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Group Assignment 7

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Other than hw and class notes, we used the one reference at the end of the document :)

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Find a series solution to $u_{tt} - 4u_{xx} = 0$ for $0 < x < 2$, given $u(0,t) = 0, u_x(2,t) = 0$

a) **Write down the ODE for $X(x)$ and $T(t)$**
We know that we want $u(x,t)$ to be in the form $u(x,t) = X(x)T(t)$

Since we have $u_t - 4u_{xx} = 0$, we know this equals $XT' - 4X''T = 0$ given that X and T only have one independent variable

Separating the X and T terms, we have $-\frac{T'}{4T} = -\frac{X''}{X}$ (the inclusion of the minus signs is arbitrary, and only with reference to the format of the textbook!)

We set $\lambda = -\frac{T'}{4T}$ which implies that $\frac{d\lambda}{dx} = 0$
We also let $\lambda = -\frac{X''}{X}$ which implies that $\frac{d\lambda}{dt} = 0$

Since both partial derivatives are 0, we know lambda must be constant and thus $\lambda = -\frac{T'}{4T} = -\frac{X''}{X}$

Then, we know that $X(x)$ ODE is $\lambda X = -X''$
Second, our $T(t)$ ODE is $T' = -4\lambda T$

b) **Write down the eigenvalues and eigenfunctions for the X ODE**
We are considering our ODE $\lambda X = -X''$. We know that we cannot have negative eigenvalues as proved in problem 5 on the individual homework, so we can let $\lambda = \beta^2$ for some $\beta \in \mathbb{R}$, $\beta \neq 0$. We will consider the case when λ is 0 later.

In question 6b) on the individual homework, we showed that the general solution to this ODE is $X(x) = C\cos(\beta x) + D\sin(\beta x)$ for some constants $C, D \in \mathbb{R}$

Now, we need to use our boundary conditions to find the arbitrary constants.

Recall $u(x,t) = X(x)T(t)$, so $u(0,t) = X(0)T(t) = 0$. This is only true if $X(0) = 0$, so we have one boundary condition for the X ODE.

$u_x(x,t) = X'T$, so $u_x(2,t) = X'(2)T(t) = 0$. This is only true if $X'(2) = 0$, so we have our second boundary condition for the X ODE.

$X(0) = C(1) + 0 = 0 \implies C = 0$
So, we now know $X(x) = D\sin(\beta x)$
$X'(x) = D\beta\cos(\beta x)$
$X'(2) = D\beta\cos(\beta 2)$

We don't want $D = 0$, or we would have the trivial solution $X(x) = 0$. So, we want to consider when the cos term is equal to 0. We know that $\cos\left(\frac{(2n-1)\pi}{2}\right) = 0$ for any $n \in \mathbb{Z}^+$, or basically any odd multiple of $\pi/2$.
So, we can let $2\beta = \frac{(2n-1)\pi}{2}$, so $\beta = \frac{(2n-1)\pi}{4}$
So, we have $X'(x) = D\frac{(2n-1)\pi}{4}\cos\left(\frac{(2n-1)\pi x}{4}\right)$
So, we know $X(x) = D\sin\left(\frac{(2n-1)\pi x}{4}\right)$ for some arbitrary constant D. However, we know that any scalar multiple of an eigenfunction is also an eigenfunction, so we don't really need to include the constant. Also, we know that each eigenfunction is dependent on n, so we can re-write:

$X_n(x) = \sin\left(\frac{(2n-1)\pi x}{4}\right)$ for $\lambda_n = \left(\frac{(2n-1)\pi}{4}\right)^2$, since we know $\lambda = \beta^2$. These eigenfunctions and eigenvalues hold for $n = 1, 2, 3, ...$ However, we need to consider when $\lambda = 0$ and therefore $n = 0$.

When $\lambda = 0$, we have the ODE $X'' = 0$. So, by doing some quick integration, we have $X_0(x) = c_1x + c_2$ for some constants $c_1, c_2 \in \mathbb{R}$.
Now, let's plug in our initial values.
$X_0(0) = c_2 = 0 \implies c_2 = 0$, so we now have $X_0(x) = c_1x$.
$X'(x) = c_1$
$X'(2) = c_1 = 2 \implies c_1 = 2$, so we have $X_0(x) = 2x$ for $\lambda = 0$

In summary, we have

$X_n(x) = \sin\left(\frac{(2n-1)\pi x}{4}\right)$ and $\lambda_n = \left(\frac{(2n-1)\pi}{4}\right)^2$ for $n = 1, 2, 3, ...$

$X_0(x) = 2x$ and $\lambda = 0$ for $n = 0$

c) **Find the general solution to the T(t) ODE**
We now look to solve $T' = -4\lambda T$, which is a first order ODE
We can solve this ODE using separation of variables. We can re-write the ODE as $\frac{dT}{dt} = -4\lambda T$
Then, we can see $-4\lambda dt = \frac{dT}{T}$. Now, we should integrate both sides.
Then, we get that $\ln(T) = -4\lambda t + C$ for some arbitrary constant of integration C
Finally, we raise both sides to the power e and arrive at:

$T = e^{-4\lambda t + C}$
$\implies T = Ce^{-4\lambda t}$
We include the arbitrary factor of 1/2 in front of C to get:

$$T(t) = \frac{C}{2}e^{-4\lambda t}$$

We know our values of λ from part b)! So, when $\lambda > 0$, we have:

$$T_n(t) = \frac{C_n}{2}e^{-\left(\frac{(2n-1)\pi}{4}\right)^2 4t}$$

$$T_n(t) = \frac{C_n}{2}e^{-\frac{((2n-1)\pi)^2}{4}t} \text{ for } n = 1, 2, 3...$$

When $\lambda = 0$, we have:

$$T_0(t) = \frac{C_0}{2} \text{ for } n = 0$$

d) **Make the appropriate linear combination to obtain a series solution**
We first want to take a look at what our X and T equations are when $n = 0$

Recall $X_0(x) = 2x$ and $T_0(t) = \frac{C_0}{2}$ for some constant $C_0 \in \mathbb{R}$
Now, taking a linear combination of solutions we get:

$$u(x,t) = X(x)T(t) = X_0(x)T_0(t) + X_n(x)T_n(t)$$

(Note that arbitrary constants are built into our products of X and T.)

Plugging in our values for $X_n(x)$ and $T_n(t)$ and our values for $X_0(x)$ and $T_0(t)$, we get:

$$u(x,t) = \frac{C_0}{2} \cdot 2x + \sum_{n=1}^{\infty} \frac{C_n}{2} \sin\left(\frac{(2n-1)\pi x}{4}\right) e^{-\frac{((2n-1)\pi)^2}{4}t}$$

$$u(x,t) = \frac{C_0}{2}x + \sum_{n=1}^{\infty} \frac{C_n}{2} \sin\left(\frac{(2n-1)\pi x}{4}\right) e^{-\frac{((2n-1)\pi)^2}{4}t} \text{ for } n = 0, 1, 2, 3, ...$$

References