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Due: September 12, 2018
Assignment 1

Elements of Probability

(1.1) Five students have been randomly chosen from a class of 20 students. Find the probability that
(a) At least one of them is born on Sunday.
(b) At least two of them are born on the same day of the week.
(c) All five are born on the weekend.

(1.2) A number is called a *palindrome* if it reads the same from left and right. For instance, 13631 is a palindrome, while 435734 is not. A 5-digit number n is randomly chosen. Find the probability of the event that
(a) The chosen number n is a palindrome.
(b) The chosen number n is even and a palindrome.
(c) The chosen number n is even or a palindrome.

(1.3) (a) Suppose A and B are two events. Let S be the event that A or B occur, but not both. Show that

$$\mathbb{P}[S] = \mathbb{P}[A] + \mathbb{P}[B] - 2\,\mathbb{P}[A \cap B].$$

(b) Suppose A, B, and C are three events in a sample space. Let T denote the event that exactly two of these three events occur. Deduce from the axioms that

$$\mathbb{P}[T] = \mathbb{P}[A \cap B] + \mathbb{P}[A \cap C] + \mathbb{P}[B \cap C] - 3\,\mathbb{P}[A \cap B \cap C].$$

Hint: Draw a Venn diagram and use it to describe S and T as Boolean combination of the given events.

(1.4) Suppose A and B are certain two events, that is, assume that

$$\mathbb{P}[A] = \mathbb{P}[B] = 1.$$

Use the axioms of probability to show that

$$\mathbb{P}[A \cap B] = 1.$$

Now suppose that A and B are "almost certain" in the sense that

$$\mathbb{P}[A] = \mathbb{P}[B] = 0.99.$$

Show that

$$\mathbb{P}[A \cap B] \geq 0.98.$$

(1.5) Let S be a random sequence of 0 and 1 of length $2n$.
(a) Find the probability p_n that the sequence contains exactly n zeros and n ones.
(b) Use Stirling's formula to show that for large value of n we have

$$p_n \sim \frac{1}{\sqrt{\pi n}}.$$

(c) Use part (b) to compute p_{100} approximately.

(1.6) (Bonus) Suppose $A_1, \ldots, A_n$ are events in a sample space. Show that

$$\sum_{1 \le i \le n} \mathbb{P}[A_i] - \sum_{1 \le i < j \le n} \mathbb{P}[A_i \cap A_j] \le \mathbb{P}\left[\bigcup_{i=1}^{n} A_i\right] \le \sum_{1 \le i \le n} \mathbb{P}[A_i].$$