JMTS-12: Probability and Random Processes

Fall 2020

M. Bode

Lecture 7

Recap …

Chapter 2: Random Variables

Conditional and Joint PDFs, pdfs

Independent random variables:

Recall, events A abd B are independent iff P[A,B]=P[A]P[B].

Two r.v.s X and Y are called independent iff

$$F_{XY}(x,y) = F_X(x)F_Y(y)$$

that is, iff

$$P[X \leq x, Y \leq y] = P[X \leq x]\ P[Y \leq y]$$

… digest the slight abuse of notation, here.

Consequence:

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_X(x)F_Y(y) = f_X(x)f_Y(y)$$

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Recap …

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2.3 Probability Distribution Functions (PDF)

2.4 Probability Density Functions (pdf)

2.5 Continuous, Discrete, Mixed Cases …

2.6 Conditional and Joint PDFs, pdfs

2.7 Failure Rates

Conditional and Joint PDFs, pdfs

Conditional pdfs … via Bayes's rule:

We saw: $P[B|X=x] = \frac{f_{X|B}(x|B)}{f_X(x)} P[B], \text{ if } f(x) \neq 0$

In particular: $P[Y \leq y|X=x] = \frac{f_{X|Y}(x|Y \leq y)}{f_X(x)} P[Y \leq y]$

$$= \frac{\frac{\partial}{\partial x} F_{XY}(x,y)}{f_X(x)}$$

Also derive wrt *y*:

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}, \qquad \text{if } f(x) \neq 0$$

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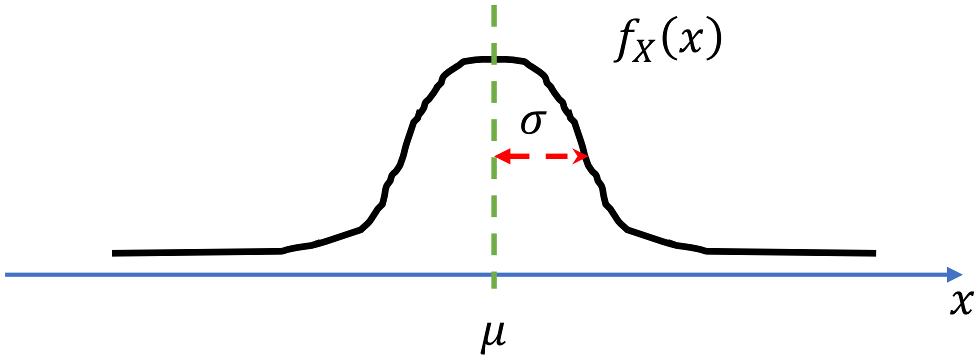

Conditional PDFs & pdfs

Example 1:

Consider two independent normally distributed random variables X and Y with means $\mu_X = \mu_X = 0$, and variances $\sigma_X^2 = \sigma_Y^2 = \sigma^2$.

Find the probability that (X,Y) falls into the unit circle.

1) Find joint pdf

$$f_{XY}(x,y) = f_X(x) f_Y(y)$$

$$= \frac{1}{\sqrt{2\pi \cdot \sigma^2}} \exp\left[-\frac{x^2}{2\sigma^2}\right] \frac{1}{\sqrt{2\pi \cdot \sigma^2}} \exp\left[-\frac{y^2}{2\sigma^2}\right]$$

$$= \frac{1}{2\pi \cdot \sigma^2} \exp\left[-\frac{x^2+y^2}{2\sigma^2}\right]$$

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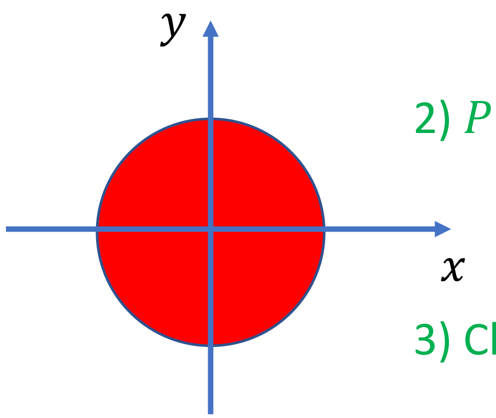

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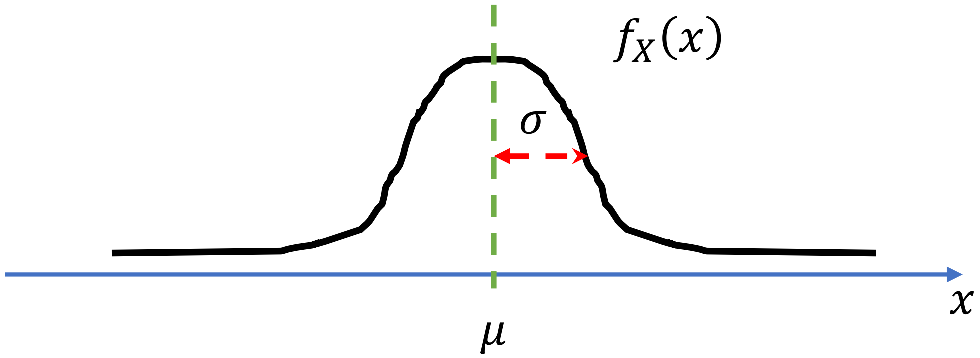

Conditional PDFs & pdfs

2) $P[(X,Y)\,\epsilon\ unit\ circle] = \iint\limits_{\substack{unit\\ -circle}} \frac{1}{2\pi\cdot\sigma^2}\exp\left[-\frac{x^2+y^2}{2\sigma^2}\right]dxdy$

3) Change integration variables:

$$\begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix} \Rightarrow dxdy \to rdrd\theta$$

$$P[(X,Y)\,\epsilon\ unit\ circle] = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \frac{r}{\sigma^2}\exp\left[-\frac{r^2}{2\sigma^2}\right]drd\theta$$

$$= -\exp\left[-\frac{r^2}{2\sigma^2}\right]\Bigg|_0^1 = 1 - \exp\left[-\frac{1}{2\sigma^2}\right]$$

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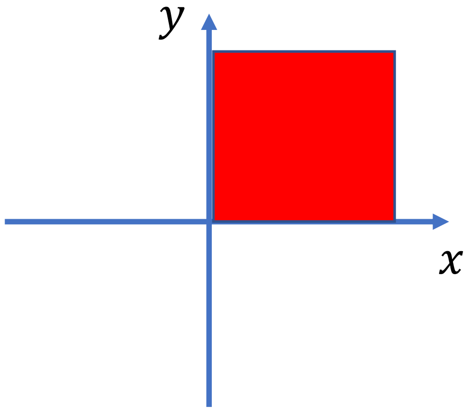

Conditional PDFs & pdfs

Example 2:

$$\text{Let } f_{XY}(x,y) = \begin{cases} A(x+y)\,; & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0\,; & otherwise \end{cases}$$

Show: R.v.s X and Y are not independent.

1) Find coefficient A:

$$1 = \int_0^1 \int_0^1 f_{XY}(x,y)dxdy = A \underbrace{\int_0^1 x\,dx}_{=1/2} \underbrace{\int_0^1 dy}_{=1} + A \underbrace{\int_0^1 y\,dy}_{=1/2} \underbrace{\int_0^1 dx}_{=1} = A$$

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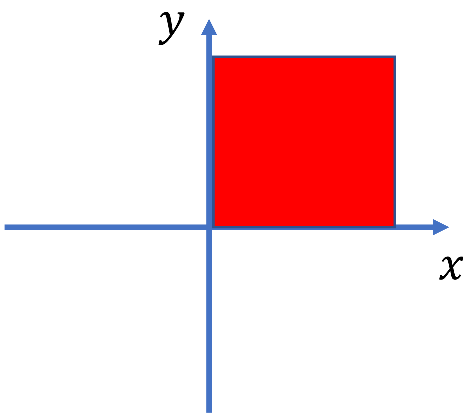

Conditional PDFs & pdfs

Example 2:

Let $f_{XY}(x,y) = \begin{cases} A(x+y)\,; & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0\,; & otherwise \end{cases}$

2) Find the marginal densities:

$$f_X(x) = \begin{cases} \int_0^1 (x+y)dy\,; 0 \le x \le 1 \\ 0; otherwise \end{cases}$$

$$= \left\{ \begin{matrix} \left.\left(xy + \frac{1}{2}y^2\right)\right|_0^1 ; 0 \le x \le 1 \\ 0; otherwise \end{matrix} \right\} = \begin{cases} x + \frac{1}{2}; 0 \le x \le 1 \\ 0; otherwise \end{cases}$$

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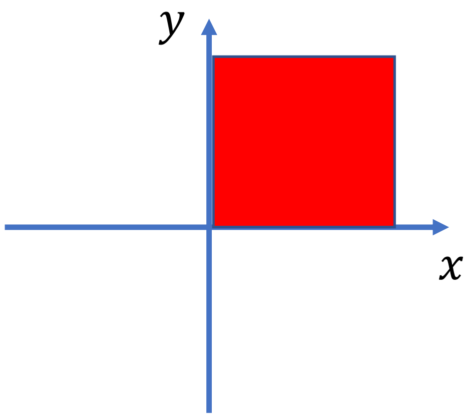

Conditional PDFs & pdfs

Example 2:

$$\text{Let } f_{XY}(x,y) = \begin{cases} A(x+y)\,; & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0\,; & otherwise \end{cases}$$

2) Find the marginal densities:

$$\text{Hence: } f_X(x) = \begin{cases} x + \frac{1}{2}; 0 \le x \le 1 \\ 0; otherwise \end{cases}$$

$$\text{Similarly: } f_y(y) = \begin{cases} y + \frac{1}{2}; 0 \le y \le 1 \\ 0; otherwise \end{cases}$$

Realize: $f_X(x)f_y(y) = \left(x+\frac{1}{2}\right)\left(y+\frac{1}{2}\right) \neq x + y \Rightarrow$ not independent

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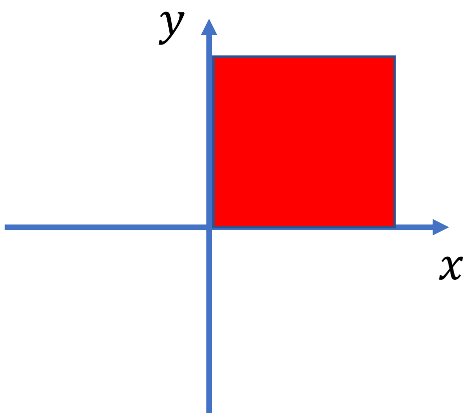

Conditional PDFs & pdfs

Example 2:

$$\text{Let } f_{XY}(x,y) = \begin{cases} A(x+y)\,; & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0\,; & otherwise \end{cases}$$

$$\Rightarrow f_X(x) = \begin{cases} x + \frac{1}{2}; 0 \le x \le 1 \\ 0; otherwise \end{cases}$$

3) Conditional density:

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

This makes sense if $f(x) \neq 0$, that is, **for** $\mathbf{0 \le x \le 1}$:

$$f_{Y|X}(y|x) = \begin{cases} \dfrac{x+y}{x+\frac{1}{2}}; & 0 \le y \le 1 \\ 0\,; & otherwise \end{cases}$$

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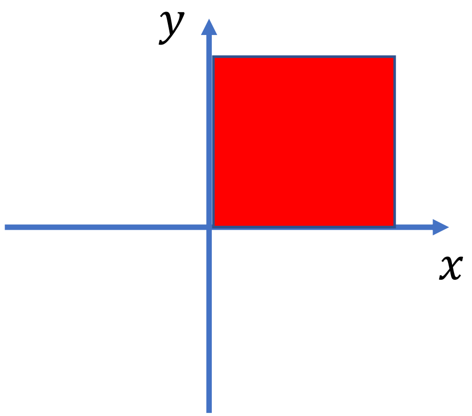

Conditional PDFs & pdfs

Example 2:

$$\text{Let } f_{XY}(x,y) = \begin{cases} A(x+y)\,; & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0\,; & otherwise \end{cases}$$

$$\Rightarrow f_X(x) = \begin{cases} x + \dfrac{1}{2}; & 0 \le x \le 1 \\ 0; & otherwise \end{cases}$$

If $f(x) \neq 0$, that is, **for $\boldsymbol{0 \le x \le 1}$:**

$$f_{Y|X}(y|x) = \begin{cases} \dfrac{x+y}{x+\frac{1}{2}}; & 0 \le y \le 1 \\ 0\,; & otherwise \end{cases}$$

4) Check whether

$$\int_{-\infty}^{\infty} f_{Y|X}(y|x)dy = 1$$

as it should be … calculate … also plot the conditional densities!

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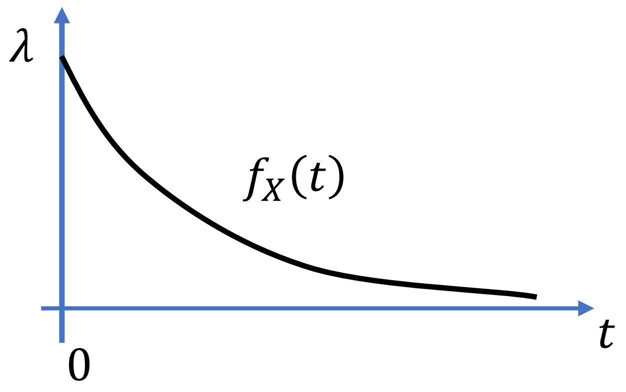

Failure Rates

Random variable: X: time of failure *or* failure time

Example 1 (Radioactive decay):

$$f_X(t) = \lambda \exp[-\lambda t], \quad F_X(t) = 1 - \exp[-\lambda t], \quad t > 0$$

Probability to decay within $t < X \leq t + dt$

$$P[t < X \leq t + dt] \approx f_X(t) dt$$

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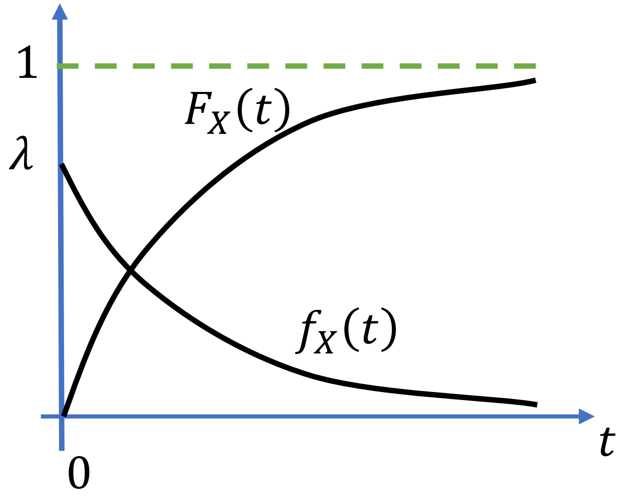

Failure Rates

Random variable: X: time of failure *or* failure time

Example 1 (Radioactive decay):

$$f_X(t) = \lambda \exp[-\lambda t]\,, \quad F_X(t) = 1 - \exp[-\lambda t]\,, \; t > 0$$

Suppose, we care only about those ``atoms´´ that are still ``alive´´:

$$P[t < X \le t + dt | X > t] = \frac{P[t < X \le t + dt, X > t]}{P[X > t]}$$

$$= \frac{P[t < X \le t + dt]}{1 - P[X \le t]} \approx \frac{f_X(t)dt}{1 - F_X(t)} := \alpha(t)dt$$

failure rate

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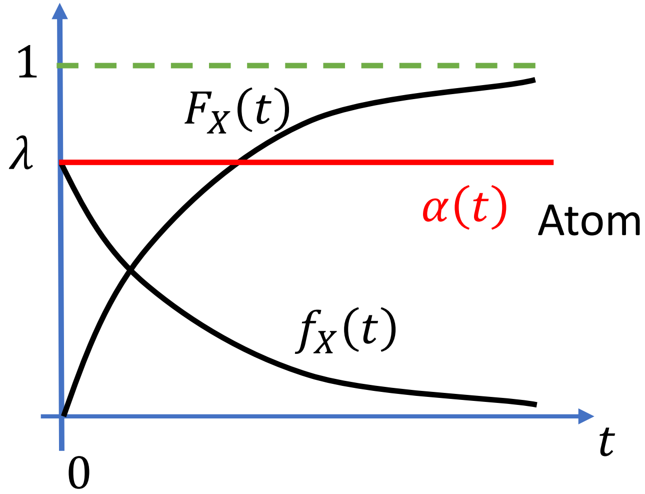

Failure Rates

Random variable: X: time of failure *or* failure time

Example 1: Radioactive decay … $f_X(t) = \lambda \exp[-\lambda t]\,, t > 0$

For an atom, find the probability to decay within $t < X \leq t + dt$

$$P[t < X \leq t + dt] \approx f_X(t)dt$$

(Conditional) Failure (decay) rate:

$$\alpha(t) = \frac{f_X(t)}{1 - F_X(t)}, \text{relative to those ``alive''}$$

Atoms:

$$\text{Here: } \alpha(t) = \frac{f_X(t)}{1 - F_X(t)} = \lambda$$

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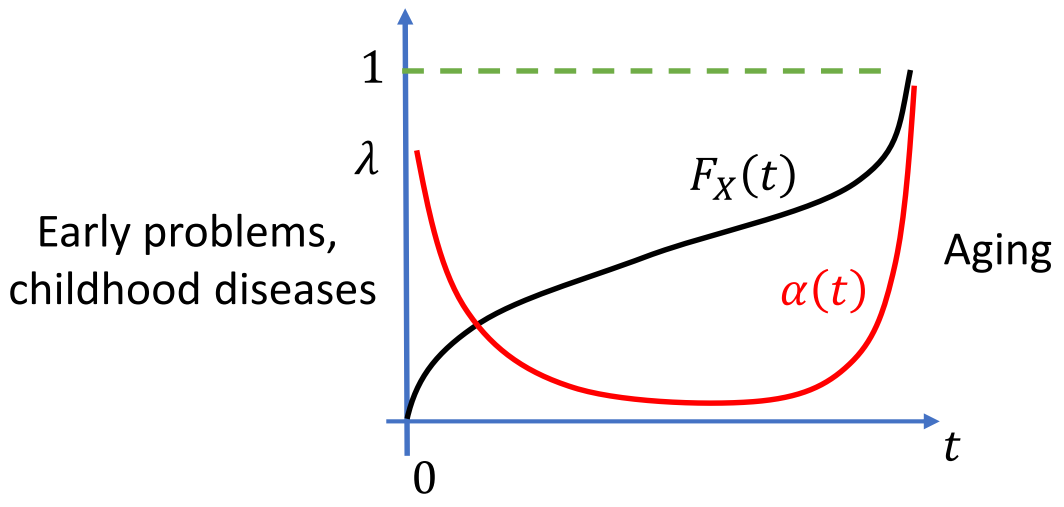
Failure Rates

Random variable: X: time of failure *or* failure time

Compare: Living organisms, manufactured devices

Bathtub curve



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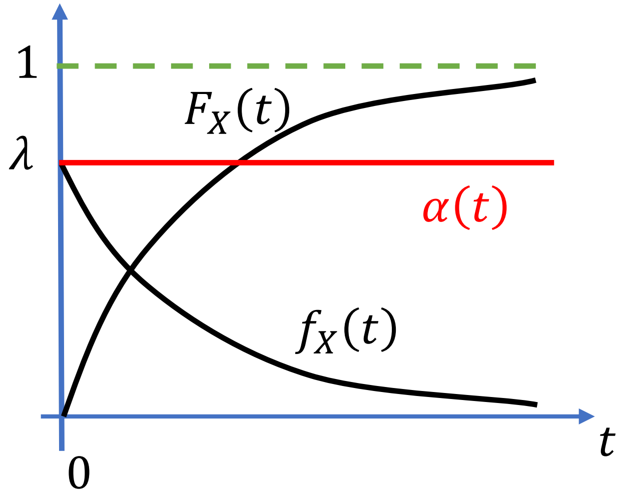

Failure Rates

Failure rate:

$$\alpha(t) = \frac{f_X(t)}{1 - F_X(t)}$$

Find $f_X(t)$ based on $\alpha(t)$

$$\alpha(t) = -\frac{d}{dt}\ln\big(1 - F_X(t)\big) \Rightarrow \ln\big(1 - F_X(t)\big) = -\int_0^t \alpha(t')dt'$$

$$F_X(t) = 1 - \exp\left(-\int_0^t \alpha(t')dt'\right)$$

$$\Rightarrow f_X(t) = \alpha(t)\exp\left(-\int_0^t \alpha(t')dt'\right)$$

■

The End

Next time: continue Chp. 2