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Homework 2

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1 Notes for Week 2

V5 First order PDEs with variable coefficients

We now consider the case where a, b are functions instead of constants, ie $a(x,y)u_x + b(x,y)u_y = 0$ given $a, b \neq 0$
Example: $u_x + yu_y = 0$
Now, $a = 1, b = y$
This implies that the characteristic curves have slope y, so $\frac{dy}{dx} = y \implies y = e^x \cdot C \implies C = ye^{-x}$
We call this equation the characteristic curve, along which $u(x,y)$ does not change
So then $f(C) = f(ye^{-x}$ which is the general solution

Rule for variable coeffiecients

$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$ then use this to find general solution

V6 Coordinate Method

Our goal is to solve equations of the form $au_x + bu_y = h(u,x,y)$
Consider $au_x + bu_y = 0$
Then, we can represent the curve $x' = ax + by$ and $y' = bx - ay$ and know that these lines are perpendicular, and that these coordinates have an x axis parallel to $v =< a, b >$

Cross Product

Now we want to take the cross product to compute the partial x and y with our new coordinates

1. $u_x(x'(x,y), y'(x,y) = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x}$
2. $u_y(x'(x,y), y'(x,y) = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y}$

Then, we can simplify $au_x + bu_y$ in terms of these partial derivatives to get:
$a(au_{x'} + bu_{y'}) + b(bu_{x'} - au_{y'} = (a^2 + b^2)u_{x'} = 0$
and then since we know $a, b \neq 0$ we have $u_{x'} = 0 \implies u(x', y') = f(y')$ then simply replace with our regular varibles to get $u(x,y) = f(bx - ay)$

V7 Coordinate Method continued

Coordinate method summary:

1. Rewrite with new coordinates $x' = ax + by$ and $y' = bx - ay$
2. Solve in new coordinates
3. Transform solution back to original coordinates (x, y)

Try: $u_x + u_y = 2$
So then $x' = x + y$ and $y' = x - y$ which gives us $2u_{x'} = 2$
Now we have $u(x', y') = x' + f(y') \implies u(x, y) = (x + y) + f(x - y)$

V8 Summary of 1st order PDEs

We know how to solve 1st order PDEs with variable coefficients, constant coefficients, and nonhomogenous PDEs set equal to a third function

Geometric Method

We use this for functions with variable coefficients

1. Write down ODE $\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$ then solve the ODE
2. Solve for C in ODe
3. General solution is then $u(x, y) = f(C)$
4. replace C with solution to ODE

Coordinate Method

Refer to V7 summary

2 Practice Problems

V5

2. Suppose that $u = u(x, y)$ **satisfies the PDE** $xu_x + yu_y = 0$
We then know $\frac{dy}{dx} = \frac{y}{x} \implies \frac{1}{y}dy = \frac{1}{x}dx \; ln(y) = ln(x) + C \implies e^{ln(y)} = e^{ln(x+C)} \implies y = Cx \implies C = y/x$

So, we know $f(y/x) = u(x, y)$ we also know $u(1, 1) = f(1) = 3$ and $u(1, 2) = f(2) = 4$

(a) $u(2, 2) = 3$ is TRUE because $f(2/2) = f(1)$ which we know equals 3

(b) $u(2,3) = 3$ is might be true because $u(2,3) = f(3/2)$ and we don't know anything about $f(3/2)$

(c) $u(2,4) = 3$ is FALSE because $f(4/2) = f(2)$ which we know is equal to 4

3. Problem 6 in 1.2 – Solve $\sqrt{1-x^2}u_x + u_y = 0$ given $u(0,y) = y$
Now, we know $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

Then, this implies that $dy = \frac{1}{\sqrt{1-x^2}}dx \implies y = sin^{-1}(x) + C \implies C = y - sin^{-1}(x)$
We then have $u(x,y) = f(y - sin^{-1}(x)$
Finally, we apply our initial condition $u(0,y) = f(y-0) = f(y) = y$
This implies that $u(x,y) = y - sin^{-1}(x)$

V6

4. Confirm x' and y' are orthogonal
$x' = ax + by \implies by = x' - ax \implies y = \frac{x'}{b} - \frac{a}{b}x$
$y' = bx - ay \implies ay = bx - y' \implies y = \frac{b}{a}x - \frac{y'}{a}$
The slopes b/a and $-a/b$ are negative reciprocals, so the lines are indeed orthogonal

5. Demonstrate that the x' axis is parallel to the vector $v =< a, b >$
We know the gradient of x' is $\nabla x' =< a, b >$ which is exactly the vector we have

6. Compute u_y using chain rule for $u(x', y') = \sin(x') + y'^2$ and $x' = x + y$ and $y' = x - y$
$u_y = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial u}{\partial y}$
$\implies u(x', y') = (u_{x'} \cdot 1) - (u_{y'} \cdot 1)$
Therefore, $u(x', y') = \cos(x') - 2y'$

V7

7. Solve $au_x + bu_y + cu = 0$
We can use the coordinate method to simplify this to be $(a^2 + b^2)u_{x'} + cu = 0$
This implies that $(a^2 + b^2)u_{x'} = -cu \implies u = e^{\frac{-cx'}{a^2+b^2}} \cdot f(y')$
Finally, we replace the coordinate variables to get our final answer:
$u(x,y) = e^{\frac{-c(ax+by)}{a^2+b^2}} \cdot f(ax - by)$

V8

8. Graph b corresponds to $u_x + 2u_y = 0$ because the characteristic curves are linear
9. Graph a corresponds to $u_x + 2xy^2 u_y = 0$ because the characteristic curves are $C = -x^2 - \frac{1}{y}$
We got this since $dy/dx = 2xy^2 \implies dy = 2xy^2 dx \implies -1/y = x^2 + C \implies C = -x^2 - \frac{1}{y}$