Elements of Probability

(4.1) Suppose X is a random variable with normal distribution with $\mu = 2$ and $\sigma = 2$. Compute the following probabilities in terms of the function Φ (the distribution function of a standard normal distribution).
(a) $\mathbb{P}[0 \le X \le 3]$.
(b) $\mathbb{P}[X > 2]$.
(c) $\mathbb{P}[X < 1]$.

Solution. Denote

$$Z = \frac{X-\mu}{\sigma} = \frac{X-2}{2} = \frac{X}{2} - 1.$$

$$\mathbb{P}[0 \le X \le 3] = \mathbb{P}\left[-1 \le Z \le \frac{1}{2}\right] = \Phi(1/2) - \Phi(-1).$$

$$\mathbb{P}[X > 2] = \mathbb{P}[Z > 0] = \frac{1}{2}.$$

$$\mathbb{P}[X < 1] = \mathbb{P}[Z < -1/2] = \Phi(-1/2).$$

In the calculations we have also used the fact that since X is continuous, we have $\mathbb{P}[X = x] = 0$ for every x.

(4.2) A continuous random variables has the density function given by

$$f_X(x) = \begin{cases} k(2-x) & \text{if } 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine the value of k.
(b) Compute the probabilities $\mathbb{P}\left[X > \frac{3}{2}\right]$ and $\mathbb{P}\left[\frac{3}{2} < X \le \frac{7}{4}\right]$.
(c) Compute $\mathbb{E}[X]$.
(d) Compute $\mathrm{Var}[X]$.

Solution. (a) Note that

$$1 = \int_1^2 k(2-x)\, dx = 2kx - \frac{kx^2}{2}\bigg|_1^2 = \frac{k}{2}$$

implying that $k = 2$. This implies that the density function for $1 < x < 2$ is given by

$$f_X(x) = 4 - 2x.$$

Otherwise, $f_X(x) = 0$. Hence for $1 < t < 2$, we have

$$F_X(t) = \int_1^t (4-2x)\, dx = 4x - x^2\bigg|_1^t = 4t - t^2 - 3.$$

From here we have

$$\mathbb{P}\left[X > \frac{3}{2}\right] = 1 - F_X(3/2) = \frac{1}{4}.$$

In the same fashion

$$\mathbb{P}\left[\frac{3}{2} < X \le \frac{7}{4}\right] = F_X(7/4) - F_X(3/2) = \frac{3}{16}.$$

(c) From the definition of the expected value we have

$$\mathbb{E}[X] = \int_1^2 x(4-2x)\ dx = \frac{4}{3}.$$

Similarly, we have

$$\mathbb{E}\left[X^2\right] = \int_1^2 x^2(4-2x)\ dx = \frac{11}{6}.$$

From here we have

$$\mathrm{Var}[X] = \frac{11}{6} - \frac{16}{9} = \frac{1}{18}.$$

(**4.3**) The probability density function of a continuous random variable is given by

$$f_X(t) = \begin{cases} 3t^2 & \text{if } 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

Compute $\mathbb{E}\left[X + \frac{1}{X}\right]$.

Solution. Setting $h(x) = x + \frac{1}{x}$, we have

$$\mathbb{E}\left[X + \frac{1}{X}\right] = \int_0^1 (x + \frac{1}{x})3x^2\ dx = \frac{9}{4}.$$

(**4.4**) A die has been rolled twice. Let X denote the outcome of the first throw and Y denote the smaller of the two outcomes. For instance, if the outcomes are $2, 3$ then $X = 2$ and $Y = 2$ and if the outcomes are $4, 3$ then $X = 4$ and $Y = 3$.

(a) Describe the joint probability mass function of X and Y by drawing a table.

(b) Compute the marginal probability mass functions of X and Y.

(c) What are the possible values of $X - Y$? Compute the probability mass function of $Z = X - Y$ and use it to find $\mathbb{E}[Z]$.

Solution. Note that X can take values $1, 2, 3, 4, 5, 6$ and Y can also take the same values. It is clear that no matter the outcome we have $Y \le X$. Let us compute $\mathbb{P}[X = i, Y = j]$. For this to be possible, we must have $j \le i$. Suppose this condition is satisfied. If $j < i$, then this is only possible if the first die is i and the second one is j. This has probability $1/36$. If $i = j$. Then there is one option for the outcome of the first die (namely i) and exactly $7 - i$ options for the outcome of the second die. Hence

$$\mathbb{P}[X = i, Y = i] = \frac{7-i}{36}.$$

We can now form the table

	$Y=1$	$Y=2$	$Y=3$	Y=4$	$Y=5$	$Y=6$
$X=1$	6/36	0	0	0	0	0
$X=2$	1/36	5/36	0	0	0	0
$X=3$	1/36	1/36	4/36	0	0	0
$X=4$	1/36	1/36	1/36	3/36	0	0
$X=5$	1/36	1/36	1/36	1/36	2/36	0
$X=6$	1/36	1/36	1/36	1/36	1/36	1/36

(**4.5**) A commercial airplane used for a flight from Frankfurt to New York has 590 seats. For this flight 625 tickets have been sold. Assume further that the probability that a passenger does not show up for the flight is 0.04. Denote by N the random variable that counts the number of passengers who show up for the flight.

(a) What are possible values for N? Describe the probability mass function for N.
(b) Show that $\mu := \mathbb{E}[N] = 600$ and $\sigma := \sqrt{\mathrm{Var}[N]} = \sqrt{24} \approx 5$.
(c) Use the Central limit theorem to approximately compute the probability that the flight is full or overbooked.

Solution. It is clear that N can take values $0, 1, \ldots, 625$ as each one of the passengers may show up. Also, it is clear that N has binomial distribution with parameters $n = 625$ and $p = 0.96$. From here we have

$$\mathbb{E}[N] = np = 625 \times 0.96 = 600.$$

We also have

$$\mathrm{Var}[N] = np(1-p) = 24.$$

Hence $\sqrt{\mathrm{Var}[N]} = \sqrt{24} \approx 5$. Let X_i be the Bernoulli random variable which is 1 when passenger i shows up. Hence we have

$$N = X_1 + \cdots + X_{625}.$$

Note that the flight is overbooked precisely when $N > 590$. Note that by the central limit theorem we know that the random variable

$$\frac{X_1 + \cdots + X_{625} - \mathbb{E}[N]}{\sqrt{625}\sigma}$$

can be approximated by the standard normal random variable. Hence we have

$$\mathbb{P}[N > 590] = \mathbb{P}\left[\frac{N-600}{5} > \frac{590-600}{5}\right] = \mathbb{P}[Z > -2] = 1 - \Phi(-2) = 0.97.$$

Here, Z is a standard normal random variable. As the computation shows the flight will be overbooked with probability 0.97.