Elements of Probability

(4.1) Consider a discrete random variable X with the probability mass function given by

$$p_X(x) = \begin{cases} k|x| & \text{if } x = -1, 1, -2, 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the value of k.
(b) Compute $\mathbb{E}[X]$ and $\text{Var}[X]$.

Solution. We have

$$1 = \sum_{x=1}^{4} p_X(x) = 6k$$

which implies that $k = \frac{1}{6}$. We have thus the following table:

X	-2	-1	1	2
$\mathbb{P}[X = x]$	1/3	1/6	1/6	1/3
X^2	4	1	1	4

(b)

$$\mathbb{E}[X] = \sum_{x=1}^{4} \frac{1}{6} x|x| = 0.$$

In order to compute the variance, first we compute

$$\mathbb{E}\left[X^2\right] = 4.\frac{2}{3} + 1.\frac{1}{3} = 3.$$

Hence

$$\text{Var}[X] = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 = 3.$$

(4.2) A continuous random variables has the density function given by

$$f_X(x) = \begin{cases} k(1-x) & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine the value of k.
(b) Compute the probabilities $\mathbb{P}\left[X > \frac{1}{2}\right]$ and $\mathbb{P}\left[\frac{1}{2} < X \leq \frac{2}{3}\right]$.
(c) Compute $\mathbb{E}[X]$.
(d) Compute $\text{Var}[X]$.

Solution. (a) Note that

$$1 = \int_0^1 k(1-x)\, dx = kx - \frac{kx^2}{2}\Big|_0^1 = \frac{k}{2}$$

implying that $k = 2$.

From here we have

$$\mathbb{P}\left[X > \frac{1}{2}\right] = \int_0^1 2(1-x)\ dx = 2x - x^2\Big|_{1/2}^{1} = \frac{1}{4}.$$

$$\mathbb{P}\left[\frac{1}{2} < X \leq \frac{2}{3}\right] = 2x - x^2\Big|_{1/2}^{2/3} = \frac{5}{36}.$$

(c) From the definition of the expected value we have

$$\mathbb{E}[X] = \int_0^1 2x(1-x)\ dx = \frac{1}{3}.$$

Similarly, we have

$$\mathbb{E}\left[X^2\right] = \int_0^1 2x^2(1-x)\ dx = \frac{1}{6}.$$

From here we have

$$\operatorname{Var}[X] = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}.$$

(**4.3**) Suppose X is a continuous random variable with the uniform distribution over the interval $[1,2]$ and $Y = X^2$.

(a) Compute $\mathbb{P}[Y \leq t]$ as a function of t. You need to distinguish three different cases.

(b) Find the probability density function of Y and use it to compute $\mathbb{E}[Y]$.

Solution. Since $1 \leq X \leq 2$, we have $1 \leq Y \leq 4$. Clearly for $t < 1$ we have $F_Y(t) = 0$ and for $t > 4$ we have $F_Y(t) = 1$. For $1 \leq t \leq 4$ we have

$$F_Y(t) = \mathbb{P}\left[X^2 \leq t\right] = \mathbb{P}\left[X \leq t^{1/2}\right] = \begin{cases} 0 & \text{if } t < 1 \\ t^{\frac{1}{2}} - 1 & \text{if } 1 \leq t \leq 4. \\ 1 & \text{if } t > 4 \end{cases}$$

(b) In order to compute the probability density function of Y, we differentiate $F_Y(t)$:

$$f_Y(t) = \begin{cases} 0 & \text{if } t < 1 \\ \frac{1}{2}t^{-1/2} & \text{if } 1 \leq t \leq 4 \\ 0 & \text{if } t > 4 \end{cases}$$

From here we have

$$\mathbb{E}[Y] = \mathbb{E}\left[X^2\right] = \int_1^2 t^2\ dt = \frac{t^3}{3}\Big|_1^2 = \frac{7}{3}.$$

(**4.4**) Let X be a random variable with the density function

$$f(x) = \begin{cases} \lambda x^{-3} & \text{if } x > 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda > 0$.

(a) Compute the value of λ.

(b) Find $\mathbb{P}[-1 < X < 2]$.

(c) Compute $\mathbb{E}[X]$.

Solution. (a) From the definition of probability density functions we have

$$1 = \int_1^{\infty} \lambda x^{-3}\, dx = \lambda \frac{x^{-2}}{-2}\bigg|_1^{+\infty} = \frac{\lambda}{2}.$$

From here we obtain $\lambda = 2$.

(b) Note that

$$\mathbb{P}[-1 < X < 2] = \mathbb{P}[1 < X < 2] = \int_1^{2} 2x^{-3}\, dx = -x^{-2}\bigg|_1^{2} = \frac{3}{4}.$$

(c) Finally we have

$$\mathbb{E}[X] = \int_1^{\infty} x f_X(x)\, dx = \int_1^{\infty} 2x^{-2}\, dx = -2x^{-1}\bigg|_1^{\infty} = 2.$$

(4.5) The joint probability mass function of discrete random variables X and Y is given by

$$p_{X,Y}(x,y) = \begin{cases} kxy & \text{if } 1 \le x, y \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine the value of constant k.

(b) Determine the probability mass functions of X and Y.

(c) Find $\mathbb{P}[X \ge Y]$.

Solution. We know that $\sum_{x,y} p(x,y) = 1$. A simple computation shows that this is equivalent to

$$k = \frac{1}{36}$$

This provides the following table:

	Y=1	Y=2	Y=3
X=1	1/36	2/36	3/36
X=2	2/36	4/36	6/36
X=3	3/36	6/36	9/36

For part (b), write

$$p_X(x) = \frac{1}{36}(x + 2x + 3x) = \frac{x}{6}.$$

Hence we have

$$p_X(x) = \begin{cases} 1/6 & \text{if } x = 1 \\ 1/3 & \text{if } x = 2 \\ 1/2 & \text{if } x = 3 \end{cases}$$

By symmetry, the same formula applies for $p_Y(y)$.

For part (c), note that the probability in question corresponds to the red cells in the table below:

	Y=1	Y=2	Y=3
X=1	1/36	2/36	3/36
X=2	2/36	4/36	6/36
X=3	3/36	6/36	9/36

Adding the respective probabilities we obtain

$$\mathbb{P}[X \geq Y] = \frac{25}{36}.$$