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Homework 8

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1 Resources

Other than the textbook and class notes, nothing

2 Notes for Week 8

V30 Music!

The wave equation is related to tension in a string
Last video we found that the solution to the wave equation with boundary conditions is:

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos(\tfrac{n\pi}{l}ct) + B_n \sin(\tfrac{n\pi}{l}ct) \right) \sin(\tfrac{n\pi}{l}x)$$

Now, let's assume $A_1 = 1, A_n = 0, B_n = 0$ otherwise
Then our B_n term goes to zero so we have:

$$u(x,t) = \cos(\tfrac{\pi ct}{l}) \sin(\tfrac{\pi x}{l})$$

Then, the cosine part is the time part and the sin part is the shape
We know that **period and frequency** describe oscillations in time.

Let's say we have $\cos(\omega t)$ then ω is the **angular frequency**
Then, if $\omega = 2\pi f$, then f is the frequency and $T = \frac{1}{f}$ is the period
So, if $c = 523.2$ and $l = 1$, then we have
$\cos(\frac{\pi t}{l}t) \implies \omega = \frac{\pi c}{l} \implies 2\pi f = \omega \implies f = \frac{\omega}{2\pi} = \frac{\pi c}{l} \cdot \frac{1}{2\pi} \implies f = 523.3/2$
If we think of T in seconds, then the period is in seconds, so $f = \frac{1}{T}$ is the frequency which is s^{-1} which is Hertz.

What is sound?

As the string vibrates, it causes the air waves to compress and move. The string oscillates at roughly 261.2 Hz, so these oscillations are going to move according to the frequency. The compressed air somehow makes it into your ear, and the compressed air (waves) cause your ear drum to vibrate. As a side note, a vibrating membrane is described by $u_{tt} - c^2(\triangle u) = 0$ Your brain interprets these vibrations as sound!So, sound is just vibrations of different frequencies

So, what does the frequency of oscillation at 261.2 Hz sound like? Modeled by $u(x,t) = \cos(\pi 523.2t)\sin(2\pi x)$
This is middle C!
What about $u(x,t) = \cos(2\pi 523.2t)\sin(2\pi x)$?

The harmonics are just multiplying the fundamental note by an integer! This was discovered by Euler in the 1700s.

Summary

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi ct}{\ell} + B_n \sin \frac{n\pi ct}{\ell} \right) \sin \frac{n\pi x}{\ell}$$

Each term in this series correspond to a note with frequency $\dfrac{n\pi\sqrt{T}}{\ell\rho}$

The fundamental note is $\dfrac{\pi\sqrt{T}}{\ell\rho}$

The first overtone is $\dfrac{2\pi\sqrt{T}}{\ell\rho}$

V31 Eigenvalues!

1. Defining an eigenvalue problem with operators
2. Connection to Dirichlet B.C.
3. Neumann BC case

An eigenvalue problem is:

$$Ax = \lambda x, \text{ where } x \text{ is a vector}$$

We also have A is a matrix and lambda is a scalar
We can think about the matrix being given and we have to find lambda and vector x
Maybe we have N pairs of eigenvalues and eigenvectors
We also know the zero vector is never an eigenvector and any scalar multiple of an eigenvector is an eigenvector

Eigenvalue problem for linear algebra:

$$-\frac{d^2}{dx^2}X = \lambda X, X'(0) = X'(l) = 0$$

Where $-\frac{d^2}{dx^2}X$ is an operator, $X = X(x)$ is a function and λ is a scalar. So, its like an eigenvalue problem but we also have boundary conditions specified as well (dirchlet b.c since we are specifying the function itself and not its derivative)
We also have that λ is an eigenvalue and $X(x)$ is an eigenfunction. We have the same rules:

- the zero function is never an eigenfunction
- any scalar multiple of an eigenfunction is an eigenfunction

We have the same problem as last week:
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Example: Find the positive eigenvalues and associated eigenfunctions to the problem:

$$-\frac{d^2}{dx^2}X =, X(0) = X(l) = 0$$

So, let's do it! Let's break this up to give us a structure:

a) Let $\beta^2 = \lambda$ and $\beta \neq 0$ and $\beta \in \mathbb{R}$
$-\frac{d^2}{dx^2}X = \beta^2 X$

b) Solve the ODE: $X(x) = C\cos(\beta x) + D\sin(\beta x)$
Note that the bcs are different, they are neumann B.C!

c) Use the boundary condition to determine the constants: $X'(0) = 0, X'(l) = 0$
$X'(x) = -\beta c\sin(\beta x) + \beta D\cos(\beta x)$
$X'(0) = \beta D\cos(0) = \beta D = 0 \implies D = 0$
$X'(l) = -\beta C\sin(\beta l) = 0 \implies \sin(\beta l) = 0$ We know this since we don't want $C = 0$
$\implies \beta l = \frac{n\pi}{l}$
We have $\lambda = \beta^2 = (\frac{n\pi}{l})^2$
$X(x) = C\cos(\beta x) = C\cos(\frac{n\pi}{l}x)$ and **this is our answer**

Now, let's check if $\lambda = 0$ is an eigenvalue of the problem (same as above)

a) $\lambda = 0 \implies -\frac{d}{dx^2}X = 0$

b) Solution is $X(x) = c_1 x + c_2$

c) Use b.c. so $X' = c_1 \implies X'(0) = 0, c_1 = 0$
Now, c_2 can be anything, and what we have is $\lambda = 0, X(x) = c_2$ so yes!

Convenient summary!

Problem $-\frac{d^2}{dx^2}X = \lambda X, \quad X(0) = X(\ell) = 0$

Eigenvalues $\lambda_n = \left(\frac{n\pi}{\ell}\right)^2$ $\quad n = 1,2,3,\ldots$

Eigenfunctions $X_n(x) = \sin\left(\frac{n\pi}{\ell}x\right)$

Problem $-\frac{d^2}{dx^2}X = \lambda X, \quad X'(0) = X'(\ell) = 0$

Eigenvalues $\lambda_n = \left(\frac{n\pi}{\ell}\right)^2$ $\quad n = 0,1,2,3,\ldots$

Eigenfunctions $X_n(x) = \cos\left(\frac{n\pi}{\ell}x\right)$

What was different this time with neumann bc was that 0 was an eigenvalue (ex we just did). The eigenfunctions are different (sin vs cos)
Also, with neumann, $n = 0 \implies \lambda_0 = 0 \implies X_0 = \cos(0) = 1$, so this is the eigenvalue with a constant as an eigenfunction, which is why we dont need the arbitrary constants!

V32: Wave eq. with Neumann B.C.

Goal: find solution to the wave equation with Neumann boundary conditions

So, we have $u_{tt} - c^2u_{xx} = 0$, given $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$, and $u_x(0,t) = 0$ and $u_x(l,t) = 0$. The last two are our Neumann b.c.s
We will be goin through the first 3 steps of separation of variable method to solve PDE on unbounded domain

1. Separate variables to obtain 2 ODEs
   Assume $u(x,t) = X(x)T(t)$ and plug in to get $\frac{T''}{c^2T} = -\frac{X''}{X} = \lambda$
   We know lambda is constant so we get $-T'' = \lambda c^2 T$ and $-X'' = \lambda X$

2. Solve X ODE subject to bcs
   $u_x(0,t) = 0 \implies u_x(x,t) = X'(x)T(t) = X'(0)T(t) = 0$, only possible if $X'(0) = 0$
   Similarly, $u_x(l,t) = X'(l)T(t) = 0$, only possible if $X'(l) = 0$
   Thus, we have $-X'' = \lambda X$ with these two conditions, $X'(0) = 0$ and $X'(l) = 0$ which is an eigenvalue problem we solved in the last video

3. Solve the t ODE

$-T'' = \lambda C62T$ with $\lambda_n = (\frac{n\pi}{l})^2$
$-T'' = (\frac{n\pi}{l})^2 c^2 t, n = 1, 2, 3....$
We've seen this already, with solution $T(t) = (A\cos(\frac{n\pi ct}{l} + B\sin(\frac{n\pi ct}{l}))$ for $n = 1, 2, 3....$
We do have to think about the $n = 0$ case since we did not consider it before. If $n = 0$, then $\lambda_0 = 0$, so what we have is $T'' = 0$ or $T_0(t) = A_0 + B_0 t$
Since this was the special case of $n = 0$,we make sure to denote it as such (above)
We will also divide this special solution by 2 for later...
$T_0(t) = \frac{A_0}{2} + \frac{B_0 t}{2}$

V33: Wave Equation Neumann BC steps 4/5

Remember the **goal** was to find a solution to the wave eq with Neumann boundary conditions. Last time, we have that:
$T_n(t) = (A_n\cos(\frac{n\pi ct}{l} + B_n\sin(\frac{n\pi ct}{l}))$ for $n = 1, 2, 3....$ with $X_n(x) = \cos(\frac{n\pi}{l}x)$ and $T_0(t) = \frac{A_0}{2} + \frac{B_0 t}{2}$ with $X_o(x) = 1$

1. Step 4 is take a linear combination of all the solutions:

$$u(x,t) = \frac{A_0}{2} + \frac{B_0 t}{2} + \sum_{n=1}^{\infty}\left((A_n\cos(\frac{n\pi ct}{l} + B_n\sin(\frac{n\pi ct}{l})\right)\cos(\frac{n\pi}{l})$$

We also consider initial conditions, $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$

$u(x,0) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n\cos(\frac{n\pi}{l}) = \phi(x)$

$u_t = \frac{B_0}{2} + \sum_{n=1}^{\infty}\left(-A_n\frac{n\pi c}{l}\sin(\frac{n\pi ct}{l}) + B_n\frac{n\pi c}{l}\cos(\frac{n\pi ct}{l})\right)\cos(\frac{n\pi x}{l})$

We see that $\sin(\frac{n\pi c\cdot 0}{l}) = 0$ and the term with cosine and t goes to 1

$\implies u_t(x,0) = \frac{B_0}{2} + \sum_{n=1}^{\infty} B_n\frac{n\pi c}{l}\cos(\frac{n\pi x}{l}) = \psi(x)$

3 Problems for Week 8

V30 Problems

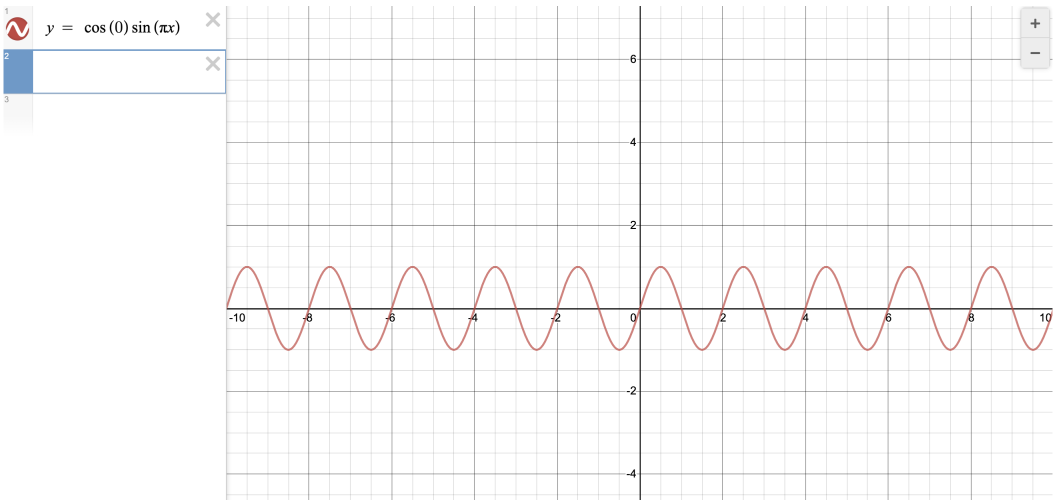
3. If you pluck the "top" string on a guitar, you will hear the note E2, which corresponds to a frequency of 82.4 Hz, roughly. Suppose someone places their finger exactly in the middle of the string, and presses down, effectively making the string length half its original length, and then plucks it once again. What frequency will the person hear? Which note that does frequency correspond to?

The frequency will be twice the frequency of the note E2, so 164.8 Hz, roughly. This corresponds to E3

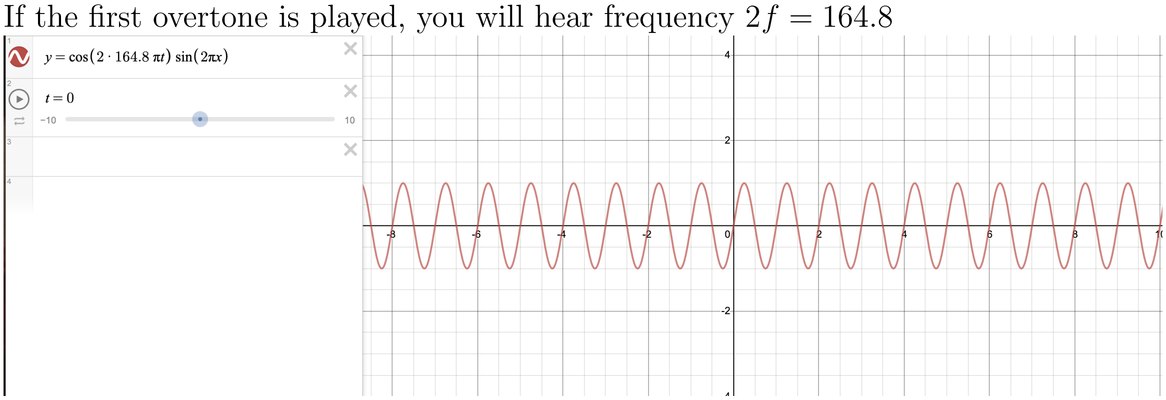
4. Consider a guitar string that is two feet long on a guitar. The fundamental note of the top string is E2, which corresponds to a frequency of 82.4 Hz.

a) **Write down the solution $u(x,t)$ of the wave equation that corresponds to the fundamental note E2.**
We know that $f = \frac{c}{2}$, and since we know that $f = 82.4$ Hz, so $c = 164.8$. Thus, we have $u(x,t) = \cos(164.8\pi t)\sin(\pi x)$

b) **Sketch the solution at $t = 0$**



c) **Sketch the solution at the first overtone at $t = 0$**
If the first overtone is played, you will hear frequency $2f = 164.8$



As we can see, the cosine part of the equation is the time part and the sine part is the shape. We see this since we only change the sine part (as $t = 0$), and we can see that the shape of the second graph is different. The higher pitched note has a higher frequency due to the factor of 2 in the cosine term, which we see as the waves finish a period twice as fast in the second graph.

V31 Problems

5. Consider the eigenvalue problem $-\frac{d}{dx^2}X = \lambda X, X'(0) = X'(l) = 0$ where $X(x)$ is a function and λ is a scalar. In the video I found the positive $\lambda > 0$ eigenvalues and corresponding eigenfunctions, and I found the $\lambda = 0$ eigenvalue and corresponding eigenfunction. Show that there are no negative eigenvalues

If we have $\lambda < 0$, then we let $-\beta^2 = \lambda$ so we get the ODE $-X'' = -\beta^2 X \implies X'' = \beta^2 X$
Then, we have:

$$X'' - \beta^2 X = 0$$

Recall from ODEs that this equation has discriminant $(-\beta^2)^2 - 4(1)(0) = \beta^4$ and thus has two real roots, 0 and β^2
Then, we have that the general solution is $X(x) = Ae^{\beta^2 x} + Be^{0x}$ which simplifies to:

$$X(x) = Ae^{\beta^2 x} + B$$

Applying our boundary conditions, we have:

$$X' = \beta^2 Ae^{\beta^2 x}$$

$$X'(0) = \beta^2 Ae^{\beta^2 0} = \beta^2 A$$

$$\implies 0 = \beta^2 A$$

This implies that either $\beta^2 = 0$ or $A = 0$, but we know that $-\beta^2 = \lambda$ for $\lambda < 0$, so we know it can't be that $\beta^2 = 0$. We can conclude that $A = 0$

So, we know this would yield a trivial solution for $X(x)$ since it would be:
$X(x) = 0 \cdot e^{\beta^2 x} + B \implies X(x) = B$

Now, we consider $X'(l) = 0$

$$X'(l) = \beta^2 Ae^{\beta^2 \cdot l} = 0$$

$$0 = \beta^2 Ae^{\beta^2 \cdot l}$$

From the first boundary condition we know A must be 0, so this brings us to a constant solution again of $X(x) = B$.

Since the eigenfunctions would just be constant, $X(x) = B$ for any B, we know that the only eigenvalue that would satisfy the original $-\frac{d}{dx^2}X = \lambda X$ would be $\lambda = 0$.

Thus, there are no negative eigenvalues

V32 Problems

6. Consider the diffusion equation with Neumann boundary conditions, $u_t - ku_{xx} = 0, u(x,0) = \phi(x), u_x(0,t) = 0, u_x(l,t) = 0$ **Our goal is to find a series solution using separation of variables. Some of the steps are the same as in the video for the wave equation! So you don't need to repeat that work, but you do need go find that information.**

a) **Write down the ODE for** $X(x)$ **and** $T(t)$
We know that we want $u(x,t)$ to be in the form $u(x,t) = X(x)T(t)$, which implies that $XT' - kX''T = 0$
Separating the X and T terms, we get that $\frac{T'}{kT} = \frac{X''}{X}$
We set $\lambda = \frac{T'}{kT}$ and $\lambda = \frac{X''}{X}$. Since $\frac{d\lambda}{dt} = 0$ and $\frac{d\lambda}{dx} = 0$, we know lambda must be constant.
Then, we know that X ODE is $\lambda X = X''$
Second, our $T(t)$ ODE is $T' = \lambda kT$

b) **Write down the eigenvalues and eigenfunctions for the X ODE**
The eigenvalues for the X ODE are $\lambda_n = (\frac{n\pi}{l})^2$ for $n \geq 0$ such that $n \in \mathbb{N}$ (from last week's videos)
The eigenfunction is $X_n(x) = \cos(\frac{n\pi}{l}x)$, also from the video

We now want to consider our initial condition, $u_x(0,t) = u_x(l,t) = 0$
We know that $u_x = X'T \implies u_x(0,t) = X'(0)T(t)$ and $u_x(l,t) = X'(l)T(t)$
This implies that $X'(0) = 0$ and $X'(l) = 0$

We have that $X' = -\frac{n\pi}{l}\sin(\frac{n\pi}{l}x)$

$\implies 0 = -\frac{n\pi}{l}\sin(\frac{n\pi}{l} \cdot 0)$
and $0 = -\frac{n\pi}{l}\sin(\frac{n\pi}{l} \cdot l)$
These are our initial conditions

We also consider the case that $n = 0$. When $n = 0$, we have $X_0(x) = \cos(0) = 1$

c) **Find the general solution to the T(t) ODE**
We now look to solve $T' = \lambda kT$, which is a first order ODE
Rearranging and integrating on both sides, we get $\int \frac{T'}{T}dt = \int \lambda k dt$
We know from ODEs that $\frac{u'}{u} = \ln'(u)$
$\implies \int \ln'(T)dt = \int \lambda k dt$
Then, we get that $\ln(T) = \lambda kt + c$
Finally, we raise both sides to the power e and arrive at:
$T = e^{\lambda kt + c}$ for some arbitrary constant of integration c
$\implies T = Ce^{\lambda kt}$

We include the random 1/2 in front of C to get:

$$T(t) = \frac{C}{2}e^{\lambda kt}$$

We replace lambda with the value we got in step b for our eigenvalues and add n to denote which solution we have:

$$T_n(t) = \frac{C_n}{2}e^{(\frac{n\pi}{l})^2 kt}$$

V33 Problems

7. Continued from above..

a) **Take the information you found in problem 6 and make the appropriate linear combination to obtain a series solution.**

We first want to take a look at $T_0(t) = \frac{C}{2}e^{(\frac{0\cdot\pi}{l})^2 kt}$
$\implies T_0(t) = \frac{C_0}{2}e^0 = \frac{C_0}{2}$
We already know from the video notes that $X_0(x) = 1$
Now, taking a linear combination of solutions we get:

$$u(x,t) = X(x)T(t) = X_0(x)T_0(t) + X_n(x)T_n(t)$$

Plugging in our values for $X_n(x)$ and $T_n(t)$ from problem 6 and our values for $X_0(x)$ and $T_0(t)$, we get:

$$u(x,t) = \frac{C_0}{2} + \sum_{n=1}^{\infty} \frac{C_n}{2}e^{(\frac{n\pi}{l})^2 kt} \cdot \cos(\frac{n\pi}{l}x)$$

b) **Find a formula that relates the initial value to the arbitrary constants in your general solution from part 7a. You DO NOT need to solve for the arbitrary constants.**
We consider the Neumann b.c.s one at a time

(a) $u(x,0) = \phi(x)$

$$\implies \phi(x) = \frac{C_0}{2} + \sum_{n=1}^{\infty} \frac{C_n}{2}e^{(\frac{n\pi}{l})^2 k\cdot 0} \cdot \cos(\frac{n\pi}{l}x)$$

$$\implies \phi(x) = \frac{C_0}{2} + \sum_{n=1}^{\infty} \frac{C_n}{2} \cdot \cos(\frac{n\pi}{l}x)$$

(b) $u_x(0,t) = 0$
Taking the x derivative of u we get:

$$u_x = \sum_{n=1}^{\infty} -\frac{C_n}{2}e^{(\frac{n\pi}{l})^2 kt} \cdot \sin(\frac{n\pi x}{l}) \cdot \frac{n\pi}{l}$$

$$\implies u_x = \sum_{n=1}^{\infty} -\frac{C_n n\pi}{2l} e^{(\frac{n\pi}{l})^2 kt} \cdot \sin(\frac{n\pi x}{l})$$

Looking at our bcs $u_x(0,t) = 0$, we have:

$u_x(0,t) = \sum_{n=1}^{\infty} -\frac{C_n n\pi}{2l} e^{(\frac{n\pi}{l})^2 kt} \cdot \sin(\frac{n\pi \cdot 0}{l})$

Since $\sin(0) = 0$, we know that the entire sum just is 0, no matter what the value of C_n.

(c) We now consider $u_x(l,t) = 0$

$\implies u_x(l,t) = \sum_{n=1}^{\infty} -\frac{C_n n\pi}{2l} e^{(\frac{n\pi}{l})^2 kt} \cdot \sin(\frac{n\pi \cdot l}{l})$

$\implies u_x(l,t) = \sum_{n=1}^{\infty} -\frac{C_n n\pi}{2l} e^{(\frac{n\pi}{l})^2 kt} \cdot \sin(n\pi)$

Since $\sin(n\pi) = 0 \ \forall n \in \mathbb{N}$, this verifies that $u_x(l,t) = 0$

References