JMTS-12: Probability and Random Processes

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M. Bode

Orga

Textbook:

Henry Stark & John W. Woods

Probability and Random Processes with Applications to Signal Processing

Chapters 1-4 … parts of 5+6 if time permits

Main platform: campusnet … course page !!!

For the online meetings, use headsets if possible…

Streaming might also help in the lecture halls.

Lecture 3

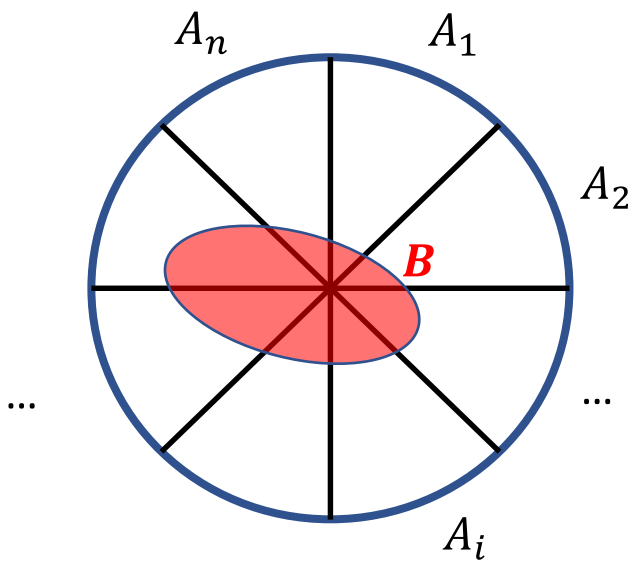
Chapter 1

Bayes's Theorem ...

Simply apply the total probability from before:

$$P[A_j|B] = \frac{P[B|A_j]P[A_j]}{P[B]}$$

$$= \frac{P[B|A_j]P[A_j]}{\sum_{i=1}^{n} P[B|A_i]\,P[A_i]}$$



Practice with communication channel example in your textbook!

Lecture 3

Chapter 1

1.7 Bayes's Theorem and Applications
1.8 Combinatorics
1.9 Bernoulli Trials - Binomial Law ...
1.10 Asymptotic ... Poisson Law

Combinatorics …

Four basic experiments: Draw r balls from a box (urn) with n … and record the result.

Draw r balls

Urn with
n balls

Record …
1) As they come out … in drawing order
2) Or … ignore that order

Then …
a) Replace them …
b) Or … do not replace

Lecture 3

Chapter 1

Experiment		Drawing Order	
		Yes	No
Replace	Yes	?	?
	No	?	?

Combinatorics ...

Four basic experiments: Draw r balls from a box (urn) ... and record the result.

Record ...

1) As they come out ... in **drawing order**
2) Or ... **ignore** that order

Then ...

a) **Replace** them ...
b) Or ... **do not replace** ...

How many different results are there ... in each type?

Let‘s count ... ☺

Lecture 3

Chapter 1

Experiment		Drawing Order	
		Yes	No
Replace	**Yes**	n^r	?
	No	?	?

Combinatorics ... COUNT

Four basic experiments: Draw r balls from a box (urn) with n ... and record the result.

1) Record in drawing order

a) Replace

How many different results are there ... in this type?

Idea: Draw a ball ... record its name ... replace.

Like (3,5,3,9) ... tuple

1. n options
2. n options
3. n options

...

Total = n^r

Understand: Multiply!

Do *not* add!

Lecture 3

Chapter 1

1.7 Bayes's Theorem and Applications
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Experiment		Drawing Order	
		Yes	No
Replace	Yes	n^r	?
	No	$\frac{n!}{(n-r)!}$	?

Combinatorics ... COUNT

Four basic experiments: Draw r balls from a box (urn) with n ... and record the result.

1) Record in drawing order

b) Do not replace

How many different results are there ... in this type?

Idea: Draw a ball ... record its name ... do not replace.

Like (3,5,9,7) ... tuple

1. n options
2. n-1 options
3. n-2 options

...

Total = $n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$

Lecture 3

Chapter 1

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Experiment		Drawing Order	
		Yes	**No**
Replace	Yes	n^r	?
	No	$\frac{n!}{(n-r)!}$	$\boldsymbol{\binom{n}{r}}$

Combinatorics ... COUNT

Four basic experiments: Draw r balls from a box (urn) with n ... and record the result.

2) Ignore drawing order

b) Do not replace

How many different results are there ... in this type?

<u>Idea:</u> Draw a ball ... record its name ... do not replace.

Like {3,5,7,9} ... subset

1. Keep drawing order
2. Then re-arrange: identify (3,5,7,9) and (3,7,9,5) etc.

...

Total = $\frac{n!}{(n-r)!r!} = \binom{n}{r}$

``n choose r´´ ... binomial coefficient

Lecture 3

Chapter 1

Experiment		Drawing Order: Yes	Drawing Order: **No**
Replace	**Yes**	n^r	**?**
	No	$\frac{n!}{(n-r)!}$	$\binom{n}{r}$

Combinatorics ... COUNT

Four basic experiments: Draw r balls from a box (urn) with n ... and record the result.

2) Ignore drawing order

a) Replace

How many different results are there ... in this type?

Idea: Draw a ball ... record its name ... replace. Like

1: III
2: I
3:
4: IIII
5: II
6:
7: III

7 balls, 13 draws:
How can we count
the different results, here?

Lecture 3

Chapter 1

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Experiment		Drawing Order	
		Yes	**No**
Replace	**Yes**	n^r	**?**
	No	$\frac{n!}{(n-r)!}$	$\binom{n}{r}$

Combinatorics ... COUNT

Start easy ... draw r balls from an urn with **n=1** ball.

2) Ignore drawing order
a) Replace

How many different results are there ... in this type?

1: r times

Just one possible case: ball 1, ball 1, ball 1 , ... r times

Now draw r balls from an urn with **n=2** balls.

Ball 1:	**0 times**	**1**	**...**	$r-1$	**r**
Ball 2:	r times	$r-1$		1	0
Cases:	1	1	1	1	1

Total:

$r+1$ **cases:**

Lecture 3

Chapter 1

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Experiment		Drawing Order: Yes	Drawing Order: No
Replace	Yes	n^r	?
	No	$\frac{n!}{(n-r)!}$	$\binom{n}{r}$

Combinatorics ... COUNT

2) Ignore drawing order

a) Replace

This leads to a beautiful induction (over n) based on Pascal‘s triangle ... TRY! The sums we need appear there ... just a bit lengthy. ☹

Here is the elegant way ... re-arrange the table:

1: *** (3 times)
2: *
3:
4: ****
5: **
6:
7: ***

Ball 1:	2	3	4	5	6	7
***	*		****	**		***

Lecture 3

Chapter 1

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Experiment		Drawing Order	
		Yes	**No**
Replace	**Yes**	n^r	$\binom{r+n-1}{r}$
	No	$\frac{n!}{(n-r)!}$	$\binom{n}{r}$

Combinatorics ... COUNT

2) Ignore drawing order

a) Replace

Ball 1:	2	3	4	5	6	7
***	*		****	**		***

Interprete:

A total of r = 13 balls (**stars**) to be distributed over n = 7 boxes (**rooms**)

How? ... Think of a hotel!

Use vertical **bars** to represent the walls between the rooms:

Fixed wall   *** | * | | **** | ** | | ***   fixed wall

Each such sequence of r stars and (n-1) bars represents exactly one result ... 1-to-1

Count:

Draw r star positions from an urn with r+n-1 numbers: Total = $\binom{r+n-1}{r}$

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Experiment		Drawing Order	
		Yes	**No**
Replace	**Yes**	n^r	$\binom{\boldsymbol{r+n-1}}{\boldsymbol{r}}$
	No	$\frac{n!}{(n-r)!}$	$\binom{n}{r}$

Combinatorics ... COUNT

2) Ignore drawing order

a) Replace

Ball 1:	2	3	4	5	6	7
***	*		****	**		***

Example (old exam):

A total of r = 15 butterflies comes to a meadow with 6 flowers.
Each butterfly lands on one of the flowers.

You count the number of butterflies on flowers 1,2,...,6
Possible result: (2,4,5,1,0,3) ... this is one result (pattern)
How many different patterns are there?

Total = $\binom{15+6-1}{15}$ = $\binom{20}{15} = 15504$

Lecture 3

Chapter 1

Bernoulli Trials - Binomial Law ...

Basic experiment: Toss a coin … an unfair one!

$P[\{H\}] = p, P[\{T\}] = q = 1 - p$

$$\Omega = \{H, T\}, \mathcal{F} = \{\emptyset, \Omega, \{\mathrm{H}\},\{\mathrm{T}\}\}$$

Now, repeat that experiment:

$$\Omega_2 = \Omega \times \Omega = \{HH, HT, TH, TT\}$$

Repeat n times (toss the coin n times):

$$\Omega_n = \Omega \times \Omega \times \cdots \times \Omega$$

Cardinality: $|\Omega_n| = 2^n$

Lecture 3

Chapter 1

Bernoulli Trials - Binomial Law ...

Toss a coin three times! $P[\{HHT\}] = ?$

Suppose, the tosses are independent.

➔ $P[\{HHT\}] = P[\{H\}]P[\{H\}]P[\{T\}] = ppq = p^2q$

So far ... so easy! ☺

Now, what if we care about the number of heads (H) in an outcome, only ... not their positions?

That is, we identify HHT, HTH, THH, and just say ``heads come 2 times´´.

P[``we get k = 2 heads´´] = ?

Lecture 3

Chapter 1

Bernoulli Trials - Binomial Law ...

All the three cases, HHT, HTH, and THH, are equally likely: p^2q
... and disjoint.

➔P[``we get k = 2 heads´´] = 3 p^2q

For n Bernoulli trials, we obtain:

$$P[k\ heads]\ =?\ p^k q^{n-k}$$

How many of those cases with k heads are there?

→Choose k out of n positions for the H: $\binom{n}{k}$ cases.

Binomial probability law (n trials):

$$P[k\ heads] = \binom{n}{k} p^k q^{n-k} =: b(k; n, p)$$

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Chapter 1

Extension: Multinomial Probability Law

Example: Roll a 6-faced (unfair) die with probabilities $p_1, p_2, \dots, p_6$ for the six possible outcomes.

Like $P[\{4\}] = p_4$ etc.

As above, for n trials, $n = r_1 + r_2 + \cdots r_6$:

$$P[``r_1\ Ones, r_2\ Twos, \dots, r_6\ Sixes''] = ?$$

Choose r_1 out of *n* positions for the Ones: $\binom{n}{r_1}$ cases.

Then, r_2 out of (n-r_1) positions for the Twos: $\binom{n-r_1}{r_2}$ cases, ...

$$\boldsymbol{P[r_1, r_2, \dots, r_6]} = \binom{n}{r_1}\binom{n-r_1}{r_2} \dots \binom{r_6}{r_6} p_1^{r_1} p_2^{r_2} \dots p_6^{r_6}$$

$$= \boldsymbol{\frac{n!}{r_1!\, r_2! \dots r_6!} p_1^{r_1} p_2^{r_2} \dots p_6^{r_6}}$$

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Chapter 1

Asymptotics of the Binomial Law ... The Poisson Law

Example:

Jacobs has about 1500 students.
The years has 365 days (simplified ☺)

P[k birthdays, today] = ?

$$b\left(k; n = 1500, p = \frac{1}{365}\right) = \binom{1500}{k}\left(\frac{1}{365}\right)^k\left(\frac{364}{365}\right)^{n-k} \ldots ☹$$

Poisson's approximation for $p \ll 1, n \gg 1, np = a$:

$b(k; n, p) \approx \frac{a^k}{k!}e^{-a}$, here, $a = 1500/365$

k	0	1	2	3	4	5	6
$P[k]$	0.016	0.068	0.139	0.190	0.195	0.160	0.110

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Chapter 1

Asymptotics of the Binomial Law ...

Poisson's approximation for $p \ll 1, n \gg 1, np = a$:

$$b(k; n, p) \approx \frac{a^k}{k!} e^{-a}$$

Why?

Consider $p \to 0,\ q \to 1, n \to \infty, np = a,\ k$ fixed:

$$\binom{n}{k} p^k q^{n-k} = \frac{n!}{k!\,(n-k)!} p^k q^{n-k}$$

$$= \frac{1}{k!} n(n-1)\cdots(n-k+1) p^k q^n q^{-k} \approx \frac{n^k p^k}{k!}(1-p)^n$$

$$= \frac{n^k p^k}{k!}\left(1 - \frac{a}{n}\right)^n \to \frac{a^k}{k!} e^{-a} \ \blacksquare$$

The End

Next time: cont. chp. 1