Kim Hancock

Homework 6

October 14, 2020

1 Resources

Other than the textbook and class notes, nothing

2 Notes for Week 6

V20 Well Posed Problems

A PDE is well-posed if:

1. **Existence** $\implies$ there exists at least one solution satisfying all the auxiliary conditions
2. **Uniqueness** $\implies$ there is at MOST 1 solution
3. **Stability** $\implies$ the unique solution depends in a stable manner on the data of the problem. This means if the data are changed a little, the corresponding solution changes only a little

If 1-3 are not **all** satisfied, the problem is not well posed

Example of 1 not being ssatisfied:
$u_x + u_t = 0 \implies u_x = -tu_t$
Solving this, we see since $\frac{1}{y}dy = dx \implies ln|y| = x + c$ the solution is thus:
$y = Ae^x$ and so $u(x, y) = f(ye^-x$

Example of 2 not being satisfied: $u(x, 0) = f(0) = x$
Here, we know $f(0)$ is constant for all x, which clearly isn't true, so there is no solution!
Now let's try the same general solution with $u(x, 0) = 1$
Thus $u(x, 0) = f(0) = 1$, but there are infinitely many functions such that $f(0) = 1$, so this fails uniqueness!

Example of 2 not being satisfied:
Concept: if we start with the same 2 populations, then a few days later one is extinct and one is thriving.
Now, consider $u_{xx} + u_{yy} = 0$ such that $u(x, 0) = 0$, $u_y(x, 0) = e^{-\sqrt{n}}sin(nx)$, which is laplace eq in 2D domain
Solution: $u(x, y) = \frac{1}{n}e^{-\sqrt{n}}sin(nx)sinh(ny)$

If we were to graph this, we would see that the laplace eq. on an infinite domain is ill-posed!

V21 Plan for proving uniqueness in wave equation

Showing that a problem is well-posed is difficult! Worth a million dollars, actually...
Plan to prove uniqueness in wave eq: (proving existence is hard)

- Define an energy and show it is constant
- Prove uniqueness of wave equation with trivial initial conditions
- Prove uniqueness of wave equation with non trivial conditions

V22: Defining the concept of energy

Recall the Wave Equation: $u_{tt} = c^2 u_{xx}$
Energy is defined as:

$$E(t) = \int_{-\infty}^{\infty} \frac{(u_t)^2}{2} + \frac{c^2}{2}(u_x)^2 dx$$

This is energy in the physical sense, we won't worry about the details too much.

Claim: E(t) is constant with respect to time Assumptions:

- $d/dt \int f(x,t)dx = \int d/dt f(x,t)dt$
- $u_{xt} = u_{tx}$
- $\lim x \longrightarrow \infty \ u_t u_x = 0$
- u and all its derivatives are continuous

We also want to recall integration by parts, ie $\int u(x)v'(x)dx = [u(x)v(x)] - \int u'(x)v(x)dx$

Proof: We will show $\frac{dE}{dt} = 0$ which implies that energy function is cts
First, we have:

$$dE/dt = d/dt \int_{-\infty}^{\infty} \frac{(u_t)^2}{2} + \frac{c^2}{2}(u_x)^2 dx$$

Using (1), we can bring the d/dt into the integral to get:

$$dE/dt = \int_{-\infty}^{\infty} u_t u_{tt} + c^2 u_x u_{xt} dx$$

By def of the wave equation, we know $u_{tt} = c^2 u_{xx}$, giving us:

$$dE/dt = \int_{-\infty}^{\infty} u_t c^2 u_{xx} + c^2 u_x u_{xt} dx$$

Now, we let $'u' = u_t$ and $'dv' = c^2u_{xx}$ in our integration by parts. Thus, we have:

$$dE/dt = \left[^{\infty}_{-\infty} u_t c^2 u_x\right] - \int_{-\infty}^{\infty} c^2 u_x u_{tx} dx + \int_{-\infty}^{\infty} c^2 u_x u_{tx} dx$$

Note that we switched the order of partial integration in the last part by (2)
Finally, we see that the last two terms cancel, leaving us with:

$$dE/dt = \left[^{\infty}_{-\infty} u_t c^2 u_x\right]$$

Now, using (3) and taking the limit as $u_t c^2 u_x$ goes to positive and negative infinity, we see these are both zero and thus $dE/dt = 0 - 0 = 0$ and thus the energy function is constant ✓

V23 Strategy to prove uniqueness

Suppose u_1, u_2 are any two solutions to the wave equation. We want to show that $u_1 = u_2$
Claim: There exists only one solution to the following:
$u_{tt} = c^2 u_{xx}$ such that $u(x, 0) = 0$, $u_t(x, 0) = 0$, for $-\infty < x < \infty$
Assume 1-4 from the last video hold, and the vanishing thm holds, ie:

If $f(x) \geq 0$ **and** $\inf^{\infty}_{-\infty} f(x)dx = 0$ **then** $f(x) = 0$

Define $w(x, t) = u_1(x, t) - u_2(x, t)$. Our goal is to show $w(x, t) = 0$

From the last, video, we know that energy is constant. We also know that:

1. $w_{tt} = c^2 w_{xx}$ since w is a linear combination of solutions
2. $w(x, 0) = 0$ since $w(x, 0) = 0 - 0$
3. $w_t(x, 0) = 0$

Now, plugging zero into our energy equation, we get:

$$E(0) = \int_{-\infty}^{\infty} \frac{w_t(x, 0))^2}{2} + \frac{c^2}{2}(w_x(x, 0))^2 dx$$

Since we know both partials are zero, this gives:

$$\int_{-\infty}^{\infty} 0 + 0dx$$

In fact, this means that our energy equation is equal to zero for any value of t!
Now, we know that in the energy equation, the two terms inside the integral are both squared, and thus always positive. This means that if we let the sum of these terms be our f in the

vanishing theorem. Then, by the vanishing theorem, $\frac{w_t^2}{2} + \frac{c^2}{2}wx^2 = 0 \implies w_t(x,t) = 0$ and $w_x(x,t) = 0$ for all (x,t).
This can only be true if $w(x,t) = c$, but $w(x,0) = 0$, so $c = 0$ and thus $w(x,t) = 0$

This implies that $u_1(x,t) - u_2(x,t) = 0 \implies u_1 = u_2$ ✓

V24 Uniqueness of Wave Eq with non trivial initial conditions

Let's consider $u_{tt} - c^2u_{xx} = f(x,t)$ such that $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$ from negative infinity to infinity
Our same assumptions as the last video hold
Now, again define $w(x,t) = u_1 - u_2$
We will now show that $w_{tt} - c^2w_{xx} = 0$:

- $w_{tt} - c^2w_{xx} = (u_{1tt} - u_{2tt} - c^2(u_{1xx} - u_{2xx}$
- $= u_{1tt} - c^2u_{1xx} - (u_{2tt} - c^2u_{2xx}$
- $= f(x,t) - f(x,t) = 0$
- $\implies w_{tt} - c^2w_{xx} = 0$

Similarly, we can think about the initial conditions, ie:
$w(x,0) = u_1(x,0) - u_2(x,0) = \phi(x) - \phi(x) = 0$
$w_t(x,0) = u_{1t}(x,0) - u_{2t(x,0)} = \psi(x) - \psi(x) = 0$
We showed in the last video that the unique solution to $w_{tt} - c^2w_{xx} = 0$ such that $w(x,0) = 0$ and $w_t(x,0) = 0$ is $w = 0$
Since we have **the same assumptions here**, we know that $w =$ is a unique solution! ✓

3 Problems

V20 Problems

3

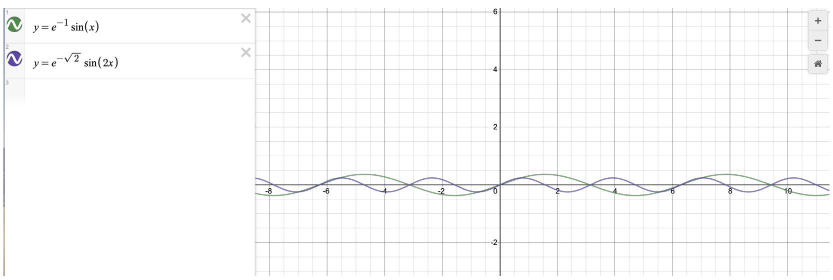
a) **Show** $u(x,y) = 1/ne^{-\sqrt{n}}\sin(nx)\sinh(ny)$ **satisfies the PDE**

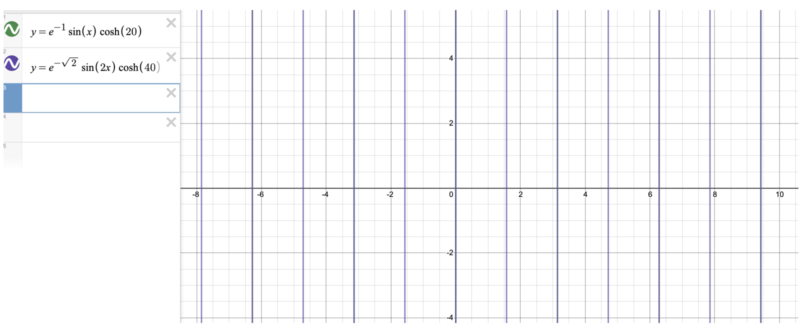
We have the following partial derivatives:
$u_{xx} = -n^2\sin(nx)\sinh(ny) \cdot \frac{1}{n}e^{-\sqrt{n}}$
$u_{yy} = n^2\sin(nx)\sinh(ny) \cdot \frac{1}{n}e^{-\sqrt{n}}$

Thus, from the above two bullet points we see that:

$$u_{xx}+u_{yy} = u_{xx} = -n^2\sin(nx)\sinh(ny)\cdot\frac{1}{n}e^{-\sqrt{n}}+u_{xx}+n^2\sin(nx)\sinh(ny)\cdot\frac{1}{n}e^{-\sqrt{n}} = 0 \checkmark$$

Now, we look at our first I.C. :
$u(x,0) = \frac{1}{n}e^{-\sqrt{n}}\sin(nx)\sinh(0) = 0$
And our second I.C:
$u_y(x,0) = \frac{1}{n}e^{-\sqrt{n}}\cdot n\cdot\sin(nx)\cdot\cosh(0) = e^{-\sqrt{n}}\sin(nx)$
Thus, the PDE and auxiliary conditions are satisfied

b) 

c) 

d) We can see that stability is violated by seeing how the solutions when $y = 20$ are a completely different type of graph than when $t = 0$. This is because $\sinh(0) = 0$ so thus our equation when $y = 0$ becomes $u_1(x,y) = e^{-1}sin(x)$ when $n = 1$ and $y = 0$ and $u_2(x,y) = e^{-\sqrt{2}}sin(2x)$ when $n = 2$. We see that when $y \neq 0$, we have the $\sinh(ny)$ term in the equation, which gives the graph a completely different shape and thus shows the instability of the equation. Going from $y = 0$ to $y = 20$ shows two completely different graphs.

V21 Problems

4. Write plan for proving uniqueness

- Define an energy and show it is constant
- Prove uniqueness of wave equation with trivial initial conditions
- Prove uniqueness of wave equation with non trivial conditions

V22 Problems

5. Let u be a solution to the problem $u_{tt}2u_{xx} + u = 0$, for $-\infty < x < \infty$. Construct an energy function such that $E(t)$ is constant wrt t and if $u(x,t) = 0$ for all x, t, then $E(t) = 0$

Let's construct our energy function to be as follows:

$$E(t) = \int_{-\infty}^{\infty} \frac{(u)^2}{2} + \frac{(u_t)^2}{2} + (u_x)^2 dx$$

Now, we know that:

$$dE/dt = d/dt \int_{-\infty}^{\infty} \frac{(u)^2}{2} + \frac{(u_t)^2}{2} + (u_x)^2 dx$$

Using A1, we can bring the d/dt inside the integral and differentiate to get:

$$dE/dt = \int_{-\infty}^{\infty} u \cdot u_t + u_t \cdot u_{tt} + 2u_{xt}u_x dx$$

We can factor u_t to get:

$$dE/dt = \int_{-\infty}^{\infty} u_t(u + u_{tt}) + 2u_{xt}u_x dx$$

Now, since we know $2u_{xx} = u + u_{tt}$ from our given equation, we can write our energy function as:

$$dE/dt = \int_{-\infty}^{\infty} u_t(2u_{xx}) + 2u_{xt}u_x dx$$

Using integration by parts, we have $'u' = u_t$ and $'dv' = 2u_{xx}$ which implies that $'v' = 2u_x$ and $'du' = u_{tx}$, giving us:

$$dE/dt =_{-\infty}^{\infty} [u_t \cdot 2u_x] - \int_{-\infty}^{\infty} 2u_x \cdot u_{tx} dx + \int_{-\infty}^{\infty} 2u_{xt}u_x dx$$

By A2, we can write this as:

$$dE/dt =_{-\infty}^{\infty} [u_t \cdot 2u_x] - \int_{-\infty}^{\infty} 2u_x \cdot u_{xt} dx + \int_{-\infty}^{\infty} 2u_{xt}u_x dx$$

We see that the last two terms are the same, so this leaves us with:

$$dE/dt =_{-\infty}^{\infty} [u_t \cdot 2u_x]$$

Now, by A3 we know that $\lim b \to \infty u_t(a)2u_x(a) - \lim b \to -\infty u_t(b)2u_x(b) = 0 - 0 = 0$
Thus, we have that $dE/dt = 0$ which implies that $E(t)$ is constant with respect to t.
Finally, looking at our energy equation, we see that if $u(x,t) = 0$ for all x, t, we have:

$$E(t) = \int_{-\infty}^{\infty} \frac{u(x,t)^2}{2} + \frac{u_t(x,0)^2}{2} + (u_x(x,t))^2 dx = \int_{-\infty}^{\infty} 0 + 0 + 0 = \int_{-\infty}^{\infty} 0 = 0$$

Now, if we define f to be the sum on the inside of the integral, we see that it is always positive since each term is squared. Since the integral from negative to positive infinity is also 0 (as we just showed), we can apply the vanishing theorem and thus if $u(x,t) = 0 \forall x, t$, then $E(t) = 0$.

V23 Problems

6. Show there is exactly one solution to the problem $u_{tt} - 2u_{xx} + u = 0, u(x,0) = 0 u_t(x,0) = 0$ **for** $-\infty < x < \infty$

Let u_1, u_2 be solutions to the PDE. Our goal is to show that $u_1 = u_2$.

Define $w(x,t) = u_1 - u_2$.We know this is also a solution to the PDE since it is a linear combination of solutions. This then implies that $w_{tt} = 2w_{xx} - w$, $w(x,0) = 0$, and $w_t(x,0) = 0$ Thus, since we have the same initial conditions as the last problem, we can invoke our energy function, ie:

$$E(t) = \int_{-\infty}^{\infty} \frac{(w)^2}{2} + \frac{(w_t)^2}{2} + (w_x)^2 dx$$

Since we have shown this equation is constant, we know $E(0) = c$. Evaluating our energy function at $t = 0$ gives us:

$$E(0) = \int_{-\infty}^{\infty} \frac{(w(x,0))^2}{2} + \frac{(w_t(x,0))^2}{2} + (w_x(x,0))^2 dx$$

Now, know this is equal to 0 since $w(x,0) = 0 \implies w_x(x,0) = 0$ and $w_t(x,0) = 0$
Thus, we know the second condition of the vanishing theorem holds for $E(t)$. We know the first condition holds since each term in the integral is squared and thus the sum is greater than 0.
Then by the vanishing theorem we know that $E(t) = 0$ for all t.
This then implies that both w_t and w_x are zero for all values of x, t, since otherwise the energy function would not hold. Since we know $w(x,0) = 0$ and we also know that energy is always 0, this implies that $w(x,t) = c$, but since $w(x,0) = 0$, we know $c = 0$ and thus$w(x,t) = 0$.

Since $w(x,t) = 0$, we then know that $u_1(x,t) - u_2(x,t) = w(x,t) = 0 \implies u_1 = u_2$

Therefore, using our energy function, we have proven that there exists one unique solution to $u_{tt} - 2u_{xx} + u = 0, u(x,0) = 0 u_t(x,0) = 0$ for $-\infty < x < \infty$ ✓

V24 Questions

7. Show there is exactly one solution to the problem $u_{tt} - 2u_{xx} + u = f(x,t), u(x,0) = \phi(x), u_t(x,0) = \psi(x), -\infty < x < \infty$**. You may refer to the result of any previous prob-**

lem you solved so far

Suppose u_1, u_2 satisfy the PDE and auxiliary conditions.
Now, we will define:
$w(x,t) = u_1 - u_2$
Thus, we have:
$w_{tt} - 2u_{xx} + u = (u_{1tt} - u_{2tt}) - (2u_{1xx} - 2u_{2xx}) + (u_1 - u_2)$
$\implies w_{tt} - 2u_{xx} + u = (u_{1tt} - 2u_{1xx} + u_1) - (u_{2tt} - 2u_{2xx} + u_2)$
$\implies w_{tt} - 2u_{xx} + u = f(x,t) - f(x,t) = 0$
$\implies w_{tt} - 2u_{xx} + u = 0$

Now, let's think about our initial conditions:
$w(x,0) = u_1(x,o) - u_2(x,0) = \phi(x) - \phi(x) = 0$
$w_t(x,0) = u_{1t}(x,o) - u_{2t}(x,0) = \psi(x) - \psi(x) = 0$
We know this is true since both u_1 and u_2 satisfy the auxiliary conditions. Thus, we have the following PDE and auxiliary conditions in terms of w:

- $w_{tt} - 2u_{xx} + u = 0$
- $w(x,0) = 0$
- $w_t(x,0) = 0$

These are the same initial conditions as the last problem, which we proved had a unique solution of $w = 0$, since we have our assumptions A1-A4 and th evanishing theorem holds. Thus, we have that $w(x,t) = u_1(x,t) - u_2(x,t) = 0 \implies u_1 = u_2$ and therefore we have a unique solution to the problem $u_{tt} - 2u_{xx} + u = f(x,t), u(x,0) = \phi(x), u_t(x,0) = \psi(x), -\infty < x < \infty$ ✓