Jacobs University Bremen
Keivan Mallahi-Karai

Due: October 9, 2017
Assignment 3

Elements of Probability

Solve only 5 out of the following 6 problems.

(3.1) Consider a discrete random variable X with the probability mass function given by

$$p_X(x) = \begin{cases} kx & \text{if } x = 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the value of k.
(b) Compute $\mathbb{E}[X]$ and $\mathrm{Var}[X]$.

(3.2) A continuous random variables has the density function given by

$$f_X(x) = \begin{cases} k(1-x^3) & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine the value of k.
(b) Compute $\mathbb{P}[X > 0]$.

(3.3) Suppose X is a random variable with the uniform distribution over the interval $[1, 2]$ and $Y = X^4$.
(a) Compute $\mathbb{P}[Y \leq t]$ as a function of t. You need to distinguish three different cases.
(b) Find the probability density function of Y and use it to compute $\mathbb{E}[Y]$.

(3.4) Let X be a random variable with the density function

$$f(x) = \lambda \frac{e^{-\lambda|x|}}{2}$$

where $\lambda > 0$.
(a) Verify that f is indeed a probability density function.
(b) Find $\mathbb{P}[-1 < X < 2]$.

(3.5) Suppose X is a random variable with the uniform distribution over $[1, 2]$.
(a) What is the probability density function of X?
(b) Find the probability density function of $Y = e^X$.
(c) Compute $\mathbb{E}[Y]$.
Hint: One of the integrals that show up can be dealt with using integration by parts.

(3.6) Alice and Bob have utility functions given by is given by

$$u_A(x) = x, \qquad u_B(x) = \log x,$$

where the log in in base 2. They are faced with a lottery with n positive outcomes $x_1, \ldots, x_n$, where each can be realized with probability $p = 1/n$.
(a) Compute the expected utility of Alice and Bob. In other words, find $\mathbb{E}[u_A(X)]$ and $\mathbb{E}[u_B(x)]$. The answer must depend on $x_1, \ldots, x_n$.

(b) Let $x_1 = 1, x_2 = 2, x_3 = 4$. What is the smallest amount of C_a (respectively, C_b) such that Alice (respectively, Bob) prefers a sure amount of C_a (respectively, C_b) to the lottery?

(c) (Bonus) Show that independent of the values of $x_1, \ldots, x_n$, Bob is always more risk averse than Alice.