JTMS-12: Probability and Random Processes

Fall 2020

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Lecture 16

Recap

Chapter 4: **Expectation and Introduction to Estimation**

4.1 Expected Value of a R.V.

4.2 Conditional Expectations

4.3 Moments

4.4 Chebyshev & Schwarz

4.5 Moment Generating Functions

4.6 Chernoff Bound

4.7 Characteristic Functions & Central Limit Theorem

4.8 Estimators for Mean and Variance

Moments

rth moment, $r = 0, 1, 2, \ldots$ (if the integral exists):

$$\boxed{E[X^r] = \int_{-\infty}^{\infty} x^r f_X(x)dx}$$

rth central moment, $r = 0, 1, 2, \ldots$ (if the integral exists):

$$\boxed{E[(X-\mu)^r] = \int_{-\infty}^{\infty} (x-\mu)^r f_X(x)dx}$$

where $\mu = E[X]$

Lecture 16

Recap

Chapter 4: **Expectation and Introduction to Estimation**

Moments

In particular, the 2nd central moment, the variance:

$$\boxed{\mathrm{Var}(X) = \sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)dx}$$

Notice the so-called moment formula:

$$\sigma^2 = E[X^2] - E[2\mu X] + E[\mu^2]$$

$$= E[X^2] - 2\mu \underbrace{E[X]}_{=\mu} + \mu^2 = E[X^2] - \mu^2 = E[X^2] - E[X]^2$$

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Example (the hard way …)

Calculate the variance of a binomial r.v. X with

$$P[X=k] = \binom{n}{k} p^k q^{n-k}$$

$$\mathrm{Var}(X) = \sigma^2 = E[X^2] - E[X]^2$$

First, find

$$E[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k}$$

Trick: Treat p and q as separate parameters

$$= p \frac{\partial}{\partial p} \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}$$

$$= p \frac{\partial}{\partial p} (p+q)^n = \begin{Bmatrix} pn(p+q)^{n-1} & , n > 0 \\ 0 & , n = 0 \end{Bmatrix} = np$$

End-of-Trick: Use $p + q = 1$

Lecture 16

Chapter 4: **Expectation and Introduction to Estimation**

Now, same idea,

$$E[X^2] = \sum_{k=0}^{n} k^2 \binom{n}{k} p^k q^{n-k}$$

Same trick: Treat p and q as separate parameters

$$= p\frac{\partial}{\partial p} p\frac{\partial}{\partial p}\sum_{k=0}^{n}\binom{n}{k} p^k q^{n-k} = p\frac{\partial}{\partial p} p\frac{\partial}{\partial p}(p+q)^n = \cdots$$

1) $n \geq 2$

$$\ldots = p\frac{\partial}{\partial p}\left[pn(p+q)^{n-1}\right] =$$

End-of-Trick: Use $p + q = 1$

$$= p\left[n(p+q)^{n-1} + pn(n-1)(p+q)^{n-2}\right]$$

$$= p[n + pn(n-1)] = p[n - np + n^2 p] = pn[1 - p + np]$$

$$= pn[q + np] = n^2p^2 + npq$$

2) $n = 0$ **… OK** (formula also valid)

3) $n = 1$ **…** $\sum_{k=0}^{n} k^2 \binom{n}{k} p^k q^{n-k} = p$

Also … $n^2p^2 + npq = p^2 + pq = p(p+q) = p$ **… OK**

Lecture 16

Chapter 4: **Expectation and Introduction to Estimation**

Example (the hard way …)

Calculate the variance of a binomial r.v. X with

$$P[X = k] = \binom{n}{k} p^k q^{n-k}$$

Combine:

$$\mathrm{Var}(X) = \sigma^2 = E[X^2] - E[X]^2$$

$$= n^2p^2 + npq - (np)^2 = npq$$

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Chapter 4: **Expectation and Introduction to Estimation**

Joint Moments

$(i,j)^{th}$ moment,of X and Y:

$$\boxed{E\left[X^i Y^j\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^i y^j f_{XY}(x,y)dx\,dy}$$

$(i,j)^{th}$ central moment:

$$\boxed{E\left[(X-\mu_X)^i (Y-\mu_Y)^j\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)^i (y-\mu_Y)^j f_{XY}(x,y)dx\,dy}$$

where $\mu_X = E[X], \mu_Y = E[Y]$

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Chapter 4: **Expectation and Introduction to Estimation**

Joint Moments

Most important … Covariance of X and Y:

$$\mathrm{Cov}[X,Y] = E[(X-\mu_X)(Y-\mu_Y)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)(y-\mu_Y)f_{XY}(x,y)dx\,dy$$

The covariance is linear in both ist arguments.

Notice the so-called moment formula:

$$\mathrm{Cov}[X,Y] = E[(X-\mu_X)(Y-\mu_Y)]$$

$$= E[XY] - \mu_X \underbrace{E[Y]}_{=\mu_Y} - \mu_Y \underbrace{E[X]}_{=\mu_X} + \mu_X\mu_Y = E[XY] - \mu_X\mu_Y =$$

$$= E[XY] - E[X]E[Y]$$

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Joint Moments

Notice … if X and Y are independent, joint moments factorize, like …

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{XY}(x,y)dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_X(x) f_Y(y)dx\,dy$$

$$= \int_{-\infty}^{\infty} x f_X(x)dx \int_{-\infty}^{\infty} y f_Y(y)dy = E[X]E[Y]$$

Hence, for independent r.v.s,

$$\mathrm{Cov}[X,Y] = E[XY] - E[X]E[Y] = 0$$

Lecture 16

Chapter 4: **Expectation and Introduction to Estimation**

Notice: $\mathrm{Var}[X] = \mathrm{Cov}[X, X]$

Consider independent r.v.s $X_1, X_2, \dots, X_n$, and their sum

$$Z = \sum_{i=1}^{n} X_i$$

Find $\mathrm{Var}[Z]$

$$\mathrm{Var}[Z] = \mathrm{Var}\left[\sum_{i=1}^{n} X_i\right] = \mathrm{Cov}\left[\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} X_i\right]$$

$$= \sum_{i,j=1}^{n} \mathrm{Cov}[X_i, X_j] = \sum_{i=1}^{n} \underbrace{\mathrm{Cov}[X_i, X_i]}_{=\mathrm{Var}[X_i]} + \sum_{i \neq j}^{n} \underbrace{\mathrm{Cov}[X_i, X_j]}_{=0}$$

$$= \sum_{i=1}^{n} \mathrm{Var}[X_i]$$

Lecture 16

Chapter 4: **Expectation and Introduction to Estimation**

Old Example (the easy way …)

Calculate the variance of a binomial r.v. *Z* with

$$P[Z = k] = \binom{n}{k} p^k q^{n-k}$$

Realize: Z is a sum of independent, identically distributed Bernoulli r.v.s $X_1, X_2, \dots, X_n$

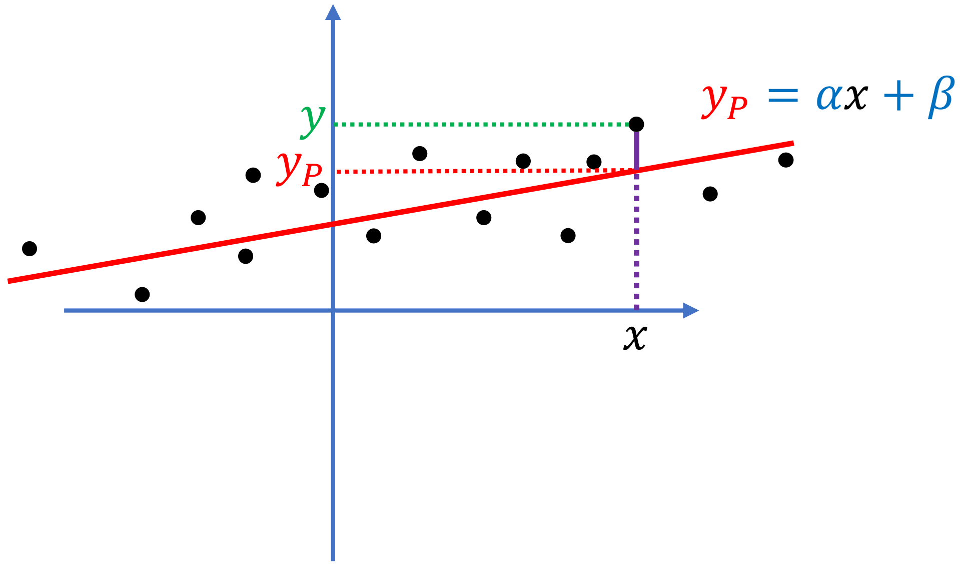
$$Z = \sum_{i=1}^{n} X_i$$

For each X_i, we have $P[X_i = 0] = q$, $P[X_i = 1] = p$, and

$$\mathrm{Var}[X_i] = E\left[X_i^2\right] - E[X_i]^2 = p - p^2 = pq$$

$$\Rightarrow \mathrm{Var}[Z] = \sum_{i=1}^{n} \mathrm{Var}[X_i] = npq$$

Lecture 16



Linear Prediction

Consider two r.v.s X and Y.

Suppose, you have early access to the X-outcomes and want to predict the Y-outcomes before they actually arrive…

Inspired by previous observations, we set up a linear model,

$$Y_P = \alpha X + \beta$$

… and try to predict with minimal quadratic error… such that

$$\varepsilon^2 = E[(Y - Y_P)^2] = E[(Y - \alpha X - \beta)^2] \to min.$$

We require

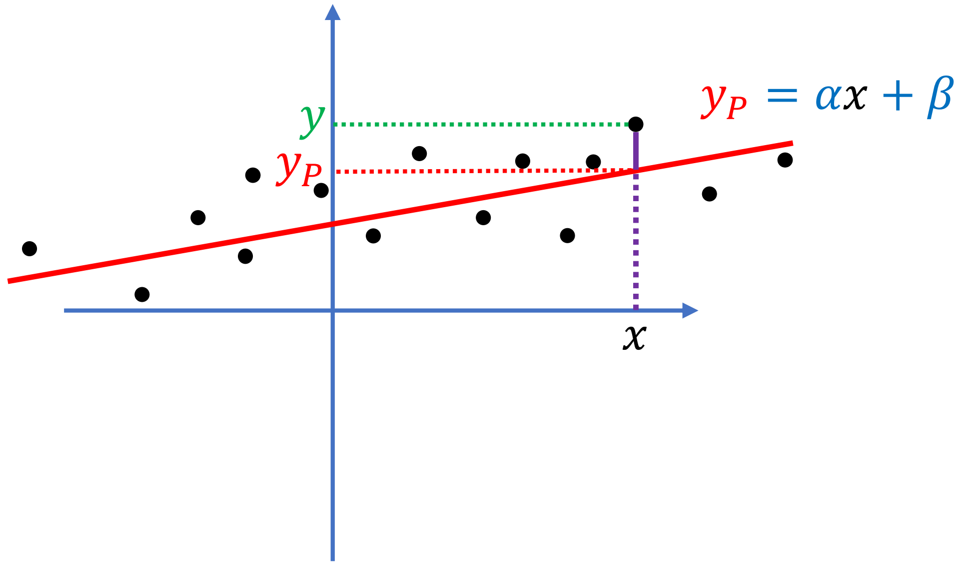
$$0 = \frac{\partial}{\partial \alpha}\varepsilon^2 = \frac{\partial}{\partial \alpha}E[(Y - \alpha X - \beta)^2]$$

and

$$0 = \frac{\partial}{\partial \beta}\varepsilon^2 = \frac{\partial}{\partial \beta}E[(Y - \alpha X - \beta)^2]$$

Lecture 16

$$\varepsilon^2 = E[(Y - Y_P)^2] = E[(Y - \alpha X - \beta)^2] \to min.$$



Hence,

$$0 = -2E[(Y - \alpha X - \beta)X]$$

$$\Leftrightarrow \alpha E[X^2] + \beta E[X] = E[XY]$$

and

$$0 = -2E[Y - \alpha X - \beta]$$

$$\Leftrightarrow \alpha E[X] + \beta = E[Y]$$

Solve:

$$\beta = E[Y] - \alpha E[X]$$

$$\Rightarrow \alpha E[X^2] + (E[Y] - \alpha E[X])E[X] = E[XY] \Rightarrow \boxed{\alpha = \frac{Cov(X, Y)}{\sigma_X^2}}$$

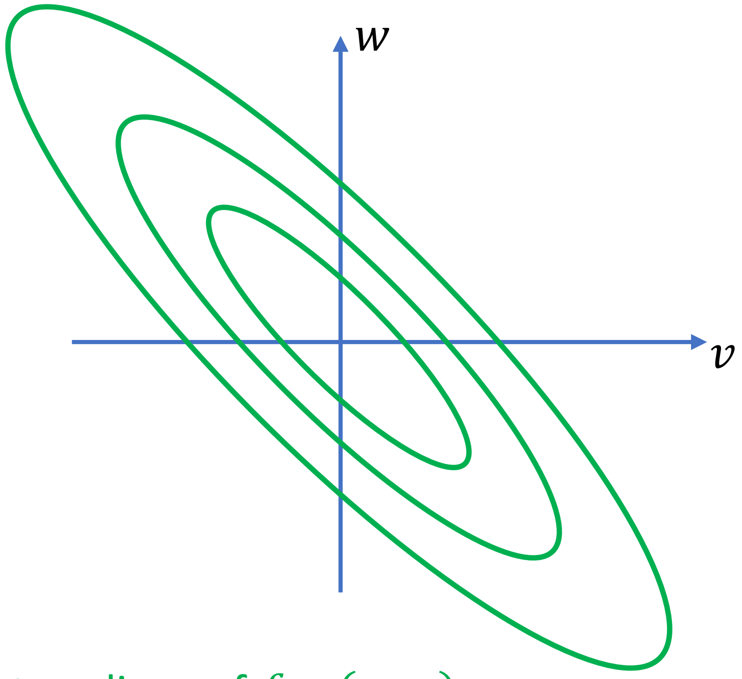
$$\Rightarrow \boxed{\beta = E[Y] - \frac{Cov(X, Y)}{\sigma_X^2} E[X]}$$

Lecture 16

Can we use the two marginals of $f_{VW}(v,w)$ to reconstruct the original joint pdf?

Obviously not … recall our example from lec. 13 about

$$f_{VW}(v,w) = \frac{1}{8\pi}\exp\left[-\frac{5v^2+6vw+5w^2}{32}\right]$$



Contour lines of $f_{VW}(v,w)$

whose marginals

$$f_W(w) = \frac{1}{\sqrt{2\pi\cdot 5}}\cdot\exp\left[-\frac{w^2}{2\cdot 5}\right]$$

$$f_V(v) = \frac{1}{\sqrt{2\pi\cdot 5}}\cdot\exp\left[-\frac{v^2}{2\cdot 5}\right]$$

are NOT independent as

$$f_V(v)\cdot f_W(w) \neq f_{VW}(v,w)$$

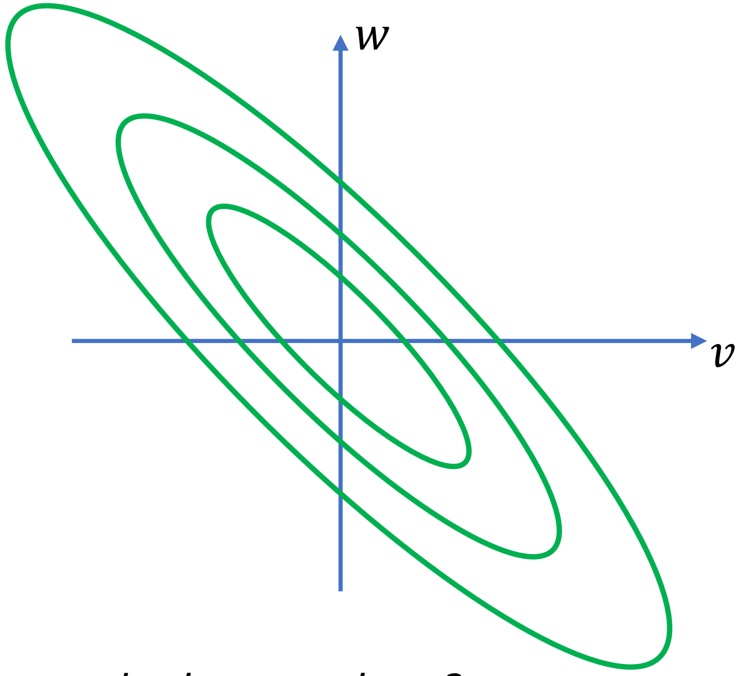
BUT, the marginals of both, the left and the right expression are the same.

Lecture 16

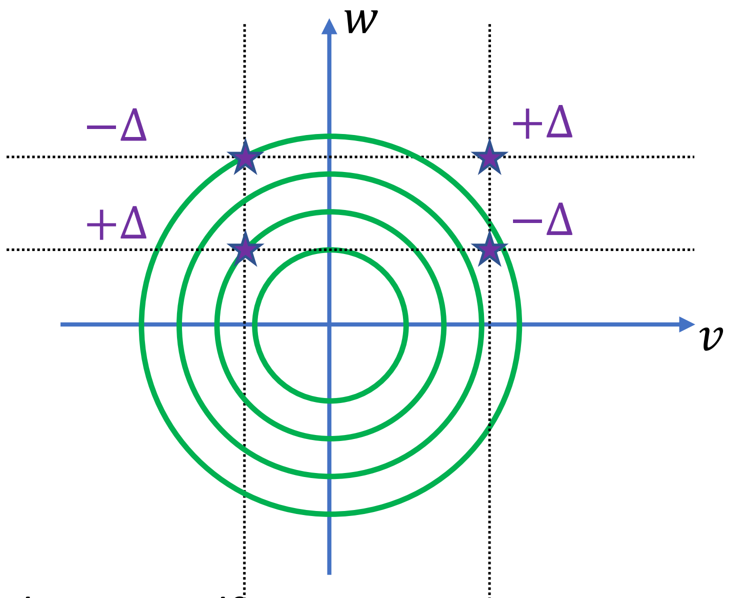
A somewhat weaker question:

Is it true that normal marginals imply a jointly normal pdf?

Not even that …



Can you do the same here?



Change the joint pdf
While keeping the marginals intact

Mind: Densities must not be negative!

The End

Next time: Continue with Chp. 4