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Group Assignment 7

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Other than hw and class notes, we used the one reference at the end of the document :)

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Find a series solution to $u_{tt} - 4u_{xx} + u_t = 0$ **for** $0 < x < 2$**, given** $u(0,t) = 0, u(2,t) = 0$

1. **Separate variables to obtain 2 ODEs**
   We want to look for solutions of the form:

   $$u(x,t) = X(t)T(t), \text{ then use this solution for } u(x,t) \text{ to plug into the PDE}$$

   We know X and T each depend on a single variable, so we know that we can take ordinary derivatives with respect to their independent variable.
   To plug this in, we get:

   $$XT'' - 4X''T + XT' = 0$$

   $$\implies XT'' + XT' = 4X''T$$

   $$\implies X(T'' + T') = 4X''T$$

   $$\implies \tfrac{(T''+T')}{4T} = \tfrac{X''}{X}$$

   We note that the LHS depends on t only and RHS depends on x only. If we want to be lazy, we can introduce a new variable and write that

   $$\lambda = \tfrac{(T''+T')}{4T} = \tfrac{X''}{X}$$

   Let's just look at the equation involving $T(t)$ and its derivatives:

   $$\implies \lambda = \tfrac{(T''+T')}{4T}$$

   If we take the derivative of this equation with respect to x we get:

   $$\implies \tfrac{d\lambda}{dx} = 0$$

   Similarly, looking at the equation involving only $X(x)$ and its derivatives we have:

   $$\lambda = \tfrac{X''}{X} \implies \tfrac{d\lambda}{dt} = 0$$

Since the derivative of λ with respect to both x and t are 0, we know the only way that this can be true is if λ is a constant value.

So, since λ is constant we have: $4\lambda T = T'' + T'$ and $\lambda X = X''$, just moving our equations around a bit to get rid of the fractions.

These are our two ODEs we now need to solve!

2. **Solve the $X(x)$ ODE subject to b.c.s**
Let's solve the $X(x)$ subject to $u(0,t) = 0$ and $u(2,t) = 0$

Recall the X ODE is $\lambda X = X''$
What we are going to do now is think about the possibilities for λ, which will determine what our general solution is and looks like. What our solution looks like will depend on what λ is. We have three cases:

(a) $\lambda < 0$
(b) $\lambda = 0$
(c) $\lambda > 0$

We will show case 1, and later we will show cases 2,3 cannot happen for Dirichlet boundary conditions.

Let's proceed to try and solve this ODE, assuming $\lambda < 0$. We will write $\lambda = -\beta^2$, since we know that β^2 must be positive for all possible values of $\beta \in \mathbb{R}$
Then, we have:

$$-\beta^2 X = X'' \tag{1}$$
$$X'' + \beta^2 X = 0$$

Recall from ODEs that solution is $X(x) = C\cos(\beta x) + D\sin(\beta x)$ for constants $C, D \in \mathbb{R}$
This is based on our rules for 2nd order ODEs [1]
Now, what we want to do is take into account our boundary conditions.
Recall that $u(0,t) = 0$ and $u(2,t) = 0$

But, I have assumed that $u(x,t) = X(x)T(t)$, which is saying that $X(0)T(t) = 0$ and $X(2)T(t) = 0$

The only way that $X(0)T(t) = 0$ can hold true for all t is if $X(0) = 0$

Similarly, the only way $X(2)T(t) = 0$ can hold is if $X(2) = 0$

Let's recall that $X(x) = C\cos(\beta x) + D\sin(\beta x)$ and use these boundary conditions to try to figure out C and D

We know that $X(0) = C\cos(0) + D\sin(0) = C$ and that $X(0) = 0$ from our boundary conditions, implying that $C = 0$

We also know that $X(2) = D\sin(2\beta) = 0 \implies D\sin(2\beta) = 0$
Let's think about this more closely...we don't want $D = 0$ since then $X(x) = 0$ and we have a trivial solution. We already know $u(x,t) = 0$ is a solution!
Thus, we need $\sin(2\beta) = 0$ to make the equation hold true.

Recall $\sin(n\pi) = 0$, where n is an integer.
That means that we need $2\beta = n\pi \implies \beta = \frac{n\pi}{2}$
Recall from the individual homework we showed why we shouldn't consider values of $n \leq 0$. So, $n = 1, 2, 3, 4, ...$
This brings us to our final answer:

$$X_n(x) = D_n \sin(\frac{n\pi}{2}x), \text{ for a given } n \in \mathbb{N}$$

We know that we don't have to consider $\lambda = 0$ and $\lambda > 0$ because on problems 5/6 of the individual homework, we showed that these values of lambda only lead to the trivial solution for $X(x)$

3. **Solve the $T(t)$ ODE**
Recall that $\lambda = \frac{T'' + T'}{4T}$ given $\lambda < 0$

We also set λ to $\lambda = -\beta^2$ so the whole thing is $\frac{T''+T'}{4T} = -\beta^2$ which is the same thing as:
$T'' + T' = -4\beta^2 T$
$\implies T'' + T' + 4\beta^2 T = 0$

And thus this is our ODE that we want to solve.

Now, we know that the determinant of this quadratic is $1 - 4(4\beta^2) < 0$ since we know that $\beta = \frac{n\pi}{2}$, which for the smallest value of $n = 1$ implies that $\beta = \frac{\pi}{2}$

Thus, we have $1 - 4\pi^2 < 0$, so we know that the determinant is always less than 0.
We therefore have two complex solutions, which are $\frac{-1 \pm \sqrt{1-4n^2\pi^2}}{2}$ so the general solution is:

$$T(t) = e^{-\frac{x}{2}}(A\cos(\frac{\sqrt{1-4n^2\pi^2}}{2}t) + B\sin(\frac{\sqrt{1-4n^2\pi^2}}{2}t))$$

Now, we can simply write:

$$T_n(t) = e^{-\frac{x}{2}}(A_n \cos(\frac{\sqrt{1-4n^2\pi^2}}{2}t) + B_n \sin(\frac{\sqrt{1-4n^2\pi^2}}{2}t))$$

which represents solutions for a given value of n.

4. **Take a linear combination of all solutions**
Now, combining our $T_n(t)$ and $X_n(x)$, we recall that $u(x,t) = X(x)T(t)$:

$$u_n(x,t) = X_n(x)T_n(t) = D_n \sin(\frac{n\pi}{2}x)\left(e^{-\frac{x}{2}}(A_n \cos(\frac{\sqrt{1-4n^2\pi^2}}{2}t) + B_n \sin(\frac{\sqrt{1-4n^2\pi^2}}{2}t))\right)$$

for a given n. However, we want to find the solution to $u(x,t)$, not just u for a given value of n!

First, we can note that we have three arbitrary constants, A, B, D, and we can absorb D_n such that $A_n \to A_nD_n$ and $B_n \to B_nD_n$. (Therefore, the constants that are part of a linear combination are still encoded in our summation below...)

Now, we want to take the linear combination for all possible values of n, so in taking the sum of all $u_n(x,t)$, this gives us:

$$u(x,t) = \sum_{n=1}^{\infty} \sin(\frac{n\pi}{2}x)\left(e^{-\frac{x}{2}}(A_n \cos(\frac{\sqrt{1-4n^2\pi^2}}{2}t) + B_n \sin(\frac{\sqrt{1-4n^2\pi^2}}{2}t))\right)$$

References

[1] Second order linear differential equations. https://www.math.utah.edu/online/1220/notes/ch12.pdf.