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Assignment 3

# Elements of Probability

**(3.1)** Consider a discrete random variable $X$ with the probability mass function given by

$$p_X(x) = \begin{cases} k \cdot 2^x & \text{if } x = -1, 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the value of $k$.
(b) Find $\mathbb{P}[X \text{ is even}]$.
(c) Find the PMF of the random variables $Y = X^2$ and $Z = X + 1$.

**Solution.** (a) We have

$$1 = \sum_{x \in \{-1,0,1,2\}} k \cdot 2^x = \frac{15}{2} k.$$

From here it follows that $k = \frac{2}{15}$.

(b) We have

$$\mathbb{P}[X \text{ is even}] = \frac{2}{15}(1 + 4) = \frac{2}{3}.$$

(c) Note that $Y = X^2$ takes values $0, 1, 4$, and

$$\mathbb{P}[Y = 1] = \mathbb{P}[X = 1] + \mathbb{P}[X = -1] = \frac{1}{2}.$$

Similarly,

$$\mathbb{P}[Y = 0] = \mathbb{P}[X = 0] = \frac{2}{15}.$$

Finally,

$$\mathbb{P}[Y = 4] = 1 - \frac{2}{15} - \frac{1}{2} = \frac{3}{5}.$$

In a similar fashion, one can see that the probability mass function of $Z$ is given by

$$p_Z(0) = \frac{1}{15}, \quad p_Z(1) = \frac{2}{15}, \quad p_Z(2) = \frac{4}{15}, \quad p_Z(3) = \frac{8}{15}.$$

**(3.2)** Suppose $X$ is a discrete random variable with $\mathbb{E}[X] = 5$ and $\mathrm{Var}[X] = 15$.
(a) Find the values of $\mathbb{E}\left[X^2\right]$, $\mathbb{E}[2 - X]$, $\mathrm{Var}[3X + 1]$.
(b) Show that $\mathbb{P}[X \geq 10] \leq \frac{3}{5}$.

**Solution.** Note that

$$\mathbb{E}\left[X^2\right] = \mathrm{Var}[X] + \mathbb{E}[X]^2 = 15 + 25 = 40.$$

Also

$$\mathbb{E}[2 - X] = 2 - \mathbb{E}[X] = -3.$$

and

$$\mathrm{Var}[3X + 1] = \mathrm{Var}[3X] = 9\mathrm{Var}[X] = 135.$$

For part (b), let us denote the subset of the sample space on which the inequality $X \geq 10$ holds by $A$. Then we have

$$15 = \mathrm{Var}[X] = \sum_{\omega\in\Omega} p(\omega)(X(\omega)-5)^2 \geq \sum_{\omega\in A} p(\omega)(X(\omega)-5)^2 \geq 25 \sum_{\omega\in\Omega} p(\omega) = 25\mathbb{P}[A].$$

This implies that

$$\mathbb{P}[A] \leq \frac{15}{25} = \frac{3}{5}.$$

**(3.3)** The number of items arriving daily in the lost-and-found section of Bremen train station has a Poisson distribution with parameter $\lambda = 4$.

(a) Find the probability of the event that on a given day no items arrive.

(b) Knowing that at least one item has arrived today, find the probability that there is at least one more item arriving today?

(c) Let $W$ denote the number of items arriving from January 1st to January 15th. What is $\mathbb{E}[W]$?

**Solution.** Let us denote the number of items arriving on a given day by $X$. Since $X$ has Poisson distribution with $\lambda = 4$ we have

$$\mathbb{P}[X = 0] = e^{-4}\frac{\lambda^0}{0!} = e^{-4}.$$

For part (b) we are interested in

$$\mathbb{P}[X \geq 2 | X \geq 1] = \frac{\mathbb{P}[X \geq 2]}{\mathbb{P}[X \geq 1]}.$$

Note that

$$\mathbb{P}[X \geq 1] = 1 - \mathbb{P}[X = 0] = 1 - e^{-4}.$$

Similarly,

$$\mathbb{P}[X \geq 2] = 1 - \mathbb{P}[X = 0] - \mathbb{P}[X = 1] = 1 - e^{-4} - 4e^{-4} = 1 - 5e^{-4}.$$

Hence

$$\mathbb{P}[X \geq 2 | X \geq 1] = \frac{1 - 5e^{-4}}{1 - e^{-4}}.$$

**(3.4)** A monkey has a bag containing 4 apples, 3 bananas and 2 pears. He heats fruits at random until he takes a fruit of a kind he has already had, and then throws away that fruit and the rest of the bag. Let $N$ denote the number of fruits eaten by the monkey.

(a) What are the possible values of $N$?

(b) Find the probability mass function of $N$.

(c) Find $\mathbb{E}[N]$ and $\mathrm{Var}[N]$.

**Solution.** It is clear that since there are three types of fruits, $N$ can take values $1, 2, 3$. Note that $N = 1$ if the second fruits is the same as the first one. The first fruit is apple, banana, or pear with probabilities $4/9$, $3/9$ and $2/9$. If the first fruit is apple, the probability of the second fruit be apple is $3/8$. Similarly the conditional probabilities of repeat for the other fruits are $2/8$ and $1/8$. Hence

$$\mathbb{P}[N = 1] = \frac{4}{9}\cdot\frac{3}{8} + \frac{3}{9}\cdot\frac{2}{8} + \frac{2}{9}\cdot\frac{1}{8} = \frac{20}{72} = \frac{5}{18}.$$

Alos, $N = 3$ occurs when the first three fruits are all different. This can be achieved in $6 \times 4 \times 3 \times 2$ ways. Since the total number of ways of choosing the fruits is $9 \times 8 \times 7$, we have

$$\mathbb{P}[N = 3] = \frac{6 \cdot 4 \cdot 3 \cdot 2}{9 \cdot 8 \cdot 8} = \frac{2}{7}.$$

Finally,

$$\mathbb{P}[N = 1] = 1 - \frac{2}{7} - \frac{5}{18} = \frac{55}{126}.$$

**(3.5)** A startup has developed a new gadget for which the demand is unknown. Assume that the demand for the product denoted by $Y$ has a uniform distribution on the set $\{1, 2, \ldots, 1000\}$. Each sold gadget will bring a profit of 12 Euros and each one made and left unsold will produce a net loss of 3 Euros.

(a) Suppose that the startup decides to produce $m$ units of this gadget. Denote the net income of the startup by $Y$. Show that

$$X = \begin{cases} 12m & \text{if } Y > m \\ 15Y - 3m & \text{if } Y \leq m \end{cases}$$

(b) Find a closed formula for $\mathbb{E}[X]$.

(c) (Bonus) How many units of this gadgets should be produced to maximize the expected income $\mathbb{E}[X]$?

*Hint:* For part (b) you may use the following identity useful:

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

**Solution.** It is clear that if the demand surpasses the number $m$ of offered products then all will be solved and hence the net profit will be $12m$. On the other hand, if the demand $Y$ is less than $m$, then the profit made by selling $Y$ items is $12Y$, while a loss of $3(m - Y)$ of all unsold items will also be incurred. The net profit will be given by

$$X = 12Y - 3(m - Y) = 15Y - 3m.$$

Now suppose that $Y$ takes one of the values $1, 2, \ldots, N$ (where $N = 1000$) with probability $1/N$. Then we have

$$\mathbb{E}[X] = \frac{1}{N}\sum_{Y=1}^{N} X = \frac{1}{N}\sum_{Y=1}^{m}(15Y - 3m) + \frac{1}{N}\sum_{Y=m+1}^{N} 12m.$$

The second sum involves $N - m$ terms each equal to $12m$, hence it is equal to $12m(N - m)$. The first sum splits as

$$\frac{15}{N}\sum_{Y=1}^{m} Y - \sum_{Y=1}^{m}\frac{3m}{N} = \frac{15m(m+1)}{2N} - \frac{3m^2}{N} = \frac{9m^2 + 15m}{2N}.$$

Combining these two we have

$$\mathbb{E}[X] = 12m(N - m) + \frac{9m^2 + 15m}{2N} =$$

**(3.6)** (Bonus) Consider a coin which lands H with probability $p$ and T with probability $1 - p$. The coin is flipped until H shows up for the *second* time. Let $N$ denote the number of required flips.

(a) For warm-up, show that $\mathbb{P}[N=0] = \mathbb{P}[N=1] = 0$, and $\mathbb{P}[N=2] = p^2$.
(b) Show that $\mathbb{P}[N=3] = 2p^2(1-p)$
(c) In general, show that the PMF of $N$ is given by

$$\mathbb{P}[N=k] = (k-1)p^2(1-p)^{k-2}, \qquad k = 2, 3, \dots$$

*Hint:* $N = k$ exactly when the $k$-th flip results in H and all but one of the previous $k-1$ flips result in $T$.

**Solution.** (a) First note that one requires at least two flips for the second H to show up, it is clear that $N=0$ and $N=1$ are impossible. Moreover, $N=2$ if and only if the first two flips result in H, hence using the independence we have $\mathbb{P}[N=2] = p^2$.

(b) Note that $N=3$ occurs exactly when the outcome of the third flip is H (which happens with probability $p$) and the outcomes of the first two flips are HT or TH. The probability of the latter is $2p(1-p)$, hence

$$\mathbb{P}[N=3] = 2p^2(1-p).$$

(c) The argument is similar to part (b). For $N=k$, one needs the outcome of the $k$-th flip to be H, and in the previous $k-1$ flips, there are exactly one H and $k-2$ tails. The probability of the latter is given by

$$\binom{k-1}{1} p(1-p)^{k-2}.$$

It follows that

$$\mathbb{P}[N=k] = (k-1)p^2(1-p)^{k-1}.$$