Lecture 4

Geometric Probability

Laplace model for a finite set of outcomes

$$\mathbb{P}(A) = \frac{\text{number of desired outcomes}}{\text{total number of outcomes}}$$

$$\Omega \text{ finite}, \quad A \text{ finite}, \quad \mathbb{P}(A) = \frac{|A|}{|\Omega|}$$

What is the number of outcomes is not finite?

Laplace model for an infinite set of outcomes

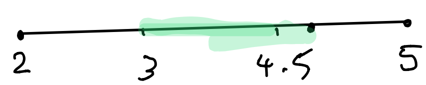
$$\Omega \rightsquigarrow \text{geometric representation}, \quad A \subseteq \Omega$$

Example A number is randomly chosen from the interval $[2,5]$. What is the probability that it lies between $3, 4.5$?

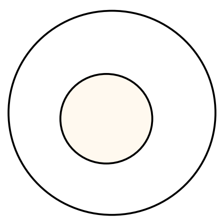
$\Omega = [2,5]$

$A = [3, 4.5]$

$\mathbb{P}(A) = \frac{1.5}{3} = \frac{1}{2}$.



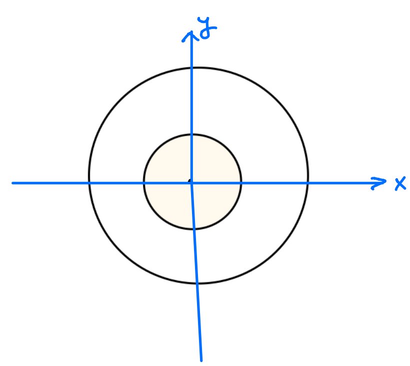
Example (Dart)



A dart is randomly thrown on a dart board, which is a disk of radius 5. Find the probability of the event that it lands at distance at most 2 from the center.

$$\Omega = \{(x,y) : x^2+y^2 \leq 25\}$$

$$A = \{(x,y) : x^2+y^2 \leq 4\}$$



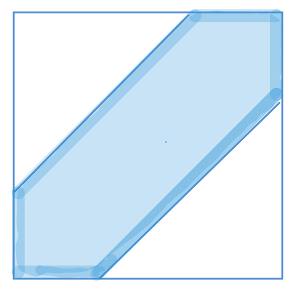
$$\mathbb{P}(A) = \frac{4}{25}$$

Example Alice and Bob have agreed to meet for lunch. Since they don't remember the time, each randomly shows up at a time between 12:00 and 13:00, waits for 20 minutes and then leaves. Find the probability of the event that they meet.

Alice and Bob meet if

$$|A - B| \leq \frac{1}{3}$$

$$-\frac{1}{3} \leq A - B \leq \frac{1}{3}$$



$$\mathbb{P}(M) = \frac{\text{area of}}{\text{total area}} = \frac{1 - \left(\frac{2}{3}\right)^2}{1} = \frac{5}{9} \approx 0.55$$

Lecture 5

Alex throws a fair coin 4 times. For every H he wins 1€ and for every T he wins nothing.

(a) what is the probability that after 4 rounds he ends up with at least 3 € ?

Ω : all possible outcomes = {HHHH, ..., TTTT}

A : at least 3 Heads

Total number of outcomes = 16

number of outcomes with 4 wins = 1

" " " 3 wins $= \binom{4}{3} = 4$

So

$$\mathbb{P}(A) = \frac{5}{16}$$

(b) Prior information : Alex has won the first two games

how to update the probability?

B: the first two games are won

$$\mathbb{P}(A|B) = \frac{3}{4} = \frac{\frac{3}{16}}{\frac{4}{16}}$$

Conditional Probability

Definition Suppose A, B are two events in a sample space Ω and $\mathbb{P}(B) \neq 0$. Then the conditional probability of A given B is defined by

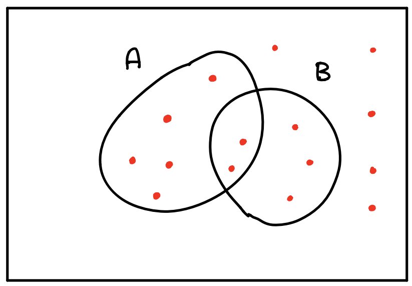
$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

<u>Interpretation in the Laplace model</u>

$|\Omega| = 14$

$|A| = 7$

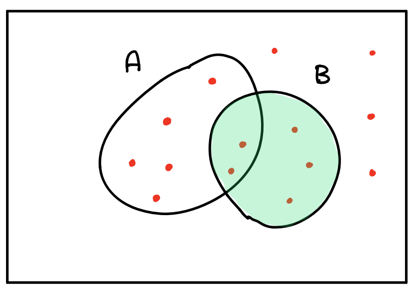
$\mathbb{P}(A) = \frac{7}{14} = 0.50$



$|B| = 5$

$|A \cap B| = 2$

$\mathbb{P}(A|B) = \frac{2}{5} = 0.40$



Example Two fair dice are rolled.

(a) what is the probability of the event that the sum of the resulting numbers is 6?

(b) Given that the first die shows a number less than 5, what is the probability of the event that the sum of resulting numbers is 6?

$\Omega = \{(1,1), (1,2), ---, (6,6)\}$, $\quad |\Omega| = 36$

$$A = \{(1,5),(2,4),(3,3),(4,2),(5,1)\} \qquad |A| = 5$$

So

$$\mathbb{P}(A) = \frac{5}{36}$$

$$B = \{(x,y) : 1 \leq x \leq 4,\ 1 \leq y \leq 6\} \qquad |B| = 24$$

$$A \cap B = \{(1,5),(2,4),(3,3),(4,2)\} \qquad |A \cap B| = 4$$

$$\mathbb{P}(A|B) = \frac{4}{24} = \frac{1}{6}$$

Conditioning

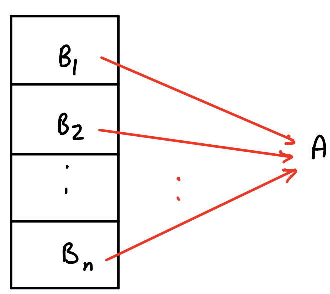
$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)} \Rightarrow \mathbb{P}(A\cap B) = \mathbb{P}(A|B)\cdot\mathbb{P}(B)$$

Suppose sample space is divided up into n disjoint events $B_1, \dots, B_n$. Then

$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}((A\cap B_1)\cup\cdots\cup(A\cap B_n)) \\ &= \mathbb{P}(A\cap B_1) + \cdots + \mathbb{P}(A\cap B_n) \\ &= \mathbb{P}(A|B_1)\,\mathbb{P}(B_1) + \cdots + \mathbb{P}(A|B_n)\,\mathbb{P}(B_n)\end{aligned}$$

Theorem Suppose sample space is divided up into n disjoint events $B_1, \dots, B_n$. Suppose A is another event. Then

$$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A|B_i)\,\mathbb{P}(B_i).$$



Example A box contains 5 red and 6 blue marbles. A marble is randomly drawn from the box, and discarded.

(a) Compute the probability that the discarded marble is blue.

(b) Without knowing the color of the first marble, a second marble is drawn. Find the prob. that the second marble is blue.

(a) B_1: first marble is blue $\qquad$ R_1: first marble is red

$$\mathbb{P}(B_1) = \frac{6}{11}, \quad P(R_1) = \frac{5}{11}$$

B_2: second marble is blue

$$\begin{aligned}
\mathbb{P}(B_2) &= \mathbb{P}(B_2 | R_1)\,\mathbb{P}(R_1) + \mathbb{P}(B_2 | B_1)\,\mathbb{P}(B_1) \\
&= \frac{6}{10} \cdot \frac{5}{11} + \frac{5}{10} \cdot \frac{6}{11} \\
&= \frac{3}{11} + \frac{3}{11} = \frac{6}{11}.
\end{aligned}$$

in

Bayes' Formula Suppose $\Omega = B_1 \cup \cdots \cup B_n$. Then

$$\mathbb{P}(B_i|A) = \frac{\mathbb{P}(B_i \cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A|B_i)\mathbb{P}(B_i)}{\mathbb{P}(A|B_1)\mathbb{P}(B_1) + \cdots + \mathbb{P}(A|B_n)\mathbb{P}(B_n)}$$

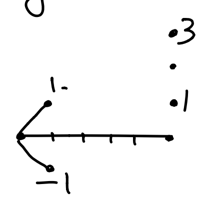
Theorem Suppose $\Omega = B_1 \cup \cdots \cup B_n$ be a splitting of the sample space into sets $B_1, \ldots, B_n$. Then,

$$\mathbb{P}(B_i|A) = \frac{\mathbb{P}(A|B_i)\ \mathbb{P}(B_i)}{\sum_{j=1}^{n} \mathbb{P}(A|B_j)\,\mathbb{P}(B_j)}.$$

Example Bob plays the game of heads and tails: every time a (fair) co land heads, Bob earns 1€, and when it lands tails, he loses 1€. Suppose after 5 rounds of play, Bob has 3€. What is the probability that he has lost the first game?

A: have 3€ after 5 rounds B: lose the first game.

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B)\ \mathbb{P}(B)}{\mathbb{P}(A|B)\mathbb{P}(B) + \mathbb{P}(A|B^c)\ \mathbb{P}(B^c)}$$



$$\mathbb{P}(A|B) = \frac{4}{2^4} = \frac{4}{16} = \frac{1}{4} \qquad \mathbb{P}(B) = 1/2$$

$$\mathbb{P}(A|B^c) = \frac{1}{2^4} = \frac{1}{16} \qquad \mathbb{P}(B^c) = 1/2$$

$$\mathbb{P}(B|A) = \frac{\frac{1}{16}\cdot\frac{1}{2}}{\frac{4}{16}\cdot\frac{1}{2} + \frac{1}{16}\cdot\frac{1}{2}} = \frac{1}{5}.$$