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Homework 3

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1 Resources

Other than the textbook and class notes, I briefly went over the problems over the phone with Kayla

2 Notes for Week 4

V13: Solving the wave equation on an unbounded domain

We know that the wave equation $u_{tt} - c^2 u_{xx} = 0$ is the wave equation, used to describe the vibration of a string, assuming $-\infty < x < \infty$ (think of a *really* long string!)
Steps to solving this:

1. Factor the PDE
2. Write down a system of 1st order PDEs
3. Solve the first order PDE

Starting with step 1, we will first factor the PDE as follows:

$$(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})u = 0$$

Now, for step 2, we will define the second part of the above equation to be v, ie

$$u_t + cu_x = v(x, y)$$

We will call this equation 1

Then, by differentiating v with respect to x and t and plugging it into our foiled equation, we get that
$v_t - cu_x = 0$, which we call equation 2

We can start by solving equation 2, ie $\int v_t dt = \int cv_x dx \implies v(x,t) = h(x + ct)$ by the coordinate method since $v(x,t) = h(bx - at)$ by the method and we have $b = 1$ and $a = -c$

Now, we need to solve equation 1, ie $u_t + cu_x = v = h(x + ct)$, where h is some arbitrary function
Now, we use coordinate method again to do step 3, solving this!
$\tilde{x} = cx + t$

$\tilde{t} = x - ct$
This implies that $(c^2 + 1)u_{\tilde{x}} = h(x + ct)$
We will write the argument of h in terms of a new function, s, ie $(c^2 + 1)u_{\tilde{x}} = h(s(\tilde{x}))$

Finally, we solve for $u_{\tilde{x}}$ and integrate to get:

$$u(\tilde{x}, \tilde{t}) = \int \frac{1}{c^2 + 1} h(s(\tilde{x}))d\tilde{x} + g(\tilde{t})$$

Simplifying, we get that $u = f(s(\tilde{x}) + g(\tilde{t})$ and now just need to replace with our old variables, ie

$$u(x, t) = f(x + ct) + g(x - ct) \text{ where f and g are arbitrary functions}$$

V14: The d'Alembert Solution

Outline of this video is as follows:

1. Define the Initial Value Problem
2. Introduce d'Alembert
3. Give an example

For the millionth time, we know the wave equation is $u_{tt} - c^2 u_{xx} = 0$
Now, we are considering this subject to the auxillary conditions $u(x, 0) = \phi(x)$ and $u_t(x, 0) = \psi(x)$ (remember that 2nd order differential equations need two auxillary conditions, which specify the initial value!)
In most cases we considerr $t = 0$ as one of our auxillary conditions, so time is 0. That's why they are called *initial* value problems

d'Alembert Equation

Okay, so we will first just write out what this equation is!

$$u(x, t) = \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds$$

Now, we will solve the IVP $u_{tt} - u_{xx} = 0$ with $u(x, 0) = e^x = \phi$ and $u_t(x, 0) = \sin(s) = \psi$

All we do is plug in and get:

$$u(x, t) = \frac{1}{2}[e^{x+ct} + e^{x-ct}] + \frac{1}{2c}\int_{x-ct} x + ct \sin(s)ds$$

Note that $c = 1$ here, so we just integrate to get a final answer:

$$u(x, t) = \frac{1}{2}[e^{x+t} + e^{x-t}] + \frac{1}{2}[-\cos(x + t) + cos(x - t)]$$

V15: Deriving the d'Alembert Equation

Recall our general solution to the wave equation, $u(x,t) = f(x-ct) + g(x+ct)$
Now, we will use our initial values to determine f, g

$\implies u(x,0) = \phi(x)$
$\implies u_t(x,0) = \psi(x)$

Then, we also know that $u(x,0) = f(x) + g(x) = \phi(x)$

Therefore, we get that $u_t(x,t) = cf'(x+ct) - g'(x-ct)$ by the chain rule
Then at $t = 0$ we have $u_t(x,0) = cf'(x) - cg'(x) = \psi(x)$
Now, we have these two equations in terms of f and g, ie

1. $\phi(s) = f(s) + g(s)$
2. $\psi(s) = cf'(s) - cg'(s)$

Differentiate (1) to get $\phi' = f' + g'$ and then divide (2) by c to get $\frac{\psi}{c} = f' - g'$

Now, add them to get $\phi' + \frac{\psi}{c} = 2f'$ and subtract them to get $\phi' - \frac{\psi}{c} = 2g'$
This implies that $f' = 0.5(\phi' + \frac{\psi}{c})$ and $g' = 0.5(\phi' - \frac{\psi}{c})$

Integrating we get:

1. $f = 0.5(\phi + \frac{1}{2c}\int_0^s ds + A)$
2. $g = 0.5(\phi - \frac{1}{2c}\int_0^s ds + B)$
3. $\implies f(s) + g(s) = \phi(s) + A + B = \phi(s)$ and also implies that $A + B = 0$

Now we recall that $u(x,t) = f(x+ct) + g(x-ct)$ and now all that's left is adding (1) and (2) together! We know that we can combine the ϕ terms and the integrals, and then remember our change of variables ($x = s$) and simplifying to get the **final equation**:

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2}c\int_{x-ct} x + ct\psi(s)ds$$

Problems for V13

3.Verify that $u(x,t) = f(x+ct) + g(x-ct)$ **satisfies** $u_{tt} = c^2 u_{xx}$

First, we will differentiate u twice with respect to t and x

1. $u_t = cf'(x+ct) - cg'(x-ct)$
2. $u_{tt} = c^2 f''(x+ct) + c^2 g''(x-ct)$
3. $u_x = f'(x+ct) + g'(x-ct)$
4. $u_{tt} = f''(x+ct) + g''(x-ct)$
5. $c^2 u_{tt} = c^2 f''(x+ct) + c^2 g''(x-ct)$

Multiplying (4) by c^2 on both sides to give equation 5 gives $u_{tt} = c^2 u_{xx}$ by the equality of 2 and 5

4. Deriving the theorem

(a) $t = \tilde{x} - cx$ and $t = \frac{x-\tilde{t}}{c} \implies \tilde{x} - cx = \frac{x-\tilde{t}}{c} \implies x = \frac{c\tilde{x}+\tilde{t}}{1+c^2}$

(b) $x = (\tilde{x} - t)/c$ and $x = \tilde{t} + ct$ implies that $(\tilde{x} - t)/c = \tilde{t} + ct$ which implies that $t = \frac{\tilde{x}-c\tilde{t}}{c^2+1}$

(c) Then, we get that $h(x+ct) = h(\frac{c\tilde{x}+\tilde{t}}{1+c^2} + c \cdot \frac{\tilde{x}-c\tilde{t}}{c^2+1})$ which simplifies to:
$h(x+ct) = h(\frac{2c\tilde{x}+\tilde{t}-c^2\tilde{t}}{c^2+1})$

(d) Now, we know that $(1+c^2)u_{\tilde{x}} = h(s(\tilde{x})) \implies s(\tilde{x}) = \frac{2c\tilde{x}+\tilde{t}-c^2\tilde{t}}{c^2+1}$

(e) Plugging our $s(\tilde{x})$ into equation (3) gives us $\int \frac{1}{1+c^2} h(\frac{2c\tilde{x}+\tilde{t}-c^2\tilde{t}}{c^2+1}) d\tilde{x}$
Now, we want to integrate and use u-substitution to let $u = \frac{\tilde{x}}{1+c^2}$ which implies that $\int h(2cu + \frac{\tilde{t}-c^2\tilde{t}}{c^2+1}) du + g(\tilde{t})$

Since we are integrating an arbitrary function h, we can integrate and get that the integral is equal to $f(2cu + \frac{\tilde{t}-c^2\tilde{t}}{c^2+1})$ where f is an arbitrary twice differentiable function which is the integral of h

Finally, we plug back in $u = \frac{\tilde{x}}{1+c^2}$ to get:

$\int \frac{1}{1+c^2} h(\frac{2c\tilde{x}+\tilde{t}-c^2\tilde{t}}{c^2+1}) d\tilde{x} = f(\frac{2c\tilde{x}+\tilde{t}-c^2\tilde{t}}{c^2+1}) + g(\tilde{t})$

Based on our definition of $s(\tilde{x})$ from part d, we arrive at the desired relation
$\int \frac{1}{1+c^2} h(s(\tilde{x})) d\tilde{x} = f(s(\tilde{x}))$

(f) We know from part e that $u(\tilde{x}, \tilde{t}) = f(\frac{2c\tilde{x}+\tilde{t}-c^2\tilde{t}}{c^2+1}) + g(\tilde{t})$
Plugging back in our original x and t give us:

$$u(x,t) = f(x+ct) + g(x-ct)$$

where f, g are arbitrary twice differentiable functions

5. Consider $u_{xx} - 2u_{xt} - 8u_{tt} = 0$

(a) **Classify the PDE as parabolic, hyperbolic or elliptic.**
This function is hyperbolic since we have:
$a_{12} = -1$
$a_{11} = 1$
$a_{22} = -8$
$\implies D = (-1)^2 - (-8) = 9 \implies D > 0$ thus it is hyperbolic

(b) **Find the general solution**

- We start by factoring the PDE to get $(\frac{\partial}{\partial x} + 2\frac{\partial}{\partial t})(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial x})u = 0$
- Now, we let $v = u_x - 4u_t$ which implies that $v_x + 2v_t = 0$ since $v_x = u_{xx} - 4u_{xt}$ and $2v_t = -8u_{tt} + 2u_{xt}$
  - Thus, the characteristic lines for $v_x + 2v_t = 0$ have direction vector $< 1, 2 >$ so the equation of the characteristic lines are $-2x + t = C$ and thus a solution is of the form $v(x,t) = h(2x - t)$ where h is an arbitrary differentiable function
  - Now, we know that $h(2x - t) = u_x - 4u_t$ by our definition of v. The last step is to figure out what $u(x, y)$ is.
  - We let $s(x) = 2x - t$ so that $u_x - 4u_t = h(s(x))$.
    Now, let's guess that $u(x, y) = f(2x - t)$
    This then implies that $u_x = 2f'$ and $u_t = f'$, so then $u_x - 4u_t = h(s(x))$
    $\implies -6f' = h(s(x))$
  - Finally, by Theorem* we know that there exists some f such that $-\frac{1}{6}\int h(s(x)) = f(s(x))$ thus verifying our guess. This first solution is thus $u_1(x, y) = f(2x-t)$
- We can now do the same thing to find another solution, with $v = u_x + 2u_t$ and thus $v_x - 4v_t = 0$
  - Then, the characteristic line for $v_x - 4v_t = 0$ have direction vector $< 1, -4 >$ so the equation of characteristic lines are $4x - t = C$ and thus a solution is of the form $v(x, t) = g(4x + t)$
  - By definition of v we then have: $g(4x + t) = u_x + 2u_t$
  - By theorem* and the same reasoning as above we now know that $u_2(x, y) = g(4x + t)$
- Finally, by linearity of u_1 and u_2 we arrive at the general solution:

$$u(x, t) = f(2x - t) + g(4x - t)$$

Problems for V14

6. Find a solution to $u_{tt} - 4u_{xx} = 0$ with initial values $u(x,0) = e^{-x^2}$ and $u(x,0) = 0$

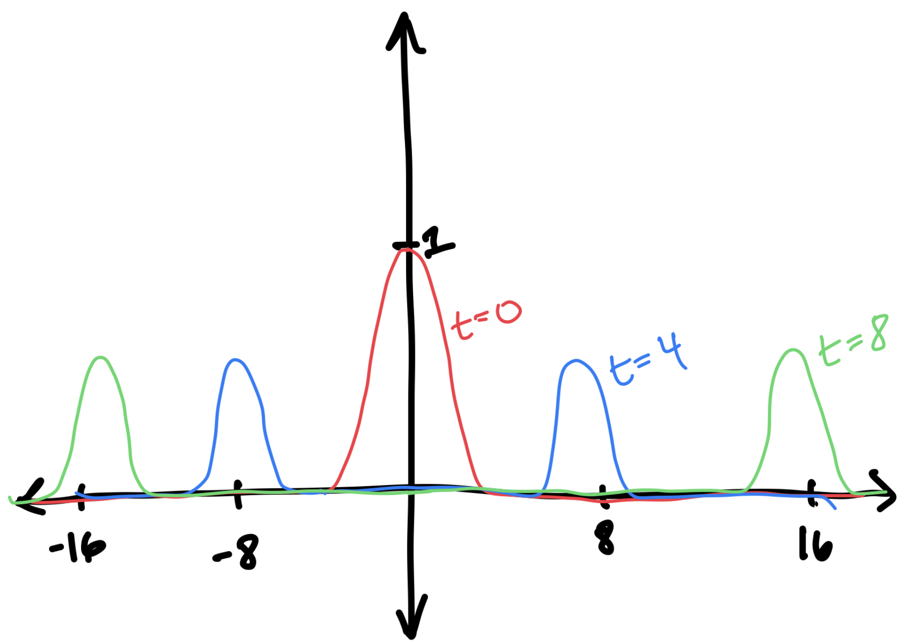
We first recognize that $c = 2$ in this wave equation. Now, plugging into d'Alembert's equation, we get that:

$$u(x,t) = \frac{1}{2}[e^{-(x+2t)^2} + e^{-(x-2t)^2}] + \frac{1}{4}\left[\int_{x-2t}^{x+2t} 0 ds\right]$$

Simplifying this, we get that:

$$u(x,t) = \frac{1}{2}[e^{-x^2-8t^2}]$$

7. Sketch a graph at $t = 0$, $t = 4$, and $t = 8$



This graph is consistent with the fact that c is a speed, since our speed is $c = 2$ units/sec. We can see that at $t = 4$ when 4 seconds have passed, the peak is at $x = 8$, indicating that the wave is travelling at the desired speed. Similarly, at $t = 8$ the peak is at $x = 16$

8. Using the d'Alembert formula, write down the solution to the following initial value problem

We have that $u_{tt} = c^2 u_{xx}$ given $u(x,0) = \log(1+x^2)$ and $u_t(x,0) = 4+x$
Plugging our values into d'Alembert's equation, we get that:

$$u(x,t) = \frac{1}{2}\left[\log(x+ct) + \log(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} 4 + sds$$

We know $\int 4 + sds = 4s + \frac{1}{2}s^2 + c$, so $\int_{x-ct}^{x+ct} 4 + sds = (4(x+ct) + 1/2(x+ct)^2 + c) - (4(x-ct) + 1/2(x-ct)^2 + c)$ and therefore simplifying, the equation equals:

$$u(x,t) = \frac{1}{2}\left[\log(x+ct) + \log(x-ct)\right] + \frac{1}{2c}\left[8ct + 2cxt\right]$$

Problems for V15

9. Solve $u_{xx} - 2u_{xt} - 8u_{tt} = 0$ **given** $u(x,0) = \phi(x)$ **and** $u_t(x,0) = \psi(x)$

First, from problem 5 we know that:
$u(x,t) = f(2x-t) + g(-4x-t)$ given arbitrary functions f, g.
Now, to find the general solution we can compute the following:

- $\phi(x) = f(2x) + g(-4x)$
- $\implies \phi' = 2f'(2x) - 4g'(-4x)$
- $\psi(x) = -f'(2x) - g'(-4x$

Then, with this, we can compute $\phi' + 2\psi$ and $\phi' - 4\psi$ to solve for g' and f'

1. $\phi' + 2\psi = -6g'(-4x) \implies g'(-4x) = \frac{-1}{6}(\phi' + 2\psi)$
2. $\phi' - 4\psi = 6f'(2x) \implies f'(2x) = \frac{1}{6}(\phi' - 4\psi)$

Now, integrating (1), we get that:

$\int g'(-4x)dx = \int \frac{-1}{6}(\phi' + 2\psi)\, ds \implies \frac{-1}{4}g(-4x) = \frac{-1}{6}\phi - \frac{1}{3}\int_{x-t/2}^{0} \psi ds + A$

Thus, $g(-4x) = \frac{2}{3}\phi + \frac{4}{3}\int_{x-t/2}^{0} \psi ds + A$

Now integrating (2), we get that:

$\int f'(2x)dx = \int (\frac{1}{6}(\phi' - 4\psi)ds \implies \frac{1}{2}f(2x) = \frac{1}{6}\phi - \frac{4}{6}\int_{0}^{x+t/4} \psi ds + B$

Thus, $f(2x) = \frac{1}{3}\phi - \frac{4}{3}\int_0^{x+t/4} \psi ds + B$

We also know that $\phi(s) = f(2s) + g(-4s)$ from our original computation of $\phi(x)$ above. Then, we see that $f(2s)+g(-4s) = \frac{1}{3}\phi+\frac{2}{3}\phi+A+B = \phi(s)+A+B$ and thus $\phi(s)+A+B = \phi(s)$ by the equivalence relation. The only way this can hold true is if $A + B = 0$

Finally, combining f and g we get that:

$u(x,t) = \frac{1}{3}\phi(2x+t) + \frac{2}{3}\phi(-4x+t) - \frac{4}{3}\int_{x-t/2}^{x+t/4} \psi ds + A + B$

Simplifying, we see that:

$$u(x,t) = \frac{1}{3}\phi(2x+t) + \frac{2}{3}\phi(-4x+t) - \frac{4}{3}\int_{x-t/2}^{x+t/4} \psi ds$$

is our final answer!