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Homework 12

December 8, 2020

1 Resources

Other than the textbook and class notes, I discussed problems 8-14 briefly with Kayla.

2 Notes for Week 13

V47: Introduction to disease modeling

- Deriving the SIR model
- Basical analysis of the SIR model
- Definition of an epidemic

Divide entire population into three groups

1. **S(t): Number of susceptible**
2. **I(t): Number of infected**
3. **R(t): Number of recovered**

The susceptible person might become infected and leave the susceptible group, but then recover and join the recovered group. We want to take this idea and put it into a mathematical model

SIR Model

$$\frac{dS}{dt} = -\alpha SI$$

The term $-\alpha SI$ is modeling the interaction of S with I. For example, twice the number of infected people will lead to twice as many interactions between the S and I populations. So, we have the minus sign because interactions between S and I will lead to more infected people, and thus cause the Susceptible population to decrease and Infected to increase. we also have α as the contagion parameter (probability a particular infected

$$\frac{dI}{dt} = \alpha SI - \beta I$$

This second equation will cause the S population to increase. The second term is the recovered people, and we know that the more people that are infected, the more will recover. We also have $\beta = 1/d$, which is the recovery paramter and d is the average length of infection. We also assume alpha and beta are both positive.

$$\frac{dR}{dt} = \beta I$$

We have the following assumptions:

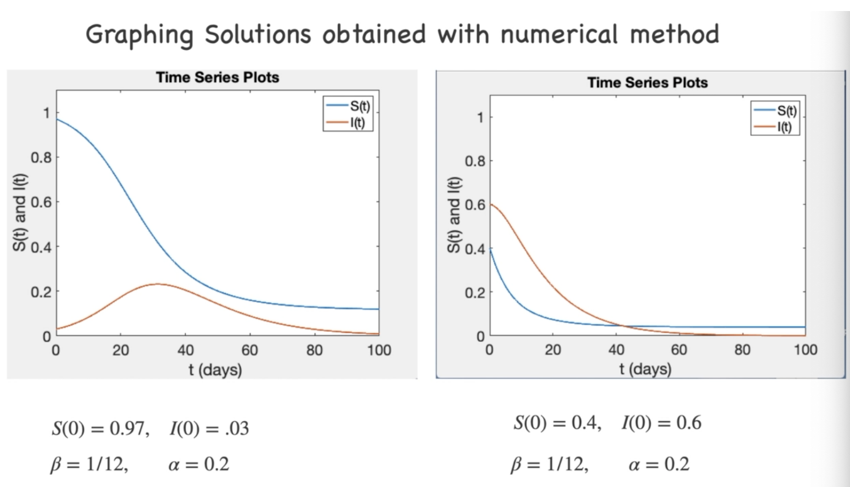
- Population is fixed (no births or deaths), or $N = S + I + R =$ constant
- No incubation period
- Susceptible and infecrted pop. are uniformly mixed
- Once a person is recovered they are permanently immune

These assumptions allow us to have a pretty simply model, but analyzing it is already pretty difficult. Note also that none of the PDEs in our system depend on R. Thus we can "solve" or analyze these first two equations only. Once we figure out what S and I are, we can use the relation $N = S + I + R$ to figure out what R is.

So we focus on our first two eq:

$$\frac{dS}{dt} = -\alpha SI$$

$$\frac{dI}{dt} - \alpha SI - \beta I$$



There is no analytical solution to this system! So we have to do other things to figure out what the model is telling us.

What does the model predict about the spread of infection?

The idea is that we can tune the graph to better fit what we want to model. We notice that these two graphs are different, where in the second we changed the initial s and i populations. Infected population is only decreasing, which is desire-able!

We say there is an epidemic if the number of infected is increasing at any given time

So, the first is an epidemic and the second is not.

V48: SIR with diffusion

1. Motivation for spatial dependence in disease models
2. Introduction to case study: rabies in foxes
3. Combining SIR and diffusion
4. Understanding parameters
5. Rescaling the system

We start with 1. Spatial location in disease spread can also be important. Aka we see the spread of the disease might be fast or slow in terms of physical space. We see a wave of red and we want to see how a disease evolves in terms of where you are.

We move onto 2, talking about the spread of rabies in Europe from 1945-1985. The front of the epizootic moved westward at an average speed of **30-60km** a year.
So, if the fox population is density is estimated at different times as the rabies epizootic passes, the wave is seen to consist of two main parts: the front through which the population is rapidly decreasing in magnitude and the much longer tail where there are essentially periodic outbreaks of the disease. We are interested in the first part.

Case Study Assumptions

- Foxes are the main carriers of rabies in the rabies epizootic considered
- The rabies virus in contained in the saliva of the rabid fox and is normally transmitted by bite
- Rabies is invariably fatal in foxes
- The rabies virus enters the central nervous system and indcues behavioral changes in its host. The foxes become aggressive, lose their sense of direction and territorial behavior and wander about in a more or less random way

Interestingly, these assumptions are similar to the SIR assumptions:

- Pop fixed
- no incubation period
- susceptible and infected uniformly mized
- once infected recovered they are immune (or dead)

Now, we consider $S(t)$: Num susceptible foxes
$I(t)$: Num infected foxes and t : time in years

$$S_t = -\alpha SI \text{ and } I_t = \alpha SI - \beta I$$

But, how do we model random motion in mathematical terms? DIFFUSION!!
So, our model for random movement is kI_{xx} and thus we have:

$$I_t = kI_{xx} + \alpha SI - \beta I$$

and where k is the diffusion constant. This is different from anything we've seen before since all 3 eq. in the PDE system have two dependent variables (S and I). Up to this point we've only been talking about one dependent variable, u. This system is also nonlinear since αSI is nonlinear. Almost everything we've seen up to this point has been linear. We now have something nonlinear and also a system...

We need to understand the parameters a bit more

Parameter estimation

- d is the life expectancy of an infected fox: field estimated 35 days
- k is the diffusion constant, in $\frac{km^2}{yrs}$
  remember $u_t = ku_{xx}$ is our diffusion eq.
  The top thing is an area and the bottom is a rate. To estimate the diffusion constant biologists think about the area for fox territory times the rate foxes leave their territory as the top part.
- α is the contagion parameter (probability that a particular infected infects a particular susceptible) but it is *hard to estimate!!!!!*

Now we get to the final topic: **Rescaled System**
We don't like all these paramters ! 3 is too many!

$$S_t = -\alpha SI \text{ and } I_t = kI_{xx}\alpha SI - \beta I$$

So we want to reduce some of them! We call this rescaling

We want to rescale the S equation.

Rescaled System

$S_t = -\alpha S I$

$I_t = k I_{xx} + \alpha S I - \beta I$

$S_0 u(x^*, t^*) = I(x,t)$

$S_0 v(x^*, t^*) = S(x,t)$

$x^* = (\alpha S_0 / k)^{1/2} x$

$t^* = \alpha S_0 t$

$r = \beta / (\alpha S_0)$

new variables

S_0 is a constant $S(x,0) = S_0$

Ex. rescale the S equation

$$\frac{\partial}{\partial t}[S(x,t)] = \frac{\partial}{\partial t}[S_0 v(x^*, t^*)] = S_0 \frac{\partial v}{\partial t^*} \frac{dt^*}{dt} = S_0 v_{t^*} \alpha S_0$$

$$\frac{dt^*}{dt} = \frac{d}{dt}[\alpha S_0 t] = \alpha S_0$$

$$\cancel{S_0^2} \cancel{\alpha} v_{t^*} = -\cancel{\alpha} \cancel{S_0} u \cancel{S_0} v$$

$$\boxed{v_{t^*} = -uv}$$

Drop the stars for notational simplicity

$$u_t = u_{xx} + u(v - r)$$
$$v_t = -uv$$

diffusion

New: SIR

$$r = \frac{\beta}{\alpha S_0}$$

only parameter!

V49 Traveling Fronts

Agenda:

1. Motivation for studying traveling fronts
2. Definition of traveling fronts

Remember that in our motivational example of the spread of rabies, we had that the front of the epizootic moved at an average speed of about 30-60km a year. Let's think about what a "front" means in terms of foxes.

suppose you have your area, and the infection of rabies starts in the left corner. This location is $t = 0$. As t increases, we say the distance from one point of infection at $t = 0$ to a point of infection at another value of $t = 1$ is about 30 km. This is how far it travels after 1 year

Now, we want to think about the number of susceptible foxes at say time 0 near $x = 0$. We are thinking about the space as well. Our space location is $x = 0$, so if the disease have started there, presumably some of the foxes are already dead and the S population is going to be smaller. If I am at a different point in space, say $x = 100$, then the S population should be the entire fox population at $x = 100$ because the disease has not gotten there yet. The population is fully healthy.

As you get closer to the region of where the infection is, the population is expected to get smaller.

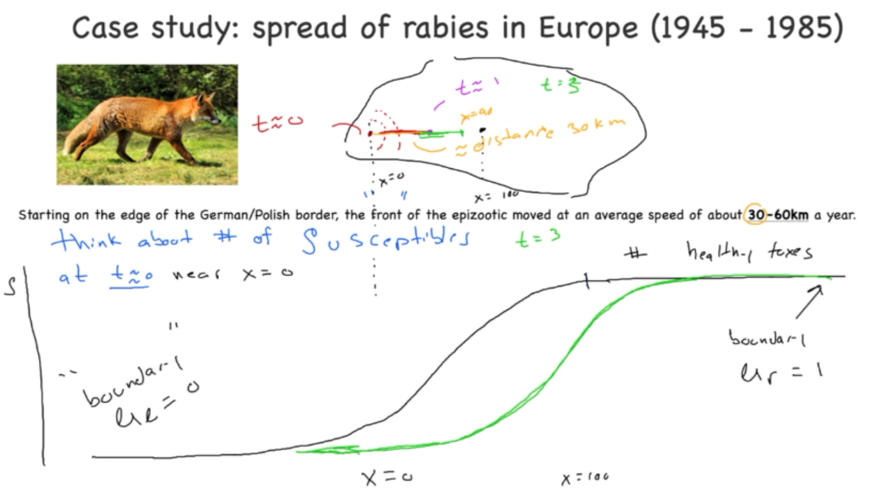
Now imagine that time increases, so say it has been three years. We are now at $x = 90$ since it travels 30km per year. There are going to be more dead foxes in the region between x=0 and x=90, since the disease has ravaged this region. We consider the boundary where we go back to a location where the disease hasn't hit, and all foxes are healthy. We call this $u_r = 1$ since 100% of foxes are healthy

Traveling Waves
Let's consider the advection equation $u_t + cu_x = 0$ and note that the shape is similar to the traveling wave modeled by the epidemic. We also remember the equation is $u(x, 0) = \frac{1}{1-e^{-x}}$ and know that we approach 0 on the LHS and 1 on the RHS.

So, the solution of this PDE is:

$$u(x, t) = U(x - ct)$$

As time passes, the 'hump' is going to travel at a given speed c. We call this a traveling wave because it has a fixed structure that's traveling across the x axis.



Some Definitions

- **Traveling wave:** solution to a PDE of the form $u(x,t) = U(x - ct) = U(z)$
  Note that $U(z)$ is a function of a single variable z
- **Traveling front:** a traveling wave that satisfies the boundary conditions $U(-\infty) = U_l$ (far left) and $U(\infty) = U_r$ (far right)
  A front is very similar to the spread of rabies, where the RHS is u_r and the LHS is u_l, so we call it a front

V50: Traveling front (math)

1. Studying a traveling front in a nonlinear PDE

Recall that last time we defined a traveling front as a traveling wave that satisfies $U(-\infty) = U_l$ and $U(\infty) = U_r$

- Traveling fronts exist in many PDEs but we can't always solve for them
- The wave speed c and the function U are **both** unknown
  Recall that $z = x - ct$ and that $u(x,t) = U(z)$....we have to figure out what c and big U are !

Example: Does the nonlinear advection-diffusion equation $u_t = u_{xx} - uu_x$ have a traveling front solution? Notice that no c is given in the equation (like in advection)

Turn it into a problem

Show that the nonlinear advection diffusion equation

$$u_t = u_{xx} - uu_x$$

has a traveling front solution $u(x,t) = U(x-ct) = U(z)$ that satisfies the boundary conditions $U(-\infty) = 1$ and $U(\infty) = 0$. Find the wave speed c.
We know what our solution should look like (a wave!)

1. Let $u(x,t) = U(z)$. Before I plug this in, we will do some derivative computations

$$\tfrac{d}{dt}u = \tfrac{d}{dt}[U(z)].$$

Remember that $z = x - ct$ so we have to do some chain rule action here. We also remember that U has a single variable

$$\tfrac{d}{dt}u = \tfrac{d}{dt}[U(z)] = U' \cdot \tfrac{dz}{dt} = U' \cdot (-c).$$

Now, we want to do the same thing wrt x

$$\tfrac{d}{dx}u = \tfrac{d}{dx}[U(z)] = U'\tfrac{dz}{dx}$$

Now, we know $\frac{dz}{dx} = 1$ and thus:

$$\tfrac{d}{dx}u = \tfrac{d}{dx}[U(z)] = U'\tfrac{dz}{dx} = U'$$

Finally, we have that $\frac{d^2}{dx^2}u = U''$

Putting it all together, we plug in and rewrite our equation as:

$$-cU' = U'' - UU'$$

Notice that this thing is an ODE! So, we have gone from a PDE to an ODE which is nice! We also remind ourselves that $U(-\infty) = 1$ and $U(\infty) = 0$
BUT, we can rewrite this in terms of U^2, ie:

$$-cU' = U'' - \tfrac{1}{2}\tfrac{d}{dz}[U^2]$$

Notice now that every term has a derivative in it, and we can thus integrate once with respect to z. **Remember** that U is a function of z so all these primes are z derivatives

$$-cU = U' - \tfrac{1}{2}U^2 + \tilde{A} \text{ where } \tilde{A} \text{ is our integration constant}$$

Now we do some algebra:

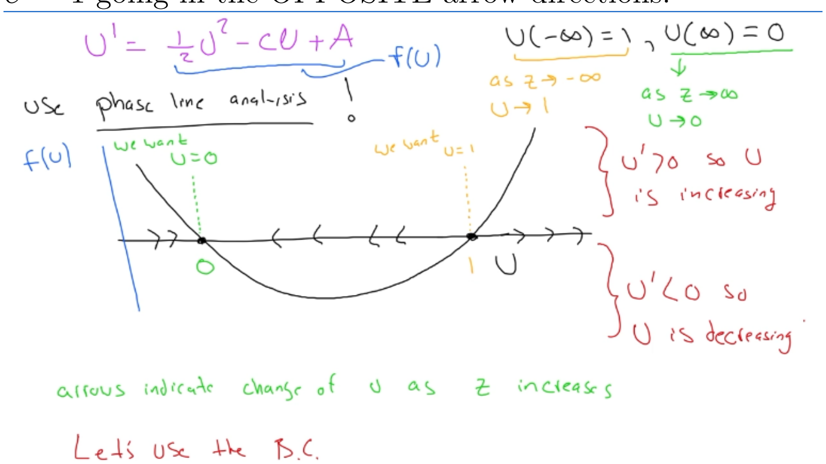
$$U' = \tfrac{1}{2}U^2 - cU + A$$

Now, this is a simple nonlinear 1st order ODE!

2. How are we going to analyze this?

$$U' = \tfrac{1}{2}U^2 - cU + A \text{ with } U(-\infty) = 1, U(\infty) = 0$$

Let's use phase line analysis! We will call $f(U) = \frac{1}{2}U^2 - cU + A$ and plot U vs $f(U)$ and draw direction arrows! Remember that U' tells us if U is increasing or decreasing, which is why we have the arrows on the U axis! The arrows indicate change of U as z increases, since z is the independent variable. We now want to use our boundary conditions. $U(\infty) = 0$ means that as z goes to infinity, we approach $U = 0$, increasing to get there.$U(-\infty) = 1$ means that as we go backwards in time we appraoch that number 1! This is what approaching negative infinity means. We see that the arrows pointing towards 0 are forward in time, so as we move backward in time we approach $U = 1$ going in the OPPOSITE arrow directions!



REMEMBER $U' = \frac{1}{2}U^2 - cU + A$ and we have our phase portrait above. The two points we have highlighted are the roots of the expression $\frac{1}{2}U^2 - cU + A$ and the roots are $r = \frac{c \pm \sqrt{c^2 - 2A}}{1}$

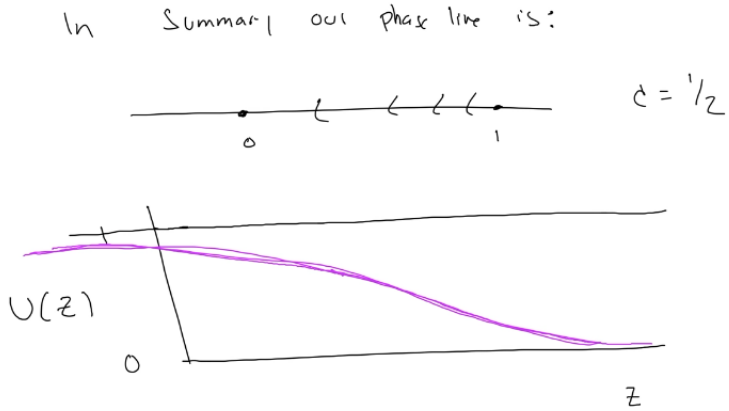
So we have the LH point is $c - \sqrt{c^2 - 2A}$ and the RH point is $c + \sqrt{c^2 - 2A}$
$\implies c - \sqrt{c^2 - 2A} = 0$ which can only happen if $A = 0$

$c + \sqrt{c^2 - 2A} - 1 \implies 2c = 1 \implies c = \frac{1}{2}$

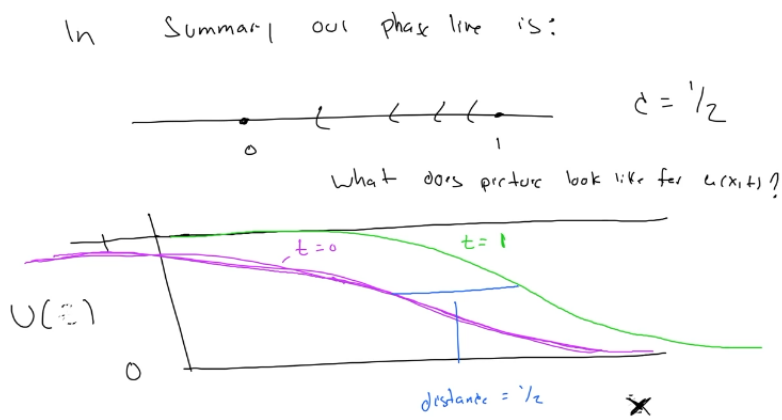
So now, our ODE is simply:

$$U' = \frac{1}{2}U^2 - \frac{1}{2}U$$

3. In summary, our phase line is:



But don't we want to go back to our original variables, $u(x,t)$? Let's look at that! It's gonna have the same form but shifted over a little bit! The distance from $t = 0$ to $t = 1$ is going to be exactly $1/2$. This is our traveling front and we are done!



Problems for Week 13

V47 Problems

3. In words, what does the parameter α represent in the SIR model? what about β?

The parameter alpha is the *contagion parameter*, meaning it is the probability that a particular infected person infects a particular susceptible person. So, if $\alpha = 0.2$, then the probability that one infected person infects one susceptible person is 20%

The parameter beta is the recovery parameter, where $\beta = 1/d$ and d is the average length of infection

4. An epidemic occurs when the number of infected individuals is increasing. According to the SIR model, this occurs when the basic reproduction number R_0 is greater than one. Mathematically, the basic reproduction number is $R_0 = S_0\frac{\alpha}{\beta}$, where S_0 is the initial number of susceptible individuals. Biologically, the basic reproduction number is the average number of secondary infections an infected individual can cause in an entirely susceptible population. For all SIR problems, assume that $S, I > 0$ which are the biologically meaningful values.

a) Justify the claim that, according to the SIR model, S is strictly decreasing with respect to time.

We know that $\frac{dS}{dt} = -\alpha SI$, which represents the rate of change with respect to time of the number of susceptible people in the population S. Since we know that $\alpha, S, I > 0$ in order to biologically make sense, we know this derivative is always negative. Thus, the number of susceptible individuals is always decreasing with respect to time. This makes sense since as people get sick they are no longer susceptible. If S could increase that would mean people went from not being susceptible to being susceptible, which wouldn't make sense for most epidemics where the only way to not be susceptible is by getting the disease, and once you get it you can never be susceptible again!

b) Using the above, show that an epidemic occurs in the SIR model if $R_0 > 1$.

Let's assume that $R_0 > 1$. We now want to use this fact to show that an epidemic must occur.

We have that $R_0 = S_0\frac{\alpha}{\beta}$. With our assumption $R_0 > 1$, this implies that $S_0\frac{\alpha}{\beta} > 1$
In order for this inequality to hold, we know it must be that $\alpha S_0 > \beta$.

We also know that $\frac{dI}{dt} = \alpha SI - \beta I$. This is the quantity we want to be greater than 0, since this would mean that the infected population is increasing
Since with our assumption we know that $\alpha S_0 > \beta$, we know $\alpha S_0 I > \beta I$

We only care about the infected population at any given time in order for it to be an epidemic, so from this we know that $\frac{dI}{dt} > 0$ when $t = 0$

$\implies \alpha S_0 I - \beta I > 0$ ✓

Thus, it must be an epidemic if $R_0 > 1$

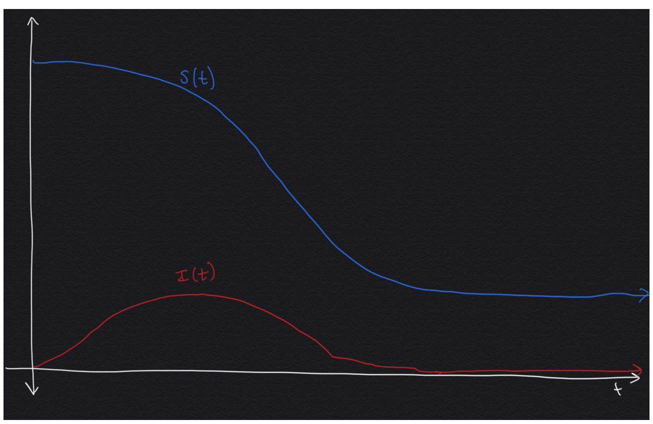
5. When an individual enters the recovered population we are assuming they obtain natural immunity after being infected. If we consider a situation where individuals are vaccinated, it is like we are putting a portion of the susceptible population into the recovered population (without ever entering the infected population). In other words, we are reducing $S(0)$. If $\beta = 0.1$ and $\alpha = 0.25$, what fraction of the susceptible population would need to be vaccinated to prevent an epidemic?

We know that an epidemic happens when $R_0 > 1$, so we want $R_0 \leq 1$.
Since $R_0 = S_0 \frac{\alpha}{\beta}$, plugging in our values for alpha and beta give us:

$$R_0 = S_0 \frac{0.25}{0.1} = 2.5 \cdot S_0$$

In order for $R_0 \leq 1$, we need $2.5S_0 \leq 1$ and thus $S_0 \leq 0.4$. This means that less than 40% of the population needs to be susceptible in order to prevent an epidemic. For this to occur, 60% of the population must be vaccinated.

6. Hand-draw a representative graph of $S(t)$ vs t and $I(t)$ vs t. Notice on all your graphs, the limiting value of $I(t)$ is zero as $t \to \infty$, but the limiting value of $S(t)$ is non-zero. What does the limiting value of S represent biologically?



The limiting value of S is the portion of the population that is still susceptible to the disease. We see that if we fix the transmission rate, then as we increase the recovery rate, S increases. This means that less people spread the disease since they recover, so less people get it and the limiting factor of S increases. Similarly, if we fix the recovery rate and increase the transmission rate, S decreases since more people are getting infected. Biologically, the limiting value of S is the portion of the population that is "immune" to the disease and will never get it. Maybe they never leave their house or maybe they have antibodies or maybe they're just super special, but they will never get the disease! We see that $I(t)$ always goes to 0 over time because when the pandemic ends there are of course no more infected individuals.

V48 Problems

7. According to the video, at what speed did the front of the rabies epizootic travel through central Europe in the mid 1900's?

The front of the rabies epizootic travelled through Europe between 30-60km per year

8. Using the rescaling variables defined in the video, show that the SIR equation with diffusion

$$S_t = -\alpha SI$$

$$I_t = kI_{xx} + \alpha SI - \beta I$$

can be transformed into the following system:

$$v_t = -uv$$

$$u_t = u_{xx} + u(v - r)$$

We recall our new variables:

1. $S_0 u(x*, v*) = I(x, t)$
2. $S_0 v(x*, t*) = S(x, t)$
3. $x* = (\alpha s_0/k)^{1/2} x$
4. $t* = \alpha S_0 t$
5. $r = \beta/(\alpha S_0)$

We now compute what S_t is in terms of these rescaled variables:

$\frac{d}{dt}[S(x,t)] = \frac{d}{dt}[S_0 v(x*, t*)] = S_0 \frac{dv}{dt*}\frac{dt*}{dt}$

Now, we also have that:

$\frac{dt*}{dt} = \frac{d}{dt}[\alpha S_0 t] = \alpha S_0$ and thus:

$$\frac{d}{dt}[S(x,t)] = S_0 v_{t*} \alpha S_0 = S_0^2 v_{t*} \alpha$$

Now, by our new variables 1) and 2) we know we can rewrite $S_t = -\alpha SI$ as:

$$S_0^2 v_{t*}\alpha = -\alpha S_0 u S_0 v$$

Now, we can divide both sides by S_0^2, which leaves us with:

$$v_{t*} = -uv$$

Now, we look to rewrite I_t in terms of our new variables, so we have:

$\frac{d}{dt}[I(s,t)] = \frac{d}{dt}[S_0 u(x*, t*)] = S_0 \frac{du}{dt*}\frac{dt*}{dt}$

We also have that:

$\frac{dt*}{dt} = \frac{d}{dt}[\alpha S_0 t] = \alpha S_0$ and thus:

$$\frac{d}{dt}[I(x,t)] = S_0 u_{t*}\alpha S_0 = S_0^2 u_{t*}\alpha$$

Now, we know $kI_{xx} = u_{x*x*}$ and $\alpha SI - \beta I = u(v-r)$ by how we defined our new variables and thus:

$$u_{t*} = u_{x*x*} + u(v-r)$$

Finally, we drop the stars from the equation for notational simplicity and are left with the desired transformed equations:

$$u_t = u_{xx} + u(v-r)$$

$$v_t = -uv$$

where $r = \frac{\beta}{\alpha S_0}$

9. How does the parameter $r = \frac{\beta}{\alpha S_0}$ relate to the basic reproduction number of the SIR model? Using only what you know from the SIR model in the previous problem, for what values of r would you anticipate an epidemic in the SIR model with diffusion?

We know the basic reproduction number is $R_0 = S_0\frac{\alpha}{\beta}$, where S_0 is the initial number of susceptible individuals. Biologically, the basic reproduction number is the average number of secondary infections an infected individual can cause in an entirely susceptible population. Thus, our parameter $r = \frac{\beta}{\alpha S_0}$ is the reciprocal of the basic reproduction number, or $R_0 = \frac{1}{r}$. Since an epidemic occurs when $R_0 > 1$, so thus we want $\frac{1}{r} > 1$.
We therefore want $r < 1$ for this inequality to hold true and for there to be an epidemic

We want the number of infected individuals to be increasing for there to be an epidemic, ie $I_t > 0$ so we want $u_{xx} + u(v-r) > 0$

10. Assuming that the average fox territory has an area of 5 km^2 and that an infected fox leaves its territory 35 days after being infected, estimate the diffusion constant k which should have units km^2/yr. Note, these aren't just made up numbers. They are taken from a journal publication on the subject of rabies in foxes [J. theor. Biol. (1985) 116, 377-393].

Since they leave 35 days after being infected, we know that the rate foxes leave the territory is 35/365 years. And thus our diffusion constant is:

(area of fox territory) * (rate foxes leave the territory)

We know the area of fox territory is 5 km^2 and that they leave at a rate of 35/365 yrs (this is because we have 365/35 as the rate foxes leave the territory which has units 1/yrs). Thus, the diffusion constant can be approximated as:

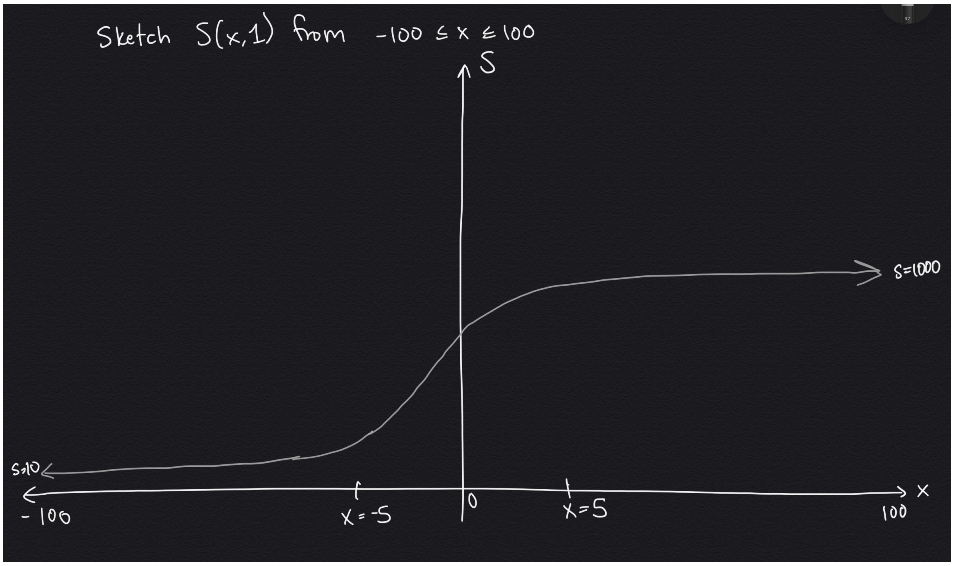
$$\frac{5}{1} \cdot \frac{365}{35} \cdot \frac{\text{km}^2}{\text{years}}$$

V49 Problems

11. A traveling front is defined mathematically in the video. It is important to understand what a traveling front is in terms of the biology. This problem will help you explore exactly this.

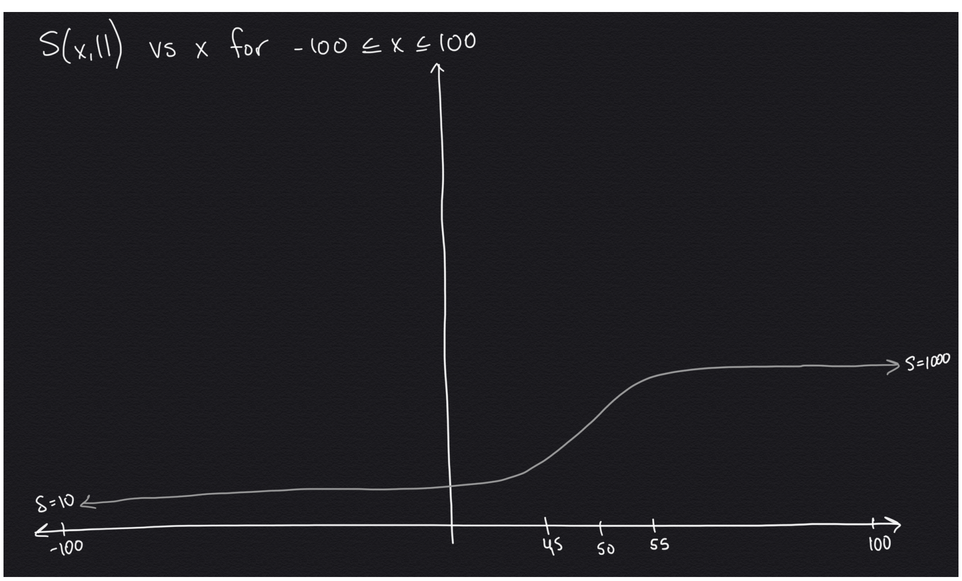
Suppose an epizootic front moves through a region at the speed 5 km a year and that there are many infected animals located near the spatial location $x = 0$ after $t = 1$ year. $x = 0$ is considered the center of the territory, and so negative x values have meaning. Let $S(x,t)$ represent the number of susceptible animals at location x (km) and time t (years). Suppose that the infection is spreading from left to right in space and that each animal belongs to the group S, I (infected) or R (removed). Suppose that there is always some non-zero fraction of the population that never gets infected at all x locations. For simplicity, assume this number is 10 for all values of x (this means for all time there at least 10 animals at every location x that are not infected). Suppose that the total animal population is $N = 1000$.

a) Sketch a rough graph of $S(x,1)$ vs x for $-100 \leq x \leq 100$. Hint: Think about what the value of $S(-\infty,1)$ should be and then think what $S(\infty,1)$ should be. Then try to "connect" these states. Remember, the epizootic is "coming" from the left. Justify in a few statements why the graph has the limiting values you have drawn.



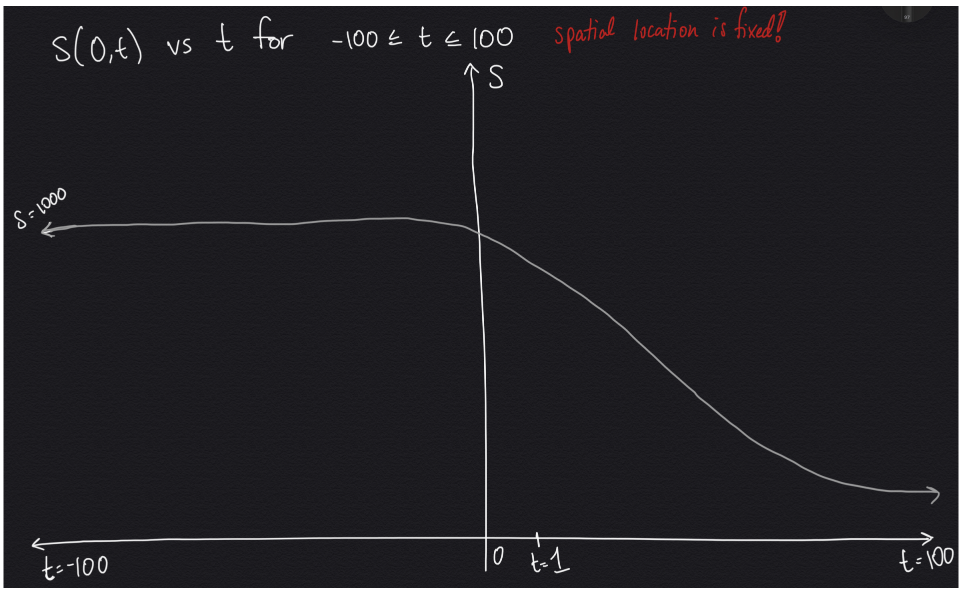
We know that at any location (x axis) there are at least 10 animals which are not infected and thus alive. Since the epizootic is coming from the left, we know that as x goes to negative infinity the number of animals approaches 10, which we see in the graph with S approaching 10 as $x \to -100$. Then, since $x = 0$ is the center of the territory and we are given that there are many infected individuals there at $t = 1$, we see that the number of susceptible animals starts to drop from 1000 at $x = 0$. The limiting value as x approaches infinity (or in this case just 100) is 1000, since the epizootic hasn't yet gotten to those spatial locations.

b) Sketch a rough graph of $S(x, 11)$ vs x for $-100 \leq x \leq 100$. How does your graph compare to the one you drew in part (a) above?



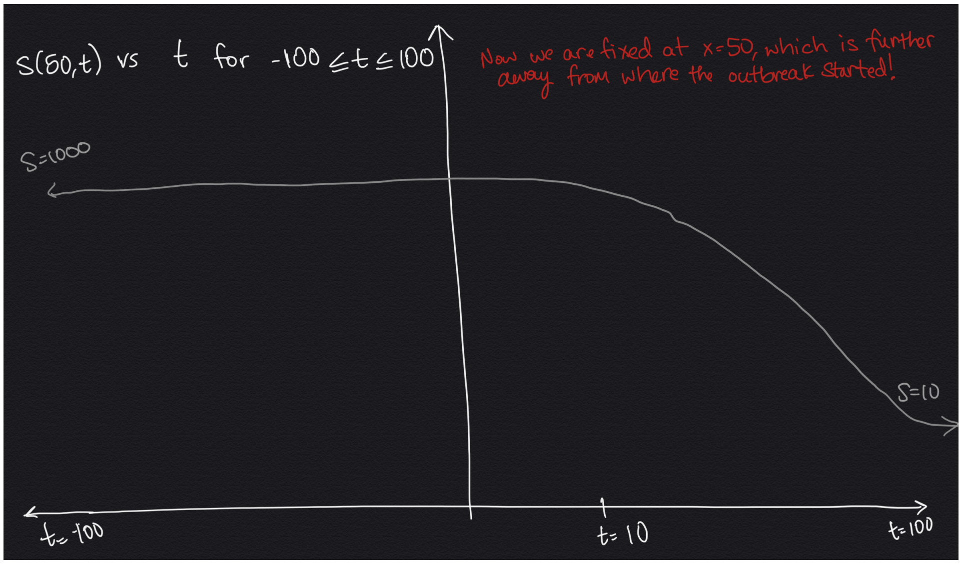
This graph is similar to the one above but we see the travelling wave property playing out, as the disease has spread to territory to the right over time. Since it moves at 5 km a year, we expect the region where the disease is newly infecting animals to be about 50 km further right after 10 years, which we see in the graph

c) Sketch a rough graph of $S(0,t)$ vs t for $-100 \leq t \leq 100$. It is strange to think of negative time, but in terms of the graph, it will be helpful later on. Hint: Think about what the value of $S(0,-\infty)$ should be and then think what $S(0,\infty)$ should be. Then try to "connect" these states. Justify in a few statements why the graph has the limiting values you have drawn.



We are now fixing the spatial location and plotting the number of susceptible animals over time. Since the epizootic hits $x = 0$ at $t = 1$, we see S start to decrease at $t = 1$ as animals are infected. It decreases to $S = 10$ at $t = 100$ because at this point all of the animals in this spatial location but 10 have been infected

d) Sketch a rough graph of $S(50,t)$ vs t for $-100 \leq t \leq 100$. How does your graph compare to the one you drew in part (c) above?



Now, we are fixed at $x = 50$, which we know will take 10 years for the disease to get to. Thus, S stays at 1000 for longer in this graph than the previous graph!

12. This problem will connect the idea of the traveling front from a mathematical perspective to the biological perspective you just considered above.

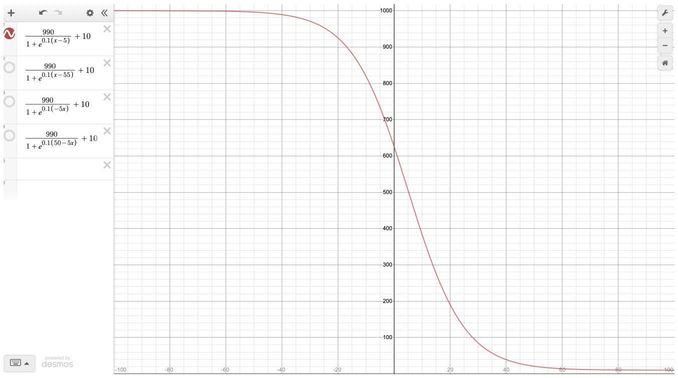
Consider the function $U(z) = \frac{990}{1+e^{0.1z}} + 10$

a) What are the values of the states $U(-\infty)$ and $U(\infty)$?

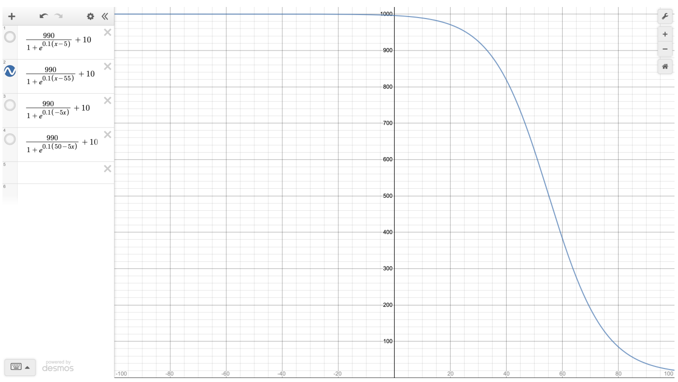
We know when $z = -\infty$, then the denominator will be approximately 1, so $U(-\infty) = 990 + 10 = 1000$

When $z = \infty$, the denominator will tend to infinity and thus the fraction will approach 0. Thus $U(\infty) = 10$

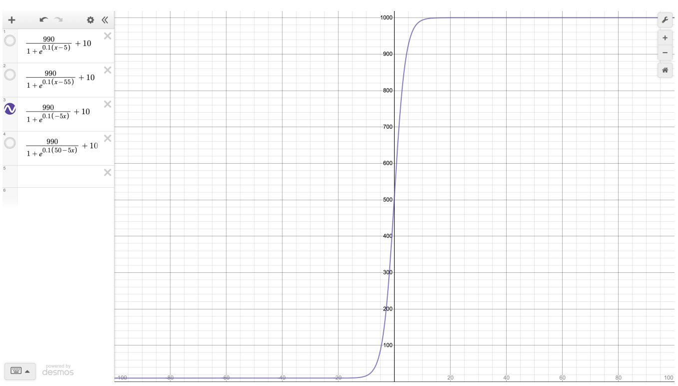
b) Suppose that $u(x,t)$ is a traveling front and it is defined as $u(x,t) = U(x - 5t) = U(z)$, where is $U(z)$ is defined above. In four separate graphs, sketch a graph of: $u(x,1), u(x,11), u(0,t)$, and $u(50,t)$. In each graph, chose the horizontal axis to have the domain -100 to 100 and chose the vertical axis to have range 0 to 1020.



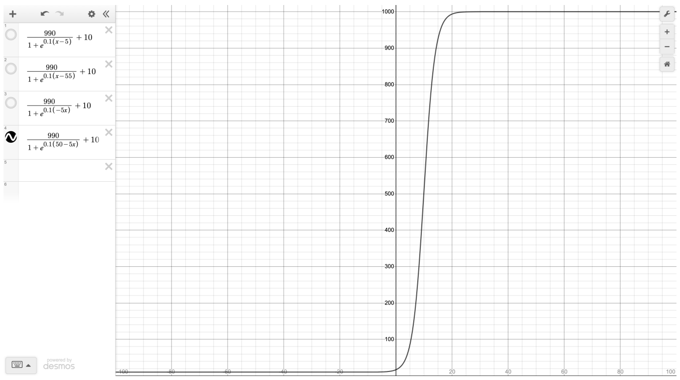
Here, we see that since it has been 1 year, the epidemic is coming from the right as S is 10 at positive infinity, meaning that the population has already had the disease at those x locations.



Now it has been 11 years, and we see that the wave is travelling to the right!



Now we are fixing x so we are considering the susceptible population at $x = 0$. We see that the number of susceptible individuals increases as time increases, meaning the population was hit early in the pandemic.



Finally, we now fix $x = 50$, so spatial location 50. As expected, the population was infected later here than at $x = 0$

V50 Problems

13. Show that the following nonlinear PDE

$$u_t = u_{xx} - u^2 u_x$$

has a traveling front solution $u(x,t) = U(x-ct) = U(z)$ that satisfies the boundary conditions $U(-\infty) = 1$ and $U(\infty) = 0$. Find the wave speed c. Hint: Recall that solutions of single first order autonomous ODEs are either: always increasing, always decreasing or are always constant (e.g. they can't change from increasing to decreasing). This means that, as a solution of an ODE approaches an equilibrium value ($U \to U_e$) then $U' \to 0$. This means that $U'(-\infty) = 0$ and $U'(\infty) = 0$. This will help you determine the value of one of the arbitrary constants that appear in your ODE solution.

1. Let $u(x,t) = U(z)$
   Now, we have:

   $$\tfrac{d}{dt}u = \tfrac{d}{dt}[U(z)]$$

   Since $z = x - ct$ we use the chain rule, knowing that $\frac{dz}{dt} = -c$.

   $$\tfrac{d}{dt}u = U'\tfrac{dz}{dt} = U \cdot (-c)$$

   We want to compute the partial derivative of u wrt x now, ie:

   $$\tfrac{d}{dx}u = \tfrac{d}{dx}[U(z)] = U'\tfrac{dz}{dx} = U' \text{ since } \tfrac{dz}{dx} = 1$$

   We also now know that $\frac{d^2}{dx^2}u = U''$

   Putting it all together, we plug in these new derivatives and get:

$$-cU' = U'' - U^2U'$$

Now, we rewrite this using the chain rule, since $\frac{d}{dz}[U^3] = 3(U)^2 \cdot U'$:

$$-cU' = U'' - \frac{1}{3}\frac{d}{dz}[U^3]$$

We can integrate this once to get:

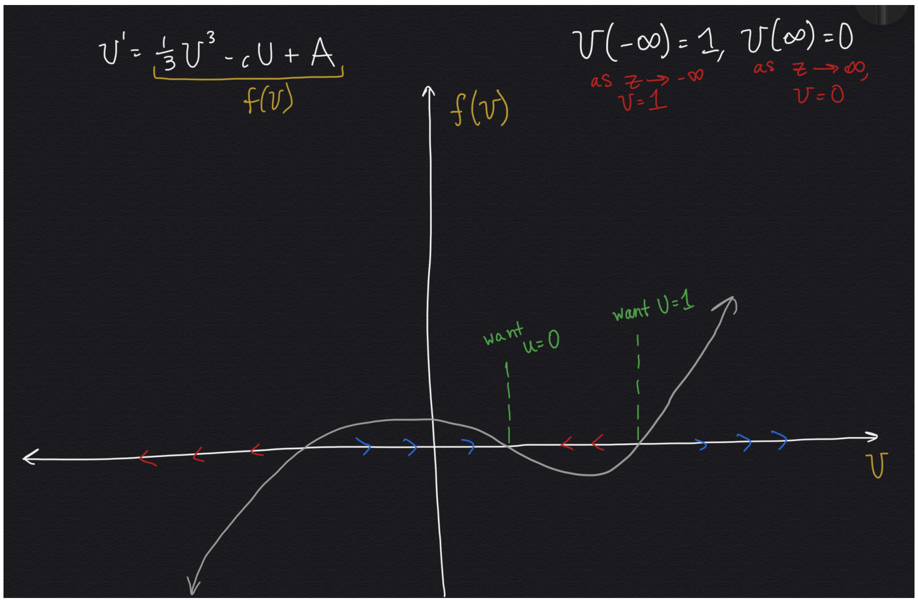
$$-cU = U' - \tfrac{1}{3}U^3 + A, \text{ where } A \text{ is a constant of integration.}$$

We rewrite this as:

$$U' = \tfrac{1}{3}U^3 - cU + A$$

This is a nonlinear 1st order ODE!

2. We now want to analyze this ODE using phase lines, knowing that we want $U = 0$ at infinity and $U = 1$ at $-\infty$

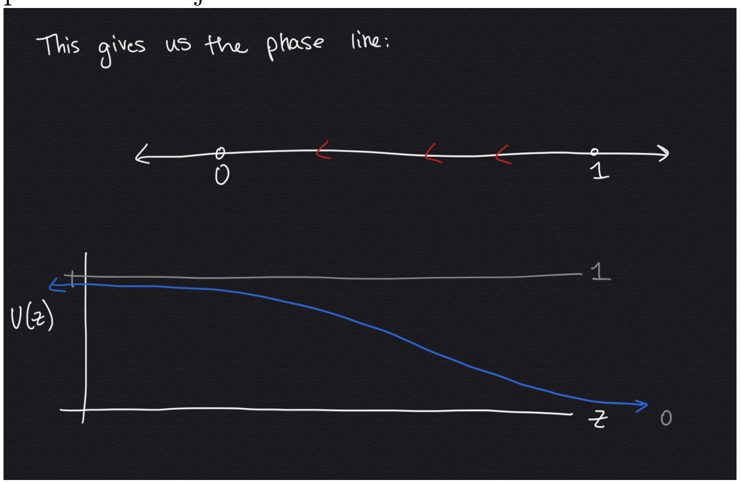


From these phase lines we see that we want $U = 0$ and $U = 1$ to be our two roots, and in between them we want the ODE to be strictly decreasing (as individuals get the disease). Thus, we have identified the range for which we want our solutions.
We now want to analyze what happens at $\pm\infty$, using the knowledge that the solution is strictly decreasing.

- $U'(\infty) = \frac{1}{3}U(\infty)^3 - cU(\infty) + A = 0 - 0 + A$. Since we know $U' \to 0$ as we approach an equilibrium value (in this case $U = 0$) we know it must be that $0 = 0 - 0 + A \implies A = 0$

- $U'(-\infty) = \frac{1}{3}U(-\infty)^3 - cU(-\infty) = 1/3 - c$. Again, we know we are approaching the equilibrium value $U = 1$ and thus $U' \to 0$, so thus $0 = \frac{1}{3} - c \implies c = \frac{1}{3}$
- Thus, we have the ODE $U' = \frac{1}{3}U^3 - \frac{1}{3}U + A$

We know there is thus a traveling front solution since we can analyze the following phase line we just derived:



If we were to increase t, we would see the wave "travel" right, with an increase of 1 in t leading to a distance of 1/3 between the wave peaks.

14. Consider the same problem as above, but now suppose only the condition $U(\infty) = 0$ is given. What are the possible states at $x = -\infty$?

With only one boundary condition, we know that the possible states at $x = -\infty$ are either the root of the cubic function we derived in the last question, or the third root of the cubic function, corresponding to $c = \pm 1/3$. Thus, the states at $x = -\infty$ are either $U = 1$ or $U = -1$. In the case that $U = -1$, our solution is strictly increasing up to $U = 0$. In the case that $U = 1$, we are strictly decreasing to $U = 0$.

References