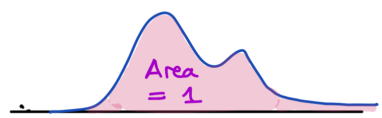
Lecture 11

Review:

A continuous random variables is given by its probability density function: $f(x)$, which satisfies two properties:

1) $f(x) \geq 0$ for every x
2) $\displaystyle\int_{-\infty}^{+\infty} f(x) = 1.$



The probability density function is used to compute probabilities of the form:

$$F_X(t) = \mathbb{P}(X \leq t) = \int_{-\infty}^{t} f(x)\,dx$$

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f(x)\,dx = F_X(b) - F_X(a)$$

$$\mathbb{P}(X \geq a) = \int_a^\infty f(x)\,dx = 1 - F_X(a)$$

Moreover:

The expected value of X is defined by

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x \cdot f(x)\,dx.$$

In many applications we may be interested in the <u>average</u> of a quantity that depends on X.

Example: N the first time that H shows up in successive flips of a coin.

Payoff $= 2^N$

We are interested in $\mathbb{E}[\text{Payoff}] = \mathbb{E}[h(N)]$ $\quad h(x) = 2^x.$

Expectation of a function of a random variable

Suppose X is a continuous random variable with the density function $f_X(x)$.

given also : $h: \mathbb{R} \longrightarrow \mathbb{R}$

wanted : $\mathbb{E}[h(X)]$

Theorem Suppose X is a continuous random variable with the density function $f_X(x)$, and $h: \mathbb{R} \to \mathbb{R}$ a function. Then

$$\mathbb{E}[h(X)] = \int_{-\infty}^{+\infty} h(x) \cdot f_X(x)\, dx$$

Example Suppose X has the density function

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $\mathbb{E}[X^2+1]$, $\mathbb{E}[\frac{1}{X}]$.

$$\mathbb{E}[X^2+1] = \int_0^1 (x^2+1)(2x)\, dx$$

$$= \int_0^1 (2x^3+2x)\, dx = \frac{2x^4}{4} + x^2 \Big|_0^1 = \frac{3}{2}.$$

$$\mathbb{E}[\tfrac{1}{X}] = \int_0^1 \frac{1}{x} \cdot 2x\, dx = \int_0^1 2\, dx = 2x \Big|_0^1 = 2$$

Variance if X is a continuous random variable, then

$$\text{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

Example. Suppose X has a uniform distribution over $[0,1]$. Find $\text{Var}[X]$.

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X] = \tfrac{1}{2}, \quad \mathbb{E}[X^2] = \int_0^1 x^2\, dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$Var[X] = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

<u>Example</u> Consider the random variable X with the PDF given by

$$f_X(x) = \begin{cases} \frac{c}{x^4} & x > 1 \\ 0 & x \le 1 \end{cases}$$

(a) Find the value of (c).
(b) Determine $\mathbb{E}[X]$, and $Var[X]$.

$$\int_1^{\infty} c \cdot x^{-4}\, dx = c \cdot \frac{x^{-3}}{-3}\Big|_1^{\infty} = \frac{c}{3} = 1 \Rightarrow c = 3$$

So

$$f_X(x) = \begin{cases} \frac{3}{x^4} & x > 1 \\ 0 & x \le 1 \end{cases}$$

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x\, f_X(x)\, dx = \int_1^{\infty} \frac{3}{x^3}\, dx = 3\frac{x^{-2}}{-2}\Big|_1^{\infty} = \frac{3}{2}$$

$$\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x)\, dx = \int_1^{\infty} \frac{3}{x^2}\, dx = 3\frac{x^{-1}}{-1}\Big|_1^{\infty} = 3$$

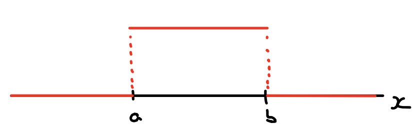
hence

$$Var[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 3 - \frac{9}{4} = \frac{3}{4}.$$

<u>Other important random variables</u>

- Recall that a random variable X has uniform distribution over the interval $[a,b]$ when its probability density function is given by

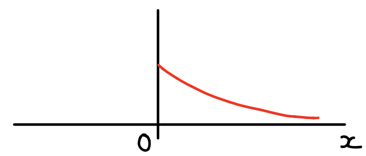
$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$



- exponential random variable with parameter λ

A random variable X has exponential random variable with parameter λ when

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$



$$F_X(t) = \int_0^t \lambda e^{-\lambda x}\, dx = 1 - e^{-\lambda t} \quad t \geq 0$$

$$\mathbb{P}(X > t) = 1 - F_X(t) = 1 - \left(1 - e^{-\lambda t}\right) = e^{-\lambda t}$$

What random quantities in real life can be modeled using exponential random variables?

1) Interarrival time in a Poisson process

e.g. time between two car accidents

number of words between two typoes in a book.

2) lifetime of an object that does not age.

An important property of exponential random variable, Then

$$\mathbb{P}[X \geq s+t \mid X \geq s] = \mathbb{P}[X \geq t].$$

- For an exponential random variable with parameter λ we have

$$\mathbb{E}[X] = \frac{1}{\lambda}, \qquad \mathrm{Var}[X] = \frac{1}{\lambda^2}.$$