Lecture 12

Recall

The NORMAL DISTRIBUTION

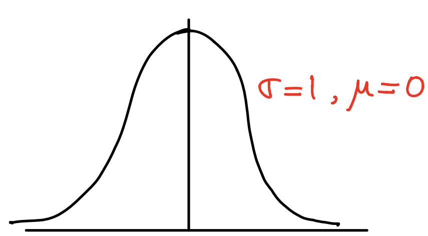
A random variable X has normal (Gaussian) distribution with parameters μ and σ if its PDF is given by

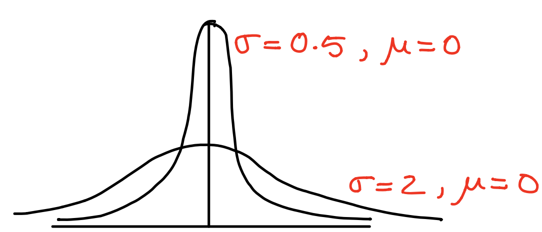
$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

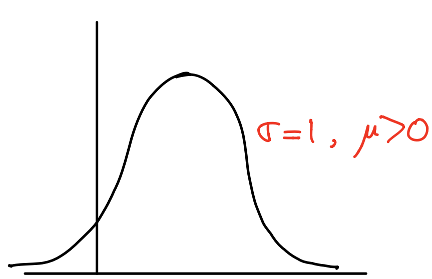
μ: mean $\quad$ σ: standard deviation $\quad$ $\sigma > 0$, μ arbitrary

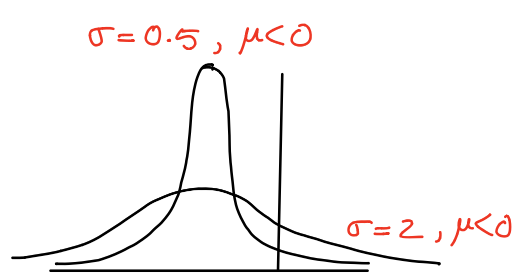
X has standard normal distribution, when $\mu = 0$, $\sigma = 1$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$





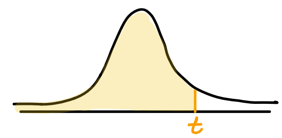




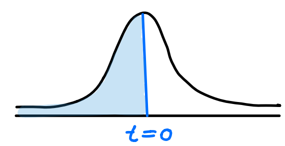
The distribution function of a normal random variable.

- Standard normal random variable

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \implies F_X(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

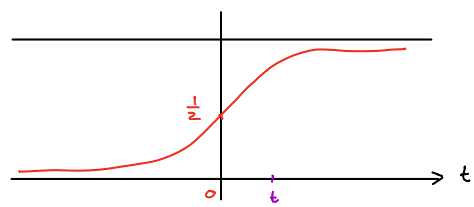


- $F_X(0) = \frac{1}{2}$



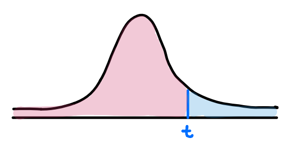
- There is no closed formula for $F_X(t)$

- The graph of



$$\Phi(t) = \mathbb{P}(X \leq t)$$

$$\mathbb{P}(X \geq t) = 1 - \Phi(t)$$



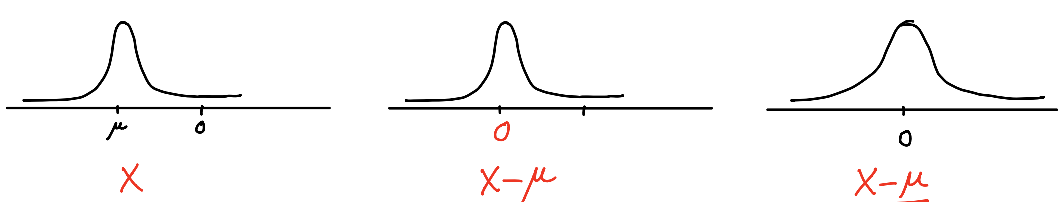
$$\mathbb{P}(t_1 \leq X \leq t_2) = \Phi(t_2) - \Phi(t_1)$$

Suppose X has a normal distribution with parameters μ and σ.

Set

$$Z = \frac{X - \mu}{\sigma}$$

Then Z has a standard normal distribution



σ

<u>Example</u> Suppose X is a normal random variable with $\mu = 3$, $\sigma = 2$. Find the value of $\mathbb{P}(1 \leq X \leq 7)$.

$$X \sim N(3,2) \Longrightarrow Z = \frac{X-3}{2} \sim N(0,1).$$

$$\mathbb{P}(1 \leq X \leq 7) = \mathbb{P}(1-3 \leq X-3 \leq 7-3) = \mathbb{P}\left(-1 \leq \frac{X-3}{2} \leq 2\right)$$
$$= \mathbb{P}(-1 \leq Z \leq 2) = \Phi(2) - \Phi(-1).$$

Why the normal distribution?

Answer: The Central limit theorem

The Central limit theorem:

Suppose $X_1, X_2, \ldots$ are independent random variables with the same distribution. Assume

$$\mathbb{E}[X_i] = \mu, \qquad \mathrm{Var}[X_i] = \sigma$$

Set

$$Z_n = \frac{(X_1-\mu) + \cdots + (X_n-\mu)}{\sqrt{n}\,\sigma}$$

Then as $n \to \infty$ the distribution of Z_n tends to the distribution of a standard normal random variable, that is,

$$\lim_{n\to\infty} \mathbb{P}(Z_n \leq t) = \Phi(t).$$

Joint distribution

Suppose X and Y are two random variables defined on some probability space. Suppose, further, that X and Y are both discrete. The joint probability mass function of X and Y is defined by

$$p_{X,Y}(x,y) = \mathbb{P}[X=x, Y=y]$$

<u>Example</u> Suppose X takes values $1, 2, 3$, and Y takes values $0, 1$

	$Y=0$	$Y=1$
$X=1$	$1/6$	$1/6$
$X=2$	$1/12$	$5/12$
$X=3$	$1/12$	$1/12$

	$Y=0$	$Y=1$	
$X=1$	$1/6$	$1/6$	$1/3$
$X=2$	$1/12$	$5/12$	$1/2$
$X=3$	$1/12$	$1/12$	$1/6$

X	1	2	3
$p_X(x)$	$1/3$	$1/2$	$1/6$

	$Y=0$	$Y=1$
$X=1$	$1/6$	$1/6$
$X=2$	$1/12$	$5/12$
$X=3$	$1/12$	$1/12$
	$1/3$	$2/3$

Y	0	1
$p_Y(y)$	$1/3$	$2/3$

<u>Example 2</u> A pair of fair coins are flipped.

outcomes $\Omega = \{HH, HT, TH, TT\}$

	H	T	
H	p	$\frac{1}{2}-p$	$\frac{1}{2}$
T	$\frac{1}{2}-p$	p	$\frac{1}{2}$
	$\frac{1}{2}$	$\frac{1}{2}$	

	H	T
H	$\frac{1}{4}$	$\frac{1}{4}$
T	$\frac{1}{4}$	$\frac{1}{4}$

$p = \frac{1}{4}$

	H	T
H	$\frac{1}{2}$	0
T	0	$\frac{1}{2}$

$p = \frac{1}{2}$

	H	T
H	0	$\frac{1}{2}$
T	$\frac{1}{2}$	0

$p = 0$

Back to example 1 Suppose that the joint PMF of X, Y is given by:

	$Y=0$	$Y=1$
$X=1$	$1/6$	$1/6$
$X=2$	$1/12$	$5/12$
$X=3$	$1/12$	$1/12$

Find the PMF of the random variable $Z = X+Y$, $T = XY$

$X+Y$ takes values: $1, 2, 3, 4$

	$Y=0$	$Y=1$
$X=1$	$1/6$	$1/6$
$X=2$	$1/12$	$5/12$
$X=3$	$1/12$	$1/12$

	$Y=0$	$Y=1$
$X=1$	$1/6$	$1/6$
$X=2$	$1/12$	$5/12$
$X=3$	$1/12$	$1/12$

	$Y=0$	$Y=1$
$X=1$	$1/6$	$1/6$
$X=2$	$1/12$	$5/12$
$X=3$	$1/12$	$1/12$

	$Y=0$	$Y=1$
$X=1$	$1/6$	$1/6$
$X=2$	$1/12$	$5/12$
$X=3$	$1/12$	$1/12$

Z	1	2	3	4
$P_Z(z)$	$1/6$	$1/4$	$1/2$	$1/12$

T takes values $0, 1, 2, 3$

	$Y=0$	$Y=1$
$X=1$	$1/6$	$1/6$
$X=2$	$1/12$	$5/12$
$X=3$	$1/12$	$1/12$

	$Y=0$	$Y=1$
$X=1$	$1/6$	$1/6$
$X=2$	$1/12$	$5/12$
$X=3$	$1/12$	$1/12$

	$Y=0$	$Y=1$
$X=1$	$1/6$	$1/6$
$X=2$	$1/12$	$5/12$
$X=3$	$1/12$	$1/12$

	$Y=0$	$Y=1$
$X=1$	$1/6$	$1/6$
$X=2$	$1/12$	$5/12$
$X=3$	$1/12$	$1/12$

T	0	1	2	3
P_T	$1/3$	$1/6$	$5/12$	$1/12$

Example Suppose X, Y are randomly chosen from the set $1, 2, 3, 4$, such that each pair (i,j) has the same probability $\frac{1}{16}$ of being chosen. Set $U = \max(X,Y)$, $V = \min(X,Y)$.
Find the joint PMF of U and V.

$U \backslash V$	$V=1$	$V=2$	$V=3$	$V=4$	
$U=1$	$\frac{1}{16}$	0	0	0	$\frac{1}{16}$
$U=2$	$\frac{2}{16}$	$\frac{1}{16}$	0	0	$\frac{3}{16}$
$U=3$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	0	$\frac{5}{16}$
$U=4$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{7}{16}$
	$\frac{7}{16}$	$\frac{5}{16}$	$\frac{3}{16}$	$\frac{1}{16}$	

So U has distribution

U	1	2	3	4
P	$\frac{7}{16}$	$\frac{5}{16}$	$\frac{3}{16}$	$\frac{1}{16}$

V has distribution

V	1	2	3	4
	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$

Example From a set of 15 batteries containing 3 new, 5 used and 7 defective batteries a sample of two batteries is chosen. Let X denote the number of new and Y denote the number of defective batteries.
(a) compute the joint PMF of X, Y (b) Compute $E[X]$.

X takes values $0,1,2$ Y: takes values $0,1,2$

	$Y=0$	$Y=1$	$Y=2$
$X=0$			
$X=1$			0
$X=2$		0	0

Definition Two random variables X and Y are called independent if for every x, y we have

$$P_{X,Y}(x,y) = P_X(x)\, P_Y(y).$$

Example. Suppose X and Y have the following PMF:

X	1	2	3
$P_X(x)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

Y	0	1
$P_Y(y)$	$\frac{1}{3}$	$\frac{2}{3}$

If X and Y are independent, then the joint PMF of X and Y is given by

	$Y=0$	$Y=1$
$X=1$	$\frac{1}{9}$	$\frac{2}{9}$
$X=2$	$\frac{1}{6}$	$\frac{1}{3}$
$X=3$	$\frac{1}{18}$	$\frac{2}{18}$

Definition For two random variables X, Y, the covariance is defined by

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

Example Suppose X, Y have the joint PMF given by

$X \backslash Y$	1	2	
1	$1/8$	$1/4$	$3/8$
2	$1/4$	$3/8$	$5/8$
	$3/8$	$5/8$	

Then

$$\mathbb{E}[XY] = 1 \cdot \frac{1}{8} + 2 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{3}{8} = \frac{21}{8}$$

$$\mathbb{E}[X] = 1 \cdot \frac{3}{8} + 2 \cdot \frac{5}{8} = \frac{13}{8}$$

$$\mathbb{E}[Y] = 1 \cdot \frac{3}{8} + 2 \cdot \frac{5}{8} = \frac{13}{8}$$

$$\mathrm{Cov}(X,Y) = \frac{21}{8} - \frac{169}{64} = \frac{168-169}{64} = \frac{-1}{64}$$