Elements of Probability

(2.1) Alice throws a fair coin 6 times.

(a) If the outcome of the first throw is heads, find the probability that she will end up with 3 heads at the end.

(b) Given that she ends up with 3 heads at the end, find the probability that the outcome of the first throw was heads.

Solution. Let us denote by B the event that the outcome of the first row is H. It is clear that $\mathbb{P}[B] = \frac{1}{2}$. Now if A denotes the event that Alice ends up with 3 heads at the end, then $A \cap B$ entails that she obtains H in the first round and then two heads in the remaining 5 rounds. Hence

$$\mathbb{P}[A \cap B] = \frac{\binom{5}{2}}{2^6} = \frac{5}{32}$$

Hence

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \frac{5/32}{1/2} = \frac{5}{16}.$$

(b) First note that

$$\mathbb{P}[A] = \frac{\binom{6}{3}}{2^6} = \frac{10}{32}.$$

$$\mathbb{P}[B|A] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[A]} = \frac{1}{2}.$$

(2.2) Belgium, Croatia, England and France have reached the semifinal of the World Cup. The betting agency ProFoot has announced their estimated probabilities for the games between these four teams in the following table:

	Belgium	Croatia	England	France
Belgium		0.4	0.6	0.4
Croatia	0.6		0.5	0.3
England	0.4	0.5		0.1
France	0.6	0.7	0.9	

For instance, the entry 0.9 indicates that the ProFoot believes that the probability that France defeats Croatia is 90 percent.

(a) Suppose that the semifinals are played by Belgium vs. England and Croatia vs. France. What is the probability (according to ProFoot) that France wins the cup?

(b) Suppose that the semifinals are played by Belgium vs. Croatia and England vs. France. What is the probability that France wins the world cup?

(c) Suppose that the semifinal games are determined randomly. In other words, assume that the opponent of Belgium is randomly chosen from remaining three teams, each with probability 1/3, and the other two teams face each other. What is the probability that Belgium makes it to the final?

Solution. Let us denote the countries by B,C,E,F, and the event that country X defeats country Y by $X \gg Y$. So, we write $\mathbb{P}[F \gg C] = 0.7$ to indicate that with probability 0.7 France will beat Croatia.

(a) Note that France will win the world cup, if it first defeats Croatia, and then the winner of the game Belgium-England.

Hence

$$\mathbb{P}[\text{ France wins the cup}] = \mathbb{P}[F \gg C]\left(\mathbb{P}[F \gg E]\,\mathbb{P}[E \gg B] + \mathbb{P}[F \gg B]\,\mathbb{P}[B \gg E]\right).$$

Using the given numbers we have

$$\mathbb{P}[\text{ France wins the cup}] = 0.50.$$

(b) The argument is similar to part (a). We have

$$\mathbb{P}[\text{ France wins the cup}] = \mathbb{P}[F \gg E]\left(\mathbb{P}[F \gg C]\,\mathbb{P}[C \gg B] + \mathbb{P}[F \gg B]\,\mathbb{P}[B \gg C]\right).$$

Using the values given in the problem we have

$$\mathbb{P}[\text{ France wins the cup}] \approx 0.59.$$

(c) The opponent of Belgium is one of E,F,C. Hence, we have

$$\begin{aligned}\mathbb{P}[\text{ Belgium is in final}] &= \mathbb{P}[B \gg E]\,\mathbb{P}[\text{Opponent is E}] + \mathbb{P}[B \gg F]\,\mathbb{P}[\text{Opponent is F}]\\ &\quad + \mathbb{P}[B \gg C]\,\mathbb{P}[\text{Opponent is C}]\\ &= \frac{1}{3}(0.6 + 0.4 + 0.4) \approx 0.46.\end{aligned} \tag{4}$$

(2.3) (a) Suppose A, B, C are three events such that $\mathbb{P}[A|\,C] > \mathbb{P}[B|\,C]$ and $\mathbb{P}[A|\,C^c] > \mathbb{P}[B|\,C^c]$. Show that $\mathbb{P}[A] > \mathbb{P}[B]$. What is the interpretation of this fact?

(b) Suppose A, B, C are three events such that $\mathbb{P}[A] > \mathbb{P}[B]$. Is it true that for every event C we have $\mathbb{P}[A|\,C] > \mathbb{P}[B|\,C]$?

Solution. (a) Using the law of total probability we have

$$\mathbb{P}[A] = \mathbb{P}[A|C]\,\mathbb{P}[C] + \mathbb{P}[A|C^c]\,\mathbb{P}[C^c] \geq \mathbb{P}[B|C]\,\mathbb{P}[C] + \mathbb{P}[B|C^c]\,\mathbb{P}[C^c] = \mathbb{P}[B]\,. \tag{5}$$

(b) The answer in general is no. For instance, suppose that $C = A^c$. Then we have $\mathbb{P}[A|C] = \mathbb{P}[A|A^c] = 0$, so the inequality $\mathbb{P}[B|C] < \mathbb{P}[A|C]$ cannot hold.

(2.4) Suppose M is an integer randomly chosen from the set $\{1, 2 \ldots, 8\}$. Once M is chosen, the integer N is chosen from the set $\{1, 2, \ldots, M\}$. For instance if it turns out that $M = 5$, then N can take one of the values $1, \ldots, 5$, each with probability $1/5$.

(a) Find the probability that $N = 7$.

(b) Find the probability of the event $M = N$.

Solution. (a) It is clear that always $N \leq M$. Hence if $N = 7$, then $M = 7$ or $M = 8$. This gives

$$\mathbb{P}[N = 7] = \mathbb{P}[N = 7|M = 7]\,\mathbb{P}[M = 7] + \mathbb{P}[N = 7|M = 8]\,\mathbb{P}[M = 8]\,.$$

Now, if $M = 7$, then there are seven options for N, hence $\mathbb{P}[N = 7|M = 7] = \frac{1}{7}$. By a similar argument, we have $\mathbb{P}[N = 7|M = 8] = \frac{1}{8}$. These imply

$$\mathbb{P}[N = 7] = \frac{1}{7} \cdot \frac{1}{8} + \frac{1}{8} \cdot \frac{1}{8} = \frac{15}{448}.$$

(b) Again, we have

$$\mathbb{P}[M = N] = \sum_{k=1}^{8} \mathbb{P}[N = M|M = k] \, \mathbb{P}[M = k].$$

If $M = k$, then there are k options for N, one of which is k. Hence

$$\mathbb{P}[N = M|M = k] = \frac{1}{k}.$$

From here we have

$$\mathbb{P}[M = N] = \sum_{k=1}^{8} \frac{1}{8} \cdot \frac{1}{k} = \frac{761}{2240}.$$

(2.5) A light is switched on sometime at random between noon and 1 pm. Then it is switched off again on a time randomly chosen between the time it is switched on and 1 pm.

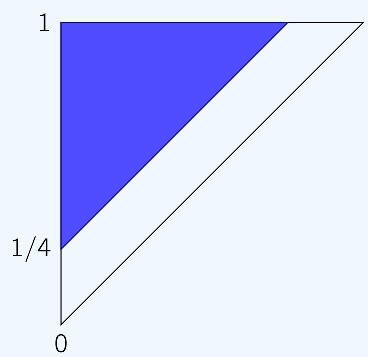
(a) Describe the sample space for this probabilistic situation by a region in the plane.

(b) Compute the probability that the light remains on for more than 15 minutes.

Solution. Let us denote by x the time that the light is switched on and by y the times that it is switched off again. If we identify the time period between the noon and 1 p.m. by the interval $[0, 1]$ (for instance, the number 0.25 will correspond to 12:15) then the sample space consists of all pairs (x, y) with

$$0 \leq x \leq y \leq 1.$$

This set can be described by the triangle below.



Note that the light will be left on for more than 15 minutes if and only if

$$y \geq x + \frac{1}{4}.$$

This set is described by the blue region. Hence the probability of the event A is given by ratio of the areas of the triangles:

$$\mathbb{P}[A] = \frac{\frac{1}{2}\left(\frac{3}{4}\right)^2}{\frac{1}{2}} = \frac{9}{16} \approx 0.56.$$

(2.6) (Bonus) You have 6 white and 8 black balls to distribute among two boxes. Your friend will randomly choose one of the boxes and takes a ball randomly out of it. How should the balls be distributed so that the probability of getting a black ball is as large as possible.

Solution. Let us assume that one of the boxes contains k balls, out of which r are black. The other box will contain the rest, that is $14 - k$ balls, out of which $8 - r$ are black. We will also assume that the first box is the one with fewer balls, hence $k \leq 7$. Hence, the probability in question is given by

$$p(r, k) = \frac{1}{2}\left(\frac{r}{k} + \frac{8-r}{14-k}\right) = \frac{8k + r(14-2k)}{2k(14-k)}.$$

For a fixed k, the denominator of the fraction only depends on k, and the numerator is a linear function in r with a non-negative slope (since $k \leq 7$). It follows that the maximum value of r will maximize the probability, hence we must have $r = k$. This shows that we need to find the maximum of

$$p(k, k) = \frac{11-k}{14-k} = 1 - \frac{3}{14-k}.$$

Clearly this quantity is maximum when $k = 1$, that is, when one box contains one black ball and the rest of the balls are in the other box.