

RESEARCH STATEMENT

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1. INTRODUCTION

Algebraic geometry originated in the study of zero sets of polynomials. An important question in algebraic geometry is about the classification of *projective varieties*. It is remarkable that a collection of certain projective varieties itself forms an algebraic variety, called a *moduli space*. A well-known example is the moduli space \mathcal{M}_g of smooth projective curves of fixed genus g . A point of a moduli space \mathcal{M} corresponds to an object in the classification. More generally, a map from a space V to \mathcal{M} corresponds to a family of objects parametrized by V . In this way, \mathcal{M} not only gives ways to understand individual objects, but also naturally gives ways to understand families of objects.

My works focus on geometry and arithmetic of moduli spaces. More precisely, I focus on moduli spaces of surfaces. In contrast to curves, the geometry of moduli spaces of surfaces is less understood. In fact, there are very few examples with known geometric properties. For this reason, my projects involve geometric and arithmetic properties of two moduli spaces of surfaces. See § 1.1 and § 1.2 below.

1.1. Birational geometry of moduli spaces of log surfaces. (See Section 2) In a work with Anand Deopurkar [8], we construct an explicit compactification of a moduli space of ‘almost K3’ stable log quadrics.

Theorem 1.1 (Moduli space of ‘almost K3’ stable log quadrics (see Theorem 2.1 for the details)). *There is a compact moduli space X parametrizing ‘almost K3’ stable log quadrics. X is a coarse moduli space of an irreducible smooth Deligne-Mumford stack \mathfrak{X} of dimension nine with Picard rank four. Moreover, there is a classification of ‘almost K3’ stable log quadrics (S, C) appearing in X .*

As an application, we also show that X contains the blowup of the hyperelliptic locus of the moduli space M_4 of smooth genus four curves as an open dense subspace. This means that X is a *birational model* of M_4 , i.e. a compact space containing an open dense subset isomorphic to an open dense subset of M_4 .

1.2. Topology and arithmetic of moduli spaces of semistable elliptic surfaces. (See Section 3) In a work with Jun-Yong Park [14], we consider moduli spaces $\mathcal{L}_{1,12n} := \text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ parametrizing stable elliptic fibrations with discriminant degree $12n$.

Theorem 1.2 (Motive of moduli spaces of stable elliptic fibrations (see Theorem 3.1 for the details)). *The motive of $\mathcal{L}_{1,12n}$ is $\mathbb{L}^{10n+1} - \mathbb{L}^{10n-1}$, where \mathbb{L} is the motive of an affine line \mathbb{A}^1 .*

From this, we deduce that the number of nonsingular semistable elliptic surfaces over a finite field \mathbb{F}_q is $2(q^{10n+1} - q^{10n-1})$. By using the *global fields analogy*, which says that $\mathbb{P}_{\mathbb{F}_q}^1$ and $\text{Spec } \mathbb{Z}$ have similar arithmetic properties, we formulate a new conjecture about the asymptotic growth of the number of semistable elliptic curves E over \mathbb{Z} with bounded height $ht(E) \leq B$.

2. BIRATIONAL GEOMETRY OF MODULI SPACES OF LOG SURFACES

2.1. Compactifying moduli spaces of curves and surfaces. A crucial step in the study of moduli spaces is the notion of a *modular compactification*. A moduli space $\overline{\mathcal{M}}$ is a *modular compactification* of a moduli space \mathcal{M} if $\overline{\mathcal{M}}$ is compact and contains \mathcal{M} as a dense open subset. Since understanding compact spaces are much easier than studying noncompact spaces, it is useful to have a modular compactification. Note

that moduli spaces of smooth varieties usually are not compact. To compactify a moduli space of smooth varieties, one can enlarge the class of varieties by allowing singularities on varieties. If we carefully select singularity conditions on varieties, then we often get a compact moduli space. For example, enlarging smooth projective curve by Deligne-Mumford stable curves, they constructed in [6] a compact moduli space $\overline{\mathcal{M}}_g$ of stable curves as a modular compactification of \mathcal{M}_g . $\overline{\mathcal{M}}_g$ led to advances in various subfields of algebraic geometry, such as the birational geometry, the Gromov-Witten theory, and the irreducibility of $\overline{\mathcal{M}}_g$ (hence \mathcal{M}_g as well). For instance, Harris and Mumford showed in [15] that the moduli space \mathcal{M}_g is of general type for $g \geq 24$ by using divisors on $\overline{\mathcal{M}}_g$, Kontsevich used intersection theory on $\overline{\mathcal{M}}_g$ to construct a Kontsevich space of stable maps, leading to an algebraic formulation of Gromov-Witten theory (see [11] for a survey), and Fulton gave an algebraic proof in [10] of the irreducibility of \mathcal{M}_g (in fact, $\overline{\mathcal{M}}_g$) on all characteristic, by using the geometry of Hurwitz space and the existence of $\overline{\mathcal{M}}_g$.

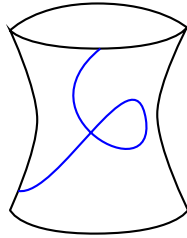
The geometry of compact moduli spaces of curves is well-understood, but not so much for moduli spaces of surfaces. Even constructing a compact moduli space of surfaces is a recent result by Kollár, Shepherd-Barron, and Alexeev (see [16] for a survey). Unlike the case of curves, we do not know the boundary members for compact moduli spaces of surfaces. By Murphy's law in algebraic geometry [17], the geometry of moduli spaces of surfaces in general can get arbitrarily bad. The broad goal of a work with Anand Deopurkar [8] is to compactify a moduli space of "almost" log K3 quadrics, then find geometric properties of this moduli space. In the next subsection, I describe the contents of this work more carefully.

2.2. Moduli of stable log quadrics: results. Consider a moduli space \mathfrak{U} of log surfaces $(\mathbb{P}^1 \times \mathbb{P}^1, C)$ where C is a curve of bidegree $(3, 3)$ in the smooth quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$. Since general canonical curves of genus 4 lie in a smooth quadric surface, \mathfrak{U} is birational to \mathcal{M}_4 . Motivated by ideas of Hacking as in [13], we consider $(\mathbb{P}^1 \times \mathbb{P}^1, C)$ as a stable pair $(\mathbb{P}^1 \times \mathbb{P}^1, D := (2/3 + \epsilon)C)$ of a surface and a \mathbb{Q} -divisor D for a small positive rational $\epsilon \leq 1/30$. Since $K_{\mathbb{P}^1 \times \mathbb{P}^1} + (2/3)C$ is linearly equivalent to 0, the pair $(\mathbb{P}^1 \times \mathbb{P}^1, (2/3)C)$ can be thought of as a K3 pair. Then, \mathfrak{U} is interpreted as a moduli space of 'almost K3' smooth log quadrics, an idea originating from Hacking [13]. I prove the following theorem in [8] jointly with Anand Deopurkar:

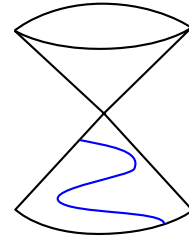
Theorem 2.1. *The moduli \mathfrak{X} of 'almost K3' stable log quadrics is a smooth and irreducible Deligne-Mumford stack of dimension 9, with Picard rank 4. It admits a projective coarse moduli space X , and is a compactification of \mathfrak{U} . Moreover, \mathfrak{X} has the following properties:*

- (1) *The boundary $\mathfrak{X} \setminus \mathfrak{U}$ is a union of 4 irreducible divisors \mathfrak{Z}_0 , \mathfrak{Z}_2 , \mathfrak{Z}_4 , and $\mathfrak{Z}_{3,3}$, which generates the Picard group of \mathfrak{X} . The log surfaces corresponding to their generic points are as follows:*

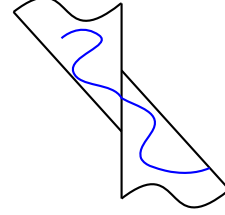
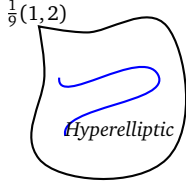
\mathfrak{Z}_0 *S is a smooth quadric surface in \mathbb{P}^3 and $D \subset S$ is a generic singular curve of bidegree $(3, 3)$.*



\mathfrak{Z}_2 *S is an irreducible singular quadric surface in \mathbb{P}^3 and D is a complete intersection of S and a cubic surface in \mathbb{P}^3 .*



- \mathfrak{Z}_4 S is a \mathbb{Q} -Gorenstein smoothing of the A_1 singularity of $\mathbb{P}(9, 1, 2)$ and D is a smooth hyperelliptic curve away from the singular point of type $\frac{1}{9}(1, 2)$.
- $\mathfrak{Z}_{3,3}$ S is a union $\text{Bl}_u \mathbb{P}(3, 1, 1) \cup \text{Bl}_v \mathbb{P}(3, 1, 1)$ along a \mathbb{P}^1 and D is a nodal union of two non-Weierstrass genus 2 tails.



- (2) There is an open substack \mathfrak{X}_0 parametrizing stable log quadrics (S, D) where S is nonsingular. In fact, $\mathfrak{X}_0 \cong \text{Bl}_{\mathcal{H}_4} \mathcal{M}_4$, the blowup of \mathcal{M}_4 along a closed substack \mathcal{H}_4 parametrizing hyperelliptic curves of genus 4.
- (3) There is a generically $2 : 1$ surjective morphism $\Phi : \overline{\mathcal{H}}_4^3(1/5) \rightarrow \mathfrak{X}$, where $\overline{\mathcal{H}}_4^3(1/5)$ is a compactification of the Hurwitz space \mathcal{H}_4^3 parametrizing trigonal maps from curves of genus 4 to rational curves (see [7]).
- (4) For any $\alpha \in (2/3 - \epsilon, 1]$, $\overline{M}_4(\alpha)$ is not isomorphic to any divisorial contraction of X , where $\overline{M}_4(\alpha)$ is an alternative compactification of M_4 coming from the Hassett-Keel program (see [9] for a survey).

This theorem is noteworthy for several reasons:

- (i) all boundary members of \mathfrak{X} are known,
- (ii) \mathfrak{X} is one of very few examples of moduli spaces of surfaces whose general member is a toric surface but a member of the boundary (for instance, a general member of \mathfrak{Z}_4) is not toric.
- (iii) intertwines moduli spaces of curves, surfaces, and branched coverings.

2.3. Future directions.

In the future, I plan to work on the following projects:

- **moduli spaces of ‘almost K3’ stable log del Pezzo surfaces:** To consider moduli spaces of other log surfaces, I plan to generalize the idea from Hacking [12] further. In other words, I want to investigate a general classification of semi-log canonical degenerations of $\mathbb{P}^1 \times \mathbb{P}^1$ in order to obtain moduli spaces of log surfaces degenerating from a smooth quadric surface and a curve of bidegree (d, d) . By using this idea and modern tools in [1], I plan to work with Anand Deopurkar to obtain moduli spaces of ‘almost K3’ stable log quadrics where curves have bidegree (d, d) for $d > 2$. In fact, we already have a theorem in progress:

Theorem 2.2 (in progress). *When $d > 2$ is odd, then the moduli of ‘almost K3’ stable log quadrics (X, C) with $K_X^2 = 8$ and $K_X + (2/d)C \sim 0$ is an irreducible, smooth and proper Deligne-Mumford stack of dimension $(d + 1)^2 - 7$. For each fixed d , there is also a finite list of possible singularities of stable log quadrics (X, C) arising in this way.*

As a next case, I would like to investigate the case of ‘almost K3’ stable log del Pezzo surfaces, where the pair (S, C) is a degeneration of a pair (H, C') where H is any del Pezzo surface of fixed degree.

- **Compactifying moduli spaces of K3 surfaces:** I plan to use the moduli space \mathfrak{X} from Theorem 2.1 to construct a compact moduli space of K3 surfaces with special automorphism groups. For example, consider when $d = 4$ from the setup above. As pointed out by Paul Hacking, a double cover T of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along a smooth curve of bidegree $(4, 4)$ is a K3 surface with a lattice polarization on $\text{Pic}(T)$ induced by pullback of $(1, 0)$ and $(0, 1)$ classes on $\mathbb{P}^1 \times \mathbb{P}^1$. A moduli space of such K3 surfaces form 18 dimensional families, forming a divisor in the moduli

space of quartic K3 surfaces. By [2, Lemma 2.2], a moduli space of stable log quadrics for $d = 4$ gives a moduli space of stable lattice polarized quartic K3 surfaces. My goal is to understand such moduli spaces, which potentially gives a good definition of stable polarized K3 surfaces (without relying on the fact that the surface is a cyclic cover of a del Pezzo surface).

This is why $d = 3$ is a good test case to understand degenerations of K3 surfaces. Notice that a triple cover T of $(\mathbb{P}^1 \times \mathbb{P}^1, D) \in \mathfrak{X}$ totally branched over D is already a lattice polarized K3 surface. By working with Anand Deopurkar and Patricio Gallardo, I plan to extend ideas and techniques of [2]. One advantage of this approach is that a general lattice polarized K3 surface of degree 6 can be thought of as a triple cover of $\mathbb{P}^1 \times \mathbb{P}^1$ simply branched over a curve C of bidegree $(6, 6)$ containing 12 cusps. To work on this setup, I plan to build up the theory of branched covering of surfaces, which is a higher-dimensional analogue of Hurwitz spaces.

- **Relationship with birational geometry of $\overline{\mathcal{M}}_4$:** Consider the birational geometry of \mathcal{M}_4 . Since Theorem 2.1 (4) says that \mathfrak{X} is not a blowup of the hyperelliptic locus of $\overline{\mathcal{M}}_4(\alpha)$'s, I plan to investigate the birational relations between them. With Anand Deopurkar, we show in a work in progress that there is no divisorial contraction of \mathfrak{Z}_4 in \mathfrak{X} . We would like to construct a new compactification of \mathcal{M}_4 , by flipping a sublocus of \mathfrak{Z}_4 then contracting proper transform of \mathfrak{Z}_4 . This involves finding a suitable line in $N^1(\mathfrak{X})$ that corresponds to the above birational transformations.

3. TOPOLOGY AND ARITHMETIC OF MODULI SPACES OF SEMISTABLE ELLIPTIC SURFACES

3.1. Counting elliptic curves and surfaces via global fields analogy. Consider an elliptic curve (E, e) over the field \mathbb{Q} . It can be written as a curve defined by a *Weierstrass equation* $y^2 = x^3 + ax + b$ with $a, b \in \mathbb{Q}$. By a suitable change of variables, we can assume that $a, b \in \mathbb{Z}$, obtaining an elliptic curve with integral coefficients (i.e. curve over $\text{Spec } \mathbb{Z}$). Then a classical problem in number theory asks how many elliptic curves are there with bounded height $ht(\Delta(E)) := ht(-16(4a^3 + 27b^2)) \leq B$? As a function of B , some asymptotic properties of the counting function are known by [5] and [3], but they do not give a complete answer.

The problem of counting stable elliptic curves over \mathbb{Z} is equivalent to counting $\text{Spec } \mathbb{Z}$ -points of the moduli space $\overline{\mathcal{M}}_{1,1}$ of stable elliptic curves. In other words, counting $\text{Spec } \mathbb{Z}$ -points is the same as counting the set $\text{Hom}(\text{Spec } \mathbb{Z}, \overline{\mathcal{M}}_{1,1})$ of maps from $\text{Spec } \mathbb{Z}$ to $\overline{\mathcal{M}}_{1,1}$. Observe that counting families of elliptic curves parametrized by $\text{Spec } \mathbb{Z}$ is similar to counting those parametrized by an affine line $\mathbb{A}_{\mathbb{F}_q}^1$ over a finite field \mathbb{F}_q by the *global fields analogy*. Since the projective line $\mathbb{P}_{\mathbb{F}_q}^1$ is a closure of $\mathbb{A}_{\mathbb{F}_q}^1$, counting points on $\text{Hom}(\mathbb{P}_{\mathbb{F}_q}^1, \overline{\mathcal{M}}_{1,1})$, the moduli space of elliptic curves over $\mathbb{P}_{\mathbb{F}_q}^1$, is related to counting stable elliptic curves over \mathbb{Z} . Such a family of elliptic curves X parametrized by $\mathbb{P}_{\mathbb{F}_q}^1$ is called a *stable elliptic fibration* over $\mathbb{P}_{\mathbb{F}_q}^1$. This is an example of an *elliptic surface* $\pi : Y \rightarrow \mathbb{P}^1$, where a general fiber of π is a smooth elliptic curve. The goal of a work with Jun-Yong Park [14] is to formulate a new conjecture about elliptic curves over \mathbb{Z} by counting certain elliptic surfaces over \mathbb{F}_q .

3.2. Counting elliptic surfaces over \mathbb{F}_q : results. Complex geometers have extensively studied elliptic surfaces $X \rightarrow \mathbb{P}^1$ with nonsingular total space X (see [4]). Hence, we restrict our attention on nonsingular *semistable* elliptic surfaces $\pi : X \rightarrow \mathbb{P}_{\mathbb{F}_q}^1$ over \mathbb{F}_q as an analogy. Since the log canonical model of π gives a stable elliptic fibration over $\mathbb{P}_{\mathbb{F}_q}^1$ with possibly singular total space, we show that π is nevertheless equivalent to a \mathbb{F}_q -point of $\mathcal{L}_{1,12n} := \text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ where the discriminant $\Delta(\pi)$ has degree $12n$. By slicing and dicing on the moduli space $\mathcal{L}_{1,12n}$ based on Weierstrass normal forms of π , we obtain the following theorem as in [14]:

Theorem 3.1. $\mathcal{L}_{1,12n}$ is a tame Deligne-Mumford stack over any field K of characteristic neither 2 nor 3. Moreover,

- (1) The motive of $\mathcal{L}_{1,12n}$ is $[\mathcal{L}_{1,12n}] = \mathbb{L}^{10n+1} - \mathbb{L}^{10n-1}$, where $\mathbb{L} := [\mathbb{A}^1]$.
- (2) The number of nonsingular semistable elliptic surfaces over \mathbb{F}_q with discriminant degree $12n$ is $2(q^{10n+1} - q^{10n-1})$.
- (3) The number of nonsingular semistable elliptic surfaces E over \mathbb{F}_q with bounded height $ht(E) = q^{\deg \Delta(E)} \leq B$ satisfies the following inequality:

$$\mathcal{Z}_{\mathbb{F}_q(t)}(B) \leq 2 \frac{q^{11} - q^9}{q^{10} - 1} \left(B^{\frac{5}{6}} - 1 \right).$$

Since Theorem 3.1 estimates that the asymptotic behavior of $\mathcal{Z}_{\mathbb{F}_q(t)}(B)$ has a leading term of order $\mathcal{O}(B^{\frac{5}{6}})$ and the lower order term of zeroth order (i.e. a constant), we expect the same to be true on elliptic curves with integral coefficients via global fields analogy:

Conjecture 3.2. *The counting function $\mathcal{Z}_{\mathbb{Q}}(B)$ of semistable elliptic curves over $\text{Spec } \mathbb{Z}$ with $ht(\Delta) \leq B$ has the leading term of order $\mathcal{O}(B^{\frac{5}{6}})$ and the lower order term of zeroth order (i.e. a constant).*

Since $\overline{\mathcal{M}}_{1,1}$ is isomorphic to a $(4, 6)$ -weighted projective line $\mathcal{P}(4, 6)$ with the same condition as in Theorem 3.1, there is an analogous result for a stack $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b))$ of maps to the (a, b) -weighted projective line, which behaves similarly to $\mathcal{L}_{1,12n}$.

3.3. Future directions. There are three applications that I want to investigate:

- **Topology and arithmetic of moduli spaces of unstable elliptic surfaces:** I plan to extend Theorem 3.1 to unstable elliptic surfaces with J-Y Park. As pointed out by Dori Bejleri and Kenny Ascher, We found two ways to describe moduli spaces of unstable surfaces as moduli spaces of twisted maps and moduli spaces of quasimaps, as a work in progress with J-Y Park. For counting points, it is easier to interpret the moduli space as parametrizing rational maps from \mathbb{P}^1 to $\overline{\mathcal{M}}_{1,1}$ with indeterminacy locus $\{x_1, \dots, x_k\}$ and vanishing conditions on x_i 's. Then the slice and dice technique used in the proof of Theorem 3.1 gives an algorithm to compute the number of such unstable surfaces.
- **Topology and arithmetic of K3 fibered threefolds:** I plan to apply this program to K3 fibered threefolds, leading to conjectures on counting K3 surfaces over \mathbb{Z} . Unfortunately, modular compactification of moduli spaces of K3 surfaces are not known over arbitrary characteristic, so some restrictions are necessary. I would like to see how well the moduli space of K3 surfaces (or its sublocus) interpreted as covers of del Pezzo surfaces (see near the end of § 2.3) fit into this framework.
- **Counting elliptic curves with semistable reductions:** I would like to see how to prove Conjecture 3.2. I could start by localizing each elliptic curve E over \mathbb{Z} at each primes $\mathfrak{p} \subset \mathbb{Z}$ and see whether there is a version of local-to-global principles for elliptic fibrations over mixed characteristic. To achieve this, I plan to find an analogous statement of Theorem 3.1 for primes $\mathfrak{p} = 2, 3$.

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