JMTS-12: Probability and Random Processes

Fall 2020

M. Bode

Lecture 8:
Recap and Exam-type questions

Lecture 8

Chapter 2: Random Variables

2.2 Random Variables

2.3 Probability Distribution Functions (PDF)

2.4 Probability Density Functions (pdf)

2.5 Continuous, Discrete, Mixed Cases …

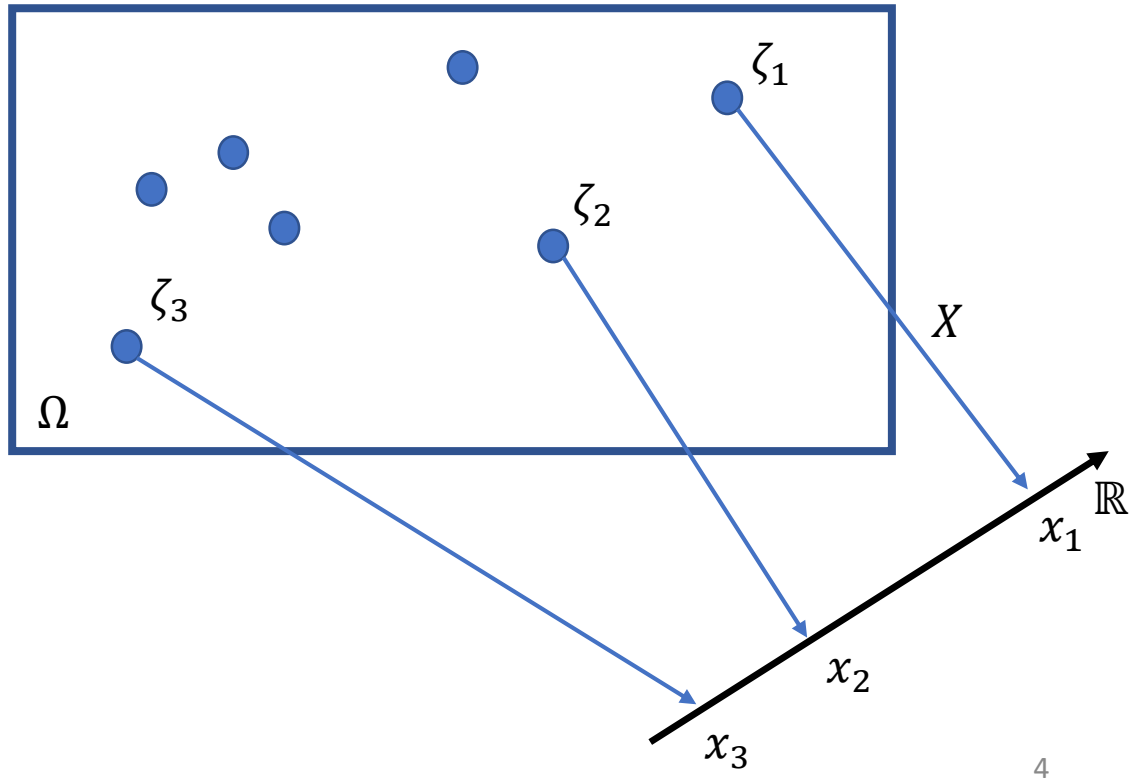
2.6 Conditional and Joint PDFs, pdfs

2.7 Failure Rates

Recap & Exam type questions

Idea: Map outcomes to (real) numbers.

The random variable X: $\Omega \to \mathbb{R}$ maps all outcomes from the sample description space to a real number.



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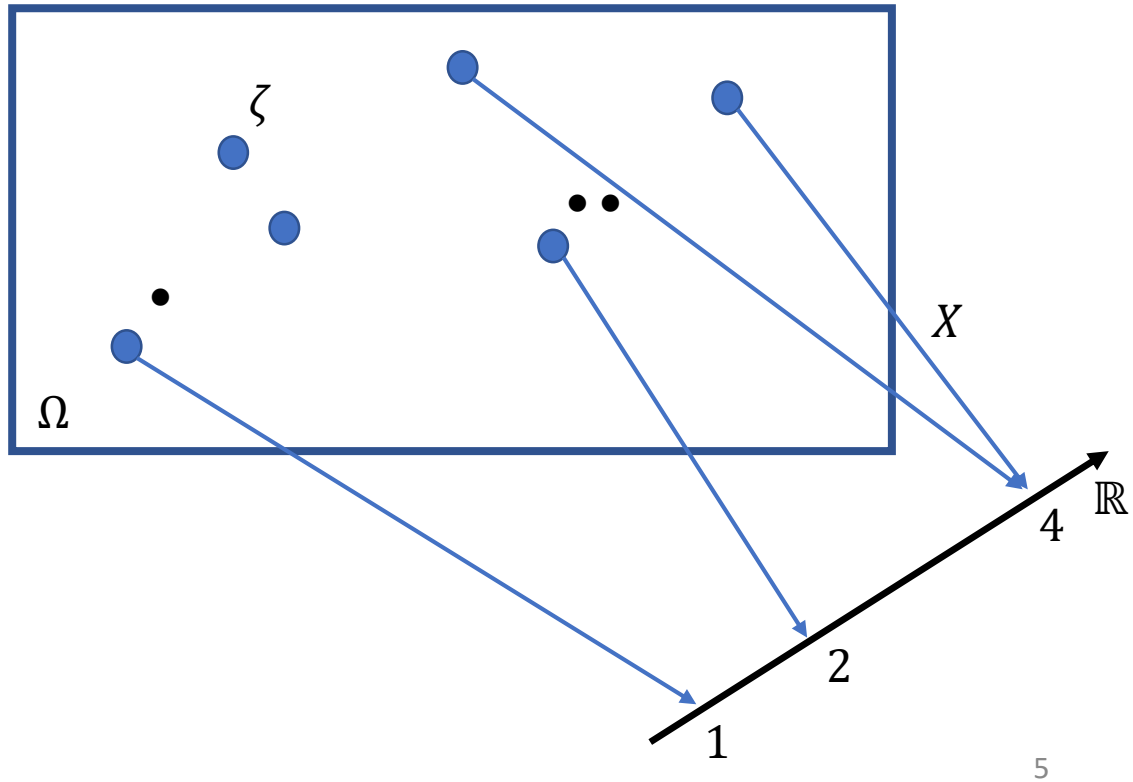
Recap & Exam type questions

TASK 4:

Consider a **fair six-faced die** and the following random variable:

X(•) = 1, X(• •) = 2, X(ζ) = 4, else.

Find $F_X(x)$ and $f_X(x)$.



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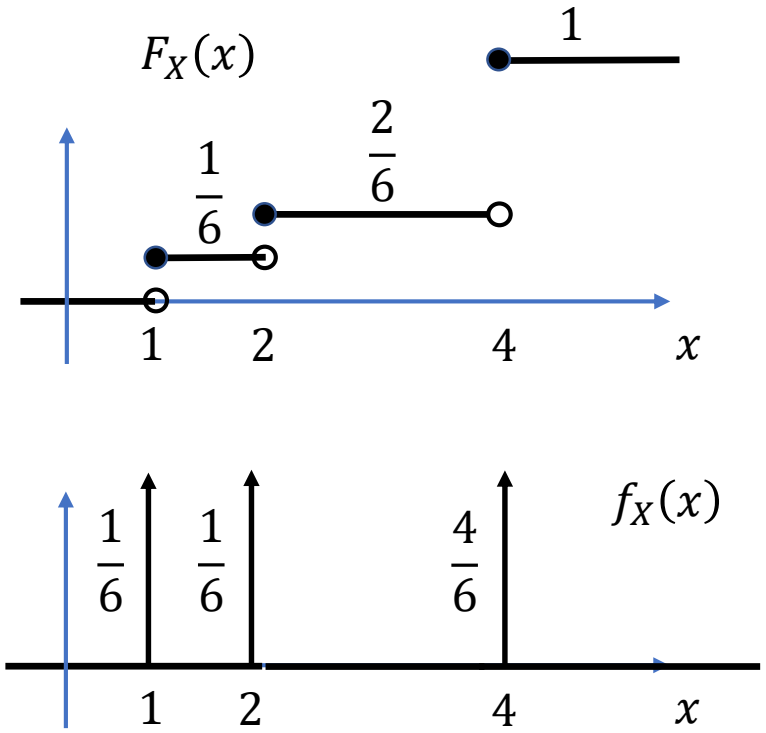
Solve:

Consider a **fair six-faced die** and the following random variable:

X(•) = 1, X(• •) = 2, X(ζ) = 4, else.

Find $F_X(x)$ and $f_X(x)$.



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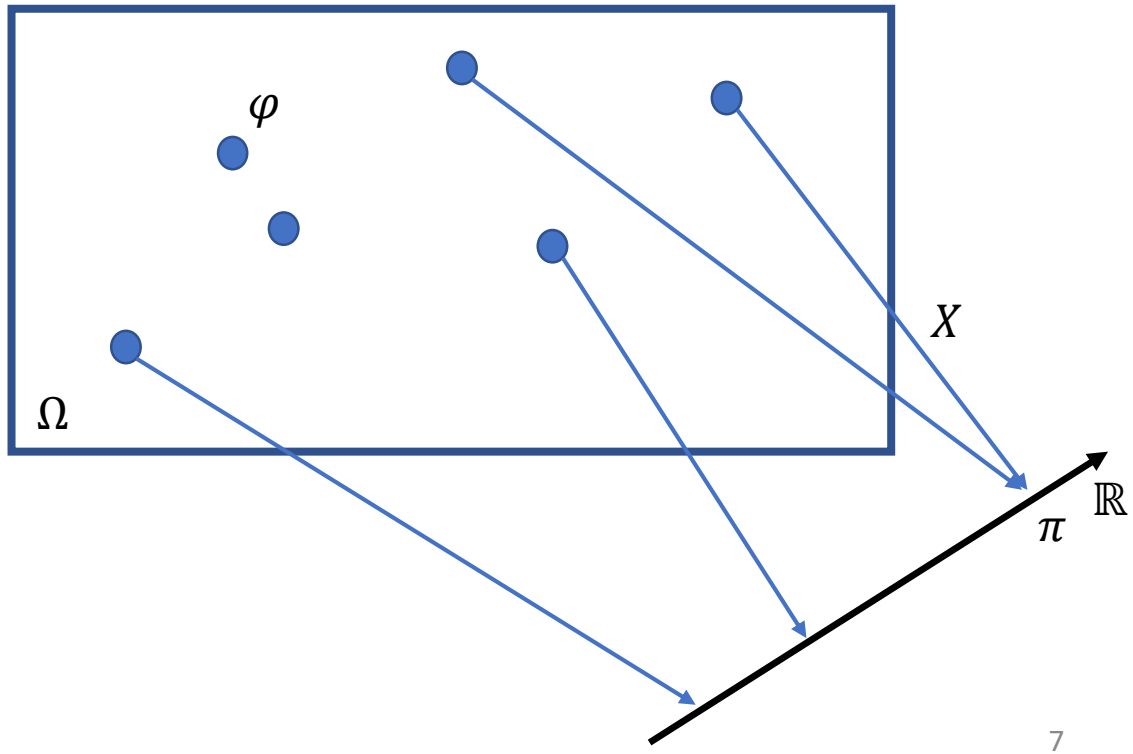
Recap & Exam type questions

TASK 5: Consider a **fair spinning wheel** with outcomes in terms of the angle $\varphi \in [0,2\pi)$, and the following random variable:

$$X(\varphi) = \begin{cases} \varphi & \text{, if } \varphi \leq \pi \\ \pi, & \text{otherwise} \end{cases}$$

Find $F_X(x)$ and $f_X(x)$.



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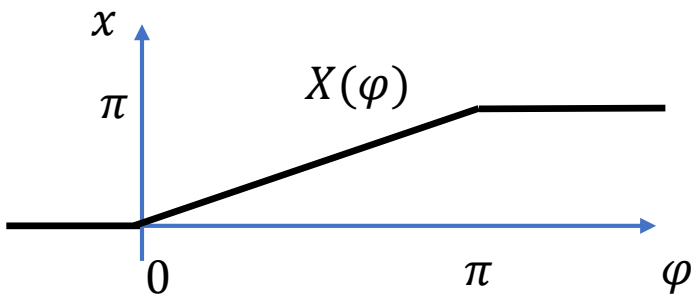

Recap & Exam type questions

Solve:

Consider a **fair spinning wheel** with outcomes in terms of the angle $\varphi \in [0,2\pi]$, and the following random variable:

$$X(\varphi) = \begin{cases} \varphi & \text{, if } \varphi \leq \pi \\ \pi & \text{, otherwise} \end{cases}$$

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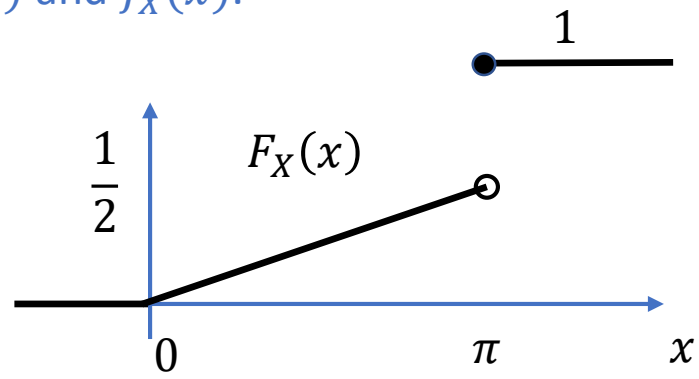
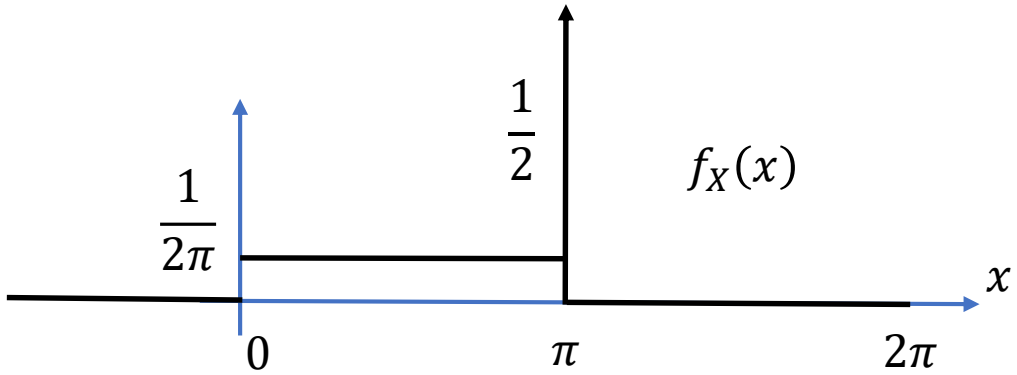




TASK 6*: Individual unstable atomic nuclei (of a given type) decay according to an exponential pdf:

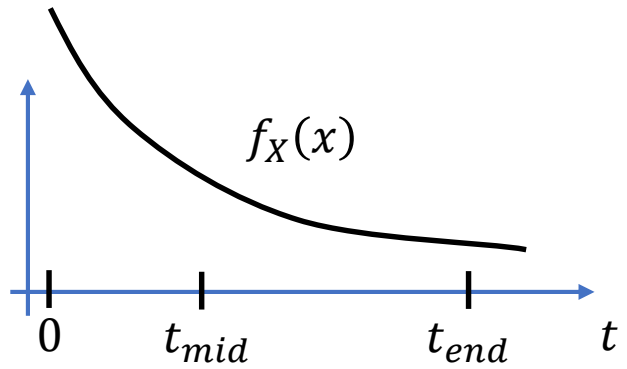
$$f_X(t) = \lambda \exp[-\lambda t], \qquad F_X(t) = 1 - \exp[-\lambda t], \qquad t > 0$$

Their decays are mutually independent.

Consider time intervals, $(0, t_{mid}]$, and $(t_{mid}, t_{end}]$.

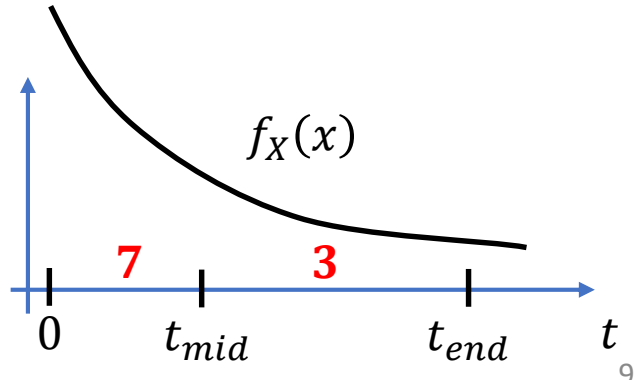
Your colleagues told you that out of your original set of n atoms 10 decayed before t_{end}. What is the probability that 7 out of those 10 decayed before t_{mid}?

Visualize



Develop a strategy!

Next: Sequence of steps



Recap & Exam type questions

TASK 6*: Individual unstable atomic nuclei (of a given type) decay according to an exponential pdf:

$$f_X(t) = \lambda \exp[-\lambda t], \qquad F_X(t) = 1 - \exp[-\lambda t], \qquad t > 0$$

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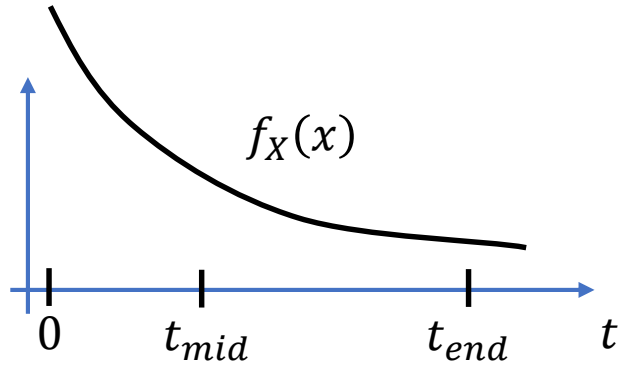


Your strategy: Visualize … Next: Sequence of steps

1) $P[\text{nucleus decays before } t_{mid}] = P[X \leq t_{mid}]$

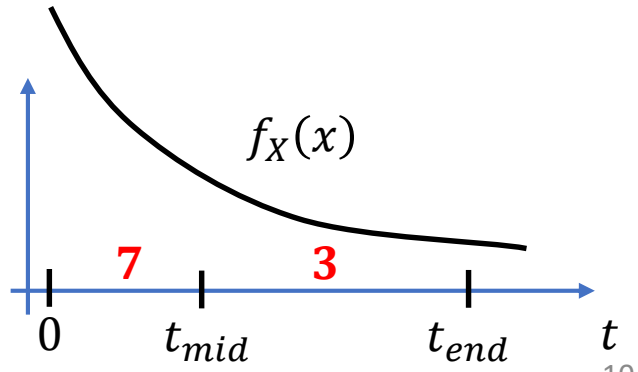
2) $P[t_{mid} < X \leq t_{end}]$

3) $P[X \leq t_{end}]$

4) $P[t_{mid} < X \leq t_{end} | X \leq t_{end}]$

5) $P[X \leq t_{mid} | X \leq t_{end}]$

Then solve: Your colleagues told you that out of your original set of n atoms 10 decayed before t_{end}. What is the probability that 7 out of those 10 decayed before t_{mid}?



Atoms decay …

Solve:

1) $P[X \leq t_{mid}] =$?

2) $P[t_{mid} < X \leq t_{end}] =$?

3) $P[X \leq t_{end}] =$?

4) $P[t_{mid} < X \leq t_{end} | X \leq t_{end}]$

$=$? ?

5) $P[X \leq t_{mid} | X \leq t_{end}]$

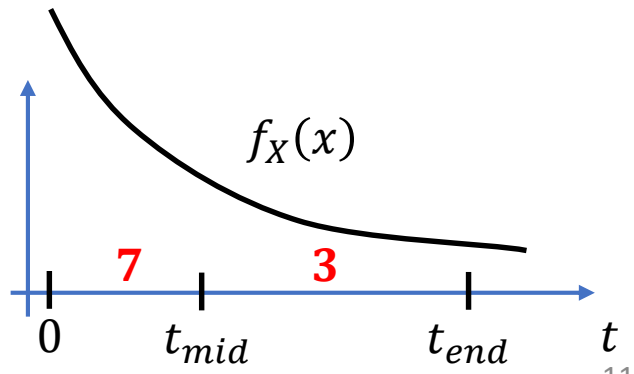
$=$? ?

6) 10 atoms decayed before t_{end} . Find the probability that 7 out of those 10 decayed before t_{mid}

Interpret … ?

Make your results explicit for:

$f_X(t) = \lambda \exp[-\lambda t]$, $F_X(t) = 1 - \exp[-\lambda t]$, $t > 0$



Atoms decay …

Solve:

1) $P[X \leq t_{mid}] = F_X(t_{mid})$

2) $P[t_{mid} < X \leq t_{end}] = F_X(t_{end}) - F_X(t_{mid})$

3) $P[X \leq t_{end}] = F_X(t_{end})$

4) $P[t_{mid} < X \leq t_{end} | X \leq t_{end}]$

$$= \frac{F_X(t_{end}) - F_X(t_{mid})}{F_X(t_{end})} = 1 - p = q$$

5) $P[X \leq t_{mid} | X \leq t_{end}]$

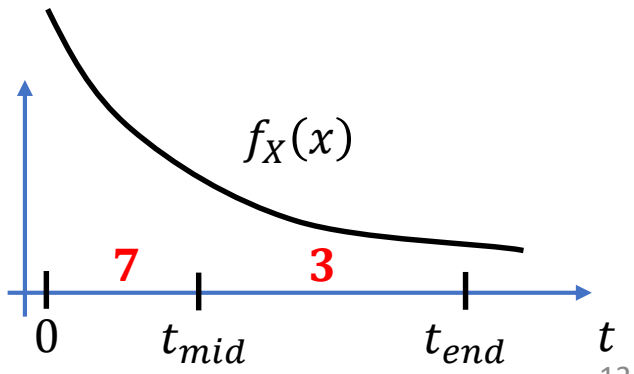
$$= \frac{F_X(t_{mid})}{F_X(t_{end})} = p$$

6) 10 atoms decayed before t_{end} . Find the probability that 7 out of those 10 decayed before t_{mid}

Interpret … Binomial law

Make your results explicit for:

$f_X(t) = \lambda \exp[-\lambda t]$, $F_X(t) = 1 - \exp[-\lambda t]$, $t > 0$



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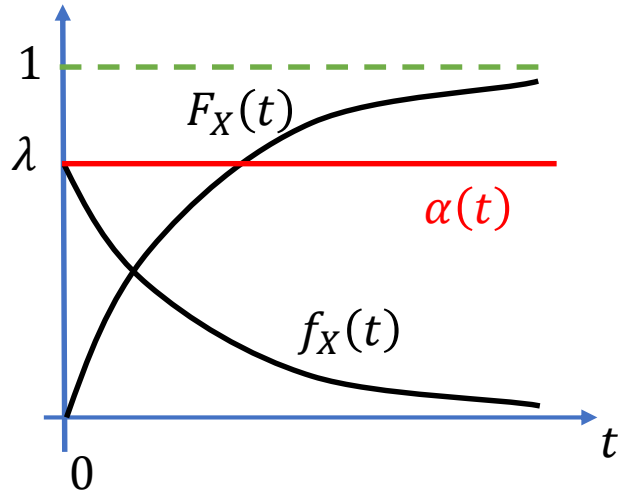

Recap & Exam type questions

Failure rate:

$$\alpha(t) = \frac{f_X(t)}{1 - F_X(t)}$$

Find $f_X(t)$ based on $\alpha(t)$

$$\alpha(t) = -\frac{d}{dt}\ln\big(1 - F_X(t)\big) \Rightarrow \ln\big(1 - F_X(t)\big) = -\int_0^t \alpha(t')dt'$$

$$F_X(t) = 1 - \exp\left(-\int_0^t \alpha(t')dt'\right)$$

$$\Rightarrow f_X(t) = \alpha(t)\exp\left(-\int_0^t \alpha(t')dt'\right)$$

Recap & Exam type questions

TASK 7: Consider a system with failure rate $\alpha(t) = t^2$. Find the corresponding PDF and pdf.

Sketch your results.

Solve: Consider a system with failure rate $\alpha(t) = t^2$. Find the corresponding PDF and pdf.

$$F_X(t) = 1 - \exp\left(-\int_0^t \alpha(t')dt'\right)$$

$$= 1 - \exp\left(-\int_0^t (t')^2 dt'\right)$$

$$= 1 - \exp\left(-\frac{t^3}{3}\right)$$

$$f_X(t) = \alpha(t)\exp\left(-\int_0^t \alpha(t')dt'\right)$$

$$= t^2 \exp\left(-\frac{t^3}{3}\right)$$

■

The End

Next time: Chp. 3