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Homework 1

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Notes for Week 1

V1

What is a PDE?

Remember, a regular ODE looks like $du/dx = 1$ and $u = u(x)$

How do we solve this? Simple separation of variables

$du = dx \implies u(x) = x + C$

Example PDE: $u(x, y) = xy + x^2 + y^2$

$$\frac{\partial u}{\partial x} = y + 2x \frac{\partial u}{\partial y} = x + 2y \tag{1}$$

Going back to original equation, we want to guess a solution with $u = u(x, y)$:
Is $u(x, y) = x + C$ still valid? yes
Welll, we don't care about the y in the equation, so our general solution is $u(x, y) = x + g(y)$

In Summary: if a pde contains only partial derivatives of one independent variable

- you can treat it like an ODE
- replace arbitrary constants with arbitrary functions

V2

We deal with systems that change in space and time–much more realistic than ODEs
Examples:

- advection $u_t + u_x = 0$
- diffusion $u_t - u_{xx} = 0$

Order: Highest derivative present in the PDE
Linear: If each term is linear with respect to the dependent variable
$\implies$ REMINDER u is the dependent variable
Non-linear examples: $u_t + u_x + u^2 = 0, u_t + u_x + u_x u, u_t + u_x + u = x^2$
Linear examples: $u_t + u_x = 0, u_t + u_x + xu$
Homogeneous: Every term contains the dependent variable, u

These classes are important for analysis!

V3

Recap

Gradient is both partial derivatives, direction of steepest ascent
Directional derivative is similar, gradient dot product direction vector

Geometric method

Equation $\nabla_v u = 0$
This means that u does not change along lines in (x, y) plane parallel to $v \implies$ the family of lines parallel to vector $v =< a, b >$ is $bx - ay = c$
SO, if we fix c, then $u(x, y)$ is constant, so $f(c) = u(x, y)$
Tells us if c is arbitrary, any (x, y) is possible so it is the whole plane, and c is $f(bx - ay)$
$\implies u(x, y) = f(bx - ay)$ is $au_x + bu_y = 0$

V4

Find general solution to $4u_x - 3u_y = 0$
So, we have to look at the characteristic line, $-3x - 4y = c$ which implies that the solution is $u(x, y) = f(-3x - 4y)$ where f is an arbitrary function

Auxiliary Conditions

We need to satisfy the PDE as well as the condition $u(0, y) = y^3$

1. find general solution of PDE
2. use auxiliary condition to determine the arbitrary function

We know general solution is $f(-3x - 4y)$ from the intro example
Now, we know $u(0, y) = y^3$ and $u(0, y) = f(-4y) \implies y^3 = f(-4y)$
Define w to be the argument, so $w = -4y \implies y = -w/4 \implies (-w/4)^3 = f(w)$
In place of w in $f(w)$ we want to use $-3x - 4y$, so we get $u(x, y) =$
$f(-3x - 4y) = (-(-3x - 4y)/4)^3 = (3x - 4y)^3/64$
Auxilary conditions are initial or boundary

Practice Problems

V1

Find general solution to $u_x = 5u$ where $u = u(x, y)$

- $du/5u = dx \implies ln(5u) = x + C$
- $5u = e^x + C \implies u = \frac{(e^x)}{5} + C'$
- We replace C' with an arbitrary function of y to give us our final answer of $u(x,y) = \frac{(e^x)}{5} + g(y)$

V2

What is one possible advantage to using a PDE model over an ODE model to describe a real life system? Can you think of a disadvantage? A PDE allows you to have more than one independent variable, which means we can model both time and space, for example. This is really useful for real life modeling, such as diffusion and elasticity. A potential disadvantage could be that we won't be able to solve the PDE analytically, but we could solve the corresponding ODE much more easily.

Consider the following PDEs:

1. $u_t + xu_{xx} + x^2 = 0$
2. $u_t + xu_x = u^2$

Equation 1 is linear
Equation 2 is homogenous
Equation 2 is first order

V3

6. We know that $u(x,y) = f(bx - ay) \implies u_x(x,y) = b \cdot f(bx - ay)$ and $u_y(x,y) = -a \cdot f(bx - ay)$
Thus, $a(u_x) + b(u_y) = ab \cdot f(ax - by) - ab \cdot f(ax - by) = 0$, verifying the assertion

7. Suppose u satisfies PDE $u_x + 2u_y = 0$ and $u(0,0) = 3$ and $u(0,1) = 4$
(a) We know from equation 5 that $a = 1$ and $b = 2$ so the characteristic line that crosses $(x,y) = (0,0)$ is $2x - y = 0$
(b) We want to check if $2x - y$ is 0 for all of the examples. We also know that the characteristic line for $u(0,1)$ is $2x - y = -1$

- $u(1,2) = 3$ is definitely true
- $u(1,3) = 3$ is definitely false

- $u(2,4) = 3$ is definitely true
- $u(2,6) = 3$ might be true, since the point $(2,6)$ is not on the characteristic line that crosses $(0,0)$ or $(0,1)$.

V4

8. Solve $u_x + 2u_y = 0$ with auxiliary condition $u(x,0) = \sin(x)$
We know that $u(x,y) = f(2x - y)$ and that $u(x,0) = f(2x) \implies f(2x) = sin(x)$
Now, let $w = 2x \implies x = w/2$, so we get:
$f(w) = \sin(w/2)$ which implies that $u(x,y) = \sin(x - (y/2))$