[redacted]n, Desmos for graphs

(3) a. Initially, the heat is distributed linearly through the rod, with the temperature increasing as x does. There is no temperature change at the ends of the rod.

b. We know from V39 that the series solution to the bounded diffusion equation with Neumann b.c. is:

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{\ell}\right)^2 kt} \cos\left(\frac{n\pi x}{\ell}\right)$$

Applying our ℓ we get: $u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-(n\pi)^2 kt} \cos(n\pi x)$

The initial value is $u(x,0) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x) = x$

This is the same formula as the Fourier cosine series, so we can use this to determine the coefficients.

On HW I9 #5c, we saw the coefficients of the Fourier cosine series are given by: $A_m = \frac{2}{\ell}\int_0^\ell \phi(x)\cos\left(\frac{m\pi x}{\ell}\right)dx \qquad m = 0, 1, 2, \ldots$

Now, we integrate separately for $m=0$ and $m>0$ using our ℓ and $\phi(x)$

$A_m = 2\int_0^1 \underbrace{x}_{u} \underbrace{\cos(m\pi x)\,dx}_{dv}$ $\quad$ integ. by parts $\quad$ $u = x \quad v = \frac{1}{m\pi}\sin(m\pi x)$; $\quad du = dx \quad dv = \cos(m\pi x)$

$A_m = 2\left[x \cdot \sin(m\pi x) \cdot \frac{1}{m\pi}\Big|_0^1 - \int_0^1 \frac{1}{m\pi}\sin(m\pi x)\,dx\right]$

$= 2\left[\overset{0}{\cancel{\sin(m\pi)}} \cdot \frac{1}{m\pi} - 0 - \left[-\frac{1}{(m\pi)^2}\cos(m\pi x)\right]_0^1\right]$

$= 2\left[-\left(-\frac{1}{(m\pi)^2}\cos(m\pi) + \frac{1}{(m\pi)^2}\cos(0)\right)\right] = \frac{2}{(m\pi)^2}\left(\cos(m\pi) - 1\right) = \frac{2}{(m\pi)^2}\left[(-1)^m - 1\right]$

~~$A_m = 0$ when m even~~

~~$A_m = \frac{-4}{(m\pi)^2}$ when m odd~~

$A_0 = 2\int_0^1 x \cdot \cos(0)\,dx = 2\int_0^1 x\,dx = 2\left[\frac{1}{2}x^2\right]_0^1 = 2\left(\frac{1}{2}\right) - 2(0) = \boxed{1}$

Finally, plug in to series solution:

$$u(x,t) = \frac{1}{2}(1) + \sum_{n=1}^{\infty} \frac{2}{(n\pi)^2}\left[(-1)^n - 1\right] e^{-(n\pi)^2 kt} \cos(n\pi x)$$

$$u(x,t) = \frac{1}{2} + \frac{-4}{(\pi)^2} e^{-(\pi)^2 kt} \cos(\pi x) + \frac{-4}{(3\pi)^2} e^{-(3\pi)^2 kt} \cos(3\pi x) + \frac{-4}{(5\pi)^2} e^{-(5\pi)^2 kt} \cos(5\pi x) + \ldots$$

(3) c. The first series term is dominant because it has the smallest negative exponent and the largest constant scalar. The denominator of its constant is not scaled, allowing that term to be more significant. Additionally, a smaller negative exponent on the e means that term can contribute more to the solution without being essentially zero.

e. The heat is contained in the rod, and it flows from the x=l position to x=0 in order to equilibrate. This decreases the temperature difference between the ends of the rod, and will continue until the rod is at one uniform temperature.

(4) Split top and bottom of fraction, evaluate, and put back together:

Top: $\int_0^{l} \underbrace{x}_{u} \underbrace{\sin\left(\frac{(2n-1)\pi x}{2l}\right) dx}_{dv}$

$u = x \qquad v = -\cos\left(\frac{(2n-1)\pi x}{2l}\right)\cdot\frac{2l}{(2n-1)\pi}$

$du = dx \qquad dv = \sin\left(\frac{(2n-1)\pi x}{2l}\right) dx$

$= \left[x\cdot -\cos\left(\frac{(2n-1)\pi x}{2l}\right)\cdot\frac{2l}{(2n-1)\pi}\right]_0^{l} - \int_0^{l} \frac{-2l}{(2n-1)\pi}\cos\left(\frac{(2n-1)\pi x}{2l}\right) dx$

$= \frac{-2l^2}{(2n-1)\pi}\cos\left(\frac{(2n-1)\pi}{2}\right) - 0 - \left[-\left(\frac{2l}{(2n-1)\pi}\right)^2 \sin\left(\frac{(2n-1)\pi x}{2l}\right)\right]_0^{l}$

$= \cancel{\frac{-2l^2}{(2n-1)\pi}\cos\left(\frac{(2n-1)\pi}{2}\right)}^{0} - \left(-\left(\frac{2l}{(2n-1)\pi}\right)^2 \sin\left(\frac{(2n-1)\pi}{2}\right) + 0\right) = \frac{4l^2}{(2n-1)^2\pi^2}\underbrace{\sin\left(\frac{(2n-1)\pi}{2}\right)}_{=(-1)^{n+1}}$

Bottom: $\int_0^{l} \sin\left(\frac{(2n-1)\pi x}{2l}\right)\sin\left(\frac{(2n-1)\pi x}{2l}\right) dx$

~~$\sin a \sin b$~~ $= \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b)$

$= \int_0^{l} \frac{1}{2}\cos\left(\frac{(2n-1)\pi x}{2l} - \frac{(2n-1)\pi x}{2l}\right) - \frac{1}{2}\cos\left(\frac{(2n-1)\pi x}{2l} + \frac{(2n-1)\pi x}{2l}\right) dx$

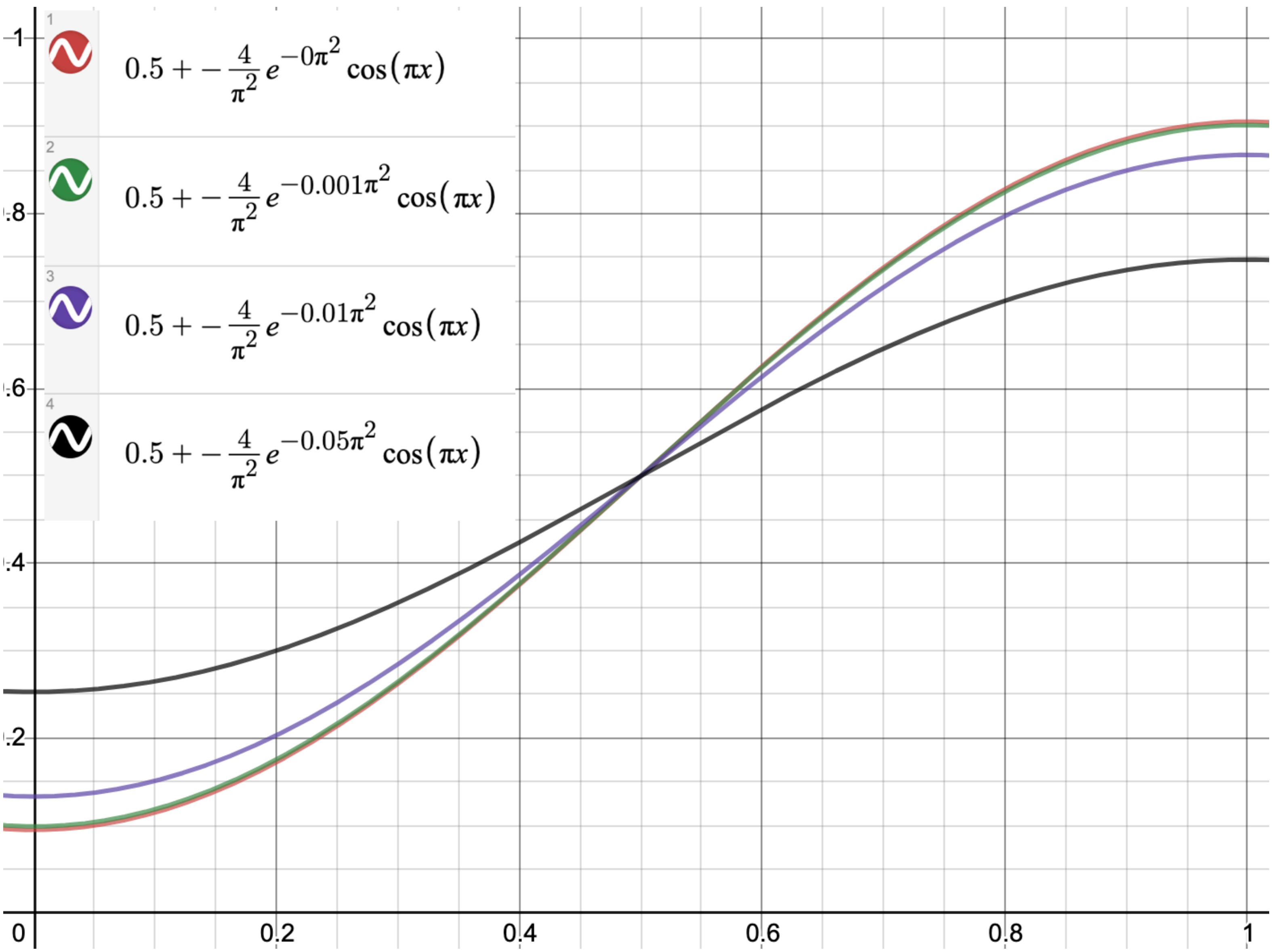
$= \int_0^{l} \frac{1}{2} - \frac{1}{2}\cos\left(\frac{(2n-1)\pi x}{l}\right) dx = \left[\frac{1}{2}x - \frac{1}{2}\sin\left(\frac{(2n-1)\pi x}{l}\right)\cdot\frac{l}{(2n-1)\pi}\right]_0^{l}$

$= \frac{1}{2}l - \frac{1}{2}\cancel{\sin((2n-1)\pi)}^{0}\cdot\frac{l}{(2n-1)\pi} - 0 + \frac{1}{2}\cancel{\sin(0)}^{0}\cdot\frac{l}{(2n-1)\pi}$

$=0$ b/c $(2n-1)$ is an integer

$= \frac{1}{2}l$

Together: $\dfrac{\frac{4l^2}{(2n-1)^2\pi^2}\cdot(-1)^{n+1}}{\frac{1}{2}l} = \boxed{\frac{8l}{(2n-1)^2\pi^2}(-1)^{n+1} \quad n=1,2,3,\ldots}$



1
$0.5+-\frac{4}{\pi^2}e^{-0\pi^2}\cos(\pi x)$
2
$0.5+-\frac{4}{\pi^2}e^{-0.001\pi^2}\cos(\pi x)$
3
$0.5+-\frac{4}{\pi^2}e^{-0.01\pi^2}\cos(\pi x)$
4
$0.5+-\frac{4}{\pi^2}e^{-0.05\pi^2}\cos(\pi x)$
1
.8
.6
.4
.2
0
0.2
0.4
0.6
0.8
1

⑤ a. Initially, the heat is distributed linearly, with temperature increasing with x. The left end of the rod is fixed cold (heat = 0), while the right end sees no temperature change.

b. This problem is the same as group HW #8, with k instead of 4 in the PDE and $l=1$ instead of 2.

From this, we know $\beta = \frac{(2n-1)\pi}{2}$ for $n=1,2,3,\ldots$

and $X_n(x) = \sin\left(\frac{(2n-1)\pi x}{2}\right)$ for $n=1,2,3,\ldots$

Applying k instead of 4 when solving for T gives

$$T_n(t) = A_n e^{-\left(\frac{(2n-1)\pi}{2}\right)^2 kt} \quad \text{for } n=1,2,3,\ldots$$

So the general series solution with $l=1$ is:

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{(2n-1)\pi x}{2}\right) e^{-\left(\frac{(2n-1)\pi}{2}\right)^2 kt}$$

The initial value is: $u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{(2n-1)\pi x}{2}\right) = x$

To find A_n for this series we use the formula from V35:

$$A_n = \frac{(\phi, X_n)}{(X_n, X_n)}$$

Here, $\phi = x$ and $X_n = \sin\left(\frac{(2n-1)\pi x}{2}\right)$, so we get:

$$A_n = \frac{\int_0^1 x \sin\left(\frac{(2n-1)\pi x}{2}\right) dx}{\int_0^1 \sin\left(\frac{(2n-1)\pi x}{2}\right) \sin\left(\frac{(2n-1)\pi x}{2}\right) dx}$$

Then, by #4, $A_n = \frac{8}{\pi^2 (2n-1)^2} (-1)^{n+1}$ $n=1,2,3,\ldots$

Finally, we plug back in to the series solution and get:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8}{\pi^2(2n-1)^2} (-1)^{n+1} \sin\left(\frac{(2n-1)\pi x}{2}\right) e^{-\left(\frac{(2n-1)\pi}{2}\right)^2 kt}$$

$$u(x,t) = \frac{8}{\pi^2} \sin\left(\frac{\pi x}{2}\right) e^{-\left(\frac{\pi}{2}\right)^2 kt} + \frac{-8}{9\pi^2} \sin\left(\frac{3\pi x}{2}\right) e^{-\left(\frac{3\pi}{2}\right)^2 kt} + \frac{8}{25\pi^2} \sin\left(\frac{5\pi x}{2}\right) e^{-\left(\frac{5\pi}{2}\right)^2 kt} + \ldots$$

5) c. Very similar to problem 3, the first term dominates because of its smaller negative exponent and larger constant. Both of these create smaller denominator values which increase the term's contribution to the solution.

e. The left end of the rod remains fixed at 0 for all t, but heat dissipates from the right end of the rod as t increases. This works to decrease the temperature difference between the ends of the rod.

6) $\frac{d}{dx}\left[-X_m' X_n + X_m X_n'\right] = -X_m'' X_n - \cancel{X_m' X_n'} + \cancel{X_m' X_n'} + X_m X_n''$

$= \boxed{-X_m'' X_n + X_m X_n''}$ ✓

7) Show $-X_m'(\ell) X_n(\ell) + X_m(\ell) X_n'(\ell) - [-X_m'(0) X_n(0) + X_m(0) X_n'(0)] = 0$

Given boundary conditions:

$-X_m'(\ell)\cdot 0 + 0 \cdot X_n'(\ell) - [-X_m'(0)\cdot 0 + 0 \cdot X_n'(0)]$

$= 0 + 0 + 0 - 0 = \boxed{0}$ ✓

8) Show $-X_m'(\ell) X_n(\ell) + X_m(\ell) X_n'(\ell) - [-X_m'(0) X_n(0) + X_m(0) X_n'(0)] = 0$

Given boundary conditions:

$0 \cdot X_n(\ell) + X_m(\ell)\cdot 0 - [0 \cdot X_n(0) + X_m(0)\cdot 0]$

$= 0 + 0 - 0 - 0 = \boxed{0}$ ✓

9) This problem is very similar to videos 25-29, with a slightly different boundary condition.

The general solution for the X ODE is $X(x) = C\cos(\beta x) + D\sin(\beta x)$

Using our boundary conditions, we know:

$u(0,t) = X(0)T(t) = 0$, so $X(0) = 0$

$u_x(\ell,t) = X'(\ell)T(t) = 0$, so $X'(\ell) = 0$

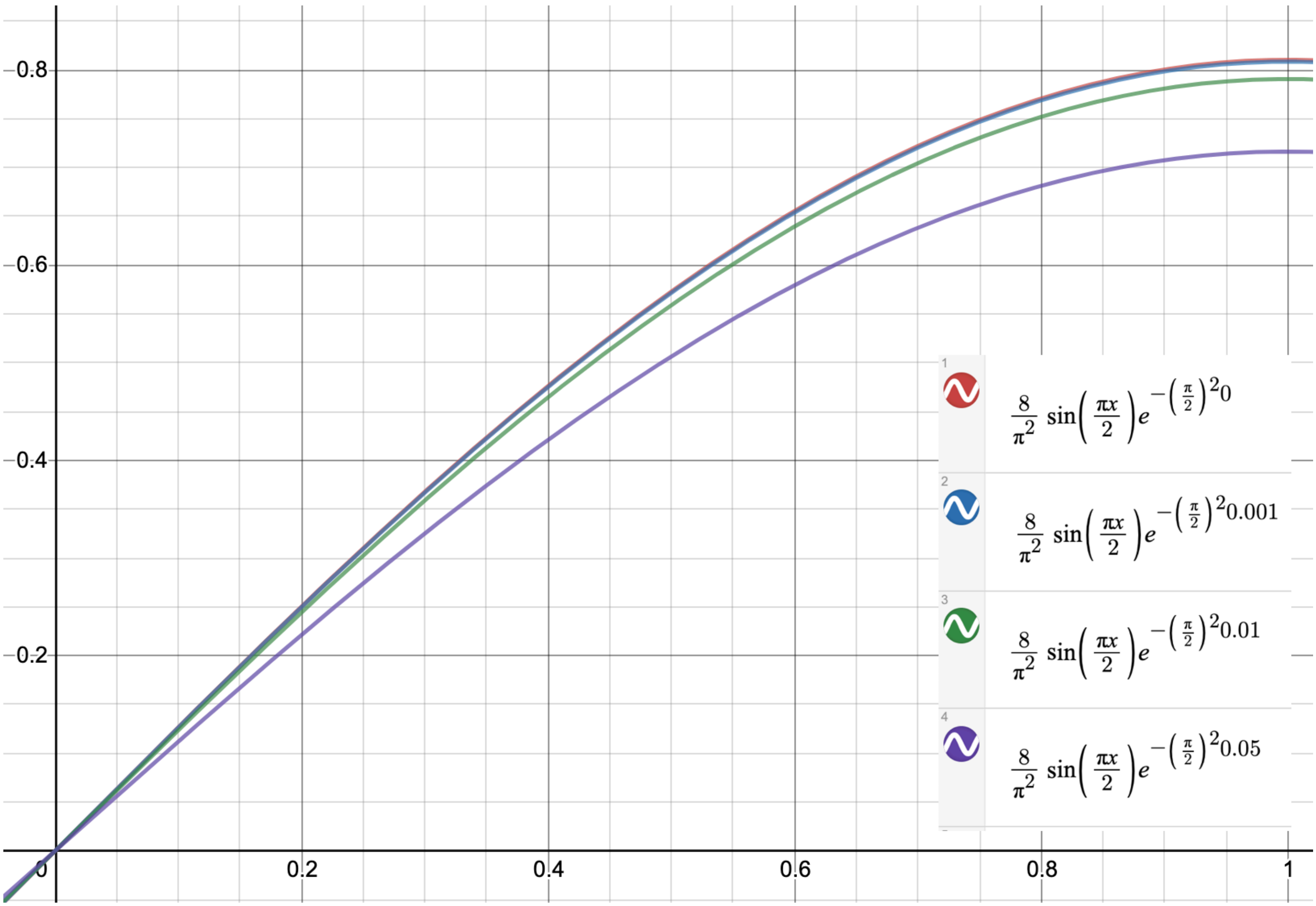
Then, $X(0) = C\cos(0) + D\cancel{\sin(0)} = 0$, $C = 0$

And $X'(x) = -\beta C\sin(\beta x) + \beta D\cos(\beta x)$

$X'(\ell) = 0 + \beta D\cos(\beta\ell) = 0$, so $\cos(\beta\ell) = 0$ → $\beta\ell = \frac{(2n-1)\pi}{2}$

$\beta = \frac{(2n-1)\pi}{2\ell}$ w/ $n = 1, 2, 3, \ldots$

So $X_n(x) = D_n \sin\left(\frac{(2n-1)\pi x}{2\ell}\right)$


0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1
$\frac{8}{\pi^2}\sin\left(\frac{\pi x}{2}\right)e^{-\left(\frac{\pi}{2}\right)^2 0}$
$\frac{8}{\pi^2}\sin\left(\frac{\pi x}{2}\right)e^{-\left(\frac{\pi}{2}\right)^2 0.001}$
$\frac{8}{\pi^2}\sin\left(\frac{\pi x}{2}\right)e^{-\left(\frac{\pi}{2}\right)^2 0.01}$
$\frac{8}{\pi^2}\sin\left(\frac{\pi x}{2}\right)e^{-\left(\frac{\pi}{2}\right)^2 0.05}$

⑨ cont. The general solution to the T ODE is also the same as the videos
$T(t) = A\cos(\beta c t) + B\sin(\beta c t)$

With our β, this yields: $T_n(t) = A_n\cos\left(\frac{(2n-1)\pi c t}{2l}\right) + B_n\sin\left(\frac{(2n-1)\pi c t}{2l}\right)$ w/ $n = 1, 2, \ldots$

The series Solution is then:

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{(2n-1)\pi x}{2l}\right)\left[A_n\cos\left(\frac{(2n-1)\pi c t}{2l}\right) + B_n\sin\left(\frac{(2n-1)\pi c t}{2l}\right)\right]$$

D_n can be dropped because the scalar of an eigenfunction is an eigenfunction

Because we're interested in harmonics, let's investigate the $u_t(x,0) = 0$ initial condition.

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{(2n-1)\pi x}{2l}\right)\left[-A_n\sin\left(\frac{(2n-1)\pi c t}{2l}\right)\cdot\frac{(2n-1)\pi c}{2l} + B_n\cos\left(\frac{(2n-1)\pi c t}{2l}\right)\cdot\frac{(2n-1)\pi c}{2l}\right]$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \sin\left(\frac{(2n-1)\pi x}{2l}\right)\left[0 + B_n\frac{(2n-1)\pi c}{2l}\right] = 0$$

Notice here that the only way for this sum to $= 0$ is if $B_n = 0$. This is very similar to the example seen at the top of page 111 in the book. The constant and sine term do not equal 0, or else $u = 0$. It is true that the sine term alternates in sign and so could cancel to 0, but the presence of the constant means that these opposite values will be differently scaled for each value of n. Hence, $B_n = 0$.

Now, our series solution is

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{(2n-1)\pi x}{2l}\right) A_n\cos\left(\frac{(2n-1)\pi c t}{2l}\right)$$

The frequency of the fundamental note is $\frac{\pi\sqrt{T}}{2l\sqrt{\rho}}$ where $c = \sqrt{T/\rho}$

The frequency of the first overtone is $\frac{3\pi\sqrt{T}}{2l\sqrt{\rho}}$

These values come from the t coefficient in the cosine term. Because of the $(2n-1)$, all overtones will be odd multiples of the fundamental note. Therefore, this clarinet cannot generate even harmonics.