hand, one can use the analytic (or combinatorial) properties of $\mathcal{M}$ and construct non-modular compactifications: a key example is the Baily-Borel compactification of the moduli of K3 surfaces (see [33] for details), which led to significant advances in Hodge theory and differential geometry (such as [17, 28]).

In joint work in progress with Valery Alexeev, Anand Deopurkar, and Philip Engel [3, 12, 13], we compare and contrast different choices of compactifications of moduli spaces of K3 surfaces, leading to a better understanding of their geometric properties. Since the distinct compactifications $\overline{M}_1, \overline{M}_2$ of given moduli space M are birational (i.e., they share a common open dense subset M), I am interested in understanding the underlying rational map $\overline{M}_1 \dashrightarrow \overline{M}_2$ geometrically for interesting choices of M, and apply this knowledge to explicitly describe $\overline{M}_2$ whenever $\overline{M}_1$ is well-known but not $\overline{M}_2$. For instance, we show that modular compactifications of moduli of certain K3 surfaces (including Kondō's example above) are very close to combinatorially-constructed compactifications of K3 surfaces (which are typically non-modular).

1.2. **Inflection points on hyperelliptic and superelliptic curves.** (See Section 3) A hyperelliptic curve X is a double cover of $\mathbb{P}^1$. The Weierstrass points of X, which are ramification points of the degree 2 map $\pi : X \to \mathbb{P}^1$, play a key role in the geometry of X. The characteristic property of a Weierstrass point p in X is that any nonzero global section of the sheaf of differentials Ω^1_X has vanishing order not equal to 1 at p. As a generalization, an inflection point p of a linear series $V \subset H^0(\mathcal{L})$ of a line bundle $\mathcal{L}$ on X is a point where sections of V have nongeneric vanishing. For example, the fact that any plane cubic has nine flexes is a special case of Plücker's formula, which counts the number of inflection points of $(X, V \subset H^0(\mathcal{L}))$ (see [15, §7.5] for a quick survey).

Although the study of inflection points of a single linear series $(X, V \subset H^0(\mathcal{L}))$ is relatively well-known over the complex numbers, it is more obscure over other fields, such as $\mathbb{R}$ and finite fields $\mathbb{F}_q$. On one hand, there is a generalization of oriented intersection theory for counting points on manifolds with $\pm$ sign (over $\mathbb{R}$): in the case of finite fields $\mathbb{F}_q$, the $\pm$ sign is replaced by certain Legendre symbols for quadratic reciprocity (see [24, 29]). In joint work with Ethan Cotterill and Ignacio Darago [10], we characterize inflection points explicitly by using this technique, giving arithmetic meaning to them.

On the other hand, given a family of curves and their linear series, it is interesting to consider the loci of inflection points. For example, the number of $\mathbb{F}_q$-rational points of special modular curve $X_1(2M)$ is indeed equivalent to the number of $\mathbb{F}_q$-rational points in the loci of inflection points of the universal family of elliptic curves. With this motivation in mind, it makes sense to ask about the distribution of $\mathbb{F}_q$-inflection points in fibers of Legendre or Weierstrass families of n-superelliptic curves $y^n = f(x)$. In joint work with Ethan Cotterill, Ignacio Darago, Cristhian Garay López, and Tony Shaska [10, 9], we obtain interesting results and conjectures about the arithmetic behavior of inflection points for such families.

1.3. **Topology and arithmetic of moduli spaces.** (See Section 4) A key question in number theory is that of determining how many arithmetic varieties there are defined over a number field K (e.g. cubic field extensions, elliptic curves, etc). In order to obtain a finite number, it is necessary to impose an upper bound on an arithmetic invariant of curves such as the height of the minimal discriminant or the conductor. Using the geometric framework of moduli spaces, it is equivalent to count the number of K-rational points of the corresponding arithmetic moduli space $\mathcal{M}$ with bounded heights. This count is typically given as an asymptotic function: for example, in counting cubic fields, the asymptotic is $a\mathcal{B} + b\mathcal{B}^{\frac{5}{6}} + o(\mathcal{B}^{\frac{5}{6}})$ where $\mathcal{B}$ denotes the upper bound on the height of the discriminant.

Then given the asymptotic count, the main issue is interpreting the leading and lower-ordered terms. The meaning of the leading terms is relatively well-known, but that of the lower-ordered terms are mysterious as in [19, Problem 5]. Since the number fields and function fields (such as $\mathbb{F}_q(t)$) share many common properties, one can instead ask the similar questions for counting the number of arithmetic varieties defined over $\mathbb{F}_q(t)$: this is the so-called global fields analogy. The main advantage of this approach is that counting arithmetic curves defined over a number field K is analogous to counting arithmetic curves over $\mathbb{F}_q(t)$; we show that this is analogous to counting genus g fibrations $Y \to \mathbb{P}^1$ defined over $\mathbb{F}_q$.

Using these observations, in joint work with Jun-Yong Park [21, 22], we compute an asymptotic upper bound for the number of hyperelliptic genus g fibrations $X \to \mathbb{P}^1$ defined over $\mathbb{F}_q$. The main idea is to first approximate the moduli space of hyperelliptic genus g curves by a weighted projective stack $\mathcal{P}(\vec{\lambda})$, and then count the $\mathbb{F}_q$-rational points of the space $\mathrm{Hom}(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ (defined in [31]) for a fixed discriminant degree. This approach not only finds the asymptotic upper bound for the count, but also uncovers topological properties of $\mathrm{Hom}(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}))$ which are related to topological properties of the original moduli $\mathcal{H}_{g,1}$ of hyperelliptic genus g fibrations. In the end, these results lead back into interesting conjectures: by using global fields analogy, there are new conjectures on asymptotics of the count of hyperelliptic genus g curves over $\mathbb{Z}$ with discriminants of bounded height.

2. Birational geometry of moduli spaces of log surfaces

2.1. **Compactifying moduli spaces of K3 surfaces.** K3 surfaces, named after Kummer, Kähler, Kodaira and the K2 mountain in Kashmir, became one of the most important types of algebraic surfaces starting from the 20th century: as a higher-dimensional analog of an elliptic curve, an algebraic K3 surface X is a complex projective surface with trivial canonical bundle and irregularity zero. A polarized K3 surface of degree $2d$ is a pair $(X, \mathcal{L})$ of a K3 surface X with a semiample line bundle $\mathcal{L}$ of degree $2d$. Then, the moduli space F_{2d} of polarized K3 surfaces of degree $2d$, is described as a quotient $\mathbb{D}_{2d}/\Gamma_{2d}$ where the period domain $\mathbb{D}_{2d}$ parameterizes certain pure Hodge structures of weight 2 on a fixed K3 lattice $L_{K3} := H^{\oplus 3} \oplus E_8^{\oplus 2}$ and Γ_{2d} is the group of Hodge isometries between them. There are many known important compactifications of F_{2d} that come from analytic and combinatorial techniques: for example, the Baily-Borel compactification $\overline{F}_{2d}^{\mathrm{BB}}$ by [6] as a generalization of Satake's method in [32] to compactify the moduli space $\overline{M}_1$ of elliptic curves, and a toroidal compactification $\overline{F}_{2d}^{\Sigma}$ of F_{2d} by [5] that depends on a choice of a collection Σ of admissible fans, which is a combinatorial data that roughly describes how boundary divisors meet.

However, both Baily-Borel and toroidal compactifications of F_{2d} are in general not modular; especially, $\overline{F}_{2d}^{\mathrm{BB}}$ admits bad singularities (not normal) wheras singularities of toroidal compactifications are nice (i.e., normal). To find modular compactifications of F_{2d}, let us consider generalizations of both compactifications, called semitoric compactifications by Looijenga in [28]: a semitoric compactification $\overline{F}_{2d}^{\Sigma}$ now depends on a collection Σ of semifans. Using this, Alexeev and Engel showed in [2] that a stable slc compactification (also called stable pair compactification by Kollár in [25]) $\overline{F}_{2d}^{slc}$ of F_{2d}, which parameterizes stable slc pairs $(X, \epsilon R)$ for canonically chosen effective divisor R based on the given polarization $\mathcal{L}$, coincides, up to normalization, with a semitoric compactification $\overline{F}_{2d}^{\Sigma}$ for certain collection Σ of semifans. Moreover, there is a refinement $\bar{\Sigma}$ of Σ as a collection of fans, which implies that $\overline{F}_{2d}^{\Sigma}$ sits in between toroidal and Baily-Borel compactifications:

$$\overline{F}_{2d}^{\bar{\Sigma}} \to \overline{F}_{2d}^{\Sigma} \cong \overline{F}_{2d}^{slc} \to \overline{F}_{2d}^{\mathrm{BB}}$$

Using similar ideas, it is known that certain toroidal compactifications of elliptic K3 surfaces are stable slc compactifications by [1].

Generalizing the observation that an elliptic curve comes equipped with an involution (the quotient being a rational curve), it is interesting to consider the moduli of K3 surfaces with nonsymplectic group actions (see [14] and § 2.2 below for examples). Let $\rho \in O(L_{K3})$ be an automorphism of order n, and a K3 surface X is ρ-markable if there is a marking $\phi : H^2(X;\mathbb{Z}) \to L_{K3}$ such that $\phi^{-1}\rho\phi = f^*$ for some nonsymplectic automorphism $f \in \mathrm{Aut}(X)$. Then, in joint work with Valery Alexeev and Philip Engel [3], we observe that as long as the fixed locus of f in X contains a smooth curve C of genus $g \geq 2$, then the choice of C is unique and $\mathcal{O}_X(C)$ is a polarization. By generalizing strategies in [2] shown above, we show in the joint work [3] that the stable slc compactification $\overline{F}_{\rho}^{slc}$ of the moduli F_ρ of ρ-markable K3 surfaces is a semitoric compactification; in particular, $\overline{F}_{\rho}^{slc}$ satisfies many nice properties coming from semitoric compactifications.

2.2. **Moduli of stable log quadrics.** In the study of moduli of surfaces, the stable slc compactifications (see for example K3 examples above) constitute important classes of modular compactifications; they are higher-dimensional analogs of the Deligne-Mumford compactifications of moduli of (pointed) curves.

As a generalization of the pairs $(X, \epsilon R)$ of K3 surfaces X with semiample divisors R, there has been a significant recent advances in the moduli of polarized Calabi-Yau pairs, first by Hacking [20] and the generalizations by Kollár and Xu in [26]. As a direct application of Hacking's idea, we show in joint work with Anand Deopurkar [12] that the stable slc compactification (also sometimes called KSBA compactification) $\mathfrak{X}$, of moduli of $(\mathbb{P}^1 \times \mathbb{P}^1, (\frac{2}{3}+\epsilon)C)$ with C curves of bidegree $(3,3)$, is projective with finite quotient singularities, and is also a compactification of the blowup $\mathrm{Bl}_{H_4} M_4$ of the hyperelliptic locus H_4 in the moduli M_4 of genus 4 curves. We obtained these results via the explicit classification of the boundary members of $\mathfrak{X}$.

Afterwards, I noticed a work by Kondō [27], which describes the birational period map $M_4 \dashrightarrow \mathbb{B}_9/\Gamma_\rho$, where $\mathbb{B}_9/\Gamma_\rho$ is a moduli of K3 surfaces with nonsymplectic automorphisms of order 3. He claims that a general genus 4 curve C sits inside $\mathbb{P}^1 \times \mathbb{P}^1$ as a bidegree $(3,3)$ curve uniquely, and the cyclic degree 3 cover X of $\mathbb{P}^1 \times \mathbb{P}^1$ totally branched over C is a K3 surface. Combining the two joint works [3, 12], $\mathfrak{X}$ must coincide with $\overline{F}_\rho^{slc}$ for some $\rho \in O(L_{K3})$ that describes the cyclic order 3 action on general such cyclic triple cover X; this compactifies Kondō's birational period map. In joint work in progress with Anand Deopurkar [13], we not only provide the explicit description of $\mathfrak{X}$ as a semitoric compactification, but also describe explicit classifications of the boundary members. Furthermore, using the idea that a bidegree $(3,3)$ curve C in $\mathbb{P}^1 \times \mathbb{P}^1$ induces two trigonal maps $C \to \mathbb{P}^1$ via projections, we also show in [13] that a compactification of the Hurwitz space H_4^3, which parameterizes such trigonal maps, coincides with a toroidal compactification $\overline{F}_{M,\rho}^{\mathrm{tor}}$ of moduli $F_{M,\rho}$ of ρ-markable K3 surfaces with extra data; this gives a rather explicit description of H_4^3 and its compactification $\overline{F}_{M,\rho}^{\mathrm{tor}}$, which is difficult to obtain for Hurwitz spaces of degree $d \geq 3$ maps in general.

2.3. **Future directions.** In the future, I plan to explore the role of period domains and their analytic compactifications in other moduli problems. For example, the Tschirnhausen bundles above is a Mori fiber space, i.e., a Fano fibration over a Fano variety. I would like to consider the role of period domains on Calabi-Yau threefolds that are cyclic covers of del Pezzo fibrations over $\mathbb{P}^1$. On the other hand, with the emergence of K-stability of log Fano pairs by [4], I am interested in generalizing above ideas to the compactified K-moduli of log del Pezzo surfaces (as cyclic quotients of K3 surfaces), henceforth drawing birational relations between the three theories: K-moduli, Baily-Borel compactification, and stable slc compactification.

3. Inflection points on hyperelliptic and superelliptic curves

3.1. **Oriented/$\mathbb{A}^1$ intersection theory and inflection points.** Many enumerative problems in algebraic geometry and topology, specifically counting the number of points in a 0-dimensional subvariety, are described as problems in intersection theory (see [15] for examples). An important class of enumerative problems in topology coincides with an Euler class $e(\mathcal{L}) := c_d(\mathcal{L})$ of a line bundle $\mathcal{L}$ of rank d on a projective variety X of dimension d: for example, in topology over $\mathbb{R}$, an Euler characteristic $\chi(X)$ of a compact manifold X of dimension d is indeed the degree of $c_d(T_X)$. The Euler class $e(\mathcal{L})$ in classical topology is obtained by looking at the cohomology class of oriented intersection $\sigma \cap Z$ in X, where σ is any nontrivial section of L and Z is the zero section of L. By using the $\mathbb{A}^1$-homotopy theory, Kass and Wickelgren in [24] extended the notion of the Euler class $e(\mathcal{L})$ to any field, now called the $\mathbb{A}^1$-enumerative geometry; signs on the points of intersection are replaced by arithmetic invariants related to quadratic reciprocity when the base field $\mathbb{F}_q$ is finite (see [29] for details). The main advantage of this theory is that the "signs" correspond geometric properties of the objects in question: for the number of lines on a cubic surface, "signs" describe how lines are contained in the cubic surface.

In joint work with Ethan Cotterill and Ignacio Darago [10], we apply Kass-Wickelgren's program to study inflection points of complete linear series $V = H^0(L)$ on a hyperelliptic curve X given by a Weierstrass equation $y^2 = f(x)$: we furthermore assume that X is defined on a perfect field F of characteristic not 2, and that $L = \mathcal{O}_X(2\ell\infty)$ where $\infty \in X$ is a F-rational Weierstrass point at infinity.

We first observe that the inflection points of $(X, V = H^0(L))$ coincides with Wronskian determinant of an evaluation map $ev : V \to J^r(L)$, where $\dim V = r+1$ and $J^r(L)$ is the r^{th}-Jet bundle [34] of L; this corresponds to a section w of $L^{\otimes r+1} \otimes K_X^{\otimes \binom{r+1}{2}}$. Using this, we immediately apply both local and global Euler class, explicitly computing the loci of inflection points; see [10] for explicit interpretations of "signs" in fields such as $\mathbb{R}$, $\mathbb{F}_q$, $\mathbb{C}(\!(t)\!)$.

3.2. **Inflectionary curves.** Given a family of hyperelliptic curves $\mathcal{X} \to \mathbb{A}^1_\lambda$ defined by $y^2 = f(\lambda, x)$, the locus of inflection points play a key role in understanding the arithmetic of the family. To study this locus effectively, Cotterill and Garay-López in [8] showed that over the real numbers, the locus of inflection points can be recovered from a determinant of a matrix with entries being inflection polynomials $P_m^\ell(\lambda, x)$, which are defined by $D^m y^\ell = f^{-m} y^\ell P_m^\ell$. The main advantage of studying inflection curves C_m^ℓ, defined by vanishing locus of P_m^ℓ, is that they are relatively simple to describe via recursive formulas for P_m^ℓ by loc.cit., and that they still capture the key properties of the families of hyperelliptic curves.

In joint work with Ethan Cotterill, Ignacio Darago, Cristhian Garay-López and Tony Shaska [10, 9], we generalized the theory of inflection curves and polynomials to both families of hyperelliptic and superelliptic curves defined by $y^n = f(\lambda, x)$ over any perfect field of characteristic not two. The key definitions and properties are all analogous as above. We computed genus of C_m^ℓ for low values of ℓ and m in the case of Weierstrass families, and when the genus of C_m^ℓ one, we also found the distribution of $\mathbb{F}_q$-rational points as q changes. Moreover, by using the Newton polygons, we found both results and conjectures on singularities of inflection curves C_m^ℓ.

3.3. **Future directions.** In the future, I plan to extend the above results to related problems in families of curves. First, given a degeneration of a superelliptic curve, it is a prior not obvious how the monodromy of the inflectionary curves occur. In this regard, I would like to consider both the monodromy and the limiting inflections on reducible curves. Some aspects of this has been worked out over $\mathbb{R}$ by [8], but it is less clear on how this appears over finite fields. On the other hand, by working with Ethan Cotterill, we would like to consider the instead the Brill-Noether locus over general fields F, which is a generalization of inflectionary locus. It would be interesting to apply Kass-Wickelgren's program on this setting.

4. Topology and arithmetic of moduli spaces of semistable elliptic surfaces

4.1. **Approximating moduli spaces.** Fix a field K of characteristic not 2, and then consider any stable hyperelliptic genus $g \geq 1$ fibration $\pi : X \to \mathbb{P}^1_K$ given by $y^2 = f(t, x)$ where $\deg_x f = 2g+1$; this enforces π to have a K-rational section at infinity. By the definition of the moduli space $\overline{\mathcal{H}}_{g,\underline{1}}$ of stable odd hyperelliptic genus g curves, π corresponds to a map $\phi : \mathbb{P}^1_K \to \overline{\mathcal{H}}_{g,\underline{1}}$. Then, the moduli of stable hyperelliptic fibrations is the Hom stack $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{H}}_{g,\underline{1}})$. However, it is difficult to extract explicit properties of the moduli stack $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{H}}_{g,\underline{1}})$ for $g \geq 2$, because the stack $\overline{\mathcal{H}}_{g,\underline{1}}$ itself does not have a satisfactory explicit description via equations.

Therefore, in order to extract a desired invariant of the moduli stack $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{H}}_{g,\underline{1}})$ when $g \geq 2$, one should instead consider a carefully chosen birational moduli stack with the birational map between them. In joint work with June Park [22], we are interested in approximating the collection (i.e., groupoid) $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{H}}_{g,\underline{1}})(K)$ of K-rational points of $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{H}}_{g,\underline{1}})$, which corresponds to the collection of stable hyperelliptic genus g fibrations defined over K. To do so, we consider the quasi-admissible curves of genus g defined by Weierstrass equations $y^2 = f(x)$. Then, the moduli of quasi-admissible curves is isomorphic to a weighted projective stack $\mathcal{P}(\vec{\lambda}_g)$ with $\lambda_g := (4, 6, 8, \ldots, 2g+4)$ by [16, 22]. Since each stable curve $(C, \infty) \in \overline{\mathcal{H}}_{g,\underline{1}}$ has a unique log-canonical model (C', ∞) as a quasi-admissible curve by [16], we obtain the inclusion of the collection $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{H}}_{g,\underline{1}})(K)$ of stable hyperelliptic genus g fibrations over K into the collection $\mathrm{Hom}(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}_g))(K)$ of quasi-admissible hyperelliptic genus g fibrations over K. We show in § 4.2 below that the latter can be counted when $K = \mathbb{F}_q$.

4.2. **Counting hyperelliptic fibrations over $\mathbb{P}^1_{\mathbb{F}_q}$.** Before discussing about counting quasi-admissible fibrations, let us first discuss the general problem of counting the collection $\mathcal{M}(\mathbb{F}_q)$ of $\mathbb{F}_q$-points (up to $\mathbb{F}_q$-isomorphisms) of a stack $\mathcal{M}$. It is tempting to apply the Grothendieck-Lefschetz fixed formula developed in [30, 7] to $\mathcal{M}$, but it instead gives the number $\#_q(\mathcal{M}) := M(\mathbb{F}_q)$ of $\mathbb{F}_q$ points of the coarse moduli space M of $\mathcal{M}$. In joint work with June Park [22], we show that the cardinality of $\mathcal{M}(\mathbb{F}_q)$ is the number $\#_q(\mathcal{I}(\mathcal{M}))$ corresponding to the inertia stack $\mathcal{I}(\mathcal{M})$ of $\mathcal{M}$; roughly, points of $\mathcal{I}(\mathcal{M})$ corresponds to a pair (x, g) of $x \in \mathcal{M}$ and a chosen automorphism $g \in \mathrm{Aut}(x)$. When the stack $\mathcal{M}$ is a weighted projective stack, then we show in [22] that the inertia stack $\mathcal{I}(\mathcal{M})$ is a disjoint union of weighted projective stacks; in this case, $\#_q(\mathcal{I}(\mathcal{M}))$ can be explicitly computed.

With a slight tweak to the above strategy, we count in [22] the number $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}_g))(\mathbb{F}_q)$ of quasi-admissible hyperelliptic genus g fibrations defined over $\mathbb{F}_q$ with a fixed discriminant degree $\Delta(g, n)$. Using this, we found that the number $\mathcal{Z}'_{g,\mathbb{F}_q(t)}(\mathcal{B})$, of quasi-admissible hyperelliptic genus g fibrations over $\mathbb{F}_q$ with a bounded discriminant degree $\mathcal{B}$, is a polynomial whose powers of $\mathcal{B}$ are rational numbers and whose coefficients are rational functions of q. This is naturally an upper bound for the number $\mathcal{Z}_{g,\mathbb{F}_q(t)}(\mathcal{B})$, of stable hyperelliptic genus g fibrations over $\mathbb{F}_q$ with a bounded discriminant degree $\mathcal{B}$ by § 4.1; in fact, we show in [21] that those two numbers are equal when $g = 1$. Finally, we obtain interesting conjectures for lower-order terms on counting the number $\mathcal{Z}_{g,\mathbb{Q}}(\mathcal{B})$ of stable hyperelliptic genus g curves over $\mathbb{Q}$ via global fields analogy, which states that the geometry of $\mathbb{P}^1_{\mathbb{F}_q}$ and $\mathrm{Spec}\,\mathbb{Z}$ are closely related.

4.3. **Future directions.** In the future, I plan to apply the counting techniques (via inertia stack) to various modulis. For instance, it would be interesting to find the actual count of $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{H}}_{g,\underline{1}})(\mathbb{F}_q)$ of stable hyperelliptic genus g fibrations by describing $\overline{\mathcal{H}}_{g,\underline{1}}$ as a sequence of nontoric blowups of a weighted projective stack via the minimal model program in [16]. Another case that I am interested is for counting the number of lines/planes of hypersurfaces or Grassmannian varieties: some aspects of this is already done when the corresponding Fano variety of lines/planes are 0-dimensional. Instead, the number of $\mathbb{F}_q$-rational points of positive-dimensional Fano varieties of lines/planes of a given variety has not been fully worked out. This will be successful as long as the inertia stacks of the Fano varieties are not too complicated.

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