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Due: October 5, 2018
Assignment 3

Elements of Probability

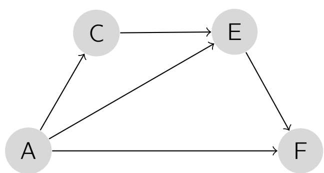
(3.1) Consider a discrete random variable X that with the PMF given by

$$p_X(x) = \begin{cases} k|x| & \text{if } x = -3, -1, 1, 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the value of k.
(b) Find $\mathbb{P}[|X| > 2]$.
(c) Find the PMF of the random variables $Y = X^2$ and $Z = X^3$.

(3.2) Theresa May is proposing a Brexit deal to the House of Commons which consists of 650 MPs.
(a) Assume first that each MP decides individually and independent of the rest of MPs to vote for the proposal with probability $p = 0.52$. Assuming that the prime minister needs more than half of the votes for passing the bill, find the probability that the bill is passed. You can use the special widget of *Wolfram's alpha* for doing the calculation.
(b) A group of 20 Pro-Brexit MPs have decided to vote against the bill. Compute the probability in part (a).
(c) Repeat part (a) and (b) for $p = 0.50$ and $p = 0.48$ and compare the results.

(3.3) A network connects computers A and F via intermediate nodes C, E as shown below. For each pair of directly connected nodes, there is a probability $p = 3/4$ that the connection from i to j is up. Assume that the link failures are independent events.
(a) Find the probability that the connection from A to F through at least one of the paths is up.
(b) Due to weather condition, connections AC, CE, EF, AE are simultaneously on or off, with probability $p = 3/4$. The connection AF which is not affected by weather is independency open with probability $p = 3/4$. Under this assumption, compute the probability that the connection from A to F through at least one of the paths is up, and compare the result to part (a).



(3.4) Suppose X is a random variable with a geometric distribution with parameter p.
(a) Show that

$$\mathbb{P}[X > k] = (1-p)^k.$$

(b) Show that for all $n, k > 0$, we have

$$\mathbb{P}[X = n + k | X > k] = \mathbb{P}[X = n].$$

(3.5) Consider a coin which lands H with probability p and T with probability $1 - p$. The coin is flipped until H shows up for the *second* time. Let N denote the number of required flips.

(a) For warm-up, show that $\mathbb{P}[N=0] = \mathbb{P}[N=1] = 0$, and $\mathbb{P}[N=2] = p^2$.
(b) Show that $\mathbb{P}[N=3] = 2p^2(1-p)$
(c) In general, show that the PMF of N is given by

$$\mathbb{P}[N=k] = (k-1)p^2(1-p)^{k-2}, \qquad k = 2, 3, \ldots$$

Hint: $N = k$ exactly when the k-th flip results in H and all but one of the previous $k-1$ flips result in T.

(3.6) (Bonus) Suppose that the probability of a coin landing heads is p, and that outcome of successive throws of the coin are independent. Let E denote the event that first HH appears before the first TT. Denote by X the outcome of the first throw.
(a) Show that

$$\mathbb{P}[E|X=H] = p + (1-p)\mathbb{P}[E|X=T].$$

(b) Find a similar formula for $\mathbb{P}[E|X=T]$.
(c) Use parts (a) and (b) to compute $\mathbb{P}[E]$.