Optimization

Class 11: Reduction

1 Reduction Method

This approach for dealing with constraints builds them directly into the objective function by solving them for the dependent variables and substituting; thus reducing the number of variables. Linear equation constraints can often be solved for dependent variables in terms of other (independent) variables, and these expressions can be used to substitute for the dependent variables in the objective function.

Group Exercise 6.2.3

(a) Solve

$$\begin{aligned} x + y + z &= 30 \\ x - y + 2\,z &= 10. \end{aligned}$$

for x and y in terms of the independent variable z only.

(b) How many independent variables would you typically expect for a system of m equations involving n variables?

In this example, we can incorporate the equation constraints via

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 - 3\,z/2 \\ 10 + z/2 \\ z \end{bmatrix};$$

so to optimize an objective $f(x, y, z)$ under these constraints amounts to optimizing the *reduced objective function*

$$rf(z) = f(20 - 3\,z/2, 10 + z/2, z)$$

without any constraints! We can thus apply Newton's optimization method or a Quasi-Newton method directly to the reduced objective function to solve the original constrained optimization problem.

2 Warning: Nonlinear constraints

Direct substitution does not always work when the constraints are nonlinear, since it is not always possible to solve for the dependent variables in that case (e.g., $\cos(x\,y) = x\,y$); and even when we can use nonlinear constraints to reduce variables, things can go horribly wrong.

Group Problem 6.3

(a) Solve the optimization problem

$$\min x^2 + y^2 \text{ over pairs } (x, y) \text{ obeying } y^2 = x - 1$$

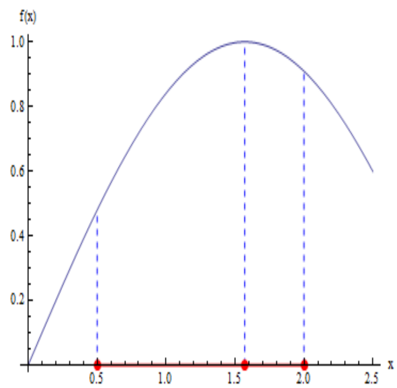
by directly substituting y^2 from the constraint into the objective function, and minimizing the resulting function of one variable.

(b) Now solve the same problem by solving $y^2 = x - 1$ for x, and directly substituting that into the objective function.

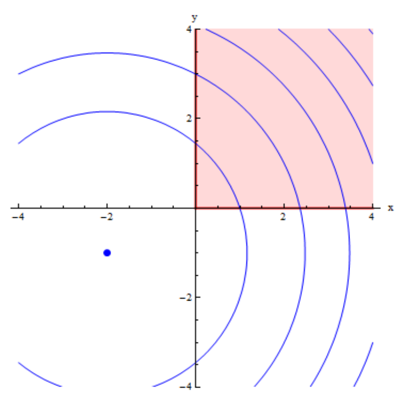
3 Inequalities

We can't solve inequalities for dependent variables, so we need to convert any inequalities to equations somehow. Following the approach from calculus, we choose the optimizer from the following candidates:

- stationary points of the (unconstrained) objective that happen to satisfy the constraints.
- points on the constraint-region boundary (endpoints in single-variable calculus).



For a two-variable example, consider minimizing $(x+2)^2 + (y+1)^2$ over the simple constraint region $x \geq 0$ and $y \geq 0$:



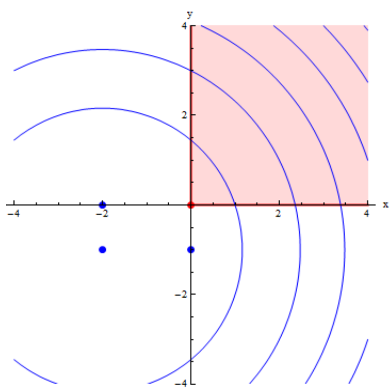
We can find the stationary point $(-2,-1)$ of the unconstrained objective, but the constraint-region boundary contains an infinite number of points to compare (not simply a few end-points). To narrow candidates, we ignore one of the constraints, and find all local optimizers on the boundary of the other. For example,

$$\min\,(x+2)^2+(y+1)^2 \text{ over pairs } (x,y) \text{ obeying } x=0 \quad \to (x,y)=(0,-1).$$

Then switch the roles to cover the other boundary.

$$\min\,(x+2)^2+(y+1)^2 \text{ over pairs } (x,y) \text{ obeying } y=0 \quad \to (x,y)=(-2,0).$$

Unfortunately, none of these candidates satisfies the constraints!



We missed the boundary of the boundaries, at their intersection $x=0$ and $y=0$:

$$\min\,(x+2)^2+(y+1)^2 \text{ over pairs } (x,y) \text{ obeying } x=0 \text{ and } y=0 \quad \to (x,y)=(0,0);$$

which solves the original constrained problem.

Approach for inequalities: Solve all combinations of equation subproblems

Consider all the combinations of equations corresponding to each component of the boundary of the constraint region (including the "combination" of none, which corresponds to unconstrained optimization). Separately find the local optimizers for each of the subproblems generated by a different combination of equation constraints, and test these local optimizers to see if they obey the original constraints.

Q: How many different equation constraint combinations are generated by the boundary of $x \geq 0$, $y \geq 0$, and $z \geq 0$?

Q: How many different equation constraint combinations are generated by the boundary of $x_1 \geq 0, \ldots, x_{30} \geq 0$?

4 Lab Exercise

- Use Mathematica's `ContourPlot` command to generate a contour plot of the objective function x^2+y^2 for inputs $x \in [0,2]$ and $y \in [-1,1]$. Store this image by putting `cp =` in front of the command and "Entering".

- Use the same command to generate a plot of the constraint curve $y^2 = x - 1$ for y from -1 to 1. Hint: use $y^2 == x - 1$ (with double equal signs) as the "function" to plot. Store this image as above, but use `constraint = ` in front.

- "Enter" `Show[cp,constraint]` to overlay the two images; and then locate the true constrained minimizer for this problem.

- Compare your solution via direct substitution with your graphical solution and explain what went wrong with direct substitution in the first case.

HINT: To learn about any of these commands, you can type the command in your Mathematica notebook, highlight it, and select `Find Selected Function` under the `Help` menu item at the top.