JTMS-12: Probability and Random Processes

Fall 2020

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Lecture 14:
Expectation and Introduction to Estimation

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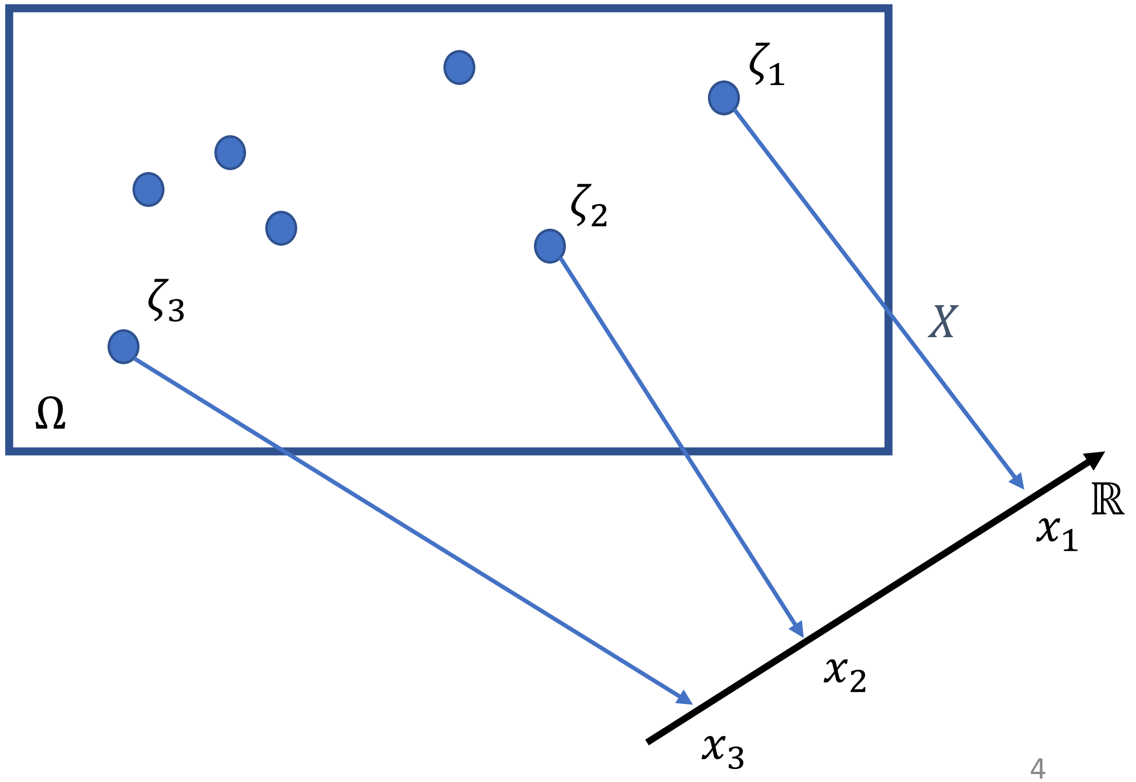
Chapter 4: **Expectation and Introduction to Estimation**

Expectation …

Expected Value of a Random Variable

Try to characterize a new chance experiment / a random variable.

What would you do? – Discuss!



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Chapter 4: **Expectation and Introduction to Estimation**

Expectation …

Expected Value of a Random Variable

Try to characterize a new chance experiment / a random variable X

What would you do? – Discuss!

Find the pdf $f_X(x)$?

Hard to achieve!

Find some aggregate description?

Mean … What is the average or the center?

Variance … How far does it scatter around the mean?

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Expectation …

Expected Value of a Random Variable

Idea of a center:

Suppose, you have a set of sample values (data): $d_1, \ldots, d_n$

Best description of a center? … Well, what is best ???

Maybe: Find the value that is (simultaneously) ``closest´´ to all the sample data.

$$D^2 = \sum_{i=1}^{n} (d_i - c)^2 \to \min$$

Derive D^2 wrt c:

$$-2\sum_{i=1}^{n} (d_i - c) = 0 \Rightarrow c = \frac{1}{n}\sum_{i=1}^{n} d_i$$

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Expectation …

Set- or sample-average (sample-mean)

Average of a set of numbers (tuple would be a better word, actually)

$$\boxed{\mu_s = \frac{1}{n}\sum_{i=1}^{n} d_i}$$

What about the range over which the values scatter?

Set- or sample-variance

$$\boxed{\sigma_s^2 = \frac{1}{n}\sum_{i=1}^{n} (d_i - \mu_s)^2}$$

σ_s is the standard deviation of the set/sample

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Example (six-faced die)

Data: $(d_1, \dots, d_n) = (3, 5, 6, 2, 4, 1, 5, 4, 6, 4)$

Set- or sample-average (mean)

$$\mu_s = \frac{1}{n}\sum_{i=1}^{n} d_i = \frac{1}{10} \cdot 40 = 4.0$$

Set- or sample-variance

$$\sigma_s^2 = \frac{1}{n}\sum_{i=1}^{n} (d_i - \mu_s)^2 \cong 2.67$$

Standard deviation of the set/sample

$$\sigma_s \cong 1.63$$

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… Example (six-faced die) contd.

Data: $(d_1, \dots, d_n) = (3, 5, 6, 2, 4, 1, 5, 4, 6, 4)$

Set- or sample-average (mean)

$$\mu_s = \frac{1}{n}\sum_{i=1}^{n} d_i = \frac{1}{10} \cdot 40 = 4.0$$

Re-interpret from the ``relative frequency´´ perspective: We observed

Ones: 1, Twos: 1, Threes: 1, Fours: 3, Fives: 2, Sixes: 2

$$\mu_s = \frac{1}{n}\sum_{i=1}^{n} d_i = \sum_{i=1}^{6} \frac{n_i}{n} x_i$$

$$= \frac{1}{10} \cdot 1 + \frac{1}{10} \cdot 2 + \frac{1}{10} \cdot 3 + \frac{3}{10} \cdot 4 + \frac{2}{10} \cdot 5 + \frac{2}{10} \cdot 6 = 4.0$$

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Inspires definition for the discrete case:

$$\boxed{\mu_X = E[X] = \sum_i P[X = x_i] \cdot x_i}$$

General case (if the integral exists):

$$\boxed{\mu_X = E[X] = \int_{-\infty}^{\infty} x f_X(x) dx}$$

Useful in practice: for $Y = \mathrm{g}(X)$

$$\mu_Y = E[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

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Example (normal r.v.)

Consider $X \sim \mathcal{N}(\mu, \sigma^2)$

$$E[X] = \int_{-\infty}^{\infty} x \left\{ \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2} \right] \right\} dx$$

Change of variables:

$$\mathrm{Z} = \frac{X - \mu}{\sigma} \Rightarrow X = \sigma Z + \mu, dx = \sigma dz$$

$$E[X] = \frac{\sigma}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} z \exp\left[-\frac{z^2}{2} \right] dz}_{=0\ (odd)} + \mu \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{z^2}{2} \right] dz}_{=1} = \mu$$

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Example 2 (Poisson)

Consider $X \sim$ Poisson with parameter a

Possible values are $x_k = k$

$$\Rightarrow E[X] = \sum_{k=0}^{\infty} k \cdot P[X = k] = \sum_{k=0}^{\infty} k \frac{e^{-a}}{k!} a^k = \sum_{\mathbf{k=1}}^{\infty} k \frac{e^{-a}}{k!} a^k$$

$$= a \sum_{k=1}^{\infty} \frac{e^{-a}}{(k-1)!} a^{k-1} = a \underbrace{\sum_{m=0}^{\infty} \frac{e^{-a}}{m!} a^m}_{=1} = a$$

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Example 3 (normal r.v.)

Consider $X \sim \mathcal{N}(0, \sigma^2)$, and $Y = X^2$

$$E[Y] = E[X^2] = \int_{-\infty}^{\infty} x^2 \left\{ \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{x^2}{2\sigma^2} \right] \right\} dx$$

Change of variables:

$$Z = \frac{X}{\sigma} \Rightarrow X = \sigma Z, dx = \sigma dz$$

$$E[Y] = \sigma^2 \underbrace{\int_{-\infty}^{\infty} z^2 \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2} \right] dz}_{=1} = \sigma^2$$

Use integration by parts … this is how to split …

$$z \cdot \frac{z}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2} \right]$$

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Example 4 (Cauchy pdf … integral may not exist)

Consider a r.v. with pdf with $-\infty < \alpha < \infty;\ \beta > 0$:

$$f_X(x) = \frac{1}{\pi\beta\left(1 + \left(\frac{x-\alpha}{\beta}\right)^2\right)}$$

Special case, $\alpha = 0, \beta = 1$:

$$E[X] = \int_{-\infty}^{\infty} x \cdot \frac{1}{\pi(1+x^2)} dx \ldots does\ not\ converge$$

One might use symmetry to argue for $E[X] = 0$, here (Cauchy principal value).

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\pi(1+x^2)} dx = \infty$$

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Which pdf to use?

Consider r.v.s X, Y with joint pdf $f_{XY}(x, y)$.

What about expected values relative to this joint density?

For instance …

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{XY}(x,y)dx\,dy = \int_{-\infty}^{\infty} x \underbrace{\int_{-\infty}^{\infty} f_{XY}(x,y)dy}_{=f_X(x)}\,dx$$

$$= \int_{-\infty}^{\infty} x f_X(x)dx = E[X] = \mu_X$$

Similarly,

$$\mu_Y = E[Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{XY}(x,y)dx\,dy$$

Mind: This is why people do not specify the pdf when they write $E[X]$.

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Expectations are linear operations …

Consider r.v.s X, Y with joint pdf $f_{XY}(x, y)$. Find $E[aX + bY]$

$$E[aX + bY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (ax + by) f_{XY}(x, y)\, dx\, dy$$

$$= a \int_{-\infty}^{\infty} x \underbrace{\int_{-\infty}^{\infty} f_{XY}(x, y)\, dy}_{=f_X(x)}\, dx + b \int_{-\infty}^{\infty} y \underbrace{\int_{-\infty}^{\infty} f_{XY}(x, y)\, dx}_{=f_Y(y)}\, dy$$

$$= a \int_{-\infty}^{\infty} x f_X(x) dx + b \int_{-\infty}^{\infty} y f_Y(y) dy = aE[X] + bE[Y]$$

For instance,

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

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Example:

We roll 100 fair dice, represented by random variables $X_1, \ldots, X_{100}$

Consider

$$Z = \sum_{i=1}^{100} X_i$$

Find $E[Z]$

$$E[Z] = E\left[\sum_{i=1}^{100} X_i\right] = \sum_{i=1}^{100} E[X_i]$$

$$E[X_i] = \frac{1}{6}(1 + 2 + \cdots + 6) = \frac{21}{6} = 3.5$$

$$\Rightarrow E[Z] = 350$$

The End

Next time: Continue with Chp. 4