Group Homework 7.

1. Used Math24.net for Step 3.
2. Presentation
3. Find a series solution to:

$$u_{tt} - 4u_{xx} + u_t = 0 \quad \text{for } 0 < x < 2$$

$$u(0,t) = 0 \quad , \quad u(2,t) = 0$$

Step 1) separate variables to obtain 2 ODEs

Assume $\exists$ solution of the form $u(x,t) = X(x)T(t)$

Plug in:

$$XT'' - 4X''T + XT' = 0$$

$$XT'' + XT' = 4X''T$$

$$X(T'' + T') = 4X''T$$

$$\frac{T'' + T'}{4T} = \frac{X''}{X}$$

Since the LHS only depends on T and the RHS only depends on X, The only way for this equality to hold is if:

$$\frac{T'' + T'}{4T} = \lambda = \frac{X''}{X}$$ for some constant λ (from V26).

So, we can rewrite this into 2 ODEs

$4T\lambda = T'' + T'$ and $\lambda X = X''$.

Step 2) solve the $X(x)$ ODE subject to bounds.

$u(0,t) = 0$ and $u(2,t) = 0$

$$\lambda X = X''$$

Recall, from individual homework 7,

if $\lambda \geq 0$, $u(x,t)$ is the trivial solution.

Thus, let's solve for $\lambda < 0$.

To ensure λ is negative, let $\lambda = -\beta^2$, so our ODE:

$-\beta^2 X = X''$.

We know from ODEs, that the solution to this is of the form:

$$X(x) = C\cos(\beta x) + D\sin(\beta x)$$

Now, use the boundary conditions.

since $u(x,t) = X(x)T(t)$

if $u(0,t) = 0$, we can assume $X(0)$ must be 0. Since if $T(t) = 0$ $\forall t$, this gives the trivial solution.

So $X(0) = 0 = C\cos(\beta(0)) + D\sin(\beta(0))$

$0 = C + 0 \rightarrow C = 0$

Now, use $u(2,t) = 0$. By the same logic, $X(2)$ must equal zero to avoid having the trivial solution.

$\rightarrow X(2) = 0 = D\sin(\beta \cdot 2)$.

$0 = D\sin(2\beta)$. Notice, if $D = 0$, this gives the trivial solution.

So, $\sin(2\beta) = 0$. This is true is

$2\beta = \pi n$ for an integer n

$\beta = \frac{\pi n}{2}$. Recall $-\beta^2 = \lambda$ and $\lambda < 0$,

so start at $n = 1$. $\boxed{\text{Our soln: } X(x) = D\sin\left(\frac{n\pi}{2}x\right) \text{ for } n \in \mathbb{Z}^+}$

Step 3 Solve the $T(t)$ ODE

$$\frac{T''+T'}{4T}=\lambda \quad \text{for } \lambda=-\beta^2$$

$$T''+T'=-\beta^2(4T)$$
$$T''+T'+\beta^2 4T=0$$

write the characterstic eq for this ODE.

$$x^2+x+4\beta^2=0$$

Consider $\beta=\frac{\pi n}{2}$, so $4\beta^2=4\left(\frac{\pi^2 n^2}{4}\right)=\pi^2 n^2$

So, solve $x^2+x+\pi^2 n^2=0$

$$x=\frac{-1\pm\sqrt{1-4(\pi^2 n^2)}}{2}$$

so our roots can be written as $\frac{-1}{2}\pm\frac{\sqrt{4\pi^2 n^2-1}}{2}i$

So, since n is positive, the roots will always be imaginary.

Therefore, our general soln is.

$$T(t)=e^{-1/2t}\left[A_n\cos\left(\frac{\sqrt{4\pi^2n^2-1}}{2}t\right)+B_n\sin\left(\frac{\sqrt{4\pi^2n^2-1}}{2}t\right)\right]$$

(This general form was adapted from Math24.net)

For each value of n in $\mathbb{Z}^+$, we can write:

$$T_n=e^{-1/2t}\left[A_n\cos\left(\frac{\sqrt{4\pi^2n^2-1}}{2}t\right)+B_n\sin\left(\frac{\sqrt{4\pi^2n^2-1}}{2}\right)\right]$$

Step 4 combine X and T and write linear combination of all solutions to obtain the series solution.

For each n, we have: (over)

$$u_n(x,t) = D_n \sin\left(\frac{n\pi}{2}x\right) \cdot e^{-1/2 t}\left[A_n \cos\left(\frac{\sqrt{4\pi^2 n^2 - 1}}{2}t\right) + B_n \sin\left(\frac{\sqrt{4\pi^2 n^2 - 1}}{2}t\right)\right]$$

Linear combination of all solutions for $n \in \mathbb{Z}^+$ yields:

$$u(x,t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi}{2}x\right) e^{-1/2 t}\left[A_n \cos\left(\frac{\sqrt{4\pi^2 n^2 - 1}}{2}t\right) + B_n \sin\left(\frac{\sqrt{4\pi^2 n^2 - 1}}{2}t\right)\right]$$

We can absorb D_n into A_n and B_n since these are arbitrary constants for each n.

so $$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}x\right) e^{-1/2 t}\left[A_n \cos\left(\frac{\sqrt{4\pi^2 n^2 - 1}}{2}t\right) + B_n \sin\left(\frac{\sqrt{4\pi^2 n^2 - 1}}{2}t\right)\right]$$

$\therefore$ This is our series solution.