Kim Hancock

Homework 13

December 15, 2020

1 Resources

Other than the textbook and class notes, I discussed problems 8-14 briefly with Kayla.

2 Notes for Week 14

V51: Traveling Fronts in SIR with diffusion

- **Reminder of the fox rabies case study**
  Remember, this is from central europe in mid-late 1900s, and there was the front of an epizootic moving 30-60km per yer

  We came up with the following model, where S, I are susceptible and infected populations

  $$I_t = kI_{xx} + \alpha SI - \beta I$$

  $$S_t = -\alpha SI$$

  Diffusion models the random motion for the foxes...if they have rabies they will walk around like random fools

- **Biological and mathematical research questions**

  Biological Research Questions
  So, if we have this model, which parameters (out of k, α, β can be estimated from the field? How could we estimate others?
  Is there an epizootic wave traveling through the region?
  If so, at what speed?
  What fraction of the population survives?

  Mathematical Research Questions
  Can we use analysis to infer some parameter values??
  Is there a traveling front solution to the model?
  What speeds are predicted by the model?
  What are the boundary conditions?

- **Answering some of the questions**
  We start with:
  Q1: What parameters can we estimate in the field?
  $k = 60\frac{km^2}{yr}$

$\beta = 10\frac{1}{yr}$

These come from research papers. Now, α is hard to estimate in the field (contagion parameter). BUT mortality rate is easier to measure.

Q4: What are the boundary conditions?
$S(x,t)$ where t is *fixed*. This is the susceptible population of foxes. Suppose that the epizootic is traveling to the right. If you're really far to the right the disease front has not reached you yet. Now, if we think about the infected population curve, we see it spiking where cases are rising and 0 at $\pm\infty$. Thus, we can think ahead to a traveling front solution where $z = x - ct$, so if $x \to \pm\infty$ then $z \to \pm\infty$.

Thus, we conclude:

$$S(\infty) = S_0 \text{ and } I(-\infty) = 0 \text{ and } I(\infty) = 0$$

What we don't know is $S(-\infty)$, which is the number of foxes that survived! **This is related to the mortality rate and is important.**

- **To do for next week**
  We realized the folowwing for biological questions:

  1. We cannot estimate the contagion parameter α in the field
  2. Is there an epizootic wave traveling through the region??
  3. If so, at what speed?
  4. We can estimate the mortality rate

  Then, for mathematical questions:

  1. Can we use analysis to infer alpha?
  2. is there a traveling front solution
  3. what speeds are predicted by the model
  4. We have a sense of each boundary condition

V52: Estimate α

- Recalling the recalled model equations
- Looking for a traveling front
- Simplifying equations to get a relationship between parameters and boundary conditions

1. Rescaled system
We were able to show that we got the rescaled system:

$$u_t = u_{xx} + u(v - r)$$

$$v_t = uv$$

Recall that $r < 1$, and then there is an epidemic. We assume this is true for the remaining videos. this little r is the reciprocal of the R in the SIR model.
That being said, let's just focus on the rescaled system, which makes more sense for the math we want to do.

2. introduce traveling front
We will introduce $z = x - ct$, and recall that c is the speed parameter we want to try and find.
To find a traveling front define:

$$u(x, t) = f(x - ct) = f(z)$$

$$v(x, t) = g(x - ct) = g(z)$$

Boundary Conditions:

$$f(-\infty) = 0 \text{ and } f(\infty) = 0 \text{ and } g(-\infty) \text{ and } g(\infty) = 1$$

Now, we are going to plug these f, g into where we have u, v above, ie:

$$u = f(z) = f(x - ct)$$

$$\implies u_t = f' \cdot \tfrac{dz}{dt} = f' \cdot -c$$

$$\implies u_x = f' \cdot \tfrac{dz}{dx} = f'$$

$$u_{xx} = f''$$

We also have:

$$v = g(z) \implies v_t = g' \cdot -c$$

Plugging these in,

$$-cf' = f'' + f(g - r)$$

$$-cg' = -fg$$

These are two nice equations! Great!
So, now another thing is that the top equation can be rewritten as:

$$\frac{f''}{c} + f' + g' - \frac{g'}{g}r = 0$$

at this stage, this should not seem obvious...so its a hw problem!
Why would we write the equation in this way? Well, by doing this, we can use a simplifying trick to remove one of the derivatives. We integrate wrt to independent variable z

$$\frac{f'}{c} + f + g - \ln(g(z))r = A$$

We recall from calc that $\frac{d}{dz}\ln[g(z)] = \frac{g'}{g}$

Now, using our boundary conditions, recall that $f(\infty) = 0$ and $g(\infty) = 1$ and $f'(\infty) = 0$
With all these assumptions at $z = \infty$, the first two terms vanish, the third term goes to 1, and the fourth term is 0. All that survives is $A = 1$, our constant.

$$\frac{f'}{c} + f + g - \ln(g(z))r = 1$$

Let's use the other boundary condition $g(-\infty) = a$ and $f(-infty) = 0$ and $f'(-\infty) = 0$

$$\implies \text{ at neg infinity, } a - r\ln(a) = 1$$

So, what have we actually shown?
Remember we were trying to infer α, which was the goal of this video. Remember also that r is related to α, which we will use on the homework to relate the two.

V53: Analysis of traveling fronts pt 1

1. Recalling the traveling front equation

2. Doing some ODE analysis

1. Last time, we got our rescaled SIR model with diffusion and looked for traveling front solutions by making a substitution and got to the following:

Last time

$u_t = u_{xx} + u(v - r)$

$v_t = uv$

$u(x,t) = f(z)$ $\qquad z = x - ct$ $\qquad$ } Traveling front

$v(x,t) = g(z)$

$g(-\infty) = a, \quad g(\infty) = 1, \quad f(-\infty) = 0, \quad f(\infty) = 0$

$-cf' = f'' + f(g - r)$ $\qquad \boxed{\frac{f'}{c} + f + g - r\ln(g) = 1}$

$-cg' = -fg$

$$\frac{f'}{c} + f + g - r\ln(g) = 1$$

$$\text{and } -cg' = -fg$$

We now have a system of ODEs! Recall that c, r are parameters.
And, we have $f = f(z)$ and $g = g(z)$
We can solve for f' and g':

$$f' = c(r\ln(g) - f - g + 1$$

$$g' = \frac{fg}{c}$$

To analyze this, we will remember *phase plan analysis!*

1. Find equilibrium points
2. Compute Jacobian matrix
3. Evaluate Jacobian matrix at an eq. pt and find eigenvalues
4. Classify equilibrium point based on eigenvalues

So, let's think about pts 1-3:
1. Find eq. pts. These are constant solutions, ie $f' = 0$ and $g' = 0$

$$0 = c(r\ln(g) - f - g + 1$$

$$0 = \frac{fg}{c}$$

The solutions are $(f_e, g_e) = (0, 1)$ and $(f_e, g_e) = (0, a)$
Recall that a is the fraction of the population that survives!
It is also our boundary, $g(-\infty) = a$

2. Compute Jacobian matrix

$$f' = c(r\ln(g) - f - g + 1 = F(f, g)$$

$$g' = \tfrac{fg}{c} = G(f, g)$$

The jacobian is a generalization of the derivative when we think about multiple variables. What we have is that $J = \begin{bmatrix} \frac{\partial F}{\partial f} & \frac{\partial F}{\partial g} \\ \frac{\partial G}{\partial f} & \frac{\partial G}{\partial g} \end{bmatrix}$

$$\implies \begin{bmatrix} -c & \frac{cr}{g} + c \\ \frac{g}{c} & \frac{f}{c} \end{bmatrix}$$

We will show in hw that computing these partials gives us this matrix. We will now just do $\frac{\partial F}{\partial f}$
We will take the derivative of $c(r\ln(g) - f - g + 1$ wrt f. We know $r\ln(g)' = 0$ and $-f' = -1$ and $-g' = 0$ and $1' = 0$ so our derivative is $-c$

3. Evaluate at eq. pt and find eigenvalues
Now, this is the jacobian for our system of 2 ODEs. All we want to do is substitute the point $(f_e, g_e) = (0, 1)$ into J

$$\begin{bmatrix} -c & \frac{cr}{1} + c \\ \frac{1}{c} & \frac{0}{c} \end{bmatrix}$$

Now, we need to find eigenvalues! Well, we do this by solving $\det(J - \lambda I) = 0$

$$\det\begin{pmatrix} -c - \lambda & -cr + c \\ \frac{1}{c} & -\lambda \end{pmatrix} = 0$$

The determinant is:

$$(-c - \lambda)(-\lambda) - (-cr + c)\tfrac{1}{c} = 0$$

Now, we just have to rearrange a little bit:

$$\lambda^2 + c\lambda - (-r + 1) = 0$$

Then, this is a simple quadratic equation we can solve!

$$\lambda = -\tfrac{1}{2}\left(c \pm \sqrt{c^2 - 4(1 - r)}\right)$$

V54: Traveling fronts in SIR + Diffusion pt 2

Here, we will do an overview of our work on rabies case study and answer our final research questions
In the group hw, we will look at how PDE and the coronavirus intersect

Can we answer these questions:

1. Can we use analysis to infer alpha?
2. is there a traveling front solution to the model?
3. what speeds are predicted by the model?

In the last two videos, we started with these two equations and did a lot of things to them to get a system of ODEs

$$I_t = kI_{xx} + \alpha SI - \beta I$$
$$S_t = -\alpha SI$$

Flow chart of our work

rescale

$$u_t = u_{xx} + u(v - r)$$
$$v_t = uv$$

assume a traveling front

$$u(x,t) = f(z) \qquad z = x - ct$$
$$v(x,t) = g(z)$$
$$g(-\infty) = a, \quad g(\infty) = 1, \quad f(-\infty) = 0, \quad f(\infty) = 0$$

Plug in, do some tricks, use BCs and simplify

$$f' = c(r\ln(g) - f - g + 1)$$
$$g' = \frac{fg}{c}$$

Last time, we analyzed this using the jacobian and equilibrium points:

1. Find equilibrium points
2. Compute Jacobian matrix
3. Evaluate Jacobian matrix at an equilibrium point and find eigenvalues
4. Classify equilibrium point based on eigenvalues

$$(f_e, g_e) = (0,1)$$
$$(f_e, g_e) = (0,a)$$

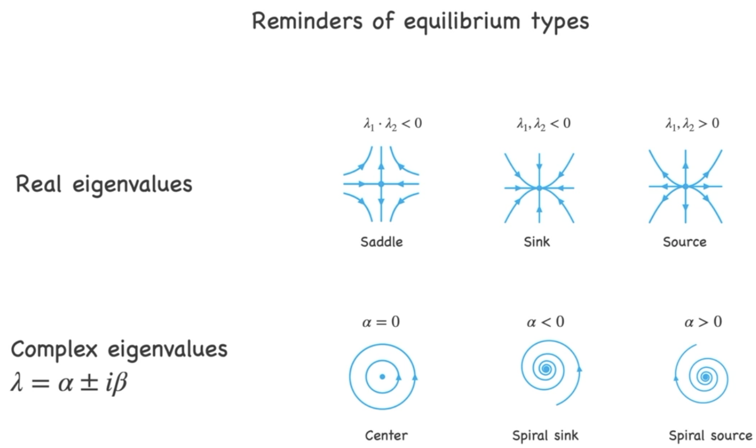
$$J = \begin{pmatrix} -c & \frac{cr}{g} + c \\ \frac{g}{c} & \frac{f}{c} \end{pmatrix}$$

$(f_e, g_e) = (0,1)$

$$J = \begin{pmatrix} -c & -cr + c \\ \frac{1}{c} & 0 \end{pmatrix}$$

$$\lambda = \frac{1}{2}\left(-c \pm \sqrt{c^2 - 4(1 - r)}\right)$$

What's left is classifying eq. pts.



We now want to classify $(f_e, g_e) = (0, 1)$

$$\lambda = \tfrac{1}{2}(-c \pm \sqrt{c^2 - 4(1-r)}$$

We know that $c > 0$ and $r < 1$ since in order for an epidemic to occur this needs to be true. So, we know we are subtracting a positive number from c^2 under the square root because $r < 1$.
$\implies c^2 - 4(1-r) < c^2$

So, $\implies -c + \sqrt{c^2 - 4(1-r)} < 0$
Thus, both eigenvalues have negative real part, and we can immediately eliminate center, source, and spiral source.

Now, we know we get a spiral sink if λ is complex. So let's assume that's true. The trajectory of this goes into territory where $f < 0$, but we know that this does not make physical sense! So, if we allow complex eigenvalues, we get something that doesn't make biological sense. Since complex eigenvalues lead to nonphysical populations, we will ignore them.

Thus, λ is real **if** $c^2 - 4(1-r) \geq 0$ aka stuff under the radical is positive.
$\implies c \geq \sqrt{2}(1-r)$

Thus, the front speed satisfies $c \geq 2\sqrt{1-r}$

So, we just answered our third research question.

It turns out that if we did more analysis, we could show that $(f_e, g_e) = (0, 1)$ is a **sink** and $(f_e, g_e) = (0, a)$ is a **saddle**

So, our mathematical answers are:

1. $a - r\ln(a) = 1$
2. Yes
3. $c \geq 2\sqrt{1-r}$

3 Week 14 Problems

V51 Problems

3. Suppose that, at each x, there are a total of $S_0 = 1000$ foxes. Mortality rate is something that can be estimated in the field (by counting dead foxes). For our rabies case study, the mortality rate was measured to be about 80%, which implies the survival rate is 20%. This means that 20% of the population does not get infected with rabies at all, since rabies is fatal in foxes. Let $S(x,t)$ be the number of susceptible (i.e. healthy) foxes at location x and time t. Assume that the infection "comes" from the left. Using this information, what is the value of $S(-\infty,t)$?

Since the disease comes from the left, we know that it has already fully infected the populations at $x \approx -\infty$. Thus, near/at negative infinity, we know that 80% of the population has been killed and 20% did not get infected. Thus, $S(-\infty,t) = 200$

4. Of the parameter values α, β, k of the model in the video, which is difficult to measure in the field?

The contagion parameter α is hard to measure in the field, although mortality, which is related to contagion, is easier.

V52 Problems:

5. Let a be the fraction of the fox population that survives the epizootic front (namely, the fraction of foxes that do not catch rabies). If the mortality rate is measured to be 80%, what is the value of a?

It is simply $a = 0.2$ since 20% of foxes do not die from the epizootic

6. Let a be the fraction of the fox population that survives the epizootic front (namely, the fraction of foxes that do not catch rabies). Write a short sentence or calculation justifying the following boundary conditions in the video:

$$f(-\infty) = 0 \text{ and } f(\infty) = 0$$

$$g(-\infty) = a \text{ and } g(\infty) = 1$$

Note, f and g are defined in terms of u and v, which are defined in terms of I and S via the rescaling (shown in V48).

Since $u = f(x - ct)$, and we defined $S_0 u = I$ in rescaling, we know that f is thus related to the number of infected individuals in the population. The number of infected individuals at locations which have already had the epidemic pass through (values of x tending to negative infinity) is obviously 0, giving us $f(-\infty) = 0$. Then, way way ahead in the future $f(\infty)$ we know the epidemic will have ended and thus there will be no more infected indivuals left, implying that $f(\infty) = 0$

Similarly, we know that $v = g(x-ct)$ and we defined $S_0 v = S$ in our rescaling. We also know S is the number of susceptible individuals in the population. Again, we know the disease is coming from the left, so towards $x = -\infty$ the disease has already ravaged the population. Since the survival rate is a, we know that the number of susceptible animals left (the lucky ones...) is a. Then, way far ahead in space and time towards ∞ we know that everyone is susceptible since no one has gotten the diease yet, implying that $g(\infty) = 1$

7. Suppose $a = 0.2$. Using the formula derived in the video that relates r to a, estimate r

In the video, we derived the following two equations for our traveling front by defining $u = f(x - ct)$ and $v = g(x - ct)$ and then plugging these into our rescaled system $u_t = u_{xx} + u(v - r), v_t = -uv$. This gave us:

$$-cf' = f'' + f(g - r)$$

$$-cg' = -fg$$

In the video, we found that $a - r\ln(a) = 1$. We can use this to estimate r, but first we want to review the steps of how we got there.

$$-cf' = f'' + f(g - r) \implies -cf' = f'' + fg - fr$$

Substituting $cg' = fg$ from the second equation gives us:

$$-cf' = f'' + cg' - fr$$

Dividing everything by c we get:

$$-f' = \frac{f''}{c} + g' - \frac{fr}{c}$$

Now, from the second equation we also have that $f = \frac{cg'}{g}$ from dividing both sides by g. Thus,

$$-f' = \frac{f''}{c} + g' - \frac{g'}{g}r$$

This looks almost like what we want! all we have to do is bring $-f'$ over to the other side,

$$0 = \frac{f''}{c} + g' + f' - \frac{g'}{g}r$$

Now, we can integrate everything, as we did in the video, to get:

$$A = \frac{f'}{c} + g + f - \ln(g(z))r$$

We also showed in the video that $A = 1$ because of our boundary conditions $g(\infty) = 1$ and $f(\infty) = 0$ and $f'(\infty) = 0$

$$1 = \frac{f'}{c} + g + f - \ln(g(z))r$$

Our other boundary conditions are $g(-\infty) = a$ and $f(-\infty) = 0$ and $f'(-\infty) = 0$. This leads us to the survival rate a being estimated by:

$$a - r\ln(a) = 1$$

$$\implies a - 1 = r\ln(a)$$

$$\implies r = \frac{a-1}{\ln(a)}$$

Plugging in $a = 0.2$ we get that $r = \frac{-0.8}{\ln(0.2)} \approx 0.497$
We know that if $r < 1$ then we have an epidemic. Therefore we do have an epidemic here!

8. In an earlier homework, the diffusion constant was estimated to be $k = 60$ and the death rate constant was estimated to be $\beta = 10$. Using these values and the survival value $a = 0.2$ and $S_0 = 1000$, estimate the contagion parameter α. (recall, that r is related to α, β and S_0, see V48).

From video 48 we have that $r = \frac{\beta}{S_0\alpha}$
Thus, we have $0.497 = \frac{10}{1000\alpha}$ since from problem 6 we estimated r and were given $\beta = 10$ and $S_0 = 1000$

$$\implies \alpha = \frac{10}{497} = \frac{1}{49.7}$$

V53 Problems

9. Verify that $(0, 1)$ and $(0, a)$ are equilibrium solutions of the ODE system. For each equilibrium point, you can do this by plugging in the equilibrium point and seeing that equations are satisfied

Remember our ODE system is:

$$f' = c(r\ln(g) - f - g + 1)$$

$$g' = \frac{fg}{c}$$

And setting both equal to 0, we have:

$$0 = c(r\ln(g) - f - g + 1)$$

$$0 = \frac{fg}{c}$$

The first eq. pt. is easy to verify. Plugging in $(f, g) = (0, 1)$ we have:

$$0 = c(r\ln(1) - 0 - 1 + 1) = c(0 - 0) = 0 \checkmark$$

$$0 = \frac{0}{c} = 0 \checkmark$$

Now, plugging in $(f, g) = (0, a)$ we have:

$$0 = c(r\ln(a) - 0 - a + 1) = c(r\ln(a) - a + 1)$$

From the video, we know that $a - r\ln(a) = 1$ so thus $0 = r\ln(a) - a + 1$ and we have:

$$0 = c(0) = 0 \checkmark$$

Further, we know that $S(-\infty) = a$ and that $S = S_0 v$, so therefore S is related to g since we let $v = g(z)$. Since a is the fraction of the population that survives, we know that $g = a$ is the limit as $x \to -\infty$. Thus, we know that at $x = -\infty$ (which is true in this case since we defined $z = x - ct$) we reach a limit a. We also know that as we approach a limit point the derivative is 0, so thus it must be that at $g = a$ the derivative wrt f is 0.

$$0 = \frac{0}{c} = 0 \checkmark$$

10. Assume that $a = 0.2$ and $r = 0.5$. (Note I have rounded the value of r to make the numbers nicer and to make it consistent with the reading). Also assume that $c = \sqrt{3}$. Write down the Jacobin matrix associated to the equilibrium point (0, 1) and compute the eigenvalues.

We first recall our system of equations:

$$f' = c(r\ln(g) - f - g + 1) = F(f, g)$$

$$g' = \tfrac{fg}{c} = G(f, g)$$

Now, we plug in the given values of c, r:

$$f' = \sqrt{3}(0.5\ln(g) - f - g + 1) = F(f, g)$$

$$g' = \tfrac{fg}{\sqrt{3}} = G(f, g)$$

What we have is that $J = \begin{bmatrix} \frac{\partial F}{\partial f} & \frac{\partial F}{\partial g} \\ \frac{\partial G}{\partial f} & \frac{\partial G}{\partial g} \end{bmatrix}$

$$\implies J = \begin{bmatrix} -\sqrt{3} & \frac{0.5\sqrt{3}}{g} - \sqrt{3} \\ \frac{g}{\sqrt{3}} & \frac{f}{\sqrt{3}} \end{bmatrix}$$

We will show the computation for these partials:

- $\frac{\partial F}{\partial f} \implies$ We know we want to take the derivative of $\sqrt{3}(0.5\ln(g) - f - g + 1)$ wrt f.
  We know $0.5\ln(g)_f = 0$ and $-f_f = -1$ and $-g_f = 0$ and $1_f = 0$
  So, our derivative is $-\sqrt{3}$
- $\frac{\partial F}{\partial g} \implies$ We know we want to take the derivative of $\sqrt{3}(0.5\ln(g) - f - g + 1)$ wrt g.
  We know $0.5\ln(g)_g = \frac{0.5}{g}$ and $-f_g = 0$ and $-g_g = -1$ and $1_f = 0$
  So our derivative is $\frac{0.5\sqrt{3}}{g} - \sqrt{3}$
- $\frac{\partial G}{\partial f} \implies$ We know we want to take the derivative of $\frac{fg}{\sqrt{3}}$ wrt f.
  We know $\frac{fg}{\sqrt{3}}_g = \frac{g}{\sqrt{3}}$ since we hold g constant and $f_f = 1$
  So our derivative is $\frac{g}{\sqrt{3}}$
- $\frac{\partial G}{\partial g} \implies$ We know we want to take the derivative of $\frac{fg}{\sqrt{3}}$ wrt g.
  We know $\frac{fg}{\sqrt{3}}_g = \frac{f}{\sqrt{3}}$ since we hold f constant and $g_g = 1$
  So our derivative is $\frac{f}{\sqrt{3}}$

We now want to evaluate at eq. pt (0,1) and find eigenvalues
Now, this is the jacobian for our system of 2 ODEs. All we want to do is substitute the point $(f_e, g_e) = (0, 1)$ into J

$$\implies J = \begin{bmatrix} -\sqrt{3} & 0.5\sqrt{3} - \sqrt{3} \\ \frac{1}{\sqrt{3}} & 0 \end{bmatrix}$$

Now, we need to find eigenvalues! Well, we do this by solving $\det(J - \lambda I) = 0$

$$\det\begin{pmatrix} -\sqrt{3} - \lambda & 0.5\sqrt{3} - \sqrt{3} \\ \frac{1}{\sqrt{3}} & -\lambda \end{pmatrix} = 0$$

The determinant is:

$$(-\sqrt{3} - \lambda)(-\lambda) - (0.5\sqrt{3} - \sqrt{3})\frac{1}{\sqrt{3}} = 0$$

Now, we just have to rearrange a little bit:

$$\lambda^2 + \sqrt{3}\lambda + 0.5 = 0$$

Then, this is a simple quadratic equation we can solve!

$$\lambda = -\tfrac{1}{2}\left(\sqrt{3} \pm \sqrt{3-2}\right)$$

$$\lambda = -\tfrac{1}{2}\left(\sqrt{3} \pm 1\right)$$

11. Assume that $a = 0.2$ and $r = 0.5$ and $c = \sqrt{3}$. Write down the Jacobin matrix associated to the equilibrium point $(0, a)$ and compute the eigenvalues.

We repeat the same process as above. Since we already showed the computation for J, we start with

$$J = \begin{bmatrix} -\sqrt{3} & \frac{0.5\sqrt{3}}{g} - \sqrt{3} \\ \frac{g}{\sqrt{3}} & \frac{f}{\sqrt{3}} \end{bmatrix}$$

and then evaluate at $(0, 0.2)$ and find eigenvalues.

$$\implies J = \begin{bmatrix} -\sqrt{3} & \frac{5\sqrt{3}}{2} - \sqrt{3} \\ \frac{0.2}{\sqrt{3}} & 0 \end{bmatrix}$$

Now, we need to find eigenvalues! Well, we do this by solving $\det(J - \lambda I) = 0$

$$\det\begin{pmatrix} -\sqrt{3}-\lambda & \frac{5\sqrt{3}}{2}-\sqrt{3} \\ \frac{0.2}{\sqrt{3}} & -\lambda \end{pmatrix} = 0$$

The determinant is:

$$(-\sqrt{3}-\lambda)(-\lambda) - (\tfrac{5\sqrt{3}}{2}-\sqrt{3})\tfrac{0.2}{\sqrt{3}} = 0$$

Now, we just have to rearrange a little bit:

$$\lambda^2 + \sqrt{3}\lambda - (\tfrac{1}{2} - 0.2) = 0$$

$$\lambda^2 + \sqrt{3}\lambda - 0.3 = 0$$

Then, this is a simple quadratic equation we can solve!

$$\lambda = -\tfrac{1}{2}\left(\sqrt{3} \pm \sqrt{3-4(-0.3)}\right)$$

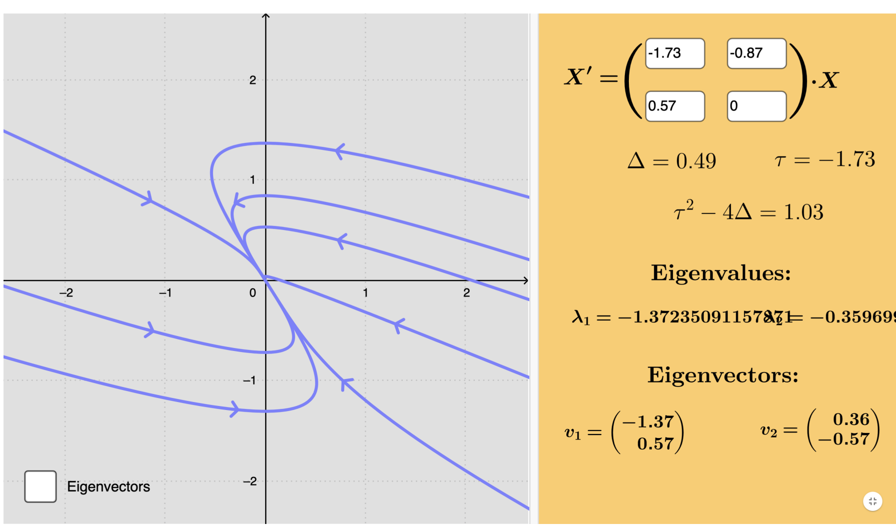
$$\lambda = -\tfrac{1}{2}\left(\sqrt{3} \pm \sqrt{3+1.2}\right)$$

$$\lambda = -\tfrac{1}{2}\left(\sqrt{3} \pm \sqrt{4.2}\right)$$

V54 Problems

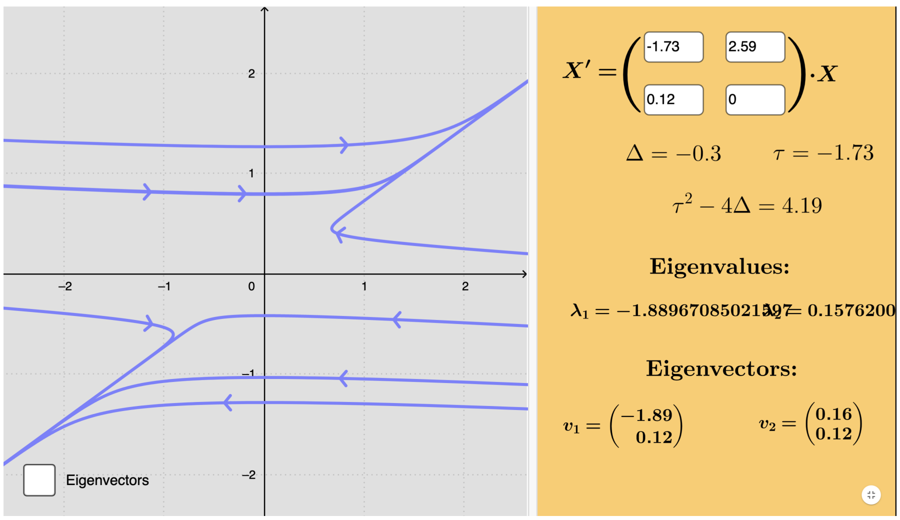
12. Consider your answer from 10. Based on the eigenvalues you computed, classify the equilibrium solution (0, 1) as a saddle, sink, source, center, spiral sink, or spiral source. Draw the local phase portrait in the (f, g) plane (here local means what the phase portrait looks like near the equilibrium point).

Since $\lambda_1 = -0.5(\sqrt{3}+1) < 0$ and $\lambda_2 = -0.5(\sqrt{3}-1) < 0$ we know $\lambda_1, \lambda_2 < 0$ and thus it is a sink.

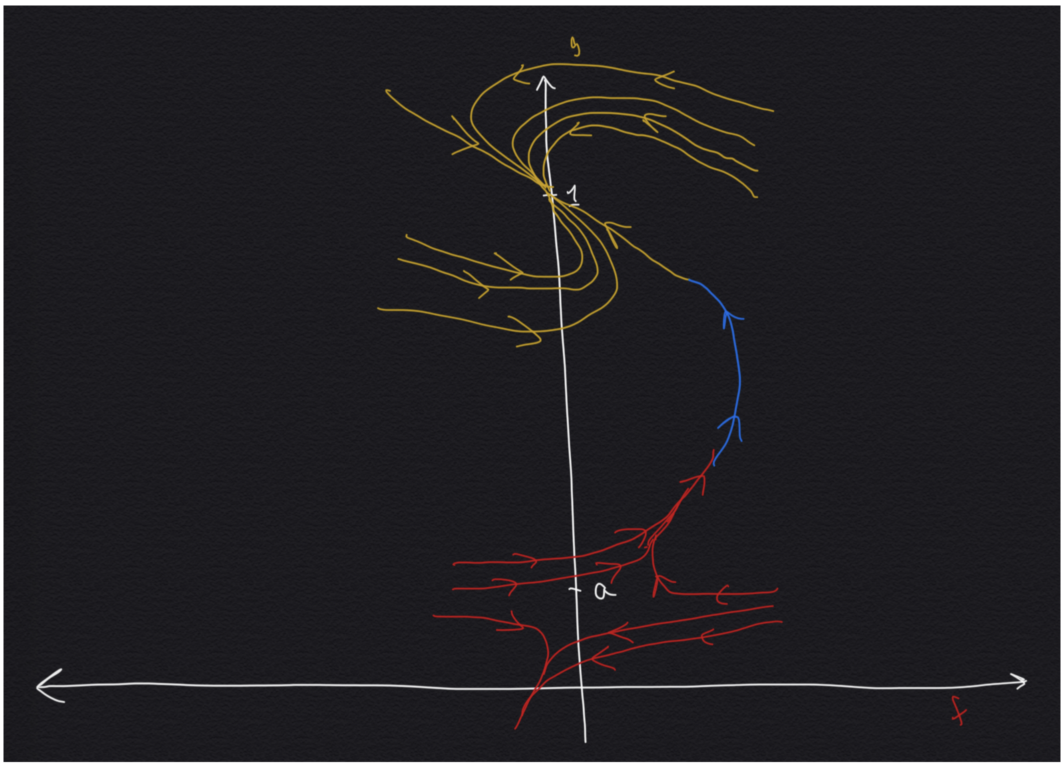


13. Consider your answer from 11. Based on the eigenvalues you computed, classify the equilibrium solution $(0, a)$ as a saddle, sink, source, center, spiral sink, or spiral source. Draw the local phase portrait in the (f, g) plane.

We know $\lambda_1 = -0.5\left(\sqrt{3} + \sqrt{4.2}\right) < 0$ and $\lambda_2 = -0.5\left(\sqrt{3} - \sqrt{4.2}\right) > 0$, so thus ${}_1 \cdot \lambda_2 < 0$ and it is a saddle point.



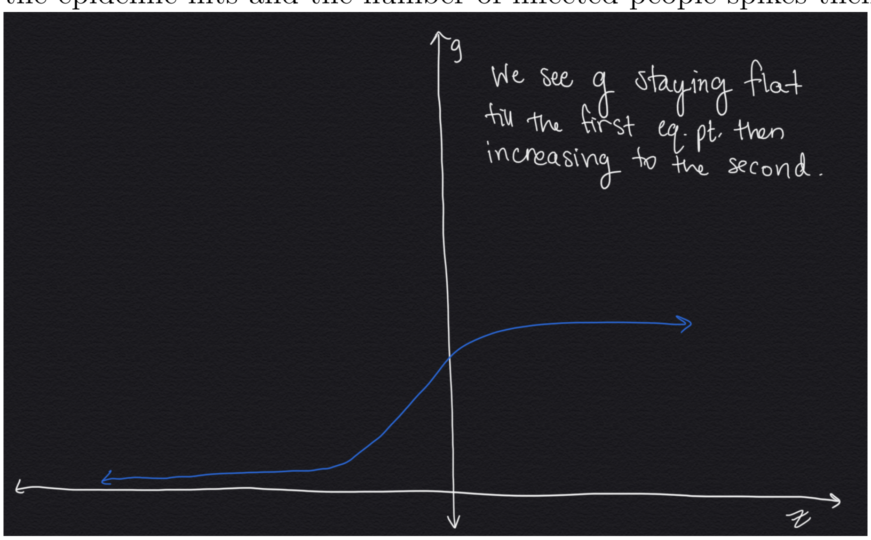
14. Combine the two phase portraits you drew in problem 12 and 13 into one phase portrait. One solution in the phase portrait is of particular importance to us. It is the one that connects $(0, a)$ with $(0, 1)$ in the positive quadrant of the (f, g) plane. Such a solution is called a heteroclinic connection. Be sure to draw the heteroclinic connection in the phase portrait.



We have the heteroclinic solution in blue, connecting both equilibrium solutions.

15. Consider the heteroclinic connection of the previous problem. Infer what f vs z and g vs z look like from the phase portrait picture of the heteroclinic connection

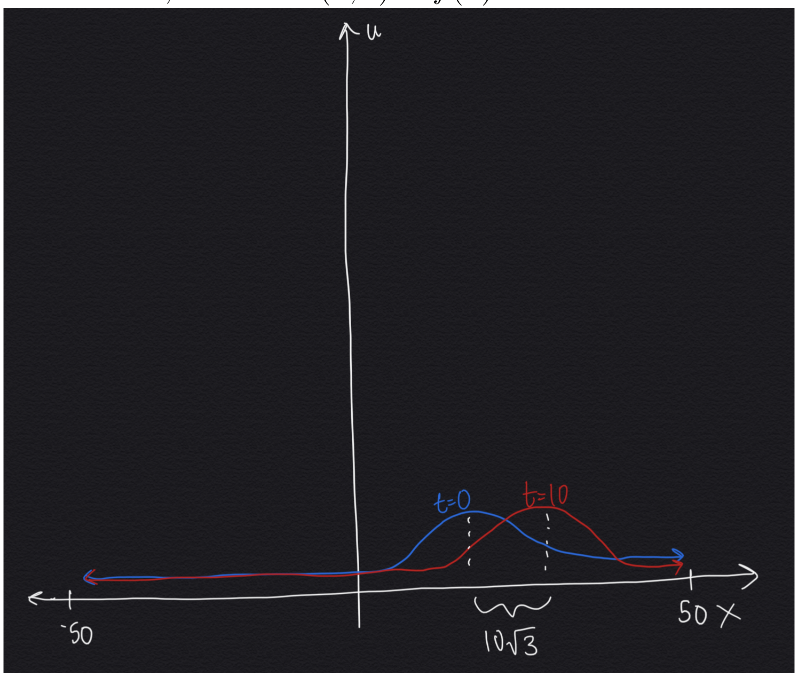
We see that f increases then decreases between eq. pts (heteroclinic solution). This is when the epidemic hits and the number of infected people spikes then falls.



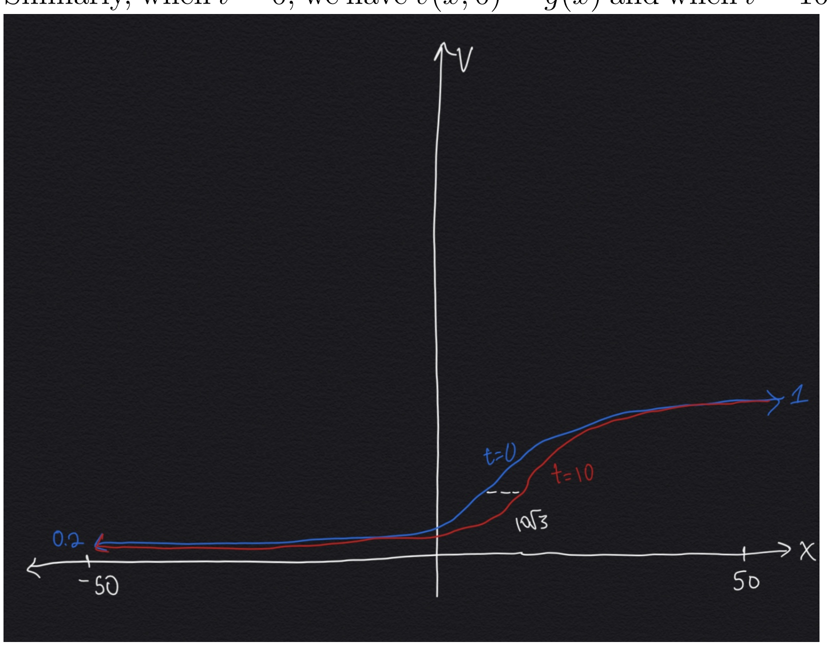
We see that g increases from the first to second eq point. This is because g models the number of susceptible individuals and where the disease has already been, there are less susceptible individuals (lower z values).

16. Using your answers from above, sketch a graph of $u(x,t)$ and $v(x,t)$ for $t = 0$ and $t = 10$ for the domain $-50 < x < 50$. Put the u graphs in one plot and put the v graphs in another. Explain how your graphs represent an "epidemic" wave (also called an epizootic front).

When $t = 0$, we have $u(x, 0) = f(x)$ and when $t = 10$ we have $u(x, 10) = f(x - \sqrt{3}10)$



Similarly, when $t = 0$, we have $v(x, 0) = g(x)$ and when $t = 10$ we have $v(x, 10) = g(x - \sqrt{3} \cdot 10)$



This represents an epidemic wave as we see the "hump" of the number of infected individuals traveling right as time increases. In other words, over time the epidemic spreads, leaving values of x (locations) that were hit in earlier time steps. We also see the number of susceptible individuals increasing at later values of x as time increases, since the disease has already hit locations further to the left. This is a traveling front as we can identify how for every time step t we move $\sqrt{3}$ in the x direction.

17. In order to obtain a physical speed, we need to undo the rescaling. It turns out, we must multiply c by the factor $\sqrt{\frac{k\beta}{r}}$. Using the field estimates of $k = 60$, $\beta = 10$ and $a = 0.2$, estimate the minimum front speed by using the formula

$$c = \sqrt{\tfrac{k\beta}{r}}2\sqrt{1-r}$$

How does your answer compare to the field estimate reported on in the videos (30-60 km a year) ?

We plug in all the given values:

$$c = \sqrt{\tfrac{60 \cdot 10}{0.5}}2\sqrt{1-0.5}$$

$$c = \sqrt{\tfrac{60 \cdot 10}{0.5}}2\sqrt{1-0.5}$$

Thus, we get that $c = 48.98$, which falls within the range of the field estimates.

References