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Due: September 12, 2018
Assignment 1

Elements of Probability

(1.1) Five students have been randomly chosen from a class of 20 students. Find the probability that
(a) At least one of them is born on Sunday.
(b) At least two of them are born on the same day of the week.
(c) All five are born on the weekend.

Solution. Denote the event in parts (a), (b), (c) by A, B, C, respectively. Then

$$\mathbb{P}[A] = 1 - \mathbb{P}[A^c] = 1 - \frac{6^5}{7^5} \approx 0.53$$

For (b), we use the same argument as in the birthday problem:

$$\mathbb{P}[B] = 1 - \mathbb{P}[B^c] = 1 - \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{7^5} \approx 0.85$$

For (c), note that there are 2 option for each student. This implies

$$\mathbb{P}[C] = \frac{2^5}{7^5} \approx 0.002$$

(1.2) A number is called a *palindrome* if it reads the same from left and right. For instance, 13631 is a palindrome, while 435734 is not. A 5-digit number n is randomly chosen. Find the probability of the event that
(a) The chosen number n is a palindrome.
(b) The chosen number n is even and a palindrome.
(c) The chosen number n is even or a palindrome.

Solution. Since there are 9 options for the first digit from the left of n, and 10 for each one of the remaining digits, the sample space consists of 9×10^4 elements. Write

$$n = \overline{n_1 n_2 n_3 n_4 n_5}$$

where $n_1, \dots, n_5$ are digits of n. There are 9 options for n_1, and since $n_1 = n_5$, this also determines the value of n_5. There are 10 possible choices for n_2 which will determine n_3. Finally, there are 10 choices for n_4. This implies that the probability of the event A that n is a palindrome is

$$\mathbb{P}[A] = \frac{9 \times 10 \times 10}{9 \times 10^4} = \frac{1}{100}.$$

(b) Denote by E the event that the randomly chosen number is even. Note that n is even when $n_5 = 0, 2, 4, 6, 8$. On the other hand, since $n_1 = n_5$, we cannot have $n_5 = 0$. Hence, there are 4 options for n_5 which determines n_1. Hence, by continuing the argument as in part (a), we have

$$\mathbb{P}[A \cap E] = \frac{4 \times 10 \times 10}{9 \times 10^4} = \frac{4}{900} \approx 0.004.$$

(c) We have

$$\mathbb{P}[A \cup E] = \mathbb{P}[A] + \mathbb{P}[E] - \mathbb{P}[A \cap E].$$

It is easy to see that $\mathbb{P}[E] = \frac{1}{2}$. Hence

$$\mathbb{P}[A \cup E] = \frac{455}{900} \approx 0.505.$$

(1.3) (a) Suppose A and B are two events. Let S be the event that A or B occur, but not both. Show that

$$\mathbb{P}[S] = \mathbb{P}[A] + \mathbb{P}[B] - 2\,\mathbb{P}[A \cap B].$$

(b) Suppose A, B, and C are three events in a sample space. Let T denote the event that exactly two of these three events occur. Deduce from the axioms that

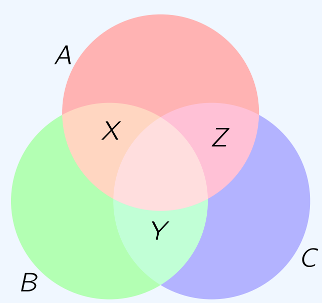
$$\mathbb{P}[T] = \mathbb{P}[A \cap B] + \mathbb{P}[A \cap C] + \mathbb{P}[B \cap C] - 3\,\mathbb{P}[A \cap B \cap C].$$

Hint: Draw a Venn diagram and use it to describe S and T as Boolean combination of the given events.

Solution. It is clear that S consists of those elements of $A \cup B$ which are *not* in $A \cap B$. Hence

$$\begin{aligned} \mathbb{P}[S] &= \mathbb{P}[A \cup B] - \mathbb{P}[A \cap B] = (\mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]) - \mathbb{P}[A \cap B] \\ &= \mathbb{P}[A] + \mathbb{P}[B] - 2\mathbb{P}[A \cap B]. \end{aligned} \tag{1}$$

For (b) Let X be the event that only A and B, but not C occur. Similarly, let Y and Z denote, respectively, the events that only A and C, and only B and C occur.



It is clear that $T = X \cap Y \cup Z$ and moreover

$$X \cap Y = X \cap Z = Y \cap Z = \emptyset.$$

This implies that

$$\mathbb{P}[T] = \mathbb{P}[X] + \mathbb{P}[Y] + \mathbb{P}[Z].$$

On the other hand, note that

$$\mathbb{P}[X] = \mathbb{P}[A \cap B] - \mathbb{P}[A \cap B \cap C].$$

Similarly, we have

$$\mathbb{P}[Y] = \mathbb{P}[A \cap C] - \mathbb{P}[A \cap B \cap C], \quad \mathbb{P}[Z] = \mathbb{P}[B \cap C] - \mathbb{P}[A \cap B \cap C].$$

Combining these we have

$$\mathbb{P}[T] = \mathbb{P}[X] + \mathbb{P}[Y] + \mathbb{P}[Z] = \mathbb{P}[A \cap B] + \mathbb{P}[A \cap C] + \mathbb{P}[B \cap C] - 3\,\mathbb{P}[A \cap B \cap C].$$

(1.4) Suppose A and B are certain two events, that is, assume that

$$\mathbb{P}[A] = \mathbb{P}[B] = 1.$$

Use the axioms of probability to show that

$$\mathbb{P}[A \cap B] = 1.$$

Now suppose that A and B are "almost certain" in the sense that

$$\mathbb{P}[A] = \mathbb{P}[B] = 0.99.$$

Show that

$$\mathbb{P}[A \cap B] \geq 0.98.$$

Solution. It is clear that $\mathbb{P}[A \cap B] \leq 1$. On the other hand, in view of

$$\mathbb{P}[A \cap B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cup B] = 2 - \mathbb{P}[A \cup B] \geq 1,$$

we have $\mathbb{P}[A \cap B] = 1$.

(b) The argument is similar. We have

$$\mathbb{P}[A \cap B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cup B] \geq 0.99 + 0.99 - 1 = 0.98.$$

(1.5) Let S be a random sequence of 0 and 1 of length $2n$.

(a) Find the probability p_n that the sequence contains exactly n zeros and n ones.

(b) Use Stirling's formula to show that for large value of n we have

$$p_n \sim \frac{1}{\sqrt{\pi n}}.$$

(c) Use part (b) to compute p_{100} approximately.

Solution. There are clearly 2^{2n} $0-1$ sequences of length $2n$. Hence $|\Omega| = 2^{2n}$. Let A be the event that the chosen sequence has exactly n zeros and n ones. Since the locations of zeros can be chosen in

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$$

ways, we have

$$p_n = \frac{1}{2^{2n}} \frac{(2n)!}{(n!)^2}.$$

Using Stirling's formula we can write

$$(2n)! = \sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}, \qquad n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}.$$

Substituting this in the above equation we obtain

$$p_n \sim \frac{1}{2^{2n}} \frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}{2\pi n\left(\frac{n}{e}\right)^{2n}} = \frac{1}{\sqrt{\pi n}}.$$

For part (c), we have

$$p_{100} \approx \frac{1}{10\pi} \approx 0.03.$$

(1.6) (Bonus) Suppose $A_1, \ldots, A_n$ are events in a sample space. Show that

$$\sum_{1\le i\le n} \mathbb{P}[A_i] - \sum_{1\le i<j\le n} \mathbb{P}[A_i \cap A_j] \le \mathbb{P}\left[\bigcup_{i=1}^{n} A_i\right] \le \sum_{1\le i\le n} \mathbb{P}[A_i].$$

Solution. We will prove the statement by induction on n. For $n = 2$, we have

$$\mathbb{P}[A_1 \cup A_2] = \mathbb{P}[A_1] + \mathbb{P}[A_2] - \mathbb{P}[A_1 \cap A_2]$$

from which both inequalities follow.

Suppose the statement has been proven for n. We will now establish it for $n+1$. For the left-hand side, we have

$$\begin{aligned}\mathbb{P}\left[\bigcup_{i=1}^{n+1} A_i\right] &= \mathbb{P}\left[\left(\bigcup_{i=1}^{n} A_i\right) \cup A_n\right] \le \mathbb{P}\left[\bigcup_{i=1}^{n} A_i\right] + \mathbb{P}[A_{n+1}] \\ &\le \sum_{1\le i\le n} \mathbb{P}[A_i] + \mathbb{P}[A_{n+1}] = \sum_{1\le i\le n+1} \mathbb{P}[A_i].\end{aligned} \tag{2}$$

For the other inequality, again using the inequality for n and the right-hand side inequality proven above, we have

$$\begin{aligned}\mathbb{P}\left[\bigcup_{i=1}^{n+1} A_i\right] &= \mathbb{P}\left[\left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n+1}\right] = \mathbb{P}\left[\bigcup_{i=1}^{n} A_i\right] + \mathbb{P}[A_{n+1}] - \mathbb{P}\left[A_{n+1} \cap \bigcup_{i=1}^{n} A_i\right] \\ &\ge \sum_{1\le i\le n} \mathbb{P}[A_i] - \sum_{1\le i<j\le n} \mathbb{P}[A_i \cap A_j] + \mathbb{P}[A_{n+1}] - \sum_{1\le i\le n} \mathbb{P}[A_i \cap A_{n+1}] \\ &= \sum_{1\le i\le n+1} \mathbb{P}[A_i] - \sum_{1\le i<j\le n+1} \mathbb{P}[A_i \cap A_j].\end{aligned} \tag{3}$$