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Group Assignment 2

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2

a Amount of substance between locations $x = a$ and $x = b$ is: $\int_a^b u(x,t)A\ dx$, since the area of the tube is A and we are integrating from location a to b

b Rate of change of 2a) with respect to time: $\int_a^b u_t(x,t)A\ dx$, since we took the derivative of $u(x,t)$ with respect to time
Units: (grams/second $\cdot$ cm^2)$\cdot$ cm^2 = grams/second

c $\phi(a,t)$ since we are at position $x = a$
Units: grams/(cm^2 sec)

d $\phi(b,t)$ since we are at position $x = b$
Units: grams/(cm^2 sec)

e The amount of substance that is added is the integral between locations a and b times the area over which it's created $\implies \int_a^b h(x,t)A\ dx$
Units: (grams/second $\cdot$ cm^2)$\cdot$ cm^2 = grams/second

f $\int_a^b u_t(x,t)A\ dx = A(\phi(a,t) - \phi(b,t)) + \int_a^b h(x,t)A\ dx$
(grams/sec) = (cm^2 * (grams/cm^2 sec)) + (grams/sec)
We can see that all of the elements in our equation have units grams/second!

g $\int_a^b u_t(x,t)\ dx = \phi(a,t) - \phi(b,t) + \int_a^b h(x,t)dx$, dividing both sides by A
$\phi(a,t) - \phi(b,t) = \int_a^b u_t(x,t)dx - \int_a^b h(x,t)dx = \int_a^b u_t(x,t) - h(x,t)dx$
$-\int_a^b \phi_x(x,t)dx = \int_a^b u_t(x,t) - h(x,t)dx$, using Fundamental Theorem of Calculus

Finally, we have $0 = \int_a^b u_t(x,t) + \phi_x(x,t) - h(x,t)dx$

h Since $a, b \in \mathbb{R}$, so a and b are arbitrary, the only way our equation in g) can be true is if $u_t(x,t) + \phi_x(x,t) - h(x,t) = 0$, or $u_t + \phi_x = h(x,t)$. ✓

i $\phi(x,t) = ku(x,t)$, where $k \in \mathbb{R}$, k is an arbitrary constant.
So, we know that $\phi_x(x,t) = ku_x(x,t)$.
We can re-write our equation derived in h) using this information, so we know that $u_t(x,t) + k * u_x(x,t) = h(x,t)$.
Now, our equation is in the form of a familiar PDE that could be solved using the coordinate method.

3

10. Solve $u_x + u_y + u = e^{x+2y}$ **with** $u(x,0) = 0$.

First, we can use the coordinate method, letting $\tilde{x} = x + y$ and $\tilde{y} = x - y$, since we know that $a = 1$ and $b = 1$ from our equation.

$u_x + u_y = e^{x+2y} - u = (a^2 + b^2)u_{\tilde{x}} = (1^2 + 1^2)u_{\tilde{x}} = 2u_{\tilde{x}}$ from coordinate method

Now, we need to rewrite $x + 2y$ in terms of $\tilde{x}$ and $\tilde{y}$.

We can see that $x+2y = \frac{1}{2}(3\tilde{x}-\tilde{y}) = \frac{3}{2}\tilde{x}-\frac{1}{2}\tilde{y} = \frac{3}{2}(x+y)-\frac{1}{2}(x-y) = \frac{3}{2}x-\frac{1}{2}x+\frac{3}{2}y+\frac{1}{2}y = x+2y$ (yay!)

Finally, we can rewrite the PDE in terms of our coordinate variables to get the ODE

$$2u_{\tilde{x}} + u = e^{\frac{1}{2}(3\tilde{x}-\tilde{y})}$$

Now, we will use the integrating factor method from ODE to solve this first-order, linear sneaky ODE.

First, we can rewrite our equation as $u_{\tilde{x}} + \frac{1}{2}u = \frac{1}{2}e^{\frac{1}{2}(3\tilde{x}-\tilde{y})}$. So, our integrating factor is $e^{\int \frac{1}{2}d\tilde{x}} = e^{\frac{1}{2}\tilde{x}}$

Then, we can use the formula for the integrating factor method and know our general solution is $u(\tilde{x},\tilde{y}) = e^{-\frac{1}{2}\tilde{x}} \int \frac{1}{2}e^{\frac{1}{2}(3\tilde{x}-\tilde{y})}e^{\frac{1}{2}\tilde{x}}d\tilde{x}$

Then, $u = e^{-\frac{1}{2}\tilde{x}}(\frac{1}{4}e^{2\tilde{x}-\frac{1}{2}\tilde{y}} + f(\tilde{y}))$
$\implies u = \frac{1}{4}e^{\frac{3}{2}\tilde{x}-\frac{1}{2}\tilde{y}} + f(\tilde{y})e^{-\frac{1}{2}\tilde{x}}$
Now, we need to re-write our equation in terms of x and y.
$u = \frac{1}{4}e^{\frac{3}{2}(x+y)-\frac{1}{2}(x-y)} + f(x-y)e^{-\frac{1}{2}(x+y)} = \frac{1}{4}e^{x+2y} + f(x-y)e^{-\frac{1}{2}(x+y)}$
Now, we need to apply our initial condition $u(x,0) = 0$
Thus, $0 = \frac{1}{4}e^{x} + f(x)e^{-\frac{1}{2}x}$
Therefore, $f(x) = -\frac{1}{4}e^{x} * e^{\frac{1}{2}x} = \frac{-1}{4}e^{\frac{3}{2}x}$
So, our final solution, plugging $f(x)$ into our general solution, is:

$$u(x,y) = \frac{1}{4}e^{x+2y} + -\frac{1}{4}e^{\frac{3}{2}(x-y)} * e^{-\frac{1}{2}(x+y)} = \frac{1}{4}e^{x+2y} - \frac{1}{4}e^{\frac{3}{2}(x-y)-\frac{1}{2}(x+y)} = \frac{1}{4}e^{x+2y} - \frac{1}{4}e^{x-2y} = \frac{1}{4}(e^{x+2y} - e^{x-2y})$$

$$u(x,y) = \frac{1}{4}(e^{x+2y} - e^{x-2y})$$