Kim Hancock

Homework 9

November 12, 2020

1 Resources

Other than the textbook and class notes, nothing

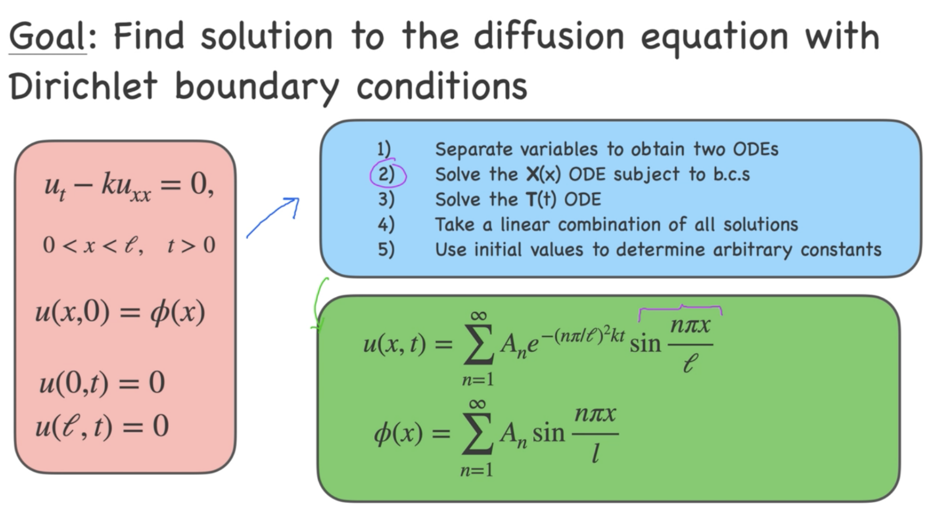
2 Notes for Week 8

V34 Fourier Series Introduction

1. Big picture goal
2. Defining fourier series

Section 5.1 in reading

Our goal is to find a solution to the diffusion equation with Dirichlet boundary conditions. We apply the 5 steps we know and love and then get our series equation.



Now, we want to focus on getting a solution for wht A_n is. We want to focus on the $\phi(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l})$
Let's think about what this formula is telling us in words.... Any function $\phi(x)$ can be written as a linear combination of sine functions. This seems crazy, right?

As an example, if this statement is true, we have $1 = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l})$ so this corresponds to $\phi(x) = 1$
The A_n is the amplitude of the sine function, and actually this works!
Turns out that most functions can be written in this way, and it was a game changer in mathematics. He discovered this and opened up a whole new branch of mathematics. So

once we are able to understand this formula, we can come back to the original formula and get rid of the phi part.

V35: Abstract formula for Fourier

Our goal is to find a formula for A_n in the fourier sine series, $\phi(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l})$
We will first get a strategy for how to do this....
Let's first call the sine part X_n so we get the sum $\phi(x) = \sum_{n=1}^{\infty} A_n X_n$

We want to keep in mind that we are taking a linear combination of all the sine functions. What we are going to do is bring the idea of linear combinations back in. So we want to write X as a vector:

$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + A_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

These As are constants, and we call each vector X_1 and X_2. So basically we just change the functions into vectors, and we do know how to take a linear combination of this.

So, now we want to solve for A_1 and A_2.

We have that $A_1 + A_2 = 2$ and $-A_1 + A_2 = 3$.
We add these together to get $2A_2 = 5 \implies A_2 = 5/2$
Then we know $\implies A_1 = -0.5$

Alternatively, we could take the dot product of $X_1 \cdot X_2 = (1 \cdot 1) + (1 \cdot (-1)) = 0$, so they are othogonal!

We can also dot product with themselves:
$X_1 \cdot X_1 = 2$ and $X_2 \cdot X_2 = 2$

So, why is this useful? Let's call $\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \phi$

Now, we dot $\phi \cdot X_1 = A_1 X_1 \cdot X_1 + A_2 X_2 \cdot X_1$ then we know the second term goes to zero so we have $\phi \cdot X_1 = A_1 X_1 \cdot X_1$

Now, solving for A_1 we get $A_1 = \frac{\phi \cdot X_1}{x+1 \cdot x_1} = \frac{2 \cdot 1}{3 \cdot -1} 2 = -\frac{1}{2}$
this is the same thing we got above! By dotting ϕ with X_1, we were able to solve for A_1.

Then, to generalize, we have:

$$A_m = \frac{\phi \cdot X_m}{X_m \cdot X_m}$$

Now let's see if we can do the same thing for the original goal of this lecture.

REMEMBER that $\phi(x) = A_1X_1 + A_2X_2 + A_3X_3 + ...$

Let's suppose we have the same structure as the previous example, so we want a "dot product" for functions, but we have to figure out what that is... We want them to be orthogonal, ie $X_m \cdot X_n = 0$
We also want $X_m \cdot X_m \neq 0$

Now suppose we have $\phi(x) = A_1X_1 + A_2X_2 + A_3X_3 + ...$ and the above two things *happen* to be true. So, we are going to dot both sides with the function X_m, which gives us:

$$\phi(x) \cdot X_m = A_1X_1 \cdot X_m + A_2X_2 \cdot X_m + A_3X_3 \cdot X_m + ... + A_mX_m \cdot X_m +$$

HOWEVER, we note that every term goes to zero except the last term, since that is dotted with itself, which gives us:

$$\phi \cdot X_m = A_mX_m \cdot X_m$$

$$\implies A_m = \frac{\phi \cdot X_m}{x_m \cdot X_m}$$

The key was that we made **a lot of assumptions** to allow this to follow from the simple calculation.

Before, we said that "dot product" for functions looks like $X_m \cdot X_n$, where both are functions of x. However, this is not the notation used. Instead we write:

(X_m, X_n) which we call the **inner product**

Summary

Given

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}$$

we showed

$$A_m = \frac{(\phi, X_m)}{(X_m, X_m)}$$

Fourier coefficients

where (f,g) is the **inner product** of the functions f and g [to be defined later]

$$X_m = \sin\frac{m\pi x}{l}$$

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V36: Concrete formula for fourier coefficients

1. Defining the inner product
2. Showing orthogonality
3. Showing non-degeneracy

Last time, we saw that given some function $\phi(x)$ we can find the values of A_m, called fourier coefficients.
Remember that $X_m = \sin(\frac{m\pi x}{l})$

TODO

- Define inner product
- show $(X_m, X_m) = 0$ if $m \neq n$
- show $(X_m, X_m) \neq 0$ for $m = 1, 2, 3, ...$

Now, last video we assumed all of these to be true. Now we need to actually show it.
So what would an inner product be for a function?
First we recall the dot product, which is the inner product for vectors.
Now, assume we have a function $X_1(x)$, which we have a value for each value of x, so in this way it is an infinite vector, ie $X_1(x) = \begin{pmatrix} . \\ . \\ . \\ . \\ . \end{pmatrix}$, and we have one of these for X_2, X_3, etc...

So, for a fixed value of x, we can think of multiplying the components of $X_1(x)X_{2(}x)$, which gives us an infinite number of products we have to add up. But, the thing is there is not a discrete set of numbers! So we take the intergral!

$$\int_0^l X_1(x)X_2(x)dx$$

We take it from 0 to l because that's how we have been defining the boundary. So this integral is our inner product!

Now, onto step 2.
Recall $X_m = \sin(\frac{m\pi x}{l}$ so then we have:

$$(X_m, X_n) = \int_0^l \sin(\frac{m\pi x}{l}) \sin(\frac{n\pi x}{l})dx$$

We rewrite this as:

$$(X_m, X_n) = \int_0^l \frac{1}{2}\cos(\frac{m\pi x}{l} - \frac{n\pi x}{l}) - \frac{1}{2}\cos(\frac{m\pi x}{l} + \frac{n\pi x}{l})dx$$

$$(X_m, X_n) = \int_0^l \frac{1}{2}\cos(\frac{(m-n)\pi x}{l}) - \frac{1}{2}\cos(\frac{(m+n)\pi x}{l})dx$$

Now, we just need to integrate this:

$$(X_m, X_n) = |_0^l \frac{1}{2}\sin(\frac{(m-n)\pi x}{l})\frac{l}{(m-n)\pi} - \frac{1}{2}(\frac{(m+n)\pi x}{l})\frac{l}{(n+m)\pi}dx$$

$$\frac{1}{2}\sin(\frac{(m-n)\pi l}{l})\frac{l}{(m-n)\pi} - \frac{1}{2}\sin(\frac{(m+n)\pi l}{l})\frac{l}{(n+m)}\frac{1}{2} - \sin(\frac{(m-n)\pi 0}{l})\frac{l}{(m-n)\pi} - \frac{1}{2}\sin(\frac{(m+n)\pi 0}{l})\frac{}{(n}$$

$$\frac{1}{2}\sin((m-n)\pi)\frac{l}{(m-n)\pi} - \frac{1}{2}\sin((m+n)\pi)\frac{l}{(n+m)} - \frac{1}{2}\sin(0)\frac{l}{(m-n)\pi} - \frac{1}{2}\sin(0)\frac{l}{(n+m)}$$

And then the $\sin((n \pm m)\pi)$ terms go to zero so we get zero as desired! And this is our 'dot product'

Now we want to move to step 3, showing $(X_m, X_m) \neq 0$ for $m = 1, 2, 3....$

We have: $(X_m, X_m) = \int_0^l \sin(\frac{m\pi x}{l})\sin(\frac{m\pi x}{l})dx$

Now we use our trig identity again,

$$(X_m, X_m) = \int_0^l \frac{1}{2}\cos(\frac{m\pi x}{l} - \frac{m\pi x}{l}) - \frac{1}{2}\cos(\frac{m\pi x}{l} + \frac{m\pi x}{l})dx$$

$$\implies (X_m, X_m) = \int_0^l \frac{1}{2}\cos(0) - \frac{1}{2}\cos(\frac{m\pi x}{l} + \frac{m\pi x}{l})dx$$

$$(X_m, X_m) = \int_0^l \frac{1}{2} - \frac{1}{2}\cos(\frac{2m\pi x}{l})dx$$

$$(X_m, X_m) = |_0^l \frac{1}{2}x - \frac{1}{2}\sin(\frac{2m\pi x}{l})\frac{l}{2m\pi}$$

$$(X_m, X_m) = |_0^l \frac{1}{2}x$$

Evaluating our bounds we just get $(X_m, X_m) = \frac{1}{2}l$

Fourier Sine Series

We have

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$$

Where A_m is given by the formula

$$A_m = \frac{2}{\ell}\int_0^{\ell} \phi(x) \sin \frac{m\pi x}{\ell} dx$$

V37 Examples Computing Fourier Series

Let's calculate the fourier sine series of $\phi(x) = 1$, with the same problem setup as the last video

$A_m = \frac{2}{l}\int_0^l \sin(\frac{m\pi x}{l} dx = \frac{2}{l}|_0^l - \cos(\frac{m\pi x}{l}\frac{l}{m\pi}$

Now, we just need to evaluate: $\frac{2}{l}\left[-\cos(\frac{m\pi l}{l})\frac{l}{m\pi} - -\cos(\frac{m\pi 0}{l})\frac{l}{m\pi}\right]$

Simplifying, we get:

$\frac{2}{l}\left[-\cos(m\pi)\frac{l}{m\pi} + \frac{l}{m\pi}\right]$

This just simplifies to:

$$\frac{2}{m\pi}\left[1 - \cos(m\pi)\right]$$

We want to simplify this even further, with the knowledge that $\cos(\pi m) = 1$ if m is even, and -1 if m is odd. So, in other words, it's equal to $(-1)^m$.

$$\implies \frac{2}{m\pi}\left[1 - (-1)^m\right]$$

If m is even, the whole thing becomes zero, and if m is odd, the whole thing becomes $A_m = \frac{4}{m\pi}$

Now, we know the coefficient is 0 if m is even, so we rewrite our sum as:

$$1 = A_1 \sin(\frac{\pi x}{l}) + A_3 \sin(\frac{3\pi x}{l}) + A_5 \sin(\frac{5\pi x}{l}) + ...$$

$$1 = \frac{4}{\pi} \sin(\frac{\pi x}{l}) + \frac{4}{3\pi} \sin(\frac{3\pi x}{l}) + \frac{4}{5\pi} \sin(\frac{5\pi x}{l}) + ...$$

We want to make sure this is valid between x and l

Let's look at how the book solves $\phi(x) = x$ on page 109
We note that they skipped doing integration by parts, they let $x = u$ and $\sin(\frac{m\pi x}{l} = dv$. They also use the trick where $-\cos(m\pi) = (-1)^{m+1}$

V38: Putting it all together!

Let's think about the diffusion equation again with Dirichlet boundary conditions.

Find solution to the diffusion equation with Dirichlet boundary conditions

$$u_t - ku_{xx} = 0, \qquad 0 < x < \ell, \quad t > 0$$

$$u(0,t) = 0 \qquad u(x,0) = 1$$
$$u(\ell, t) = 0$$

We know the start could be initially hot everywhere except at the boundaries, where it is really cold....so our question is, how is this going to diffuse heat?
We know what $u(x,t)$ is using separation of variables steps 1-4, then we have step 5 which is using our boundary conditions to simplify it, then finally we have our fourier coefficients.

Let's compute the first three terms of $u(x,t)$

1. $u(x,t) = \frac{4}{\pi} e^{-(\frac{\pi}{l})^2 kt} \sin(\frac{\pi x}{l}) + ...$ (all of this so far is when $n = 1$
2. We skip $n = 2$ because it's just zero
3. Adding in $n = 3$, we get:
   $u(x,t) = \frac{4}{\pi} e^{-(\frac{\pi}{l})^2 kt} \sin(\frac{\pi x}{l}) + \frac{4}{3\pi} e^{-(\frac{3\pi}{l})^2 kt} \sin(\frac{3\pi x}{l})$

3 HW i9 problems

V34 Problems

3. Write down the formula for the Fourier Sine series of a function $\phi(x)$. What does the formula mean in plain words?

The Fourier sine series is $\phi(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l})$. In plain words, this means that any (maybe?) function $\phi(x)$ can be written as a linear combination of sine functions.

V35 Problems

4. In order to understand the procedure to compute Fourier coefficients, it is instructive to consider a similar problem but for vectors. Consider the three basis vectors

a) **Show that $X_n \cdot X_m = 0$ where $n \neq m$ for $n = 1, 2, 3$ and $m = 1, 2, 3$. This is orthogonality.**

- $\vec{X_1} \cdot \vec{X_2} = 1 + 0 - 1 = 0$
- $\vec{X_1} \cdot \vec{X_3} = 1 + 0 - 1 = 0$
- $\vec{X_2} \cdot \vec{X_3} = 1 - 2 + 1 = 0$

b) **Compute $X_m \cdot X_m$ where $m = 1, 2, 3$.**

- $\vec{X_1} \cdot \vec{X_1} = 1 + 0 + 1 = 2$
- $\vec{X_2} \cdot \vec{X_2} = 1 + 2 + 1 = 4$
- $\vec{X_3} \cdot \vec{X_3} = 1 + 2 + 1 = 4$

c) **Using the procedure described in the video, find values for the coefficients A_1, A_2 and A_3 such that** $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = A_1\vec{X_1} + A_2\vec{X_2} + A_3\vec{X_3}$

We let $\vec{\phi} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Then, we know from the video that $A_m = \frac{(\phi, X_m)}{(X_m, X_m)}$

Plugging in our three values for X_m, we get:

- $A_1 = \frac{\phi \cdot \vec{X_1}}{\vec{X_1} \cdot \vec{X_1}} = \frac{1+0-3}{2} = \frac{-2}{2} = -1$

- $A_2 = \frac{\phi \cdot \vec{X_2}}{X_2 \cdot \vec{X_2}} = \frac{1+2\sqrt{2}+3}{4} = \frac{4+\sqrt{2}}{4}$
- $A_3 = \frac{\phi \cdot \vec{X_3}}{X_3 \cdot \vec{X_3}} = \frac{1-2\sqrt{2}+3}{4} = \frac{4-2\sqrt{2}}{4}$

d) **Write down a formula for the coefficients A_m. Show work to justify your formula.**
Suppose we are hoping to find one of the constants, A_m. Since all the vectors are orthogonal, we can multiply both sides of the equation by X_m and know that everything except the term that has X_m in it will be zero, since we are multiplying every term by X_m, which is the same as taking the dot product and we know that is 0 for orthogonal vectors.

$$\vec{X_m} \cdot \vec{\phi} = \vec{X_m} \cdot \sum_{n=1}^{N} A_n \vec{X_n}$$

$$\implies \vec{X_m} \cdot \vec{\phi} = \vec{X_m} \cdot A_1 \vec{X_1} + \vec{X_m} \cdot A_2 \vec{X_2} + ... + \vec{X_m} \cdot A_m \vec{X_m} + \vec{X_{m+1}} \cdot A_{m+1} \vec{X_{m+1}} + ... + \vec{X_m} \cdot A_n \vec{X_n}$$

Again, every term on the RHS except $\vec{X_m} \cdot A_m \vec{X_m}$ has a dot product of zero, giving us:

$$\implies \vec{X_m} \cdot \vec{\phi} = \vec{X_m} \cdot \vec{X_m} A_m$$

Solving for A_m, we get:

$$A_m = \frac{\vec{X_m} \cdot \vec{\phi}}{\vec{X_m} \cdot \vec{X_m}}$$

V36 Problems

5. The Fourier cosine series of a function $\phi(x)$ is defined as $\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{l})$ where $n \geq 0$

a) **Show that $(X_m, X_n) = 0$ if $n \neq m$ where m and n are non-negative integers and $X_n = \cos(\frac{n\pi x}{l})$**

In order to compute the inner product, we take the integral of the product of X_m, X_n. We also assume $m \neq n$ for integers $m, n \geq 0$. Then, we have:

$(X_m, X_n) = \int_{n=0}^{l} \cos(\frac{n\pi x}{l}) \cos(\frac{m\pi x}{l})$

Using the trig identity for $\cos(a)\cos(b)$, we get:

$(X_m, X_n) = \int_{n=0}^{l} \frac{\cos(\frac{n\pi x}{l} + \frac{m\pi x}{l}) + \cos(\frac{n\pi x}{l} - \frac{m\pi x}{l})}{2}$

Simplifying and computing the integral gives us:

$(X_m, X_n) = \frac{1}{2}\int_{n=0}^{l} \cos(\frac{(n+m)\pi x}{l}) + \cos(\frac{(n-m)\pi x}{l})$

$(X_m, X_n) = \frac{1}{2}|_{n=0}^{l} \sin(\frac{(n+m)\pi x}{l})\frac{l}{(n+m)\pi} + \sin(\frac{(n-m)\pi x}{l})\frac{l}{(n-m)\pi}$

Plugging in our bounds:

$(X_m, X_n) = \frac{1}{2}\left(\sin(\frac{(n+m)\pi l}{l}) + \sin(\frac{(n-m)\pi l}{l}) - \left(\sin(\frac{(n+m)\pi 0}{l}) + \sin(\frac{(n-m)\pi 0}{l})\right)\right)$

$(X_m, X_n) = \frac{1}{2}\left(\sin((n+m)\pi) + \sin((n-m)\pi) - 0\right)$

$(X_m, X_n) = \frac{1}{2}(0 - 0)$

$\implies (X_m, X_n) = 0$

b) **Compute** (Xm, Xm)

We first assume $m \geq 0$.

$(X_m, X_m) = \int_{n=0}^{l} \cos(\frac{m\pi x}{l})\cos(\frac{m\pi x}{l})$

$(X_m, X_m) = \int_{n=0}^{l} \cos^2(\frac{m\pi x}{l})$

Using our trig identity $\cos(2x) = 2\cos^2(x) - 1$ gives us:

$(X_m, X_m) = \frac{1}{2}\int_{n=0}^{l} \cos(\frac{2m\pi x}{l}) + \frac{1}{2}\int_{n=0}^{l} dx$

$\implies (X_m, X_m) = \frac{1}{2}|_{n=0}^{l} \sin(\frac{2m\pi x}{l})\frac{l}{2m\pi} + x$

Plugging in our bounds we get:

$(X_m, X_m) = \frac{1}{2}\left(\sin(\frac{2m\pi l}{l})\frac{l}{2m\pi} + l) - (\sin(\frac{2m\pi 0}{l})\frac{l}{2m\pi} + 0)\right)$

$\implies (X_m, X_m) = \frac{1}{2}\left(\sin(2m\pi)\frac{l}{2m\pi} + l) - (\sin(0)\frac{l}{2m\pi} + 0)\right)$

$\implies (X_m, X_m) = \frac{1}{2}\left(\sin(2m\pi)\frac{l}{2m\pi} + l - 0\right)$

Finally, we know that $\sin(m\pi) = 0\ \forall m \in \mathbb{Z}$, giving us:

$(X_m, X_m) = \frac{1}{2}(0 + l) = \frac{1}{2}l$

We now assume $m = 0$

Then, we have:

$(X_m, X_m) = \int_{n=0}^{l} \cos(\frac{m\pi x}{l}) \cos(\frac{m\pi x}{l}) dx$

Plugging in 0 for m gives us:

$(X_m, X_m) = \int_{n=0}^{l} \cos(\frac{0\pi x}{l}) \cos(\frac{0\pi x}{l}) dx$

$(X_m, X_m) = \int_{n=0}^{l} \cos(0) \cos(0) dx$

$(X_m, X_m) = \int_{n=0}^{l} 1 dx$

Integrating 1, we get:

$(X_m, X_m) = |_{n=0}^{l} x$

Finally, plugging in our bounds gives us our final answer:

$(X_m, X_m) = l - 0 = l$

c) **Use the formula from the video to find a formula for A_m if $\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{l})$**

We first want to denote $X_n = \cos(\frac{n\pi x}{l})$ and then we have $\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n X_n$

Our goal is to find an equation for any A_m such that $m \leq n$
Then, expanding this expression, we have $\phi(x) = \frac{1}{2}A_0 + A_1X_1 + A_2X_2 + ... + A_nX_n$

We want to take the 'dot product' of both sides with X_m, which is the inner product. This gives us:

$(\phi(x), X_m) = (\frac{1}{2}A_0, X_m) + A_1(X_1, X_m) + A_2(X_2, X_m) + .. + A_m(X_m, X_m).. + A_n(X_n, X_m)$

Now, we know from part b that $(X_m, X_n) = 0$ if $n \neq m$, so all the terms where $n \neq m$ go to zero. For now, we will consider $m > 0$ and go back to the case that $m = 0$ later.

$\implies (\phi(x), X_m) = A_m(X_m, X_m)$

Solving for A_m we get:

$\implies A_m = \frac{(\phi(x), X_m)}{(X_m, X_m)}$

Now, we know from part b that $(X_m, X_m) = \frac{1}{2}l$ if $m > 0$, so we get:

$A_m = \frac{2((\phi(x), X_m)}{l}$

We know that the inner product is just an integral, giving us: $(\phi(x), X_m) = \int_0^l \phi(x)\cos(\frac{n\pi x}{l})dx$, giving us:

$$A_m = \frac{2}{l}\int_0^l \phi(x)\cos(\frac{m\pi x}{l})dx, \text{ for } m \neq 0$$

We now consider the case that $m = 0$.

We go back to the previous step: $\phi(x) = \frac{1}{2}A_0 + A_1X_1 + A_2X_2 + ... + A_nX_n$

We know that $X_0 = \cos(\frac{0\cdot\pi x}{l}) = \cos(0) = 1$ so when we take the inner product of every term with X_0, we get:

$(\phi(x), 1) = (\frac{1}{2}A_0, 1) + A_1(X_1, 1) + A_2(X_2, 1) + ... + A_n(X_n, 1)$
Then, know that $(X_m, 1) = \int_0^l \cos(\frac{m\pi x}{l})dx$

This implies that $(X_m, 1) = |_0^l \sin(\frac{m\pi x}{l})\frac{l}{m\pi} = \sin(\frac{m\pi l}{l})\frac{l}{m\pi} - \sin(0)m\pi = \sin(m\pi)\frac{l}{m\pi} - \sin(0)m\pi = 0$
So, all the terms except the first in our expression for ϕ go to zero, leaving us with:

$(\phi(x), 1) = (\frac{1}{2}A_0, 1)$
These inner products are again just integrals, giving us

$\implies \int_0^l \phi(x)dx = \frac{1}{2}A_0\int_0^l 1dx$

Evaluating the RHS integral:

$\implies \int_0^l \phi(x)dx = \frac{1}{2}lA_0$

Finally, solving for A_0, we get:

$\implies A_0 = \frac{2}{l}\int_0^l \phi(x)dx$

Since $\cos(\frac{m\pi x}{l}) = 1$ when $m = 1$, our equation for A_0 is the same as our equation for A_m (this is why we added the random $1/2$ to the A_0 coefficient)! Thus, for all non-negative integers m, we have the following formula for coefficients A_m:

$$A_m = \frac{2}{l}\int_0^l \phi(x)\cos(\frac{m\pi x}{l})dx$$

V37 Problems

Consider the function $\phi(x) = 2x$ with domain $0 < x < 1$

a) **Write what the Fourier cosine coefficients are for this function. Use the formula you derived in the previous problem. For this example you can explicitly compute any integrals. Please show lots of detail**

In the last problem, we got:

$$A_m = \frac{2}{l}\int_0^l \phi(x)\cos(\frac{m\pi x}{l})dx$$

Now, we want to replace $\phi(x)$ with $2x$, and consider the interval $(0, 1)$ giving us:

$$A_m = 2\int_0^1 2x\cos(\frac{m\pi x}{1})dx$$

Now, we let $2x = u$ and $dv = \cos(m\pi x)$ to use integration by parts $\int udv = uv - \int vdu$, so then $du = 2$ and $v = \sin(m\pi x)\frac{1}{m\pi}$

$$A_m = 2\left(\frac{2x}{m\pi}\sin(m\pi x)|_0^1 - \int_0^1 2\sin(m\pi x)\frac{1}{m\pi}dx\right)$$

We now compute the integral:

$$A_m = 2\left(\frac{2x}{m\pi}\sin(m\pi x)|_0^1 + \cos(m\pi x)\frac{2}{m^2\pi^2}|_0^1\right)$$

$$A_m = \frac{4x}{m\pi}\sin(m\pi x)|_0^1 + \cos(m\pi x)\frac{4}{m^2\pi^2}|_0^1$$

Plugging in our bounds, we get:

$$A_m = (\frac{4}{m\pi}\sin(m\pi) - \frac{4}{m\pi}\sin(0)) + (\cos(m\pi)\frac{4}{m^2\pi^2} - \cos(0)\frac{4}{m^2\pi^2}))$$

We know $\sin(0)$ and $\sin(m\pi) = 0$, and $\cos(0) = 1$ so we get:

$$A_m = (0 - 0) + (\cos(m\pi)\frac{4}{m^2\pi^2} - \frac{4}{m^2\pi^2}))$$

$$A_m = \frac{4}{m^2\pi^2}\left[\cos(m\pi) - 1\right])$$

$$A_m = \frac{4}{m^2\pi^2}\left[(-1)^m - 1\right]$$

If m is even, the equation goes to zero and we are left with:

$$A_m = \frac{4}{m^2\pi^2}\left[1 - 1\right] = 0$$

If m is odd, we have:

$$A_m = \frac{4}{m^2\pi^2}\left[-1 - 1\right] = \frac{-8}{m^2\pi^2}$$

b) **Write down the Fourier cosine series of the function $\phi(x) = 2x$ with domain $0 < x < 1$ using the coefficients you just computed**
We know the Fourier sine series is of the form:

$$2x = \sum_{n=1}^{\infty} \frac{-8}{n^2\pi^2} \cos(n\pi x)$$

c) **Write the first three terms explicitly**
Then, the first three terms are:

$$2x = \frac{-8}{\pi^2} \cos(\pi x) + \frac{-2}{\pi^2} \cos(2\pi x) + \frac{-8}{9\pi^2} \cos(3\pi x) + ...$$

References