**Math 3209** Fall 2020
Final Exam

## Exam Instructions

(i) You have 24 hours from the time you download the exam to answer all questions and to upload your solutions to gradescope

(ii) You must SHOW your work to receive full credit. You have plenty of time, so do your work on scrap paper, then re-write your answers neatly and with good detail

(iii) Put your answers on separate pieces of paper, writing only on one side. Be sure to tag the location of the answers, just like you do in homework submissions

* **When using separation of variables:*** You must show all steps of any work you do. You should check for zero, positive and negative eigenvalues.

## What is allowed on exam

- You may refer to our book, the videos and your notes.
- You may use any software you like, but indicate which software you used and for what reason in your answer.
- ****While you may use your notes/videos/the book/software to aid you (or check work), you may not reference such results in your solution. You must work everything out by hand, unless otherwise stated****
- You are allowed to google prerequisite material (if e.g. you need an integration reminder), but you are required to cite any webpages/ or other sources not listed above that you used in your answer.
- You are free to email me any questions, but I can only answer clarifying questions. I cannot give hints on exam problems.

## What is NOT allowed on exam

(a) If you use a source besides the book, course videos or your notes and do not cite it, it will be considered plagiarism. Example sources you must cite include webpages and software

(b) You are not allowed to communicate with other students, or any person other than Prof. Chong, about the content of the course from the time you download the exam until the end of finals week

1. Write down the following statement as your answer to this problem. "I, <u>(name)</u>, have read the rules listed on the cover sheet of the exam and I agree to follow them. I realize I may lose points for not following (i)-(iii). I realize action with the judiciary committee will be pursued if I do not follow (a)-(b) "

2. Find the steady state of the following problem

$$u_t - 5u_{xx} = 0 \quad \text{for } 0 < x < 2$$
$$u_x(0,t) = 0, \qquad u(2,t) = 1$$

3. Find the solution to the following problem

$$u_t - \frac{2}{9}u_{xx} = 0 \quad \text{for } 0 < x < 2$$
$$u(x,0) = 8\cos\left(\frac{9\pi}{4}x\right)$$
$$u_x(0,t) = 0, \qquad u(2,t) = 0$$

4. Find the solution to the following problem

$$u_{xx} + u_{yy} = 0 \quad \text{for } 0 < x < 1 \qquad 0 < y < 1$$

$$u_x(0,y) = 0, \qquad u(1,y) = y$$

$$u_y(x,0) = 0, \qquad u_y(x,1) = 0$$

Use hyperbolic cosine (cosh) or hyperbolic sine (sinh) to simplify your answer. Write your answer as a series and explicitly write out the Fourier coefficients in your series. You may use any result on page 108-110 of the book without redoing the calculation.

**Problems continue on next page!**

5. The goal of this problem is to show that the following nonlinear PDE

$$u_t = u_{xx} + u(1-u) \tag{1}$$

has a traveling front solution. We will also find any restrictions on the wave speed.

(a) Let $u(x,t) = U(x-ct) = U(z)$ and define $V = U'$, where the prime indicates the derivative with respect to $z$, namely $U' = \frac{dU}{dz}$. Show that the following nonlinear ODE system is satisfied

$$U' = V \tag{2}$$

$$V' = -cV - U(1-U) \tag{3}$$

(b) Find formulas for the two equilibrium points of system (2)-(3).

(c) For each equilibrium point you found in 5b: linearize the equations (2)-(3) about the equilibrium point and find the eigenvalues by hand

(d) Suppose that $u$ represents a population, and hence we only want to consider solutions with $u \geq 0$ (and hence $U \geq 0$). Use this fact, and your answers from part 5c to find the minimum wave speed $c$.

(e) Assume $c = 3$. For each equilibrium point you found in 5b: draw the phase portrait in a region close to the equilibrium point. Put $U$ on the x-axis and put $V$ on the y-axis. You will need to compute the eigenvalues and associated eigenvectors for the Jacobian matrix evaluated at the equilibrium in order to do this. You may use software to compute eigenvalues and eigenvectors of a matrix for this part. Be sure to write down which matrix you are computing eigenvalues of eigenvectors of.

(f) Assume $c = 3$. Using your answers from part 5e infer what the phase portrait of system (2)-(3) looks like. Put $U$ on the x-axis and put $V$ on the y-axis. Only consider parts of the phase portrait with $U \geq 0$. Be sure to draw the heteroclinic orbit and use arrows to indicate how the solution is changing with respect to $z$.

(g) For the heteroclinic orbit, draw $U$ vs $z$, and be sure to indicate the limiting behavior for $z \to -\infty$ and $z \to \infty$.