Optimization

Class 3: Impractical Optimization

1 Monthly payment example

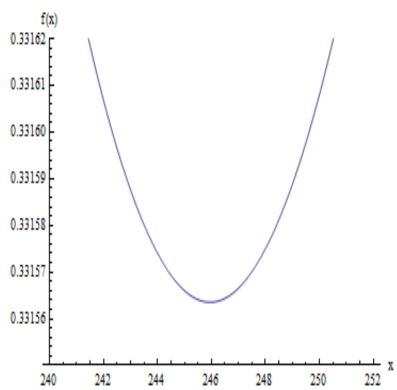
Suppose we want to minimize monthly payments on a $\$30K$ loan for a term of x months (with annual interest rate 5%):

$$f(x) = \frac{30\,(1+0.05)^{\frac{x}{12}}}{x}$$

Q: How might we find the minimizing term length?

A: The minimizing term isn't obvious since a longer term decreases the value of f via the denominator, but increases the value of f via the numerator (since more compounding).

We could plot the monthly payment function:



Maybe the minimizing term is $x = 246$ months. It certainly looks like a *local* minimizer (no nearby inputs generate lower output value), but we can't be sure it's a *global* minimizer (no inputs generate lower output value) from just this piece of the graph. In fact, this function has no global minimizer since the function values approach negative infinity as the variable x approaches zero from the negative side. Of course, in this application it makes sense to enforce $x \geq 0$, so we'll assume that.

We can use calculus to find the stationary point(s) that are non-negative (where $f'(x) = 0$) since those include all local minimizers (and global minimizers, if they exist).

Group Problem 1

(a) Use calculus to find the stationary point of the monthly payment function. (Recall $\frac{\mathrm{d}(a^x)}{\mathrm{d}x} = \ln(a)\,a^x$.)

(b) Assuming a minimizing term length $x \geq 0$ exists, what is it? Explain.

Group Problem 1 (solution)

(a) Differentiating the monthly payment function, we get

$$f'(x) = -\frac{(30)1.05^{x/12}}{x^2} + \frac{5\ \ln(1.05)\,1.05^{x/12}}{2\,x}.$$

Setting this equal to zero and solving for x gives

$$\begin{aligned}
0 &= -\frac{(30)1.05^{x/12}}{x^2} + \frac{5\ \ln(1.05)\,1.05^{x/12}}{2\,x} \\
\text{(multipliying by } -x^2\text{)} \quad & \Updownarrow \\
0 &= (30)1.05^{x/12} - x\left(\frac{5\ \ln(1.05)\,1.05^{x/12}}{2}\right) \\
&= 1.05^{x/12}\left(30 - \frac{5\ \ln(1.05)}{2}\,x\right) \\
& \Updownarrow \\
x &= \frac{30}{\frac{5\ \ln(1.05)}{2}} = \frac{12}{\ln(1.05)} \approx 245.951
\end{aligned}$$

(b) We could approach this problem by analyzing the global behavior of this function on $x \geq 0$. However, knowing that there is a minimizer allows us to conclude that it must be among the critical points (stationary points and undefined points of f) or at the endpoint of $x \geq 0$. Since the monthly payment function approaches infinity as $x \to 0$ (from the positive side), the minimizer must be at the stationary point 245.951.

2 Trouble: Rebates and variable interest rates

Now suppose that the interest rate z changes depending on the term and rebate:

term\rebate	\$1,000	\$2,000	\$3,000	\$4,000	\$5,000	\$6,000
36 months	0.94%	1.1%	1.3%	1.6%	2.0%	2.4%
48 months	1.4%	1.6%	1.9%	2.3%	2.8%	3.5%
60 months	2.1%	2.5%	3.0%	3.7%	4.4%	5.5%
72 months	3.8%	4.5%	5.4%	6.5%	7.9%	9.7%

We can model this via the interest-rate function

$$z = \frac{10^9}{(108-x)^2\,(30-y)^5}.$$

It follows in this case that the monthly payment function depends on term x and rebate y according to:

$$f(x, y) = \frac{(30-y)\left(1 + \frac{10^9}{(108-x)^2\,(30-y)^5}\right)^{\frac{x}{12}}}{x}.$$

Group Problem 2

(a) Explain the first factor $(30 - y)$ in the monthly payment function with variable interest rate.

(b) Explain how the interest rate function favors rebate values y that avoid 30. Why is this desirable?

(c) Find the stationary points of this function (i.e. where $\nabla f(x, y) = \vec{0}$). Hint, its two partial derivatives are:

$$f_x(x, y) = \frac{1}{x^2} \left(1 - \frac{1000000000}{(-108 + x)^2(-30 + y)^5}\right)^{x/12} (30 - y) \left(-1 + \frac{1}{12}x\left(\frac{2000000000x}{(108 - x)^3\left(1 - \frac{1000000000}{(-108+x)^2(-30+y)^5}\right)(30 - y)^5} + \ln\left(1 - \frac{1000000000}{(-108 + x)^2(-30 + y)^5}\right)\right)\right)$$

and

$$f_y(x, y) = \frac{1}{3} \left(-\frac{3}{x} + \frac{1250000000}{(-108 + x)^2\left(1 - \frac{1000000000}{(-108+x)^2(-30+y)^5}\right)(30 - y)^5}\right) \left(1 - \frac{1000000000}{(-108 + x)^2(-30 + y)^5}\right)^{x/12}.$$

Group Problem 2 (solution)

(a) The first factor $(30 - y)$ in the monthly payment function with variable interest rate corresponds to the principal after rebate, which is the amount of principal still owed to the dealer after the sale.

(b) The interest rate function with variable interest rate favors rebate values y that avoid 30 via the quotient

$$\frac{1}{(30 - y)^5}$$

which becomes very large when y approaches 30. This is desirable since the price of the car is 30 (thousand dollars), and the car dealer would not want to give the car away for free by rebating the full price.

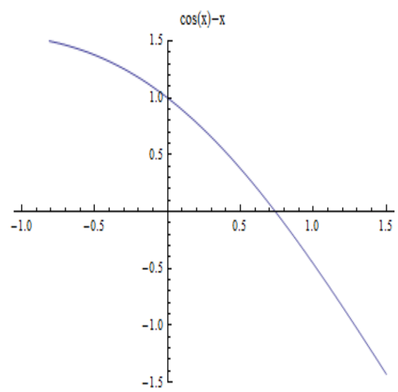
(c) Setting these equal to zero and solving simultaneously for (x, y) is not something anyone (including your software) will enjoy. **See With Variable Interest Rate in Class3.nb for Mathematica's attempt.**

3 Algebraic Failure

Solving equations by algebra is rarely possible in practice:

- *Unsolvable*: The equation has no solution among the variables of interest. E.g., $x^2 = -1$ has no real-valued solutions.
- *Implicit*: Algebraic manipulations can't reveal the solution explicitly. E.g. $\cos(x) - x = 0$ is called a "transcendental" equation because it transcends our ability to solve it algebraically.

It is clear from the following graph that there is a solution to $\cos(x) - x = 0$:



We'll see later that the stationary-point equations for the monthly payment function with variable interest rate also have a solution (that cannot be revealed explicitly via algebra).

Sometimes algebraic manipulation is theoretically possible, but the size of the problem makes this impractical. E.g., airlines with hundreds of flights $\Rightarrow$ hundreds of variables $\Rightarrow$ hundreds of simultaneous stationary point equations. The following section illustrates an important construction that often generates such large-scale optimization problems.

4 Aerodynamics example

Find the surface with least resistance to downward motion among all surfaces generated by vertical-axis revolution of graphs functions $x(t)$ passing through $(0, 0)$ and $(1, 1)$ like

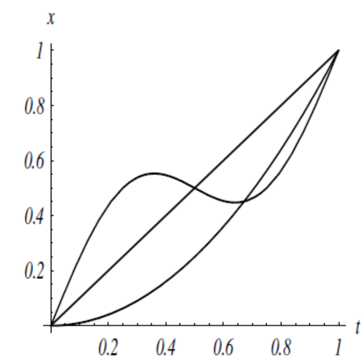


Figure 1: Graphs of three surface-generating $x(t)$

The resistance of such surfaces is a constant times Newton's integral

$$\int_0^1 x(t)\,\frac{x'(t)^3}{1+x'(t)^2}\,\mathrm{d}t.$$

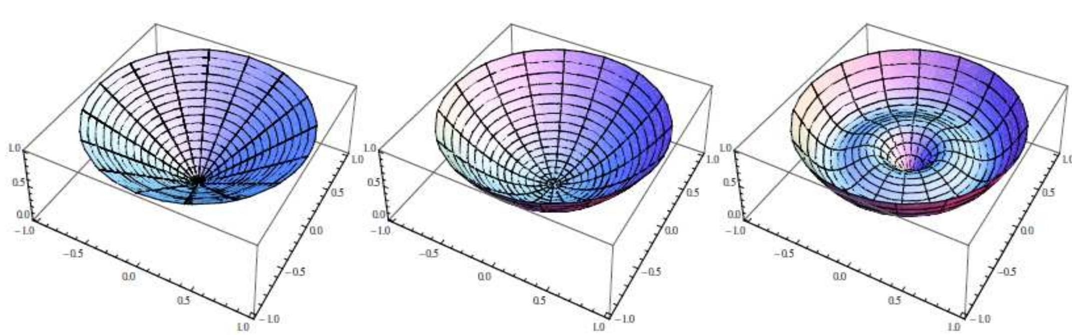
Group Problem 3

(a) Sketch the three surfaces of revolution corresponding to the three surface-generating functions whose graphs are shown in Figure 1.

(b) Give the formulas $x(t) = ?$ for two surface-generating functions, and compute the value of Newton's integral for each.

Group Problem 3 (solution)

(a) Using Mathematica (**see Surfaces of Revolution in Class3.nb**):



(b) Two possibilities are $x(t) = t$ (with integral value 0.25) and $x(t) = t^2$ (with integral value 0.3506). One that you're unlikely to guess is $x(t) = 1/4\sin(2\,\pi\,t) + t$ (with integral value 0.406286). These are the three surface generating functions pictured in Figure 1.

4.1 Integral optimization

- Variables: Functions $x(t)$ inside the integral, representing a continuum of input; and not simply n-tuples (i.e. pairs, triples, etc.) of numbers.
- Objective: Integral "functional", so-called because it is a function of inputs $x(t)$ which are functions themselves.

This is beyond the tools of traditional calculus (e.g., how to differentiate the integral with respect to the function $x(t)$?).

4.2 Discretization

Similar to rectangle approximations of integrals. Turn the integral optimization problem into a traditional one with a discrete (i.e., finite) number of variable inputs. A simple example:

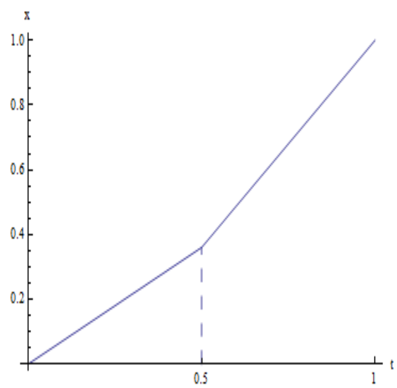


Figure 2: Choose only the x-coordinate above $t = 0.5$

where two linear pieces connect the endpoints through the middle point. Now the only variable is the middle height of the graph (a number, we'll just call it x). **See Discretization in Class3.nb.**

4.3 Approximating the integral functional

One version of discretization would simply break the integral objective function at the chosen t-values, and substitute the linear piece for $x(t)$ over each subinterval. For simple integrals, that may lead to a reasonable discretized objective function. However, in order to be able to handle more complex integrals, we often approximate instead in ways that are (loosely) related to the integral approximations seen in calculus.

For instance, the "right-endpoint" version uses

$$\begin{array}{ll} \text{for } t \in [0, 0.5] \text{ replace} & \begin{array}{l} x(t) \to x \quad \text{(the middle height of the graph)} \\ x'(t) \to 2\,x \quad \text{(the slope of the first line segment)} \end{array} \\ \text{for } t \in (0.5, 1] \text{ replace} & \begin{array}{l} x(t) \to 1 \quad \text{(the rightmost height of the graph)} \\ x'(t) \to 2\,(1 - x) \quad \text{(the slope of the second line segment)} \end{array} \end{array}$$

Note, this replaces the function $x(t)$ with a constant on each t-subinterval, but does not use the slope (zero) of that constant function to replace the derivative $x'(t)$.

The right-endpoint replacements result in the discretized Newton integral

$$\begin{aligned} \text{RENewton}(x) &= \int_0^{\frac{1}{2}} x \left(\frac{(2\,x)^3}{1+(2\,x)^2} \right) \mathrm{d}t + \int_{\frac{1}{2}}^1 1 \left(\frac{(2\,(1-x))^3}{1+(2\,(1-x))^2} \right) \mathrm{d}t \\ &= x \left(\frac{(2\,x)^3}{1+(2\,x)^2} \right) \frac{1}{2} + 1 \left(\frac{(2\,(1-x))^3}{1+(2\,(1-x))^2} \right) \frac{1}{2} \end{aligned}$$

as the function of the single-variable x (a number). Notice that our replacements mean the integrand is constant with respect to t, so each integration is just the interval width times the corresponding integrand.

Group Problem 4

Give the discretized Newton integral function $\text{LENewton}(x)$ that results from using left-endpoints instead of right.

Group Problem 4 (solution)

$$\text{LENewton}(x) = x \left(\frac{(2\,(1-x))^3}{1+(2\,(1-x))^2} \right) \frac{1}{2}.$$

Its stationary point equation is

$$\frac{4\,(1-x)^3}{1+4\,(1-x)^2} - \frac{12\,(1-x)^2\,x}{1+4\,(1-x)^2} + \frac{32\,(1-x)^4\,x}{(1+4\,(1-x)^2)^2} = 0,$$

so the stationary points are $x = 1$ and $x = 0.362$. The second of these corresponds to a surface-generating function like the one pictured in Figure 2.

4.4 Discretizing at more points

Q: How many variables are there in a discretization problem using the t-values of 1/4, 1/2, and 3/4?

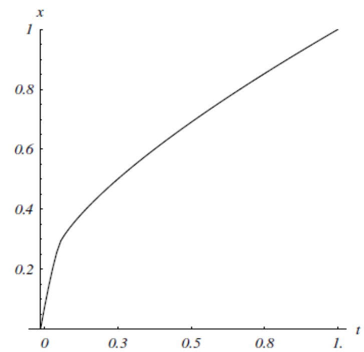
A: Three; one for each interior point (the endpoints are locked in).

Q: How can we be sure that the resulting set of surface-generating functions includes all of the surface-generating functions from the discretization using just $t = 1/2$?

A: Because $t = 1/2$ is included in the set of new t-values: 1/4, $\boxed{1/2}$, and 3/4. **See Discretization at more points in Class3.nb.**

Moral: We improve the approximation to our solution by discretizing at more points (just like rectangles and integrals); but the price we pay is needing more variables (which can make algebraic manipulation infeasible).

4.5 Solution



Q: Can this be obtained by discretization?

A: Only with a lot of intermediate t-values (thus a lot of variables), since the graph doesn't appear to be made up of line segments. The graph above was generated with 73 intermediate t-values.