Elements of Probability

(2.1) Alice throws a fair coin 6 times.

(a) If the outcome of the first throw is heads, find the probability that she will end up with 3 heads at the end.

(b) Given that she ends up with 3 heads at the end, find the probability that the outcome of the first throw was heads.

(2.2) Belgium, Croatia, England and France have reached the semifinal of the World Cup. The betting agency ProFoot has announced their estimated probabilities for the games between these four teams in the following table:

	Belgium	Croatia	England	France
Belgium		0.4	0.6	0.4
Croatia	0.6		0.5	0.3
England	0.4	0.5		0.1
France	0.6	0.7	0.9	

For instance, the entry 0.9 indicates that the ProFoot believes that the probability that France defeats Croatia is 90 percent.

(a) Suppose that the semifinals are played by Belgium vs. England and Croatia vs. France. What is the probability (according to ProFoot) that France wins the cup?

(b) Suppose that the semifinals are played by Belgium vs. Croatia and England vs. France. What is the probability that France wins the world cup?

(c) Suppose that the semifinal games are determined randomly. In other words, assume that the opponent of Belgium is randomly chosen from remaining three teams, each with probability $1/3$, and the other two teams face each other. What is the probability that Belgium makes it to the final?

(2.3) (a) Suppose A, B, C are three events such that $\mathbb{P}[A|\,C] > \mathbb{P}[B|\,C]$ and $\mathbb{P}[A|\,C^c] > \mathbb{P}[B|\,C^c]$. Show that $\mathbb{P}[A] > \mathbb{P}[B]$. What is the interpretation of this fact?

(b) Suppose A, B, C are three events such that $\mathbb{P}[A] > \mathbb{P}[B]$. Is it true that for every event C we have $\mathbb{P}[A|\,C] > \mathbb{P}[B|\,C]$?

(2.4) Suppose M is an integer randomly chosen from the set $\{1, 2 \dots, 8\}$. Once M is chosen, the integer N is chosen from the set $\{1, 2, \dots, M\}$. For instance if it turns out that $M = 5$, then N can take one of the values $1, \dots, 5$, each with probability $1/5$.

(a) Find the probability that $N = 7$.

(b) Find the probability of the event $M = N$.

(2.5) A light is switched on sometime at random between noon and 1 pm. Then it is switched off again on a time randomly chosen between the time it is switched on and 1 pm.

(a) Describe the sample space for this probabilistic situation by a region in the plane.

(b) Compute the probability that the light remains on for more than 15 minutes.

(**2.6**) (Bonus) You have 6 white and 8 black balls to distribute among two boxes. Your friend will randomly choose one of the boxes and takes a ball randomly out of it. How should the balls be distributed so that the probability of getting a black ball is as large as possible.