JTMS-12: Probability and Random Processes

Fall 2020

M. Bode

Lecture 15

Recap

Chapter 4: **Expectation and Introduction to Estimation**

Expected Value … discrete case:

$$\boxed{\mu_X = E[X] = \sum_i P[X = x_i] \cdot x_i}$$

General case (if the integral exists):

$$\boxed{\mu_X = E[X] = \int_{-\infty}^{\infty} x f_X(x) dx}$$

Useful aspect in practice: For $Y = \mathrm{g}(X)$,

$$\mu_Y = E[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

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Recap

Chapter 4: **Expectation and Introduction to Estimation**

4.1 Expected Value of a R.V.

4.2 Conditional Expectations

4.3 Moments

4.4 Chebyshev & Schwarz

4.5 Moment Generating Functions

4.6 Chernoff Bound

4.7 Characteristic Functions & Central Limit Theorem

4.8 Estimators for Mean and Variance

General question: Which pdf to use in order to find $E[X]$?

Consider r.v.s X, Y with joint pdf $f_{XY}(x, y)$. Notice:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{XY}(x,y)dx\,dy = \int_{-\infty}^{\infty} x f_X(x)dx = E[X] = \mu_X$$

Also, expectations are linear operations …

$$E[aX + bY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(ax + by)f_{XY}(x,y)dx\,dy$$

$$= aE[X] + bE[Y]$$

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Chapter 4: **Expectation and Introduction to Estimation**

Conditional Expectations

Consider an event B and a r.v. X with conditional pdf $f_{X|B}(x|B)$…

$$E[X|B] \overset{\text{def}}{=} \int_{-\infty}^{\infty} x f_{X|B}(x|B)dx$$

Like

``What is the average temperature, given it is a sunny day?´´

Or, consider r.v.s X, Y with conditional pdf $f_{X|Y}(x|y)$

$$E[X|Y] \overset{\text{def}}{=} \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx$$

Like

``What is the average temperature, given that the relative humidity is 27%?´´

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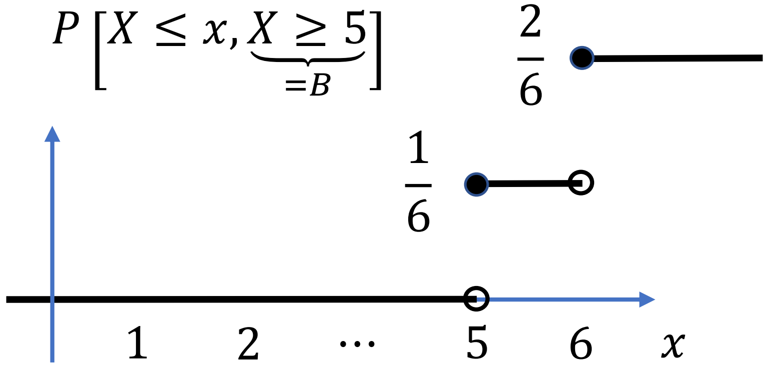
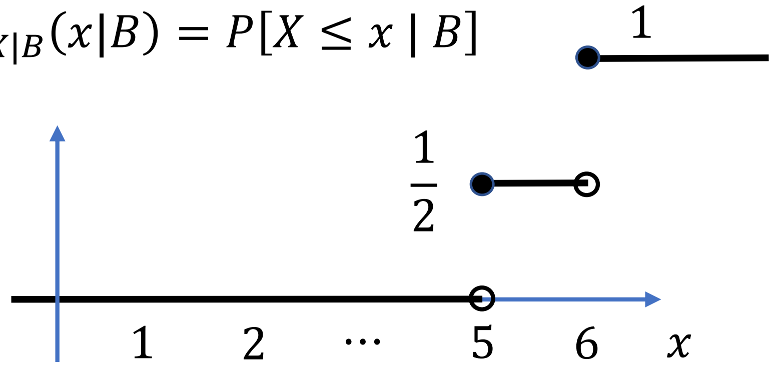
$P\left[X \le x, \underbrace{X \ge 5}_{=B}\right]$



$F_{X|B}(x|B) = P[X \le x \mid B]$



Example (fair die)

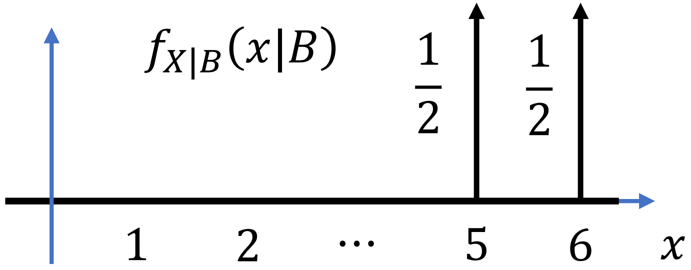
Consider a fair 6-faced die represented by a r.v. $X\colon \Omega \to \mathbb{R}$ with the standard image values, and an event $B\colon\ X \ge 5$

Find $E[X|B]$

$$E[X|B] = \int_{-\infty}^{\infty} x f_{X|B}(x|B)dx$$

Steps:

$$f_{X|B}(x|B) = \frac{d}{dx}F_{X|B}(x|B) = \frac{d}{dx}\frac{P[X \le x, B]}{P[B]} = \frac{d}{dx}\frac{P[X \le x, X \ge 5]}{P[X \ge 5]}$$



$E[X|B] = 5.5$

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Chapter 4: **Expectation and Introduction to Estimation**

Conditional Expectations

Again, consider r.v.s X, Y with joint pdf $f_{XY}(x, y)$ …

Different perspective:

$$\mu_X = E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{XY}(x, y) dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \frac{f_{XY}(x, y)}{f_Y(y)} f_Y(y) dx\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X|Y}(x|y) f_Y(y) dx\, dy$$

$$= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx\right\} f_Y(y) dy = E[E[X|Y]]$$

Mind: Being a function of Y, this is also a random variable.

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$\Rightarrow f_{V|W}(v|w) =$

$$= \frac{1}{\sqrt{2\pi \frac{16}{5}}} \cdot \exp\left[-\frac{50v^2 + 60vw - 18w^2}{320}\right]$$

$$= \frac{1}{\sqrt{2\pi \frac{16}{5}}} \cdot \exp\left[-\frac{5\left(v + \frac{6w}{10}\right)^2}{32}\right]$$

$$= \frac{1}{\sqrt{2\pi \frac{16}{5}}} \cdot \exp\left[-\frac{\left(v + \frac{6w}{10}\right)^2}{2 \cdot \frac{16}{5}}\right]$$

Identify: Normal with $\mu = -\frac{6w}{10}$, $\sigma^2 = \frac{16}{5}$

Can you sketch the joint pdf … understand our result?

Example 2 (Normals)

Consider two jointly normal r.v.s V, W with joint pdf

$$f_{VW}(v, w) = \frac{1}{8\pi} \exp\left[-\frac{5v^2 + 6vw + 5w^2}{32}\right]$$

Find the conditional density $f_{V|W}(v|w)$

$$f_{V|W}(v|w) = \frac{f_{VW}(v, w)}{f_W(w)}$$

Before (by completing the square in lec.13), we found

$$f_W(w) = \frac{1}{\sqrt{2\pi \cdot 5}} \cdot \exp\left[-\frac{w^2}{2 \cdot 5}\right]$$

$$\Rightarrow f_{V|W}(v|w) = \frac{\sqrt{2\pi \cdot 5}}{\sqrt{64\pi^2}} \exp\left[-\frac{5v^2 + 6vw + 5w^2}{32} + \frac{w^2}{2 \cdot 5}\right]$$

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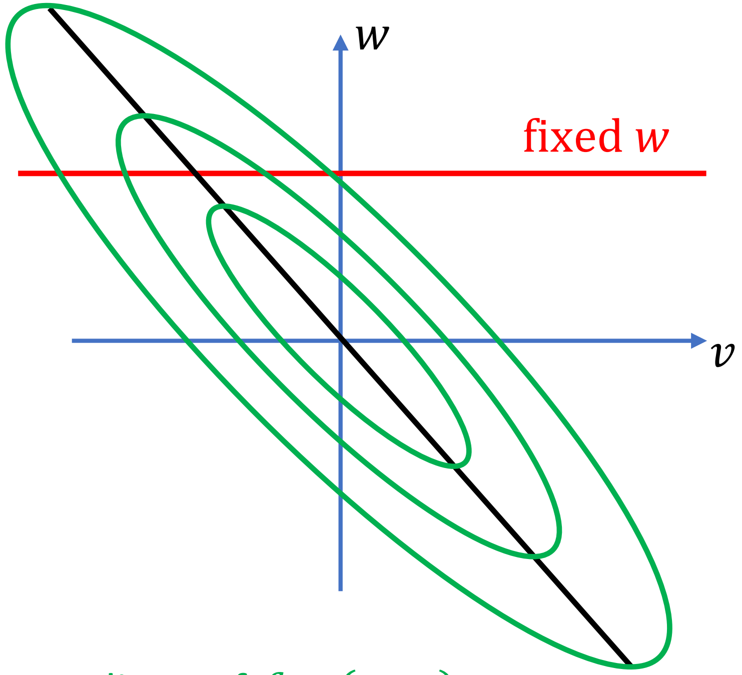
Sketch the joint pdf … understand our result?



Contour lines of $f_{VW}(v,w)$

Example 2 (Normals) contd …

Where are the conditional expected values ?

$$f_{VW}(v,w) = \frac{1}{8\pi}\exp\left[-\frac{5v^2+6vw+5w^2}{32}\right]$$

For a fixed w, the maxium density occurs when

$$\frac{\partial}{\partial v}\left(-\frac{5v^2+6vw+5w^2}{32}\right) = 0$$

That is at $10v + 6w = 0$ or $v = -6w/10$ ….

In accordance with our result

$$f_{V|W}(v|w) = \frac{1}{\sqrt{2\pi\frac{16}{5}}}\cdot\exp\left[-\frac{\left(v+\frac{6w}{10}\right)^2}{2\cdot\frac{16}{5}}\right] \Rightarrow E[V|W] = -6w/10$$

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Conditional Expectations

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x f_{X|YZ}(x|y,z)dx \right\} f_{Z|Y}(z|y)dz$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x \frac{f_{XYZ}(x,y,z)}{f_{YZ}(y,z)} dx \right\} \frac{f_{YZ}(y,z)}{f_Y(y)} dz$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x \frac{f_{XYZ}(x,y,z)}{f_Y(y)} dx \right\} dz$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f_{XYZ}(x,y,z)dz \right\} \frac{x}{f_Y(y)} dx$$

Notice: If X and Y are independent, then

$$E[X|Y] = E[X]$$

Intuition says: Yes! – But can you prove it?

$$\int_{-\infty}^{\infty} x \frac{f_{XY}(x,y)}{f_Y(y)} dx = \int_{-\infty}^{\infty} x \frac{f_X(x) f_Y(y)}{f_Y(y)} dx = \int_{-\infty}^{\infty} x f_X(x)dx$$

Also, there can be fancy expressions … try to understand:

$$E[E[X|Y,Z]|Y] = \ ???$$

$$= \int_{-\infty}^{\infty} x \frac{f_{XY}(x,y)}{f_Y(y)} dx = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx = E[X|Y]$$

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Chapter 4: **Expectation and Introduction to Estimation**

Moments

rth moment, $r = 0, 1, 2, \ldots$ (if the integral exists):

$$\boxed{E[X^r] = \int_{-\infty}^{\infty} x^r f_X(x) dx}$$

rth central moment, $r = 0, 1, 2, \ldots$ (if the integral exists):

$$\boxed{E[(X-\mu)^r] = \int_{-\infty}^{\infty} (x-\mu)^r f_X(x) dx}$$

where $\mu = E[X]$

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Chapter 4: **Expectation and Introduction to Estimation**

Moments

In particular, the 2nd central moment, the variance:

$$\boxed{\mathrm{Var}(X) = \sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)dx}$$

Notice the so-called moment formula:

$$\sigma^2 = E[X^2] - E[2\mu X] + E[\mu^2]$$

$$= E[X^2] - 2\mu \underbrace{E[X]}_{=\mu} + \mu^2 = E[X^2] - \mu^2 = E[X^2] - E[X]^2$$

The End

Next time: Continue with Chp. 4