Lecture 8

Recall

- A random variable is a function $X:\Omega \to \mathbb{R}$
- A discrete random variable is a random variable with a finite or countable set of values $x_1, x_2, \ldots$

Definition Let X be discrete random variable. the probability mass function (PMF) of X is defined via

$$p_X(x) = \mathbb{P}[X=x]$$

value $x \longmapsto$ Probability that x is attained.

Representing a random variable

- Values of X: $x_1, x_2, x_3, \ldots$ (may be finitely or infinitely many)
- Choose probabilities $p_i := \mathbb{P}[X = x_i]$

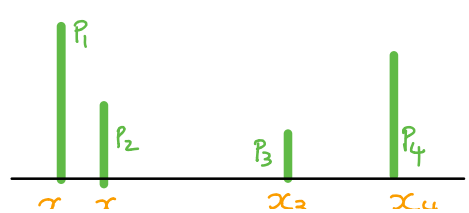
with the constraints:

$$p_i \geq 0 \quad , \quad p_1 + p_2 + \cdots = 1 .$$

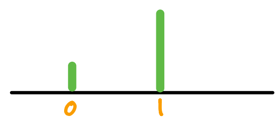
Representing by a table

x	x_1	x_2	x_3	x_4	$\cdots$
$p_X(x)$	p_1	p_2	p_3	p_4	

Representing by a diagram

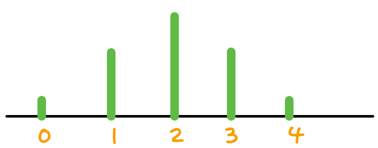


Bernoulli $(p = 2/3)$

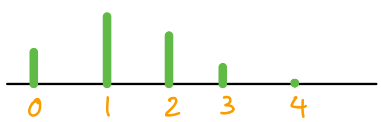


X	0	1
$P_X(x)$	$\frac{1}{3}$	$\frac{2}{3}$

Binomial, $p = \frac{1}{2}$, $n = 4$



X	0	1	2	3	4
$P_X(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$



X	0	1	2	3	4
$P_X(x)$	$\frac{16}{81}$	$\frac{32}{81}$	$\frac{24}{81}$	$\frac{8}{81}$	$\frac{1}{81}$
	≈ 0.2	≈ 0.4	≈ 0.3	≈ 0.1	≈ 0.01

Definition We say that X is a geometric random variable with parameter p when

$$P_X(x) = \mathbb{P}[X = k] = p(1-p)^{k-1} \cdot \quad k = 1, 2, 3, 4, \ldots$$

Example. Suppose X has geometric distribution with parameter p.

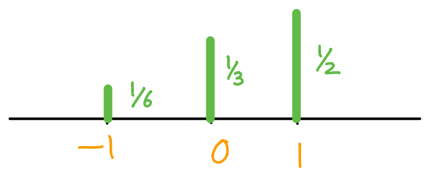
$$\mathbb{P}[X > k] = p(1-p)^k + p(1-p)^{k+1} + \cdots =$$
$$= p(1-p)^k \left(1 + (1-p) + (1-p)^2 + \cdots\right) = (1-p)^k.$$

- What is the interpretation of this?
- Second success? third success?

Example X discrete random variable, taking values $0, 1, -1$

with $\mathbb{P}[X=1] = \frac{1}{2}$, $\mathbb{P}[X=0] = \frac{1}{3}$, $\mathbb{P}[X=-1] = \frac{1}{6}$

x	-1	0	1
$P_X(x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$



$$\mathbb{P}[X \neq 0] = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

$$\mathbb{P}[X \geqslant 0] = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

Example Suppose X is a discrete random variable with the PMF

$$P_X(1) = P_X(-1) = \frac{1}{4}, \quad P_X(2) = P_X(-2) = \frac{1}{6} \quad P_X(3) = \frac{1}{6}$$

Let $Y = X^2$, $Z = X+1$. Compute the PMF of Y, Z.

X	-1	1	-2	2	3
$P_X(x)$	$1/4$	$1/4$	$1/6$	$1/6$	$1/6$
Y	1	1	4	4	9
Z	0	2	-1	3	4

$P_Y(1) = 1/2$, $P_Y(4) = 1/3$ $P_Y(9) = 1/6$

$P_Z(0) = P_Z(2) = 1/4$, $P_Z(-1) = P_Z(3) = P_Z(4) = 1/6$.

y	1	4	9
$P_Y(y)$	$1/2$	$1/3$	$1/6$

z	-1	0	2	3	4
$P_Z(z)$	$1/6$	$1/4$	$1/4$	$1/6$	$1/6$

Definition For a random variable X, the distribution function of X is defined by

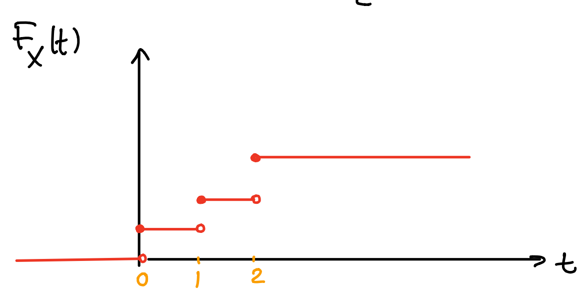
$$F_X(t) = \mathbb{P}[X \leq t]$$

Example Consider a discrete random variable with PMF given by

$$P_X(0) = 1/3, \quad P_X(+1) = 1/4, \quad P_X(+2) = 5/12$$

The distribution function of X is given by

$$F_X(t) = \begin{cases} 0 & t<0 \\ 1/3 & 0 \leq t < 1 \\ 1/3 + 1/4 & 1 \leq t < 2 \\ 1 & 2 \leq t \end{cases}$$



Consider a random variable X with the distribution function given by

$$F_X(t) = \begin{cases} 0 & t<0 \\ t/2 & 0 \leq t < 1 \\ 2/3 & 1 \leq t < 2 \\ 1 & 2 \leq t \end{cases}$$

Compute $\mathbb{P}[X \leq 1]$, $\mathbb{P}[X = 1]$, $\mathbb{P}[1/2 < X \leq 3/2]$

$$\mathbb{P}[X \leq 1] = F_X(1) = 2/3$$

$$\mathbb{P}[X < 1] = \lim_{t \to 1-} F_X(t) = 1/2 \implies \mathbb{P}[X=1] = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

$$\mathbb{P}[\tfrac{1}{2} < X \leq \tfrac{3}{2}] = \mathbb{P}[X \leq \tfrac{3}{2}] - \mathbb{P}[X \leq \tfrac{1}{2}] = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}.$$

<u>Example</u> Consider a random variable X with

$$p_X(x) = \begin{cases} k \cdot x & x = 1,2,3,4 \\ 0 & \text{otherwise} \end{cases}$$

(a) determine k

(b) Compute $\mathbb{P}[X \text{ is even}]$

(c) Plot $F_X(t)$.

<u>Solution</u>

x	1	2	3	4
$p_X(x)$	k	$2k$	$3k$	$4k$
	0.1	0.2	0.3	0.4

(a)

$$p_X(1) + p_X(2) + p_X(3) + p_X(4) = 1$$

$$k + 2k + 3k + 4k = 1$$

$$k = \frac{1}{10}$$

$$\mathbb{P}[X \text{ is even}] = \mathbb{P}[X=2, X=4] = \frac{1}{10} \cdot 2 + \frac{1}{10} \cdot 4 = 0.6$$

x	1	2	3	4
$p_X(x)$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$

$$\mathbb{P}[X \leq t] = \begin{cases} 0 & t < 1 \\ 1/10 & 1 \leq t < 2 \\ 3/10 & 2 \leq t < 3 \\ 6/10 & 3 \leq t < 4 \\ 1 & 4 \leq t \end{cases}$$