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Homework 5

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1 Resources

Other than the textbook and class notes, I briefly went over the problems over the phone with Kayla

2 Notes for Week 5

V16 Diffusion Equation

- revisit diffusion eq
- special solution to diffusion eq

1. We know the solution graphs to the diffusion equation will flatten as t increases..graph becomes wider and shorter
So, our guess for a solution could be:
$\frac{1}{t}e^{-\frac{x^2}{t}}$
Turns out this isn't exactly it, the actual solution is messier:

$$u(x,t) = \frac{1}{\sqrt{\pi 4kt}}e^{-\frac{x^2}{4kt}}$$

We call this eq. $s(x,t)$, our **special solution**
This k is the same as in the diffusion equation!
We thus know that $\int_{\infty}^{\infty} s(x,t)dx = 1$ aka mass of special solution is always 1
Example, when t is close to zero, graph will be very tall and very skinny, and flatten out over time because mass is conserved!

V17 Deriving a solution formula for IVP

We first want to talk about invariances of the diffusion eq.

- If I have a solution $u(x,t)$ then $u(x-y,t)$ is also a solution. Called **translate**
- A linear combination of solutions is also a solution

Now we can create many solutions starting from just s!

Let $\phi(y)$ be some function of y
RECALL: $\phi(1)\cdot s(x-1,t)$ is a solution by translation and ALSO $\phi(1)s(x-1,t)+\phi(2)s(x-2,t)+ ....$ is a solution!

So, we can write this as a sum:

$$\sum_{i=-\infty}^{\infty} s(x-y_i,t)\phi(y_i)\delta y = \int_{-\infty}^{\infty} s(x-y,t)\phi(y)dy$$

We get this by thinking about what happens when the change in y goes to 0 –¿ we approach the derivative! Adding up all these narrow rectangles gives us an integral

This is a solution! Plugging in our special solution gives us:

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)dy$$

This is just a linear combination of **all** the translates of s!
This solves the diff. eq!

V18

Recalling the eq. from the end of V17, we see that when $k = 1$ then we treat everything except y as constant! Then, if the amplitude is $\phi(y)$, then that turns out to be our initial value!
So, for example, let's say we have $u_t - u_{xx} = 0$ given $u(x,0) = e^{-x}$

1. First, we just 'plug in' $\phi(x) = e^{-x}$ into the formula, remembering thaat in the equation it's actually $\phi(y)$
   $u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t}e^{-y}dy$
2. Now, we can take the powwers of e and simplify the exponent, by multiplying the $-y$ by $4t/4t$ to combine the fraction. Then, we expand the $(x-y)^2$ and then pull out $-1/4t$ to get:
   $\frac{-(x^2-2xy+y^2)-4ty}{4t}$
   Now, we can complete the square knowing $e = c - b^2/4a$ which simplifies to $e = t - x$. Now, we get that the equation is:
   $u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(y+2t-x)^2/4t}e^{t-x}dy$
3. Remembering that we only care about y, we can do a p substitution, letting $p = (y+2t-x)/\sqrt{4t}$ which finally simplifies to $1/sqrt\pi \int_{-\infty}^{\infty} e^{-p^2}dye^{t-x} = e^{t-x}$ and THIS IS OUR SOLUTION!

V19 Using the error solution to IVP

In solving the previous eq, we had to do a lot of nasty work....many times we would get stuck with different IV.

Now, we want to revisit the error function from stats (normal distrubution!) and we can use this to simplify our answers to the diffusion equation.

$$Erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2} dp$$

Now, let's consider $u_t - u_{xx} = 0$, with $u(x, 0) = 1$ if $x > 0$ and 0 otherwise.
We first plug in ϕ to get: $u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_0 \infty e^{-(x-y)^2/4t} dy$
We got this since we know u is 0 when $x < 0$, so we only care about the integral from zero to infinity!
We can make a similar substitution as before to get $p = x - y/\sqrt{4t}$ and thus $dp = -dy/\sqrt{4t}$
The dp cancels out the $1/\sqrt{4t}$ in the front of the integral, leaving us with an integral from $x/\sqrt{4t}$ (p(0)) to negative infinity $p(\infty) = -\infty$

So, we rewrite the integral from $-\infty$ to $x/\sqrt{4t}$ which we can split into two integrals with 0. Finally, we see that from $-\infty$ to 0 is half the Gaussian distrubution, and the other integral is almost the error function, so our final answer is:

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^0 e^{-p^2} dp + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4t}} e^{-p^2} = 1/\sqrt{\pi} \cdot \sqrt{\pi}/2 + 1/2Erf(x/\sqrt{4t} = \frac{1}{2} + \frac{1}{2} Erf(\frac{x}{\sqrt{4t}})$$

3 Problems

V16 Problems

3. In this video, it was asserted that $S(x,t) = \frac{1}{\sqrt{\pi 4kt}} e^{-x^2/4kt}$ solves the diffusion equation $u_t - ku_{xx} = 0$

a) We have that $S_t = \frac{1}{\sqrt{\pi 4kt}} e^{-x^2/4kt} \cdot \frac{x^2}{4kt^2} + e^{-x^2/4kt} \cdot \frac{-1}{2}(4kt\pi)^{\frac{-3}{2}} \cdot \pi 4k$

This thus implies that $S_t = \frac{x^2 e^{\frac{-x^2}{4kt}}}{4kt^2\sqrt{\pi 4kt}} = \frac{x^2 e^{-x^2/4kt} - 2kte^{-x^2/4kt}}{4kt^2\sqrt{4kt\pi}} = \frac{(x^2 - 2kt) \cdot e^{-x^2/4kt}}{4kt^2\sqrt{4kt\pi}}$

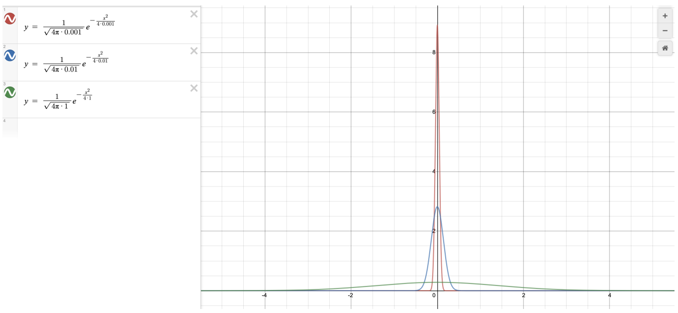
Now, we compute $u_x = \frac{1}{\sqrt{4kt\pi}} \cdot \frac{-2x}{4kt} \cdot e^{-x^2/4kt}$
Then, $u_{xx} = \frac{-2x}{4kt\sqrt{4kt\pi}} \cdot \frac{-2x}{4kt} \cdot e^{-x^2/4kt} + e^{-x^2/4kt} \cdot \frac{-2}{4kt\sqrt{4kt\pi}} = \frac{x^2 e^{-x^2/4kt}}{4k^2t^2\sqrt{4kt\pi}} \cdot \frac{-e^{-x^2/4kt}}{2kt\sqrt{4kt\pi}}$

Multiplying by $2kt/2kt$ and simplifying, we get that $u_{xx} = \frac{(x^2-2kt) \cdot e^{-x^2/4kt}}{4k^2t^2\sqrt{4kt\pi}}$

Finally, multiplying this by k, we see that the k^2 in the denominator cancels, giving us $ku_{xx} = \frac{(x^2-2kt) \cdot e^{-x^2/4kt}}{4kt^2\sqrt{4kt\pi}}$

Replacing these quantities into $u_t - ku_{xx} = 0$ for $S = u$, we get that:

$\frac{(x^2-2kt)\cdot e^{-x^2/4kt}}{4kt^2\sqrt{4kt\pi}} - \frac{(x^2-2kt)\cdot e^{-x^2/4kt}}{4kt^2\sqrt{4kt\pi}} = 0$, thus verifying our solution

b) 

c) We know that the density S is diffusing because the graph is getting flatter and wider as t increases, while the area remains the same

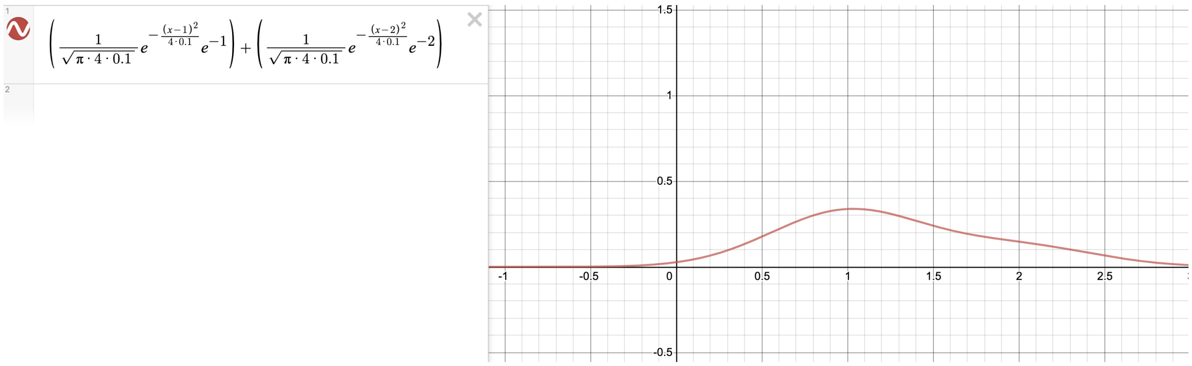
V17 Problems

4. Verify the two invariance properties shown in the video

a) **Translate Invariance**
We know that $u_t - ku_{xx} = 0$. To show that the translate invariance holds, let's consider $(u(x-y,t))_t = u_t(x-y,t)$ and $(u(x-y,t))_{xx} = u_{xx}(x-y,t)$
Then, since we know the original equation holds, this shows that $u_t(x-y,t) = ku_{xx}(x-y,t)$ thus verifying the translate invariance

b) **Linear combination**
Assume u_1 and u_2 are solutions. This implies that $u_{1t} = ku_{1xx}$ and $u_{2t} = ku_{2xx}$
Now, let $u(x,t) = c_1u_1(x,t) + c_2u_2(x,t)$ for some $c_1, c_2 \in \mathbb{R}$
We want to verify that this is a solution
$\implies u_t = \frac{\partial}{\partial t}[c_1u_1 + c_2u_2]$
$\implies u_t = c_1\frac{\partial u_1}{\partial t} + c_2\frac{\partial u_2}{\partial t} = c_1k\frac{\partial^2 u_1}{\partial x^2} + c_2k\frac{\partial^2 u_2}{\partial x^2}$
$\implies u_t = k\frac{\partial^2}{\partial x^2}[c_1u_1 + c_2u_2]$
Finally, we see that this implies that $u_t = ku_{xx}$ ✓

5. Let S(x, t) be defined as before

a) 

b) We know this is a solution because it is a linear combination of translate invariances of the original equation.

6. Based on everything you have done in this problem and seen in the video, explain, in words, why $u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{(xy)^2}{4kt}} \phi(y)dy$ ought to solve the diffusion equation where $\phi(x)$ is some function.

We can see that the integral is really just "adding" up all the solutions for every value of y, taking the limit as our change in y gets infinitely small. In other words, the rectangles under the curve that we are considering for each linear combination term are getting skinnier and skinnier, giving us the integral. Since each translate and linear combination is a solution, so is the sum of all of them.

V18 Problems

7. Solve $u_t - u_{xx} = 0$ given $u(x,0) = e^{2x}$
We first want to plug in $\phi(y) = e^{2y}$ by our initial condition, which gives us:

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} e^{2y} dy$$

Now, we can do the following to simplify the exponent:

1. $\frac{-(x-y)^2}{4t} + 2y = \frac{-x^2+2xy-y^2+8ty}{4t} = \frac{1}{4t}(-y^2 + y(8t+2x) - x^2)$
   To complete the square, we see that $a = -1$, $b = 8t + 2x$ and $c = -x^2$
   Now, this gives us: $\frac{1}{4t}(-y+4t+x)^2 + \frac{-x^2}{4t} - (\frac{(8t+2x)^2}{-16t})$
   Now, this equals $\frac{1}{4t}(-y+4t+x)^2 - \frac{x^2}{4t} + 4t + 2x + \frac{x^2}{4t} = 4t + 2x$

2. Thus, we have that:

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{(-y+4t+x)^2/4t} e^{(4t+2x} dy$$

3. If we let $p = -y + 4t + x/\sqrt{4t}$ then we see that $dp = dy/\sqrt{4t}$ and thus:

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} dy \cdot e^{(4t+2x} = \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} \cdot e^{4t+2x} = e^{4t+2x}$$

4. Then, $u_t = -4e^{-4t-2x}$, $u_{xx} = 4e^{4t+2x}$

5. Finally, we verify that this solution is correct with the following:
$u_t = 4e^{4t+2x}$
$u_{xx} = 4e^{4t+2x}$
$\implies u_t - u_{xx} = 4e^{4t+2x} - 4e^{4t+2x} = 0$, thus verifying that this is a solution ✓

V19 Problems

8. Using the solution formula Eq. (2), solve the following IVP,
$u_t - u_{xx} = 0, u(x,0) = 1$ **for** $-1 < x < 1$ **and 0 otherwise**

- We first note that when $-1 < x < 1$, we have that $\phi(y) = 1$
- This thus implies that our solution is: $u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-1}^{1} e^{-(x-y)^2/4t} dy$ since for all values of x outside $-1 < x < 1$, we know the integral is 0.
- Now, let's do a change of variables and let $p(y) = \frac{x-y}{\sqrt{4t}}$ which implies that $p(1) = \frac{x-1}{\sqrt{4t}}$ and $p(-1) = \frac{x+1}{\sqrt{4t}}$
- We also know that $dp = \frac{-dy}{4t}$
- Thus, we know that $u = \frac{-1}{\sqrt{\pi}} \int_{\frac{x+1}{\sqrt{4t}}}^{\frac{x-1}{\sqrt{4t}}} e^{-p^2} dp$
- Flipping the order of integration, we get:
  $u = \frac{1}{\sqrt{\pi}} \int_{\frac{x-1}{\sqrt{4t}}}^{\frac{x+1}{\sqrt{4t}}} e^{-p^2} dp$
- We can break this into two integrals now, ie:
  $u = \frac{-1}{\sqrt{\pi}} \int_{0}^{\frac{x-1}{\sqrt{4t}}} e^{-p^2} dp + \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x+1}{\sqrt{4t}}} e^{-p^2} dp$
- Finally, we have to use our error function in place of the integrals to give us our final answer:

$$u(x,t) = -\frac{1}{2}\mathscr{E}rf(\frac{x-1}{\sqrt{4t}}) + \frac{1}{2}\mathscr{E}rf(\frac{x+1}{\sqrt{4t}})$$

9. Graph your solution at t=0,1,2,5,10