Elements of Probability

(2.1) Alice and Bob are supposed to meet in the cafeteria. Alice arrives at a random time between noon and 1pm, and wait for 15 minutes upon her arrival and then leaves. Bob also also arrives at a random time between noon and 1 pm, but waits up to 20 minutes and then leaves.

(a) What is the probability that Bob arrives before 12:20?

(b) What is the probability that Alice and Bob meet?

(c) If Bob arrives later than Alice, what is the probability that they meet?

(d) Suppose that Alice and Bob have managed to meet. What is the probability that Bob has arrived before 12:20?

Solution. Let us represent the time between the noon and 1 pm. by the interval $[0, 1]$. Then the event A that Bob arrives before 12:20 corresponds to the interval $[0, 1/3]$ and hence has probability $1/3$.

For part (b), the sample space can be described by

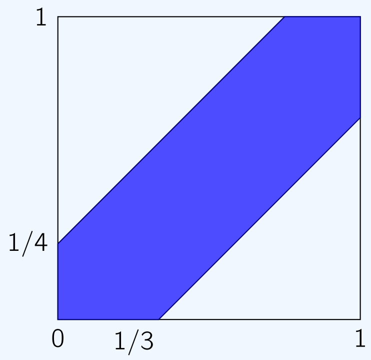
$$\Omega = \{(t_1, t_2) \big| \ 0 \le t_1 \le 1, \quad 0 \le t_2 \le 1\}$$

where t_1, t_2 are the times that Alice and Bob show up. If M is the event that they meet, then

$$M = \{(t_1, t_2) \big| \ t_1 \le t_2 \le t_1 + \frac{1}{4} \text{ or } t_2 \le t_1 \le t_2 + \frac{1}{3}\}.$$

This can be seen as the shaded area in the square:

$$\mathbb{P}[M] = \text{Area}(M) = 1 - \frac{1}{2}(\frac{2}{3})^2 - \frac{1}{2}(\frac{3}{4})^2 = \frac{143}{288}.$$



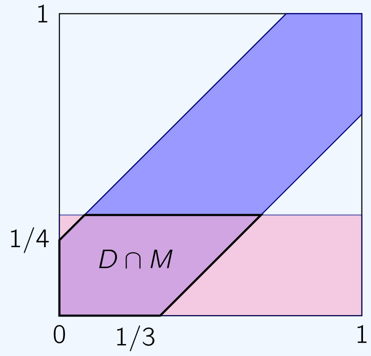
Note that the probability of the event B that Bob arrives after Alice is $1/2$. IClearlu

$$B \cap M = \{(t_1, t_2) \big| \ t_1 \le t_2 \le t_1 + \frac{1}{4}\}.$$

Hence

$$\mathbb{P}[M|B] = \frac{\mathbb{P}[B \cap M]}{\mathbb{P}[B]} = \frac{\frac{1}{2} - \frac{1}{2}(\frac{3}{4})^2}{1/2} = \frac{7}{16}$$

(d) The area corresponding to Alice and Bob meeting is in blue. The event D that Bob arrives before 12:20 is colored in magenta:



We would like to compute $\mathbb{P}[D|M]$. In order to compute $\mathbb{P}[D \cap M]$, we need to compute the area of the region

$$\mathbb{P}[D \cap M] = \frac{1}{3} - \frac{1}{2}(\frac{1}{12})^2 - \frac{1}{9} - \frac{1}{18} = \frac{47}{288} \approx 0.16.$$

This gives

$$\mathbb{P}[D|M] = \frac{\mathbb{P}[D \cap M]}{\mathbb{P}[M]} = \frac{47/288}{143/288} = \frac{47}{143} \approx 0.33.$$

(2.2) A bias coin has the probability 2/3 of turning up heads. The coin is thrown 4 times.

(a) What is the probability that the total number of heads shown is 3?

(b) Suppose that we know that outcome of the first throw is a head. Find the probability that the total number of heads shown is 3.

(c) If we know that the total number of heads shown is 3, find the probability that the outcome of the first throw was a head.

Solution. Denote the event that the total number of heads shown is 3 by A. Then

$$\mathbb{P}[A] = \binom{4}{3}(2/3)^3(1/3) = \frac{32}{81}.$$

Suppose B denotes the event that outcome of the first throw is a head, so $\mathbb{P}[B] = 2/3$. Then $A \cap B$ is the event that the outcome of the first throw is a head and there was a total of 3 heads implying 2 heads in the remaining 3 throws. Hence

$$\mathbb{P}[A \cap B] = (2/3)\binom{3}{2}(2/3)^2(1/3) = \frac{8}{27}.$$

This gives

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \frac{8/27}{2/3} = \frac{4}{9}.$$

For (c), note that

$$\mathbb{P}[B|A] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[A]} = \frac{8/27}{32/81} = \frac{3}{4}.$$

(2.3) Suppose that 15 percent of the messages arriving at a mailxbox are spam and that 20 percent of spam messages arriving there contain the word "winner". Suppose also that the probability that the word "winner" appears in a non-spam message is 5 percent.

(a) What percentage of the received emails contain the word "winner"?

(b) Suppose that a message is tagged as spam based on containing the word "winner". Find the probability that the message is indeed a spam.

Solution. Denote the event that a message is spam by S and the event that it contains the word "winner" by W. The problem gives

$$\mathbb{P}[S] = 0.15, \quad \mathbb{P}[W|S] = 0.2, \quad \mathbb{P}[W|S^c] = 0.05.$$

Then

$$\mathbb{P}[W] = \mathbb{P}[W|S]\,\mathbb{P}[S] + \mathbb{P}[W|S^c]\,\mathbb{P}[S^c] = 0.2 \times 0.15 + 0.05 \times 0.85 = 0.03 + 0.0425 = 0.0725.$$

Using Bayes' formula, we can write

$$\mathbb{P}[S|W] = \frac{\mathbb{P}[W|S]\,\mathbb{P}[S]}{\mathbb{P}[W]} = \frac{0.2 \times 0.15}{0.0725} \approx 0.41.$$

(**2.4**) Suppose M is an integer randomly chosen from the set $\{1, 2 \ldots, 10\}$. Once M is chosen, the integer N is chosen from the set $\{1, 2, \ldots, M\}$. For instance if it turns out that $M = 7$, then N can take one of the values $1, \ldots, 7$, each with probability $1/7$.

(a) Find the probability that $N = 7$.

(b) Find the probability of the event $M = N$.

Solution. (a) It is clear that always $N \leq M$. Hence if $N = 7$, then M can take one of the values $7, 8, 9, 10$. This gives

$$\mathbb{P}[N = 7] = \sum_{i=7}^{10} \mathbb{P}[M = i]\,\mathbb{P}[N = 7|M = i] = \frac{1}{10}\left(\frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}\right).$$

(b) Again, we have

$$\mathbb{P}[M = N] = \sum_{i=1}^{10} \mathbb{P}[N = M|M = i]\,\mathbb{P}[M = i].$$

If $M = i$, then there are i options for N, one of which is i. Hence

$$\mathbb{P}[N = M|M = i] = \frac{1}{i}.$$

From here we have

$$\mathbb{P}[M = N] = \sum_{i=1}^{10} \frac{1}{10} \cdot \frac{1}{i}.$$

(**2.5**) A defect switch turns the light on automatically at a random time between noon and 1 pm. Charlotte checks the light at a random time between noon and 1 pm, and if the light is on, switches it off and leaves immediately.

(a) Describe the sample space for this probabilistic situation by a two-dimensional region.

(b) Find the probability that the light is on at $12:45$.

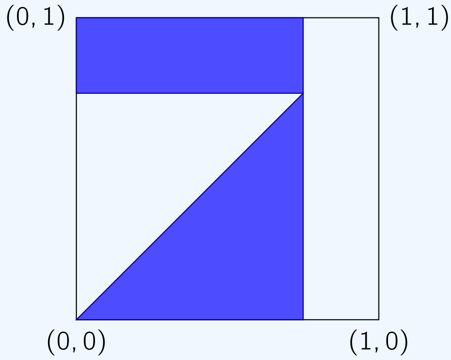
(c) Find the probability that the light is on at $12:30$.

Solution. Let us denote by x the time that the light is switched on and by y the times that Charlotte arrives.

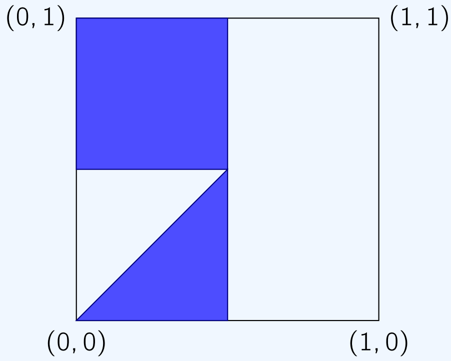
If we identify the time period between the noon and 1 p.m. by the interval $[0, 1]$ (for instance, the number 0.25 will correspond to 12:15) then the sample space consists of all pairs (x, y) with

$$0 \leq x \leq 1, 1 \leq y \leq 1.$$

In order to find the probability that the light is on at $12:45$, observe that this will be the case if either (a) the light goes on before 12:45 and Charlotte arrives before the light goes on, or (b) the light goes on before 12:45 and Charlotte arrives after 12:45. A similar description holds for part (b). The corresponding regions are given by



Hence $\mathbb{P}[A] = \frac{1}{2}(3/4)^2 + (1/4)(3/4) = \frac{15}{32}$.



For part (b), we have $\mathbb{P}[B] = \frac{1}{2}(1/2)^2 + (1/2)(1/2) = \frac{3}{8}$.