Final Exam

Ordinary Differential Equations -- Math 2208

Subhadip Chowdhury

Due: Noon, Dec 16, 2019

Instructions:

- This is a take-home exam. As such, your written arguments will be held to a higher standard than on a sit-down in-class exam. Please submit clear and carefully composed solutions, and explain the concepts you are using and the connections among them. As always, points may be deducted for any unjustified steps, and generous partial credit will be given if you explain your thought process to me.
- You may consult and use our course materials while taking this exam, including the textbook, class notes, your problem sets, and any of the worksheets on Blackboard. You are not allowed to use the web links (for eignevalues or TD plane etc.) on worksheets.
  You may **NOT** use a calculator or any graphing tool.
  You may **NOT** use `dfield` or `pplane` or `Octave`.
  You may **NOT** consult the internet or discuss problem specifics with other people.
  You may email me to ask questions.
  If you are not sure if some resource is allowed, please ask!
- When submitting your exam, print and staple this first page on top, and sign the "Honor Signature" to indicate that you followed Bowdoin's Honor Code with respect to this exam.

Full Name: ______________________________

Honor Signature: ______________________________

Section Number	A	B	C	D	E	Total
Available Points	**30**	**25**	**35**	**35**	**20**	**145**
Your Score						

§A. Air Resistance and Terminal Velocity

A paratrooper jumps out of an airplane at a sufficiently high altitude (with initial downward velocity $v(0) = 0$), falls freely for 20 seconds and then opens her parachute. Assume her mass is m.

Her vertical motion is subject to two forces:

- a downward gravitational force $\mathbf{F_G} = mg$ and
- a force $\mathbf{F_R}$ of air resistance that is proportional to velocity (so that $\mathbf{F_R} = kv$) and of course directed opposite to the direction of motion of the body (i.e. upward).

Newton's law of motion says that the net force acting on the paratrooper is equal to her mass times her acceleration.

■ Question 1 (5 points).

Show that at time t, the velocity $v(t)$ of the paratrooper can be found by solving the ODE

$$\frac{dv}{dt} = -\rho v + g$$

where $\rho = \frac{k}{m}$.

■ Question 2 (10 points).

For the first 20 seconds, without the parachute opened, ρ is given by 0.5. Find $v(20)$. Assume $g = 10\, m/s^2$.

■ Question 3 (10 points).

With the parachute open, ρ increases to 1.5. Find a formula for $v(t)$ for $t > 20$.

■ Question 4 (5 points).

Show that as $t \to \infty$, the paratrooper's velocity does not increase indefinitely. Instead, it approaches a finite limiting velocity, called the *terminal velocity*. Find the terminal velocity of the paratrooper.

§B. Boundary Value Problems

We know that the solution of a second-order linear differential equation is uniquely determined by two initial conditions. In particular, the solution of the initial value problem

$$y'' + py' + qy = 0, \qquad y(a) = 0, y'(a) = 0 \tag{1}$$

has a unique solution $y(t) \equiv 0$. The situation is quite radically different for a problem such as

$$y'' + py' + qy = 0, \qquad y(a) = 0, y(b) = 0 \tag{2}$$

The difference between the problems in equations (1) and (2) is that in (2) the two conditions are imposed at two different points a and b with (say) $a < b$. In (2) we are to find a solution of the differential equation on the interval (a, b) that satisfies the conditions $y(a) = 0$ and $y(b) = 0$ at the endpoints of the interval. Such a problem is called an endpoint or *boundary value problem*.

Consider the Boundary Value Problem (BVP)

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y(\pi) = 0$$

Question 1 (5 points).

Let $\lambda = 3$. Show that the only solution to above BVP in this case is the trivial solution $y(t) = 0$ for all t.

Question 2 (4 points).

Show that if $\lambda = 0$, then the only solution to the BVP is the trivial solution.

Question 3 (6 points).

Show that if $\lambda < 0$, then the only solution to the BVP is the trivial solution.

Question 3 (10 points).

Show that if $\lambda > 0$, then the BVP has a non-trivial solution if and only if λ is of the form

$$\lambda = n^2, \qquad n = 1, 2, 3, \ldots$$

§C. Competing Species

The blue crab is native to the US Atlantic coast, but there is concern that the population is on the decline. The European green crab is an invasive species (recently introduced to the US in the ballast waters of ships), that competes with the blue crab. Assume that the interaction between the blue crab, x, and the green crab, y, is modeled by (up to some scaling):

$$\frac{dx}{dt} = x(100 - x) - 2xy$$

$$\frac{dy}{dt} = y(400 - 6y) - xy$$

Question 1 (8 points).

Find equation of the nullclines and the four equilibrium solutions (three of which are non-negative, representing biologically relevant values).

Question 2 (7 points).

Assume x and y are non-negative. Draw the direction field in the first quadrant of the phase plane. what does the model predict about the long-term fate of the two species?

Question 3 (8 points).

Suppose an intervention effort is launched to preserve the blue crab by harvesting a proportion h of the green crabs, so the equations modeling the system become (this is up to scaling, so h can be any positive number):

$$\frac{dx}{dt} = x(100 - x) - 2xy$$
$$\frac{dy}{dt} = y(400 - 6y) - xy - hy$$

It would be nice if we could find harvesting values h so that the species can co-exist. Calculate the new equilibrium solutions and observe that for some values of h, the fourth equilibrium solution can be found at a point in the first quadrant (it will represent co-existence of the species, since both populations will have positive values). For what range of h values does the fourth equilibrium solution have positive coordinates? Find the exact h values.

Question 4 (12 points).

Suppose that you found in the previous problem that for $h_1 < h < h_2$ there is a fourth equilibrium solution in first quadrant of the phase plane. Show that, if $h_1 < h < h_2$ then the equilibrium solution where both blue and green crab co-exist is the **only** stable equilibrium solution among the four possible options. Thus, no mater what the initial value (as long as it is positive), the green crab and blue crab will co-exist. (yay!)

For this problem, you can assume that the Jacobian (associated to linearization) at each equilibrium point always has two real eigenvalues.

[Hint: Try to answer using the $(\mathbf{T},\mathbf{D})$-plane. Don't calculate the eigenvalues.]

[Note: The algebra will be a little messy in this question. So be patient during your calculation.]

§D. Damped Harmonic Oscillator with Sinusoidal Forcing

Consider a damped harmonic oscillator with sinusoidal forcing whose equation is given by

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + y = \cos(\omega t) \tag{$\star$}$$

where

- $y(t)$ denotes the displacement at time t,
- $p > 0$ is the damping constant,
- the spring constant is 1, and
- $\omega > 0$ is the forcing frequency.

Question 1 (9 points).

Find a particular solution $y_0(t)$ to above differential equation ($\star$) using the *method of undetermined coefficient*. Your solution should be in terms of ω and p.

Question 2 (6 points).

Suppose your solution $y_0(t)$ is of the form

$$y_0(t) = a\cos(\omega t) + b\sin(\omega t)$$

(a) Suppose the polar form of the complex number $z = a + ib$ is given by $re^{i\theta}$. What's the polar form of $a - ib$ in terms of r and θ?

(b) Calculate the real part of $(a - ib)e^{i\omega t}$ in two different ways to show that the solution $y_0(t)$ can be also written as

$$y_0(t) = r\cos(\omega t - \theta)$$

Question 3 (10 points).

Show that regardless of initial conditions, all general solutions of the differential equation ($\star$) converge to $y_0(t)$ for large values of t.

[Hint: Show that the two-dimensional system of first order differential equations corresponding to the *associated homogeneous equation* always has a sink of some type at the origin. What do the general solutions of the nonhomogeneous system ($\star$) look like?]

Question 4 (3 points).

We conclude that if damping is present, in the long-term, every solution of ($\star$) oscillates with frequency ω and amplitude r. Write r in terms of a and b and consequently as a function of ω and p.

Question 5 (7 points).

Fix p and let $r(\omega)$ be the amplitude of the particular solution when the system is forced at frequency ω. We say that *Practical resonance* occurs when $r(\omega)$ achieves its maximum as a function of ω.

(a) Show that if $p < \sqrt{2}$, then practical resonance occurs at $\omega = \sqrt{1 - \frac{p^2}{2}}$.

(b) Show that if $p > \sqrt{2}$, then no practical resonance occurs.

§E. Existence and Uniqueness

Consider the differential equation $y' = f(t,y)$ where $f(t,y)$ is continuously differentiable in t and y i.e. the partial derivatives are continuous functions. Suppose $f(t,y)$ is a periodic function of t with period T i.e.

$$f(t+\mathrm{T},y) = f(t,y) \qquad \text{for all } t \text{ and } y$$

and suppose there are constants p,q with $p < q$ such that

$$f(t,p) > 0, \quad f(t,q) < 0 \qquad \text{for all } t$$

Question 1 (10 points).

(a) If $y_p(t)$ is a solution curve such that $y_p(0) = p$, then explain why $y_p(\mathrm{T}) > p$.

(b) If $y_q(t)$ is a solution curve such that $y_q(0) = q$, then explain why $y_q(\mathrm{T}) < q$.

[Hint: What does the direction field look like?]

Question 2 (10 points).

Show that there is a periodic solution $\tilde{y}(t)$ with period T such that $p < \tilde{y}(0) < q$.

[Hint: Why is it enough to find $\tilde{y}$ such that $\tilde{y}(\mathrm{T}) = \tilde{y}(0)$?]

Note: An example of such a function is $f(t,y) = \sin(t) - y$. However, you are not allowed choose a fixed example to prove above question statements. You have to prove that the statements are true for all functions satisfying the given assumptions.