RESEARCH STATEMENT

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1. Introduction

Algebraic geometry originated in the study of shapes defined by zero sets of polynomials, i.e., algebraic varieties. Since functions can be approximated by polynomials, problems in other subjects such as differential geometry, number theory, mathematical physics, statistics, and geometric optimization, can be modeled by problems in algebraic geometry. I am interested in finding geometric properties of such shapes and how they change as coefficients of the underlying polynomials vary. More precisely, I am interested in problems arising from the classification of projective varieties: remarkably, it suffices to investigate a *moduli space* $\mathcal{M}$, which is an algebraic variety that parameterizes certain collections of projective varieties of interest. Moreover, a map from a space V to $\mathcal{M}$ corresponds to a family of objects parameterized by V. For example, as a zero set of polynomial equations, such a family parametrized by V corresponds to the case when coefficients of polynomials are functions on V instead of being fixed numbers. In this way, $\mathcal{M}$ not only gives ways to understand individual objects, but also naturally gives ways to understand families of objects. My work focuses on the geometry and arithmetic of moduli spaces that parameterizes projective varieties, particularly moduli spaces of curves and surfaces. Some of my projects involve producing explicit geometric and arithmetic descriptions of moduli spaces in various settings. The other projects instead focus on understanding invariants of curves and how they vary in special families. See Section 2, Section 3, and Section 4 below for snapshots of my research projects.

2. Birational geometry of moduli spaces of surfaces

2.1. **Compactifications of moduli of K3 surfaces with cyclic nonsymplectic automorphisms.** A K3 surface, as a higher-dimensional analog of an elliptic curve, is a compact complex surface with trivial canonical bundle and irregularity zero. Generalizing the observation that an elliptic curve comes equipped with an involution (the quotient being a rational curve), it is interesting to consider the moduli of K3 surfaces with nonsymplectic group actions (see [17] and § 2.2 below for examples). Let $\rho \in O(L_{K3})$ be a nonsymplectic automorphism of order n (so that $\rho^*\omega_X \neq \omega_X$, where ω_X is a nondegenerate holomorphic 2-form on X), and a K3 surface X is ρ-markable if there is a marking $\phi : H^2(X;\mathbb{Z}) \to L_{K3}$ such that $\phi^{-1}\rho\phi = f^*$ for some nonsymplectic automorphism $f \in \mathrm{Aut}(X)$. Then, in joint work with Valery Alexeev and Philip Engel [2], we observe that as long as the fixed locus of f in X contains a smooth curve C of genus $g \geq 2$, then the choice of C is unique and $\mathcal{O}_X(C)$ is a polarization; thus, X must be projective. This led us to construct the moduli space F_ρ of ρ-markable K3 surfaces. As an open subset of a quotient of Hermitian symmetric domains, there are many known important compactifications of F_{2d} that come from analytic and combinatorial techniques: for example, the Baily-Borel compactification $\overline{F}_\rho^{\mathrm{BB}}$ by [7] as a generalization of Satake's method in [29] to compactify the moduli space $\overline{M}_1$ of elliptic curves; and a toroidal compactification $\overline{F}_\rho^{\Sigma}$ of F_{2d} by [6] that depends on a choice of a collection Σ of admissible fans, which is combinatorial data that roughly describes how boundary divisors meet.

However, both Baily-Borel and toroidal compactifications of F_{2d} are in general *not* modular. For the Baily-Borel compactification, points in the boundary do not correspond to a projective variety. On the other hand, some toroidal compactifications (depending on Σ) are close to being modular, except that multiple snc surfaces may correspond to the same boundary point. To find modular compactifications of F_ρ, we consider the semitoric compactifications constructed by Looijenga in [27]: a semitoric compactification $\overline{F}_\rho^{\Sigma}$ now depends on a collection Σ of semifans. Using this, we show that:

Theorem 2.1 ([2])**.** *A stable slc compactification $\overline{F}_\rho^{slc}$ of the moduli F_ρ of ρ-markable K3 surfaces exists. $\overline{F}_\rho^{slc}$ is isomorphic to a semitoric compactification, and sits in between toroidal and Baily-Borel compactifications:*

$$\overline{F}_\rho^{\bar{\Sigma}} \to \overline{F}_\rho^{\Sigma} \cong \overline{F}_\rho^{slc} \to \overline{F}_\rho^{\mathrm{BB}}.$$

Moreover, the first morphism has a geometric meaning: it sends a pair $(X, \epsilon R) \in \overline{F}_\rho^{\bar{\Sigma}}$ of a Kulikov surface X, which is a snc degeneration of K3 surfaces, and the degeneration R of ρ-fixed locus of ρ-markable K3 surfaces, to its log canonical model $(\overline{X}, \epsilon\overline{R})$.

There are several examples of $\overline{F}_\rho^{slc}$ that have been worked out explicitly. Applying our theorem to describe the birational geometry of moduli spaces of any K3 surfaces with a nonsymplectic involution has been recently carried out by [3, 1]. For cyclic nonsymplectic automorphisms of higher order, it is less known: see § 2.2 for an example in progress.

2.2. **Moduli of stable log quadrics and Kondō's ball quotients.** When the nonsymplectic cyclic automorphism ρ has order 3, there is an interesting example of F_ρ considered by Kondō [26]: any general ρ-markable K3 surface X comes from the cyclic triple cover of $\mathbb{P}^1 \times \mathbb{P}^1$ totally branched over a curve C of bidegree $(3,3)$. In *loc. cit.*, Kondō describes the birational period map $M_4 \dashrightarrow F_\rho$ by the following observation: a general genus 4 curve $C \in M_4$ sits inside $\mathbb{P}^1 \times \mathbb{P}^1$ as a bidegree $(3,3)$ curve uniquely, and the cyclic degree 3 cover X of $\mathbb{P}^1 \times \mathbb{P}^1$ totally branched over C is a K3 surface.

Using this idea, we show in [2] that $\overline{F}_\rho^{slc}$ is isomorphic to $\mathfrak{X}$, which is a stable slc (also called KSBA) compactification of moduli space parameterizing pairs $(\mathbb{P}^1 \times \mathbb{P}^1, (\frac{2}{3} + \epsilon)C)$ with C curves of bidegree $(3,3)$. This $\mathfrak{X}$ is an example of the moduli of polarized Calabi-Yau pairs, first by Hacking [20] and the generalizations by Kollár and Xu in [25]. From this viewpoint, any member $(Y, (\frac{2}{3} + \epsilon)D) \in \mathfrak{X}$ is called a stable log quadric, which is a slc degeneration of $(\mathbb{P}^1 \times \mathbb{P}^1, (\frac{2}{3} + \epsilon)C)$. In joint work with Anand Deopurkar [15], we classify members of $\mathfrak{X}$ and relate $\mathfrak{X}$ to M_4:

Theorem 2.2 ([15])**.** *As a scheme, $\mathfrak{X}$ is an irreducible of dimension 9 of Picard rank 4 with finite quotient singularities, and is also a compactification of the blowup $\mathrm{Bl}_{H_4} M_4$ of the hyperelliptic locus H_4 in the moduli space M_4 of genus 4 curves. In addition, there is a classification of stable log quadrics parameterized by $\mathfrak{X}$.*

In joint work in progress with Anand Deopurkar [16], we describe the geometry of the toroidal compactification $\overline{F}_\rho^{tor}$ appearing in Theorem 2.1 for Kondō's ball quotient moduli F_ρ from above. First, by using the isomorphism $\mathfrak{X} \cong \overline{F}_\rho^{slc}$ where a stable log quadric is mapped to its cyclic triple cover, we also classify elements of $\overline{F}_\rho^{slc}$, which do not necessarily have snc singularities. To classify members of $\overline{F}_\rho^{tor}$ which should have snc singularities, we use the idea that a bidegree $(3,3)$ curve C in $\mathbb{P}^1 \times \mathbb{P}^1$ induces two trigonal maps $C \to \mathbb{P}^1$ via projections.

Theorem 2.3 (in progress, [16])**.** *$\overline{F}_\rho^{tor}$ is isomorphic to the quotient $H_4^3(\frac{1}{6} + \epsilon)/(\mathbb{Z}/2\mathbb{Z})$, where $H_4^3(\frac{1}{6} + \epsilon)$ is a compactification of the Hurwitz space H_4^3 parameterizing trigonal maps from genus 4 curves to $\mathbb{P}^1$ by* [14]*, and $\mathbb{Z}/2\mathbb{Z}$-action is the extension of the switching the two rulings of $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ as a bidegree $(3,3)$ curve. Using this isomorphism, we obtain the classification of members of $\overline{F}_\rho^{tor}$.*

This also gives an interpretation of $H_4^3(\frac{1}{6} + \epsilon)$ as a toroidal compactification of ρ-markable K3 surfaces equipped with an elliptic fibration (without section).

2.3. **Future directions.** In the future, I plan to explore the role of period domains and their compactifications in other moduli problems. A natural line of investigation is to find further explicit geometric properties of various toroidal compactifications as in Theorem 2.3. For example, I would like to classify members of toroidal compactifications associated to the moduli of genus 3 curves (generally, such C is a plane quartic in $\mathbb{P}^2$), and draw connections to the Brill-Noether loci of g^2_3's over M_3.

Also, I would like to consider the role of period domains on Calabi-Yau threefolds that are cyclic covers of del Pezzo fibrations over $\mathbb{P}^1$. On the other hand, with the emergence of K-stability of log Fano pairs by [5], I am interested in generalizing above ideas to the compactified K-moduli of log del Pezzo surfaces (as cyclic quotients of K3 surfaces), henceforth drawing birational relations between K-moduli spaces, Baily-Borel compactifications, and stable slc compactifications. I believe that with enough guidance, a strong graduate student would be able to produce a good thesis on classifying members of these moduli spaces, and also possibly draw interesting birational relations between them.

3. Enriched counts of inflection points on (hyper/super)elliptic curves

3.1. **Enriched/$\mathbb{A}^1$ enumerative geometry.** Unlike the field of complex numbers, enumerative problems in algebraic geometry are more complicated over other fields, such as $\mathbb{R}$ and finite fields $\mathbb{F}_q$. On one hand, there is an enrichment of counting problems, which is an algebraic analogue of counting points on manifolds defined over $\mathbb{R}$ with $\pm$ "signs"; without this enrichment, the resulting count may depend on the topology of $X(\mathbb{R})$, where X is the ambient real manifold.

An important class of enumerative problems in topology coincides with an Euler class $e(E) := c_d(\mathcal{E})$ of a vector bundle $\mathcal{E}$ of rank d on a projective variety X of dimension d defined over a field F: for example, in topology over $\mathbb{R}$, an Euler characteristic $\chi(X)$ of a compact manifold X of dimension d is indeed the degree of $c_d(T_X)$. When σ is a general section of $\mathcal{E}$, the Euler class $e(\mathcal{L})$ coincides with the degree of intersection $\sigma \cap Z$ in X, where Z is the zero section of L. By using $\mathbb{A}^1$-homotopy theory, Kass and Wickelgren in [24] enriched the notion of the Euler class $e(\mathcal{E})$ to any field. In enriched enumerative geometry, the multiplicity of intersection $\sigma \cap Z$ at a closed point p of X is replaced by an element $\mathrm{ind}_p\sigma$ in the Grothendieck–Witt ring $GW(F)$, which is the ring of quadratic forms defined over F upto isometric equivalences. The enriched local index $\mathrm{ind}_p\sigma$ is a simultaneous generalization of "signs" and multiplicities when $F = \mathbb{R}$, because $GW(\mathbb{R}) \cong \mathbb{Z}\langle 1\rangle \oplus \mathbb{Z}\langle -1\rangle$ as a group, and $m\langle 1\rangle + n\langle -1\rangle$ refers to intersection multiplicity $m+n$ and signature $m-n$ (sum of "signs").

3.2. **Enriched counts of inflection points.** The Weierstrass points of a hyperelliptic curve X, which are ramification points of the degree 2 map $\pi : X \to \mathbb{P}^1$, play a key role in the geometry of X. As a generalization, an inflection point p of a linear series $V \subset H^0(\mathcal{L})$ of a line bundle $\mathcal{L}$ on X is a point where sections of V have nongeneric vanishing. For example, the fact that any plane cubic has nine flexes is a special case of Plücker's formula, which counts the number of inflection points of $(X, V \subset H^0(\mathcal{L}))$ (see [18, §7.5] for a quick survey).

In joint work with Ethan Cotterill and Ignacio Darago [12], we apply Kass-Wickelgren's enriched counting program to study inflection points of complete linear series $V = H^0(L)$ on a hyperelliptic curve X given by a Weierstrass equation $y^2 = f(x)$: we furthermore assume that X is defined on a perfect field F of characteristic not 2, and that $L = \mathcal{O}_X(2\ell\infty)$ where $\infty \in X$ is a F-rational Weierstrass point at infinity. We use the fact that the inflection points of $(X, V = H^0(L))$ comprise the vanishing locus of the Wronskian determinant of an evaluation map $ev : V \to J^r(L)$, where $\dim V = r+1$ and $J^r(L)$ is the r^{th}-Jet bundle [30] of L; this corresponds to a section w of $L^{\otimes r+1} \otimes K_X^{\otimes\binom{r+1}{2}}$. Using this, we obtain:

Theorem 3.1 ([12]). *The global enriched count of inflection points is $\frac{m}{2}\mathbb{H}$, where m is the degree of $L^{\otimes r+1} \otimes K_X^{\otimes\binom{r+1}{2}}$ and $\mathbb{H} = \langle 1\rangle + \langle -1\rangle$ is the hyperbolic form. Furthermore, these explicit formulas for local inflection indices* $\mathrm{ind}_p w$ *at inflection points p, where w is the Wronskian section as above.*

3.3. **Inflectionary curves.** On the other hand, given a family of hyperelliptic curves $\mathcal{X} \to \mathbb{A}^1_\lambda$ defined by $y^2 = f(\lambda, x)$, the locus of inflection points play a key role in understanding the arithmetic of the family. To study this locus effectively, Cotterill and Garay-López in [10] showed that the locus of inflection points can be recovered from a determinant of a matrix with entries being inflection polynomials $P^\ell_m(\lambda, x)$, which are defined by $D^m y^\ell = f^{-m} y^\ell P^\ell_m$. The main advantage of studying inflection curves C^ℓ_m, defined by vanishing locus of P^ℓ_m, is that they are relatively simple to describe via recursive formulas for P^ℓ_m by loc.cit., and that they still capture the key properties of the families of hyperelliptic curves.

In joint work with Ethan Cotterill, Ignacio Darago, Cristhian Garay-López and Tony Shaska [12, 11], we generalized the theory of inflection curves and polynomials to both families of hyperelliptic and superelliptic curves defined by $y^n = f(\lambda, x)$ over any perfect field of characteristic not two:

Theorem 3.2. *The inflection curves C^ℓ_m for a given family superelliptic curves can be recursively computed. In particular, there is an explicit formula for the genus of C^ℓ_m for low values of ℓ and m in the case of Weierstrass families. The singularities of C^ℓ_m can be analyzed under some mild conditions.*

We also found the distribution of $\mathbb{F}_q$-rational points on C^ℓ_m as q changes. Moreover, by using the Newton polygons, we found conjectures on singularities of inflection curves C^ℓ_m.

3.4. **Future directions.** In the future, I plan to extend the above results to related problems in families of curves. First, given a hyperelliptic curve X with a linear series V, the d-secant u-plane locus, consisting of a d-tuple of points of X that lie on a u-plane in $\mathbb{P}V$. This is related to the Brill-Noether theory. In a joint work in progress with Ethan Cotterill and Naizhen Zhang, we plan to compute both local and the global enriched count of such secant loci. Also, it would be interesting to get a grasp of how local inflectionary/secant indices change as the linear series and curve changes. By using the recent theory of enriched Brouwer degree in [23], we plan to obtain deeper understanding on the "variation" of local inflectionary/secant indices on a family of linear series over a family of curves. It would be an interesting area for a graduate student to dive in, as this relatively new area has many interesting explicit problems that are accessible up to black-boxing the proofs of certain computational tools.

4. Topology and arithmetic of moduli spaces of elliptic curves/surfaces

4.1. **Arithmetic of moduli spaces.** Given a global field K (i.e., a number field or a function field of a curve) and arithmetic varieties of interest defined over K, the notion of a height distinguishes the complexity of such varieties when extending them to a scheme over the spec of a ring of integers $\mathcal{O}_K$. Using the geometric framework of moduli spaces, the number of such varieties up to a bounded height B is equal to the number of K-rational points of the corresponding arithmetic moduli space $\mathcal{M}$ with the same height condition. The projects in the rest of this section aim to enumerate the set $\langle \mathcal{M}(K) \rangle$ of K-isomorphisms of K-points of $\mathcal{M}$ by using the arithmetic and geometric properties of M.

When the global field K is a function field of a curve C defined over a finite field $\mathbb{F}_q$, the set $\langle \mathcal{M}(K) \rangle$ is loosely related to $\left\langle \mathcal{M}(\mathbb{F}_q) \right\rangle$ by a "local-to-global" heuristic. When $\mathcal{M}_{\mathbb{F}_q}$ is a stack, the Grothendieck-Lefschetz fixed formula developed in [28, 8] gives the number $\#_q(\mathcal{M}) := M(\mathbb{F}_q)$ of $\mathbb{F}_q$ points of the coarse moduli space M of $\mathcal{M}$. In joint work with June Park [22], we compute $\left\langle \mathcal{M}(\mathbb{F}_q) \right\rangle$:

Theorem 4.1 ([22])**.** *We have $\# \left\langle \mathcal{M}(\mathbb{F}_q) \right\rangle = \#_q(\mathcal{I}(\mathcal{M}))$. where $\mathcal{I}(\mathcal{M})$ is the inertia stack of $\mathcal{M}$. Furthermore, this number can be explicitly computed when $\mathcal{M}$ is a weighted projective stack.*

Roughly, points of $\mathcal{I}(\mathcal{M})$ corresponds to a pair (x, g) of $x \in \mathcal{M}$ and a chosen automorphism $g \in \mathrm{Aut}(x)$.

4.2. **Counting hyperelliptic fibrations over $\mathbb{P}^1_{\mathbb{F}_q}$.** When K is a function field of the form $\mathbb{F}_q(t)$, any K-point of the moduli space $\mathcal{H}_{g,\underline{1}}$ of smooth odd hyperelliptic genus g curves extend to a hyperelliptic fibration $\pi : X \to \mathbb{P}^1_{\mathbb{F}_q}$ over $\mathbb{F}_q$; but some fibers of π could be singular. When every fiber of π is stable, this π corresponds to a map $\phi : \mathbb{P}^1_K \to \overline{\mathcal{H}}_{g,\underline{1}}$, where $\overline{\mathcal{H}}_{g,\underline{1}}$ is the moduli space of stable odd hyperelliptic genus g curves. For example, when $g=1$, $\overline{\mathcal{H}}_{g,\underline{1}}$ is isomorphic to $\overline{\mathcal{M}}_{1,1}$ the moduli space of elliptic curves. From this viewpoint, the moduli of stable hyperelliptic fibrations is then the Hom stack $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{H}}_{g,\underline{1}})$.

However, the geometry of $\overline{\mathcal{H}}_{g,\underline{1}}$ is complicated for $g \geq 2$, so we instead consider a birational model of $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{H}}_{g,\underline{1}})$: in joint work with June Park [22], we instead consider the quasi-admissible fibrations $\pi : X \to \mathbb{P}^1_{\mathbb{F}_q}$ of relative genus g defined by Weierstrass equations $y^2 = f(x,t)$. Then, using that the weighted projective stack $\mathcal{P}(\vec{\lambda}_g)$ with $\lambda_g := (4,6,8,\ldots,2g+4)$ parameterizes possible fibers of such fibrations by [19, 22], along with a log-MMP technique (extending [19]), we show that the collection $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{H}}_{g,\underline{1}})(K)$ of stable hyperelliptic genus g fibrations over F is included in the collection $\mathrm{Hom}(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}_g))(F)$ of quasi-admissible hyperelliptic genus g fibrations over F.

By defining height of π to be the $q^{\deg \Delta(\pi)}$ where $\Delta(\pi) \subset \mathbb{P}^1_{\mathbb{F}_q}$ is the discriminant divisor, we have:

Theorem 4.2 ([21, 22]). *There is an explicit formula for $\#\left\langle \mathrm{Hom}(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}_g))(\mathbb{F}_q)\right\rangle$ as a polynomial in q, which arises from the class $\left\{\mathrm{Hom}(\mathbb{P}^1, \mathcal{P}(\vec{\lambda}_g))\right\}$ in the Grothendieck ring of $\mathbb{F}_q$-stacks. As a result, the number $\mathcal{Z}'_{g,\mathbb{F}_q(t)}(\mathcal{B})$, of quasi-admissible hyperelliptic genus g fibrations over $\mathbb{F}_q$ with bounded discriminant degree $\mathcal{B}$ is a polynomial in $\mathcal{B}$ with rational exponents, whose coefficients are rational functions of q. When $g=1$, these numbers coincide with counting semistable elliptic fibrations over $\mathbb{F}_q$.*

Finally, we [21, 22] obtain interesting conjectures for lower-order terms on counting the number $\mathcal{Z}'_{g,\mathbb{Q}}(\mathcal{B})$ of quasi-admissible genus g curves over $\mathbb{Q}$ via the "global fields analogy", which posits that the geometry of $\mathbb{P}^1_{\mathbb{F}_q}$ and $\operatorname{Spec}\mathbb{Z}$ should be closely related.

4.3. **Counting elliptic curves with 5-isogenies defined over $\mathbb{Q}$.** On the other hand, in joint work in progress with Santiago Arango-Piñeros, Oana Padurariu, and Sun Woo Park [4], we instead directly count the number of rational points, upto bounded height, of a modular curve $\mathfrak{X}_0(5)$ parameterizing elliptic curves with cyclic 5-isogenies. The key observation is that the ring of modular forms of level 5 allows us to take a nicer integral model of $\mathfrak{X}_0(5)$, as a stacky curve $\mathfrak{Y}_4$ given by the generalized Fermat equation $x^2+y^2=z^4$ in the weighted projective stack $\mathcal{P}(4,4,2)$. This observation leads to the main result, where we use the height induced by the näive height of elliptic curves:

Theorem 4.3. *(in progress, [4]) The count $\mathcal{N}_{\mathfrak{X}_0(5)}(B)$ of $\langle \mathfrak{X}_0(5)\rangle$ upto the bounded näive height B is asymptotic to $cB^{1/6}(\log B)^2$ for some constant c. Furthermore, this count verifies the stacky Batyrev–Manin conjecture for $\mathfrak{X}_0(5)$ as in [13].*

Note that the counting methods implemented in [4] share some similarities with those used in counting unstable elliptic fibrations over the function field by [9].

4.4. **Future directions.** In the future, I plan to apply the counting techniques (via inertia stack) to various moduli spaces. For instance, I am interested is counting the number of lines/planes of hypersurfaces or Grassmannian varieties: some aspects of this are already settled when the corresponding Fano variety of lines/planes are 0-dimensional. On the other hand, the number of $\mathbb{F}_q$-rational points of positive-dimensional Fano varieties of lines/planes of a given variety has not been fully worked out. This will be successful as long as the inertia stacks of the Fano varieties are not too complicated. On the number fields side, I would like to count the rational points of Shimura varieties (that parameterizes abelian varieties with level structures). I think arithmetic-minded graduate students would enjoy working on these explicit counting projects.

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