Kim Hancock

Homework 11

November 24, 2020

1 Resources

Other than the textbook and class notes, I discussed problem 5 briefly with Kayla.

2 Notes for Week 12

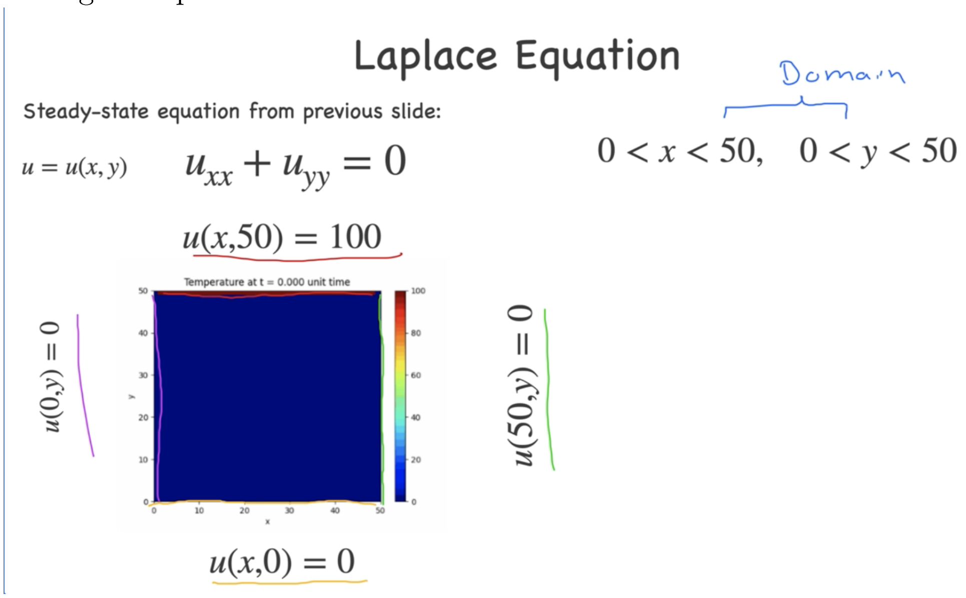
V43: LaPlace Equation intro

- Motivation for studying laplace on a bounded domain
- Example PDE
- Steps on how. to solve w sep of variables

For the first, we are considering the following PDE:

$$u_t - k(u_{xx} + u_{yy}) = 0 \text{ and } u = u(x, y, t)$$

So, here we have the heat equation which is mathematically the same as the diffusion equation, but now we are thinking about it in two spatial dimensions, x and y. THere is also a time variable. What we've been working with is this same eq without the u_{yy} (this is in 1 spatial dimension). So, now a picture of the problem is a square (metal plate) that we are heating the top of. The temperature will vary as time changes, and we see the heat diffusing through the plate.



The question we want to answer is, is there a steady state temperature for this problem? aka no change in time, $u_t = 0$ and $u_{tt} = 0$

We might see that over time this approaches the steady state $u_{xx} + u_{yy} = 0$, which is the laplace equation we know and love. We will also consider the domain and range $0 < x, y < 50$

We have both x and y bounded here. When the y value is zero, we fix the temperature to zero. When x is 50, then we have the temperature of 0. Finally, when x is zero, we have the temperature is 0. When y is 50, the temperature is 100. Note there is no time variable, so no initial condition, only boundary conditions for this problem. But, we will generatlize and say the following:

- $u(a, y) = 0$
- $u(b, y) = 0$
- $u(x, c) = 0$
- $u(x, d) = 0$

We are going to solve this with the following steps:

Separation of variable method to solve a PDE on a bounded domain with two spatial variables

1) Separate variables u = X(x)Y(y)to obtain two ODEs
2) Solve the ODE that has two homogeneous b.c.s
3) Solve the other ODE ODE
4) Take a linear combination of all solutions
5) Use the non homogeneous bc to determine arbitrary constants

Having two homogeneous bcs means that they are the two bcs which are equal to zero. Laplace equation is actually so special that we call them **harmonic functions**

V44: Solve the laplace eq steps 1-3

Reading: 6.2 pg 161-162
Remember our example was the heat equation and our steady state. We have:

$$u_{xx} + u_{yy} = 0 \text{ given}$$

$$u(0,y)=0,\ u(x,50)=100, u(50,y)=0, u(x,0)=0$$

So, we want to separate variables $u = X(x)Y(y)$ to obtain two ODEs
$\implies u = \frac{X''Y+XY''}{XY} = 0$
$\implies -\frac{X''}{X} = \frac{Y''}{Y}$

Then, we do the same thing and set both equal to λ

$$\frac{Y''}{Y} = \lambda \text{ and } -\frac{X''}{X} = \lambda$$

$$Y'' = Y\lambda \text{ and } -X'' = X\lambda$$

Now we want to consider our X ODE $-X'' = \lambda X$ and bcs $u(0,y) = 0$ and $u(50,y) = 0$

$u(0,y) = 0 \implies X(0)Y(y) = 0 \implies X(0) = 0$

$u(50,y) = 0 \implies X(50)Y(y) = 0 \implies X(50) = 0$

Now, we have $-X'' = \lambda X$ with $X(0) = 0, X(50) = 0$

However, if we try and do the same thing with our Y ODE, we end up with:

$u(x,50) = 0 \implies X(x)Y(50) = 100 \implies Y(50) = 100$ well, nothing...

So, let's take this information and go to next step:

$$-X'' = \lambda X \text{ with } X(0) = 0, X(50) = 0$$

Remember this is an eigenvalue problem we have already seen! We know that we can simply solve this ODE by integrating twice, giving us $X(x) = \sin(\frac{n\pi}{50}x)$ with eigenvalues $\lambda_n = (\frac{n\pi}{50})^2$ for $n \geq 1$

Now let's recall what we have with the Y ODE:

$$Y'' = (\tfrac{n\pi}{50})^2 Y \text{ with } Y(0) = 0$$

$$Y'' = \beta^2 Y \text{ with } Y(0) = 0$$

Now, we know $Y = e^{ry} \implies r^2 = \beta^2 \implies r = \pm\beta$, giving us:

$$Y(y) = Ae^{\beta y} + B^{-\beta y}$$

Now, we use any homogeneous bcs for this ODE, which we may or may not have. We use this to eliminate one of the constants...
$\implies Y(0) = 0 = A + B \implies A = -B$

$$Y(y) = Ae^{\beta y} - Ae^{-\beta y}$$

$$Y(y) = A(e^{\beta y} - e^{-\beta y})$$

But, we can simplify this in one more way, using a definition $\sinh(x) = \frac{e^x - e^{-x}}{2}$
We can use this to rewrite our Y as:

$$Y(y) = A\sinh(\tfrac{n\pi}{50}y)$$

Note that we absorbed the 1/2 by letting $A \to 2A$

In summary, we have:

- $X_n(x) = \sin(\frac{n\pi}{50}x)$
- $Y_n(y) = A_n \sinh(\frac{n\pi}{50}y)$
- $n = 1, 2, 3....$

V45: Solve Laplace steps 4-5

Reminder, we have:

$$u_{xx} + u_{yy} = 0 \text{ given}$$

$$u(0, y) = 0,\ u(x, 50) = 100, u(50, y) = 0, u(x, 0) = 0$$

We also found in the last video:

$$X_n(x) = \sin(\tfrac{n\pi}{50}x)$$

$$Y_n(y) = A_n \sinh(\tfrac{n\pi}{50}y)$$

Then, we make our series solution (step 4):

$$u(x, y) = \textstyle\sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{50}x)\sinh(\frac{n\pi}{50}y)$$

Now, the only boundary condition we haven't applied yet is $u(x, 50) = 100$, our inhomogeneous boundary condition.

We know $u(x, 50) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{50}x)\sinh(\frac{n\pi}{50}y)$

$$u(x, 50) = \sum_{n=1}^{\infty} A_n \sin(\tfrac{n\pi}{50}x)\sinh(n\pi) = 100$$

Recall our Fourier series $1 = \sum_{n=1} D_n \sin(\frac{n\pi}{50}x)$ for $0 < x < 50$
As we take more and more sine terms, our solution gets closer to 1 in the limit
We also know $D_m = \frac{2}{m\pi}(1-(-1)^m)$

We want to exploit this result to simplify $100 = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{50}x)\sinh(n\pi)$
So, let's first divide both sides by 100 so the LHS is the same:
Define $\tilde{A_n} = \frac{A_n \sinh n\pi)}{100}$, which we can do becuase it is just a constant

$$1 = \sum_{n=1} \tilde{A_n} \sin(\tfrac{n\pi}{50}x)$$

Thus, this is the same as the equation we want, we simply let $\tilde{A_m} = D_m$

$\implies \frac{A_n \sinh n\pi)}{100} = \frac{2}{m\pi}(1-(-1)^m)$

$\implies A_m = \frac{2}{m\pi}(1-(-1)^m)\frac{100}{\sinh(m\pi}$

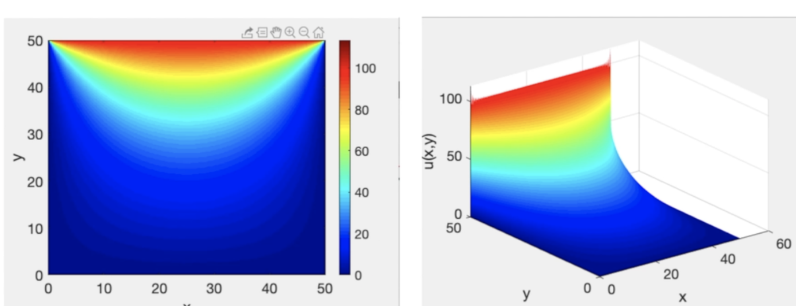
$$\implies A_m = \frac{1}{m\pi}(1-(-1)^m)\frac{200}{\sinh(m\pi})$$

Then, we have our full solution:

$$u(x, y) = \sum_{n=1} \frac{200}{n\pi\sinh(n\pi)}(1-(-1)^n)\sin(\tfrac{n\pi}{50}x)\sinh(\tfrac{n\pi}{50}y)$$

What is this telling us about the original problem?
The red is hot and the blue is cold, and with less than 10 terms, it is not consistently at $u = 10$

Plot of first 200 terms of solution



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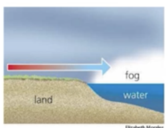
V46: Overview

Let's take a look at all the equations we have looked at so far in this class:
We started by finding solutions to advection, wave, and diffusion on an unbounded domain. We showed well-posedness, and specifically that laplace was ill-posed on an unbounded domain.
More recently, we have been thinking about bounded domains with various boundary conditions. We have solved heat and wave for dirichlet and neumann, and one type of mixed. We just solved laplace on a rectangle. We are going to show in the hw that advection on a bounded domain doesn't work!

	advection	wave	diffusion/heat	Laplace
real world				
Model	$u_t + cu_x = 0$	$u_{tt} - c^2 u_{xx} = 0$	$u_t - ku_{xx} = 0$	$u_{xx} + u_{yy} = 0$
domain		$0 < x < \ell, t > 0$	$0 < x < \ell, t > 0$	$a < x < b, \quad c < y < d$
Dirichlet		✓	✓	✓
Neumann		✓	✓	
Mixed		✓	✓	
Mixed				

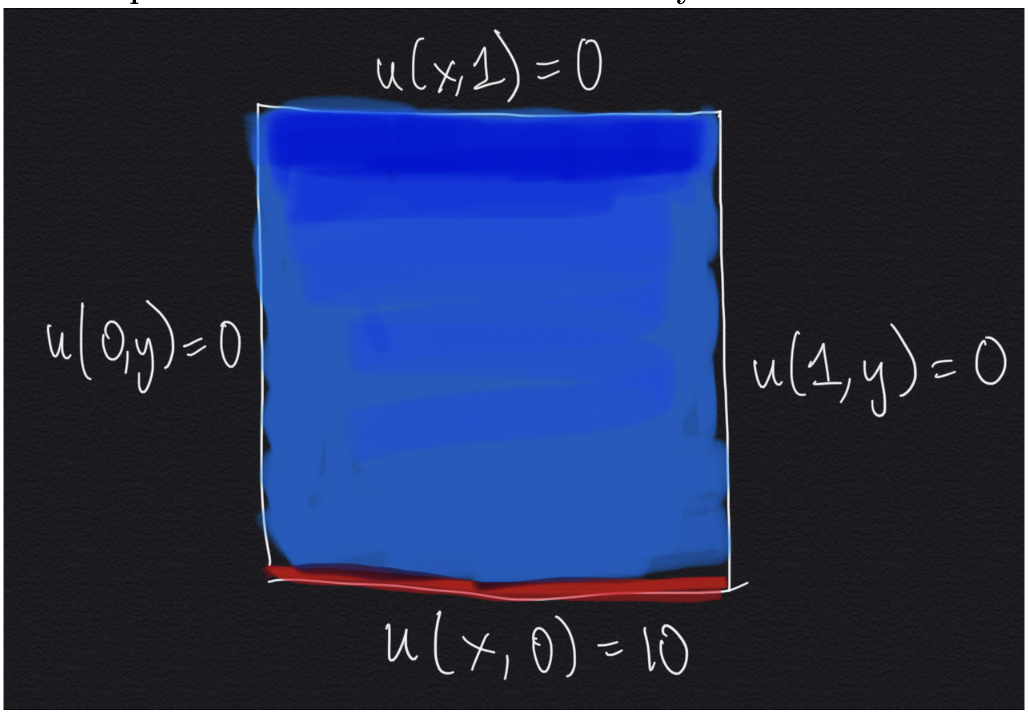
Bounded domains!

3 Problems for week 12

V43 Problems

3. Consider a square metal plate that has length 1 on each side. Suppose the bottom of the plate is being heated and held to a fixed temperature of $u = 10$ degrees. Assume that all other sides are kept cold at $u = 0$ degrees.

a) **Draw a picture that describes the boundary conditions**



As we can see, the bottom is heated while the rest of the plate remains cold.

b) **Write down a PDE and auxiliary conditions that describes the situation**

We have the PDE $u_t - k(u_{xx} + u_{yy}) = 0$ and $u = u(x, y, t)$ since we are considering heat diffusing in two spatial dimensions.
The auxiliary conditions are $u(x, 0) = 10$, corresponding to the bottom being heated, $u(1, y) = 0$, $u(0, y) = 0$ and $u(x, 1) = 0$, corresponding to the rest of the plate being cold

c) **Write down the PDE that corresponds to the steady-state of the PDE you just found.**
The steady state is the same thing as in the video since we want u_t and $u_{tt} = 0$. So, we have the steady state is $u_{xx} + u_{yy} = 0$ with the domain $0 < x < 1$ and $0 < y < 1$ with the same auxiliary conditions.

4. What is one way that the method to solve the Laplace equation on a bounded domain is different than solving the wave equation on a bounded domain?

With the laplace eq on a bounded domain, we have one ODE with homogenous bcs (don't have the dependent variable). This is the conditions that are both equal to zero. We then use the non homogeneous bcs to determine the arbitrary constants, whereas in the wave equation we always used the X ODE to determine the constants.

V44 Problems

5. Consider the following PDE and auxiliary conditions:

$$u_{xx} + u_{yy} = 0 \text{ for } 0 < x < 1, \text{ and } 0 < y < 1$$

$$u(x, 0) = 10,\ u(0, y) = 0$$

$$u(x, 1) = 0,\ u(1, y) = 0$$

a) **Carry out Step 1. Separate variables to find two ODEs. Where relevant, write down the boundary conditions for each ODE. Which of the two ODEs corresponds to the eigenvalue problem you want to solve first, and why?**
We first write our equation as $u = X(x)Y(y)$
This implies that $u = \frac{X''Y + XY''}{XY} = 0$
$\implies X''Y = -XY''$
$\implies -\frac{X''}{X} = \frac{Y''}{Y}$

Then, we do the same thing that we always do (video 27) and set both equal to eigenvalue λ

$$\frac{Y''}{Y} = \lambda \text{ and } -\frac{X''}{X} = \lambda$$

$$Y'' = Y\lambda \text{ and } -X'' = X\lambda$$

We now have our X ODE with bcs:

$$-X'' = X\lambda \text{ with } u(0, y) = 0 \text{ and } u(1, y) = 0$$

And our Y ODE with bcs:

$$Y'' = Y\lambda \text{ with } u(x, 0) = 10 \text{ and } u(x, 1) = 0$$

We want to solve the X ODE first because it has homogenous boundary conditions, meaning we will be able to plug them in and solve for our constants.

b) **Carry out Step 2: Solve the ODE that has two homogeneous boundary conditions**
We know how to solve this ODE from this video and from video 27, where we first solved this ODE. We know that $X(x) = A\sin(\beta x) + B\cos(\beta x)$ with $\lambda = \beta^2$
Then we apply our bcs $X(0) = 0 \implies A\sin(0) + B\cos(0) = 0 \implies B = 0$

Then, we know $X(x) = A\sin(\beta x)$
We also have bcs $X(1) = 0 \implies A\sin(\beta) = 0 \implies \beta = n\pi$ for any integer n and $\lambda = (n\pi)^2$. Since beta is just a constant, we know we don't care about the constant A (we can just 'absorb' A into beta) and write our equation as:

$$X(x) = \sin(n\pi x) \text{ with eigenvalues } \lambda_n = (n\pi)^2 \text{ for } n \geq 1$$

c) **Carry out Step 3: Solve the other ODE (which will have one or zero homogeneous boundary conditions). Try to simplify your answer in terms of hyperbolic sine or cosine**
We have $Y'' = \beta^2 Y$ with initial condition $u(x, 1) = 0 \implies Y(1) = 0$.
Now, we know the general solution of this from video 44 since $Y = e^{ry}$ which implies that $r^2 = \beta^2 \implies r = \pm\beta$

$$Y(y) = Ae^{\beta y} + Be^{-\beta y}$$

We now apply our bcs $Y(1) = 0 \implies Ae^{\beta} + Be^{-\beta} = 0$
We will pause here and try to rewrite our equation in the form:
$Y = A\cosh(n\pi y) + B\sinh(n\pi y)$
We know the hyperbolic sine function is $\sinh(x) = \frac{e^x - e^{-x}}{2}$
We know the hyperbolic cosine function is $\cosh(x) = \frac{e^x + e^{-x}}{2}$
So, how do we do this?

Well, we let $A = A + B$ and $B = A - B$, which we can do since A and B are just arbitrary constants. Then, we have:

$Y(y) = (A + B)e^{\beta y} + (A - B)e^{-\beta y}$

$Y(y) = Ae^{\beta y} + Be^{\beta y} + Ae^{-\beta y} - Be^{-\beta y}$

Again, since the constants are arbitrary, we divide each term by 2 and rearrange, giving us:

$Y(y) = \frac{Ae^{\beta y} + Ae^{-\beta y}}{2} + \frac{Be^{\beta y} - Be^{-\beta y}}{2}$

Thus, we can write our solution as:

$$Y(y) = A\cosh(\beta y) + B\sinh(\beta y)$$

Now, we know that $\beta = n\pi$, so we rewrite as:

$$Y(y) = A\cosh(n\pi y) + B\sinh(n\pi y)$$

Finally, we know that n is just any integer greater than zero, so we will apply a clever shift of this term by subtracting 1 from y. This way, when we plug our initial value of $Y(1)$ in, we end up with the sinh term going to zero.

$$Y(y) = A\cosh(n\pi(y-1)) + B\sinh(n\pi(y-1))$$

As we can see, we did this translation in the cosh and sinh functions so that when we apply our boundary condition of $Y(1) = 0$ the correct term goes to zero:

$$Y(1) = A\cosh(n\pi(0)) + B\sinh(n\pi(0)) = 0$$

$$Y(1) = A + 0 = 0$$

$$\implies A = 0$$

This leaves us with the following equation for $Y(y)$:

$$Y(y) = B\sinh(n\pi(y-1))$$

Alternatively, after talking to Kayla about this step of the problem, I realized that letting $A = A+B$ and $B = A-B$ would not really work since then you are assuming A and B depend on each other, which they do not have to. It is better to just start with the assumption that we want our answer to be of the form $Y = A\cosh(n\pi y) + B\sinh(n\pi y)$ and work backwards from there. Our first step now is to do the translate, since n starts at 1 and if we didn't do this, our boundary conditions would not be satisfied.

$$Y = A\cosh(n\pi(y-1)) + B\sinh(n\pi(y-1))$$

We showed above that this satisfies the bcs $Y(1) = 0$ and that this leaves us with $Y(y) = A\sinh(n\pi(y-1))$
Now, instead of doing the $A = A + B$ thing, we will just see if this solution indeed satisfies our ODE $Y'' = \lambda Y$

$$Y'(y) = A\cosh(n\pi(y-1)) \cdot n\pi$$

$$Y''(y) = A\sinh(n\pi(y-1)) \cdot (n\pi)^2$$

Since we defined $\lambda = \beta^2 = (n\pi)^2$, we have that $Y = A\sinh(n\pi(y-1))$ and $Y'' = A\sinh(n\pi(y-1)) \cdot \beta^2$, and thus:

$$Y'' = \lambda Y \implies A\sinh(n\pi(y-1)) \cdot \beta^2 = A\sinh(n\pi(y-1)) \cdot \beta^2 \text{ giving us the desired equality}$$

V45 Problems

6. Consider the previous problem. If you use an existing result anywhere in this problem, make a specific reference (e.g. we know from page XX of the book that ...)

a) **Carry out Step 4 from the previous problem.**
We want to take a linear combination of the solutions we got in the last question:

$$u(x,y) = \sum_{n=1} B_n \sinh(n\pi(y-1))\sin(n\pi x)$$

b) **) Carry out Step 5 from the previous problem** We now want to figure out what our constant B_n is. We recall that our last inhomogenous bcs is $u(x,0) = 10$, which, when we plug this in, yields:

$$10 = \sum_{n=1} B_n \sinh(-n\pi))\sin(n\pi x)$$

Recall our Fourier series $1 = \sum_{n=1} D_n \sin(\frac{n\pi}{50}x)$ for $0 < x < 50$
As we take more and more sine terms, our solution gets closer to 1 in the limit
We also know $D_m = \frac{2}{m\pi}(1-(-1)^m)$

We want to exploit this result to simplify $10 = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi}{50}x)\sinh(-n\pi)$
So, let's first divide both sides by 10 so the LHS is the same:

$1 = \frac{\sum_{n=1}^{\infty} B_n \sin(\frac{n\pi}{50}x)\sinh(-n\pi)}{10}$

Define $\tilde{B_n} = \frac{B_n \sinh(-n\pi)}{10}$, which we can do becuase it is just a constant
Then, we have $1 = \sum_{n=1}^{\infty} \tilde{B_m}\sin(\frac{n\pi}{50}x)$
This is something we can use our Fourier equation with, knowing that $D_m = \tilde{B_m}$
$\implies \tilde{B_n} = \frac{2}{m\pi}(1-(-1)^m)$
$\implies \frac{B_n\sinh(-n\pi)}{10} = \frac{2}{m\pi}(1-(-1)^m)$
Solving for B_n, we get:

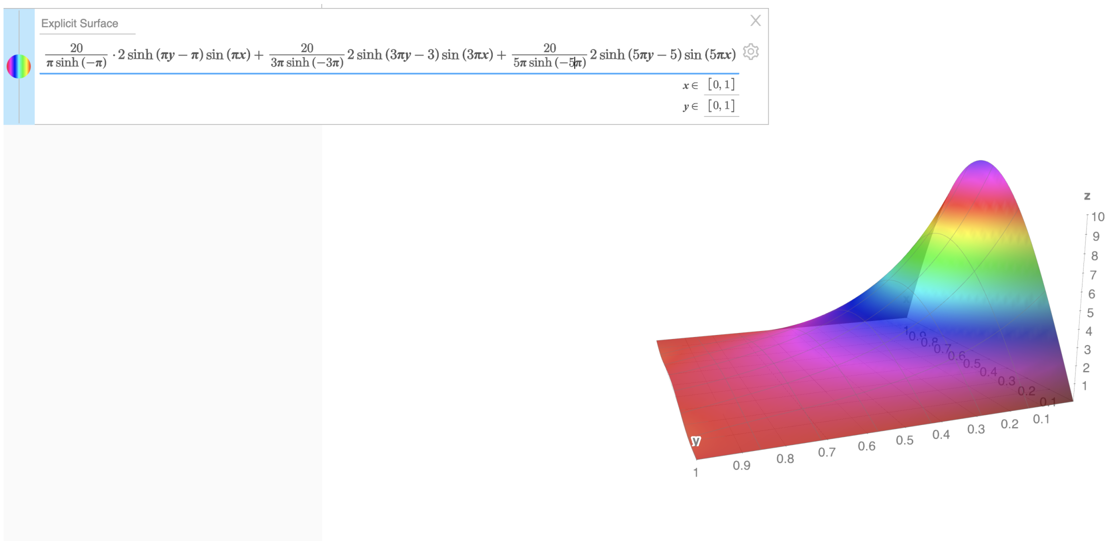
$$B_n = \frac{10}{\sinh(-n\pi)}\frac{2}{m\pi}(1-(-1)^m)$$

$$B_n = \frac{20}{m\pi \sinh(-n\pi)}(1-(-1)^m)$$

Thus, plugging this into our sum, we get:

$$u(x,y) = \sum_{n=1} \frac{20}{n\pi \sinh(-n\pi)}(1-(-1)^n)\sinh(n\pi(y-1)\sin(n\pi x)$$

c) **Plot the first three terms of your solution on the appropriate domain.**



We indeed see the equation approaching $u = 10$, and if we took more terms, we would see the top "flattening" out at $u = 10$

V46 Problems: Overview

7. Show that the only separable solution to the following advection equation on a bounded domain

$$u_t + cu_x = 0 \text{ for } 0 < x < l \text{ and } t > 0 \text{ With bcs}$$

$$u(0,t) = 0 \text{ and } u(1,t) = 0$$

is the trivial one.

We know we want to separate this equation so the solution is of the form $u(x,t) = X(x)T(t)$.

$\implies X(x)T'(t) + cX'(x)T(t) = 0$
$\implies X(x)T'(t) = -cX'(x)T(t)$
$\implies \frac{T'(t)}{cT(t)} = -\frac{X'(x)}{X(x)}$
Now, we want to set both of these equal to lambda, yielding:

$$\implies \frac{T'(t)}{cT(t)} = -\frac{X'(x)}{X(x)} = \lambda$$

We recall $\lambda = \frac{T'}{cT} \implies \frac{d}{dx} = 0$ and $\lambda = \frac{-X'}{X} \implies \frac{d\lambda}{dt} = 0$
Since x derivative of λ is 0 and so is t derivative, we know that λ has to be constant.

So, since lambda is constant we have: $\lambda cT = T'$ and $-X'(x) = \lambda X(x)$

Now, we want to solve our X ODE:

$$-\frac{X'(x)}{X(x)} = \lambda \implies -\int_0^l \frac{1}{X}dx = \int_0^l \lambda dx$$

$$-\ln(X) = \lambda x + c$$

$$X = e^{-\lambda x + c}$$

Finally, we arrive at our X ODE:

$$X(x) = Ce^{-\lambda x}$$

We also have our initial condition $u(0,t) = 0 \implies X(0) = 0$ since we do not want $T(t) = 0$
$\implies X(0) = Ce^0 = C = 0$
Thus, this implies that $C = 0$ and so we have a trivial X ODE $X(x) = 0$

Then, we have $u(x,t) = X(x)T(t) = 0 \cdot T(t) = 0$, and therefore the only solution to our PDE using the separation of variables method is the trivial solution.

8. Problem 3 in Section 6.2 of the book. [Hint: The answer is in the back of the book, which you can use to check your work. Since you will know what the answer is, I expect a glorious amount of detail here. You are still are allowed to refer to existing results, but be specific, and be careful (don't use a result that doesn't apply to the problem you are solving!)]

Find the harmonic function $u(x,y)$ in the square $D = \{0 < x < \pi, 0 < y < \pi\}$ with the boundary conditions:

$$u_y = 0 \text{ for } y = 0 \text{ and for } y = \pi,$$

$$u = 0 \text{ for } x = 0 \text{ and } u = \cos^2 y = \tfrac{1}{2}(1 + \cos 2y) \text{ for } x = \pi.$$

We will go through each of the steps to solve this:

a) **Separate variables to find two ODEs:**
We know our PDE we are solving is the laplace equation $u_{xx} + u_{yy} = 0$. Then, we know we want something of the form $u = X(x)Y(y)$ so we can separate the variables. From problem 5 part a that we can separate this to be of the form $X''Y = -XY''$.

Thus, also from problem 5 we know our X and Y ODEs are:

$$X'' = \lambda X \text{ and } Y'' = -\lambda Y$$

b) **Solve the ODE with two homogenous bcs**
We know our Y ODE has two homogenous bcs, so we solve that first
We know our answer is of the form:
$Y = A\sin(\beta y) + B\cos(\beta y)$ where $\lambda = \beta^2$
Now, we apply our bcs $u_y(x, 0) = 0$ and $u_y(x, \pi) = 0$
Since we don't want X to be trivial it must be that $Y'(0) = 0$ and $Y'(\pi) = 0$

$$Y' = A\beta\cos(\beta y) - B\beta\sin(\beta y)$$

$$\implies Y'(0) = A\beta\cos(0) - B\beta\sin(0) = A\beta = 0$$

Since we know $\beta \neq 0$, as we showed in V27 and the problems for that week that a zero value of lambda leads to the trivial solution
Thus, it must be that $A = 0$ and we are left with $Y(y) = \cos(\beta y)$
We also have the bcs that $Y'(\pi) = 0$, so we have:

$$Y'(\pi) = -\sin(\beta\pi) = 0$$

We know this is true for integer multiples of π, so we know that $\beta = n$ and we have:

$$Y'(\pi) = -\sin(n\pi) = 0 \text{ which holds true for some integer } n \geq 0$$

Then, our Y ODE is:

$$Y(y) = \cos(n\pi)$$

We note that we can drop the constant B since it is "absorbed" inside the cosine with the integer n
We also note that when $n = 0$, we are left with $Y(y) = \cos(0) = 1$

c) **Solve the other ODE**
We now consider our X ODE, $X'' = \lambda X$
We know our answer is of the form:

$X = Ae^{nx} + Be^{nx}$ where $\beta = n$, which we know from problem 5 and from video 44.

We now want to apply our bcs.

We have our one homogenous bcs for this ODE $u(0, y) = 0 \implies X(0)Y(y) = 0$. Since we don't want Y to be trivial, it must be that $X(0) = 0$
Applying this bcs, we have:

$$X(0) = Ae^{n0} + Be^{-n0}$$

$$X(0) = A + B = 0 \implies A = -B$$

Then, we can rewrite X as:

$$X = A(e^{nx} - e^{-nx})$$

Using the definition of hyperbolic sine, we rewrite this as:

$$X = A\sinh(nx)$$

We also note that when $n = 0$, we have $X'' = 0$, which has solution $X_0(x) = Ax + B$ $\implies X_0(0) = 0 \implies B = 0 \implies X_0(x) = A_0 x$

d) **Linear combination of solutions**
Taking a linear combination of solutions gives us:
$u(x, y) = A_0 x + \sum_{n=1}^{\infty} A\cos(ny)\sinh(nx)$

e) **Use inhomogenous bcs to determine A**
Our goal is now to figure out what A is
We look at our inhomogeneous bcs, $u(\pi, y) = \cos^2 y = \frac{1}{2}(1 + \cos 2y)$

Then, we have:

$$\tfrac{1}{2}(1 + \cos 2y) = A_0\pi + \textstyle\sum_{n=1}^{\infty} A\cos(ny)\sinh(n\pi)$$

We want to solve for A using this; since both sides of the equation have two terms, we can say the following:

$$\tfrac{1}{2} = A_0\pi \text{ and } \tfrac{1}{2}\cos 2y = \textstyle\sum_{n=1}^{\infty} A\cos(ny)\sinh(n\pi)$$

This implies that $A_0 = \frac{1}{2\pi}$
Now, we know the second equality will not hold for every value of n, but rather only when $n = 2$. This way, the $\cos(ny)$ on the RHS matches the $\cos(2y)$ on the LHS. Any other value of n would make this impossible to solve.

$$\implies \tfrac{1}{2}\cos 2y = A\cos(2y)\sinh(2\pi)$$

$$\implies A = \tfrac{\cos 2y}{2\cos(2y)\sinh(2\pi)}$$

$$\implies A = \tfrac{1}{2\sinh(2\pi)}$$

This boundary condition tells us that any value of n besides $n = 0$ and $n = 2$ will not satisfy the boundary condition.
Thus, we are left as our final answer with:

$$u(x, y) = \tfrac{1}{2\pi} + \tfrac{1}{2\sinh(2\pi)}\cos(2y)\sinh(2x)$$

References