JMTS-12: Probability and Random Processes

Fall 2020

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Textbook:

Henry Stark & John W. Woods

Probability and Random Processes with Applications to Signal Processing

Chapters 1-4 … parts of 5+6 if time permits

Main platform: campusnet … course page !!!

For the online meetings, use headsets if possible…

Streaming might also help in the lecture halls.

Watch out for messages from your TAs !!!

Lecture 4

Chapter 1

1.11 Normal Approximation …

1) Use Stirling's formula:

$$n! \approx \left({}^{n}/_{e}\right)^{n}\sqrt{2\pi n}$$

2) Substitute:

$$k = np + \Delta k$$
$$n - k = nq - \Delta k$$

3) For the first two factors: Take logarithms

4) Use Taylor for small $\frac{\Delta k}{k}, \frac{\Delta k}{n-k}$:

$$\ln(1+\varepsilon) \approx \varepsilon - \frac{\varepsilon^2}{2}$$

Normal Approximation to the Binomial Law

Normal Approximation for $k \approx np$:

$$b(k;n,p) = \binom{n}{k} p^k q^{n-k} \approx \frac{1}{\sqrt{2\pi \cdot npq}} \exp\left[-\frac{(k-np)^2}{2npq}\right]$$

Why?

Consider $n \to \infty$, $k \to \infty$, $(n-k) \to \infty$ but p, q fixed.

Use: $k = np + \Delta k$, $n - k = nq - \Delta k$, with $\frac{\Delta k}{k}, \frac{\Delta k}{n-k} \to 0$

$$\binom{n}{k} p^k q^{n-k} \approx \frac{\left({}^{n}/_{e}\right)^{n}}{\left({}^{k}/_{e}\right)^{k}\left({}^{(n-k)}/_{e}\right)^{n-k}} \frac{\sqrt{2\pi n}}{\sqrt{2\pi k}\sqrt{2\pi(n-k)}} p^k q^{n-k}$$

$$= \frac{n^n}{k^k (n-k)^{n-k}} p^k q^{n-k} \frac{\sqrt{n}}{\sqrt{2\pi k(n-k)}}$$

$$\approx \left(\frac{np}{k}\right)^k \left(\frac{nq}{n-k}\right)^{n-k} \frac{\sqrt{n}}{\sqrt{2\pi \cdot np \cdot nq}}$$

$$= \left(\frac{k - \Delta k}{k}\right)^k \left(\frac{n-k+\Delta k}{n-k}\right)^{n-k} \frac{1}{\sqrt{2\pi \cdot npq}}$$

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Now use

$$\ln\left[\left(\frac{k-\Delta k}{k}\right)^k \left(\frac{n-k+\Delta k}{n-k}\right)^{n-k}\right]$$

$$= \mathrm{k} \cdot \ln\left(1 - \frac{\Delta k}{k}\right) + (n-k) \cdot \ln\left(1 + \frac{\Delta k}{n-k}\right)$$

$$\approx k \cdot \left(-\frac{\Delta k}{k} - \frac{\Delta k^2}{2k^2}\right) + (n-k) \cdot \left(\frac{\Delta k}{n-k} - \frac{\Delta k^2}{2(n-k)^2}\right)$$

$$= -\frac{\Delta k^2}{2}\left(\frac{1}{k} + \frac{1}{n-k}\right)$$

$$= -\frac{\Delta k^2}{2}\frac{n}{k(n-k)} \approx -\frac{\Delta k^2}{2}\frac{n}{np \cdot nq} = -\frac{\Delta k^2}{2npq}$$

To obtain

$$\boxed{\binom{n}{k} p^k q^{n-k} \approx \frac{1}{\sqrt{2\pi \cdot npq}} \exp\left[-\frac{(k-np)^2}{2npq}\right]}$$

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Find: $\boxed{\binom{n}{k} p^k q^{n-k} \approx \frac{1}{\sqrt{2\pi \cdot npq}} \exp\left[-\frac{(k-np)^2}{2npq}\right]}$

Example: Toss 100 coins with $p = 0.5$. Find $P[k = 40\ heads]$ using the binomial law and its normal approximation.

Notice: np=50, $\Delta k = -10, \frac{\Delta k}{\mathrm{k}} = -0.25, \frac{\Delta k}{\mathrm{n-k}} = -0.167.$

$$\frac{1}{\sqrt{2\pi \cdot npq}} \exp\left[-\frac{(k-np)^2}{2npq}\right]$$

$$= \frac{1}{\sqrt{2\pi \cdot 25}} \exp\left[-\frac{(-10)^2}{2 \cdot 25}\right]$$

$$\approx 0.010798\ldots$$

Compare:

$$\binom{n}{k} p^k q^{n-k} \approx 0.010843\ldots$$

Lecture 4

Chapter 1

1.11 Normal Approximation …

Standardize:

Change variable: $\boxed{z = \frac{x-np}{\sqrt{npq}}}$

$$\Rightarrow dz = \frac{1}{\sqrt{npq}}dx$$

Also, use a change of variables to standardize the formula to find the probability for a range of k-values.

Example: Toss 100 coins with $p = 0.5$. Find $P[40 \le k \le 55]$ using the binomial law and its normal approximation.

Notice: np=50, $\Delta k = -10 \ldots + 5$,

$\frac{\Delta k}{\mathrm{k}} = -0.25 \ldots + 0.09$, $\frac{\Delta k}{\mathrm{n-k}} = -0.167 \ldots + 0.11$.

For each k, we obtain: $\binom{n}{k} p^k q^{n-k} \approx \frac{1}{\sqrt{2\pi \cdot npq}} \exp\left[-\frac{(k-np)^2}{2npq}\right]$

$$\sum_{k=40}^{55} \binom{n}{k} p^k q^{n-k} \approx \frac{1}{\sqrt{2\pi \cdot 25}} \sum_{k=40}^{55} \exp\left[-\frac{(k-50)^2}{2 \cdot 25}\right]$$

$$\approx \frac{1}{\sqrt{2\pi \cdot 25}} \int_{39.5}^{55.5} \exp\left[-\frac{(x-50)^2}{2 \cdot 25}\right] dx = \frac{1}{\sqrt{2\pi}} \int_{-2.1}^{1.1} \exp\left[-\frac{z^2}{2}\right] dz$$

Lower boundary: $\frac{39.5-50}{5} = -2.1$, upper boundary: $\frac{55.5-50}{5} = 1.1$,

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Chapter 1

1.11 Normal Approximation …

Standardize:

Change variable: $\boxed{z = \frac{x-np}{\sqrt{npq}}}$

$$\Rightarrow dz = \frac{1}{\sqrt{npq}}\, dx$$

How to evaluate that?

$$\frac{1}{\sqrt{2\pi}} \int_{-2.1}^{1.1} \exp\left[-\frac{z^2}{2}\right] dz$$

Write as

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{1.1} \exp\left[-\frac{z^2}{2}\right] dz - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-2.2} \exp\left[-\frac{z^2}{2}\right] dz$$

$$= \Phi(1.1) - \Phi(-2.2)$$

Where

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left[-\frac{z^2}{2}\right] dz$$

… find in tables: ≈ 0.8643 – (1 – 0.9861) = 0,8504

Original sum: 0.8468

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$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left[-\frac{z^2}{2}\right] dz$$

t	**+0.00**	**+0.01**	**+0.02**	**+0.03**	**+0.04**	**+0.05**	**+0.06**	**+0.07**	**+0.08**	**+0.09**
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621

https://www.normaltable.com/

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$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left[-\frac{z^2}{2}\right] dz$$

t	+0.00	+0.01	+0.02	+0.03	+0.04	+0.05	+0.06	+0.07	+0.08	+0.09
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890

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Find: $\boxed{\binom{n}{k} p^k q^{n-k} \approx \frac{1}{\sqrt{2\pi \cdot npq}} \exp\left[-\frac{(k-np)^2}{2npq}\right]}$

Where is this a good approximation?

For the expansions, we used $\Delta k \ll k$, and $\Delta k \ll n - k$

This means, we have to stay ``close´´ to the peak at *np.*

But what is ``close´´?

A typical way to look at that question is in terms of fixed multiples of the so-called standard deviation: $\Delta k = \alpha \cdot \sqrt{npq}$ where α is a constant.

As long as Δk is small, $k \approx np$, $\mathrm{n} - k \approx nq$, and $\frac{\Delta k}{k} \approx \frac{\alpha \cdot \sqrt{npq}}{np}$ which scales like $\frac{1}{\sqrt{n}}$ for large *n*.

Hence, for large n: $\frac{\Delta k}{k} \rightarrow 0$. Similarly $\frac{\Delta k}{n-k} \rightarrow 0$.

Exam-type questions

TASK 1: The River Bed walk is a pretty short street in Animal Town. There are just two houses in this street. Four rabbits and two foxes want to move in. But they do not know how to distribute. There are so many options – and they all missed their Combinatorics classes ☹… Can you help them a little?

How many different occupancy patterns are there? Mind, this is a little bit beyond what we did in class: Rabbits are rabbits and foxes are foxes. We do not distinguish individual rabbits. Nor do we distinguish individual foxes. But rabbits and foxes are different species. So we cannot ignore this difference – they care ☺.

a) Consider the special situation mentioned above with 2 foxes and 4 rabbits and 2 houses.

b) Find the general formula for f foxes, r rabbits and h houses.

*****c) How many occupancy patterns are there with at least one fox and one rabbit in each of the houses? [Suppose there are enough foxes and rabbits.]

TASK 2: A company produces lots of highly complex devices. As a consequence, these devices may not always work properly. Suppose, individual devices are defective with a probability p, and these problems occur independently. Also, the number X of devices produced on a certain day is random. It follows a Poisson probability law:

$$P[X = k] = \frac{\mu^k}{k!} e^{-\mu}$$

$$k \geq 0, \mu \geq 0$$

On a sunny day in October, how large is the probability to have no defective device at all?

The End

Next time: Start Chp. 2