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Homework 10

November 17, 2020

1 Resources

Other than the textbook and class notes, nothing

2 Notes for Week 11

V39: Neumann BC and Fourier sine series

Let's consider the heat eq. $u_t - ku_{xx} = 0$ with $0 < x < l, t > 0$, and then Neumann bcs $u_x(0,t) = 0$ and $u_x(l,t) = 0$ and finally $u(x,0) = x$

First, we have to consider what we know from the series solution of the diffusion eq. (pg 90)

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \cos(\frac{n\pi x}{l})$$

Turns out that having the 1/2 on the constant term is the most convenient, so if we take that series relation and plug in $t = 0$, then we should be able to see what we get:

$$u(x,0) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{l})$$

This is the fifth step of the separation of variables method–¿ this is the series solution evaluated at 0 so we know this is equal to x

$$u(x,0) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{l}) = \phi(x) = x$$

We know the definition of fourier cosine series from hw and page 106, and the formula for coefficients are:

$$A_m = \frac{2}{l}\int_0^l \phi(x)\cos(\frac{m\pi x}{l} dx \text{ for } m = 0, 1, 2, 3, ...$$

So, now the question is, why the random 1/2 factor? The nice thing about it is if I pull out the constant away from the sum and add a random 1/2, then the A_0 and all the other A_m are the same. If we leave the 1/2 out, we have to have a separate equation for A_0, since it would just be

$$A_m = \frac{1}{l}\int_0^l \phi(x)dx$$

So, having these different cases is awkward, so we add the random factor to make it easier.
What we know:

From our initial condition we had:

$$u(x,0) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{l})$$

This initial condition is the exact same thing as the fourier cosine series, which is what we use to solve the PDE! Since we know how to get the coefficients with the random 1/2 factor, we add that to our initial conitiions so we can use the same equation to compute A_m

V40: Mixed BC and General Fourier Series

The only difference from the last video will be that we have a **mixed boundary condition** instead of Neumann
So, we have $u(0,t) = 0$ and $u_x(l,t) = 0$
In other words, there is no change in temperature at $x = l$, and it remains cold at $x = 0$

What we know:
We have a series solution from the HW:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\left(\frac{(2n-2)\pi}{2l}\right)^2 t} \sin(\tfrac{(2n-1)\pi x}{2l})$$

Notice: no constant terms in this series solution, so every term has one of the I.V. This is different from the cosine sum

We are now going to think about the initial value:

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(\tfrac{(2n-1)\pi x}{2l})$$

Now, what is this series? This is equal to our $\phi(x) = x$, but we don't know how to solve this series.... it's not really a sine or cosine fourier series, it looks different!!!!

To try and address this issue, we will think back to when we talked about fourier in an abstract sense. We extraced it as a series of basis functions ('linear combination'), and we always assume the series converges

Earlier we showed that $\phi(x) = \sum_{n=1}^{\infty} A_n X_n(x)$ and $A_m = \frac{(\phi, X_m)}{(X_m, X_m)}$ as long as $(X_m, X_n) = 0$, they are orthogonal, and also $(X_m, X_m) \neq 0$

We will explore ocndition 1 later and for now assume they both hold

The old rules were getting this linear combination of basis functions, and what we got to today was that $\phi(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{(2n-1)\pi x}{2l})$

Now, we just need to use the sine part of this as our basis function, and we can then use our equation for A_m

$$A_m = \frac{\int_0^l \phi(x) \sin(\frac{(2n-1)\pi x}{2l}) dx}{\int_0^l \sin(\frac{(2n-1)\pi x}{2l}) \sin(\frac{(2n-1)\pi x}{2l}) dx}$$

Now we have a nice closed form expression for these coefficients. It's not a fourier series but we can still use it! We call this a **general fourier series**
This looks super nasty, but we can look at example 6 on page 110 to try and evaluate this integral

V41 Orthogonality

Earlier we talked about how an arbitrary function $\phi(x)$ can be expressed as an expansion of basis functions X_n, ie

$$\phi(x) = \sum_{n=1}^{\infty} A_n X_n(x) \text{ and } A_m = \frac{(\phi, X_m)}{(X_m, X_m)}$$

We know these formulas hold as long as we have:

- $(X_m, X_m) = 0$ if $m \neq n$
- $(X_m, X_m) \neq 0$ for $m = 1, 2, 3, 4...$

Now, we have been assuming these things hold and last time we ignored the issue of orthogonality, so today we will focus on the first condition. But first, we have to think about where these basis functions X_n come from

We have been thinking about these things as eigenfunctions of some eigenvalue problem with boundary conditions. So, the goal is to prove orthogonality in a more general situation, $(X_m, X_n) = 0$ if $m \neq n$, ie

$$\int_0^l X_m(x) X_n(x) dx = 0$$

We want to exploit the eigenvalue problem to do this

So, suppose X_n and X_m are eigenfunctions for the eigenvalue problem $-X'' = X$ with eigenvalues λ_n and λ_m
So, we want to show:

$$\int_0^l X_m(x) X_n(x) dx = 0$$

We are keeping this equation firm in our mind since it what we want to show. They are not random functions, these are both eigenfunctions to the eigenvalue problem $-X'' = \lambda X$

Now, we start by assuming $\lambda_m \neq \lambda_n$
It can be shown that $-X_m'' X_n + X_m X_n'' = [-X_m' X_n + X_m X_n']'$

We will prove in the homework that these two things are true.
So, for now we will integrate both sides with respect to x

$$\int_0^l -X_m'' X_n + X_m X_n'' dx = -X_m' X_n + X_m X_n' |_0^l$$

On the RHS we used the FTC!

Now, this RHS already looks like boundary conditions, and on the other hand we have the LHS
What do we know? We need to use the fact that we know $-X'' = \lambda X$ and we want to get to $\int X_n X_m = 0$

The left hand side can be simplified using the fact that $-X'' = \lambda X$

$$\int_0^l \lambda_m X_m X_n + X_m(-\lambda_n X_n) dx = -X'_m X_n + X_m X'_n|_0^l$$

Now, we can pull out the lambdas to get:

$$(\lambda_m - \lambda_n) \int_0^l X_m X_n dx = -X'_m(l)X_n(l) + X_m(l)X'_n(l) - [-X'_m(0)X_n(0) + X_m(0)X'_n(0)]$$

We want this LHS to be equal to zero, so we know we are almost where we want. We know that $(\lambda_m - \lambda_n) \neq 0$ since they are distinct.

Now, the role of the b.c.s comes into play...
If we have mixed bcs, for instance $X(0) = 0, X'(l) = 0$, and we aren't able to figure out the eigenvalue expansions for this bcs!
So, every time we see an $X(0)$ its going to be zero!

$$(\lambda_m - \lambda_n) \int_0^l X_m X_n dx = -X'_m(l)0 + X_m(l)0 - [-X'_m(0)0 + 0X'_n(0)]$$

$$(\lambda_m - \lambda_n) \int_0^l X_m X_n dx = 0$$

Thus, we can conclude that

$$\int_0^l X_m X_n dx = 0$$

This is precisely what we set out to demonstrate. We never have to say specifically waht the eigenfunctions are, all we need is the structure from the eigenvalue problem and the boundary conditions. But any boundary conditions that make the LHS go to zero also solve orthogonality!

In summary

Eigenfunctions that satisfy the eigenvalue problem

$$-X'' = \lambda X$$

Are mutually orthogonal (namely $(X_m, X_n) = 0$ if $m \neq n$) if

$$-X_m'(\ell)X_n(\ell) + X_m(\ell)X_n'(\ell) - \left[-X_m'(0)X_n(0) + X_m(0)X_n'(0)\right] = 0$$

and $\lambda_n \neq \lambda_m$

V42: PDEs and sound (again!)

We will get a reminder of wave equation solution, vibration and sound, and connect it to the fourier series

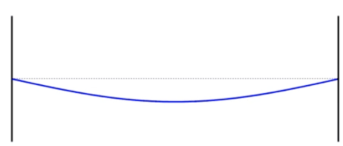
Wave equation

$$u_{tt} - c^2 u_{xx} = 0, \qquad 0 < x < \ell, \quad t > 0$$

$$u(x,0) = \phi(x) \qquad u_t(x,0) = \psi(x)$$

$$u(0,t) = 0$$

$$u(\ell, t) = 0$$



$c = \sqrt{T/\rho}$

T: string tension
ρ: string density

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi ct}{\ell} + B_n \sin \frac{n\pi ct}{\ell} \right) \sin \frac{n\pi x}{\ell}$$

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{\ell} \qquad \psi(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \sin \frac{n\pi x}{\ell}$$

We know we can determine the coefficients for the solution to the wave equation using our fourier sine series. Going back to our frequency, fundamental note, and first overtone, this is assuming dirchilet boundary conditions.

Sound and wave equation

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi ct}{\ell} + B_n \sin \frac{n\pi ct}{\ell} \right) \sin \frac{n\pi x}{\ell}$$

Each term in this series correspond to a note with frequency $\dfrac{n\pi\sqrt{T}}{\ell\rho}$

The fundamental note is $\dfrac{\pi\sqrt{T}}{\ell\rho}$ $n=1$

The first overtone is $\dfrac{2\pi\sqrt{T}}{\ell\rho}$ $n=2$

But this is assuming Dirichlet boundary conditions.....

We want to reiterate that sound is just how our brain interprets changes in air pressure. We can see that a sound wave is pockets of increased and decreased pressure (high vs low density of particles) and the sound propogates along the wave. Sound waves in the air satisfy the wave equation just like sound waves of a violin. This allows us to talk about instruments beyond string instruments

- Stringed instruments satisfty dirichlet bc
- Woodwinds/brass are mixed bc (one side is fixed, one is open)
- The flue is neumann bc (both ends are open)

So, when we hear sound, we are not just listening to one frequency, but a combination of frequencies.
If we play a perfect sine wave, it sounds and looks different in the wave form. So what's going on here in terms of the mathematics? Same note, different sine waves?

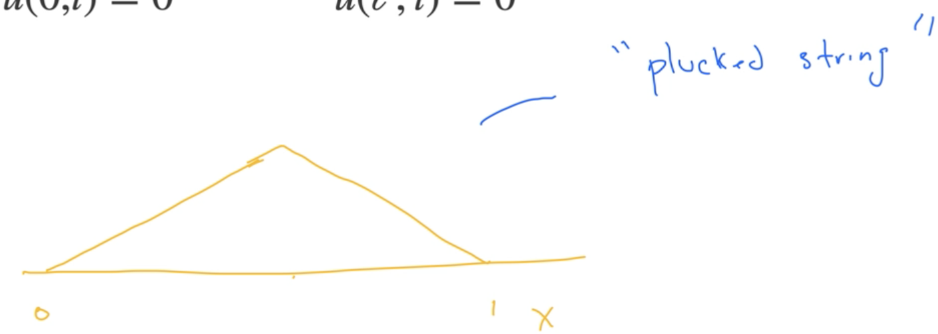
We can think of the non-perfect sine waves (violin/piano) as a sum of sine waves, or a sum of eigenfunctions, or a sum of overtones...so we are playing the fundamental note as a combination of overtones. The design of the instrument plays together to generate some combination of overtones. This is exactly the idea of the fourier series

Wave equation example

$$u_{tt} - c^2 u_{xx} = 0, \qquad 0 < x < \ell, \quad t > 0$$

$$u(x,0) = \begin{cases} x & 0 < x < 1/2 \\ 1 - x & 1/2 < x < 1 \end{cases} \qquad u_t(x,0) = 0$$

$$u(0,t) = 0 \qquad\qquad u(\ell,t) = 0$$



With a "plucked string" you generate sound by plucking the middle of the string, generating many overtones in some combination. We can solve the wave equation with these ICs and boundary conditions. So, what are the A_n and B_n? The infinite series looks crazy and confusing, but the formula is telling us something simpler without the whole sum.

3 HW i10 problems

V39 Problems

3. Consider the following PDE and auxiliary conditions

$$u_t - k u_{xx} = 0 \text{ for } 0 < x < l$$

$$u(x,0) = x$$

$$u_x(0,t) = 0, u_x(1,t) = 0$$

a) **Describe in words what this problem is describing in the context of heat in a rod**
This problem corresponds to both ends of the rod being "insulated", which is why at positions 0 and l the amount of heat is constant (x derivative is 0 at either end). As t

increases, we should see the heat diffusing through the rod, reflected in the x derivative increasing on the one end of the rod and decreasing on the other. Heat is thus flowing from one end to the other, so the rate of change of total heat will increase in a positive direction on one side and decrease in a negative direction on the other. The change in the x derivative will stop on either boundary, although the total amount of heat will still increase/decrease on either endpoint. In other words, the heat flow at the endpoints is prescribed at all times.

b) **Find the solution of this PDE using separation of variables, Fourier series stuff, etc.**
We can find the solution to the PDE in the exact same manner we did on individual hw 8 question 6/7, with the only difference being that here we have $l = 1$ So, we can just paste our answer from that problem here with $l = 1$:

$$u(x,t) = \tfrac{C_0}{2} + \textstyle\sum_{n=1}^{\infty} C_n e^{(n\pi)^2 kt} \cdot \cos(n\pi x)$$

We can now use the expression we derived in video 36 for our fourier coefficients, $A_m = \frac{2}{l}\int_0^l \phi(x) X_m dx$, to find an equation for our coefficients $C_n = A_n$.
We know $\phi(x) = x$ since we have $u(x,0) = x$ and $X_n = \cos(n\pi x)$, since we are dealing with our cosine fourier series.

So, we have $A_m = \frac{(\phi(x), X_m)}{(X_m, X_m)} = 2\int_0^1 x\cos(m\pi x)dx$

- We can use integration by parts, letting $u = x$ and $dv = \cos(m\pi x)dx$

  Therefore, we also have $du = dx$ and $v = \frac{1}{m\pi}\sin(m\pi x)$

  Thus, we have $2\int_0^1 x\cos(m\pi x)dx = 2\left[\frac{x}{m\pi}\sin(m\pi x) - \frac{1}{m\pi}\int_0^1 \sin(m\pi x)dx\right]\Big|_0^1$

  $= 2\left[\frac{x}{m\pi}\sin(m\pi x) + \frac{1}{m^2\pi^2}\cos(m\pi x)\right]\Big|_0^1$

  $= \frac{2x}{m\pi}\sin(m\pi x) + \frac{2}{m^2\pi^2}\cos(m\pi x)\Big|_0^1$

  Then, we know $\sin(m\pi) = 0 \forall m \in \mathbb{Z}$ so we plug in our bounds and are left with:

  $A_m = \frac{2}{m^2\pi^2}\cos(m\pi) - \frac{2}{m^2\pi^2}$

  Now, we know $\cos(m\pi) = -1$ when m is odd and 1 when m is even, giving us:

  $$A_m = \frac{2}{m^2\pi^2}\left[(-1)^m - 1\right]$$

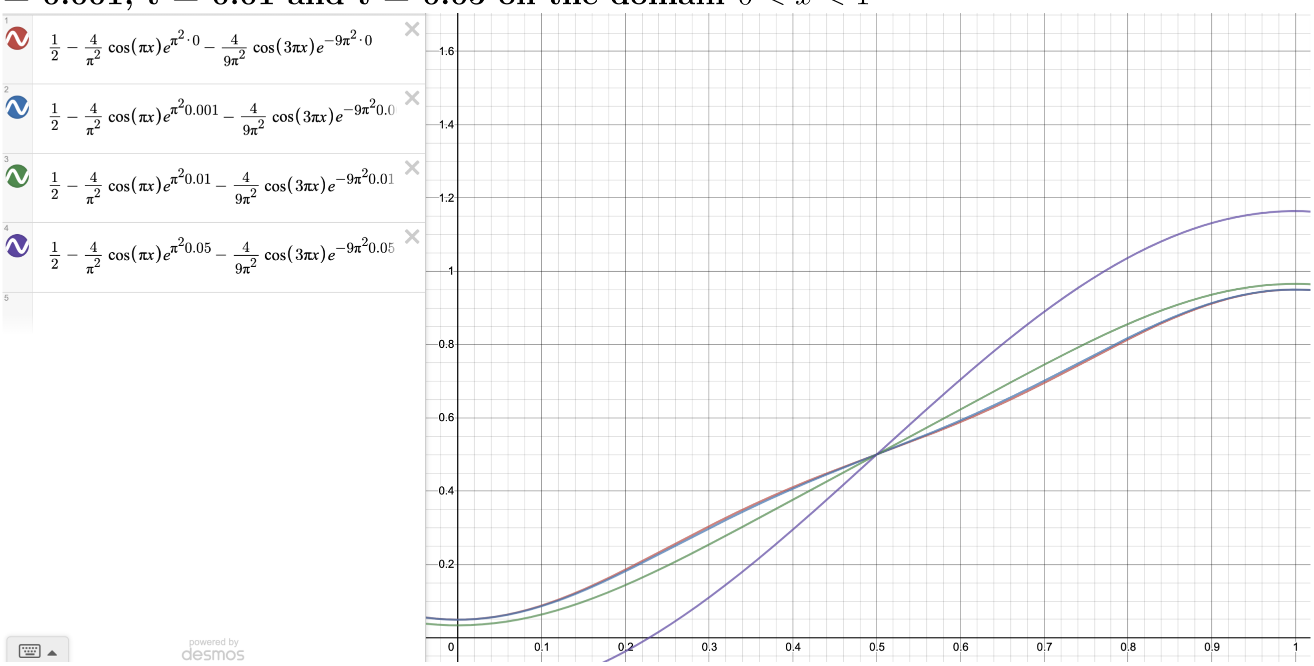
Then, we know $A_m = 0$ when m is even and $A_m = \frac{-4}{m^2\pi^2}$ when m is odd.
We also want to solve for $\frac{C_0}{2}$, so we consider the integral $A_0 = 2\int_0^1 x\cos(0\pi x)dx = 2\int_0^1 xdx = x^2|_0^1 = 1 - 0 = 1$

Finally, the first five terms of our series solution are, starting at $m = 1$:

$$u(x,t) = \frac{1}{2} - \frac{4}{\pi^2}\cos(\pi x)e^{-k(\pi)^2 t} - \frac{4}{9\pi^2}\cos(3\pi x)e^{-9k(\pi)^2 t} - \frac{4}{25\pi^2}\cos(5\pi x)e^{-25k(\pi)^2 t} + \ldots$$

c) **Argue why the first term in your solution is the most dominant.** This is a similar arrgument to the group hw problem once again. We know that the values for A_n are getting smaller and smaller in each term, which means that the terms become negligible as n increases. Thus, the first term is the largest sum and is dominant.

d) **Using $k = 1$ and $l = 1$, plot the first two terms of your solution for t = 0, t = 0.001, t = 0.01 and t = 0.05 on the domain** $0 < x < 1$



e) **Using your answers above, describe in words what the model predicts about the heat flow in the rod as time evolves.**
We can see that at both endpoints, the x derivative is indeed zero. We also see the heat evolving in the rod, with the heat flowing from one end to the other as time evolves. We also note that we can see that the derivatives on either end, $x = 0$ and $x = 1$, are both zero, as desired.

V40 Problems

4. In the video, we found a formula for the Fourier coefficients that corresponded to a particular type of mixed boundary condition. Now you should simplify it. Show that:

$$A_m = \frac{\int_0^l x\sin(\frac{(2n-1)\pi x}{2l})dx}{\int_0^l \sin(\frac{(2n-1)\pi x}{2l})\sin(\frac{(2n-1)\pi x}{2l})dx} = \frac{8l}{\pi^2(2n-1)^2}(-1)^{n+1}, n = 1, 2, 3, ...$$

We will start by considering the denominator $\int_0^l \sin^2(\frac{(2n-1)\pi x}{2l})dx$:

- We know that $\cos(2x) = 1 - 2\sin^2(x)$
  Solving for $\sin^2$, we have $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$
- Plugging this into our expression, the denominator becomes:

  $\int_0^l \frac{1}{2} - \frac{1}{2}\cos(\frac{(2n-1)\pi x}{l})dx$
- Breaking up the integral gives us:

  $\frac{1}{2}\int_0^l dx - \frac{1}{2}\int_0^l \frac{1}{2}\cos(\frac{(2n-1)\pi x}{l})dx$
- Computing the integral, we have:

  $-\frac{1}{2}\left[\frac{l}{(2n-1)\pi}\sin((\frac{(2n-1)\pi x}{l})\right]|_0^l + \frac{1}{2}l$
- When we plug in our bounds, we know that $\sin(0) = 0$ and $\sin((2n-1)\pi) = 0$, giving us:

  $-\frac{1}{2}(0-0) + \frac{1}{2}l = \frac{1}{2}l$

We now consider the numerator, $\int_0^l x\sin(\frac{(2n-1)\pi x}{2l})dx$

- We will use integration by parts, letting $u = x$ and $dv = \sin(\frac{(2n-1)\pi x}{2l})$
- Then, we have $du = 1$ and $v = \frac{-2l}{(2n-1)\pi}\cos(\frac{(2n-1)\pi x}{2l})$
- Then, the numerator becomes:

  $\frac{-2lx}{(2n-1)\pi}\cos(\frac{(2n-1)\pi x}{2l})|_0^l - \int_0^l \frac{-2l}{(2n-1)\pi}\cos(\frac{(2n-1)\pi x}{2l})dx$
- Computing the integral gives us:

  $\frac{-2lx}{(2n-1)\pi}\cos(\frac{(2n-1)\pi x}{2l})|_0^l - (-\frac{4l^2}{(2n-1)^2\pi^2}\sin(\frac{(2n-1)\pi x}{2l})|_0^l)$

- We now note that with our bound $x = 0$, both terms evaluate to zero, since there is an x in front of the cosine term and $\sin(0) = 0$. So, we plug in l for x in both terms and get:

  $\frac{-2l^2}{(2n-1)\pi}\cos(\frac{(2n-1)\pi}{2}) + \frac{4l^2}{(2n-1)^2\pi^2}\sin(\frac{(2n-1)\pi}{2})$

- Now, we know that $\cos(\frac{(2n-1)\pi}{2}) = 0$ for all n, and that $\sin(\frac{(2n-1)\pi}{2}) = 1$ if n is odd and -1 if n is even. This leaves us with:

  $\frac{4l^2}{(2n-1)^2\pi^2}(-1)^{n+1}$

Now, we need to divide our simplified expression for the numerator by our expression for the denominator, $\frac{1}{2}l$

$$\frac{\frac{4l^2}{(2n-1)^2\pi^2}(-1)^{n+1}}{\frac{1}{2}l}$$

Finally, we simplify the denominator and arrive at the desired expression:

$$A_m = \frac{8l}{(2n-1)^2\pi^2}(-1)^{n+1}$$

5. Consider the following PDE and auxiliary conditions

$$u_t - ku_{xx} = 0 \text{ for } 0 < x < l$$

$$u(x, 0) = x$$

$$u(0, t) = 0, u_x(1, t) = 0$$

a) **Describe in words what this problem is describing in the context of heat in a rod (notice this problem is similar but different than problem 3).**
In this context, the rate of change in temperature remains constant at the end $x = 1$, and the actual stays cold at a temperature of 0 at the end $x = 0$. In other words, the left end will always remain cold while the right end will get warmer over time but stay "insulated".

b) **Find the solution of this PDE using separation of variables, Fourier series stuff, etc.**
This problem is similar to the problem on group hw8, where we have mixed boundary conditions. Thus, we can use our same $X(x)$ and $T(t)$ equations, and the only difference is that we have $u_x(1, t) = 0$ instead of $u_x(2, t) = 0$. We also showed in that problem we cannot have negative or zero eigenvalues, so we do not need to show that again.

The only difference in this problem and ghw8 is the one initial condition $u_x(1,t) = 0$, which we consider now.

We know from the first I.C that since $X(x) = C\cos(\beta x) + D\sin(\beta x)$ then $X(0) = 0$ implies that $C = 0$

Then, $X'(1) = 0 \implies D\beta\cos(\beta) = 0$. Since we dont want $D = 0$, it must be that $\cos(\beta) = 0$ and thus $\beta = \frac{(2n-1)\pi}{2}$ for $n = 1, 2, 3, ...$

So, we have that $X_n(x) = \sin(\frac{(2n-1)\pi x}{2})$ for $n = 0, 1, 2, 3, ...$

Now, our $T(t)$ ODE is $T' = k\lambda T \implies \frac{T'}{T} = -k(\frac{(2n-1)\pi}{2})^2$

So, then we integrate both sides and get:

$$T_n = A_n e^{-k(\frac{(2n-1)\pi}{2})^2 t}$$

Then, our final equation becomes:

$$u(x,t) = \frac{A_0}{2}\sin(\frac{-\pi x}{2})e^{\frac{k\pi^2 t}{4}} + \sum_{n=1}^{\infty} A_n \sin(\frac{(2n-1)\pi x}{2})e^{-k(\frac{(2n-1)\pi}{2})^2 t}$$

From the last problem, we have our equation for the coefficients A_n! We know this will work since we have the same sine series and condition $u(x,0) = \phi(x) = x$. We also have that $l = 1$. Then, we can simply plug in our expression for each A_n:

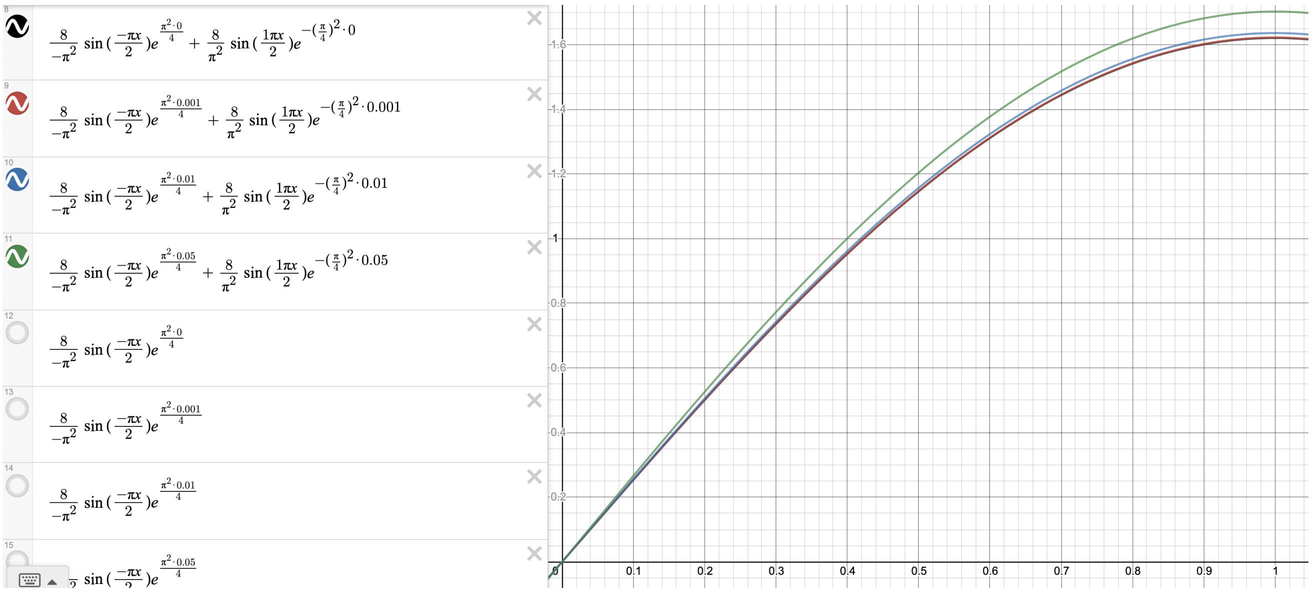
$$u(x,t) = \frac{8l}{-\pi^2}\sin(\frac{-\pi x}{2})e^{\frac{k\pi^2 t}{4}} + \sum_{n=1}^{\infty} \frac{8l}{(2n-1)^2\pi^2}(-1)^{n+1}\sin(\frac{(2n-1)\pi x}{2})e^{-k(\frac{(2n-1)\pi}{2})^2 t}$$

Then, the first three terms of our series solution are:

$$u(x,t) =$$
$$\frac{8l}{-\pi^2}\sin(\frac{-\pi x}{2})e^{\frac{k\pi^2 t}{4}} + \frac{8l}{\pi^2}\sin(\frac{1\pi x}{2})e^{-k(\frac{\pi}{4})^2 t} - \frac{8l}{9\pi^2}\sin(\frac{3\pi x}{2})e^{-k(\frac{3\pi}{4})^2 t} + \frac{8l}{25\pi^2}\sin(\frac{5\pi x}{2})e^{-k(\frac{5\pi}{4})^2 t} + ...$$

c) **Argue why the first term in your solution is the most dominant.**
The first term is again dominant as the value of A_m becomes negligible as m increases, since it is squared in the denominator of our expression for A_m.

d) **Using $k = 1$ and $l = 1$, plot the first term of your solution for t = 0, t = 0.001, t = 0.01 and t = 0.05 on the domain $0 < x < 1$**



e) **Using your answers above, describe in words what the model predicts about the heat flow in the rod as time evolves.**
As time evolves, we see the heat travelling from the end $x = 0$ to the end $x = 1$. This is because we see the heat fixed at $u(0, t) = 0$ for all values of t, and we see the heat increasing at the end $x = 1$ as time increases. We predict that this trend would continue until the heat is diffused all to the right end of the rod.

V41 Problems

6. Verify the following assertion made in the video:

$$-X''_m X_n + X_m X''_n = \frac{d}{dx}[-X'_m X_n + X_m X'_n]$$

We can use the sum rule of differentiation to rewrite this as:

$$-X''_m X_n + X_m X''_n = \frac{d}{dx}[-X'_m X_n] + \frac{d}{dx}[X_m X'_n]$$

Now, we can use the product rule, $(fg)' = f'g + fg'$, to compute each derivative on the RHS:

$$-X''_m X_n + X_m X''_n = (-X''_m X_n - X'_m X'_n) + (X'_m X'_n + X_m X''_n)$$

We see that the two middle terms cancel, leaving us with the desired equality:

$$-X''_m X_n + X_m X''_n = -X''_m X_n + X_m X''_n$$

7. Using the formula derived in the video, show that eigenfunctions of the eigenvalue problem with Dirichlet boundary conditions

$$-X'' = \lambda X \text{ with bcs } X(0) = X(l) = 0$$

are mutually orthogonal. You may assume that all eigenvalues are distinct

In the video, we found that eigenfunctions satisfying $-X'' = \lambda X$ are mutually orthogonal if

$$-X_m'(l)X_n(l) + X_m(l)X_n'(l) - [-X_m'(0)X_n(0) + X_m(0)X_n'(0)] = 0$$

Now, we want to plug in our boundary conditions, so we plug in zero wherever we have $X(0)$ or $X(l)$

$$-X_m'(l) \cdot 0 + 0 \cdot X_n'(l) - [-X_m'(0) \cdot 0 + 0 \cdot X_n'(0)] = 0$$

Then, every term is multiplied by zero, leaving us with the desired result of mutual orthogonality for eigenfunctions X_m, X_n:

$$0 + 0 - (0 + 0) = 0$$

8. Using the formula derived in the video, show that eigenfunctions of the eigenvalue problem with Neumann boundary conditions

$$X'' = \text{ with bcs } X'(0) = X'(l) = 0$$

are mutually orthogonal. You may assume that all eigenvalues are distinct

Again, in the video, we found that eigenfunctions satisfying $-X'' = \lambda X$ are mutually orthogonal if

$$-X_m'(l)X_n(l) + X_m(l)X_n'(l) - [-X_m'(0)X_n(0) + X_m(0)X_n'(0)] = 0$$

Now, we want to plug in our boundary conditions, so we plug in zero wherever we have $X'(0)$ or $X'(l)$

$$0 \cdot)X_n(l) + X_m(l) \cdot 0 - [0 \cdot X_n(0) + X_m(0) \cdot 0] = 0$$

Then, every term is multiplied by zero, leaving us with the desired result of mutual orthogonality for eigenfunctions X_m, X_n:

$$0 + 0 - (0 + 0) = 0$$

V42 Problems

9. Suppose that the pressure waves (i.e. sound waves) a clarinet produces is described by the wave equation with mixed boundary conditions:

$$u_{tt} - c^2 u_{xx} = 0 \text{ for } 0 < x < l$$

$$u(x, 0) = \phi(x), u_t(x, 0) = 0$$

$$u(0, t) = 0, u_x(l, t) = 0$$

Show that this clarinet cannot generate even harmonics (that is, even multiples of the fundamental note). The absence of even harmonics is responsible for the dark or eerier sound of the clarinet. This is in contrast to the saxophone, whose geometry allows the generation of even harmonics too, and hence a "warmer" sound.

This is a wave equation with mixed boundary conditions. In order to show that the clarinet can't generate even harmonics, we want to show that we cannot produce even multiples of the fundamental note. We know that every term corresponds to a note with frequency $\frac{n\pi\sqrt{T}}{lp}$ and thus want to find what this frequency is in the context of this problem.

So, we will solve this PDE with what we know about the wave equation.
From previous videos and page 85 in the textbook, we know that our two ODEs are:

- $X(x) = C\cos(\beta x) + D\sin(\beta x)$
- $T(t) = A\cos(\beta ct) + B\sin(\beta ct)$

We now impose our boundary conditions $X(0) = 0$ and $X'(l) = 0$
This implies that $X(0) = C\cos(0) + D\sin(0) = C$ and thus:
$X(0) = C = 0 \implies C = 0$

This leaves us with $X(x) = D\sin(\beta x)$ and then $X'(x) = \frac{1}{\beta}\cos(\beta x)$
$\implies X'(l) = 0 = \frac{1}{\beta}\cos(\beta l)$
We don't want $\beta = 0$, so thus we want $\beta = \frac{(2n-1)\pi}{2l}$ and then:

$$X(x) = \sin(\frac{(2n-1)\pi x}{2l})$$

We also know that $u_t(x, 0) = 0 \implies T'(0) = 0$ and thus we have:

$$T'(0) = -\frac{A}{\beta}\sin(\beta \cdot 0) + \frac{B}{\beta}\cos(\beta \cdot 0) = \frac{B}{\beta} = 0$$

This implies that $\frac{B}{\beta} = 0$. We know that $\beta = \frac{(2n-1)\pi}{2}$ so $\frac{2B}{(2n-1)\pi} = 0$
Then, we know that $B = 0$ since $2n - 1$ will never equal 0 because $n \in Z$.

$$\implies T(t) = A\cos(\beta ct)$$

Then, this gives us our final answer for $u_n(x,t)$:

$$u_n(x,t) = \sum_{n=1}^{\infty} A_n \cos(\frac{(2n-1)\pi}{2l}ct)\sin(\frac{(2n-1)\pi x}{2l})$$

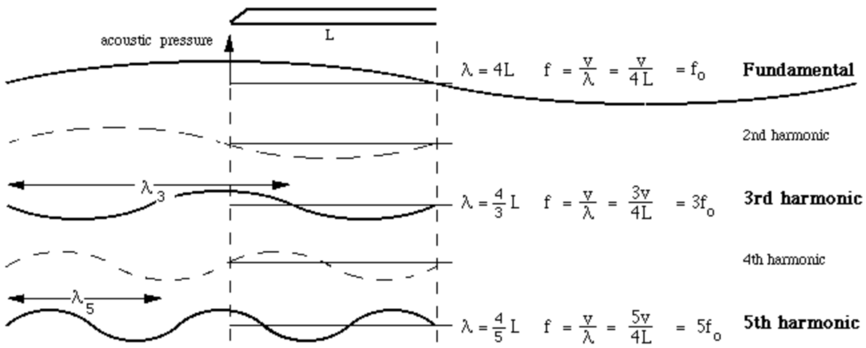
We also know that $\phi(x) = u(x,0)$

$$\implies \phi(x) = \sum_{n=1}^{\infty} A_n \cos(\frac{(2n-1)\pi}{2l}c \cdot 0))\sin(\frac{(2n-1)\pi x}{2l})$$

$$\implies \phi(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{(2n-1)\pi x}{2l})$$

Finally, we then know that $\cos(\omega t)$ holds, where omega is our angular frequency. We also know $2\pi f = \omega$, where f is the frequency.
So, we have $\cos(\frac{(2n-1)\pi}{2l}ct) \implies \omega = \frac{(2n-1)\pi}{2l}c$ and we also have that $\omega = 2\pi f$
$\implies f = \frac{\omega}{2\pi}$
$\implies f = \frac{(2n-1)\pi}{2l}c \cdot \frac{1}{2\pi}$
$\implies f = \frac{(2n-1)}{4l}c$ Hz
So, this value corresponds to our frequency, and the fundamental frequency occurs at $n = 1 \implies f = \frac{c}{4l}$
However, for the first overtone, we have $n = 2 \implies f = \frac{3}{4l}c$ Hz, which is an odd multiple of the fundamental note.

In fact, any value of n will give an odd multiple of the fundamental note. This holds because the clarinet is open to the air at the far end and closed on the other, so the acoustic pressure is zero. The mouthpiece end can have a variation in pressure, and the distance between a zero and maximum on a sine wave is 1/4 of a wavelength. This means that the longest standing wave has four times the length of the instrument. However, when we double the frequency, the low pressure end of the wave (bottom of the sine curve) will not match the bottom of the clarinet, and thus will not be an overtone. I was curious and looked this up [1], and found the following picture from the webpage to be useful in understanding what is going on:



[1]

References

[1] Clarinet acoustics: an introduction. http://newt.phys.unsw.edu.au/jw/clarinetacoustics.html.