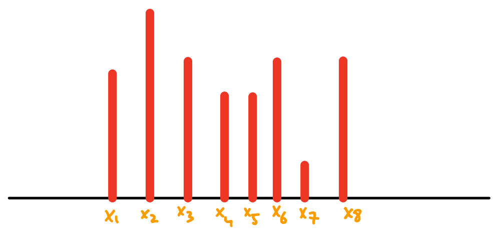
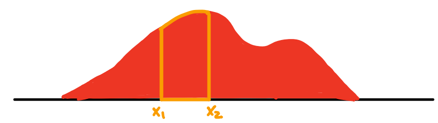
Continuous random variables





$\mathbb{P}[X = x_i] = ?$

$\mathbb{P}[x_1 \leq X \leq x_2] = ?$

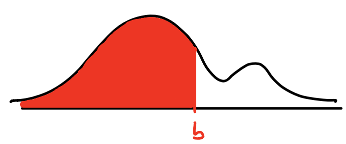
A continuous random variable is a random variable X for which there exists a non-negative function $f_X(x)$ such that

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)\,dx$$

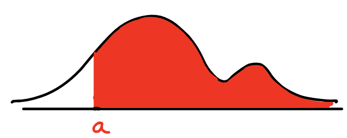
The function $f_X(x)$ is called the probability density function (PDF) of X.

More generally

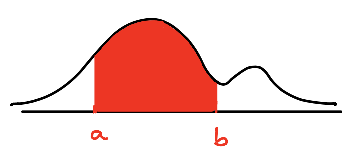
$$\mathbb{P}(X \leq b) = \int_{-\infty}^{b} f_X(x)\,dx$$



$$\mathbb{P}(X \geq a) = \int_a^{+\infty} f_X(x)\,dx$$



$$\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)\,dx$$

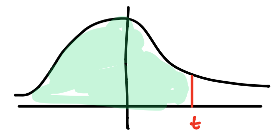


Distribution function of X

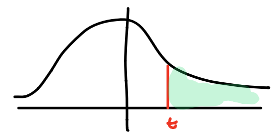
$$F_X(t) = \mathbb{P}(X \leq t)$$

Note that the knowledge of $F_X(t)$ is enough to compute other probabilities:

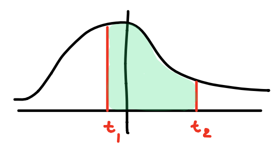
$$\mathbb{P}(X \leq t) = F_X(t)$$



$$\mathbb{P}(X \geqslant t) = 1 - F_X(t)$$



$$\mathbb{P}(t_1 \leq X \leq t_2) = F_X(t_2) - F_X(t_1)$$



The probability density function is related to the distribution function via

$$\boxed{F_X(t) = \mathbb{P}(X \leq t) = \int_{-\infty}^{t} f_X(x)\, dx}$$

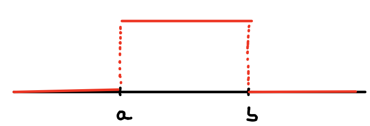
or, equivalently

$$\boxed{f_X(x) = F_X'(x)}$$

<u>Key examples</u>

1) uniform distribution over the interval $[a,b]$

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

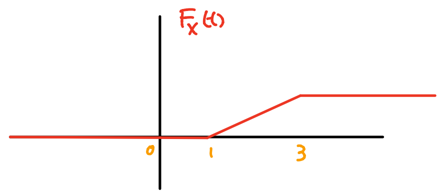


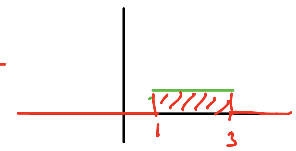
$$F_X(t) = \begin{cases} 0 & t \leqslant a \\ \dfrac{t-a}{b-a} & a \leq t \leq b \\ 1 & t \geqslant b \end{cases}$$

$$F_X(t) = \int_a^t \frac{1}{b-a}\, dt = \frac{t-a}{b-a}.$$

Example Suppose X has uniform distribution over the interval $[1,3]$. Find the probability of the events: $1 \le X \le 2$, $X \geqslant 2$, $1 \le X \le 4$.

$$F_X(t) = \begin{cases} 0 & t \le 1 \\ \frac{t-1}{2} & 1 \le t \le 3 \\ 1 & t \geqslant 3 \end{cases}$$





$$\mathbb{P}(1 \le X \le 2) = \frac{2-1}{2} = 1/2$$

$$\mathbb{P}(X \geqslant 2) = 1 - F_X(2) = 1 - 1/2 = 1/2$$

$$\mathbb{P}(1 \le X \le 4) = F_X(4) - F_X(1) = 1 - 0 = 1$$

Example: The distribution function of a random variable X is given by

$$F_X(t) = \begin{cases} 1 - e^{-t^2} & t \geqslant 0 \\ 0 & t < 0 \end{cases}$$

Find its PDF

$$f_X(t) = F_X'(t) = \begin{cases} 2t e^{-t^2} & t \geqslant 0 \\ 0 & t < 0 \end{cases}$$

Example The density function of a random variable is given by

$$f_X(x) = \begin{cases} x e^{-x} & x \geqslant 0 \\ 0 & x < 0 \end{cases}$$

Find $\mathbb{P}(X > 2)$.

$$\mathbb{P}(X > 2) = \int_2^\infty x e^{-x}\,dx = -x e^{-x}\Big|_2^\infty - \int_2^\infty 1.\,(-e^{-x})\,dx$$

$$= 2e^{-2} + e^{-2} = 3e^{-2}$$

Example: Suppose X has uniform distribution over $[0,2]$, and let $Y=X^3$.

Find the distribution and probability density function of Y.

Step 1 Find $F_Y(t)$.

$$F_Y(t) = \mathbb{P}(Y \leq t) = \mathbb{P}(X^3 \leq t) = \mathbb{P}(X \leq \sqrt[3]{t})$$

$$t \leq 0 \implies F_Y(t) = 0$$

$$t \geq 8 \implies F_Y(t) = 1$$

$$0 \leq t \leq 8 \implies F_Y(t) = \mathbb{P}(X \leq \sqrt[3]{t}) = \frac{1}{2}\sqrt[3]{t}.$$

Step 2

$$f_Y(t) = \begin{cases} 0 & t \leq 0,\ t \geq 8 \\ \frac{1}{6} t^{-2/3} & 0 < t < 8 \end{cases}$$

Example Suppose X is uniformly distributed in $[-1,1]$, and $Y=X^2$.

Compute $f_Y(x)$.

Note that : $-1 \leq X \leq 1 \implies 0 \leq Y \leq 1$.

For $0 < t < 1$:

$$F_Y(t) = \mathbb{P}(X^2 \leq t) = \mathbb{P}(-\sqrt{t} \leq X \leq \sqrt{t})$$
$$= \int_{-\sqrt{t}}^{\sqrt{t}} \frac{1}{2}\, dx = \sqrt{t}$$

$$f_Y(t) = \begin{cases} \frac{1}{2} t^{-1/2} & 0 < t < 1 \\ 0 & t < 0 \text{ or } t > 1 \end{cases}$$

Expected value and variance

Suppose X is a continuous random variable with the PDF $f_X(x)$. Then the expected value of X is defined by

$$E[X] = \int_{-\infty}^{+\infty} x\, f_X(x)\, dx$$

compare with

$$E[X] = \sum x \cdot P_X(x) \quad \text{for discrete random variables.}$$

Example. Find $E[X]$ when X has uniform distribution over an interval $[a,b]$.

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \int_a^b \frac{x}{b-a}\, dx = \frac{1}{b-a} \cdot \frac{x^2}{2}\Big|_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{b+a}{2}.$$

Example. Suppose X is a continuous random variable with the density function given by

$$f_X(x) = \begin{cases} k(1-x^2) & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find k. (b) Find $E[X]$.

Since $\int_0^1 k(1-x^2)\, dx = 1 \Rightarrow k\left(x - \frac{x^3}{3}\right)\Big|_0^1 = 1$

$\Rightarrow k \cdot \frac{2}{3} = 1 \Rightarrow k = \frac{3}{2}$

So

$$f_X(x) = \begin{cases} \frac{3}{2}(1-x^2) & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \int_0^1 \frac{3}{2} x(1-x^2)\, dx = \frac{3}{2} \cdot \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{3}{2} \cdot \frac{1}{4} = \frac{3}{8}.$$