JMTS-12: Probability and Random Processes

Fall 2020

M. Bode

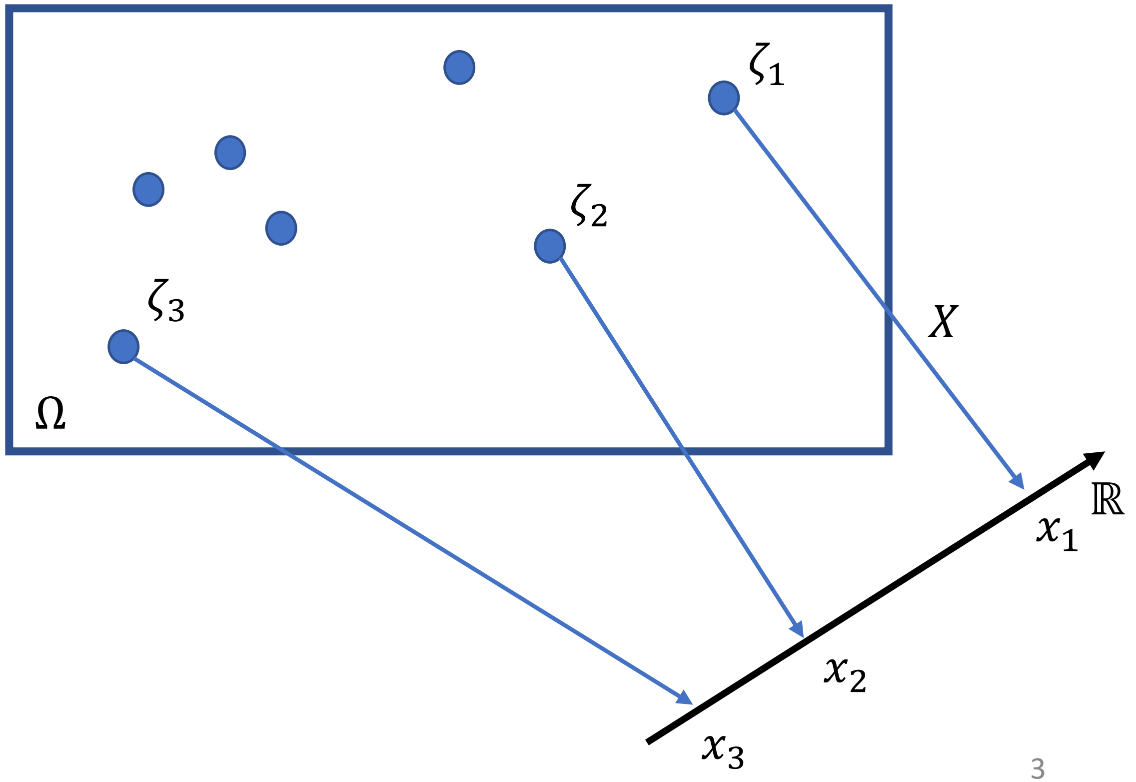
Lecture 5

Chapter 2: Random Variables

Random Variables

Idea: Map outcomes to (real) numbers.

The random variable $X\colon\ \Omega \rightarrow \mathbb{R}$ maps all outcomes from the sample description space to a real number.



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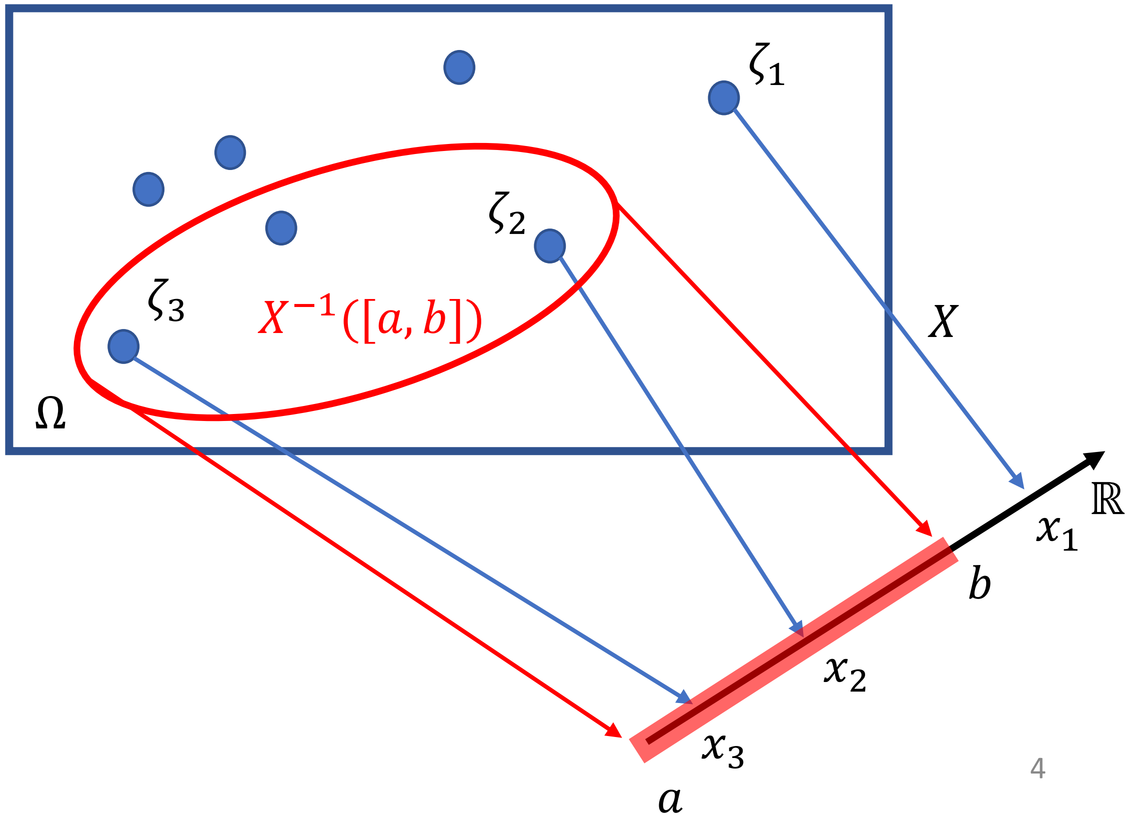
The new sample (description) space is $\mathbb{R}$.

For the events in $\mathbb{R}$ we use the Borel sets, i.e., intervals like $(-\infty, a]$ plus their (countable) unions and intersections, in particular [a,b], (a,b], [a,b), (a,b).

➔ For consistency, the inverse images, $X^{-1}((-\infty, a])$ etc., have to be events in Ω.



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2.2 Random Variables

2.3 Probability Distribution Functions (PDF)

2.4 Probability Density Functions (pdf)

2.5 Continuous, Discrete, Mixed Cases …

2.6 Conditional and Joint PDFs, pdfs

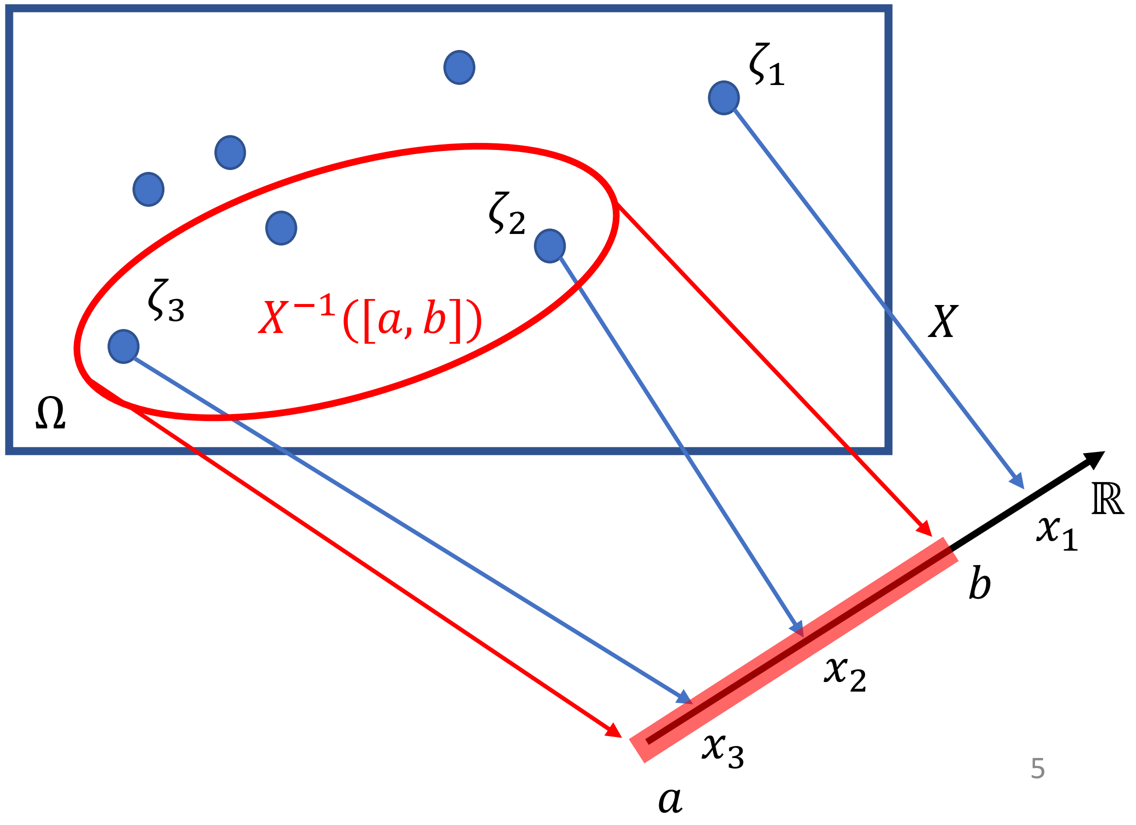
2.7 Failure Rates

Random Variables

New probability: $P_X: \mathcal{B} \to [0,1]$ as induced …

$P_X[B \in \mathcal{B}] = P[X^{-1}(B)]$.

New probability space: $(\mathbb{R}, \mathcal{B}, P_X)$



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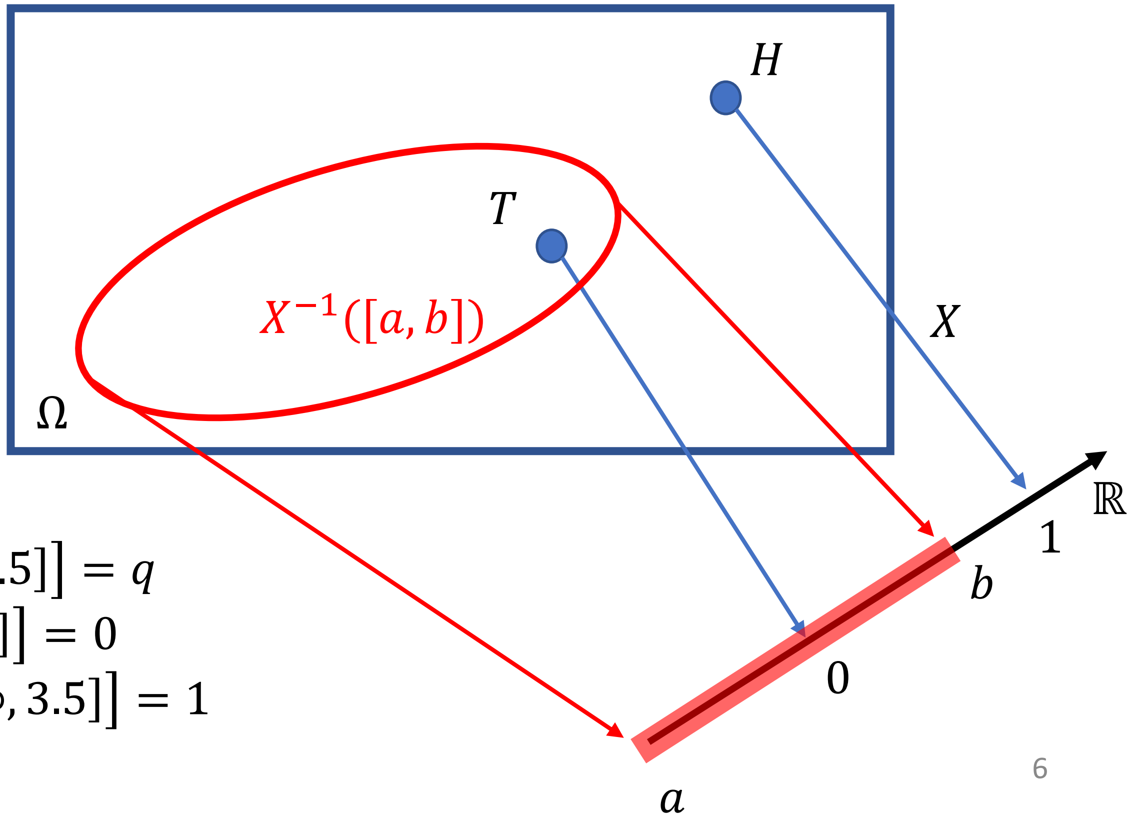
Random Variables

Example:

Toss a coin with $P[H] = p, P[T] = q$

Choose, $X(H) = 1, X(T) = 0$



Induced probabilities:

$$X^{-1}\big((-\infty, 0.5]\big) = \{T\} \Rightarrow P_X\big[(-\infty, 0.5]\big] = q$$
$$X^{-1}\big((-\infty, -2]\big) = \emptyset \Rightarrow P_X\big[(-\infty, -2]\big] = 0$$
$$X^{-1}\big((-\infty, 3.5]\big) = \{H, T\} \Rightarrow P_X\big[(-\infty, 3.5]\big] = 1$$

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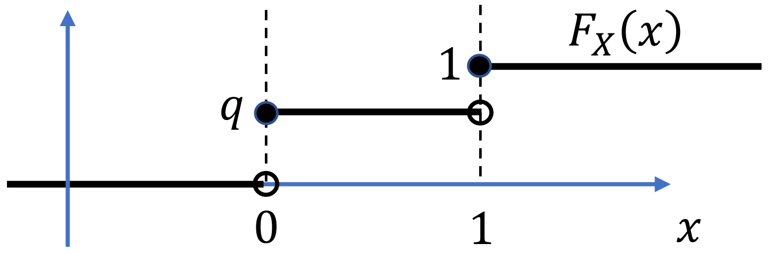
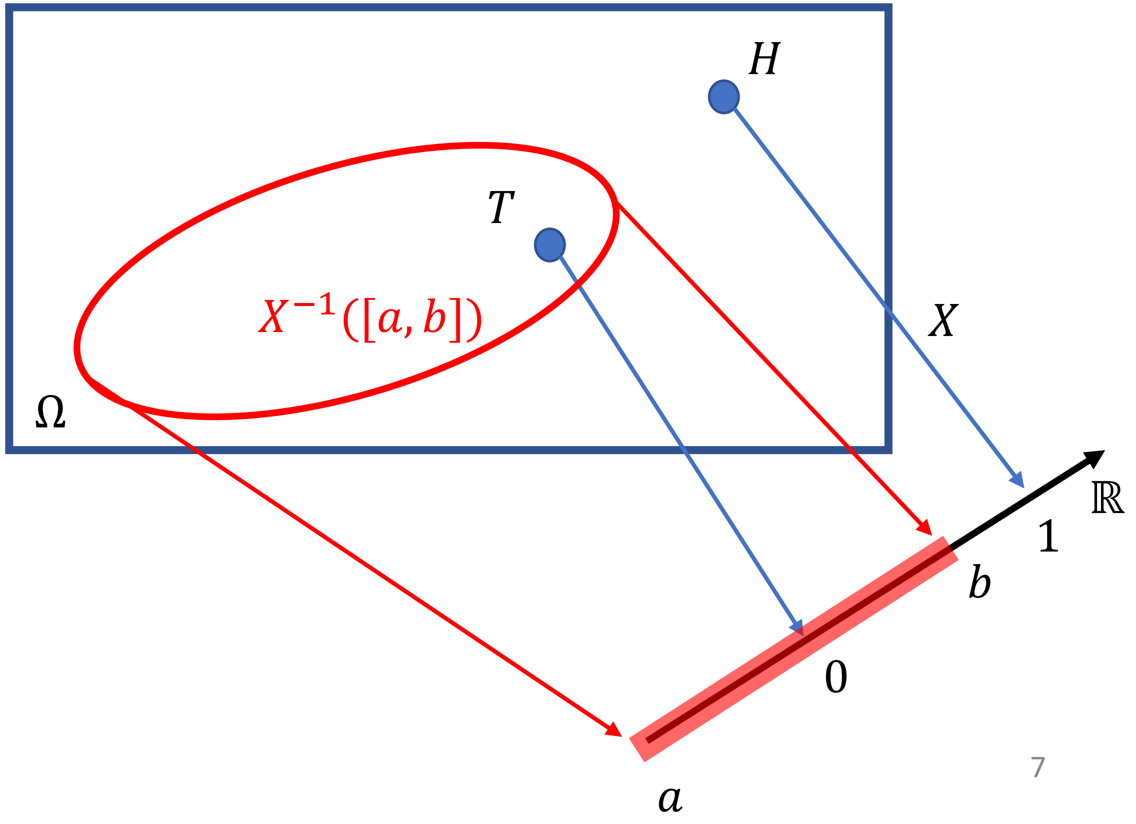
Probability Distribution Functions (PDF)

$$F_X(a) := P_X\big[(-\infty, a]\big] = P\big[X^{-1}\big((-\infty, a]\big)\big] = P[\{\zeta \in \Omega : X(\zeta) \le a]$$

For a coin:





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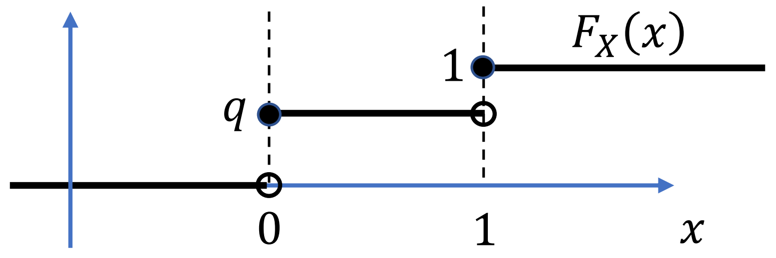
For a coin:



Properties of $F_X(x)$:

(i) $F_X(-\infty) = 0$, $F_X(\infty) = 1$

(ii) $x_1 < x_2 \Rightarrow F_X(x_1) \le F_X(x_2)$, nondecreasing function

(iii) $F_X(x)$ is continuous from the right.

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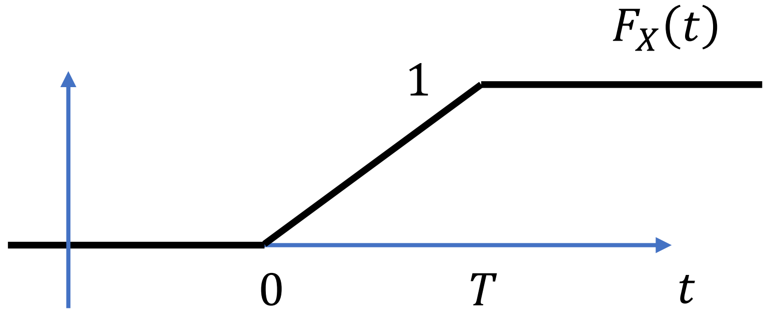
Probability Distribution Functions (PDF)

$$F_X(a) := P_X[(-\infty, a]] = P[X^{-1}((-\infty, a])] = P[\{\zeta \in \Omega : X(\zeta) \leq a]$$

Example: Bus arrival

Suppose, the bus arrives at any time t between 0 and T, equally likely… uniform distribution.



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Probability Distribution Functions (PDF)

Properties of $F_X(x)$:

(i) $F_X(-\infty) = 0,\ F_X(\infty) = 1$

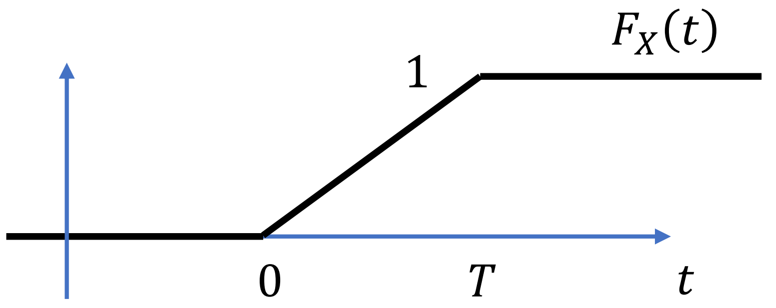
(ii) $x_1 < x_2 \Rightarrow F_X(x_1) \leq F_X(x_2)$, nondecreasing function

(iii) $F_X(x)$ is continuous from the right.

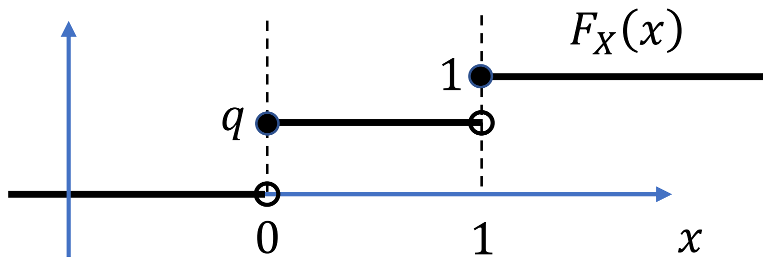
If $F_X(x)$ is continuous (not only from the right), then also:

$$F_X(x) = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} F_X(x-\varepsilon) = F_X(x^-)$$

Think! What‘s that: $P_X[X = x] = F_X(x) - F_X(x^-)$?



Continuous case



DIscontinuous case

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Probability Density Function (pdf)

For a start: Assume that $F_X(x)$ is differentiable …

The basic idea: $f_X(x) = \frac{d}{dx} F_X(x)$

Properties of $f_X(x)$:

(i) $f_X(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f_X(\xi) d\xi = F_X(\infty) - F_X(-\infty) = 1$

(iii) $F_X(x) = \int_{-\infty}^{x} f_X(\xi) d\xi = P[X \leq x]$

(iv) $F_X(x_2) - F_X(x_1) = \int_{-\infty}^{x_2} f_X(\xi) d\xi - \int_{-\infty}^{x_1} f_X(\xi) d\xi = \int_{x_1}^{x_2} f_X(\xi) d\xi = P[x_1 < X \leq x_2]$

Also, $P[x < X \leq x + \Delta x] \approx f_X(x) \Delta x$ if $f_X(x)$ is continuous.

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Continuous, Discrete, Mixed Cases …

Continuous case:

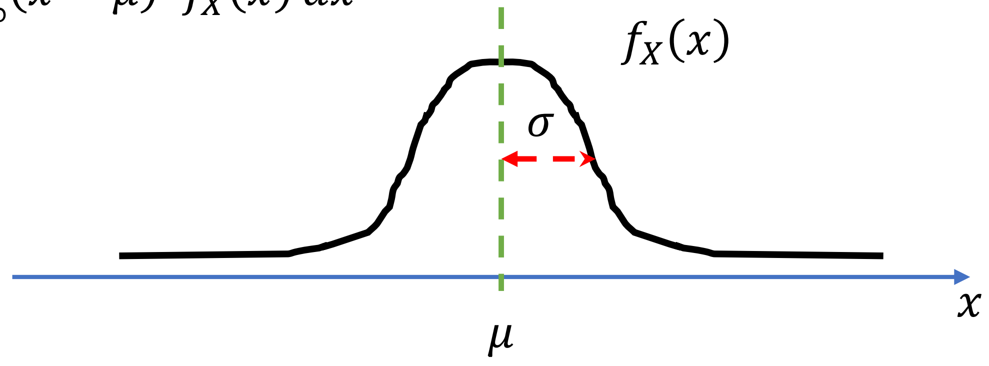
Example (normal or Gaussian pdf):

$$f_X(x) = \frac{1}{\sqrt{2\pi \cdot \sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

Parameters:

mean: $\mu = \int_{-\infty}^{\infty} x f_X(x)\, dx$

variance: $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\, dx$



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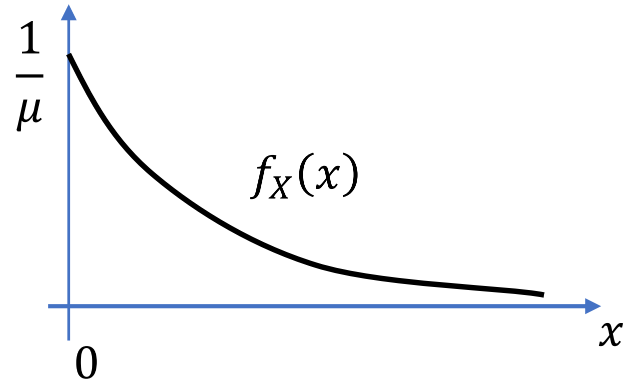
Continuous case:

Example 2 (exponential pdf):

$$f_X(x) = \frac{1}{\mu}\exp\left[-\frac{x}{\mu}\right]u(x)$$

Parameters:

mean: $\mu = \int_{-\infty}^{\infty} x f_X(x)\,dx$



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Continuous, Discrete, Mixed Cases …

Not so smooth PDFs … but still continuous.

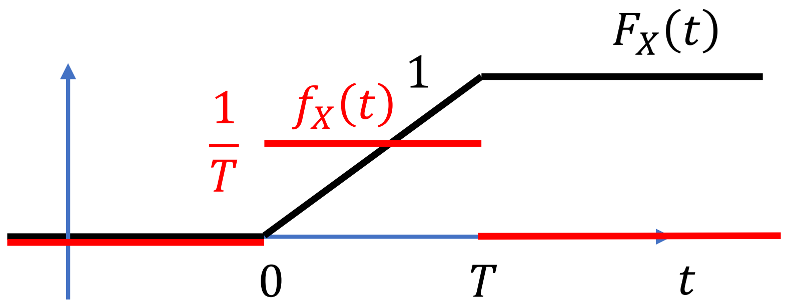
Suppose, $F_X(x)$ is continuous … but derviatives

$f_X(x)$ from left and right are different for a countable set of points.

The basic idea, $f_X(x) = \frac{d}{dx} F_X(x)$, basically survives … the ``few´´ exceptions do not hurt (ignore … or use arbitrary (finite values):

$F_X(x) = \int_{-\infty}^{x} f_X(\xi) d\xi = P[X \leq x]$ still works.

Discuss (uniform pdf):



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Continuous, Discrete, Mixed Cases …

Discrete random variables …

Suppose, $F_X(x)$ looks like a staircase.

The probability is concentrated at certain points.

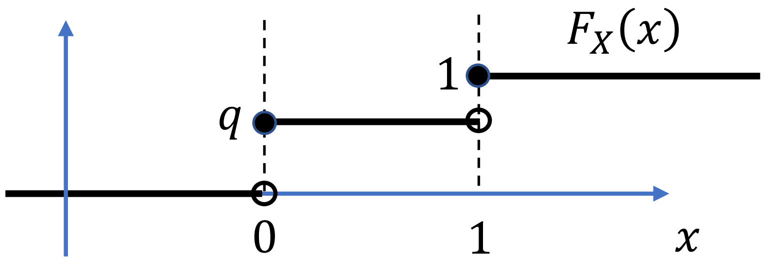
Probability mass function: $P_X(x) = P[X \leq x] - P[X < x]$

(Notice the sloppy notation, here!)

The basic idea, $f_X(x) = \frac{d}{dx} F_X(x)$, survives only in the sense of ``delta functions.´´

$F_X(x) = \int_{-\infty}^{x} f_X(\xi)d\xi = P[X \leq x]$ still works: Use $\int_{-\infty}^{x^+} f_X(\xi)d\xi$ (limit from the right).

Discuss (coin):



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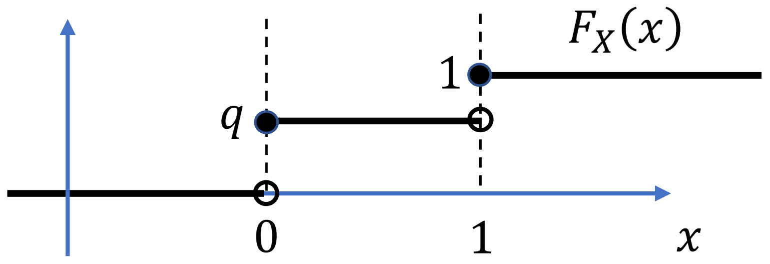
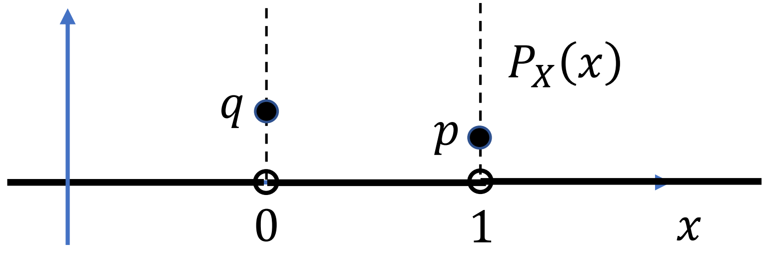
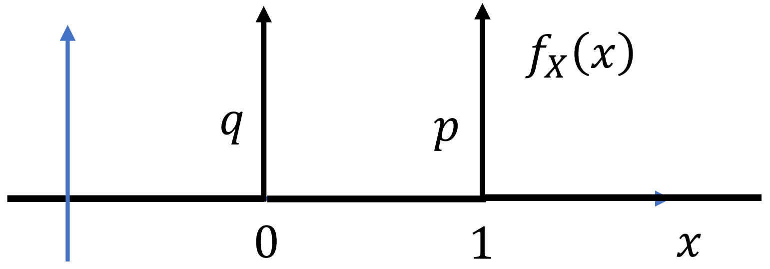
Coin:



Probability mass function: $P_X(x) = P[X \leq x] - P[X < x]$



Density $f_X(x) = \frac{d}{dx} F_X(x)$, … ``delta functions´´



$F_X(x) = \int_{-\infty}^{x^+} f_X(\xi) d\xi = P[X \leq x]$ still works.

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Continuous, Discrete, Mixed Cases …

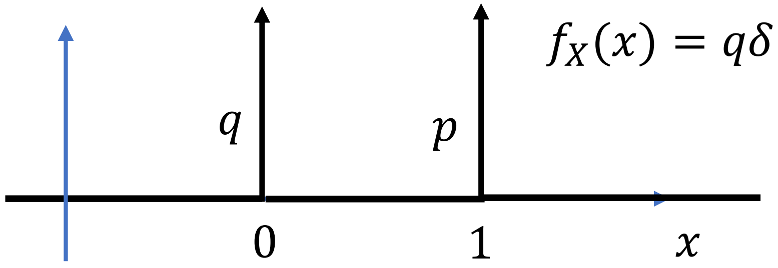
Delta ``functions´´…

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0),$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a),$$

Density $f_X(x) = \frac{d}{dx}F_X(x)$, … ``delta functions´´



$$f_X(x) = q\delta(x) + p\delta(x-1)$$

$F_X(x) = \int_{-\infty}^{x^+} f_X(\xi)d\xi = P[X \leq x]$ still works.

The End

Next time: continue Chp. 2