Elements of Probability

Fall semester 2019

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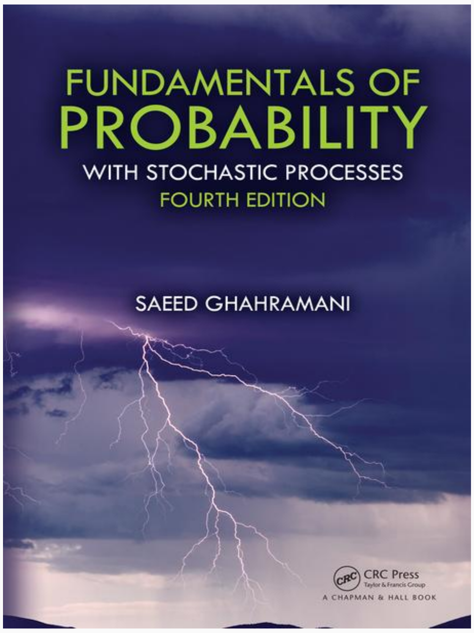
Jacobs University

Basic information

Textbook:
Fundamentals of Probability, with Stochastic Processes
Author: Saeed Ghahramani
Pearson; 3 edition, 2004



- No need to purchase the book. Problem sets will be posted online.
- Multiple copies are reserved at IRC. Ask at the circulation desk.

Basic information

Grading:

Weekly problem sets: 30 percent

Final exam: 70 percent

Webpage of the course: `https://sites.google.com/site/kmallahikarai/teaching/elements-of-probability-2019`

Mailing list: course-eop17 @ lists.jacobs-university.de

Mass subscription on 09.09.2019. You need to subscribe only if you join the course later!

Office hours: Thursdays, 16:00-17:00, Research I, 108.

Probability in the real world

Problems long before the theory (compare with arithmetic and geometry)

Historically: games of chance (today: Las Vegas)

Betting: Sports, politics.

Dealing with *risk* in daily life: medical interventions, insurances, eduction.

Mathematics of financial markets, actuarial science.

Desired randomness (Random number generators, randomized algorithms)

Understanding Causality: Bayesian thinking.

World of *Big data*: signal vs. noise.

Is intuition a good guide?

- Which one is more probable? Getting between 45 and 55 Heads in 100 throws of a fair coin or between 4500 and 5500 Heads in 10000 throws?
- Suppose 100 applicants will be interviewed for a position. How many of them we need to interview before deciding to hire in order to maximize the odds of hiring the best applicant?
- If the outcome of a reliable test for a rare disease is positive, should we get worried?

Answers

The probability of getting between 45 and 55 Heads in 100 throws is less than 0.70.

The probability of getting between 4500 and 5500 Heads in 10000 throws is more than 0.99999

The best strategy is to interview 37 applicants and then choose the first candidate who is better than the rest of them.

We don't need to worry if the test result is positive!

A bit of history: Sources of randomness

- Coin

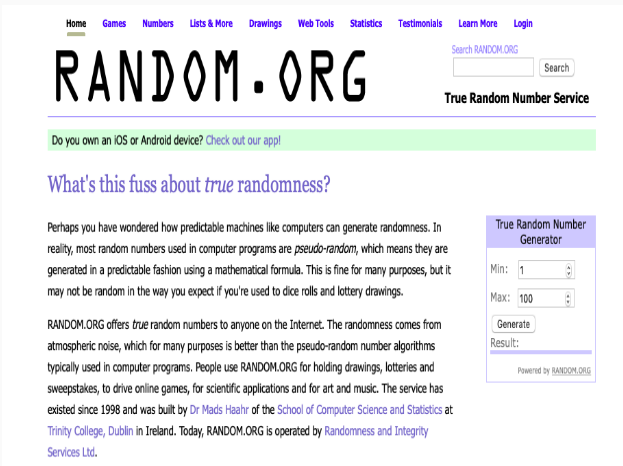
- Die (they come in different shapes)
  The latin word *alia* meaning die is the root of words such as *aleatory* or *aléatoire*.
  The word *hazard* is derived from an Arabic word meaningdie.

A bit of history: Sources of randomness

- Deck of cards

2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠	A♠
2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥	A♥
2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣	A♣
2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦	A♦

- Modern days: random number generators



Ideal fair coin

What do we expect from a coin?

- When thrown, it has a propensity to land Heads or Tails with the same *frequency*.
- (No memory) The outcomes of different throws have nothing to do with another.

Frequentist interpretation of probability

Definition
Probability of an event A denoted by $\mathbb{P}[A]$ is the limit of its relative frequency in a large number of trials.

Example

A coins is thrown n times, with Heads occuring h times. Then we define the probability of Heads as

$$\frac{h}{n}$$

when n is large.

Problems with the frequentist view

1. How many times should we through a coin? How large is large?
2. A coin has landed Heads 50 times in 100 throws, but none in the next 50 throws. Do you count it as fair?
3. More serious: a coin which is fair is thrown 1000 times. The probability of getting 500 heads is very small.

Problems with the frequentist view

Consider the following one-off events.

1. Germany wins the World Cup in 2022.
2. The UK leaves the European Union without a deal.
3. The exchange rate between Euro and Dollar drops below 1.

Question: How to access the probability of each one of these events?

Subjective Probability of Ramsey-De Finetti

Probability $\leftrightarrow$ Degree of belief $\leftrightarrow$ Willingness to place certain kind of bets.

Alice believes that a coin is fair $\leftrightarrow$ Alice is willing to enter the following bet:

Bet: the coin is thrown. If the outcome is H then Alice gets 1 dollar. If the outcome is T, then Alice pays 1 dollar.

Alice believes that H is twice as likely as T (i.e. the probability of H is 2/3) then Alice is likely to enter the following bet:

Bet: the coin is thrown. If the outcome is H then Alice gets 1 dollar. If the outcome is T, then Alice pays 2 dollar.

More generally: If Alice declares an event to have probability p then Alice is willing to enter either one of these bets:

1. Alice pays p dollars, and receives 1 dollar if the event takes place.
2. Alice receives p dollar, and pays 1 dollar if the event takes place.

Alice is indifferent between N dollars if the event takes place and Np dollars upfront.

Betting on Brexit

1. Alice: The UK leaves the EU on 31.10.19 with probability 1/3.
2. Alice: The UK does not leave EU on 31.10.19. with probability 1/3.
   What is the problem with this system?
   Give Alice 1/3 dollar. She will pay 1 dollar if UK leaves.
   Give Alice 1/3 dollar. She will pay 1 dollar if UK does not leave.
   Alice is paid $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$ dollar, but she will pay a sure amount of 1 dollar, which is more!

Betting on Brexit

1. Alice: The UK leaves the EU on 31.10.19 with probability 1/3.
2. Alice: The UK does not leave the EU on 31.10.19. with probability 1/3.
   What is the problem with this system?
   Ask Alice 2/3 dollar. We will pay her 1 dollar if UK leaves.
   Ask Alice 2/3 dollar. We will pay 1 dollar if UK does not leave.
   Alice has paid $\frac{2}{3} + \frac{2}{3} = \frac{4}{3}$ dollar, but she will receive 1 dollar, independent of the outcome.

Probability of certain events

Suppose A is an event that will certainly happen. Then its subjective probability is 1.

Alice is indifferent between p dollars upfront and 1 dollar in case the event happens.

Since the event definitely happens, we must have $p = 1$.

Properties of subjectivist probability theory

Denote by Ω the set of all possible outcomes of an experiment. An assignment of probabilities is an assignment

$$A \to \mathbb{P}[A].$$

We say that the assignment is *coherent* if the following must hold:

(i) $\mathbb{P}[A] \geq 0$, for all $A \subseteq \Omega$;

(ii) $\mathbb{P}[\Omega] = 1$;

(iii) $\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B]$, given that $A \cap B = \varnothing$.

One can show that if the assignments are *not* coherent then one can organize a system of bets with a certain negative payoff.

The equiprobable model of Pascal

Ω: set of all possible outcomes of an experiment. We call Ω the sample space of the experiment. We will assume that Ω is finite,.

Every subset of Ω is called an event.

Definition (Uniform Probability)
Let Ω be a finite set. The uniform probability on Ω is defined by

$$\mathbb{P}[A] = \frac{|A|}{|\Omega|},$$

for every subset $A \subseteq \Omega$.

Intuitively: probability is the ratio of favorable outcomes to all outcomes.

The equiprobable model of Pascal: examples

Example
A coin is flipped. The sample space is

$$\Omega = \{H, T\}.$$

$$\mathbb{P}\left[\{H\}\right] = \frac{1}{2}.$$

Example
For two coins:

$$\Omega = \{HH, HT, TH, TT\}$$

$$\mathbb{P}\left[\{HH, HT, TH\}\right] = \frac{3}{4}.$$

One can generalize this to more than two coins:

If the experiment consists of throwing n coins, then we consider sequences of length n as the sample space.

Examples of events

Example
A die is rolled. What is the probability that the outcome is an even number.

$$\Omega = \{⚀, ⚁, ⚂, ⚃, ⚄, ⚅\}.$$

The event A is defined by

$$A = \{⚁, ⚃, ⚅\}.$$

$$\mathbb{P}\left[A\right] = \frac{3}{6} = 0.5$$

Let B be the event that the outcome is at most 4. Then

$$B = \{⚀, ⚁, ⚂, ⚃\}.$$

$$\mathbb{P}\left[B\right] = \frac{4}{6} = 0.67.$$

Example

A random card is dealt from a well-shuffled deck of cards. What is the probability that the card is (a) an ace (b) red (c) an ace or red.

2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠	A♠
2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥	A♥
2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣	A♣
2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦	A♦

$$\mathbb{P}[A] = \frac{4}{52} = \frac{1}{13}.$$

Example

A random card is dealt from a well-shuffled deck of cards. What is the probability of the event the card is (a) an ace (b) red (c) an ace or red.

2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠	A♠
2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣	A♣
2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥	A♥
2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦	A♦

$$\mathbb{P}[R] = \frac{26}{52} = \frac{1}{2}.$$

Example

A random card is dealt from a well-shuffled deck of cards. What is the probability of the event the card is (a) an ace (b) red (c) an ace or red.

2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠	A♠
2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣	A♣
2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥	A♥
2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦	A♦

$$\mathbb{P}[A \cup R] = \frac{4 + 36 - 2}{52} = \frac{38}{52}.$$

The union law

Theorem (The Union law)
Suppose A and B are two events. Then

$$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B].$$

Proof.
Proof using Venn diagram:

□

Counting

Many problems in probability boil down to finding out how many elements are in a set. This turns out to be an art, but there are also methods.

Example

A 3-digit number x is chosen randomly. What is the probability that x is at least 200.

Sample space is $\Omega = \{100, 101, \ldots, 999\}$.

$$1, 2, \ldots, 999$$

The total number is

$$999 - 99 = 900.$$

If $a < b$ are integers, the integers in the list

$$a, a+1, \ldots, b$$

are

$$1, 2, \ldots, (a-1), a, a+1, \ldots, b-1, b$$

So their number is

$$b - (a-1) = b - a + 1.$$

Counting

It follows that

$$|\Omega| = 999 - 100 + 1 = 900.$$

$$|A| = 999 - 200 + 1 = 800.$$

$$\mathbb{P}[A] = \frac{800}{900} = \frac{8}{9}.$$

Counting

Example

A 3-digit number x is chosen randomly. Find the probability of the event that the given number is even.

To write this number the following three boxes have to be filled:

Use a decision tree to count.

$$|A| = 9 \times 10 \times 5 = 450.$$

$$|\Omega| = 9 \times 10 \times 10 = 900.$$

$$\mathbb{P}[A] = \frac{450}{900} = \frac{1}{2}.$$

Counting

Example

A 3-digit number x is chosen randomly. Find the probability of the event that the sum of the digits of the given number is even.

$$\mathbb{P}[A] = \frac{450}{900} = \frac{1}{2}.$$

A different approach

To write a 3-digit number N, we need to fill in these boxes:

When is the sum of the digit of N an even number?

Factorials

In how many ways can one order a set with n elements?

$$n = 2 : \qquad 12, 21$$

$$n = 3 : \qquad 123, 132, 231, 213, 312, 321$$

For general n the total number is equal to

$$n! := n(n-1)(n-2)\cdots 1.$$

Properties of $n!$

The sequence $n!$ grows very quickly. In fact it grows faster than any exponential function.

Theorem (Stirling's formula)

For large values of n, one can use the following asymptotic formula to approximate $n!$:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

A neat trick

A fair die is thrown 4 times. What is the probability that the score 5 appears at least once.

The sample space is

$$\Omega = \{(x_1, x_2, x_3, x_4) : 1 \leq x_i \leq 6\}.$$

$$A = \Omega = \{(x_1, x_2, x_3, x_4) : x_i = 6 \text{ for some } i.\}.$$

The event A^c, indicating that A did not happen consists of those outcomes that consist only of $1, 2, 3, 4, 6$. So,

$$\mathbb{P}[A^c] = \frac{5^4}{6^4} = 0.48.$$

Hence

$$\mathbb{P}[A] = 1 - 0.48 = 0.52.$$

Counting the number of subsets of a given set

Consider a set A with n elements.

The total number of subsets of A is equal to 2^n:

The total number of subsets with k elements is given by

$$\begin{aligned}\binom{n}{r} &= \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} \\ &= \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}\frac{(n-r)!}{(n-r)!} \\ &= \frac{n!}{r!(n-r)!}\end{aligned} \tag{1}$$

Tie breaking

Example
A committee with an odd number of members (say, $N = 2n + 1$) is voting to choose one of the plans. Assume that the two plans are equally popular and each committee member votes for one plan with probability $1/2$. What is the probability that the last vote is a tie-breaker?

This happens when the first $2n$ votes are split equally between the candidates. Hence:

$$\mathbb{P}[A] = \binom{2n}{n}\frac{1}{4^n} \approx \frac{1}{\sqrt{\pi n}},$$

For $N = 1001$ then the probability is approximately $p = 0.018$.