Math 3209 - Fall 2020
Partial Differential Equations - Individual Homework i9

Please include LOTS of detail in your solutions!!!!!

Easy point problems :-)

1. If you used any sources other than the book, the videos, discussions with me, or your brain, please include them here. Examples include other people you talked to in class, or web pages. If not applicable, simply write NA.

2. Include pictures of your notes from the video lectures from this week. You will get credit just for submitting these.

Problems for video lecture V39: Neumann BC and Fourier Cosine Series

3. Consider the following PDE and auxiliary conditions

$$\begin{aligned} u_t - ku_{xx} &= 0 \quad \text{for } 0 < x < \ell \\ u(x,0) &= x \\ u_x(0,t) = 0, &\qquad u_x(1,t) = 0 \end{aligned} \tag{1}$$

   (a) Describe in words what this problem is describing in the context of heat in a rod
   (b) Find the solution of this PDE using separation of variables, Fourier series stuff, etc. You can refer to any result in the book, the videos, or old HW, but **be specific about your reference** (e.g. we know from Video XX that, or we know from page XX in the book that)
   (c) Argue why the first term in your solution is the most dominant.
   (d) Using $k = 1$ and $\ell = 1$, plot the sum of first and second term of your solution for $t = 0$, $t = 0.001$, $t = 0.01$ and $t = 0.05$ on the domain $0 < x < 1$
   (e) Using your answers above, describe in words what the model predicts about the heat flow in the rod as time evolves.

Problems for video lecture V40: Mixed BC and General Fourier Series

4. In the video, we found a formula for the Fourier coefficients that corresponded to a particular type of mixed boundary condition. Now you should simplify it. Show that:

$$\frac{\int_0^\ell x \sin\frac{(2n-1)\pi x}{2\ell}\,dx}{\int_0^\ell \sin\frac{(2n-1)\pi x}{2\ell}\sin\frac{(2n-1)\pi x}{2\ell}\,dx} = \frac{8l}{\pi^2(2n-1)^2}(-1)^{n+1}, \qquad n = 1, 2, 3, \ldots$$

   showing all steps of your work along the way. Hint: The approach to evaluate the integrals appearing here is the same as those described in earlier videos and the book, see page 105 and 109 for example.

5. Consider the following PDE and auxiliary conditions

$$\begin{aligned} u_t - ku_{xx} &= 0 \quad \text{for } 0 < x < \ell \\ u(x,0) &= x \\ u(0,t) = 0, &\qquad u_x(1,t) = 0 \end{aligned} \tag{2}$$

   (a) Describe in words what this problem is describing in the context of heat in a rod (notice this problem is similar but different than problem 3).
   (b) Find the solution of this PDE using separation of variables, Fourier series stuff, etc. You can refer to any result in the book, the videos, or old HW, but **be specific about your reference** (e.g. we know from Video XX that, or we know from page XX in the book that). In particular, you will want to use the result of the previous problem, 4
   (c) Argue why the first term in your solution is the most dominant.
   (d) Using $k = 1$ and $\ell = 1$, plot the first term of your solution for $t = 0$, $t = 0.001$, $t = 0.01$ and $t = 0.05$ on the domain $0 < x < 1$
   (e) Using your answers above, describe in words what the model predicts about the heat flow in the rod as time evolves.

Problems continue on next page!

Problems for video lecture V41: Orthogonality and General Fourier Series

6. Verify the following assertion made in the video:

$$-X_m''X_n + X_mX_n'' = \frac{d}{dx}\left[-X_m'X_n + X_mX_n'\right]$$

7. Using the formula derived in the video, show that eigenfunctions of the eigenvalue problem with Dirichlet boundary conditions

$$-X'' = \lambda X \qquad X(0) = X(\ell) = 0$$

are mutually orthogonal. You may assume that all eigenvalues are distinct

8. Using the formula derived in the video, show that eigenfunctions of the eigenvalue problem with Neumann boundary conditions

$$-X'' = \lambda X \qquad X'(0) = X'(\ell) = 0$$

are mutually orthogonal. You may assume that all eigenvalues are distinct

Problems for video lecture V42: PDEs and sound (again!)

9. Suppose that the pressure waves (i.e. sound waves) a clarinet produces is described by the wave equation with mixed boundary conditions:

$$\begin{aligned} u_{tt} - c^2u_{xx} = 0 &\qquad \text{for } 0 < x < \ell \\ u(x,0) = \phi(x), &\qquad u_t(x,0) = 0 \\ u(0,t) = 0, &\qquad u_x(\ell,t) = 0 \end{aligned} \tag{3}$$

Show that this clarinet cannot generate even harmonics (that is, even multiples of the fundamental note). The absence of even harmonics is responsible for the dark or eerier sound of the clarinet. This is in contrast to the saxophone, whose geometry allows the generation of even harmonics too, and hence a "warmer" sound.