Math 2208: Ordinary Differential Equations

Assignment 9

Fall 2019 **Subhadip Chowdhury** **Due: Nov 6**

Reading

Section 3.3 and 3.4 from the textbook.

Exercises

Don't forget to be neat and thorough. No fringe, and please use the cover page.

Question 1.

Book problem 3.3.24. Be sure to read the instructions for this problem that starts on page 295.

Question 2.

Book problems 3.4.(2, 4, 6). For part (e) of problem 4 and 6, use `pplane` to create the phase plane, and use `ODE45_system_Example.m` to graph the $x(t)$ vs t and $y(t)$ vs t graphs for $0 \leq t \leq 2\pi$.

Question 3.

Consider the Economics model from project 2. Come up with a model that takes into account the following:

- Paul's (i.e. x) profits are hurt by Bob's (i.e. y) profit.
- Bob's profits benefit from Paul's profit.
- **There is no other factor affecting either ones' profit.**
- Assume all nonzero parameters have magnitude 1.

Then use either analytical (by finding the general solution), or qualitative (by finding Eigenvalues and the phase portrait) or numerical (using ODE45) tool to find out what happens in the long term based on the model.

Question 4.

Book problem 3.4.19.

Question 5: Summarizing Some Results.

In class we showed the following result:

Theorem 1. *Suppose $\vec{\mathbf{R}}_1(t)$ and $\vec{\mathbf{R}}_2(t)$ are two solutions to the ODE*

$$\frac{d\vec{\mathbf{R}}}{dt} = \mathbf{A}\vec{\mathbf{R}}.$$

If $\vec{\mathbf{R}}_1(0)$ and $\vec{\mathbf{R}}_2(0)$ are linearly independent, then given any initial condition $\vec{\mathbf{R}}(0)$, we can find constant k_1 and k_2 such that

$$\vec{\mathbf{R}}(0) = k_1\vec{\mathbf{R}}_1(0) + k_2\vec{\mathbf{R}}_2(0)$$

By the Existence and Uniqueness Theorem for systems, we know that each initial-value problem for a linear system has exactly one solution.

By the Linearity principle, $k_1\vec{\mathbf{R}}_1(t) + k_2\vec{\mathbf{R}}_2(t)$ is a solution to the ODE for all constants k_1 and k_2.

Hence by combining both ideas, given any two solutions $\vec{\mathbf{R}}_1(t)$ and $\vec{\mathbf{R}}_2(t)$ of a linear system with linearly independent initial conditions $\vec{\mathbf{R}}_1(0)$ and $\vec{\mathbf{R}}_2(0)$, every general solution of the system belongs to the two-parameter family $k_1\vec{\mathbf{R}}_1(t) + k_2\vec{\mathbf{R}}_2(t)$.

In other words, this shows that the solution space is $\mathbf{Span}\left\{\vec{\mathbf{R}}_1(t), \vec{\mathbf{R}}_2(t)\right\}$. Next, recall that the book problem 3.1.35 gave us the following theorem:

Theorem 2. *Suppose that $\vec{\mathbf{R}}_1(t)$ and $\vec{\mathbf{R}}_2(t)$ are solutions to the system*

$$\frac{d\vec{\mathbf{R}}}{dt} = \mathbf{A}\vec{\mathbf{R}}.$$

If $\vec{\mathbf{R}}_1(0)$ and $\vec{\mathbf{R}}_2(0)$ are linearly independent, then $\vec{\mathbf{R}}_1(t)$ and $\vec{\mathbf{R}}_2(t)$ are also linearly independent for every t.

this theorem along with the fact that the solution space is $\mathbf{Span}\left\{\vec{\mathbf{R}}_1(t), \vec{\mathbf{R}}_2(t)\right\}$ shows that the solution space is *two dimensional* with $\left\{\vec{\mathbf{R}}_1(t), \vec{\mathbf{R}}_2(t)\right\}$ as a *basis*.

While both result hold even in the complex eigenvalue case, it is preferable to express the general solution in terms of real-valued functions. In class we showed that if $\mathbf{A}$ has complex eigenvalues (in fact they will be complex conjugates) then $\vec{\mathbf{R}}_{\mathfrak{Re}}(t)$ and $\vec{\mathbf{R}}_{\mathfrak{Im}}(t)$ are real solutions where $\vec{\mathbf{R}}_{\mathfrak{Re}}(t)$ and $\vec{\mathbf{R}}_{\mathfrak{Im}}(t)$ are the real and imaginary part of some complex valued solution $\vec{\mathbf{R}}_0(t)$. Since the solution space is two-dimensional, the expression

$$\vec{\mathbf{R}}(t) = k_1\vec{\mathbf{R}}_{\mathfrak{Re}}(t) + k_2\vec{\mathbf{R}}_{\mathfrak{Im}}(t)$$

will represent a general solution as long as $\vec{\mathbf{R}}_{\mathfrak{Re}}(t)$ and $\vec{\mathbf{R}}_{\mathfrak{Im}}(t)$ are linearly independent for all t (in which case, they form a basis). However, we never proved the linear independence of $\vec{\mathbf{R}}_{\mathfrak{Re}}(t)$ and $\vec{\mathbf{R}}_{\mathfrak{Im}}(t)$, which is the goal of this problem:

The Actual Problem Statement: Let $\lambda = \alpha + i\beta$ (where $\beta \neq 0$) be an eigenvalue of $\mathbf{A}$ with associated eigenvector $\vec{v} = \vec{v}_1 + i\vec{v}_2$, where $\vec{v}_1$ and $\vec{v}_2$ are real valued vectors. Let $\vec{\mathbf{R}}_{\mathfrak{Re}}(t)$ and $\vec{\mathbf{R}}_{\mathfrak{Im}}(t)$ be the real and imaginary parts of the complex valued solution

$$\vec{\mathbf{R}}_0(t) = e^{\lambda t}\vec{v}.$$

Prove that $\vec{\mathbf{R}}_{\mathfrak{Re}}(t)$ and $\vec{\mathbf{R}}_{\mathfrak{Im}}(t)$ are linearly independent for all t.

You may use the theorems given in the summary above and your result from problem 3.4.19 in the book. This problem is more about connecting some dots, so if the problem seems easy, that is OK (it is also OK if it is not easy)!