Kim Hancock
Kayla Snyder

Group Assignment 9

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Desmos

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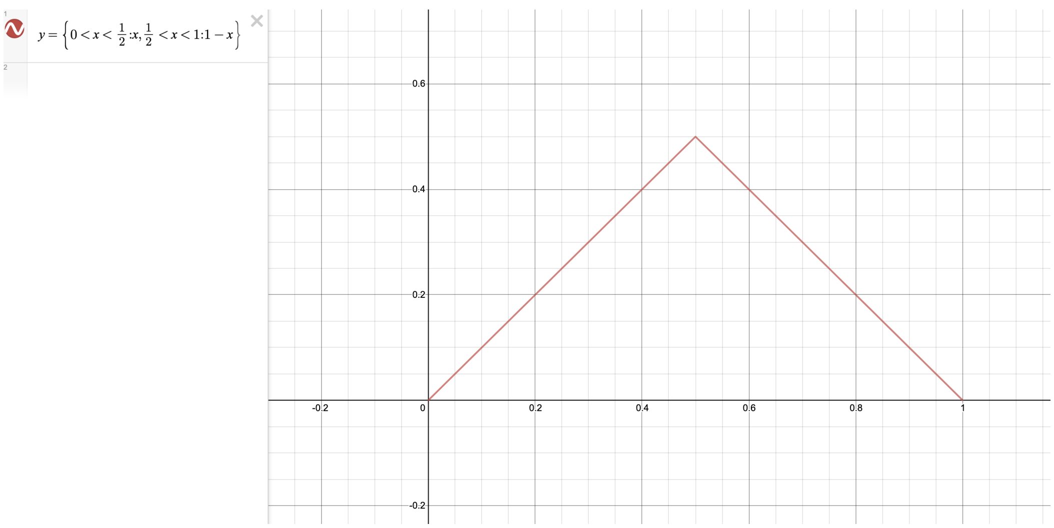
Consider the following problem:

$$u_t - 4u_{xx} = 0 \text{ for } 0 < x < 1$$

$$u(x,0) = \begin{cases} x & 0 < x < 1/2 \\ 1 - x & 1/2 < x < 1 \end{cases}$$

$$u(0,t) = 0, \ u(1,t) = 0$$

a) **Plot the initial condition**



b) **Find the solution of the PDE that satisfies that initial condition and boundary conditions**

(a) **Write down the ODE for $X(x)$ and $T(t)$**
We know that we want $u(x,t)$ to be in the form $u(x,t) = X(x)T(t)$

Since we have $u_t - 4u_{xx} = 0$, we know this equals $XT' - 4X''T = 0$ given that X and T only have one independent variable

Separating the X and T terms, we have $-\frac{T'}{4T} = -\frac{X''}{X}$ (the inclusion of the minus signs is arbitrary, and only with reference to the format of the textbook!)

We set $\lambda = -\frac{T'}{4T}$ which implies that $\frac{d\lambda}{dx} = 0$
We also let $\lambda = -\frac{X''}{X}$ which implies that $\frac{d\lambda}{dt} = 0$

Since both partial derivatives are 0, we know lambda must be constant and thus $\lambda = -\frac{T'}{4T} = -\frac{X''}{X}$

Then, we know that $X(x)$ ODE is $\lambda X = -X''$
Second, our $T(t)$ ODE is $T' = -4\lambda T$

(b) **Write down the eigenvalues and eigenfunctions for the X ODE**
We are considering our ODE $\lambda X = -X''$.

First, we will consider when $\lambda > 0$, so when $\lambda = \beta^2$ for some $\beta \in \mathbb{R}$

So, we have $\beta^2 X = -X''$, or $X'' + \beta^2 X = 0$

From ODEs, we know that the general solution to this ODE is $X(x) = C\cos(\beta x) + D\sin(\beta x)$ for some constants $C, D \in \mathbb{R}$

Now, we need to use our boundary conditions to find the arbitrary constants.

Recall $u(x,t) = X(x)T(t)$, so $u(0,t) = X(0)T(t) = 0$. This is only true if $X(0) = 0$, so we have one boundary condition for the X ODE.

Then, $u(1,t) = X(1)T(t) = 0$. This is only true if $X(1) = 0$, so we have our second boundary condition for the X ODE.

Now, let's plug in our boundary conditions.

$X(0) = C(1) + 0 = 0 \implies C = 0$
So, we now know $X(x) = D\sin(\beta x)$

We also know $X(1) = D\sin(\beta) = 0$

We don't want $D = 0$, or we would have the trivial solution $X(x) = 0$. So, we want to consider when the sin term is equal to 0. We know that $\sin(n\pi) = 0$ for any $n \in \mathbb{Z}$. So, we can define $\beta = n\pi$ for any $n \in \mathbb{Z}$

So, we know $X(x) = D\sin(n\pi x)$ for some arbitrary constant D. However, we know that any scalar multiple of an eigenfunction is also an eigenfunction, so we don't really need to include the constant. Also, we know that each eigenfunction is dependent on n, so we can re-write:

$X_n(x) = \sin(n\pi x)$ for $\lambda_n = (n\pi)^2$, since we know $\lambda = \beta^2$. These eigenfunctions and eigenvalues hold for $n = 1, 2, 3, ...$ We don't need to consider when n is negative, because we showed in problem 7b) on IHW7 that we can ignore them due to repeat solutions.

Now, we need to consider when $\lambda = 0$.

When $\lambda = 0$, we have the ODE $-X'' = 0$, or equivalently, $X'' = 0$. So, by doing some quick integration, we have $X_0(x) = c_1x + c_2$ for some constants $c_1, c_2 \in \mathbb{R}$.
Now, let's plug in our initial values.
$X_0(0) = c_2 = 0 \implies c_2 = 0$, so we now have $X_0(x) = c_1x$.
$X_0(1) = c_1 = 0 \implies c_1 = 0$, so we have $X_0(x) = 0$ for $\lambda = 0$

So, the only solution to $X_0(x)$ is the trivial solution. Since we know that the zero function is never an eigenfunction, we know that the $\lambda = 0$ also cannot be an eigenvalue!

Finally, we need to address when $\lambda < 0$, so $\lambda = -\beta^2$ for some $\beta \in \mathbb{R}$

So, we have the problem $-X'' = -\beta^2 X$, or equivalently, $X'' = \beta^2 X$

In problem 5 on IHW8, we showed that the only solution to this problem is the trivial solution $X(x) = 0$, and since we know that the zero function is never an eigenfunction, we know that there are no negative eigenvalues.

Therefore, in summary, we have:

$$X_n(x) = \sin(n\pi x) \text{ and } \lambda_n = (n\pi)^2 \text{ for } n = 1, 2, 3, ...$$

(c) **Find the general solution to the T(t) ODE**
We now look to solve $T' = -4\lambda T$, which is a first order ODE.
We can solve this ODE using separation of variables. We can re-write the ODE as $\frac{dT}{dt} = -4\lambda T$

Then, we can see $-4\lambda dt = \frac{dT}{T}$. Now, we should integrate both sides.
Then, we get that $\ln(T) = -4\lambda t + C$ for some arbitrary constant of integration C
Finally, we raise both sides to the power e and arrive at:
$T = e^{-4\lambda t + C}$
$\implies T = Ce^{-4\lambda t}$

$$T(t) = Ce^{-4\lambda t}$$

We know our values of λ from part b)! So we have:

$$T_n(t) = C_n e^{-(n\pi)^2 4t} \text{ for } n = 1, 2, 3...$$

(d) **Make the appropriate linear combination to obtain a series solution**

Now, taking a linear combination of solutions we get:

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin(n\pi x)\, e^{-(n\pi)^2 4t}$$

c) **Write down explicitly the first five terms of the series solution (which includes having explicit expressions for any Fourier coefficients)**

We now want to consider our boundary condition $\phi(x) = u(x,0)$

We know that $\phi(x) = \sum_{n=1}^{\infty} C_n \sin(n\pi x)$ since the exponential term in $u(x,t)$ becomes equal to 1 when we plug in 0 for t. We can also let $C_n = A_n$ since the letter is arbitrary.

Now, we know that this sum equals x if $0 < x < 1/2$:

$$x = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \text{ for } 0 < x < 1/2$$

The sum equals $1 - x$ if $1/2 < x < 1$

$$1 - x = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \text{ for } 1/2 < x < 1$$

We now define the function $X_m(x) = \sin(m\pi x)$. In video 36, we showed that for $X_m = \sin(m\pi x)$, $(X_m, X_m) \neq 0$ and $(X_m, X_n) = 0$ if $m \neq n$. Now, we can use the equation we found in video 36 to compute the Fourier coefficients A_m.

Thus, we can use the expression $A_m = \frac{2}{l}\int_0^l \phi(x) X_m dx$

So, we have $A_m = \frac{(\phi(x), X_m)}{(X_m, X_m)} = 2\int_0^1 \phi(x)\sin(m\pi x)dx$

We now have to break this up into two integrals because we are dealing with a piece-wise function. So, we have:

- $A_m = 2\int_0^{1/2} x \sin(m\pi x)dx$

  We can use integration by parts, letting $u = x$ and $dv = \sin(m\pi x)dx$

  Therefore, we also have $du = dx$ and $v = -\frac{1}{m\pi}\cos(m\pi x)$

Thus, we have $2\int_0^{1/2} x\sin(m\pi x)dx = 2\left[-\frac{x}{m\pi}\cos(m\pi x) + \frac{1}{m\pi}\int_0^{1/2}\cos(m\pi x)dx\right]\Big|_0^{1/2}$

$= 2\left[-\frac{x}{m\pi}\cos(m\pi x) + \frac{1}{m^2\pi^2}\sin(m\pi x)\right]\Big|_0^{1/2}$

$= -\frac{2x}{m\pi}\cos(m\pi x) + \frac{2}{m^2\pi^2}\sin(m\pi x)\Big|_0^{1/2}$

$= -\frac{1}{m\pi}\cos(\frac{m\pi}{2}) + \frac{2}{m^2\pi^2}\sin(\frac{m\pi}{2}) + 0 - 0$

$A_m = -\frac{1}{m\pi}\cos(\frac{m\pi}{2}) + \frac{2}{m^2\pi^2}\sin(\frac{m\pi}{2})$

Now, when m mod $4 = 0$, the cosine term is 1 and the sine term is 0
$\implies A_m = -\frac{1}{m\pi}$

When m mod $4 = 2$, we know the cosine term is -1 and the sine term is 0
$\implies A_m = \frac{1}{m\pi}$

When m mod $4 = 3$, we know the sine term is -1 and the cosine term is 0
$\implies A_m = -\frac{2}{m^2\pi^2}$

When m mod $4 = 1$, we know the sine term is 1 and the cosine term is 0
$\implies A_m = \frac{2}{m^2\pi^2}$

- $A_m = 2\int_{1/2}^1 (1-x)\sin(m\pi x)dx$

Let's solve by integration by parts. Let $u = 1 - x$ and $dv = \sin(m\pi x)dx$, so therefore $du = -dx$ and $v = -\frac{1}{m\pi}\cos(m\pi x)$

Thus, we have $2\int_{1/2}^1 (1-x)\sin(m\pi x)dx = 2\left[\frac{x-1}{m\pi}\cos(m\pi x) - \frac{1}{m\pi}\int_{1/2}^1\cos(m\pi x)dx\right]\Big|_{1/2}^1$

$= \frac{2(x-1)}{m\pi}\cos(m\pi x) - \frac{2}{m^2\pi^2}\sin(m\pi x)\Big|_{1/2}^1$

$= 0 - \frac{2}{m^2\pi^2}\sin(0) + \frac{1}{m\pi}\cos(\frac{m\pi}{2}) + \frac{2}{m^2\pi^2}\sin(\frac{m\pi}{2})$

$A_m = \frac{1}{m\pi}\cos(\frac{m\pi}{2}) + \frac{2}{m^2\pi^2}\sin(\frac{m\pi}{2})$

Now, when m mod $4 = 0$, the cosine term is 1 and the sine term is 0
$\implies A_m = \frac{1}{m\pi}$

When m mod $4 = 2$, we know the cosine term is -1 and the sine term is 0
$\implies A_m = -\frac{1}{m\pi}$

When $m \bmod 4 = 3$, we know the sine term is -1 and the cosine term is 0
$\implies A_m = -\frac{2}{m^2\pi^2}$

When $m \bmod 4 = 1$, we know the sine term is 1 and the cosine term is 0
$\implies A_m = \frac{2}{m^2\pi^2}$

Now, finally, we can begin to write the first five terms of the series solution.

Recall our series solution $u(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)\, e^{-(n\pi)^2 4t}$. Now, we can plug in values of A_n!

For $0 < x < 1/2$:

$$u(x,t) = \frac{2}{\pi^2}\sin(\pi x)e^{-4(\pi)^2 t} + \frac{1}{2\pi}\sin(2\pi x)e^{-16(\pi)^2 t} - \frac{2}{9\pi^2}\sin(3\pi x)e^{-36(\pi)^2 t} - \frac{1}{4\pi}\sin(4\pi x)e^{-64(\pi)^2 t} + \frac{2}{25\pi^2}\sin(5\pi x)e^{-100(\pi)^2 t} + \ldots$$

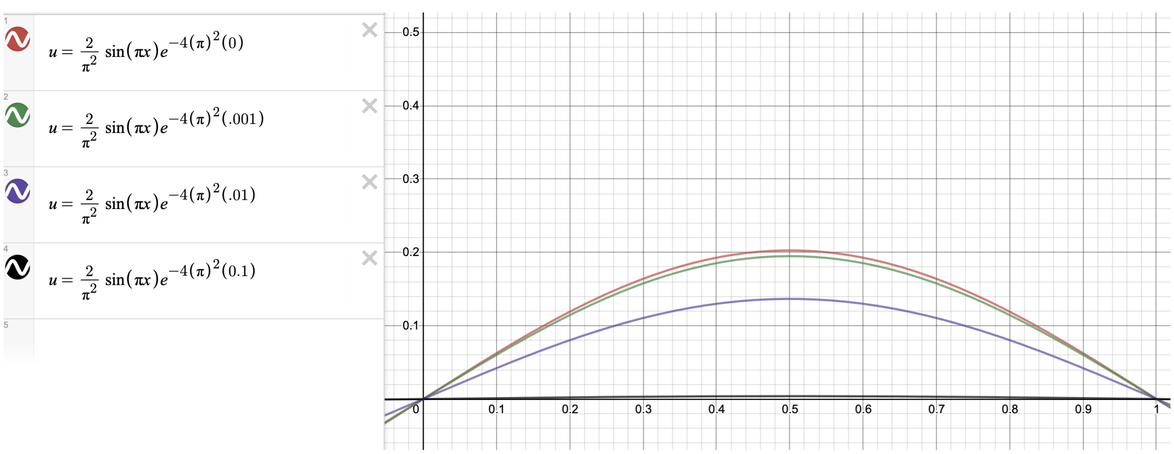
For $1/2 < x < 1$:

$$u(x,t) = \frac{2}{\pi^2}\sin(\pi x)e^{-4(\pi)^2 t} - \frac{1}{2\pi}\sin(2\pi x)e^{-16(\pi)^2 t} - \frac{2}{9\pi^2}\sin(3\pi x)e^{-36(\pi)^2 t} + \frac{1}{4\pi}\sin(4\pi x)e^{-64(\pi)^2 t} + \frac{2}{25\pi^2}\sin(5\pi x)e^{-100(\pi)^2 t} - \ldots$$

d) **Argue why the first term in your solution is the most dominant**

We can see that the exponential gets increasingly smaller as n increases, and thus each term in the series solution gets smaller and smaller. In other words, as n increases, the terms become negligible.

e) **Plot the first term of your solution for $t = 0$, $t = 0.001$, $t = 0.01$ and $t = 0.1$.**



References