JTMS-12: Probability and Random Processes

Fall 2020

M. Bode

Lecture 10:
Functions of Random Variables

Lecture 10

Reminder:

Chapter 3: Functions of Random Variables

3.1 Functions of Random Variables

3.2 $Y = g(X)$

3.3 $Z = g(X, Y)$

3.4 $V = g(X, Y)$, $W = h(X, Y)$

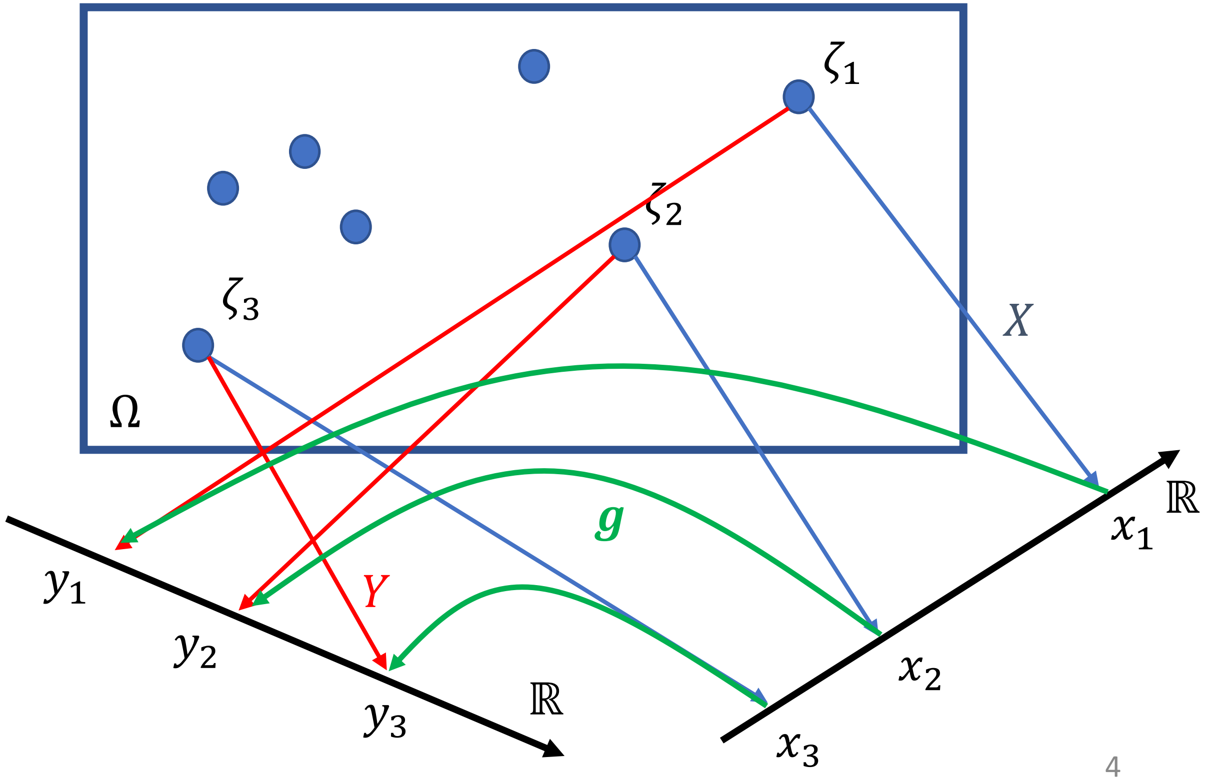
Functions of Random Variables

Idea: Map outcomes to (real) numbers.

The random variable $X\colon\ \Omega \to \mathbb{R}$ maps all outcomes from the sample description space to a real number.

Re-lable: $y = g(x)$

Re-interpret: $Y\colon\ \Omega \to \mathbb{R}$, $Y(\zeta) = g(X(\zeta))$



Lecture 10

Chapter 3: Functions of Random Variables

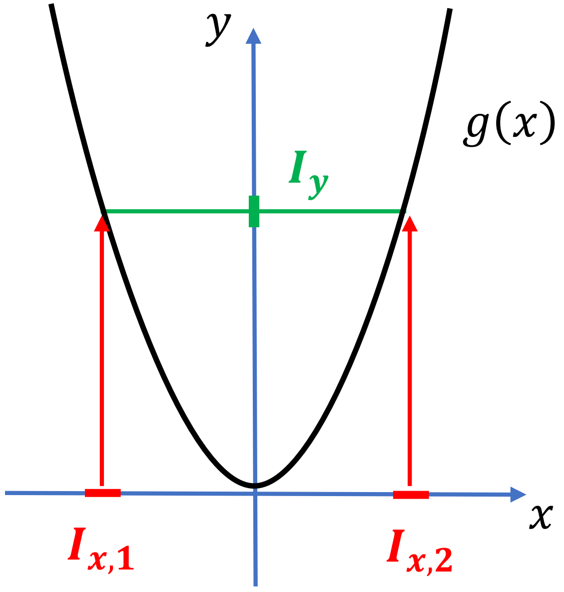
3.1 Functions of Random Variables

3.2 $Y = g(X)$

3.3 $Z = g(X,Y)$

3.4 $V = g(X,Y)$, $W = h(X,Y)$

Functions of Random Variables



Quick way to find the density:

The direct formula (for cont. r.v.s)

$$P[Y \in I_y] = P[X \in I_{x,1}] + P[X \in I_{x,2}]$$

$$f_Y(y)|\Delta y| \approx f_X(x_1)|\Delta x_1| + f_X(x_2)|\Delta x_2|$$

Mind the relative orientations of $\boldsymbol{I_y}$ and its (partial) pre-images $\boldsymbol{I_{x,1}}$ and $\boldsymbol{I_{x,2}}$.

$$\Rightarrow f_Y(y) \approx f_X(x_1)\left|\frac{\Delta x_1}{\Delta y}\right| + f_X(x_2)\left|\frac{\Delta x_2}{\Delta y}\right|$$

Hence, in the limit $\Delta y \rightarrow 0$:

$$f_Y(y) = f_X(x_1)\left|\frac{1}{g'(x_1)}\right| + f_X(x_2)\left|\frac{1}{g'(x_2)}\right|$$

…sum over all pre-images

Lecture 10

Chapter 3: Functions of Random Variables

3.1 Functions of Random Variables

3.2 $Y = g(X)$

3.3 $Z = g(X, Y)$

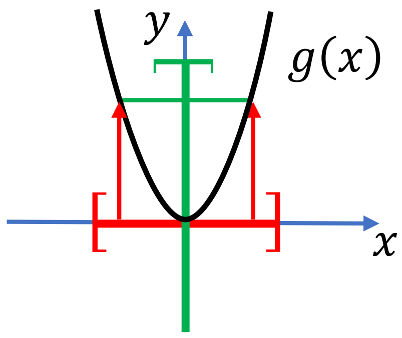
3.4 $V = g(X, Y)$, W $= h(X, Y)$

Functions of Random Variables

Example:

Consider a continuous r.v. X, and

$$Y = X^2 = g(X)$$



$$f_Y(y) = f_X(x_1)\left|\frac{1}{g'(x_1)}\right| + f_X(x_2)\left|\frac{1}{g'(x_2)}\right|$$

$$\Rightarrow f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y})$$

Special case: Normal distribution $X \sim \mathcal{N}(0,1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

$$\Rightarrow f_Y(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi y}} \exp\left(-\dfrac{y}{2}\right) \\ 0; y < 0 \end{cases}$$

Lecture 10

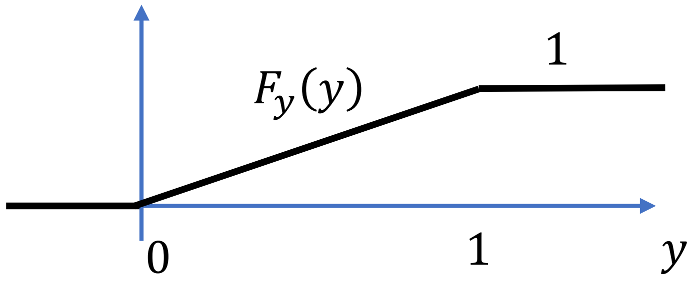
Chapter 3: Functions of Random Variables

3.1 Functions of Random Variables

3.2 $Y = g(X)$

3.3 $Z = g(X,Y)$

3.4 $V = g(X,Y)$, $W = h(X,Y)$



$\Rightarrow Y$ is uniform over [0,1]

Functions of Random Variables

Application: Random number generator …

Consider a continuous r.v. X, with (strictly increasing) distribution $F_X(x)$, and a mapping

Mind: F_X is non-decreasing, anyway.

$$Y = F_X(X)$$

Find $F_Y(y)$, $f_Y(y)$.

Standard approach …

$$F_Y(y) = P[Y \leq y] = P[F_X(X) \leq y]$$

Can you do this in reverse direction?

$$= P[X \leq F_X^{-1}(y)] = F_X\left(F_X^{-1}(y)\right) = ?$$

$$F_Y(y) = \begin{cases} 1 & ; \quad y \geq 1 \\ y & ; \quad 0 < y < 1 \\ 0 & ; \quad y \leq 0 \end{cases}$$

Lecture 10

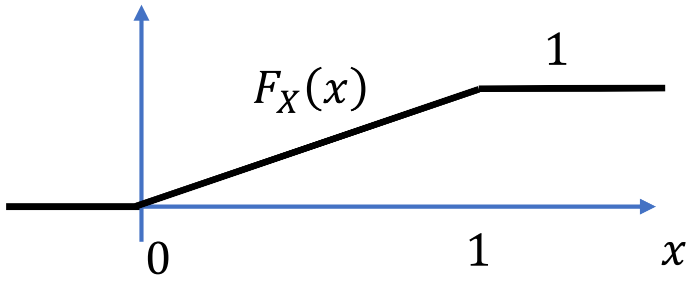
Chapter 3: Functions of Random Variables

3.1 Functions of Random Variables

3.2 $Y = g(X)$

3.3 $Z = g(X, Y)$

3.4 $V = g(X, Y)$, W $= h(X, Y)$



Functions of Random Variables

Application: Random number generator … the inverse direction

Suppose, we wish to generate random numbers W distributed according to a PDF Wish(w).

So far, we only have a random variable X, uniformly distributed over [0,1].

Consider:

$$W = Wish^{-1}(X)$$

Standard approach …

$$F_W(w) = P[W \leq w] = P[Wish^{-1}(X) \leq w]$$

$$= P[X \leq Wish(w)] = ?$$

$$F_W(w) = F_X\big(Wish(w)\big) = Wish(w)$$

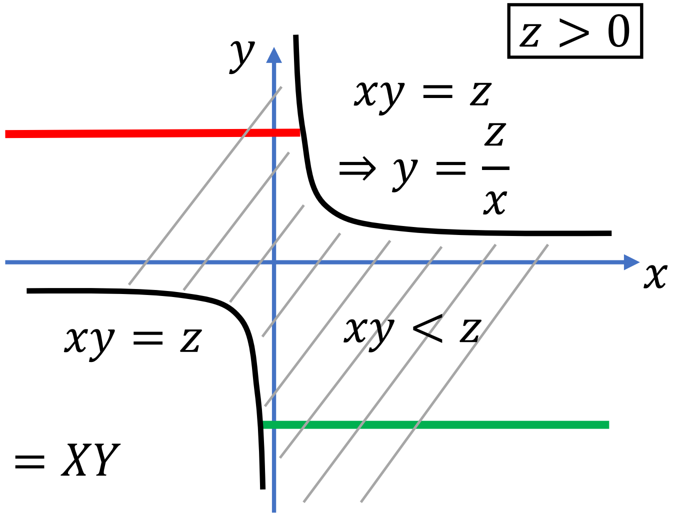
☺

Functions of Two Random Variables

Lecture 10

Chapter 3: Functions of Random Variables

Functions of Random Variables



Example 1:

Consider continuous r.v.s X, Y, and

$$Z = g(X,Y) = XY$$

Find $F_Z(z)$ and $f_Z(z)$.

Standard approach …

$$F_Z(z) = P[Z \leq z] = P[g(X,Y) \leq z] = P[XY \leq z]$$

For $z > 0$:

$$= \iint_{\substack{shaded \\ area}} f_{XY}(x,y)dxdy$$

$$= \int_0^{\infty} \left(\int_{-\infty}^{z/y} f_{XY}(x,y)dx \right) dy + \int_{-\infty}^{0} \left(\int_{z/y}^{\infty} f_{XY}(x,y)dx \right) dy$$

Lecture 10

Chapter 3: Functions of Random Variables

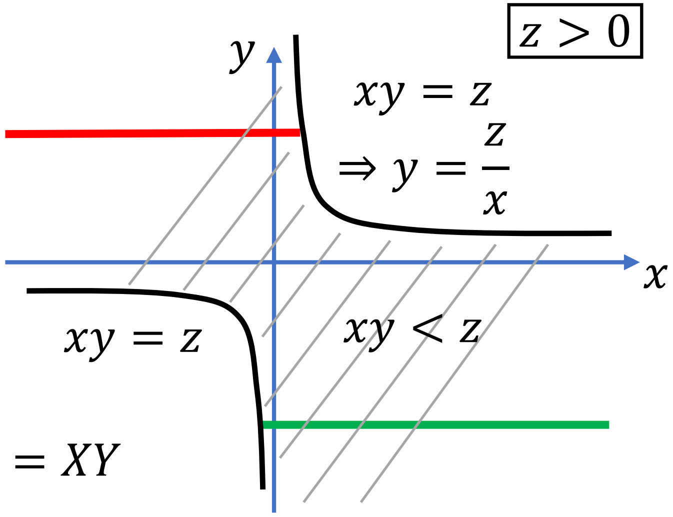
3.1 Functions of Random Variables

3.2 $Y = g(X)$

3.3 $Z = g(X,Y)$

3.4 $V = g(X,Y)$, W $= h(X,Y)$

Functions of Random Variables



solve:

Consider continuous r.v.s X, Y, and

$$Z = g(X,Y) = XY$$

Find $F_Z(z)$ and $f_Z(z)$.

Standard approach …

For $z > 0$:

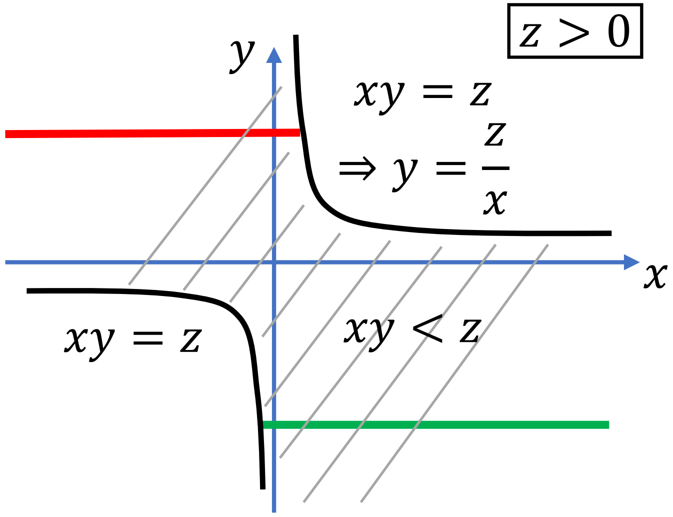
$$F_Z(z) = \int_0^\infty \left(\int_{-\infty}^{z/y} f_{XY}(x,y)dx \right) dy + \int_{-\infty}^0 \left(\int_{z/y}^{\infty} f_{XY}(x,y)dx \right) dy$$

$$\Rightarrow f_Z(z) = \frac{d}{dz} F_Z(z) = \int_{-\infty}^{\infty} \frac{1}{|y|} f_{XY}\left(z/y, y\right) dy$$

Similarly for $z < 0$ …

Lecture 10

Functions of Random Variables



Chapter 3: Functions of Random Variables

3.1 Functions of Random Variables

3.2 $Y = g(X)$

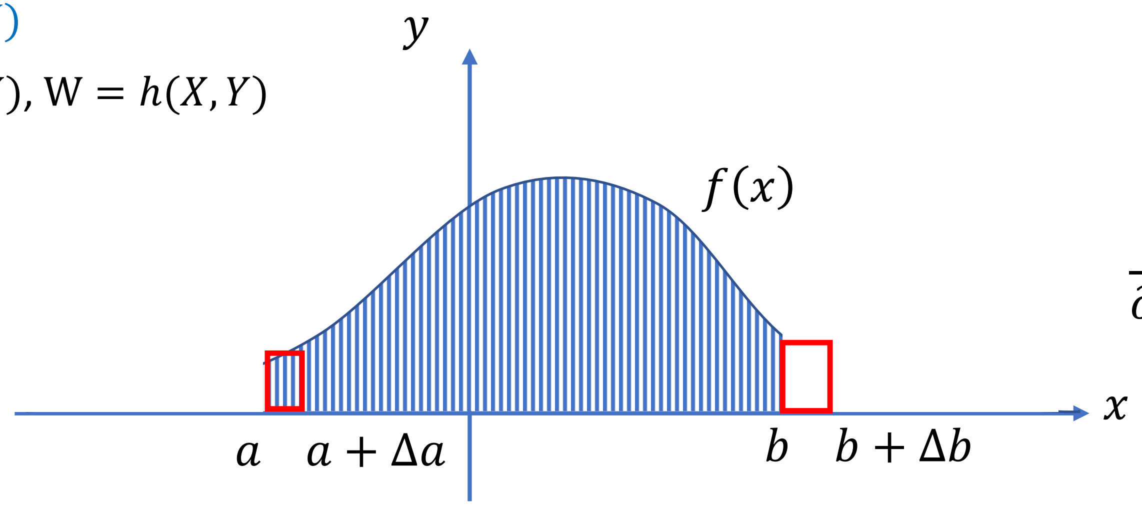
3.3 $Z = g(X, Y)$

3.4 $V = g(X, Y)$, $W = h(X, Y)$

Think: Derive an integral with regard to its boundary …

$$\frac{\partial}{\partial b}\int_a^b f(x)dx = f(b)$$

$$\frac{\partial}{\partial a}\int_a^b f(x)dx = -f(a)$$



Increase lower boundary:
➔ Area shrinks

Increase upper boundary:
➔ Area grows

Lecture 10

Chapter 3: Functions of Random Variables

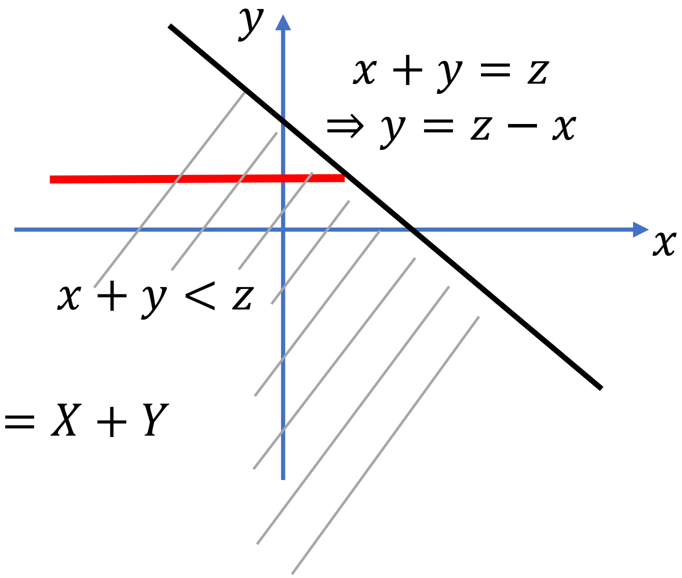
3.1 Functions of Random Variables

3.2 $Y = g(X)$

3.3 $Z = g(X,Y)$

3.4 $V = g(X,Y)$, W $= h(X,Y)$

Functions of Random Variables



Example 2:

Consider continuous r.v.s X, Y, and

$$Z = g(X,Y) = X + Y$$

Find $F_Z(z)$ and $f_Z(z)$.

Standard approach …

$$F_Z(z) = P[Z \le z] = P[g(X,Y) \le z] = P[X + Y \le z]$$

$$= \iint_{\substack{shaded \\ area}} f_{XY}(x,y)dxdy$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{z-y} f_{XY}(x,y)dx \right) dy$$

Lecture 10

Chapter 3: Functions of Random Variables

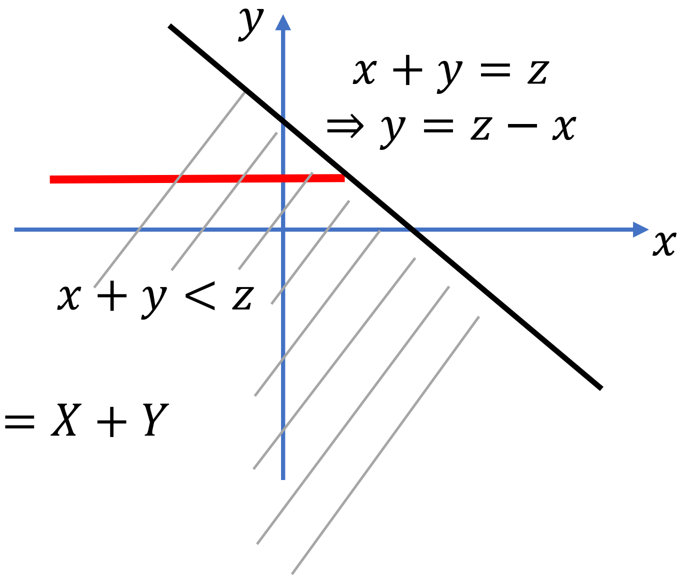
3.1 Functions of Random Variables

3.2 $Y = g(X)$

3.3 $Z = g(X,Y)$

3.4 $V = g(X,Y)$, $W = h(X,Y)$

Functions of Random Variables



solve:

Consider continuous r.v.s X, Y, and

$$Z = g(X,Y) = X + Y$$

Find $F_Z(z)$ and $f_Z(z)$.

Standard approach …

$$F_Z(z) = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{z-y} f_{XY}(x,y)dx\right)dy$$

$$\Rightarrow f_Z(z) = \frac{d}{dz}F_Z(z) = \int_{-\infty}^{\infty} f_{XY}(z-y,y)dy$$

Lecture 10

Chapter 3: Functions of Random Variables

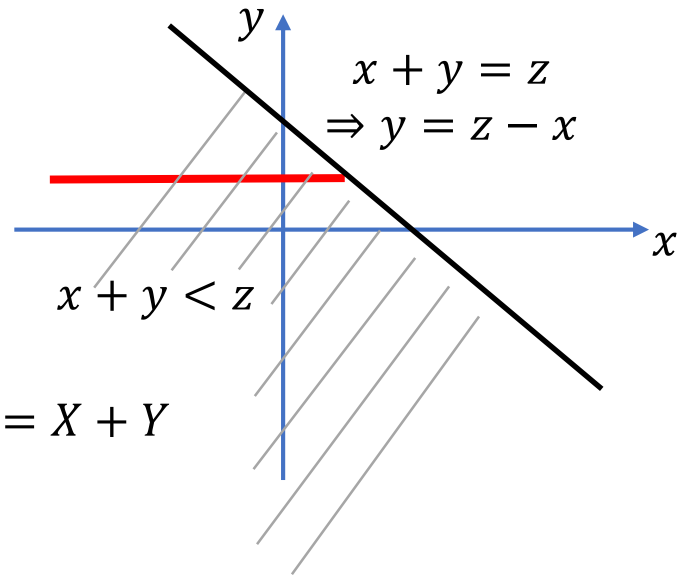
3.1 Functions of Random Variables

3.2 $Y = g(X)$

3.3 $Z = g(X,Y)$

3.4 $V = g(X,Y)$, $W = h(X,Y)$

Functions of Random Variables



solve:

Consider continuous r.v.s X, Y, and

$$Z = g(X,Y) = X + Y$$

Find $F_Z(z)$ and $f_Z(z)$.

Standard approach …

Special case: X and Y are independent…

$$f_{XY}(x,y) = f_X(x) f_Y(y)$$

$$\Rightarrow f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(z-y,y)dy = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)dy$$

Convolution!

… does that ring a bell?

The End

Next time: Chp. 3