Kim Hancock
Kayla Snyder

Group Assignment 10 November 18, 2020

1

Desmos

3

Consider someone plucking the E string on a guitar near the center of the string. We want to understand which overtones we will "hear" when we play the string in this particular way. The fundamental note in this situation has a frequency of about 83 Hz. For simplicity, we will assume the string has length of 1 meter (not realistic, sorry). The relevant model is,

$$u_{tt} - (166)^2 u_{xx} = 0 \text{ for } 0 < x < 1$$

$$u(x,0) = \begin{cases} .04x & 0 < x < 1/2 \\ .04(1x) & 1/2 < x < 1 \end{cases}$$

$$u_t(x,0) = 0$$

$$u(0,t) = 0, u(1,t) = 0$$

a) **Find the solution of the PDE that satisfies that initial condition and boundary conditions. Hint: You have dealt with the Fourier sine series of this initial condition before, so use that result. The only difference is the constant factor of .04 which you can easily incorporate into your answer, if you think about it.**

From problem 9 on Individual Homework 10, we showed that we get the two ODEs $-\lambda c^2 T = T''$ and $-\lambda X = X''$ for $\beta^2 = \lambda$ for $\beta \in \mathbb{R}$, $\beta \neq 0$. (In problems 5 and 6 on Individual Homework 7, we showed that we cannot have negative or zero values of λ.) In problem 9, we showed these $X(x)$ and $T(t)$ ODEs have the general form:

- $X(x) = C\cos(\beta x) + D\sin(\beta x)$
- $T(t) = A\cos(\beta 166t) + B\sin(\beta 166t)$

for constants $A, B, C, D \in \mathbb{R}$

Now, we consider our boundary conditions:
$u(0,t) = 0 = X(0)T(t) \implies X(0) = 0$ since we don't want $T(t)$ to be trivial.
$u(1,t) = 0 = X(1)T(t) \implies X(1) = 0$ since we don't want $T(t)$ to be trivial.

$X(0) = C\cos(0) + D\sin(0) = C$
$\implies C = 0$

Now, we are left with $X(x) = D\sin(\beta x)$
We now impose our second boundary condition.
$\implies X(1) = D\sin(\beta) = 0$
Since we don't want D to be zero, we know we want $\sin(\beta) = 0$
This occurs at integer multiples of π, so we thus know $\beta = n\pi$ for $n = 1, 2, 3, ...$ (since λ can't be 0 and negative values of n would result in repeat solutions as shown in problem 7 on Individual Homework 7) and thus our X ODE is:

$$X_n(x) = \sin(n\pi x)$$

(We don't need to include the constant since we know that any scalar multiple of an eigenfunction is also an eigenfunction, and this formula for $X(x)$ is the solution to an eigenvalue problem.)

Now, we know what β is, so we know that $T(t) = A\cos(n\pi 166t) + B\sin(n\pi 166t)$

Now, we can compose the series solution for $u(x,t)$:

$$u(x,t) = \textstyle\sum_{n=1}^{\infty}(A_n\cos(n\pi 166t) + B_n\sin(n\pi 166t))\sin(n\pi x)$$

Let's consider our initial condition, $u_t(x,0)$. We can see:

$$u_t(x,t) = \textstyle\sum_{n=1}^{\infty}(-A_n(n\pi 166)\sin(n\pi 166t) + B_n(n\pi 166)\cos(n\pi 166t))\sin(n\pi x)$$

Plugging in $t = 0$, we have:

$$u_t(x,0) = \textstyle\sum_{n=1}^{\infty}(-A_n(n\pi 166)\sin(0) + B_n(n\pi 166)\cos(0))\sin(n\pi x) = \\ \textstyle\sum_{n=1}^{\infty}(B_n(n\pi 166))\sin(n\pi x)$$

We know that $u_t(x,0) = 0$, so, simplifying further, we have:

$$u_t(x,0) = \textstyle\sum_{n=1}^{\infty}(B_n(n\pi 166))\sin(n\pi x) = 0$$

Therefore, we know that $B_n = 0$ in order to make this statement true for all x. So, we can re-write $u(x,t)$ as:

$$u(x,t) = \textstyle\sum_{n=1}^{\infty} A_n\cos(n\pi 166t)\sin(n\pi x)$$

Now, let's consider our other initial condition $u(x,0)$. We can plug in $t = 0$ to get:

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) = \phi(x)$$

From the initial condition $u(x,0)$ in the problem, we know that:

$$\sum_{n=1}^{\infty} A_n \sin(n\pi x) = 0.04x \text{ from } 0 < x < 1/2$$
$$\sum_{n=1}^{\infty} A_n \sin(n\pi x) = 0.04(1-x) \text{ from } 1/2 < x < 1$$

In Group Homework 9, we showed that the Fourier Coefficients A_m for this problem, where $X_m = \sin(m\pi x)$, took the form $A_m = \frac{2}{l}\int_0^1 \phi(x)\sin(m\pi x)dx$. Note that in this case, we know $l = 1$. (This information was from video 36.)

Now, we have two different equations for $\phi(x)$, giving us:

$$A_m = 2\int_0^{1/2} 0.04x\sin(m\pi x)dx + 2\int_{1/2}^1 0.04(1-x)\sin(m\pi x)dx$$

This is the same equation that we saw in Group Homework 9, except the coefficient of 0.04 is included in each integral. We can pull this factor out in front to get:

$$A_m = 0.08\int_0^{1/2} x\sin(m\pi x)dx + 0.08\int_{1/2}^1 (1-x)\sin(m\pi x)dx$$

Following the same procedure as Group Homework 9, we can use integration by parts for both integrals, letting $u = x$, $du = dx$ and $u = 1-x$, $du = -dx$, respectively, and then $dv = \sin(m\pi x)$ so $v = -\frac{1}{m\pi}\cos(m\pi x)$ for both integrals. Thus, we get:

$$A_m =$$
$$0.08\left[(-\tfrac{x}{m\pi}\cos(m\pi x) + \int \tfrac{1}{m\pi}\cos(m\pi x)dx)\big|_0^{1/2} + (-\tfrac{1-x}{m\pi}\cos(m\pi x) - \int \tfrac{1}{m\pi}\cos(m\pi x)dx)\big|_{1/2}^1\right]$$

$$= 0.08\left[(-\tfrac{x}{m\pi}\cos(m\pi x) + \tfrac{1}{m^2\pi^2}\sin(m\pi x))\big|_0^{1/2} + (-\tfrac{1-x}{m\pi}\cos(m\pi x) - \tfrac{1}{m^2\pi^2}\sin(m\pi x))\big|_{1/2}^1\right]$$

$$= 0.08[-\tfrac{1}{2m\pi}\cos(\tfrac{m\pi}{2}) + \tfrac{1}{m^2\pi^2}\sin(\tfrac{m\pi}{2}) + \tfrac{0}{m\pi}\cos(0) - \tfrac{1}{m^2\pi^2}\sin(0) - \tfrac{0}{m\pi}\cos(m\pi) - \tfrac{1}{m^2\pi^2}\sin(m\pi) + \tfrac{1}{2m\pi}\cos(\tfrac{m\pi}{2}) + \tfrac{1}{m^2\pi^2}\sin(\tfrac{m\pi}{2})]$$

We know that $\sin(0) = 0$ and $\sin(m\pi) = 0 \ \forall m \in \mathbb{Z}$, so simplifying, we get:

$$A_m = 0.08\left[-\tfrac{1}{2m\pi}\cos(\tfrac{m\pi}{2}) + \tfrac{1}{m^2\pi^2}\sin(\tfrac{m\pi}{2}) + \tfrac{1}{2m\pi}\cos(\tfrac{m\pi}{2}) + \tfrac{1}{m^2\pi^2}\sin(\tfrac{m\pi}{2})\right]$$

We multiply every term by 0.08 and get:

$$A_m = -\tfrac{0.04}{m\pi}\cos(\tfrac{m\pi}{2}) + \tfrac{0.08}{m^2\pi^2}\sin(\tfrac{m\pi}{2}) + \tfrac{0.04}{m\pi}\cos(\tfrac{m\pi}{2}) + \tfrac{0.08}{m^2\pi^2}\sin(\tfrac{m\pi}{2})$$

$$A_m = \frac{0.16}{m^2\pi^2}\sin(\frac{m\pi}{2})$$

We have different cases: $A_m = \begin{cases} 0 & \text{when m is even} \\ \frac{0.16}{m^2\pi^2} & \text{when } m \mod 4 = 1 \\ -\frac{0.16}{m^2\pi^2} & \text{when } m \mod 4 = 3 \end{cases}$

Now, we can finally begin to plug in our Fourier Coefficients. Recall our formula for $u(x,t)$:

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos(n\pi 166t)\sin(n\pi x)$$

Now, let's plug in the coefficients for a few of the first terms in our solution.

$u(x,t) = \frac{0.16}{\pi^2}\cos(166\pi t)\sin(\pi x) - \frac{0.16}{9\pi^2}\cos(498\pi t)\sin(3\pi x) + \frac{0.16}{25\pi^2}\cos(830\pi t)\sin(5\pi x) - ...$

b) We want to calculate the frequency of the fundamental note for our wave equation, so we can note what multiples of it appear in our solution. To do so, we know that $\cos(\omega t)$ holds, where omega is our angular frequency.
We also know $2\pi f = \omega$, where f is the frequency.
Then, we know that in our equation we have $\cos(n166\pi t)$ which implies that $\omega = n166\pi$

Then, $f = \frac{\omega}{2\pi}$
We plug in our value for omega, giving $f = \frac{n166\pi}{2\pi} = 83n$ Hz

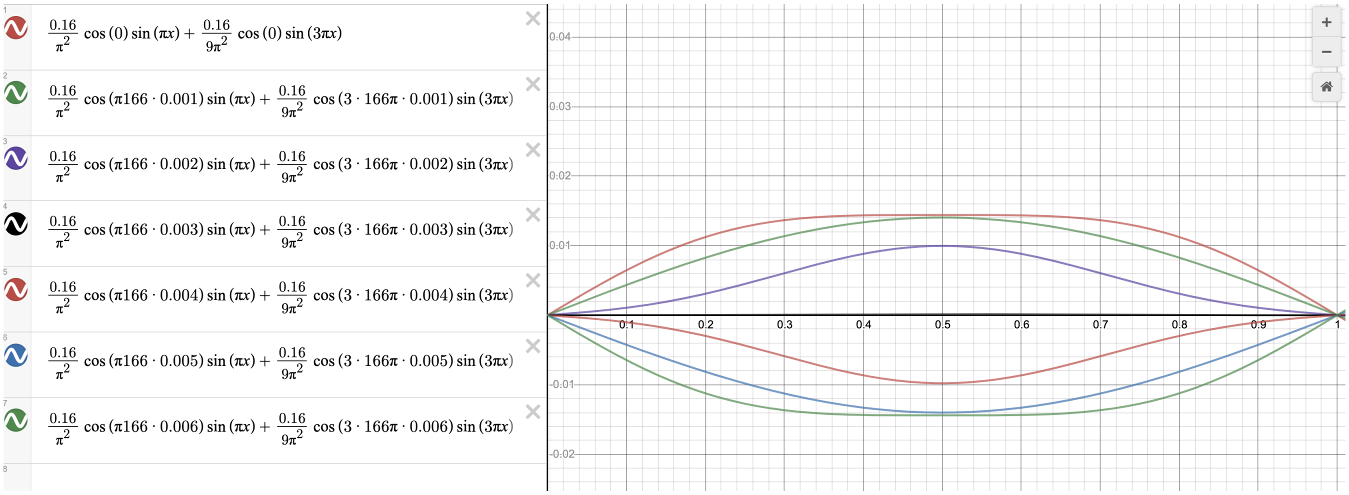
So, the first overtone occurs at $n = 2$, which is 166 Hz.
We notice that we do not have this frequency, since our second term has $\omega = 3 \cdot 166\pi$
This implies that $f = \frac{3 \cdot 166\pi}{2\pi} = 3 \cdot 83 = 249$ Hz.

Thus, we are missing odd multiples of the fundamental note

c) We only want the first two terms of the solution, since our first three coefficients are:

- $A_1 = 0.16/\pi^2 \approx 0.01211$
- $A_3 = 0.16/9\pi^2 \approx 0.0018$
- $A_5 = 0.16/25\pi^2 \approx 0.00064 < 0.001$

Thus, we would hear $u(x,t) = \frac{0.16}{\pi^2}\cos(\pi 166t)\sin(\pi x) + \frac{0.16}{9\pi^2}\cos(3 \cdot 166\pi t)\sin(3\pi x)$

d) 

References