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Homework 3

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1 Notes for Week 3

V9 The advection equation

1. Deriving the equation
2. Graphing a solution

1. We know the conservation law is $u_t + \phi_x = 0$, where $u(x,t)$ is density and $\phi(x,t)$ is flux. So, if all particles are moving at a constant speed, what should the flux be? Let's think about the units of flux, g/ cm^2 s
$\implies$ This means the number of particles at any given time in a square centimeter as it moves through the tube, sort of like density over time
Flux is larger when there are more particles
So, we know that flux needs to be proportional to density if we assume speed is constant, ie $\phi(x,t) = cu$ which brings us to the transport equation:

$$u_t + cu_x = 0 \tag{1}$$

We know the general solution to this is $u(x,t) = f(x - ct)$

V10 The diffusion equation

We know we have density and flux in the transport equation, and that when they are proportional, its called **advection**
Now, this assumption is okay for something viscous like water, but really we want to introduce the concept of diffusion, where particles will be travelling from high concentration to low concentration.

How can we relate flux to density to describe diffusion?
We know that when flux is greater than 0, we are moving to the RIGHT, and that flux is g/cm^2
So, a greater difference in the number of particles between two x values implies a greater flux, since there are more particles moving across the x axis and therefore greater density. Thus, a small flux implies a smaller derivative of u and a smaller negative slope, and a larger flux implies a larger derivative and a larger negative slope
$\implies$ $\phi(x,t) = -ku_x(x,t)$ where k is our diffusion constant.
IMPORTANT k is negative because we know the derivative is negative but we want the flux to be a greater positive number for a greater flux, so we need to multiply by -1

Now, putting everything together we have

$$u_t - ku_{xx} = 0, \text{ with } k > 0 \tag{2}$$

As we can see this is very similar to the transport equation!

V11 Laplace Equations

1. Diffusion equation in more spacial dimensions
   We know that in one spacial dimension the diffusion equation is $u_t - ku_{xx} = 0$, so in two it is $u_t - k(u_{xx} + u_{yy}) = 0$
   generalizing, in more spacial dimensions the diffusion equation is $u_t - k\triangle u = 0$

2. **the laplace equation**
   A steady state is achieved when the solution *does not* depend on time, ie $u_t = 0$
   So, this gives us $\triangle u = 0$, since we can divide by k
   the above is known as the laplace equation

V12 Summary of Models

Classifying 2nd order PDEs
So far, we've done the following:

- advection: $u_t + cu_x = 0$ which is 1st order
- diffusion: $u_t - ku_{xx} = 0$ 2nd order and parabolic because it has algebraic cousin $y - x^2 = 1$
- laplace: $\triangle u = 0$ which is 2nd order and elliptic because it's algebraic cousin is $ay^2 + bx^2 = 1$
- wave: $u_{tt} - c^2 u_{xx} = 0$ which is hyperbolic because its algebraic cousin is $y^2 - x^2 = 1$

Diffusion, laplace, and wave are the big three 2nd order PDEs!

Theorem 1.1. *Any second order PDE is of the form:* $a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1 u_x + a_2 u_y + a_0 u = 0$

This theorem can be transformed into one of the following three cases, with $D = a_{12}^2 - a_{11}a_{22}$

- Elliptic if $D < 0$
- Hyperbolic if $D > 0$
- Parabolic if $D = 0$

2 Homework Problems

V9 Problems

2.

(a) **What are the units of c?**
The units of c are cm/s, which is simply the velocity of the particles. I arrived at this conclusion since we know that the units of flux are g/cm$^2\cdot$ sec, and the units of u are g/cm^3. We also know that flux is proportional to density, so their units must match. Thus, in order for the units to match in the relationship $\phi = cu$, c must have units cm/sec

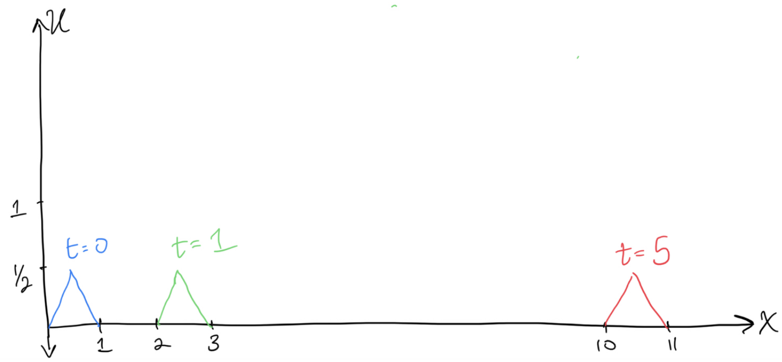
(b) First, we know that $u(x,t) = f(x-2t)$
Now, we want to solve for f for x between 0 and 1/2.
$u(x,0) = x \implies f(x) = x$
$\implies u(x,t) = x - 2t$, given $0 + 2t < x + 2t < 1/2 + 2t$

For $1/2 \leq x - 2t \leq 1$ we have $u(x,0) = f(x) = 1 - x \implies f(x) = 1 - x$
$\implies u(x,t) = 1 - x + 2t$ for $1/2 + 2t \leq x + 2t \leq 1 + 2t$

Finally, we have $u(x,0) = 0$ for all other values of x, $\implies u(x,0) = f(x) = 0$
$\implies u(x,t) = 0$ for all other values of $x - 2t$

(c) When $t = 0$, we have $u(x,0) = x$ when $0 < x < 1/2$, $u(x,0) = 1 - x$ when $1/2 \leq x \leq 1$ and $u(x,0) = 0$ for all other x
When $t = 1$, the corresponding $u(x,1)$ are $u(x,1) = x - 2$, $u(x,1) = -x + 1$ and $u(x,1) = 0$
When $t = 5$, the corresponding $u(x,5)$ are $u(x,5) = x - 10$, $u(x,5) = -x + 9$ and $u(x,5) = 0$

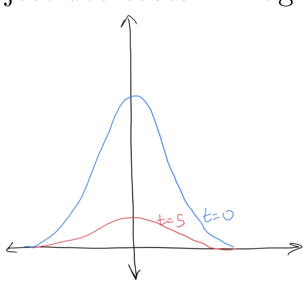
The graph for each of these values of t is displayed below:



(d) The 'peak' travelled 10 units, from $x = 0.5$ to $x = 10.5$, which is consistent with the fact that c is cm/sec, and we are considering a span of 5 seconds with $c = 2$

V10 Problems

3. Why is there a negative sign in front of k?
There is a negative sign because we know that the rate of change of flux is negative when particles are travelling from high concentration to low concentration. This corresponds to a negative slope–however, a greater flux will have a greater negative slope, and a smaller flux will have a smaller negative slope. Since we want the greater flux to have a greater value, not a small value (since greater negative value means it is less), we multiply by -1 so that a larger negative derivative corresponds to a greater flux.
4. Draw graph at t=5
This is the heat equation, so we know that the heat is going to dissipate through the object over time, flattening the curve of the graph as the derivative at each location throughout the object decreases in magnitude



V11 Problems

5. The steady state for $u_t - u_{xx} = 0$ is just $u_{xx} = 0$
6. Find the general solution

- We have $u_{xx} = 0$
- Therefore, $\int u_{xx} dx = \int 0 dt$ which is $u_x = f(t)$ where f is an arbitrary function of t
- Integrating again, we get that $\int u_x dt = \int f(t) dt \implies u(x,t) = x \cdot f(t) + g(t)$ where f, g are arbitrary functions of t

Therefore, $u(x,t) = x \cdot f(t) + g(t)$

V12 Problems

7. Classify $u_{xx} - 4u_{xy} + 4u_{yy} = 0$
We have the following, based on our 2nd order PDE theorem:

- $a_{11} = 1$
- $2a_{12} = -4 a_{12} = -2$
- $a_{22} = 4$
- Therefore, $D = (-2)^2 - 4 = 0$ so the equation is **parabolic**

Now, we want to verify that $u(x, y) = f(y + 2x) + xg(y + 2x)$ is a solution
We have that:
$u_{xx} = 4f_{xx}(y + 2x) + 4xg_{xx}(y + 2x) + 4g_x(y + 2x)$
$4u_{yy} = 4(f_{xx}(y + 2x) + xg_{xx}(y + 2x))$
$4u_{xy} = 4(2f_{xx}(y + 2x) + 2xg_{xx}(y + 2x) + g_x(y + 2x))$

Now, adding these together, we see that we have 8 positive f_{xx} terms in u_{xx} and $4u_{yy}$, 8 negative f_{xx} terms in $4u_{xy}$, and that all the other terms cancel in the same way, thus verifying that $u_{xx} - 4u_{xy} + 4u_{yy} = 0$ is satisfied by the given solution

8. Can you think of a situation where classifying 2nd order PDEs is useful?
In situations where the PDE cannot be solved analytically (or at least not easily), it would be useful to classify the PDE in order to understand the general behavior of the solution graph. For example, if we are trying to model something like population growth, the classification of the PDE will tell us whether the population reaches a carrying capacity.