1. NA
2. Using LaTeX
3. Consider the following problem:

$$u_t - 4u_{xx} = 0 \qquad \text{for } 0 < x < 1$$

$$u(x,0) = \begin{cases} x & 0 < x < 1/2 \\ 1 - x & 1/2 < x < 1 \end{cases}$$

$$u(0,t) = 0, \quad u(1,t) = 0$$

a) Plot the initial condition.

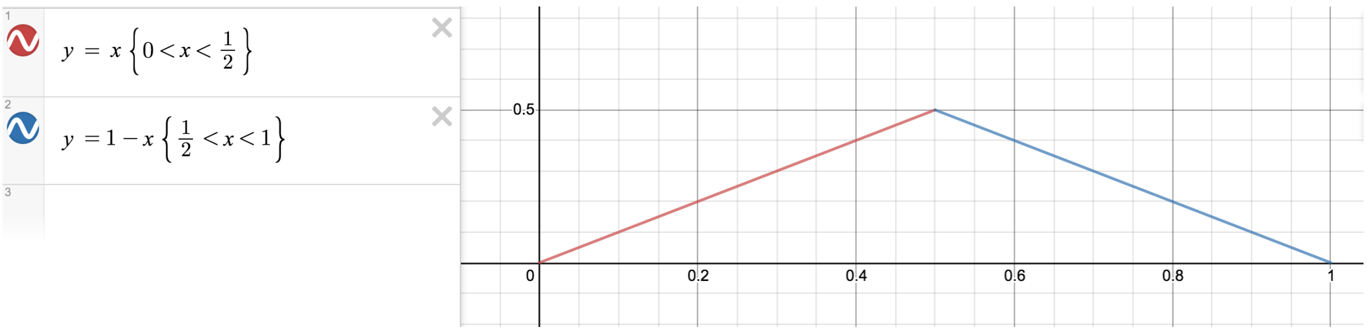

Figure 1: u(x,0)

b) Find the solution of the PDE that satisfies that initial condition and boundary conditions.
Step 1: Separate variables
Suppose $u(x,t) = X(x)T(t)$, then our PDE $u_t - 4u_{xx} = 0$ can be written as:

$$XT' - 4X''T = 0$$

$$T'X = 4X''T$$

$$\frac{T'}{4T} = \frac{X''}{X} = \lambda$$

$$T' = 4\lambda T \qquad X'' = \lambda X$$

Step 2: Solve the X ODE using the given boundary conditions
If $\lambda < 0$ we can write $\lambda = -\beta^2$. Then our ODE becomes

$$X'' + \beta^2 X = 0$$

And the characteristic equation is $r^2 + \beta^2 = 0$ so $r = \pm\beta i$. Then

$$X(x) = C\cos(\beta x) + D\sin(\beta x)$$

Now using the given boundary conditions we can solve for C and D. Since $u(0,t) = 0$, that means $X(0)T(t) = 0$, and since we don't want $T(t)$ to be 0 for all t, this means

$$X(0) = 0$$

Also, since $u(1,t) = 0$, that means $X(1)T(t) = 0$, and for the same reason

$$X(1) = 0$$

So now we know

$$X(0) = C\cos(\beta \cdot 0) + D\sin(\beta \cdot 0) = 0$$

$$C = 0$$

$$X(1) = 0 \cdot \cos(\beta) + D\sin(\beta) = 0$$

$$D\sin(\beta) = 0$$

We want $D \neq 0$ so that we don't have the zero eigenfunction, and $\beta \neq 0$ so that means $\sin(\beta) = 0$ or

$$\beta = n\pi, \quad n \in \mathbb{N}$$

So λ can be < 0, now let's check the case where $\lambda = 0$. Then our ODE becomes

$$X'' = 0$$

Double integrating leaves

$$X = Cx + D$$

Using our initial conditions we have:

$$X(0) = C \cdot 0 + D = 0$$

$$D = 0$$

$$X(1) = C \cdot 1 + 0 = 0$$

$$C = 0$$

Thus, X is the zero eigenvector so $\lambda = 0$ is NOT an eigenvalue.
Lastly, let's check $\lambda > 0$ so let $\lambda = \beta^2$. Then our ODE becomes

$$X'' - \beta^2 X = 0$$

And the characteristic equation is $r^2 - \beta^2 = 0$ so $r = \pm\beta$. Then

$$X(x) = Ce^{\beta x} + De^{-\beta x}$$

Now using the given boundary conditions we can solve for C and D.

$$X(0) = Ce^{\beta \cdot 0} + De^{-\beta \cdot 0} = 0$$

$$C = -D$$

$$X(1) = -De^{\beta} + De^{-\beta} = 0$$

$$De^{\beta} = De^{-\beta}$$

$$\beta = -\beta$$

This is only true if $\beta = 0$ which is not the case here, so there are no eigenvalues > 0.
Thus,

$$\lambda = -\beta^2 = -(n\pi)^2, \quad X(x) = D\sin(n\pi x), \ n \in \mathbb{N}$$

This solution is for a given value of n so we can write

$$\lambda_n = -\beta^2 = -(n\pi)^2, \quad X_n(x) = D_n \sin(n\pi x), \ n \in \mathbb{N}$$

Recall that any scalar multiple of an eigenfunction is an eigenfunction, so we can write the eigenfunction more simply as

$$X_n(x) = \sin(n\pi x), \ n \in \mathbb{N}$$

Step 3: Solve the T ODE

$$\begin{aligned} T' &= 4\lambda T \\ \int \frac{dT}{T} &= \int 4\lambda dt \\ ln|T| &= 4\lambda t + c \\ T &= Ae^{-4n^2\pi^2 t}, \quad n \in \mathbb{N} \end{aligned}$$

Here we've incorporated the $\pm$ into the constant.

Step 4: Take a linear combination of solutions to get the series solution
Given that $u(x,t) = X(x)T(t)$ we have

$$u_n(x,t) = \sin(n\pi x) \cdot A_n e^{-4n^2\pi^2 t}$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-4n^2\pi^2 t} \sin(n\pi x)$$

Step 5: Consider the initial conditions
Given the series solution, we can find ϕ(x)

$$\phi(x) = u(x,0) = \sum_{n=1}^{\infty} A_n e^{-4n^2\pi^2(0)} \sin(n\pi x)$$

$$\phi(x) = \sum_{n=1}^{\infty} A_n e^0 \sin(n\pi x)$$

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$

Now let's solve for the Fourier Sine coefficients. By individual homework 9 problem 5c, we know that

$$A_m = \frac{2}{l}\int_0^l \phi(x) X_m \ dx$$

For our PDE, we know l=1 and that $X_m = \sin(m\pi x)$, so we have

$$A_m = 2\int_0^1 \phi(x)\sin(m\pi x) \ dx$$

Now we can plug $\phi(x) = \begin{cases} x & 0 < x < 1/2 \\ 1-x & 1/2 < x < 1 \end{cases}$ into our equation for A_m

$$A_m = 2\Big[\int_0^{\frac{1}{2}} x\sin(m\pi x) \ dx + \int_{\frac{1}{2}}^1 (1-x)\sin(m\pi x) \ dx\Big]$$

By using integration by parts, we have

$$\begin{aligned} A_m &= 2\Big[\Big(-\frac{x}{m\pi}\cos(m\pi x) - \int_0^{\frac{1}{2}} -\frac{1}{m\pi}\cos(m\pi x)dx\Big) + \Big(-\frac{1-x}{m\pi}\cos(m\pi x) - \int_{\frac{1}{2}}^1 \frac{1}{m\pi}\cos(m\pi x)dx\Big)\Big] \\ A_m &= 2\Big[\Big(-\frac{x}{m\pi}\cos(m\pi x) + \frac{1}{m^2\pi^2}\sin(m\pi x)\Big)\Big|_0^{\frac{1}{2}} + \Big(-\frac{1-x}{m\pi}\cos(m\pi x) - \frac{1}{m^2\pi^2}\sin(m\pi x)\Big)\Big|_{\frac{1}{2}}^1 \ \Big] \end{aligned}$$

$$A_m = 2\Big[\Big(-\frac{1}{2m\pi}\cos(\frac{m\pi}{2})+\frac{1}{m^2\pi^2}\sin(\frac{m\pi}{2})-0-\frac{1}{m^2\pi^2}sin(0)\Big)+\Big(0-\frac{1}{m^2\pi^2}\sin(m\pi)+\frac{1}{2m\pi}\cos(\frac{m\pi}{2}) + \frac{1}{m^2\pi^2}\sin(\frac{m\pi}{2})\Big)\ \Big]$$

Notice that $\sin(0) = 0$, and because $m \in \mathbb{Z}$, $\sin(m\pi) = 0$. Therefore, we have

$$\begin{aligned} A_m &= 2\Big[\Big(-\frac{1}{2m\pi}\cos(\frac{m\pi}{2})+\frac{1}{m^2\pi^2}\sin(\frac{m\pi}{2})\Big)+\Big(\frac{1}{2m\pi}\cos(\frac{m\pi}{2})+\frac{1}{m^2\pi^2}\sin(\frac{m\pi}{2})\Big)\ \Big] \\ A_m &= -\frac{1}{m\pi}\cos(\frac{m\pi}{2})+\frac{2}{m^2\pi^2}\sin(\frac{m\pi}{2})+\frac{1}{m\pi}\cos(\frac{m\pi}{2})+\frac{2}{m^2\pi^2}\sin(\frac{m\pi}{2}) \\ A_m &= \frac{2}{m^2\pi^2}\sin(\frac{m\pi}{2})+\frac{2}{m^2\pi^2}\sin(\frac{m\pi}{2}) \\ A_m &= \frac{4}{m^2\pi^2}\sin(\frac{m\pi}{2}) \end{aligned}$$

Notice when m is even, $\frac{m}{2} \in \mathbb{Z}$, so $\sin(\frac{m\pi}{2}) = 0$. Thus,

$$\begin{aligned} A_m &= 0 && \text{when m is even} \\ A_m &= \frac{4}{m^2\pi^2}\sin(\frac{m\pi}{2}) && \text{when m is odd} \end{aligned}$$

c) We can now plug the Fourier Coefficients into $u(x,t) = \sum_{n=1}^{\infty} A_n e^{-4n^2\pi^2 t}\sin(n\pi x)$, where $A_n = \frac{4}{n^2\pi^2}\sin(\frac{n\pi}{2})$ when n is odd and $A_n = 0$ when n is even, to find the first five terms of the series solution.

$$u(x,t) = \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\sin(\frac{n\pi}{2})e^{-4n^2\pi^2 t}\sin(n\pi x)$$

$$u(x,t) = \frac{4}{1^2\pi^2}\sin(\frac{\pi}{2})e^{-4(1)^2\pi^2 t}\sin(\pi x)+\frac{4}{3^2\pi^2}\sin(\frac{3\pi}{2})e^{-4(3)^2\pi^2 t}\sin(3\pi x)+\frac{4}{5^2\pi^2}\sin(\frac{5\pi}{2})e^{-4(5)^2\pi^2 t}\sin(5\pi x) + \frac{4}{7^2\pi^2}\sin(\frac{7\pi}{2})e^{-4(7)^2\pi^2 t}\sin(7\pi x)+\frac{4}{9^2\pi^2}\sin(\frac{9\pi}{2})e^{-4(9)^2\pi^2 t}\sin(9\pi x)$$

$$u(x,t) = \frac{4}{\pi^2}\sin(\frac{\pi}{2})e^{-4\pi^2 t}\sin(\pi x)+\frac{4}{9\pi^2}\sin(\frac{3\pi}{2})e^{-36\pi^2 t}\sin(3\pi x)+\frac{4}{25\pi^2}\sin(\frac{5\pi}{2})e^{-100\pi^2 t}\sin(5\pi x) + \frac{4}{49\pi^2}\sin(\frac{7\pi}{2})e^{-196\pi^2 t}\sin(7\pi x)+\frac{4}{81\pi^2}\sin(\frac{9\pi}{2})e^{-324\pi^2 t}\sin(9\pi x)$$

$$u(x,t) = \frac{4}{\pi^2}(1)e^{-4\pi^2 t}\sin(\pi x)+\frac{4}{9\pi^2}(-1)e^{-36\pi^2 t}\sin(3\pi x)+\frac{4}{25\pi^2}(1)e^{-100\pi^2 t}\sin(5\pi x) + \frac{4}{49\pi^2}(-1)e^{-196\pi^2 t}\sin(7\pi x)+\frac{4}{81\pi^2}(1)e^{-324\pi^2 t}\sin(9\pi x)$$

$$u(x,t) = \frac{4}{\pi^2}e^{-4\pi^2 t}\sin(\pi x)-\frac{4}{9\pi^2}e^{-36\pi^2 t}\sin(3\pi x)+\frac{4}{25\pi^2}e^{-100\pi^2 t}\sin(5\pi x)-\frac{4}{49\pi^2}e^{-196\pi^2 t}\sin(7\pi x) + \frac{4}{81\pi^2}e^{-324\pi^2 t}\sin(9\pi x)$$

d) The first term of our series solution dominates because of both the coefficient out front and the exponent of e. This term has a significantly larger front coefficient compared to subsequent terms (n^2 is 1, so the denominator is not scaled), increasing its contribution to the solution. Its e exponent is also much less negative than other terms (again because n is 1), allowing the e term to notably contribute to the solution.

e) Plot the first term of the solution.

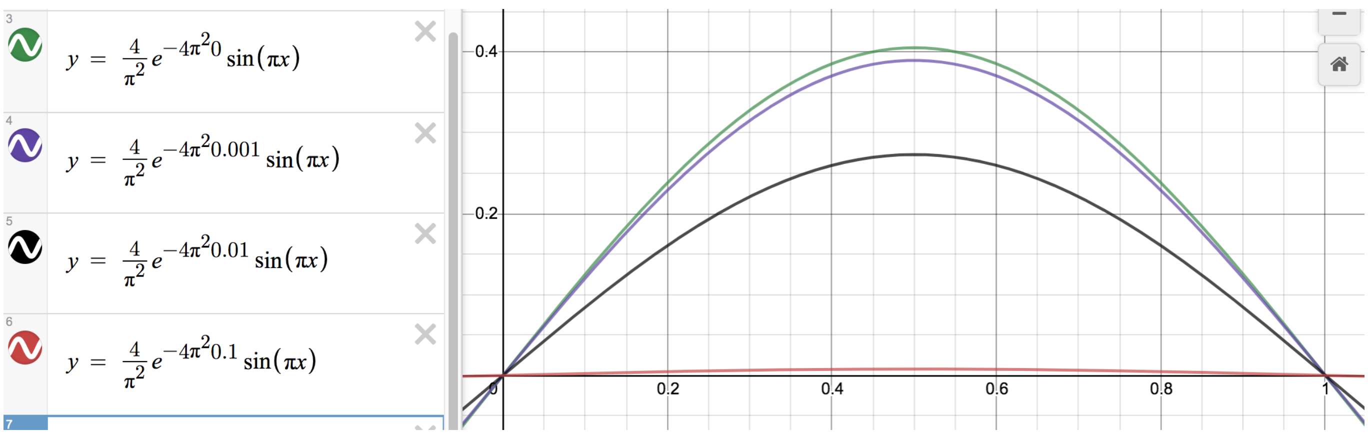
3
$y = \frac{4}{\pi^2} e^{-4\pi^2 0} \sin(\pi x)$
4
$y = \frac{4}{\pi^2} e^{-4\pi^2 0.001} \sin(\pi x)$
5
$y = \frac{4}{\pi^2} e^{-4\pi^2 0.01} \sin(\pi x)$
6
$y = \frac{4}{\pi^2} e^{-4\pi^2 0.1} \sin(\pi x)$
7
0.4
0.2
0
0.2
0.4
0.6
0.8
1

Figure 2: u(x,t)