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Sample problems for Elements of Probability

(P1) Five students have been randomly chosen from a large class. Find the probability that
(a) At least one of them is born on Sunday.
(b) At least two of them are born on the same day of the week.
(c) All five are born on the weekend.

Solution. Denote the event in parts (a), (b), (c) by A, B, C, respectively. Then

$$\mathbb{P}[A] = 1 - \mathbb{P}[A^c] = 1 - \frac{6^5}{7^5}.$$

For (b), we use the same argument as in the birthday problem:

$$\mathbb{P}[B] = 1 - \mathbb{P}[B^c] = 1 - \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{7^5}.$$

For (c), note that there are 2 option for each student. This implies

$$\mathbb{P}[C] = \frac{2^5}{7^5}.$$

(P2) A die has been thrown 6 times.
(a) Find the probability that the number 2 appears at least once.
(b) Suppose that it is given that the number 3 has appeared at least once. Find the conditional probability that the number 2 has also appeared at least once.
(c) Compare the results of part (a) and (b). Do you find the result reasonable? Why?

Solution. Let A_2 and A_3 denote, respectively, the events that 2 and 3 appear at least once. Then

$$\mathbb{P}[A_2] = 1 - \mathbb{P}[A_2^c] = 1 - \frac{5^6}{6^6} \approx 1 - 0.33 = 0.66$$

Note that the same probability also holds for $\mathbb{P}[A_3]$. For (b), note that

$$\mathbb{P}[A_2|A_3] = \frac{\mathbb{P}[A_2 \cap A_3]}{\mathbb{P}[A_3]}.$$

Note that

$$\mathbb{P}[A_2 \cap A_3] = 1 - \mathbb{P}[(A_2 \cap A_3)^c] = 1 - \mathbb{P}[A_2^c \cup A_3^c].$$

On the other hand,

$$\mathbb{P}[A_2^c \cup A_3^c] = \mathbb{P}[A_2^c] + \mathbb{P}[A_3^c] - \mathbb{P}[A_2^c \cap A_3^c] = \left(\frac{5}{6}\right)^6 + \left(\frac{5}{6}\right)^6 - \left(\frac{4}{6}\right)^6 \approx 0.58$$

This gives $\mathbb{P}[A_2 \cap A_3] = 1 - 0.58 \approx 0.41$.

$$\mathbb{P}[A_2|A_3] \approx \frac{0.41}{0.66} = 0.62.$$

(P3) Let A and B be two events. Let Z describe the event that exactly one of these two events occurs.

(a) By using a Venn diagram or otherwise, prove that

$$Z = (A \cup B) - (A \cap B).$$

(b) Deduce from part (a) that

$$\mathbb{P}[Z] = \mathbb{P}[A] + \mathbb{P}[B] - 2\mathbb{P}[A \cap B].$$

(c) A random number n is chosen from the set $\{1, 2, \ldots, 100\}$. Find the probability of the event that n is divisible by 3 or 5, but is not divisible by both of them.

Solution. Note that an element is in Z if and only if it belongs to A or B, but not in A and B. The elements that are in A or B are precisely those which are in $A \cup B$. On the other hand, the elements that are in A and B are those in $A \cap B$. It follows that the elements which are in exactly one of A and B are given by $A \cup B - A \cap B$.

(b) Using part (a), we have

$$\begin{aligned}\mathbb{P}[Z] &= \mathbb{P}[(A \cup B) - (A \cap B)] = \mathbb{P}[A \cup B] - \mathbb{P}[A \cap B] \\ &= (\mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]) - \mathbb{P}[A \cap B] \\ &= \mathbb{P}[A] + \mathbb{P}[B] - 2\mathbb{P}[A \cap B].\end{aligned}$$

(c) Let A_3 and A_5 denote the events that the randomly chosen number is divisible by 3 and 5, respectively, and D denote the event that the randomly chosen number is divisible by exactly one of them. It is easy to see that

$$\mathbb{P}[A_3] = \frac{33}{100}, \quad \mathbb{P}[A_5] = \frac{20}{100}, \mathbb{P}[A_4 \cap A_5] = \frac{6}{100}.$$

Using part (b), we obtain

$$\mathbb{P}[D] = \frac{33}{100} + \frac{20}{100} - 2 \cdot \frac{6}{100} = \frac{41}{100}.$$

(P4) A three-element subset X of the set $\{1, 2, \ldots, 10\}$ is randomly chosen.

(a) Let A be the event that the largest element of X is 6. Find $\mathbb{P}[A]$.

(b) Let B denote the event the smallest element of A is 2. Find $\mathbb{P}[B]$.

(c) Find the conditional probabilities $\mathbb{P}[A|B]$ and $\mathbb{P}[B|A]$.

Solution. Call this random subset X. Note that since the set $\{1, 2, \ldots, 10\}$ has exactly $\binom{10}{3} = 120$ subsets with 3 elements.

(a) If the largest element of X is 6, it means that the other two elements have to be chosen from the set $\{1, \ldots, 5\}$. Hence

$$\mathbb{P}[A] = \frac{\binom{5}{2}}{\binom{10}{3}} = \frac{10}{120} = \frac{1}{12}.$$

(b) If the least element of B is 2, it implies that the other two elements must be chosen from the set $\{3, \ldots, 10\}$. Hence

$$\mathbb{P}[B] = \frac{\binom{8}{2}}{\binom{10}{3}} = \frac{28}{120} = \frac{7}{30}.$$

(c) Note that $A \cap B$ means that the least element of A is 2 and the largest element of B is 6. That leave 3 options for the remaining element. Hence

$$\mathbb{P}[A \cap B] = \frac{3}{120} = \frac{1}{40}.$$

From here, we have

$$\mathbb{P}[A|B] = \frac{1/40}{7/30} = \frac{3}{28}.$$

$$\mathbb{P}[B|A] = \frac{1/40}{1/12} = \frac{12}{40}.$$

(P5) Alex goes to the bus stop at Vegesack at some random time between noon and 1 pm, and waits for 24 minutes for the bus. The bus is also supposed to arrive at a random time between noon and 1 pm, and wait there for 6 minutes before leaving.

(a) What is the probability that Alex succeeds in catching the bus?

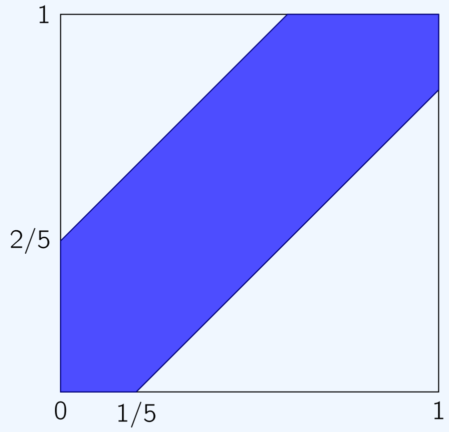
(b) Assuming that Alex has caught the bus, find the probability that his waiting time was less than 12 minutes.

Hint: Find a formulation of the problem similar to the Alice and Bob example discussed in class.

Solution. Denote by C the event C that Alex catches the bus. We will first find a geometric representation of C. In order to do this, let x denote the time that Alex arrives at the bus stop, and let y denote the time that the bus arrives. We use the interval $[0, 1]$ to represent the timeline between noon and 1 pm. First suppose $x \leq y$, that is, consider the case that Alex arrives before the bus. Alex will catch the bus if the bus arrives no later than 24 minutes after x, that is $x \leq y \leq x + \frac{2}{5}$. (Note that 24 minutes is exactly two fifth of an hour). Similarly, if the bus arrives first, then Alex is bound to arrive within 12 minutes in order to catch the bus. This translates to the equation $y \leq x \leq y + \frac{1}{5}$. These two conditions can be combined in

$$x - \frac{1}{5} \leq y \leq x + \frac{2}{5},$$

which, in turn, describe the following region:



Hence the probability of C is given by

$$\mathbb{P}[C] = 1 - \frac{1}{2}\left(\frac{4}{5}\right)^2 - \frac{1}{2}\left(\frac{3}{5}\right)^2 = \frac{1}{2}.$$

For (b), let W denote the event that Alex has waited less than 12 minutes. This means that

$$y \leq x + \frac{1}{5}.$$

Hence

$$\mathbb{P}[W \cap C] = 1 - \left(\frac{4}{5}\right)^2 = \frac{9}{25}.$$

This implies that

$$\mathbb{P}[W|C] = \frac{\mathbb{P}[W \cap C]}{\mathbb{P}[C]} = \frac{9/25}{1/2} = \frac{18}{25} = 0.72.$$

(**P6**) Three points M, N, and L are randomly chosen on a circle centered at O. Find the probability of the event that

(a) O is inside the triangle MNP.
(b) O on *on* one of the sides of the triangle MNP.
(c) O is outside the triangle MNP.

Hint: Due to the rotational symmetry of the circle, one can fix one of the points.

Solution. As the hint suggest, due to the rotational symmetry of the circle, one can fix the point M. Let α and β denote the angles that ON and OP form with the radius OM, when measured in counterclockwise direction. It will be convenient to assume that α and β vary independently in the interval $[-\pi, 2\pi]$. It is easy to see that O is inside the triangle MNP, precisely when all three angles of MNP are less than $\pi/2$. If $\alpha < \beta$, then the angles of the triangle MNP are equal to $\frac{\alpha}{2}$, $\frac{\beta-\alpha}{2}$ and $\pi - \frac{\beta}{2}$. This implies that MNP is an acute triangle when

$$\alpha < \pi, \beta - \alpha < \pi, \beta > \pi.$$

Similarly, when $\alpha > \beta$, we obtain the inequalities

$$\alpha > \pi, \alpha - \beta < \pi, \beta < \pi.$$

Let A denote the event that the triangle MNP is acute. Then

$$\mathbb{P}[A] = \frac{1}{4}.$$

It is clear that O is on one of the sides of the triangle if $\alpha = 0$ or $\beta = 0$ or $\alpha = \beta$. Each one of these equations defines a long segment, which has area zero. This implies that the probability of this event is equal to zero.

From parts (a) and (b) it follows that with probability 3/4 the triangle is obtuse.

(**P7**) Suppose $A_1, \ldots, A_n$ are events in a sample space. Show that

$$\sum_{1 \le i \le n} \mathbb{P}[A_i] - \sum_{1 \le i < j \le n} \mathbb{P}[A_i \cap A_j] \le \mathbb{P}\left[\bigcup_{i=1}^{n} A_i\right] \le \sum_{1 \le i \le n} \mathbb{P}[A_i].$$

Solution. We will prove the statement by induction on n. For $n = 2$, we have

$$\mathbb{P}[A_1 \cup A_2] = \mathbb{P}[A_1] + \mathbb{P}[A_2] - \mathbb{P}[A_1 \cap A_2]$$

from which both inequalities follow.

Suppose the statement has been proven for n. We will now establish it for $n+1$. For the left-hand side, we have

$$\mathbb{P}\left[\bigcup_{i=1}^{n+1} A_i\right] = \mathbb{P}\left[\left(\bigcup_{i=1}^{n} A_i\right) \cup A_n\right] \le \mathbb{P}\left[\bigcup_{i=1}^{n} A_i\right] + \mathbb{P}[A_{n+1}]$$
$$\le \sum_{1 \le i \le n} \mathbb{P}[A_i] + \mathbb{P}[A_{n+1}] = \sum_{1 \le i \le n+1} \mathbb{P}[A_i]. \tag{1}$$

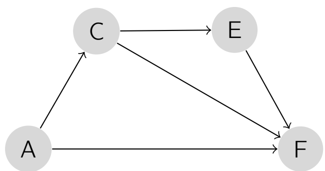
For the other inequality, again using the inequality for n and the right-hand side inequality proven above, we have

$$\begin{aligned}\mathbb{P}\left[\bigcup_{i=1}^{n+1} A_i\right] &= \mathbb{P}\left[\left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n+1}\right] = \mathbb{P}\left[\bigcup_{i=1}^{n} A_i\right] + \mathbb{P}[A_{n+1}] - \mathbb{P}\left[A_{n+1} \cap \bigcup_{i=1}^{n} A_i\right] \\ &\geq \sum_{1\leq i\leq n} \mathbb{P}[A_i] - \sum_{1\leq i<j\leq n} \mathbb{P}[A_i \cap A_j] + \mathbb{P}[A_{n+1}] - \sum_{1\leq i\leq n} \mathbb{P}[A_i \cap A_{n+1}] \qquad (2) \\ &= \sum_{1\leq i\leq n+1} \mathbb{P}[A_i] - \sum_{1\leq i<j\leq n+1} \mathbb{P}[A_i \cap A_j].\end{aligned}$$

(P8) A network connects computers A and F via intermediate nodes C, E as shown below. For each pair of directly connected nodes, there is a probability $p = 3/4$ that the connection from i to j is up. Assume that the link failures are independent events.

(a) Find the probability that the connection from A to F through at least one of the paths is up.

(b) Suppose that due to some technical work, the connections CE and CF are simultaneously on or off, with the same probability $p = 2/3$. This aside, the other connections are all independent. Under this assumption, compute the probability that the connection from A to F through at least one of the paths is up, and compare the result to part (a).



Solution. Let us denote by C_1, C_2, and C_3 the events that the paths $ACEF$, ACF, and AF are up, respectively. Note that $ACEF$ is up, when all connections AC, CE and EF are up. Since these events are assumed to be independent, we have

$$\mathbb{P}[C_1] = p^3.$$

Similarly, we have

$$\mathbb{P}[C_2] = p^2, \qquad \mathbb{P}[C_3] = p.$$

Let us now consider $C_1 \cap C_2$. Note that $C_1 \cap C_2$ entails that all four paths AC, CE, EF, CF are up, and hence $\mathbb{P}[C_1 \cap C_2] = p^4$. Similarly, we obtain

$$\mathbb{P}[C_1 \cap C_3] = p^4, \qquad \mathbb{P}[C_2 \cap C_3] = p^3, \qquad \mathbb{P}[C_1 \cap C_2 \cap C_3] = p^5.$$

From here, we obtain

$$\mathbb{P}[Connect] = \mathbb{P}[C_1 \cup C_2 \cup C_3] = (p^3 + p^2 + p) - (p^4 + p^4 + p^3) + p^5 = p + p^2 - 2p^4 + p^5.$$

(b) The only difference in part (b) is that since connections CE and CF are both on or off, we have

$$\mathbb{P}[C_1 \cap C_2] = p^3, \qquad \mathbb{P}[C_1 \cap C_3] = p^4, \qquad \mathbb{P}[C_2 \cap C_3] = p^3, \qquad \mathbb{P}[C_1 \cap C_2 \cap C_3] = p^4.$$

From here we obtain

$$\mathbb{P}[Connect] = \mathbb{P}[C_1 \cup C_2 \cup C_3] = (p^3 + p^2 + p) - (p^3 + p^4 + p^3) + p^4 = p + p^2 - p^3.$$

(P9) Alice and Bob use the following method for determining who pays for lunch. They have three non-standard dice, each with the following numbers on them:

$$(A) : 1, 1, 6, 6, 8, 8$$
$$(B) : 2, 2, 4, 4, 9, 9$$
$$(C) : 3, 3, 5, 5, 7, 7$$

Each side of each one of these dice has probability 1/6 of coming up. Alice and Bob, each, throws one of the dice, and the one who rolls the smaller number loses and will buy lunch.

(a) If Bob uses die A and Alice uses die B, show that the odds that Alice wins is more than 1/2.

(b) If Bob uses die B and Alice uses die C, show that the odds that Alice wins is more than 1/2.

(c) If Bob uses die C and Alice uses die A, show that the odds that Alice wins is still more than 1/2!

Solution. Suppose that Bob uses die A and Alice uses die B. The set of possible outcomes are given by

$$\Omega = \{(1, 2), (1, 4), (1, 9), (6, 2), (6, 4), (6, 9), (8, 2), (8, 4), (8, 9)\}.$$

Each outcomes has probability $\frac{4}{36} = \frac{1}{9}$. Note that the event that Alice wins is given by

$$A = \{(1, 2), (1, 4), (1, 9), (6, 9), (8, 9)\}$$

Hence

$$\mathbb{P}[A] = \frac{5}{9} > \frac{1}{2}.$$

Parts (b) and (c) can be dealt with in a similar fashion.

(P10) Consider a discrete random variable X that with the PMF given by

$$p_X(x) = \begin{cases} k|x| & \text{if } x = -2, -1, 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the value of k.

(b) Find $\mathbb{P}[X > 1]$.

(c) Find the PMF of the random variables $Y = X^2$ and $Z = X^3$.

Solution. (a) We have

$$1 = \sum_{x \in \{-2,-1,1,2,3\}} k|x| = 9k.$$

From here it follows that $k = \frac{1}{9}$.

(b) We have

$$\mathbb{P}[X > 1] = \mathbb{P}[X = 2] + \mathbb{P}[X = 3] = \frac{2}{9} + \frac{3}{9} = \frac{5}{9}.$$

(c) Note that $Y = X^2$ takes values $1, 4, 9$, and

$$\mathbb{P}[Y = 1] = \mathbb{P}[X = 1] + \mathbb{P}[X = -1] = \frac{1}{9} + \frac{1}{9} = \frac{2}{9}.$$

Similarly,

$$\mathbb{P}[Y = 4] = \mathbb{P}[X = 2] + \mathbb{P}[X = -2] = \frac{2}{9} + \frac{2}{9} = \frac{4}{9}.$$

And, finally, $\mathbb{P}[Y=9]=\frac{3}{9}$.

x	1	4	9
$p_Y(x)$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{3}{9}$

Similarly, $Z=X^3$ takes values $-8,-1,1,8,27$. The probabilities of these values correspond exactly to the probabilities of $-2,-1,1,2,3$. For instance, $\mathbb{P}[Z=27]=\mathbb{P}[X=3]=\frac{3}{9}$, etc. We can represent the answer in the following table

x	-8	-1	1	8	27
$p_Z(x)$	2/9	1/9	1/9	2/9	3/9

(P11) Suppose X is a random variable with a geometric distribution with parameter p.

(a) Compute $\mathbb{P}[X>k]$ for $k=0,1,2,\ldots$.

(b) Show that for all $n,k>0$, we have

$$\mathbb{P}[X=n+k|X>k]=\mathbb{P}[X=n].$$

Solution. There are two different ways of computing $\mathbb{P}[X>k]$. One can proceed directly: we have

$$\mathbb{P}[X>k]=\sum_{j=k+1}^{\infty}\mathbb{P}[X=j]=\sum_{j=k+1}^{\infty}p(1-p)^{j-1}=p(1-p)^k\sum_{j=0}^{\infty}(1-p)^j=\frac{p(1-p)^k}{p}=(1-p)^k.$$

Alternatively, we can use the fact that the geometric distribution is the distribution of the first success in a sequence of Bernoulli trials (such as flipping a coin) where the probability of success in each round is p. Hence $X>k$ if and only if the first k rounds lead to failure which has probability $(1-p)^k$.

(b)

$$\begin{aligned}\mathbb{P}[X=n+k|X>k]&=\frac{\mathbb{P}[X=n+k \text{ and } X>k]}{\mathbb{P}[X>k]}=\frac{\mathbb{P}[X=n+k]}{\mathbb{P}[X>k]}\\ &=\frac{p(1-p)^{n+k-1}}{(1-p)^k}=p(1-p)^{n-1}=\mathbb{P}[X=n].\end{aligned} \tag{3}$$

(P12) For events $A,B\subseteq\Omega$, we write $A\perp B$ if A and B are independent.

(a) Show that if $A\perp B$ then $A\perp B^c$.

(b) Show that if $A\perp B$ then $A^c\perp B^c$.

(c) Is it true that if $A\perp B$ and $C\perp D$ then $A\cup C\perp B\cup D$?

Solution.

$$\mathbb{P}[A\cap B^c]=\mathbb{P}[A-A\cap B]=\mathbb{P}[A]-\mathbb{P}[A\cap B]=\mathbb{P}[A](1-\mathbb{P}[B])=\mathbb{P}[A]\mathbb{P}[B^c].$$

(b) follows from a repeated application of (a): $A\perp B$ implies $A\perp B^c$, which is equivalent to $B^c\perp A$, and this in turn implies $B^c\perp A^c$.

(c) is false. Take $\Omega=\{00,01,10,11\}$, where each point in the sample space has probability 1/4. Take $A=D=\{00,01\}$, $B=C=\{00,11\}$. Then $\mathbb{P}[A]=\mathbb{P}[B]=\frac{1}{2}$ and $\mathbb{P}[A\cap B]=\frac{1}{4}$, hence A and B are independent. Since $C=B$ and $D=A$, we also have that C and D are independent. On the other hand

$$A\cup C=B\cup D=\{00,01,11\}$$

and it is clear that $A \cup C$ is not independent from $B \cup D$.

(P13) Consider a discrete random variable X with the probability mass function given by

$$p_X(x) = \begin{cases} kx & \text{if } x = 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the value of k.
(b) Compute $\mathbb{E}[X]$ and $\mathrm{Var}[X]$.

Solution. We have

$$1 = \sum_{x=1}^{4} p_X(x) = 10k$$

which implies that $k = \frac{1}{10}$.
(b)

$$\mathbb{E}[X] = \sum_{x=1}^{4} \frac{1}{10} x^2 = \frac{30}{10} = 3.$$

Similarluy,

$$\mathbb{E}\left[X^2\right] = \sum_{x=1}^{4} \frac{1}{10} x^3 = \frac{100}{10} = 10.$$

Hence

$$\mathrm{Var}[X] = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 = 1.$$

(P14) A continuous random variables has the density function given by

$$f_X(x) = \begin{cases} k(1 - x^3) & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine the value of k.
(b) Compute $\mathbb{P}[X > 0]$.

Solution. (a) We have

$$1 = \int_{-1}^{1} k(1 - x^3)\, dx = 1 \Rightarrow k\left(x - \frac{x^4}{4}\right)\Big|_{-1}^{1} = 2k$$

which implies that $k = \frac{1}{2}$. From here we have

$$\mathbb{P}[X > 0] = \frac{1}{2}\int_{0}^{1} (1 - x^3)\, dx = \frac{1}{2}\left(x - \frac{x^4}{4}\right)\Big|_{0}^{1} = \frac{3}{8}.$$

(P15) Suppose X is a random variable with the uniform distribution over the interval $[1, 2]$ and $Y = X^4$.
(a) Compute $\mathbb{P}[Y \leq t]$ as a function of t. You need to distinguish three different cases.
(b) Find the probability density function of Y and use it to compute $\mathbb{E}[Y]$.

Solution. Since $1 \le X \le 4$, we have $1 \le Y \le 16$. Clearly for $t < 1$ we have $F_Y(t) = 0$ and for $t > 16$ we have $F_Y(t) = 1$. For $1 \le t \le 16$ we have

$$F_Y(t) = \mathbb{P}\left[X^4 \le t\right] = \mathbb{P}\left[X \le t^{1/4}\right] = \begin{cases} 0 & \text{if } t < 1 \\ t^{\frac{1}{4}} - 1 & \text{if } 1 \le t \le 16 \\ 0 & \text{if } t > 16 \end{cases}.$$

(b) In order to compute the probability density function of Y, we differentiate $F_Y(t)$:

$$f_Y(t) = \begin{cases} 0 & \text{if } t < 1 \\ \frac{1}{4}t^{-3/4} & \text{if } 1 \le t \le 16 \\ 0 & \text{if } t > 16 \end{cases}$$

From here we have

$$\mathbb{E}[Y] = \mathbb{E}\left[X^4\right] = \int_1^2 t^4 \, dt = \left.\frac{t^5}{5}\right|_1^2 = \frac{31}{5}.$$

(P16) Alice and Bob have utility functions given by is given by

$$u_A(x) = x, \qquad u_B(x) = \log x,$$

where the log in in base 2. They are faced with a lottery with n positive outcomes $x_1, \ldots, x_n$, where each can be realized with probability $p = 1/n$.

(a) Compute the expected utility of Alice and Bob. In other words, find $\mathbb{E}[u_A(X)]$ and $\mathbb{E}[u_B(x)]$. The answer must depend on $x_1, \ldots, x_n$.

(b) Let $x_1 = 1, x_2 = 2, x_3 = 4$. What is the smallest amount of C_a (respectively, C_b) such that Alice (respectively, Bob) prefers a sure amount of C_a (respectively, C_b) to the lottery?

(c) (Bonus) Show that independent of the values of $x_1, \ldots, x_n$, Bob is always more risk averse than Alice.

Solution. Since $x_1, \ldots, x_n$ have the same probability $1/n$, we have

$$\mathbb{E}[u_A(X)] = \mathbb{E}[X] = \frac{1}{n}(x_1 + \cdots + x_n).$$

Similarly,

$$\mathbb{E}[u_B(x)] = \mathbb{E}[\log x] = \frac{1}{n}(\log x_1 + \cdots + \log x_n) = \log(x_1 \cdots x_n)^{1/n}.$$

(b)For Alice to prefer a sure amount of C_a to the lottery, we need to have

$$C_a = u_A(C_a) > \mathbb{E}[u_A(x)] = \frac{1}{3}(1 + 2 + 4) = \frac{7}{3} \Rightarrow C_a > \frac{7}{3}.$$

For Bob to prefer a sure amount of C_b to the lottery, we need to have

$$\log(C_n) = u_B(C_b) > \mathbb{E}[u_B(x)] = \log(1.2.4)^{1/3} = \log 2 \Rightarrow C_b > 2.$$

(c) We need to show that if Bob prefers a lottery to a sure amount C, then so does Alice. If Bob prefers a lottery to a sure amount C we have

$$u_B(C) < \mathbb{E}[u_B(X)] \Rightarrow \log C < \log(x_1 \cdots x_n)^{1/n} \Rightarrow C < (x_1 \cdots x_n)^{1/n}.$$

For Alice to prefers the lottery to a sure amount C we need to have

$$u_A(C) < \mathbb{E}[u_A(X)] \Rightarrow C < \frac{1}{n}(x_1 + \cdots + x_n).$$

We know (from the arithmetic-geometric mean inequality) that for all values $x_1, \dots, x_n > 0$ we have

$$\frac{1}{n}(x_1 + \cdots + x_n) \geq (x_1 \cdots x_n)^{1/n}.$$

This implies that if Bob prefers the lottery to the sure amount, so does Alice.

(**P17**) Let $n \geq 2$ be an integer and let the joint probability mass function of discrete random variables X and Y be given by

$$p_{X,Y}(x,y) = \begin{cases} k(x+y) & \text{if } 1 \leq x, y \leq n \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine the value of constant k.

(b) Determine the probability mass functions of X and Y.

(c) Find $\mathbb{P}[X \geq Y]$.

Solution. We know that $\sum_{x,y} p(x,y) = 1$. This implies that

$$k \sum_{x=1}^{n} \sum_{y=1}^{n} (x+y) = 1.$$

Simplifying this leads to $kn^2(n+1) = 1$, which leads to $k = \frac{1}{n^2(n+1)}$.

For part (b), write

$$p(x) = k \sum_{y=1}^{n} (x+y) = k(nx + \frac{n(n+1)}{2}) = \frac{2x+n+1}{2n(n+1)}.$$

Since $p(x,y) = p(y,x)$, we have

$$p(y) = \frac{2y+n+1}{2n(n+1)}.$$

For part (c), note that $\mathbb{P}[X \geq Y] + \mathbb{P}[X \leq Y] + \mathbb{P}[X = Y] = 1$. By symmetry, we have $\mathbb{P}[X \geq Y] = \mathbb{P}[Y \geq X]$. This implies that

$$\mathbb{P}[X \geq Y] = \frac{1}{2}(1 - \mathbb{P}[X = Y]) = \frac{1}{2}\left(1 - 2k\sum_{x=1}^{n} x\right) = \frac{n-1}{2n}.$$

(**P18**) A coin is flipped three times. Let X denote the number of heads and Y denote the number of streaks of heads of length 2. For instance, if the outcome is HTH, then $X = 2$ and $Y = 0$, while if the outcome is HHT, then $X = 2$ and $Y = 1$.

(a) Find the joint probability mass function of X and Y.

(b) Find the covariance of X and Y.

(c) Are X and Y independent?

Solution. It is clear that $0 \leq X \leq 3$ and $0 \leq Y \leq 2$. If $Y = 2$, then clearly $X = 3$, and this only happens when the outcome is HHH, hence

$$\mathbb{P}[X = 3, Y = 2] = \frac{1}{8}, \quad \mathbb{P}[X = j, Y = 2] = 0\, j = 0, 1, 2.$$

Consider $Y = 1$, this corresponding to two outcomes HHT and THH. Hence

$$\mathbb{P}[X = 2, Y = 1] = \frac{2}{8}, \quad \mathbb{P}[X = j, Y = 1] = 0, \quad j = 0, 1, 3.$$

Finally assume that $Y = 0$. Then X can take any of values 0,1, 2. We have $X = 0$ for TTT, we have $X = 1$ for HTT, THT, TTH, and we have $X = 2$ for THT. Hence

$$\mathbb{P}[X = 0, Y = 0] = \frac{1}{8}, \quad \mathbb{P}[X = 1, Y = 0] = \frac{3}{8}, \quad \mathbb{P}[X = 2, Y = 0] = \frac{1}{8}.$$

These numbers can be summarized in the following table:

	$Y = 0$	$Y = 1$	$Y = 2$	$Y = 3$
$X = 0$	1/8	0	0	0
$X = 1$	3/8	0	0	0
$X = 2$	1/8	1/4	0	0
$X = 3$	0	0	0	1/8