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Preparation for the final exam: fall 2017

Elements of Probability

1. Basics

(FP.1) A 3-digit positive integer N is randomly chosen. Compute the probability of the event that
(a) N is divisible by 3.
(b) N is divisible by 3 if its leftmost digit is 1.

(FP.2) Give an example of two events such that $\mathbb{P}[A \cap B] < \mathbb{P}[A]\,\mathbb{P}[B]$.

(FP.3) Suppose A and B are two event with $\mathbb{P}[A] = \frac{2}{3}$ and $\mathbb{P}[B] = \frac{1}{2}$. Is it possible that $A \cap B = \emptyset$? Why?

(FP.4) Suppose A, B are two different events. For each one of the following statements, decide whether it is always true or not. Justify your answer in each case:
(a) $\mathbb{P}[A \cap B] \leq \mathbb{P}[A|B]$.
(b) $\mathbb{P}[A|B] = \mathbb{P}[B|A]$.
(c) $\mathbb{P}[A \cap B|B] = \mathbb{P}[A|B]$.

(FP.5) Each incoming student to a college has to take two tests. Let P_1 denote the event that a student passes the first test and P_2 the event that the student passes the second test. Let Q be the event that a student is qualified (according to a certain criterion). Suppose

$$\mathbb{P}[P_1|Q] = 0.8, \quad \mathbb{P}[P_2|Q] = 0.75, \quad \mathbb{P}[P_1^c|Q^c] = 0.80, \quad \mathbb{P}[P_2^c|Q^c] = 0.90.$$

(a) Describe in words the meaning of each one of the above probabilities.
(b) Assume that 90 percent of the students are qualified. Find the probabilities $\mathbb{P}[Q^c|P_1]$ and $\mathbb{P}[Q^c|P_2]$.

(FP.6) A four-bit string (a string of 0s and 1s of length 4) is randomly chosen. Find the probability of the event that
(a) It contains at least one 1 and at least one 0.
(b) It has an event number of 1s.
(c) It has no consecutive 1s.

(FP.7) Suppose A and B are independent events and $A \subseteq B$. Show that $\mathbb{P}[A] = 0$ or $\mathbb{P}[B] = 1$.

(FP.8) Two numbers x and y are randomly and independently chosen in the interval $[0, 1]$. Find the probability of the event that
(a) $x \leq y$.
(b) $2x \leq y$.
(c) $2x \leq y$ if $x \leq y$.

(FP.9) A fair coin is tossed n times, Show that the events "at least two heads" and "one or two tails" are independent if $n = 3$, but dependent for $n = 4$.

(FP.10) 3 urns are on a table. The first one contains one red and two magenta balls, the second one contains 2 red and 1 magenta balls and the last one contains 3 red and no magenta balls. An urn is picked at random and a ball is drawn from this urn at random.

(a) Find the probability that the ball taken out is magenta.
(b) If the ball taken out is magenta, find the probability of the event that the urn chosen is the first urn.

(FP.11) From the set $\{1, 2, \ldots, 15\}$ four numbers have been randomly chosen. Find the probability of the events that

(a) The smallest chosen number is 6.
(b) The smallest chosen number is 6 and the largest is 14.
(c) All the chosen numbers are odd.

2. Random Variables

(FP.12) The distribution function for a continuous random variable X is given by

$$F_X(t) = \begin{cases} 1 - \frac{1}{x^3} & \text{if } x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the probability density function $f_X(x)$ of X.
(b) Calculate $\mathbb{E}[X]$.
(c) Determine the probability $\mathbb{P}[2 < X < 4]$.
(d) Sketch the graphs of f_X and F_X.

(FP.13) Identify which one of the following functions can be the distribution function of a random variable.

(a) $F(x) = e^{-x}$.
(b) $F(x) = 1 - \frac{1}{x}$.
(c) $F(x) = x^2$.

(FP.14) The probability density function of a random variable X is given by

$$f_X(x) = \begin{cases} cx(1-x) & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the value of c.
(b) Find the distribution function of X.
(c) What is the probability of the event that $X \geq \frac{2}{3}$.
(d) Compute $\mathbb{E}[X]$ and $\mathrm{Var}[X]$.

(FP.15) Suppose X is randomly chosen from the interval $[-1, 1]$ according to the uniform distribution. Set $Y = |X|$.

(a) Find the distribution function of Y.
(b) Find the density function of Y and compute $\mathbb{E}[Y]$.

(FP.16) Let X have the exponential distribution with parameter 1. Calculate the density function of $Y = X^2$

3. Expected Value and Variance

(FP.17) Suppose X has the density function given by

$$f_X(x) = \begin{cases} cx^2 & \text{if } 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the value of c.
(b) Find $\mathbb{E}[X^n]$ for any integer $n \geq 1$.

(FP.18) Let the probability density function of X be given by

$$f_X(x) = \begin{cases} |x-1| & \text{if } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find $\mathbb{E}\left[X^2 + X\right]$.

(FP.19) Let $a_1, \ldots, a_n$ be n real numbers. Let $X_1, \ldots, X_n$ be independent identically distributed random variables with the distribution $\mathbb{P}[X_i = 1] = \mathbb{P}[X_i = -1] = \frac{1}{2}$. Set $S = a_1X_1 + \cdots + a_nX_n$.
(a) Shows that $\mathbb{E}[S] = 0$.
(b) Compute $\mathrm{Var}[S]$ in terms of $a_1, \ldots, a_n$.

(FP.20) Alice has forgotten her password which is an 8-digit string of 0s and 1s. She tries possible passwords completely at random, discarding the unsuccessful ones. What is the expected number of attempts needed to find the correct password?

(FP.21) 12 marbles are randomly put into 6 boxes. Let X denote the number of boxes that contain at least one marble. What is $\mathbb{E}[X]$?
Hint: This is essentially the same as the elevator problem discussed in class.

(FP.22) (not for the exam, just for fun!) A box contains n ropes. At each step, two rope ends (which may or may not belong to the same rope) are randomly chosen and tied together and then put back into the box. This process is continued until there are no free ends left, i.e., the box contains only loops. Let L denote the number of loops generated by this process.
(a) Let L_j be the Bernoulli random variable which is equal to 1, when in round j a loop is made. Show that $\mathbb{P}[L_j = 1] = \frac{1}{2n-2j+1}$.
(b) Show that

$$\mathbb{E}[L] = 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}.$$

(FP.23) (also not for the exam!) 50 green and 50 red marbles are given. You can divide the marbles between two boxes as you wish. A box is randomly selected and from the selected box a marble is randomly drawn. If the chosen marble is red you win 1 million dollar. How should the marbles be distributed between two boxes so that the winning probability is as large as possible?

(FP.24) A point is selected at random and uniformly from a disk of radius 1. Let R denote the distance of the point from the center of the disk.
(a) Show that $\mathbb{P}[R \leq t] = t^2$.
(b) Find the probability density function of R.
(c) Compute $\mathbb{E}[R]$ and $\mathrm{Var}[R]$.

4. Special Distributions

(FP.25) The grades for a certain exam are normally distributed with mean 67 and variance 64. What percentage of students get at least 90 or at most 80? The answer can be given in terms of the function Φ.

(FP.26) The number of claims made in one week at a small insurance company is a Poisson random variable with parameter $\lambda = 10$.
(a) What is the probability that there are at most 5 claims made in a given week?
(b) What is the average number of claims made in three weeks?

(FP.27) A group of 11 people are supposed to make a choice. Suppose that each person's judgement is good enough to make the correct choice with probability 60 percent.
(a) Suppose that the choice is made by the majority vote among a subgroup of 3 people. Find the probability that the correct choice is made.
(b) Repeat part (a) with 7 and 11 instead of 3.
(c) What do you conclude?

5. Joint Distributions and Independence

(FP.28) The joint probability mass function of random variables X and Y is given by

$$p(x_1, x_2) = \begin{cases} \frac{x_1 x_2 + 1}{13} & \text{if } x_1 = 1, 2; \quad x_2 = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Specify the probability mass function of X_1 and X_2.
(b) Are X_1 and X_2 independent? Are they identically distributed? Explain.
(c) Find the probability of the event that $X_1 + 2X_2 \geq 3$.
(d) Find the probability of the event that $X_1 X_2 > 2$.

(FP.29) A box contains 5 green, 6 blue, and 3 red balls. Two balls are randomly chosen. Let X denote the number of green and Y the number of blue balls among the chosen balls.
(a) Compute the joint probability mass function of X and Y.
(b) Use part (a) to compute the probability mass functions of X and Y.
(c) Compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.
(d) Compute $\mathrm{Cov}(X, Y)$.

6. The Central limit theorem

(FP.30) Consider 1000 light bulbs, whose life time has an exponential distribution with expected value of 5 days. Approximate using the central limit theorem, the probability that the total lifetime of the bulbs exceeds 5200 days. The answer can be expressed using $\Phi(t) = \mathbb{P}(N(0,1) < t)$ where $N(0,1)$ is a standard normal distribution. Note that for an exponential random variable X with parameter λ the expected value and the variance are given by $\mathbb{E}[X] = \lambda^{-1}$ and $\mathrm{Var}[X] = \lambda^{-2}$.