Elements of Probability

(3.1) Consider a discrete random variable X with the probability mass function given by

$$p_X(x) = \begin{cases} k \cdot 2^x & \text{if } x = -1, 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the value of k.
(b) Find $\mathbb{P}[X \text{ is even}]$.
(c) Find the PMF of the random variables $Y = X^2$ and $Z = X + 1$.

(3.2) Suppose X is a discrete random variable with $\mathbb{E}[X] = 5$ and $\text{Var}[X] = 15$.
(a) Find the values of $\mathbb{E}[X^2]$, $\mathbb{E}[2 - X]$, $\text{Var}[3X + 1]$.
(b) Show that $\mathbb{P}[X \geq 10] \leq \frac{3}{5}$.

(3.3) The number of items arriving daily in the lost-and-found section of Bremen train station has a Poisson distribution with parameter $\lambda = 4$.
(a) Find the probability of the event that on a given day no items arrive.
(b) Knowing that at least one item has arrived today, find the probability that there is at least one more item arriving today?
(c) Let W denote the number of items arriving from January 1st to January 15th. What is $\mathbb{E}[W]$?

(3.4) A monkey has a bag containing 4 apples, 3 bananas and 2 pears. He heats fruits at random until he takes a fruit of a kind he has already had, and then throws away that fruit and the rest of the bag. Let N denote the number of fruits eaten by the monkey.
(a) What are the possible values of N?
(b) Find the probability mass function of N.
(c) Find $\mathbb{E}[N]$ and $\text{Var}[N]$.

(3.5) A startup has developed a new gadget for which the demand is unknown. Assume that the demand for the product denoted by Y has a uniform distribution on the set $\{1, 2, \ldots, 1000\}$. Each sold gadget will bring a profit of 12 Euros and each one made and left unsold will produce a net loss of 3 Euros.
(a) Suppose that the startup decides to produce m units of this gadget. Denote the net income of the startup by Y. Show that

$$X = \begin{cases} 12m & \text{if } Y > m \\ 15Y - 3m & \text{if } Y \leq m \end{cases}$$

(b) Find a closed formula for $\mathbb{E}[X]$.
(c) (Bonus) How many units of this gadgets should be produced to maximize the expected income $\mathbb{E}[X]$?

Hint: For part (b) you may use the following identity useful:

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

(3.6) (Bonus) Consider a coin which lands H with probability p and T with probability $1-p$. The coin is flipped until H shows up for the *second* time. Let N denote the number of required flips.

(a) For warm-up, show that $\mathbb{P}[N=0] = \mathbb{P}[N=1] = 0$, and $\mathbb{P}[N=2] = p^2$.

(b) Show that $\mathbb{P}[N=3] = 2p^2(1-p)$

(c) In general, show that the PMF of N is given by

$$\mathbb{P}[N=k] = (k-1)p^2(1-p)^{k-2}, \qquad k = 2, 3, \ldots$$

Hint: $N = k$ exactly when the k-th flip results in H and all but one of the previous $k-1$ flips result in T.