Jacobs University Bremen
Keivan Mallahi-Karai

Due: September 11, 2019
Assignment 1

Elements of Probability

In each one of the problems below, if you are asked to compute a probability, first identify the sample space and the event in question explicitly.

(1.1) The birthday of six random people has been checked. Find the probability that
(a) At least one of them is born in September.
(b) All six are born in the Spring. Spring here means one of the month March, April, or May.
(c) At least two of them are born in the same month.

In this problem you can assume that a year is 365 days and the probability that a person is born on a specific day of the year is exactly 1/365.

Solution. Denote the event in parts (a), (b), (c) by A, B, C, respectively. Then

$$\mathbb{P}[A] = 1 - \mathbb{P}[A^c] = 1 - \frac{11^6}{12^6} \approx 0.53$$

For (b), we use the same argument as in the birthday problem:

$$\mathbb{P}[B] = \frac{3^6}{12^6}. \approx 0.85$$

For (c), we use the compliment event:

$$\mathbb{P}[C] = 1 - \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7}{12^6} \approx$$

(1.2) A fair die is rolled three times. We say that a match has occurred if the outcome of the first throw is 1, or the outcome of the second throw is 2, or the outcome of the third throw is 3. Find the probability of the event that a match occurs.

Solution. For $i = 1, 2, 3$, let us denote the event that the outcome of throw i is i by A_i. We are interested in computing $\mathbb{P}[A_1 \cup A_2 \cup A_3]$. It is clear that

$$\mathbb{P}[A_1] = \mathbb{P}[A_2] = \mathbb{P}[A_3] = \frac{1}{6}.$$

Similarly,

$$\mathbb{P}[A_1 \cap A_2] = \mathbb{P}[A_1 \cap A_3] = \mathbb{P}[A_2 \cap A_3] = \frac{1}{36}.$$

Finally, since $\mathbb{P}[A_1 \cap A_2 \cap A_3] = \frac{1}{6^3}$, using inclusion-exclusion principle, we can write

$$\mathbb{P}[A_1 \cup A_2 \cup A_3] = \frac{3}{6} - \frac{3}{6^2} + \frac{1}{6^3} = \frac{91}{6^3}.$$

Alternatively, if no matches occur, then 5 options are left for each round. Hence

$$\mathbb{P}[A] = 1 - \frac{5^3}{6^3} = \frac{91}{6^3}.$$

(1.3) An ordinary deck of playing cards (containing 52 standard cards, 13 of each suit) is randomly divided into two parts, each containing at least one card.

(a) What is the probability that each part contains at least one ace.

(b) Find the probability that each part contains exactly two aces.

Solution. Denote the cards in the left and the right parts by D_1 and D_2. It is clear that D_1 must be a subset of the set of all cards, which is non-empty and also is not equal to the set of all cards, for otherwise, D_2 would be empty. This implies that that the sample space contains $2^{52} - 2$ elements.

Let A be the event that each part contains an ace. Note that A^c will be the event that all the aces are in D_1 or in D_2. If all aces are in D_1, then there are 48 cards left to be distributed between D_1 and D_2 with the constraint that not all of them go to D_1. This implies that the number of possibilities is $2^{48} - 1$. There is the same number of distributions in which all aces go to D_2. Hence

$$\mathbb{P}[A] = 1 - \frac{2 \times (2^{48} - 1)}{2^{52} - 2} = 1 - \frac{2^{48} - 1}{2^{51} - 1}.$$

For part (b), note that there are $\binom{4}{2} = 6$ aces to choose two out of four aces for D_1. The remaining 48 cards can be distributed arbitrarily between D_1 and D_2. Hence the probability of the event B that each part contains exactly two aces equals:

$$\mathbb{P}[B] = \frac{6 \times 2^{48}}{2^{52} - 2}.$$

(1.4) A is called a *palindrome* if it reads the same from left and right. For instance, 13631 is a palindrome, while 435734 is not. A 6-digit number n is randomly chosen. Find the probability of the event that

(a) n is a palindrome.

(b) n is odd and a palindrome.

(c) n is even and a palindrome.

Solution. Since there are 9 options for the first digit from the left of n, and 10 for each one of the remaining digits, the sample space consists of 9×10^4 elements. Write

$$n = \overline{n_1 n_2 n_3 n_4 n_5 n_6}$$

where $n_1, \ldots, n_6$ are digits of n. There are 9 options for n_1, and since $n_1 = n_6$, this also determines the value of n_6. There are 10 possible choices for n_2 which will determine n_5. Finally, there are 10 choices for n_4, which also determines the value of n_3. This implies that the probability of the event A that n is a palindrome is

$$\mathbb{P}[P] = \frac{9 \times 10 \times 10}{9 \times 10^5} = \frac{1}{1000}.$$

(b) Denote by OP the event that the randomly chosen number is odd and a palindrome. Note that n is even when $n_6 = 1, 3, 5, 7, 9$. Hence, there are 5 options for n_6 which determines n_1. Hence, by continuing the argument as in part (a), we have

$$\mathbb{P}[A \cap E] = \frac{5 \times 10 \times 10}{9 \times 10^5} = \frac{5}{9000}.$$

(c) Denote by EP the event that the randomly chosen number is even and a palindrome. Then

$$\mathbb{P}[EP] = \mathbb{P}[P] - \mathbb{P}[OP] = \frac{9}{9000} - \frac{5}{9000} = \frac{4}{9000}.$$

(1.5) (a) Suppose A and B are two events. Let S be the event that A or B occur, but not both. Show that

$$\mathbb{P}[S] = \mathbb{P}[A] + \mathbb{P}[B] - 2\,\mathbb{P}[A \cap B].$$

(b) Suppose A, B, and C are three events in a sample space. Let T denote the event that exactly one of these three events occur. Deduce from the axioms that

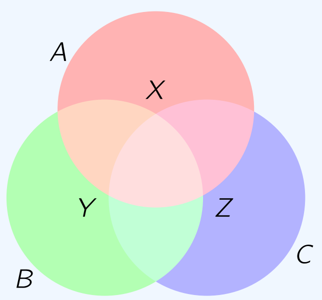
$$\mathbb{P}[T] = \mathbb{P}[A] + \mathbb{P}[B] + \mathbb{P}[C] - 2\,\mathbb{P}[A \cap B] - 2\,\mathbb{P}[A \cap C] - 2\,\mathbb{P}[B \cap C] + 3\,\mathbb{P}[A \cap B \cap C].$$

Hint: Draw a Venn diagram and use it to describe S and T as Boolean combination of the given events.

Solution. It is clear that S consists of those elements of $A \cup B$ which are *not* in $A \cap B$. Hence

$$\begin{aligned} \mathbb{P}[S] &= \mathbb{P}[A \cup B] - \mathbb{P}[A \cap B] = (\mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]) - \mathbb{P}[A \cap B] \\ &= \mathbb{P}[A] + \mathbb{P}[B] - 2\mathbb{P}[A \cap B]. \end{aligned} \tag{1}$$

For (b) Let X be the event that only A occurs. Similarly, let Y and Z denote, respectively, the events that only B and C occur.



It is clear that $T = X \cap Y \cup Z$. Moreover, using part (a) we can write

$$\mathbb{P}[X] = \mathbb{P}[A] - \mathbb{P}[A \cap B] - \mathbb{P}[A \cap C] + \mathbb{P}[A \cap B \cap C].$$

Similarly, we have

$$\mathbb{P}[Y] = \mathbb{P}[B] - \mathbb{P}[A \cap B] - \mathbb{P}[B \cap C] + \mathbb{P}[A \cap B \cap C].$$

and

$$\mathbb{P}[Z] = \mathbb{P}[C] - \mathbb{P}[A \cap C] - \mathbb{P}[B \cap C] + \mathbb{P}[A \cap B \cap C].$$

Adding these up and noting that X, Y, Z are pairwise disjoin, we obtain the result.

(1.6) (Bonus) Suppose $A_1, \ldots, A_n$ are events in some probability space. Show that

$$\mathbb{P}[A_1 \cap A_2 \cdots \cap A_n] \geq \sum_{i=1}^{n} \mathbb{P}[A_i] - (n-1).$$

Solution. Denote the complement of A_i by C_i. We have $\mathbb{P}[C_i] = 1 - \mathbb{P}[A_i]$. Moreover, since $A_1 \cap A_2 \cdots \cap A_n = (C_1 \cup \cdots \cup C_n)^c$ we have

$$\mathbb{P}[A_1 \cap A_2 \cdots \cap A_n] = 1 - \mathbb{P}[C_1 \cup \cdots \cup C_n].$$

But

$$\mathbb{P}[C_1 \cup \cdots \cup C_n] \leq \sum_{i=1}^{n} \mathbb{P}[C_i] = \sum_{1=1}^{n} (1 - \mathbb{P}[A_i]) = n - \sum_{=1}^{n} \mathbb{P}[A_i].$$

The claim follows immediately from here.