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Group Assignment 5

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No outside sources other than hw and class notes :)

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a) **Make a change of the spatial variable to be in a frame that is "moving" at the same speed as the water. That is, define $z = x - 2t$ and reformulate the PDE with independent variables z and t.**

- We wish to replace x with $z = x - 2t$ and replace t with $T = t$
- Now, we need to rewrite our PDE wit these changed variables such that this equation is true:
  $u_t(x-2t,t) + 2u_x(x-2t,t) = u_{xx}(x-2t,t)$
- Taking the change of variables into account, we then have:
  $u_t(z,T) + 2u_x(z,T) = u_{xx}(z,T)$
- Now, we can use the chain rule to write:
  $\frac{\partial}{\partial t}u(z,T) = \frac{\partial u}{\partial z}\frac{dz}{dt} + \frac{\partial u}{\partial T}\frac{dT}{dt}$

  $\frac{\partial}{\partial x}u(z,T) = \frac{\partial u}{\partial z}\frac{dz}{dx} + \frac{\partial u}{\partial T}\frac{dT}{dx} = \frac{\partial u}{\partial z}\frac{dz}{dx}$ (since T does not depend on x and thus $\frac{dT}{dx} = 0$)

  $\frac{\partial^2}{\partial x^2}u(z,T) = \frac{\partial u}{\partial z}\frac{dz}{dt} \cdot \frac{\partial u}{\partial z}\frac{dz}{dt}$
- We now note that $\frac{dz}{dt} = -2$, $\frac{dz}{dx} = 1$, and $\frac{dT}{dt} = 1$

  Simplifying, this gives us:
  $u_T = \frac{\partial u}{\partial z}(-2) + \frac{\partial u}{\partial T}(1)$
  $2u_x = \frac{\partial u}{\partial z}(2)$
  $u_{xx} = \frac{\partial^2 u}{\partial z^2}$
- Therefore, we can re-write our equation:
  $u_T(z,T) + 2u_x(z,T) = u_{xx}(z,T)$ as $-2u_z + u_T + 2u_z = u_{zz} \implies u_T - u_{zz} = 0$
- Finally, we see from the above that $u_T = u_{zz}$ is our PDE in terms of z and T. Let's note that this is now in the recognizable form of our diffusion equation which we know how to solve!

b) At this point, we can solve the PDE using the same method as HW problem 9, using the error function.

- We first note that when $-1 < x < 1$, we have that $\phi(x) = 1$. Therefore, for all other values of x, the integral would be equal to 0. So, we can change the bounds of our integral.

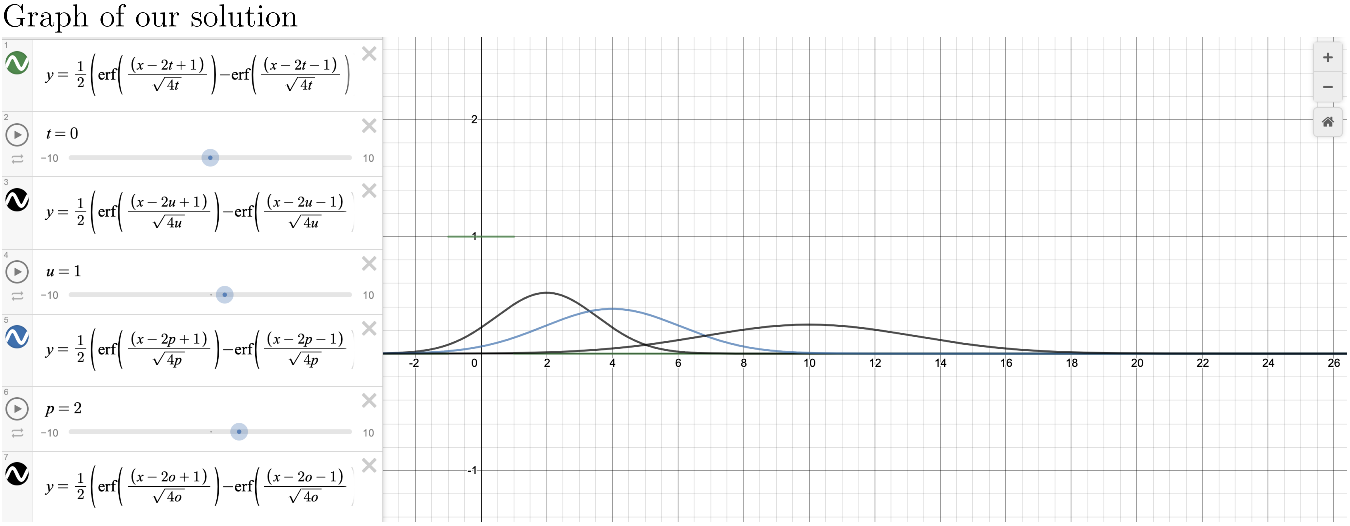
- We can also note that in this PDE, $k = 1$.
- This implies that our solution is: $u(z,T) = \frac{1}{\sqrt{4\pi T}} \int_{-1}^{1} e^{-(z-y)^2/4T} dy$
- Now, let's do a change of variables and let $p(y) = \frac{z-y}{\sqrt{4T}}$ which implies that $p(1) = \frac{z-1}{\sqrt{4T}}$ and $p(-1) = \frac{z+1}{\sqrt{4T}}$
- We also know that $dp = \frac{-dy}{\sqrt{4T}}$
- Thus, we know that $u = \frac{-1}{\sqrt{\pi}} \int_{\frac{z+1}{\sqrt{4T}}}^{\frac{z-1}{\sqrt{4T}}} e^{-p^2} dp$
- Flipping the order of integration, we get:
  $u = \frac{1}{\sqrt{\pi}} \int_{\frac{z-1}{\sqrt{4T}}}^{\frac{z+1}{\sqrt{4T}}} e^{-p^2} dp$
- We can break this into two integrals now, ie:
  $u = \frac{-1}{\sqrt{\pi}} \int_0^{\frac{z-1}{\sqrt{4T}}} e^{-p^2} dp + \frac{1}{\sqrt{\pi}} \int_0^{\frac{z+1}{\sqrt{4T}}} e^{-p^2} dp$
- Finally, we can use our error function in place of the integrals to give us our final answer in terms of z:

$$u(z,T) = -\frac{1}{2} erf\left(\frac{z-1}{\sqrt{4T}}\right) + \frac{1}{2} erf\left(\frac{z+1}{\sqrt{4T}}\right) = \frac{1}{2}\left(erf\left(\frac{z+1}{\sqrt{4T}}\right) - erf\left(\frac{z-1}{\sqrt{4T}}\right)\right)$$

c)
$$u(x,t) = \frac{1}{2}\left(erf\left(\frac{x-2t+1}{\sqrt{4t}}\right) - erf\left(\frac{x-2t-1}{\sqrt{4t}}\right)\right)$$

by replacing z with its known value, $z = x - 2t$ and replacing T with its known value, $T = t$.

d) Graph of our solution



e) As the values of t increase, we can see the curve growing flatter and wider, which is reflective of diffusion. Simultaneously, the peak of the curve is travelling along the x-axis in the positive direction as t increases. Since advection represents the flow and movement of a substance, this curve demonstrates this movement through the travelling of the peak. This is represented in our solution as the numerator changes with respect to t, which we do not see in solutions to the diffusion equation (without advection). This dependence on t effectively shifts the x variable (horizontal shift!) that is also in the numerator within the erf(!) function.