Elements of Probability

Solve only 5 out of the following 6 problems.

(4.1) [redacted] let the joint probability mass function of discrete random variables X and Y be given by

$$p_{X,Y}(x,y) = \begin{cases} k(x+y) & \text{if } 1 \leq x, y \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine the value of constant k.

(b) Determine the probability mass functions of X and Y.

(c) Find $\mathbb{P}[X \geq Y]$.

(4.2) A coin is flipped three times. Let X denote the number of heads and Y denote the number of streaks of heads of length 2. For instance, if the outcome is HTH, then $X = 2$ and $Y = 0$, while if the outcome is HHT, then $X = 2$ and $Y = 1$.
(a) Find the joint probability mass function of X and Y.
(b) Find the covariance of X and Y.
(c) Are X and Y independent?

(4.3) Suppose X and Y are independent random variables with the same probability mass function given by

$$p(-1) = \frac{1}{4}, \quad p(0) = \frac{1}{2}, \quad p(1) = \frac{1}{4}.$$

(a) Find the joint probability mass function of X and Y.
(b) Determine the probability mass function of $Z = XY$ and $T = X + Y$.

(4.4) Suppose X is a normal random variable with $\mu = 3$ and $\sigma = 2$. Compute the following probabilities in terms of the function Φ.
(a) $\mathbb{P}[-1 < X < 1]$.
(b) $\mathbb{P}[X > 3]$.
(c) $\mathbb{P}[X < -3]$.

(4.5) Suppose X and Y are randomly chosen from the set $\{-1, 0, 1\}$, in such a way that each pair (x, y) has the same probability $1/9$ of being chosen. Define $M = |X|$ and $N = |Y|$.
(a) Compute the joint probability mass function of M and N.
(b) Find $\mathbb{E}[M], \mathbb{E}[N]$, and $\mathrm{Cov}(M, N)$.
(c) Are M and N independent?

(4.6) (Bonus) Suppose X and Y are discrete random variables with mean 0 and variance 1. Let $Z = \max(X^2, Y^2)$.
(a) Show that $\mathbb{E}[Z] \leq 2$.
(b) Suppose $\rho = cov(X, Y)$. Prove that $\mathbb{E}[Z] \leq 1 + \sqrt{1 - \rho^2}$.