JMTS-12: Probability and Random Processes

Fall 2020

M. Bode

Lecture 6

Chapter 2: Random Variables

2.2 Random Variables

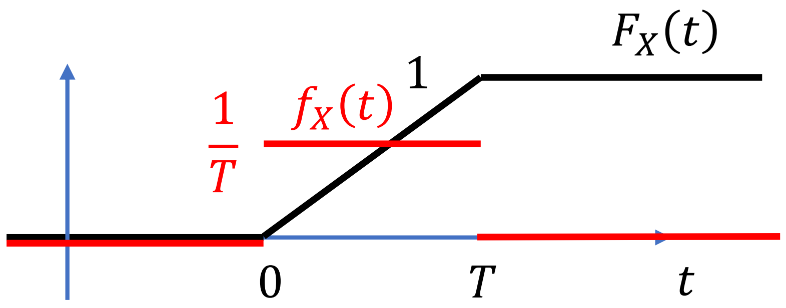
2.3 Probability Distribution Functions (PDF)

2.4 Probability Density Functions (pdf)

2.5 Continuous, Discrete, Mixed Cases …

2.6 Conditional and Joint PDFs, pdfs

2.7 Failure Rates



Conditional PDFs & pdfs

Example:

Consider our **bus example**, below … and find:

1) $F_X(a) = P[X \leq a]$ for $0 \leq a \leq T$
2) $P[X \leq a \mid X \leq T/2]$
3) Does it make sense to write this as a new PDF?

$$P[X \leq a \mid X \leq T/2] = F_X(a \mid X \leq T/2)$$

$$F_X(a) = P[X \leq a] = \frac{a}{T} \text{ for } 0 \leq a \leq T$$

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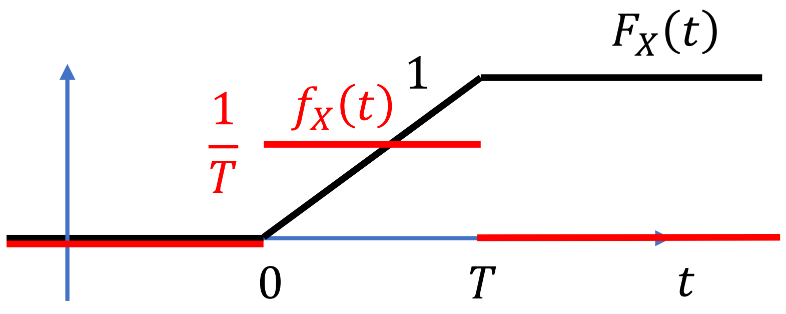


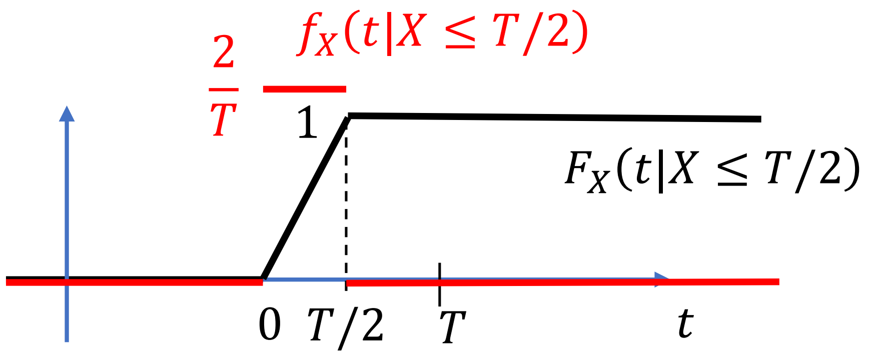
Conditional PDFs & pdfs

Consider our **bus example**, below … and find:

1) $F_X(a) = P[X \le a]$ for $0 \le a \le T$

2) $P[X \le a \mid X \le T/2] = \dfrac{P\left[X \le a,\, X \le \frac{T}{2}\right]}{P\left[X \le \frac{T}{2}\right]}$

$$= \begin{cases} \dfrac{P[X \le a]}{P\left[X \le \frac{T}{2}\right]} = 2P[X \le a]\,;\text{if } a \le \dfrac{T}{2} \\ \dfrac{P\left[X \le \frac{T}{2}\right]}{P\left[X \le \frac{T}{2}\right]} = 1\,;\text{if } a > \dfrac{T}{2} \end{cases}$$



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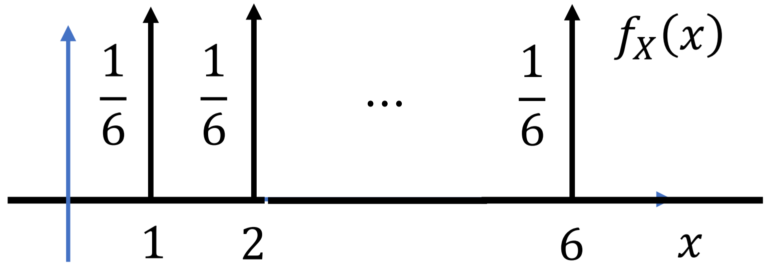


Conditional PDFs & pdfs

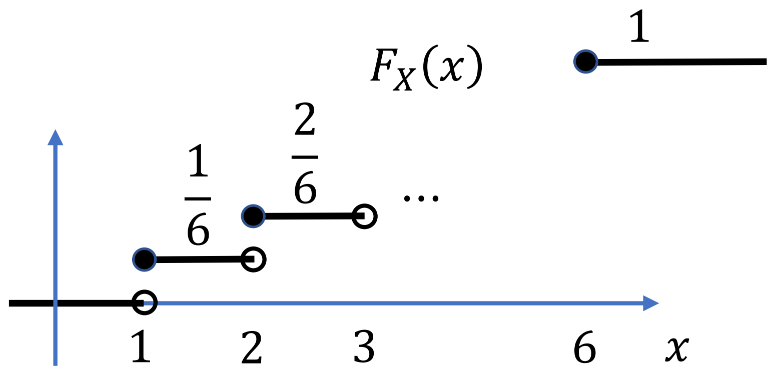
Example:

Consider a **fair six-faced die** with the natural pdf

$$f_X(x) = \frac{1}{6}\sum_{k=1}^{6} \delta(x-k)$$

… and find:

1) $F_X(x)$
2) $F_X(x \,|\, X \text{ is even})$
3) $F_X(x \,|\, X \text{ is prime})$
4) Sketch $f_X(x \,|\, X \text{ is even})$ and $f_X(x \,|\, X \text{ is prime})$



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Some useful formulas …

For a partition $\{A_i\}_{i=1,\ldots,n}$:

$$F_X(x) = \sum_{i=1}^{n} F_X(x\,|A_i)\,P[A_i]$$

Example:

Chip production with 1/6 defective and 5/6 good chips (DC,GC) (textbook)…

X = Time to failure … conditional PDFs:

$F_X(x\,|DC) = \left(1 - e^{-x/2}\right)u(x)$ (x in months)

$F_X(x\,|GC) = \left(1 - e^{-x/10}\right)u(x)$ (x in months)

A buyer cannot tell …

$F_X(x) = ?$

P[failure before six months]=?

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The partition is „all chips“ = DC ∪ GC

$$F_X(x) = F_X(x\,|DC)P[DC] + F_X(x\,|GC)P[GC]$$

$F_X(x\,|DC) = \left(1 - e^{-x/2}\right)u(x)$ (x in months)

$F_X(x\,|GC) = \left(1 - e^{-x/10}\right)u(x)$ (x in months)

$$F_X(x) = \frac{1}{6}\left(1 - e^{-x/2}\right)u(x) + \frac{5}{6}\left(1 - e^{-x/10}\right)u(x)$$

P[failure before six months] = $F_X(x = 6)$

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Conditional PDFs & pdfs

Bayes‘s formula for pdfs: $P[B|X = x] = \frac{P[B,X=x]}{P[X=x]}$

So far, conditions came with a probability greater than zero …

What if X is continous?

… $P[X = x] = 0$ … but $P[B, X = x] = 0$, too.

$$P[B|x < X \leq x + \Delta x] = \frac{P[x < X \leq x + \Delta x|B]P[B]}{P[x < X \leq x + \Delta x]}$$

$$= \frac{F_X(x < X \leq x + \Delta x|B)\ P[B]}{F_X(x < X \leq x + \Delta x)}$$

$$\approx \frac{f_{X|B}(x|B)\Delta x}{f_X(x)\Delta x} P[B] = \frac{f_{X|B}(x|B)}{f_X(x)} P[B]$$

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Bayes's formula for pdfs: $P[B|X=x] = \frac{P[B,X=x]}{P[X=x]}$

What if X is continous?

So, for $\Delta x \rightarrow 0$, we read:

$$P[B|X=x] = \frac{f_{X|B}(x|B)}{f_X(x)} P[B]$$

Multiply with $f_X(x)$ and integrate …

$$P[B] = \int_{-\infty}^{\infty} P[B|X=x] f_X(x) dx$$

A continuous partition if you like.

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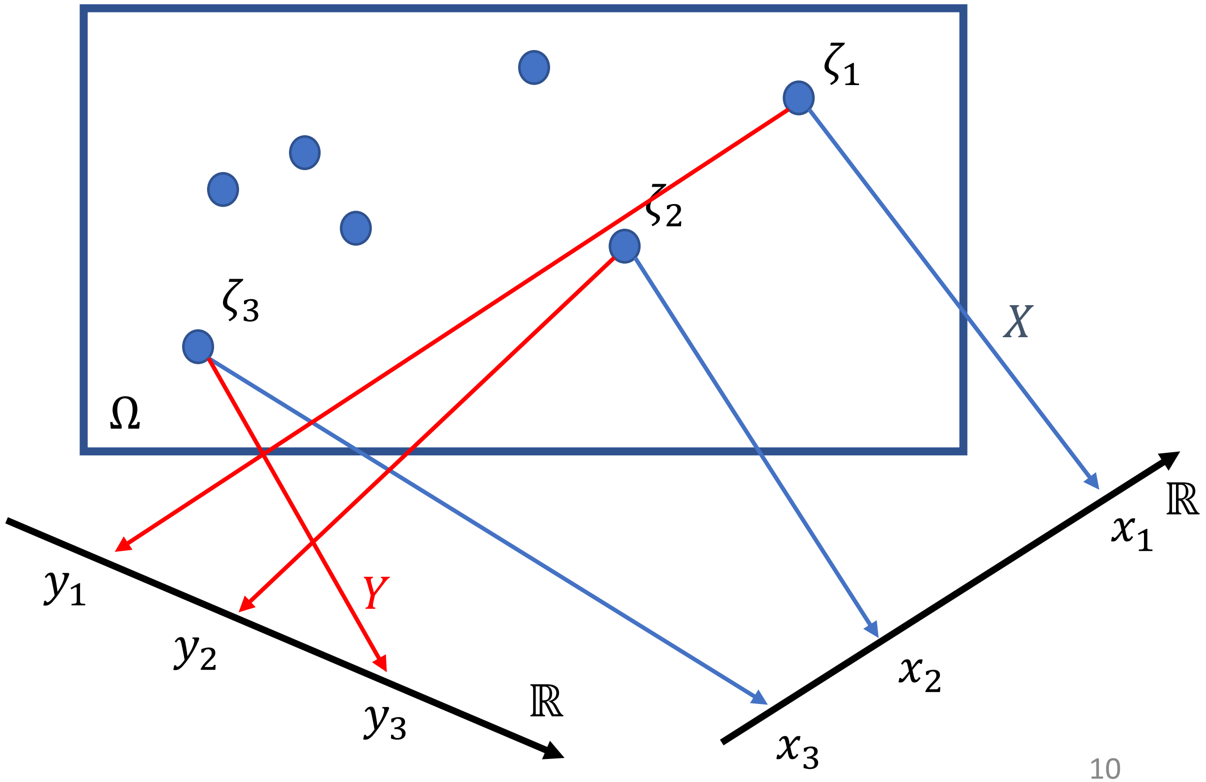
Conditional and Joint PDFs, pdfs

Two or more r.v.s … Same idea: Map outcomes to (real) numbers.

Individually, the random variables $X\colon\ \Omega \to \mathbb{R}$, Y: $\Omega \to \mathbb{R}$, etc. map all outcomes from the sample description space to a real numbers.

Jointly, this maps an outcome to a point in $\mathbb{R}^2$.



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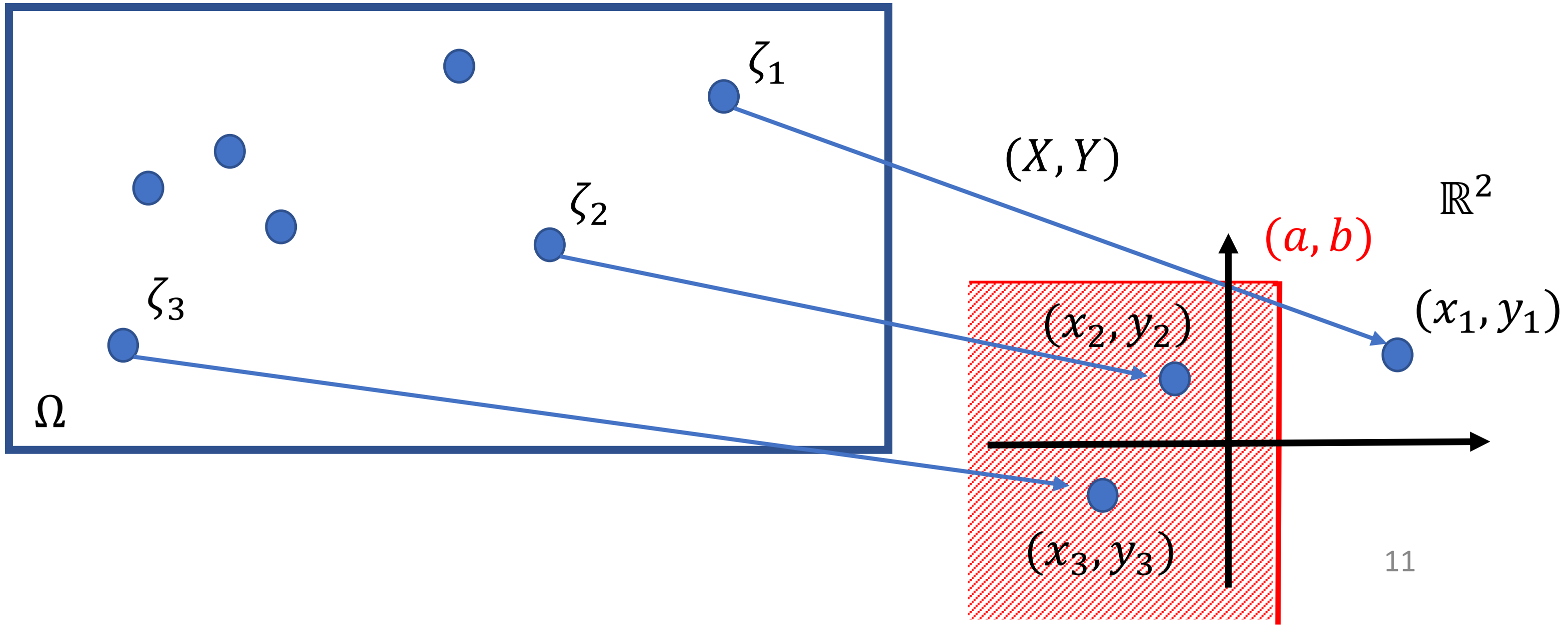
Individually, the random variables $X\colon\ \Omega \to \mathbb{R}$, Y: $\Omega \to \mathbb{R}$, etc. map all outcomes from the sample description space to real numbers.

Jointly, this maps an outcome to a point in $\mathbb{R}^2$… joint PDF:

$$F_{XY}(a,b) := P_{XY}\big[(-\infty,a]\times(-\infty,b]\big]$$

$$= P[\{\zeta \in \Omega : X(\zeta) \leq a, Y(\zeta) \leq b]$$



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Joint PDF:

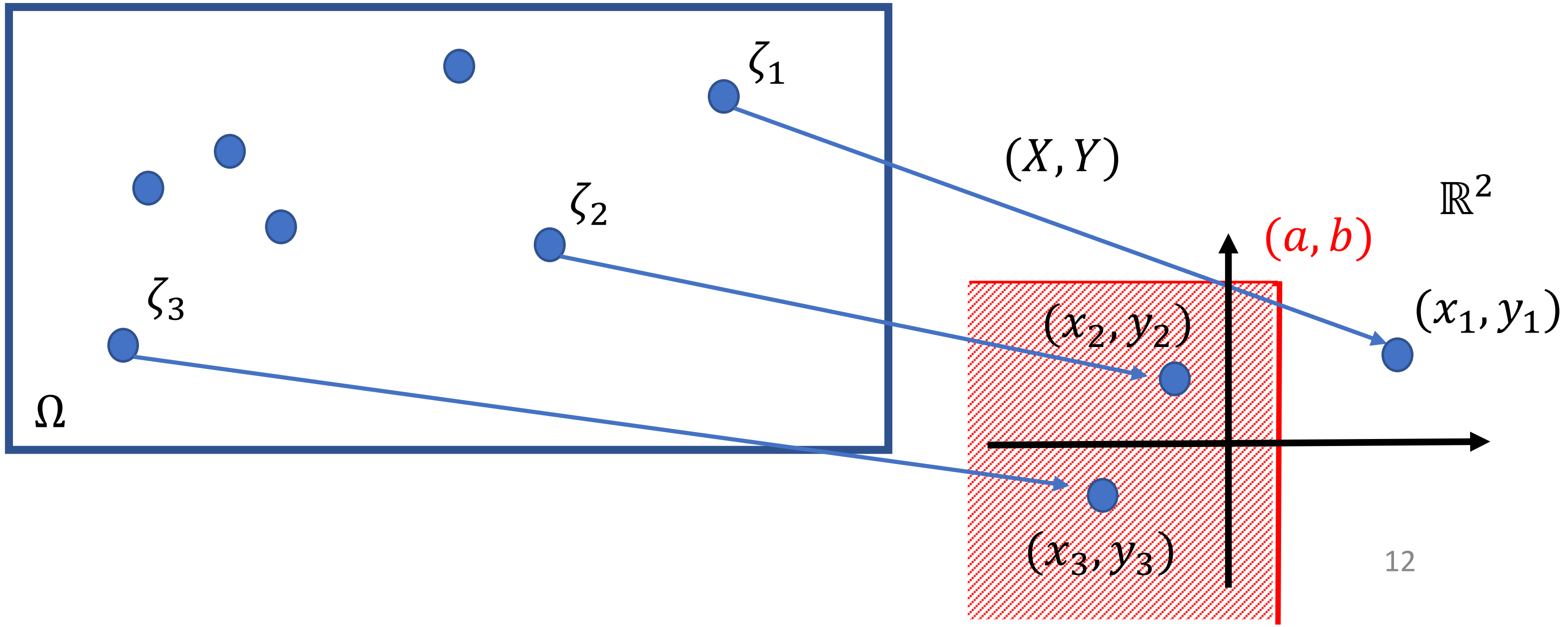
$$F_{XY}(a,b) := P_{XY}\big[(-\infty,a]\times(-\infty,b]\big]$$

$$= P[\{\zeta\in\Omega : X(\zeta)\le a, Y(\zeta)\le b]$$

Marginal PDFs:

$$F_{XY}(x,\infty) = F_X(x)$$

$$F_{XY}(\infty,y) = F_Y(y)$$



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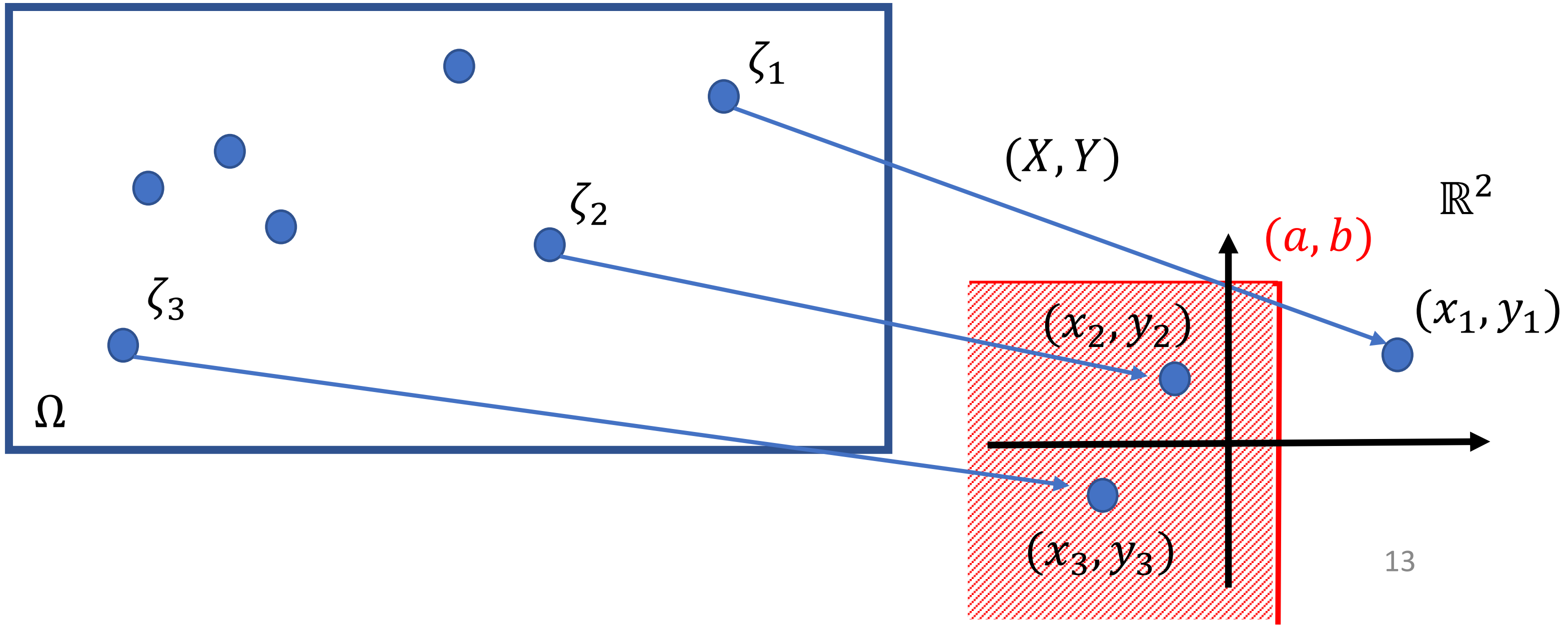
Related (joint) pdf:

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$$

$$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(\xi,\eta) d\eta d\xi$$



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Properties of $F_{XY}(x, y)$:

(i) $F_{XY}(\infty, \infty) = 1$, $F_{XY}(-\infty, y) = 0$, $F_{XY}(x, -\infty) = 0$,

$F_{XY}(x, \infty) = F_X(x)$, $F_{XY}(\infty, y) = F_Y(y)$

(ii) $x_1 \leq x_2, y_1 \leq y_2 \Rightarrow F_{XY}(x_1, y_1) \leq F_{XY}(x_2, y_2)$

(iii) $F_{XY}(x, y) = \lim\limits_{\substack{\varepsilon,\delta \to 0 \\ \varepsilon,\delta > 0}} F_{XY}(x + \varepsilon, y + \delta)$

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Independent random variables:

Recall, events A abd B are independent iff P[A,B]=P[A]P[B].

Two r.v.s X and Y are called independent iff

$$F_{XY}(x,y) = F_X(x)F_Y(y)$$

that is, iff

$$P[X \leq x, Y \leq y] = P[X \leq x]\; P[Y \leq y]$$

… digest the slight abuse of notation, here.

Consequence:

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_X(x)F_Y(y) = f_X(x)f_Y(y)$$

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Conditional pdfs … via Bayes's rule:

Above, we saw: $P[B|X=x] = \frac{f_{X|B}(x|B)}{f_X(x)} P[B]$

In particular: $P[Y \le y|X=x] = \frac{f_{X|Y}(x|Y \le y)}{f_X(x)} P[Y \le y]$

$$= \frac{\frac{\partial}{\partial x} F_{XY}(x, Y \le y)}{f_X(x)}$$

Derive wrt *y*:

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

■

The End

Next time: continue Chp. 2