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Group Assignment 6

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1

No outside sources other than hw and class notes :)

3

Show that there is exactly one solution to the following problem:
$u_t + cu_x = f(x,t)$**, given I.C.** $u(x,0) = \phi(x), -\infty < x < \infty$

1. Suppose that u_1 and u_2 are two solutions to our PDE.
   Our goal is to prove that $u_1 = u_2$, which would imply that there is only one unique solution.
   Define $w(x,t) = u_1(x,t) - u_2(x,t)$.
   $\implies w_t + cw_x = (u_{1t} - u_{2t}) + c(u_{1x} - u_{2x})$
   $\implies w_t + cw_x = (u_{1t} + cu_{1x}) - (u_{2t} + cu_{2x})$
   $\implies w_t + cw_x = f(x,t) - f(x,t) = 0$
   Thus, we have shown that $w_t + cw_x = 0$

   Now, we consider our I.C. $w(x,0) = u_1(x,0) - u_2(x,0) = \phi(x) - \phi(x) = 0$

2. Now, we will define our energy function, which we will use to prove the uniqueness of our solution.
   $$E(t) = \int_{-\infty}^{\infty} \frac{w^2}{2} dx$$
   Now, our goal is to show that this energy function is constant with respect to t. In order to prove this, we will show that $dE/dt = 0$.
   $\implies \frac{dE}{dt} = \frac{d}{dt} \int_{-\infty}^{\infty} \frac{w^2}{2} dx$

   Now, by assumption A1, we know that $\frac{dE}{dt} = \int_{-\infty}^{\infty} \frac{d}{dt} \frac{w^2}{2} dx = \int_{-\infty}^{\infty} \frac{2w \cdot w_t}{2} dx = \int_{-\infty}^{\infty} w \cdot w_t dx$
   Let's re-visit what we discovered in part 1. We know that $w_t + cw_x = 0$ and $w(x,0) = 0$. So, we can use the geometric method to solve this PDE.

   We can see that $a = c$ and $b = 1$, so the general solution to this PDE is $w(x,t) = g(x-ct)$ for an arbitrary function g. We can plug in our initial condition to get $w(x,0) = g(x) = 0$. Since we know $g(x) = 0$, we also know that $g(x-ct) = (0) = w(x,t)$ for all x, t.

   Note: We solved the problem here. Since $w(x,t) = 0$, $u_1(x,t) = u_2(x,t)$, so there is exactly one solution to the problem. However, we will continue to show the normal progression of solving this type of problem to show that we are extra knowledgeable humans and love math.

Since we know $w(x,t) = 0$, we can plug this value into our integral to get $\frac{dE}{dt} = \int_{-\infty}^{\infty} (0) \cdot w_t dx = \int_{-\infty}^{\infty} 0 dx = 0$

Finally, we have that $\frac{dE}{dt} = 0$, which implies that $E(t)$ is constant with respect to t

3. Since $E(t)$ is constant with respect to t, then we know that $E(t) = E(a)$ for all a, and therefore $E(t) = E(0)$.

   $E(0) = \int_{-\infty}^{\infty} \frac{w(x,0)^2}{2} dx = \int_{-\infty}^{\infty} \frac{(0)^2}{2} dx = 0$

   Therefore, we know that since $E(0) = 0, E(t) = 0$ for all t. So, we have $E(t) = \int_{-\infty}^{\infty} \frac{w(x,t)^2}{2} dx = 0$

   Let us express the inside of the integral as $f(x)$, since we are differentiating with respect to x so we do not need to include t. This means that $f(x) = \frac{w(x,t)^2}{2}$. We know that $f(x) \geq 0$ for all x since $w(x,t)$ is being squared and therefore cannot be negative.

   Thus, by the vanishing theorem, we know that $\frac{w(x,t)^2}{2} = 0$. So, we know that $w(x,t) = 0$, or else this expression wouldn't be true. (Yes we already showed this...but double confirmation!)

   Since $w(x,t) = 0$, we know that $w(x,t) = u_1(x,t) - u_2(x,t) = 0 \implies u_1(x,t) = u_2(x,t)$, so there is exactly one solution to this problem.