

### Elements of Probability

(5.1) Suppose  $X$  is a normal random variable with  $\mu = 3$  and  $\sigma = 2$ . Compute the following probabilities in terms of the distribution function of the standard normal distribution  $\Phi$ .

- (a)  $\mathbb{P}[-2 < X < 2]$ .
- (b)  $\mathbb{P}[X > 5]$ .
- (c)  $\mathbb{P}[X < -4]$ .

(5.2) The probability density function of a continuous random variable is given by

$$f_X(t) = \begin{cases} 2e^{-2t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Compute  $\mathbb{E}[e^X]$ .

(5.3) Let  $X$  be a random variable with the uniform distribution in  $(0, 1)$ . Find the density function of the random variables  $Y = -\log X$ .

(5.4) A two-element subset  $A$  of the set  $\{1, 2, \dots, 6\}$  is randomly chosen. Suppose  $X$  denote the smaller and  $Y$  denote the larger of the two. For instance if  $A$  turns out to be the set  $\{2, 5\}$ , then we have  $X = 2$  and  $Y = 5$ .

- (a) Explain why  $X$  can take values  $1, 2, \dots, 5$ , and  $Y$  can take values  $2, 3, \dots, 6$ . Moreover, show that their probability mass functions are given by

$$\mathbb{P}[X = x] = \frac{6-x}{15}, \quad 1 \leq x \leq 5.$$
$$\mathbb{P}[Y = y] = \frac{y-1}{15}, \quad 2 \leq y \leq 6.$$

- (b) Show that  $X$  and  $Y$  are not independent.

(5.5) A coin is flipped three times. Let  $X$  denote the number of heads and  $Y$  denote the number of streaks of heads of length 2. For instance, if the outcome is  $HTH$ , then  $X = 2$  and  $Y = 0$ , while if the outcome is  $HHT$ , then  $X = 2$  and  $Y = 1$ .

- (a) Find the joint probability mass function of  $X$  and  $Y$ .
- (b) Compute the conditional probability  $\mathbb{P}[X = 2|Y = 1]$  and  $\mathbb{P}[Y = 1|X = 2]$ .
- (c) Are  $X$  and  $Y$  independent?