Midterm 2

Ordinary Differential Equations -- Math 2208

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Due: Nov 11, 2019, 1:15 PM

Instructions:

- This is a take-home exam. As such, your written arguments will be held to a higher standard than on a sit-down in-class exam. Please submit clear and carefully composed solutions, and explain the concepts you are using and the connections among them. As always, points may be deducted for any unjustified steps, and generous partial credit will be given if you explain your thought process to me.
- You may consult and use our course materials while taking this exam, including the textbook, class notes, your problem sets, and any of the worksheets on Blackboard. You are not allowed to use the web links (for eignevalues or TD plane etc.) on worksheets.
  You may **NOT** use a calculator or any graphing tool.
  You may **NOT** use `dfield` or `pplane` or `Octave`.
  You may **NOT** consult the internet or discuss problem specifics with other people.
  You may email me to ask questions.
  If you are not sure if some resource is allowed, please ask!
- When submitting your exam, print and staple this first page on top, and sign the "Honor Signature" to indicate that you followed Bowdoin's Honor Code with respect to this exam.

Full Name: ______________________________

Honor Signature: ______________________________

Section Number	A	B	C	D	E	Total
Available Points	**35**	**25**	**25**	**50**	**00**	**135**
Your Score						

§A. A Nonautonomous Linear System

Consider a linear system of ODE as follows:

$$\frac{dx}{dt} = x + 2y + e^{-2t}$$
$$\frac{dy}{dt} = 4x - y$$

We can rewrite it in the following form:

$$\frac{d\vec{\mathbf{R}}}{dt} = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \vec{\mathbf{R}}(t) + \begin{bmatrix} e^{-2t} \\ 0 \end{bmatrix} = \mathbf{A}\vec{\mathbf{R}}(t) + \vec{f}(t) \qquad (\bigstar)$$

where $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}$ and $\vec{f}(t) = \begin{bmatrix} e^{-2t} \\ 0 \end{bmatrix}$.

■ Question 1 (6 points).

If $\vec{\mathbf{R}}_1(t)$ and $\vec{\mathbf{R}}_2(t)$ are any two solutions to system ($\bigstar$), show that $\vec{\mathbf{R}}_1(t) - \vec{\mathbf{R}}_2(t)$ is a solution to the system

$$\frac{d\vec{\mathbf{R}}}{dt} = \mathbf{A}\vec{\mathbf{R}}(t) \qquad (\dagger)$$

This is called the "associated homogeneous system".

■ Question 2 (15 points).

Find the general solution $\vec{\mathbf{R}}_h(t)$ to the associated homogeneous system ($\dagger$) using eigenvalues and eigenvectors of $\mathbf{A}$.

■ Question 3 (4 points).

Suppose $\vec{\mathbf{R}}_p(t)$ is one particular solution to the system ($\bigstar$). Then explain why all solutions to the system ($\bigstar$) must be of the form $\vec{\mathbf{R}}_p(t) + \vec{\mathbf{R}}_h(t)$.

■ Question 4 (6 points).

Show that $\vec{\mathbf{R}}_p(t) = \frac{e^{-2t}}{5}\begin{bmatrix} 1 \\ -4 \end{bmatrix}$ is a particular solution to ($\bigstar$).

■ Question 5 (4 points).

Write down the general form of the solutions to ($\bigstar$).

§B. Second Order Linear ODE

Consider the second-order linear ODE

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0$$

where p and q are constant real numbers.

Question 1 (4 points).

Write this equation as a two-dimensional first-order linear system of ODEs.

Question 2 (8 points).

Suppose λ is an eigenvalue of the matrix corresponding to above linear system. Show that the corresponding eigenvector is $\begin{bmatrix} 1 \\ \lambda \end{bmatrix}$.

Question 3 (3 points).

What conditions on p and q guarantee that the eigenvalues are complex numbers?

Question 4 (6 points).

What conditions on p and q guarantee that the origin is a spiral sink?

What conditions guarantee that the origin is a center?

What conditions guarantee that the origin is a spiral source?

Question 5 (4 points).

Assume we draw the phase plane corresponding to the linear system such that the variable y corresponds to the horizontal axis. If the eigenvalues are complex, show that solution curves in the phase plane spiral around the origin in a clockwise direction.

§C. Bifurcation in Trace-Determinant Plane

Consider the one-parameter family of linear system with real number a as the parameter:

$$\frac{d\vec{\mathbf{R}}}{dt} = \begin{bmatrix} a & \sqrt{5-a^2} \\ -1 & 0 \end{bmatrix} \vec{\mathbf{R}}$$

Question 1 (6 points).

Sketch the **D** vs. **T** curve corresponding to the family in the trace-determinant plane.

Question 2 (4 points).

Write down the values of a where the qualitative behavior of the system changes. These are the bifurcation values of a.

Question 3 (15 points).

In a couple of sentences, discuss different types of behaviors exhibited by the system as a increases from $-\sqrt{5}$ to $\sqrt{5}$. Include the boundary cases as well. If your solution curve spirals, find out whether it's clockwise or counterclockwise. Include pictures of sample phase portraits in each case.

You are being asked to identify the types of equilibria only, no analytical calculation is needed. You are NOT allowed to use a computer or any graphing tools other than pen and paper.

§D. A Modified Lotka-Volterra Model

A modified version of Lotka-Volterra equations can be used to model the population of two competing species. Because of the finiteness of resources, the reproduction rate per individual is adversely affected by high levels of its own species and the other species with which it is in competition. Denoting the two populations by x and y, the competing species system can be modeled by the ODE system

$$\frac{dx}{dt} = ax(1-x) - bxy$$
$$\frac{dy}{dt} = cy(1-y) - dxy$$

where $a, b, c,$ and d are positive numbers. For this problem we will assume

$$a = 1, b = 2, c = 1, \text{ and } d = 3$$

■ Question 1 (8 points).

Find the equations of the nullclines. Find the coordinates of the equilibrium point(s).

■ Question 2 (8 points).

Draw the nullclines on a phase plane $0 \le x \le 1, 0 \le y \le 1$. (Since x and y are population of species, they must be nonnegative).

Clearly denote the direction of the arrows along the nullclines.

■ Question 3 (8 points).

Find the direction of the arrows in other regions of above phase plane.

■ Question 4 (4 points).

Check that when $y = 2x$, we get

$$\left(\frac{dy}{dt}\right) \Big/ \left(\frac{dx}{dt}\right) = 2$$

This means all the arrows in the direction field that start on the straight line $y = 2x$, lie on the straight line. So we make an educated guess that one of the solution curves to this system is a straight line of the form $\vec{r}(t) = \begin{bmatrix} f(t) \\ 2f(t) \end{bmatrix}$.

■ Question 5 (8 points).

If $x = f(t), y = 2f(t)$ satisfies the given system, find $f(t)$ as a function of t.

Question 6 (4 points).

What does the solution curve with initial condition $x(0) = 1, y(0) = 2$ look like? Draw it on the phase plane. You do not need to write down the analytic formula for the solution.

Draw another solution curve that starts at $x(0) = 0.1, y(0) = 0.2$. Clearly label the two curves separately (with different color, or solid/dashed etc.) in your phase plane.

Question 7 (6 points).

Looking at the direction field, What can you say about the long term behavior of solution curves whose initial condition satisfies $\frac{y(0)}{x(0)} > 2$? What about $\frac{y(0)}{x(0)} < 2$? You may need to use the fact that no two solution curves cross each other.

Question 8 (4 points).

The line $y = 2x$ is called a *Separatrix*. It separates the quadrant into two regions each of which corresponds to a different long term behavior of the two populations.

Write a couple of sentences describing the long term fate of both species for different initial populations.

Is there anyway the two species can both exist peacefully in the long-term?

§E. Bonus (5 points)

Solve the initial-value problem

$$\frac{d\vec{R}}{dt} = \begin{bmatrix} \pi^2 & 37.4 \\ \sqrt{555} & 8.01234 \end{bmatrix} \vec{R}, \quad \vec{R}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$