JTMS-12: Probability and Random Processes

Fall 2020

M. Bode

Lecture 13:
Recap and Exam-type questions

Lecture 13

Recap:

Chapter 3: Functions of Random Variables

3.1 Functions of Random Variables

3.2 $Y = g(X)$

3.3 $Z = g(X, Y)$

3.4 $V = g(X, Y)$, W $= h(X, Y)$

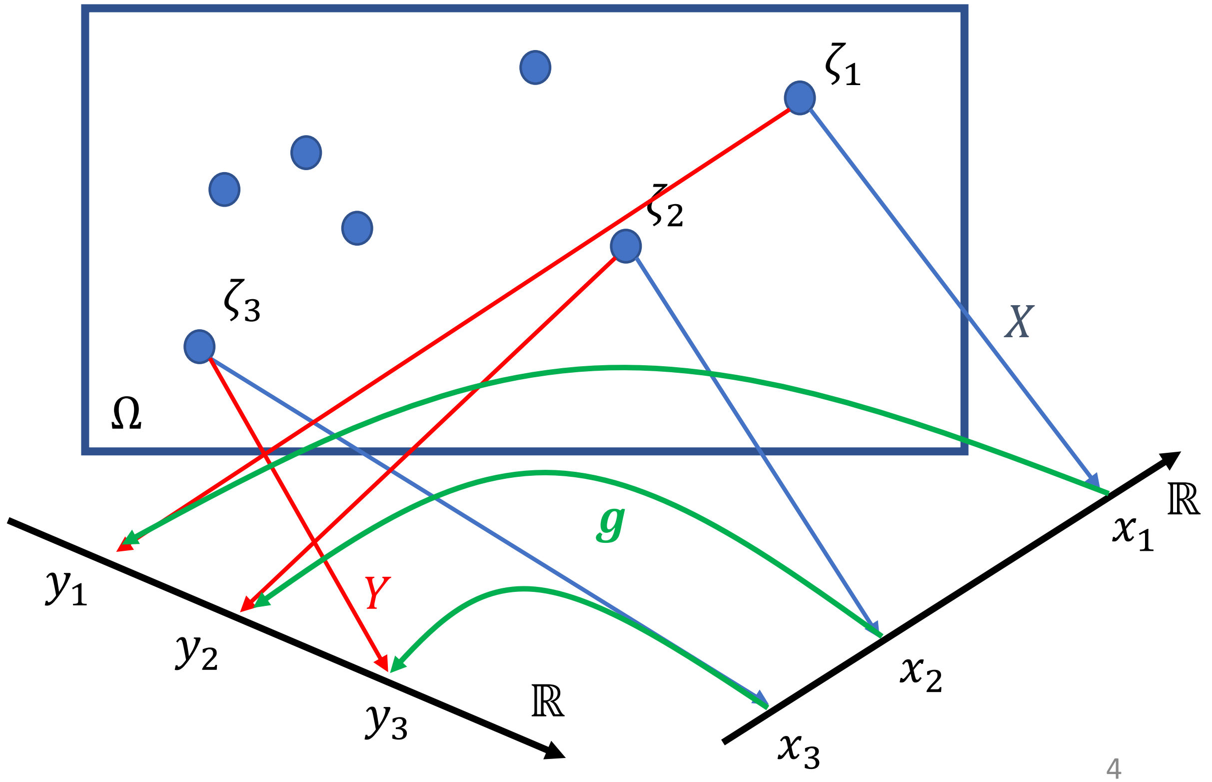
Functions of Random Variables

Idea: Map outcomes to (real) numbers.

The random variable $X\colon\ \Omega \to \mathbb{R}$ maps all outcomes from the sample description space to a real number.

Re-lable: $y = g(x)$

Re-interpret: $Y\colon\ \Omega \to \mathbb{R}$, $Y(\zeta) = g(X(\zeta))$



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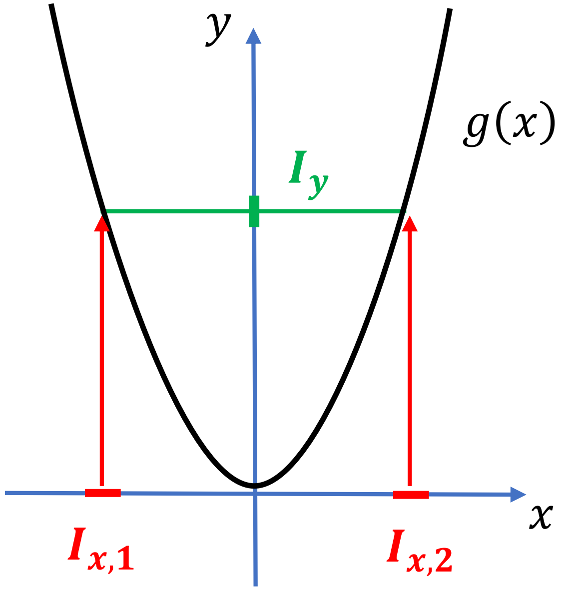
Functions of Random Variables

Quick way to find the density:

The direct formula (for cont. r.v.s)



$$P[Y \in I_y] = P[X \in I_{x,1}] + P[X \in I_{x,2}]$$

$$f_Y(y)|\Delta y| \approx f_X(x_1)|\Delta x_1| + f_X(x_2)|\Delta x_2|$$

Mind the relative orientations of I_y and its (partial) pre-images $I_{x,1}$ and $I_{x,2}$.

$$\Rightarrow f_Y(y) \approx f_X(x_1)\left|\frac{\Delta x_1}{\Delta y}\right| + f_X(x_2)\left|\frac{\Delta x_2}{\Delta y}\right|$$

Hence, in the limit $\Delta y \rightarrow 0$:

$$f_Y(y) = f_X(x_1)\left|\frac{1}{g'(x_1)}\right| + f_X(x_2)\left|\frac{1}{g'(x_2)}\right|$$

…sum over all pre-images

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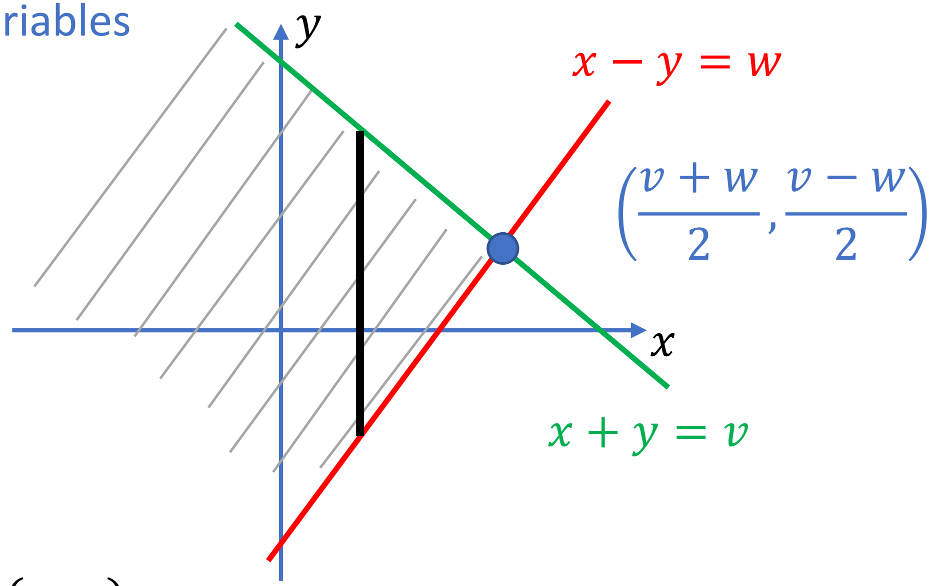
Functions of Random Variables

Example:

Consider continuous r.v.s X, Y, and

$$V = g(X,Y) = X + Y$$
$$W = h(X,Y) = X - Y$$

Find $F_{VW}(v,w)$ and $f_{VW}(v,w)$.



Standard approach …

$$F_{VW}(v,w) = P[V \le v, W \le w] = P[g(X,Y) \le v, h(X,Y) \le w]$$

$$= P[X + Y \le v, X - Y \le w] = \iint_{\substack{shaded \\ area}} f_{XY}(x,y)dxdy$$

… integrate & derive to find the density

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Jacobian matrix:

$$\frac{\Delta(v,w)}{\Delta(x,y)} = \left|\det\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right)\right| = 2$$

Functions of Random Variables

Determine densities like $f_{UV}(u,v)$ diretcly

<u>Same Example - Alternative Perspective:</u>

$$V = g(X,Y) = X + Y, \qquad W = h(X,Y) = X - Y$$

$$\Leftrightarrow \quad X = \frac{V+W}{2}, \qquad Y = \frac{V-W}{2}$$

$\Rightarrow$

$$f_{VW}(v,w) = \frac{1}{2} f_{XY}\left(\frac{v+w}{2}, \frac{v-w}{2}\right)$$

<u>**Mind:**</u>

$$\boxed{f_{VW}(v,w) = f_{XY}\big(x(v,w), y(v,w)\big)\frac{\Delta(x,y)}{\Delta(v,w)}}$$

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Recap & Exam type questions

TASK 8:

Consider a continuous r.v. X, and

$$Y = g(X) = \exp(X)$$

Find $F_Y(y)$ and $f_Y(y)$.

Standard approach …

$$F_Y(y) = P[Y \le y] = P[g(X) \le y] = P[\exp(X) \le y]$$

$$= P[X \le \ln(y)] = F_X(\ln(y))$$

For which y-range does that make sense?

$\mathrm{y} > 0$, only, $\mathrm{y} \le 0$ is impossible.

$$\Rightarrow F_Y(y) = \begin{cases} F_X(\ln(y)); & y > 0 \\ 0; else & \end{cases}$$

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Recap & Exam type questions

Solve:

Consider a continuous r.v. X, and

$$Y = g(X) = \exp(X)$$

Find $F_Y(y)$ and $f_Y(y)$.

$$F_Y(y) = \begin{cases} F_X(\ln(y)); & y > 0 \\ 0; else \end{cases}$$

$$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{y} f_X(\ln(y)); & y > 0 \\ 0; else \end{cases}$$

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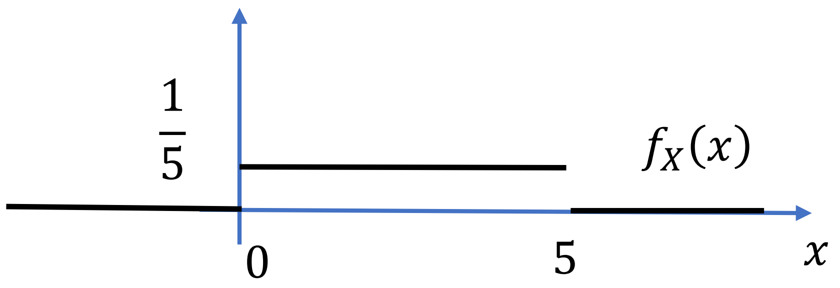
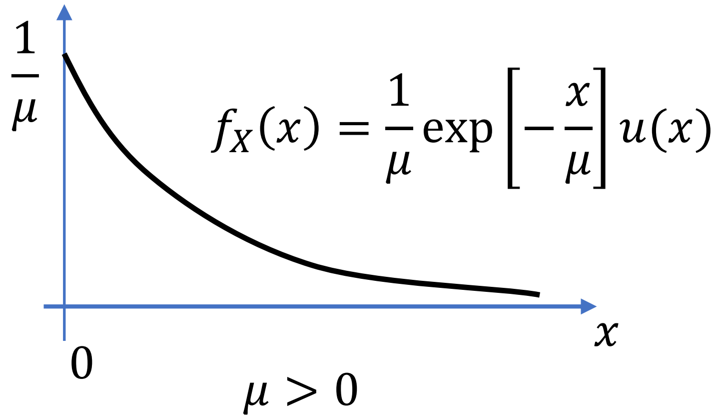
As in an exam: Find the mistakes!
Double check via integration of $f_Y(y)$





Recap & Exam type questions

Solve:

Consider a continuous r.v. X, and $Y = g(X) = \exp(X)$

Special case: uniform over (0,5)

$$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{y} f_X(\ln(y)); & y > 0 \\ 0; else \end{cases} = \begin{cases} \frac{1}{5y}; & y > 0 \\ 0; else \end{cases}$$

Special case: exponential

$$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{\mu y}\exp\left[-\frac{\ln(y)}{\mu}\right]; & y > 0 \\ 0; else \end{cases} = \begin{cases} \frac{1}{\mu y^{\mu+1}}; & y > 0 \\ 0; else \end{cases}$$

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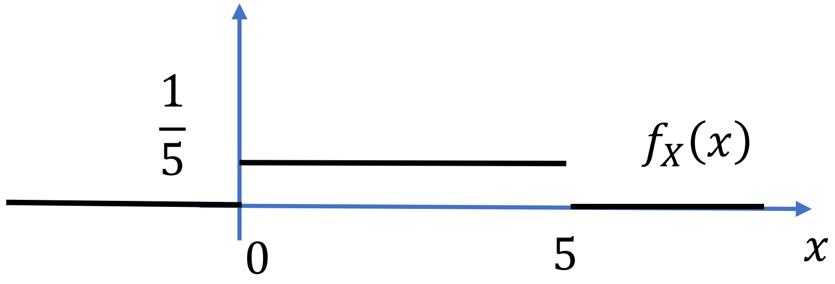
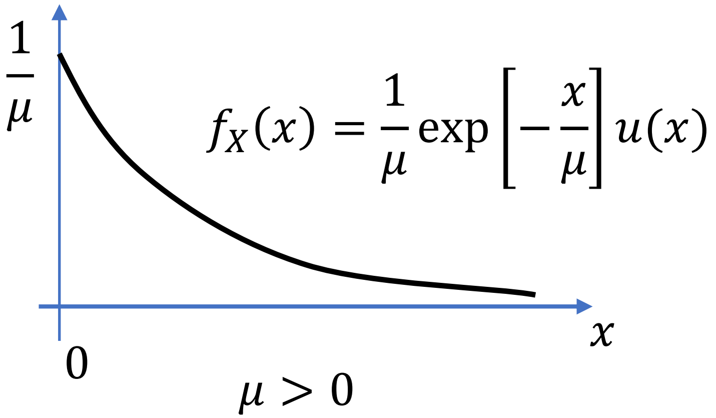
Found the mistakes?





Recap & Exam type questions

Mind the boundaries

Resulting from the support of $f_X(x)$

Solve:

Consider a continuous r.v. X, and $Y = g(X) = \exp(X)$

Special case: uniform over (0,5) ... maps to $(1, e^5)$

$$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{y} f_X(\ln(y)); & y > 0 \\ 0; else \end{cases} = \begin{cases} 0; y > e^5 \\ \frac{1}{5y}; & 1 \leq y \leq e^5 \\ 0; y \leq 1 \end{cases}$$

Special case: exponential over $(0,\infty)$ ... maps to $(1,\infty)$

$$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{\mu y}\exp\left[-\frac{\ln(y)}{\mu}\right]; & y > 0 \\ 0; else \end{cases} = \begin{cases} \frac{1}{\mu y^{\mu+1}}; & y > 1 \\ 0; y \leq 1 \end{cases}$$

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Functions of Random Variables

TASK 9:

Consider continuous two independent standard normal r.v.s X, Y

$$f_{XY}(x, y) = \frac{1}{2\pi} \exp\left[-\frac{x^2 + y^2}{2}\right]$$

$V = g(X, Y) = X + 2Y, \qquad W = h(X, Y) = X - 2Y$

Find $f_{VW}(v, w)$. Are V and W still independent?

Standard approach …

$$F_{VW}(v, w) = P[V \le v, W \le w] = P[g(X, Y) \le v, h(X, Y) \le w]$$

$$= P[X + 2Y \le v, X - 2Y \le w] \ldots$$

… We might integrate & derive to find the density

… but we don't

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3.2 $Y = g(X)$

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3.4 $V = g(X, Y)$, $W = h(X, Y)$

Functions of Random Variables

Solve:

Consider two jointly standard normal r.v.s X, Y

$$f_{XY}(x, y) = \frac{1}{2\pi} \exp\left[-\frac{x^2 + y^2}{2}\right]$$

$$V = g(X, Y) = X + 2Y, \qquad W = h(X, Y) = X - 2Y$$

Direct approach …

$$f_{VW}(v, w) = f_{XY}\big(x(v, w), y(v, w)\big) \frac{\Delta(x, y)}{\Delta(v, w)}$$

$$\frac{\Delta(v, w)}{\Delta(x, y)} = \left|\det\left(\begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}\right)\right| = 4$$

$$\Rightarrow f_{VW}(v, w) = \frac{1}{4} f_{XY}\left(\frac{v + w}{2}, \frac{v - w}{4}\right) = \frac{1}{8\pi} \exp\left[-\frac{\left(\frac{v + w}{2}\right)^2 + \left(\frac{v - w}{4}\right)^2}{2}\right] = \frac{1}{8\pi} \exp\left[-\frac{5v^2 + 6vw + 5w^2}{32}\right]$$

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Functions of Random Variables

Solve:

Are V and W still independent?

Find the marginals of

$$f_{VW}(v,w)=\frac{1}{8\pi}\exp\left[-\frac{5v^2+6vw+5w^2}{32}\right]$$

$$f_W(w)=\int_{-\infty}^{\infty} f_{VW}(v,w)dv=\frac{1}{8\pi}\int_{-\infty}^{\infty}\exp\left[-\frac{5v^2+6vw+5w^2}{32}\right]dv$$

$$=\frac{1}{8\pi}\int_{-\infty}^{\infty}\exp\left[-\frac{5v^2+10v\frac{6w}{10}+5w^2}{32}\right]dv$$

Complete the square

$$=\frac{1}{8\pi}\int_{-\infty}^{\infty}\exp\left[-\frac{\mathbf{5\left(v+\frac{6w}{10}\right)^2}-5\left(\frac{6w}{10}\right)^2+5w^2}{\mathbf{32}}\right]dv$$

Compare: Normal with $\mu=-\frac{6w}{10},\ \sigma^2=\frac{16}{5}$

$$\Rightarrow f_W(w)=$$

$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\exp\left[-\frac{\mathbf{5\left(v+\frac{6w}{10}\right)^2}}{\mathbf{32}}\right]dv\cdot\exp\left[-\frac{w^2}{10}\right]$$

Use the normalization factor of a normal

$$=\frac{\sqrt{2\pi\frac{16}{5}}}{8\pi}\cdot\exp\left[-\frac{w^2}{10}\right]$$

$$=\frac{1}{\sqrt{2\pi\cdot 5}}\cdot\exp\left[-\frac{w^2}{2\cdot 5}\right]$$

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Functions of Random Variables

Solve:

Are V and W still independent?

$$f_{VW}(v,w) = \frac{1}{8\pi}\exp\left[-\frac{5v^2+6vw+5w^2}{32}\right]$$

$$\Rightarrow f_W(w) = \frac{1}{\sqrt{2\pi\cdot 5}}\cdot\exp\left[-\frac{w^2}{2\cdot 5}\right]$$

Similarly:

$$\Rightarrow f_v(v) = \frac{1}{\sqrt{2\pi\cdot 5}}\cdot\exp\left[-\frac{v^2}{2\cdot 5}\right]$$

Hence,

$$f_v(v)f_W(w) = \frac{1}{10\pi}\cdot\exp\left[-\frac{v^2+w^2}{10}\right]$$

Consequence: R.v.s V and W are **not** independent.

Discuss:

Can we go in the reverse direction?

That is, can we use a transformation to obtain independent r.v.s?

How can we find such a transformation?

Techniques coming, soon!

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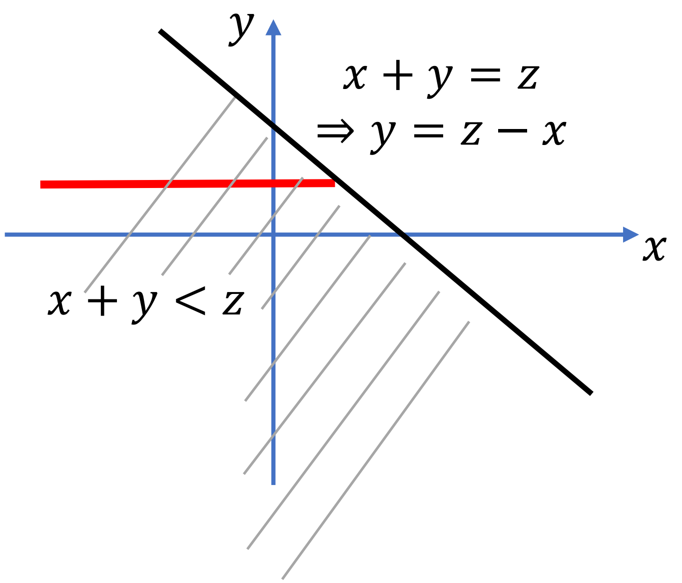
Functions of Random Variables

TASK 10 (re-visit an old example):

Consider continuous r.v.s X, Y,

and $Z = g(X,Y) = X + Y$

Find $f_Z(z)$.



Alternative approach (dummy variable):

$$Z = X + Y, \qquad U = X - Y \text{ (for instance)}$$

1) Find $f_{ZU}(z,u)$

$$f_{ZU}(z,u) = f_{XY}\big(x(z,u), y(z,u)\big)\frac{\Delta(x,y)}{\Delta(z,u)}$$

$$= f_{XY}\left(\frac{z+u}{2}, \frac{z-u}{2}\right)\cdot\frac{1}{2}$$

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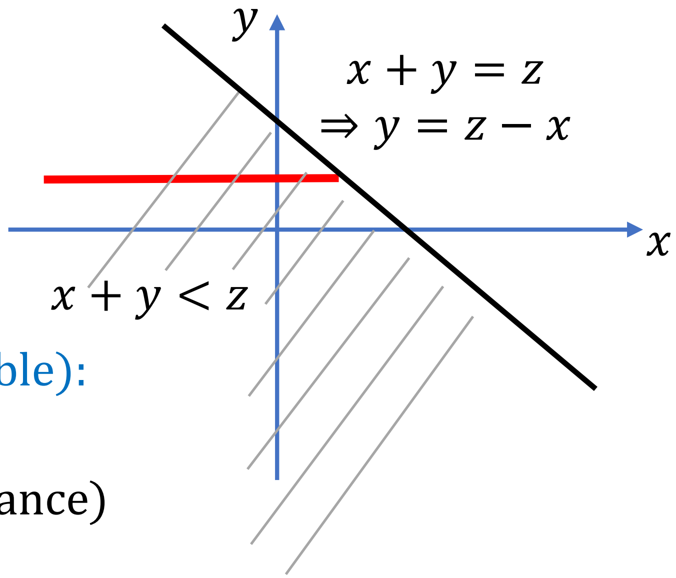
Functions of Random Variables

Solve:



Alternative approach (dummy variable):

$Z = X + Y, \qquad U = X - Y$ (for instance)

2) Marginalize to find $f_Z(z)$

$$f_{ZU}(z,u) = \frac{1}{2} f_{XY}\left(\frac{z+u}{2}, \frac{z-u}{2}\right)$$

$$\Rightarrow f_Z(z) = \int_{-\infty}^{\infty} f_{ZU}(z,u)du = \frac{1}{2}\int_{-\infty}^{\infty} f_{XY}\left(\frac{z+u}{2}, \frac{z-u}{2}\right) du$$

Hence, same thing:

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(t, z-t)dt$$

<u>Interesting Option:</u>

Another change of variables: $(z,u) \mapsto (z,t)$

That is,

keep z, and use $t = \frac{z+u}{2}$ or $u = 2t - z$

$$f_{ZT}(z,t) = f_{ZU}(z,u)\frac{\Delta(z,u)}{\Delta(z,t)}$$

$$= f_{XY}(t, z-t)$$

$$\frac{\Delta(z,u)}{\Delta(z,t)} = \left|\det\left(\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}\right)\right| = 2$$

Can you simplify this derivation?

The End

Next time: Chp. 4