HW 17 -

① JJ Taube + Aaron Carlton, Desmos for graphing, math24.net → 2nd order linear homogenous diff. eqns w/ constant coefficients

③ 1) Separate variables to get 2 ODEs

2) Solve the $X(x)$ ODE with boundary conditions

3) Solve the $T(t)$ ODE

4) Take the linear combination of all solutions

5) Use initial values to determine arbitrary constants

④ $X'' + 4X = 0$ → characteristic eqn $= k^2 + 4 = 0$

* discriminant of char. eqn. $= b^2 - 4ac = 0 - 4\cdot 1\cdot 4 = 0 - 16 = \underline{-16}$

find roots of char. eqn. : $\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{0 \pm \sqrt{-16}}{2} = 0 \pm 2i$

general solution is: $X(x) = e^{\alpha x}\left[A\cos(\beta x) + B\sin(\beta x)\right]$

$= e^{0}\left[A\cos(2x) + B\sin(2x)\right]$

$\boxed{X(x) = A\cos(2x) + B\sin(2x)}$

$X'' - 4X = 0$ → characteristic eqn $= k^2 - 4 = 0$

discriminant $= b^2 - 4ac = 0 - 4\cdot 1\cdot(-4) = 0 + 16 = 16$

roots: $k^2 - 4 = (k+2)(k-2)$, $k = \pm 2$

general solution is: $X(x) = Ae^{k_1 x} + Be^{k_2 x}$

$\boxed{X(x) = Ae^{2x} + Be^{-2x}}$

*used math24.net source for ODE refresher

⑤ $X'' = \lambda X$ → $X'' = 0X$ → $X'' = 0$

If $X'' = 0$, then $X' = c$ and $\underline{X = c_1 x + c_2}$ where c_1, c_2 constants

$X(0) = c_1(0) + c_2$

boundary condition $\left[\begin{array}{l} X(0) = c_2 \\ 0 = c_2 \end{array}\right.$

now, $X = c_1 x$

bound. cond. $\left[\begin{array}{l} X(l) = c_1(l) \\ 0 = c_1(l) \end{array}\right.$ → this is only true if $c_1 = 0$

So the solution is $X = c_1 x + c_2 = 0x + 0 = 0$.

This is the trivial solution, which we are not interested in!

⑥ $X'' = \lambda X$ let $\lambda = \beta^2$ where β is real

$X'' = \beta^2 X$

$X'' - \beta^2 X = 0$ → Know from problem 4 that general solution is:

$$X(x) = Ae^{\beta x} + Be^{-\beta x}$$

Now use boundary conditions:

$X(0) = Ae^0 + Be^0 = A+B$

$0 = A+B$

$-A = B$ →

$X(l) = Ae^{\beta l} + Be^{-\beta l}$

$0 = Ae^{\beta l} + Be^{-\beta l}$

$0 = Ae^{\beta l} + (-A)e^{-\beta l}$

$0 = Ae^{\beta l} - Ae^{-\beta l}$

$Ae^{-\beta l} = Ae^{\beta l}$ → $e^{-\beta l} = e^{\beta l}$

$-\beta l = \beta l$

$-\beta = \beta$ → only true if $\beta = 0$

So $X(x) = Ae^{0x} + Be^{0x}$

$= A+B$

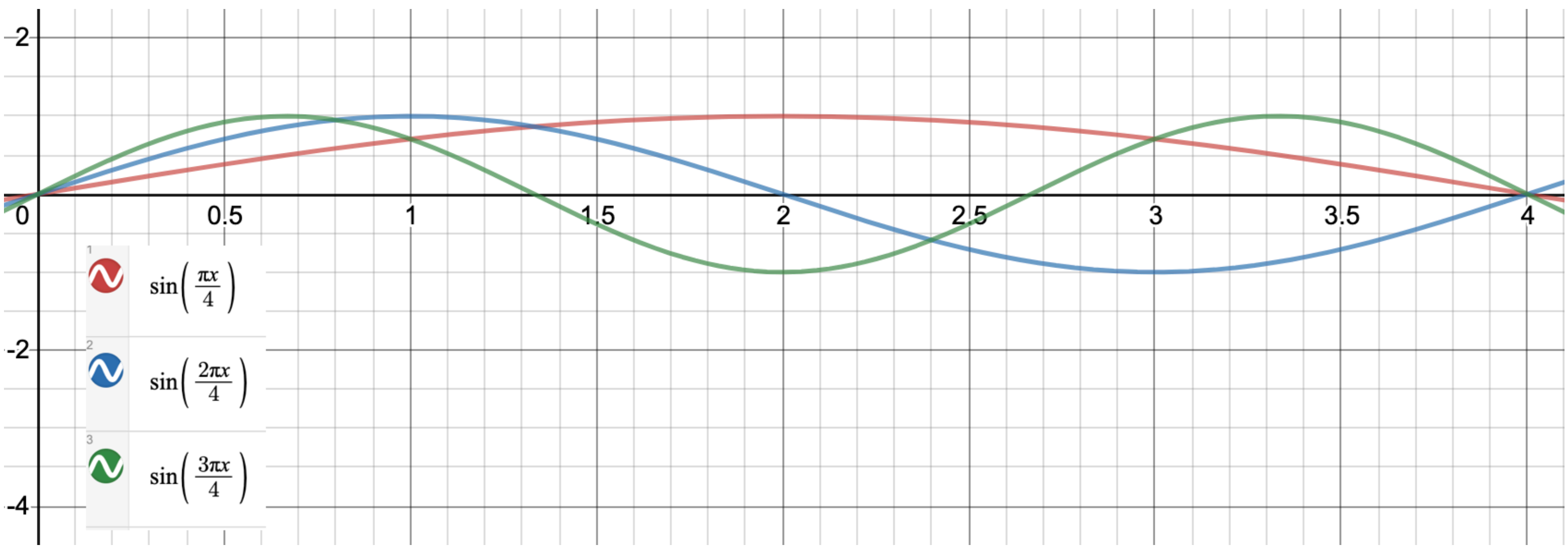
$X(x) = 0$ → This is the trivial solution!

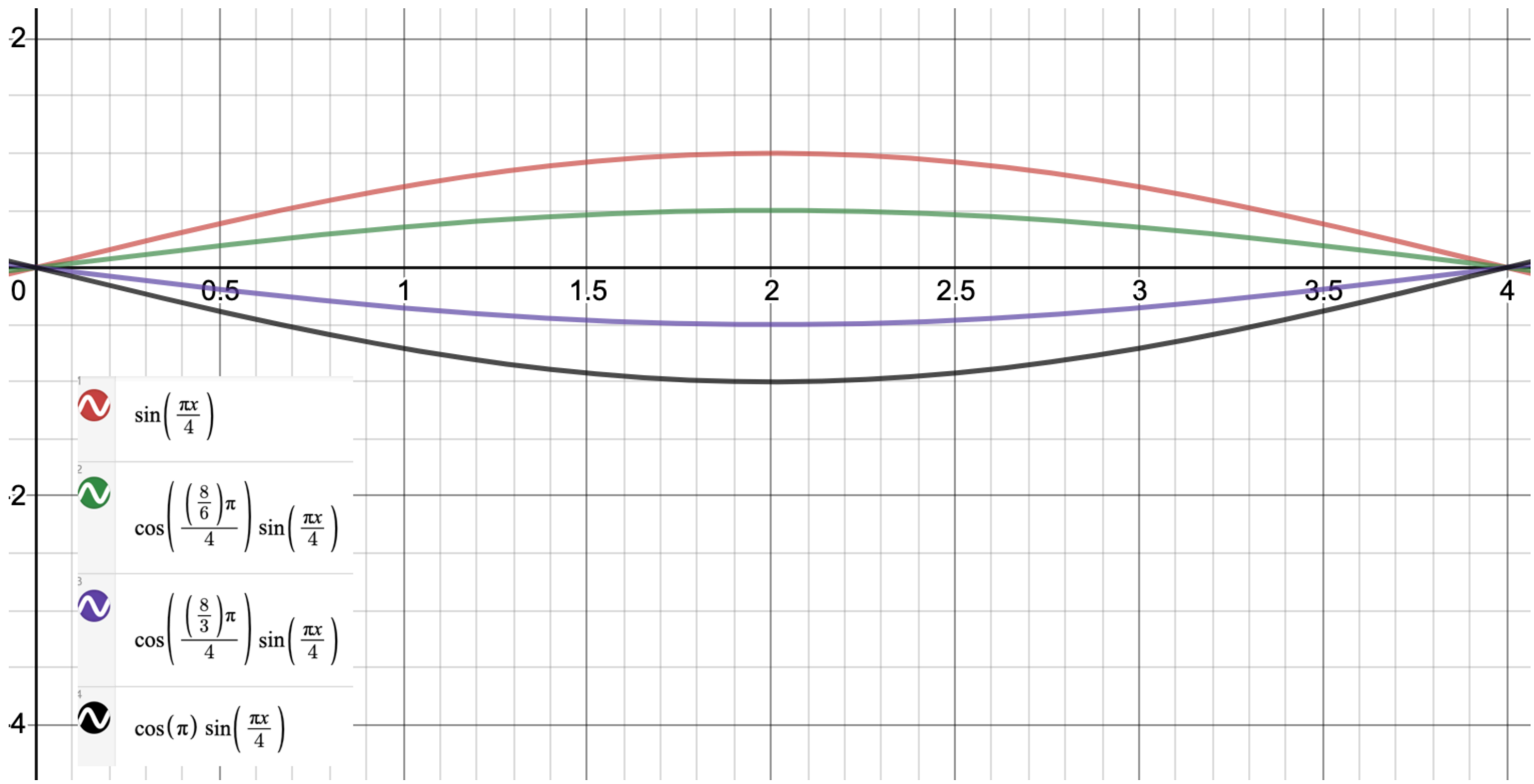
⑦ To get this solution, we used the fact that $\beta = \frac{n\pi}{l}$ and $\lambda = -\beta^2$
If $n=0$, $\beta = 0$ and then $\lambda = 0$.
But we showed in problem 5 that we are not interested in the case where $\lambda = 0$. So $n=0$ is not included in our solution.
As for negative n values, these are covered by their positive counterparts because $\lambda = -\beta^2$. If $n=-2$, $\beta = \frac{-2\pi}{l}$ and $\lambda = -\left(\frac{-2\pi}{l}\right)^2$.

But this value of lambda is the same as $-\left(\frac{2\pi}{l}\right)^2$. So we do not miss solutions by leaving out negative values of n.

⑧ Yes! $X(0)=0$ and $X(4)=0$ for all three values of n.

⑨ The solution looks like an oscillating string with fixed ends at $x=0$ and $x=4$ and maximum heights of 1 when $t=0,1,2,3,4,\ldots$ and -1 when $t = 1/2, 3/2, 5/2, \ldots$.


2
0
0.5
1
1.5
2
2.5
3
3.5
4
-2
-4
1
$\sin\left(\frac{\pi x}{4}\right)$
2
$\sin\left(\frac{2\pi x}{4}\right)$
3
$\sin\left(\frac{3\pi x}{4}\right)$



2
0
0.5
1
1.5
2
2.5
3
3.5
4
-2
-4
$\sin\left(\frac{\pi x}{4}\right)$
$\cos\left(\frac{\left(\frac{8}{6}\right)\pi}{4}\right)\sin\left(\frac{\pi x}{4}\right)$
$\cos\left(\frac{\left(\frac{8}{3}\right)\pi}{4}\right)\sin\left(\frac{\pi x}{4}\right)$
$\cos(\pi)\sin\left(\frac{\pi x}{4}\right)$

(10) $\phi(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) = \boxed{\sin\left(\frac{\pi x}{4}\right)}$

$\psi(x) = u_t(x,0)$

$u_t(x,t) = \sum_{n=1}^{\infty} \left(-A_n \sin\left(\frac{n\pi ct}{l}\right)\frac{n\pi c}{l} + \overset{0}{\cancel{B_n}} \cos\left(\frac{n\pi ct}{l}\right)\frac{n\pi c}{l}\right) \sin\left(\frac{n\pi x}{l}\right)$

↳ in this solution: $u_t(x,t) = -\sin(2\pi t)\, 2\pi \cdot \sin\left(\frac{\pi x}{4}\right)$

$u_t(x,0) = -\sin(0)\, 2\pi \cdot \sin\left(\frac{\pi x}{4}\right) = 0$

$\boxed{\psi(x) = 0}$

(11) Step 3, solving the $T(t)$ ODE, will look different because this ODE is now first order homogenous instead of second, making it easily separable.