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Homework 7

October 27, 2020

1 Resources

Other than the textbook and class notes, nothing

2 Notes for Week 6

V25: Introduction to separation of variable method for PDEs

1. Reminder of wave equation
2. Outline of how to use separation of variable method

Recall wave equation: $u_{tt} - c^2 u_{xx} = 0$**, where** $c = \sqrt{T/\rho}$
T is the string tension and ρ is the string density
We now consider a bounded domain $0 < x < l$

We will use separation of variables method to solve the bounded case
GOAL: Look for a solution of a special form:

$$u(c,t) = X(x)T(t)$$

We want to separate dependence on x and dependence on t

1. Separate variables to obtain 2 ODEs
2. Solve the $X(x)$ ODE subject to b.c.s
3. Solve the $T(t)$ ODE
4. take a linear combination of all solutions
5. Use initial values to determine arbitrary constants

V26: Separation of Variables pt 1

Readings: 4.1 pg 84-85

GOAL: find a solution to the wave eq on a bounded domain
We have the following **important** setup:

- $u_{tt} - c^2 u_{xx} = 0$
- $0 < x < l, t < 0$

- I.C $u(x,0) = \phi(x), u_t(x,0) = \psi(x)$
- $u(0,t) = 0, u(l,t) = 0$ ¡–boundary conditions

We will talk about just the first step of the process, separating variables to obtain 2 ODEs

We want to look for solutions of the form:

$$u(x,t) = X(t)T(t), \text{ plug this into the PDE}$$

We know X an T depend on a single variable, so when we take the derivative it is an ordinary derivative
To plug this in, we get:

$$XT'' - c^2X''T = 0$$

$$\implies XT'' = c^2X''T$$

$$\implies \frac{T''}{c^2T} = \frac{X''}{X}$$

We note that the LHS depends on t only and RHS depends on x only. If we want to be lazy, we write that

$\lambda = \frac{T''}{c^2T} = \frac{X''}{X}$
$\implies \lambda = \frac{T''}{c^2T} \implies \frac{d\lambda}{dx} = 0$ and $\lambda = \frac{X''}{X} \implies \frac{d\lambda}{dt} = 0$
Since x derivative of λ is 0 and so is t derivative, we know that λ has to be constant.

So, since lambda is constant we have: $\lambda c^2T = T''$

We also have: $\lambda X = X''$
These are our two equations we need to solve now

V27: Separation of variable method to solve PDE on a bounded domain pt 2

We are now considering the second step, solving the $X(x)$ ODE subject to b.c.s

Let's solve the $X(x)$ subject to $u(0,t) = 0$ and $u(l,t) = 0$

Recall the X ODE is $\lambda X = X''$
What we are going to do now is think about the possibilities for λ, which will determine what our general solution is and looks like. What our solution looks like will depend on λ

1. $\lambda < 0$
2. $\lambda = 0$
3. $\lambda > 0$

Today we will show case 1, later we will show cases 2,3 as they cannot happen for Dirchlet boundary conditions

Let's proceed to try and solve this ODE, assuming $\lambda < 0$. We will write $\lambda = -\beta^2$
Then, we have:

$$-\beta^2 X = X'' \tag{1}$$
$$X'' + \beta^2 X = 0$$

Recall from ODEs that solution is $X(x) = C\cos(\beta x) + D\sin(\beta x)$
This is based on our rules for 2nd order ODEs [1]
Now, what we want to do is take into account our boundary conditions.
Recall that $U(0,t) = 0$ and $u(l,t) = 0$

But, I have assumed that $u(x,t) = X(x)T(t)$, which is saying that $X(0)T(t) = 0$ and $X(l)T(t) = 0$

The only way that $X(0)T(t) = 0$ can hold true for all t is if $X(0) = 0$

Similarly, the only way $X(0)T(t) = 0$ can hold is if $X(l) = 0$

Let's recall that $X(x) = C\cos(\beta x) + D\sin(\beta x)$ and use these boundary conditions to try to figure out C and D using the boundary conditions.

We know that $X(0) = C\cos(0) = C$ and that $X(0) = 0$ from our boundary conditions, implying that $C = 0$

We also know that $X(l) = D\sin(\beta l) = 0 \implies D\sin(\beta l) = 0$
Let's think about this more closely...we don't want $D = 0$ since then $X(x) = 0$ and we have a trivial solution. We already know 0 is a solution!
Then, we need $\sin(\beta l) = 0$ to make the equation hold true.

Recall $\sin(n\pi) = 0$, where n is an integer.
That means that we need $\beta l = n\pi$ OR that $\beta = \frac{n\pi}{l}$
Also recall that $\lambda = -\beta^2$ assuming lambda is greater than 0. So, $n = 1, 2, 3, 4, ...$
Thus, $\beta = \frac{n\pi}{l}$
This brings us to our final answer:

$$X(x) = D\sin(\frac{n\pi}{l}x)$$

V28: Using separation of variable method pt 2

This is steps 3/4 of the process outlined in V25, also page 85 of the textbook

Let's first quickly recall what we've learned so far:
Solution has a special form $u(x,t) = X(x)T(t)$, and we also found that $X(x) = D\sin(\beta x)$, where $\beta = \frac{n\pi}{l}$, and that $n = 1, 2, 3, ...$
We have an infinite number of functions since β can be any whole number. Thus, we can think of β as being β_n
$\implies X_n(x) = D_n \sin(\frac{\beta n x}{l}) = D_n \sin(n\pi x l)$

Recall that $\frac{T''}{c^2T} = \lambda$, given $\lambda < 0$

We also set λ to $\lambda = -\beta^2$ so the whole thing is $\frac{T''}{c^2T} = -\beta^2$ which is the same thing as $T'' = -\beta^2c^2T$ or that $T'' + (\beta c)^2T = 0$
And thus this is our ODE that we want to solve

Now, we know that $(\beta c)^2 > 0$, so the general solution is:

$$T(t) = A\cos(\beta ct) + B\sin(\beta ct)$$

Now, we don't have boundary conditions for t so it is really much simpler
We've already decided β, so we can instead write:

$$T_n(t) = A_n \cos(\frac{n\pi}{l}) + B_n \sin(\frac{n\pi}{l}ct)$$

represents solutions for each value of n

So, let's go back to step 4, taking a linear comb of all solutions

$$u(x,t) = X(x)T(t) = D_n \sin(\tfrac{n\pi}{l}x)\left(A_n\cos(\tfrac{n\pi}{l}ct) + B_n\sin(\tfrac{n\pi}{l}ct)\right)$$

Thus, we have one solution for each n and that we have three arbitrary constants, A, B, D so we can absorb these constants, ie $A_n \to A_nD_n$ and $B_n \to B_nD_n$

So, we know a linear combination of solutions is a solution and since we already have constants in the eq. we can just take the sum of all of them!

$$u(x,t) = \textstyle\sum_{n=1}^{\infty}\left(A_n\cos(\tfrac{n\pi}{l}ct) + B_n\sin(\tfrac{n\pi}{l}ct)\right)\sin(\tfrac{n\pi}{l}x)$$

So this is a linear combination of all the solutions, which we call the **series solution**, because this thing here is a series

V29: Using separation of variable method part 3

Step 5, considering initial values to determine arbitrary constants

We are going to lay out how this works. We've talked about the PDE and considered the boundary conditions but now need to consider the inital conditions
Recall from the last video we have:

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos(\tfrac{n\pi}{l}ct) + B_n \sin(\tfrac{n\pi}{l}ct)\right) \sin(\tfrac{n\pi}{l}x)$$

We also have I.C.

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(\tfrac{n}{l})$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \left(-A_n \sin(\tfrac{n\pi}{l}ct)\tfrac{n\pi c}{l} + B_n \cos(\tfrac{n\pi}{l}ct)\tfrac{cn\pi}{l}\right) \sin(\tfrac{n\pi x}{l})$$

Evaluating this derivative at 0 we get:

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \sin(\tfrac{n\pi x}{l})\tfrac{cn\pi}{l}$$

So, this is our initial conditions $\phi(x)$ and $\psi(x)$ such that:

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \sin(\tfrac{n\pi x}{l})\tfrac{cn\pi}{l} = \psi(x)$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(\tfrac{n}{l}) = \phi(x)$$

This is as long as the above series converges, which we will address later.

In sum, we have done the following in the last few videos:

- Have solution to wave equation (pink box) on a bounded domain, IC and bcs
- Formula for u which is an infinite series that involves the coefficients A_n and B_n which are defined implicitly
- Call this the series solution
- How do we get specific values for the A_n and B_n? Topic for next week!

Summary:

Solution to:

$$u_{tt} - c^2 u_{xx} = 0, \qquad 0 < x < \ell, \quad t > 0$$
$$u(x,0) = \phi(x) \qquad u_t(x,0) = \psi(x)$$
$$u(0,t) = 0$$
$$u(\ell,t) = 0$$

is

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi ct}{\ell} + B_n \sin \frac{n\pi ct}{\ell} \right) \sin \frac{n\pi x}{\ell}$$

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{\ell} \qquad \psi(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \sin \frac{n\pi x}{\ell}$$

3 Problems

V25 Problems

3. What are the steps listed in the video for the separation of variable method to solve PDEs? We will be seeing a lot of details soon, so it is good to keep the road map in mind.

Refer to V25 notes for more details

1. Separate variables to obtain 2 ODEs
2. Solve the $X(x)$ ODE subject to b.c.s
3. Solve the $T(t)$ ODE
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V26 Problems

Find the general solution to the ODEs $X'' + 4X = 0$ and $X'' - 4X = 0$ where $X = X(x)$.

We first consider $X'' - 4X = 0$.
Then, the characteristic polynomial is $k^2 - 4$.
This polynomial has positive discriminant $b^2 - 4ac = 0^2 - 4(1)(-4) = 16$, so we know the roots are both real and distinct. These roots are ± 2.
Now, we know the solution to the ODE is of the form:

$$Y(x) = C_1 e^{2x} + C_2 e^{-2x}$$

Second, we consider $X'' + 4X = 0$.
The characteristic polynomial is $k^2 + 4$
This polynomial has discriminant $b^2 - 4ac = 0^2 - 4(1)(4) = -16$, so we have complex roots $\pm 2i$ [1]

Now, we know the solution to the ODE is of the form:

$$Y(x) = e^{0x} \left[C_1 \cos(2x) + C_2 \sin(2x) \right]$$

Simplifying, we have:

$$Y(x) = \left[C_1 \cos(2x) + C_2 \sin(2x) \right]$$

V27 Problems

5. In the video we assumed in the equation, $X'' = \lambda X$, (1), that $\lambda < 0$. Suppose $\lambda = 0$. Show that there is no solution of (1) with $\lambda = 0$ that satisfies the Dirichlet boundary condition $X(0) = 0$ and $X(l) = 0$ besides the trivial zero one.

If $\lambda = 0$, then we have that $0 = X''$
Then, we know this is the case of a double root, since the corresponding polynomial is $x'' = 0$, which as an ODE has one root since $b^2 - 4ac = 0 - 4(1)(0) = 0$
We know this root is $x = 0$, giving us our general solution $X = (Ax+B)e^0 x$, which simplified, is simply:

$$X(x) = (Ax + B)$$

Now, we have $X(0) = 0 \implies A(0) + B = 0 \implies B = 0$
We also have $X(l) = 0 \implies Al = 0$
This implies that either $l = 0$ or $A = 0$
Since we assume in our important setup (from video 26) that $l > 0$, it must be that $A = 0$

We can then conclude that the only solution that satisfies the boundary conditions is the trivial solution $X(x) = 0$

6. In the video we assumed in the equation, $X'' = \lambda X$, (2), that $\lambda < 0$. Suppose $\lambda > 0$. You could call it $\lambda = \beta^2$, 2 where β is real. Show that there is no solution of (2) with $\lambda = \beta^2$ that satisfies the Dirichlet boundary condition $X(0) = 0$ and $X(l) = 0$ besides the trivial zero one.

Given that $\lambda = \beta^2$, we then have:

$$\beta^2 X = X''$$

$$X'' - \beta^2 X = 0$$

Recall from ODEs that this equation has discriminant $(-\beta^2)^2 - 4(1)(0) = \beta^4$ and thus has two real roots, 0 and β^2
Then, we have that the general solution is $X(x) = Ae^{\beta^2 x} + Be^{0x}$ which simplifies to:

$$X(x) = Ae^{\beta^2 x} + B$$

Now, applying our boundary conditions, we have that $X(0) = Ae^{\beta^2 \cdot 0} + B \implies A + B = 0$
We also have $X(l) = Ae^{\beta^2 \cdot l} + B \implies Ae^{\beta^2 \cdot l} + B = 0$
Thus, we see that $A + B = Ae^{\beta^2 \cdot l} + B \implies A = Ae^{\beta^2 \cdot l}$
We note that the only way this can hold true is if $A = 0$ since we assumed $l > 0$

Then, we have $X(x) = B$, and applying our b.c.s again we see that $X(0) = B \implies B = 0$

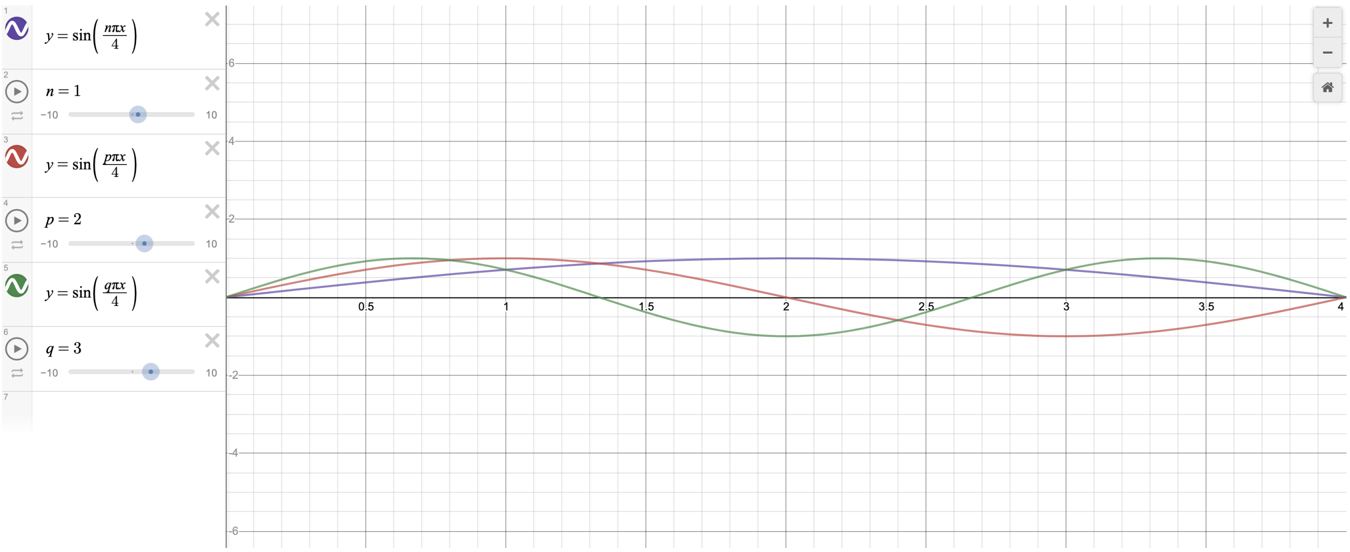
We then see that the only solution is the trivial solution $X(x) = 0$

7. In the video, it was shown that $X(x) = D\sin(\frac{n\pi x}{l})$ solves Eq. (1) with $\lambda < 0$, where $n = 1, 2, 3, ...$ Why didn't we include $n = 0$? Are we missing solutions by ignoring negative n values?

In order to make the equation $X(l) = D\sin(\beta l) = 0$ hold true without trivial solution $D = 0$, it had to be that $\sin(\beta l) = 0$
We also know that $\sin(n\pi) = 0$, where n is an integer, always holds because of the unit circle.
But, since we assumed $\lambda < 0$, and $\lambda = -\beta^2$, we know that $\beta \neq 0$

Since we need to make $\sin(\beta l) = 0$, we let $\beta = n\pi/\lambda$ for some integer $n > 0$
Thus, we are not missing solutions for this case since we know $\beta \neq 0$, and also note that if β *were* equal to 0, then we would have $\lambda = 0$, which we showed in problem 5 only has the trivial solution.

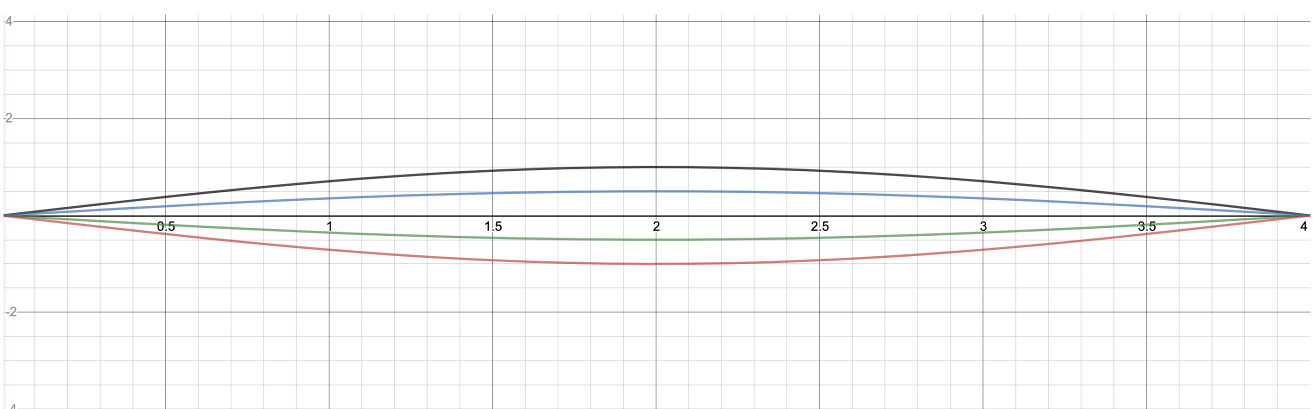
8. Set $D = 1$ and $l = 4$. Plot $X(x) = D\sin(\frac{n\pi x}{l})$ in Desmos for $n = 1, 2, 3$ on the same graph.



Yes, the graph clearly demonstrates the boundary conditions since we see $X(0) = 0$ for all three graphs and $X(4) = 0$ for all three graphs

V28 Problems

9. Consider the wave equation with $c = 8$ and $l = 4$. Consider the series solution of the wave equation, formula (9) on page 85, with $B_n = 0$ for all n and $A_1 = 1$ for $n = 1$ and $A_n = 0$ otherwise. Plot $u(x, t)$ for $0 \leq x \leq 4$ for $t = 0, t = 1/6, t = 1/3$ and $t = 1/2$ on the same graph. (If you make a t slider in Desmos you can actually make a movie if you hit the play button on the left part of where you enter the slider information). Describe the behavior of the solution by ploting and experimenting with different values of t.



In this graph, the black line represents $t = 0$, the blue $t = 1/6$, the green $t = 1/3$ and the red $t = 1/2$

The graph is oscillating between $u = 1$ and $u = -1$, with the peak at $x = 2$. The period of the oscillations are 1, meaning that the graph cycles from the black curve down to the red curve and back up to the black curve every 1 unit.

V29 Problems

10. What are $\phi(x)$ and $\psi(x)$ that correspond to the solution you considered in problem 9.

In problem 9, we had $u(x,t) = \cos(2\pi t)\sin\left(\frac{\pi x}{4}\right)$
Then, $u(x,0) = \phi(x)$, and so we have:

$$\phi(x) = \cos(2\pi \cdot 0)\sin\left(\frac{\pi x}{4}\right)$$

$$\phi(x) = \cos(0)\sin\left(\frac{\pi x}{4}\right)$$

$$\phi(x) = \sin\left(\frac{\pi x}{4}\right)$$

Also, $\psi(x) = u_t(x,0)$
We have that $u_t = -2\sin(2\pi t)\sin(\frac{nx}{4})$

$$\psi(x) = -2\sin(2\pi \cdot 0)\sin(\frac{nx}{4})$$

$$\psi(x) = -2\sin(0)\sin(\frac{nx}{4})$$

$$\psi(x) = 0$$

11. Read, starting from "the analogous problem for diffusion is ..." on page 87 of the book, to the little open box on page 88 (up until the part about eigenvalues). Here the solution for the diffusion/heat equation on a bounded domain with Dirichlet boundary conditions is discussed. Find one step from the 5 listed for separation of variables you wrote down in problem 3, whose work will look different when considering the diffusion equation instead of the wave equation, and write what is different.

The only difference is in step 3, solving the $T(t)$ ODE, since in the wave equation we have a second derivative of t, u_{tt}, while in the diffusion equation we only have a first derivative of t. Then, since the corresponding $T(t)$ was a second order ODE we used the method of solving second order ODEs which is based on the associated polynomial of the ODE. Here, however, we only have a first order ODE, so the solution to $T' = -\lambda kT$ using separation of variables is:

$$\tfrac{1}{T}T' = -\lambda k$$

$$\implies \int \tfrac{1}{T} dT = -\lambda k \int dt$$

$$\implies |\ln(T)| = -\lambda kt + a, \text{ for some arbitrary constant } a$$

Thus, we arrive at the desired answer,

$$T(t) = Ae^{-\lambda kt}$$

References

[1] Second order linear differential equations. https://www.math.utah.edu/online/1220/notes/ch12.pdf.