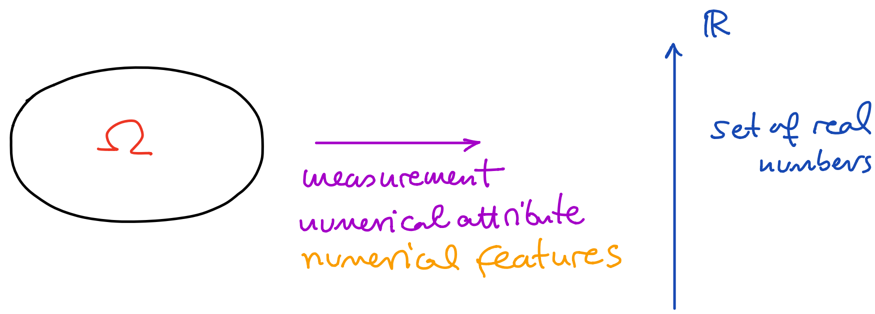
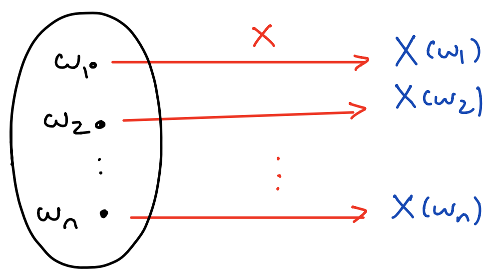
Lecture 7

Random variables



A random variable is a function

$$X : \Omega \longrightarrow \mathbb{R}$$



$X(\omega)$ is a random quantity, since ω is "random".

Example Constant function:

$$X(\omega) = c \quad \text{for all values of } \omega.$$

Example

$$\Omega = \{H, T\}$$

$$X(\omega) = \begin{cases} 1 & \omega = H \\ 0 & \omega = T \end{cases}$$

Example

$$\Omega = \{HH, HT, TH, TT\}$$

$$X(\omega) = \begin{cases} 2 & \omega = HH \\ 1 & \omega = HT, TH \\ 0 & \omega = TT \end{cases} \qquad \text{\# of heads}$$

Example A coin is thrown n times. Assume

1) each outcomes in H with probability p and T with probability $1-p$.
2) The outcomes of different throws are independent.

$\Omega =$ sequences of H and T of length n

$$\mathbb{P}(\{HTH\}) = \mathbb{P}(\underset{H}{\text{first throw}}) \cdot \mathbb{P}(\underset{T}{\text{second throw}}) \cdot \mathbb{P}(\underset{H}{\text{Third throw}}) = p(1-p)p.$$

Example: $n=3$ $\mathbb{P}(\{HHT\}) = p \cdot p \cdot (1-p) = p^2(1-p)$

$n=4$ $\mathbb{P}(\{HTHT\}) = p \cdot (1-p) \cdot p \cdot (1-p) = p^2(1-p)^2$

If σ is a sequence of H and T of length n, with k terms equal to H and $n-k$ remaining terms equal to T

then
More generally,

$$\boxed{\mathbb{P}(\sigma) = p^k (1-p)^{n-k}}$$

We may be just interested in number of heads

$$X(\omega) = \text{number of heads} = \begin{cases} 3 & \omega = HHH \\ 2 & \omega = HHT, HTH, THH \\ 1 & \omega = HTT, THT, TTH \\ 0 & \omega = TTT \end{cases}$$

Events defined in terms of random variables

exactly two heads $\{X=2\} = \{\omega : X(\omega) = 2\} = \{HHT, HTH, THH\}$

at most one head $\{X \leq 1\} = \{\omega : X(\omega) \leq 1\} = \{TTT, HTT, THT, TTH\}$

Probabilities of events defined in terms of random variables

$$\mathbb{P}(X=2) = \mathbb{P}[\{HHT, HTH, THH\}] = 3p^2(1-p)$$

$$\mathbb{P}(X\leq 1) = \mathbb{P}[\{TTT, HTT, THT, TTH\}] = 3p(1-p)^2 + (1-p)^3$$

Note that

$$\mathbb{P}(X\leq 1) = \mathbb{P}((X=0)\cup(X=1)) = \mathbb{P}(X=0)+\mathbb{P}(X=1)$$
$$= (1-p)^3 + 3p(1-p)^2.$$

Example Assume that the football team Werder Bremen wins each game with probability 10% and loses with probability 90%. What is the prob. of the event E that after 34 games, it wins exactly 4 games:

$$\mathbb{P}(E) = \binom{34}{4}\left(\frac{1}{10}\right)^4\left(\frac{9}{10}\right)^{30} \approx 0.23$$

Discrete random variables

A random variable X is called discrete if the set of values it can attain can be listed as $x_1, x_2, \ldots$ (may be finite or infinite)

Example A coin is tossed. Suppose that it has probability p of coming up heads. Let X be a random variable which takes value 1 when the outcomes is heads, and 0, otherwise.

Then

$$\mathbb{P}[X=1] = p, \quad \mathbb{P}[X=0] = 1-p$$

Definition A random variable X is said to be a Bernoulli random variable with parameter p, when

$$\mathbb{P}(X=1)=p, \quad \mathbb{P}(X=0)=1-p$$

in other words, a Bernoulli random variable is a random variable that takes only values 0 and 1.

Example A coin has probability p of showing up H, and has been flipped n times. Denote by X the number of heads

Values X can attain are: $0, 1, 2, \dots, n$. Moreover,

$$\mathbb{P}[X=k] = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & k=0,1,\dots,n \\ 0 & \text{otherwise} \end{cases}$$

Definition

A random variable X is said to be binomial with parameters (n,p) when:

- X takes values $0, 1, \dots, n$, and, further,

$$\mathbb{P}(X=k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & k=0,1,\dots,n \\ 0 & \text{otherwise} \end{cases}$$

Example A fair die has been thrown 7 times. Let X denote the number of throws where the outcome was 2 or 3.

What is $\mathbb{P}(X=5)$?

X: binomial with $n=7$, $p=\frac{2}{6}$

$$\mathbb{P}(X=5) = \binom{7}{5} \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^2.$$

<u>Example</u> (tie-branking probability)

$2n$ people vote, each voting independently with probability p for candidate A and with probability $1-p$ for candidate B. What is the probability of a tie?

$X = \#$ votes for A

$$\mathbb{P}(X=n) = \binom{2n}{n} p^n (1-p)^n$$

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$$

Recall $\quad n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$

So
$$\binom{2n}{n} \sim \frac{\sqrt{4\pi n}\,\left(\frac{2n}{e}\right)^{2n}}{\left(\sqrt{2\pi n}\,\left(\frac{n}{e}\right)^n\right)^2} = \frac{2^{2n}}{\sqrt{\pi n}}$$

So
$$\mathbb{P}(\text{tie}) = \frac{(4p(1-p))^n}{\sqrt{\pi n}}$$

$$p = \tfrac{1}{2} \Rightarrow 4p(1-p) = 1 \Rightarrow \mathbb{P}(\text{tie}) = \frac{1}{\sqrt{\pi n}}$$

$$p = 0.45 \Rightarrow 4p(1-p) = 0.99 \Rightarrow \mathbb{P}(\text{tie}) = \frac{(0.99)^n}{\sqrt{\pi n}}$$

$$p = 0.40 \Rightarrow 4p(1-p) = 0{,}96 \Rightarrow \mathbb{P}(\text{tie}) = \frac{(0.96)^n}{\sqrt{\pi n}}$$

$p \backslash n$	$n=10$	$n=20$	$n=100$	$n=1000$
0.5	0.17	0.12	0.05	0.01
0.45	0.16	0.10	0.02	7×10^{-7}
0.40	0.11	0.05	0.0009	3×10^{-20}