Can Implicit Bias Imply Adversarial Robustness?

YES, by choosing the right activation

Hancheng Min, René Vidal

Center for Innovation in Data Engineering and Science (IDEAS), University of Pennsylvania

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INTRODUCTION

- **Implicit bias of GD often benefits generalization**
- **Does it also improve adversarial robustness?**
- **If standard networks fail to be robust, can we alter the bias to favor more robust networks?**

pReLU network, $p \geq 1$; $\theta := \{w_j, v_j\}_j^h$

$$f_p(x;\theta) = \sum_{j=1}^h v_j \frac{\sigma^p(\langle x, w_j\rangle)}{\|w_j\|^{p-1}},\quad \sigma\text{: ReLU}$$

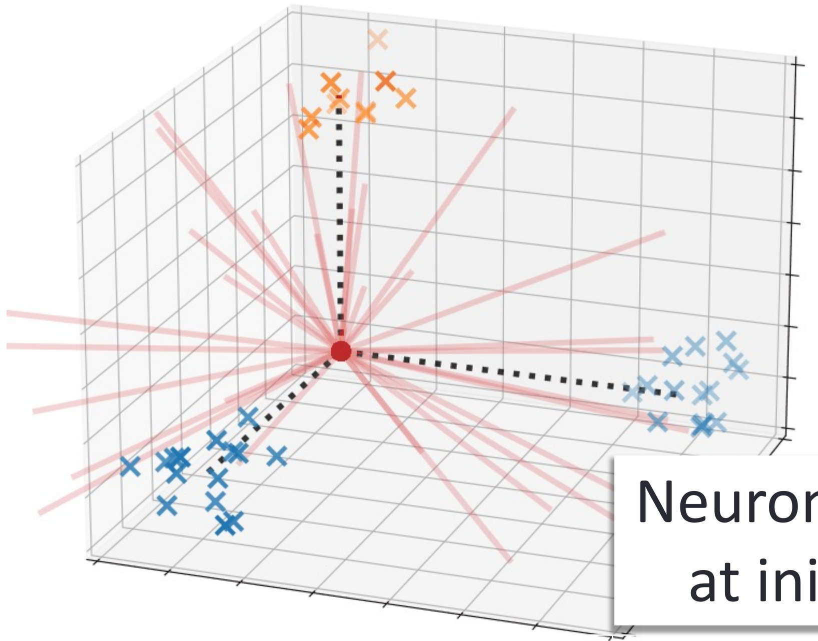
Loss: $\mathcal{L} = \sum_{i=1}^n \ell(y_i f_p(x_i;\theta))$

ℓ: exp. or logistic loss

Gradient flow (GF) with small initialization:

$$\dot{\theta} = -\nabla_\theta \mathcal{L}, \|\theta(0)\| \ll 1$$

Problem: Training shallow networks for binary classification problems with orthogonal data clusters



Neurons visualized at initialization

x : Positive data $\{x_i : y_i = +1\}$ **o** : Neurons $\{w_j\}$

x : Negative data $\{x_i : y_i = -1\}$ - : Neuron directions $\left\{\frac{w_j}{\|w_j\|}\right\}$

⋯ : Cluster centers

Data: samples from balanced mix. of Gaussians

$\mathcal{N}(\mu_1, \alpha^2 I), \cdots, \mathcal{N}(\mu_{K_1}, \alpha^2 I)$ K_1 pos. clusters

$\mathcal{N}(\mu_{K_1+1}, \alpha^2 I), \cdots \mathcal{N}(\mu_K, \alpha^2 I)$ K_2 neg. clusters

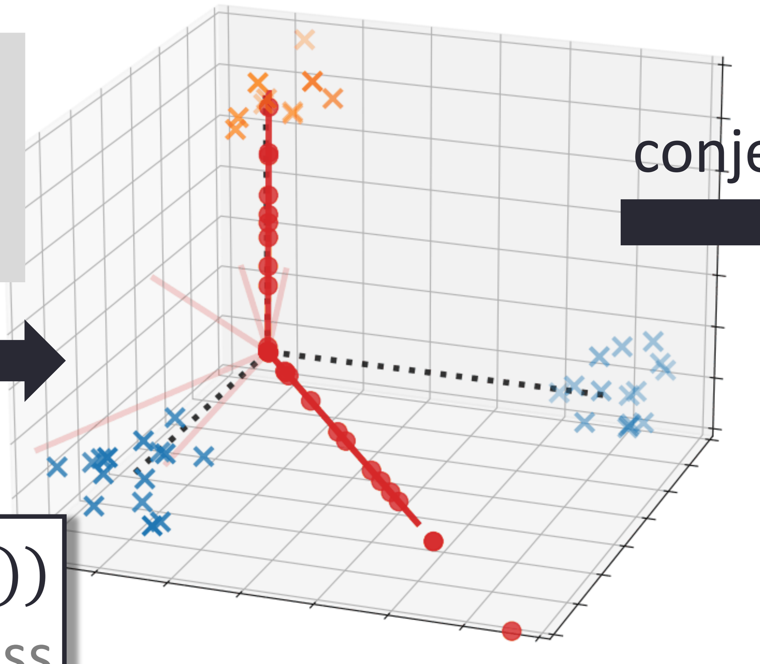
Cluster centers: $\mu_1, \cdots, \mu_K$ are orthonormal

Class average: $\mu_+ := \sum_{k=1}^{K_1} \mu_k$, $\mu_- := \sum_{k=K_1+1}^{K} \mu_k$

HARNESSING IMPLICIT BIAS FOR ADVERSARIAL ROBUSTNESS

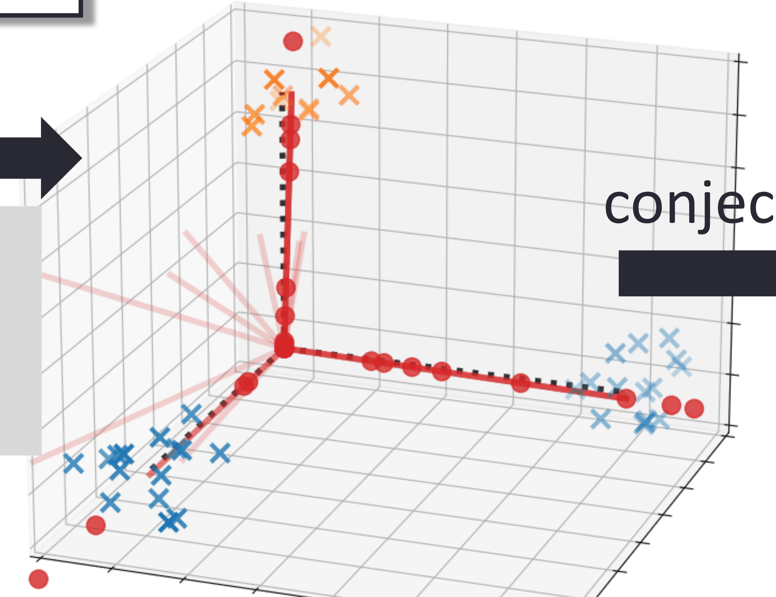
Observations

$p = 1$: ReLU Net
Neurons learn class average



Neurons visualized at the end of training

$p > 2$: pReLU Net
Neurons learn cluster centers



Conjectures

After training for sufficient time T

conjecture

$\boldsymbol{f_1(\cdot\,;\theta(T)) \approx F}$ up to a scaling factor

$F(x) = \sigma(\langle x, \mu_+\rangle) - \sigma(\langle x, \mu_-\rangle)$

- $\boldsymbol{F}$ is a ReLU network with two neurons μ_+ and μ_-
- GF on $\boldsymbol{f_1}$ converges to *ReLU classifier* $\boldsymbol{F}$

Implicit bias in shallow ReLU networks can harm adversarial robustness

Implicit bias of pReLU networks for p>3 leads to robust networks

conjecture

$\boldsymbol{f_p(\cdot\,;\theta(T)) \approx F^{(p)}}$ up to a scaling factor

$F^{(p)}(x) = \sum_{k=1}^{K_1} \sigma^p(\langle x, \mu_k\rangle) - \sum_{k=K_1+1}^{K} \sigma^p(\langle x, \mu_k\rangle)$

- $\boldsymbol{F^{(p)}}$ is a pReLU network with neurons $\mu_1, \cdots, \mu_K$
- GF on $\boldsymbol{f_p}$ ($p > 2$) converges to *pReLU classifier* $\boldsymbol{F^{(p)}}$

Main Theorems

Frei et al., 23: similar results hold for any ReLU net learned via GF/GD

New sample $(x, y) \in \mathbb{R}^D \times \{+1, -1\}$

ReLU classifier F

- ***Generalize on clean data***

$$\mathbb{P}(F(x)y > 0) \geq 1 - 2\exp\left(-\frac{CD}{4\alpha^2 K}\right)$$

- ***Non-robust against $\mathcal{O}(1/\sqrt{K})$-attack***

There exist $d_0 \in \mathbb{S}^{D-1}$, such that $\forall \rho > 0$

$$\mathbb{P}\left(F\left(x + \frac{1+\rho}{\sqrt{K}} d_0\right) y > 0\right) \leq 2\exp\left(-\frac{CD\rho^2}{\alpha^2 K}\right)$$

pReLU classifier $F^{(p)}$

- ***Generalize on clean data***

$$\mathbb{P}(F^{(p)}(x)y > 0) \geq 1 - 2(K+1)\exp\left(-\frac{CD}{\alpha^2 K}\right)$$

- ***Robust against $\mathcal{O}(1)$-attack***

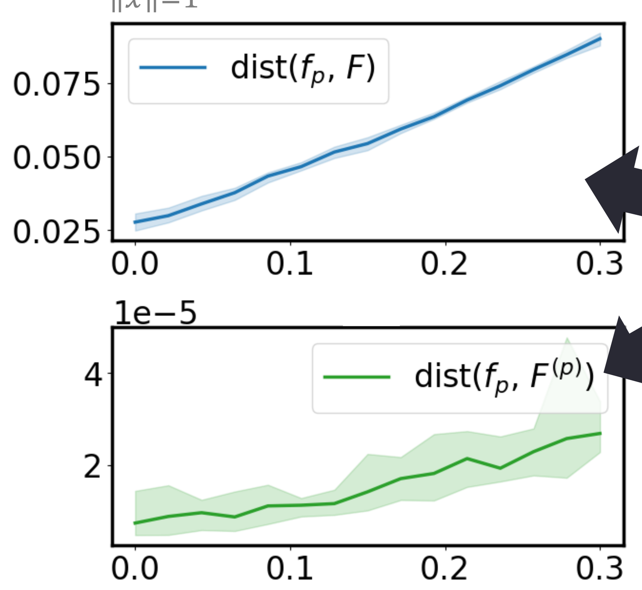
Let $p > 2$, then $\forall \delta \in (0, \sqrt{2})$,

$$\mathbb{P}\left(\min_{\|d\| \leq 1}\left[F^{(p)}\left(x + \frac{\sqrt{2}-\delta}{2} d\right) y\right] > 0\right) \geq 1 - 2(K+1)\exp\left(-\frac{CD\delta^2}{2\alpha^2 K^2}\right)$$

NUMERICAL EXPERIMENTS

Our conjectures are empirically verified

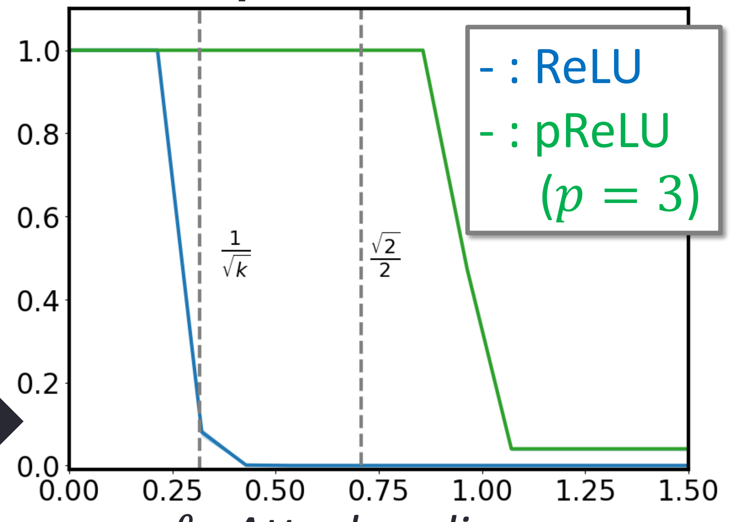
$\mathrm{dist}(f_p, F) = \inf_{c>0} \sup_{\|x\|=1} |cf_p(x) - F(x)|$



intra-cluster variance α

- $D = 1000, K_1 = 6, K_2 = 4$
- $p = 1$ (ReLU): $f_p \approx F$
- $p = 3$ (pReLU): $f_p \approx F^{(p)}$
- Robustness of f_p is well predicted by our theory

Robust Accuracy of f_p on new (x, y)



ℓ_2 Attack radius

pReLU improves the robustness of trained network on MNIST without adversarial training

Experiment details:

- Digits are centered by the mean digit of the training set
- CE loss; Kaiming initialization; Adam with batch size 1000
- Attack model: AutoAttack (Croce&Hein, 20)