Optimization

Class 5: Newton's Optimization Method

1 Introduction

Last time, we learned that the basic optimization method relied on a replacement function; and we saw that the Taylor quadratic was a good choice of replacement function. Newton's optimization method implements the basic optimization method with the Taylor quadratic as the replacement function.

2 Newton's optimization method

Uses the stationary point of TQ as a next guess at a stationary point for $f(x, y)$ (recall, the minimizers and maximizers of f can be found among its stationary points). The stationary point (x, y) of TQ solves

$$\nabla TQ(x,\, y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Updownarrow \qquad \text{(from Exercise 3.1.2)}$$

$$\nabla^2 f(\bar{x}, \bar{y}) \begin{bmatrix} x - \bar{x} \\ y - \bar{y} \end{bmatrix} + \nabla f(\bar{x}, \bar{y}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Updownarrow \qquad \text{(from matrix algebra)}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} - \nabla^2 f(\bar{x},\, \bar{y})^{-1}\, \nabla f(\bar{x},\, \bar{y})$$

Using the current guess $(x_{\text{old}}, y_{\text{old}}) = (\bar{x}, \bar{y})$ as the base point, we get our new guess $(x_{\text{new}}, y_{\text{new}}) = (x, y)$:

$$\boxed{\begin{bmatrix} x_{\text{new}} \\ y_{\text{new}} \end{bmatrix} = \begin{bmatrix} x_{\text{old}} \\ y_{\text{old}} \end{bmatrix} - \nabla^2 f(x_{\text{old}}, y_{\text{old}})^{-1} \nabla f(x_{\text{old}}, y_{\text{old}})} \quad \text{NEWTON UPDATE}$$

So the new guess is obtained by taking a full-step away from the old guess along the *step-direction* vector

$$\mathbf{d} = -\nabla^2 f(x_{\text{old}}, y_{\text{old}})^{-1} \nabla f(x_{\text{old}}, y_{\text{old}})$$

Warning: Need the second-derivative matrix to be invertible for this to work.

The Newton update is one example of an update of the form

$$\begin{bmatrix} x_{\text{new}} \\ y_{\text{new}} \end{bmatrix} = G(x_{\text{old}}, y_{\text{old}})$$

where the new guess is determined by evaluating an *iteration mapping* $G(x, y)$ at the current guess. The Newton iteration mapping is evidently given by

$$G(x,y) = \begin{bmatrix} x \\ y \end{bmatrix} - \nabla^2 f(x,y)^{-1} \nabla f(x,y).$$

3 Lab Exercise

Use **VisOpt2D.nb** (see the Visual Optimization section on Blackboard) to implement Newton's optimization method separately for

(i) the two-variable discretization function $f(x,y) = x^3 - x\,y^2 + y^3 - y + 1$, and

(ii) the monthly payment function

$$f(x,y) = \frac{(30-y)\left(1 + \frac{10^9}{(108-x)^2\,(30-y)^5}\right)^{\frac{x}{12}}}{x}.$$

See Class5.nb for solutions