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Due: October 30, 2017
Assignment 5

Elements of Probability

(5.1) For $\alpha > 1$, suppose that the continuous random variable X has the density function given by

$$f_X(t) = \begin{cases} \alpha e^{-\alpha t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Compute $\mathbb{E}\left[e^X\right]$.

(5.2) There are 100 students in a probability class. Since each lecture hall has room for only 70 students, the students have to be divided into two groups for the exam. The professor suggests the following scheme: each students uses a fair coin to decide which lecture hall to go to. Under the assumption the coin outcomes are independent, use the central limit theorem to find the approximate value of the probability that none of the lecture halls is overcrowded with students.

(5.3) A fair dice is rolled 20 times. What is the approximate probability that the sum of the outcomes is between 65 and 75? (*Hint:* Use the central limit theorem)

(5.4) A three-element subset A of the set $\{1, 2, \dots, 10\}$ is randomly chosen. Suppose X and Y denote the least and the largest element of the set A. For instance if A turns out to be the set $\{4, 7, 9\}$, then we have $X = 4$ and $Y = 9$.

(a) Explain why X can take values $1, 2, \dots, 8$, and Y can take values $3, 4, \dots, 10$. Moreover, show that their probability mass functions are given by

$$\mathbb{P}[X = x] = \frac{(10-x)(9-x)}{240}, \qquad 1 \leq x \leq 8.$$

$$\mathbb{P}[Y = y] = \frac{(y-1)(y-2)}{240}, \qquad 3 \leq y \leq 10.$$

(b) Show that X and Y are not independnet.

(5.5) Suppose X and Y are independent positive random variables with the same distribution, and the finite expected value μ.

(a) What is wrong with the following argument:

$$\mathbb{E}\left[\frac{X}{X+Y}\right] = \frac{\mathbb{E}[X]}{\mathbb{E}[X+Y]} = \frac{\mu}{2\mu} = \frac{1}{2}.$$

(b) Give a correct proof for

$$\mathbb{E}\left[\frac{X}{X+Y}\right] = \frac{1}{2}.$$