JTMS-12: Probability and Random Processes

Fall 2020

M. Bode

Lecture 12

Recap

Chapter 3: Functions of Random Variables

3.1 Functions of Random Variables

3.2 $Y = g(X)$

3.3 $Z = g(X, Y)$

3.4 $V = g(X, Y)$, $W = h(X, Y)$

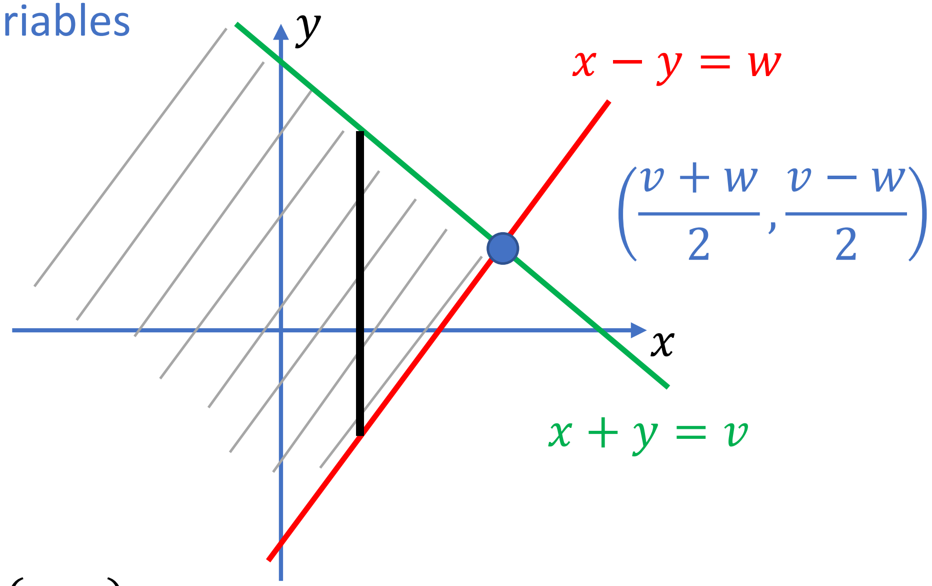
Functions of Random Variables

Example:

Consider continuous r.v.s X, Y, and

$$V = g(X,Y) = X + Y$$
$$W = h(X,Y) = X - Y$$

Find $F_{VW}(v,w)$ and $f_{VW}(v,w)$.



Standard approach …

$$F_{VW}(v,w) = P[V \leq v, W \leq w] = P[g(X,Y) \leq v, h(X,Y) \leq w]$$

$$= P[X + Y \leq v, X - Y \leq w] = \iint_{\substack{shaded\\area}} f_{XY}(x,y)dxdy$$

$$= \int_{-\infty}^{\frac{v+w}{2}} \left(\int_{x-w}^{v-x} f_{XY}(x,y)dy \right) dx$$

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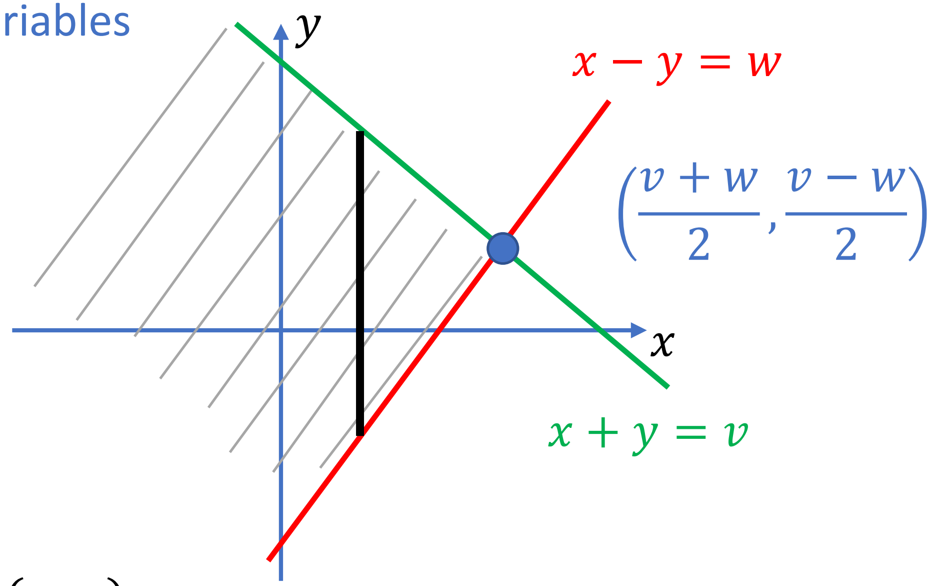
Functions of Random Variables



solve:

Consider continuous

r.v.s X, Y, and

$$V = g(X,Y) = X + Y$$
$$W = h(X,Y) = X - Y$$

Find $F_{VW}(v,w)$ and $f_{VW}(v,w)$.

Standard approach …

$$F_{VW}(v,w) = \int_{-\infty}^{\frac{v+w}{2}} \left(\int_{x-w}^{v-x} f_{XY}(x,y)dy \right) dx$$

$$\Rightarrow f_{VW}(v,w) = \frac{\partial^2}{\partial v \partial w} F_{VW}(v,w) = \frac{1}{2} f_{XY}\left(\frac{v+w}{2}, \frac{v-w}{2}\right)$$

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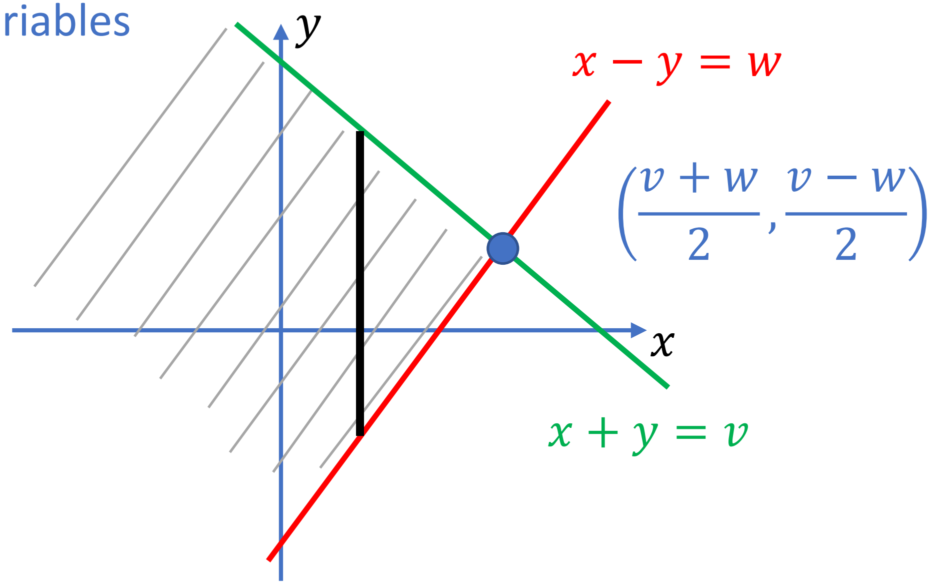
Functions of Random Variables

solve:

Consider continuous r.v.s X, Y, and

$$V = g(X,Y) = X + Y$$
$$W = h(X,Y) = X - Y$$



Details …

$$f_{VW}(v,w) = \frac{\partial^2}{\partial v \partial w} \int_{-\infty}^{\frac{v+w}{2}} \left(\int_{x-w}^{v-x} f_{XY}(x,y)dy \right) dx$$

$$= \frac{\partial}{\partial v}\left[\frac{1}{2} \int_{\frac{v-w}{2}}^{\frac{v-w}{2}} f_{XY}\left(\frac{v+w}{2}, y\right) dy + \int_{-\infty}^{\frac{v+w}{2}} f_{XY}(x, x-w) dx \right] = \frac{1}{2} f_{XY}\left(\frac{v+w}{2}, \frac{v-w}{2}\right)$$

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Chapter 3: Functions of Random Variables

3.1 Functions of Random Variables

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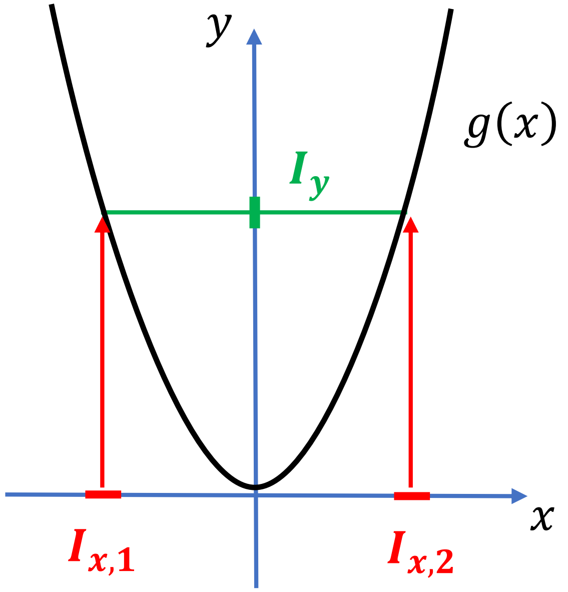
Functions of Random Variables



Determine densities like

$f_Y(y)$ directly

Recall:
$f_Y(y)|\Delta y| \approx f_X(x_1)|\Delta x_1| + f_X(x_2)|\Delta x_2|$

Basically, we try to calculate $\int f_X(x)dx$

via

$$\int f_X(x(y))\frac{\Delta(x)}{\Delta(y)}dy = \int f_Y(y)dy$$

based on the ratio $\frac{\Delta(x)}{\Delta(y)}$ of corresponding “line/area-elements”.
Possibly with a need to sum over several pre-images …

$$\left|\frac{\Delta x}{\Delta y}\right| = \left|\frac{1}{g'(x)}\right| = |\varphi'(x)|, \qquad \text{if} \quad \varphi = g^{-1}, x = \varphi(y)$$

We‘ll stick to this inverse perspective …

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Chapter 3: Functions of Random Variables

3.1 Functions of Random Variables

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Functions of Random Variables

Determine densities like $\boldsymbol{f_{UV}(u, v)}$, etc. directly

We try to calculate:

$$\iint \boldsymbol{f_{XY}}(x, y)\, dx\, dy$$

via

$$\iint \boldsymbol{f_{XY}}\big(x(u, v), y(u, v)\big) \frac{\Delta(x, y)}{\Delta(u, v)} du\, dv = \iint \boldsymbol{f_{UV}}(\boldsymbol{u}, \boldsymbol{v}) du\, dv$$

based on the ratio $\frac{\Delta(x,y)}{\Delta(u,v)}$ of corresponding “area-elements”.

So, how large is that ratio?

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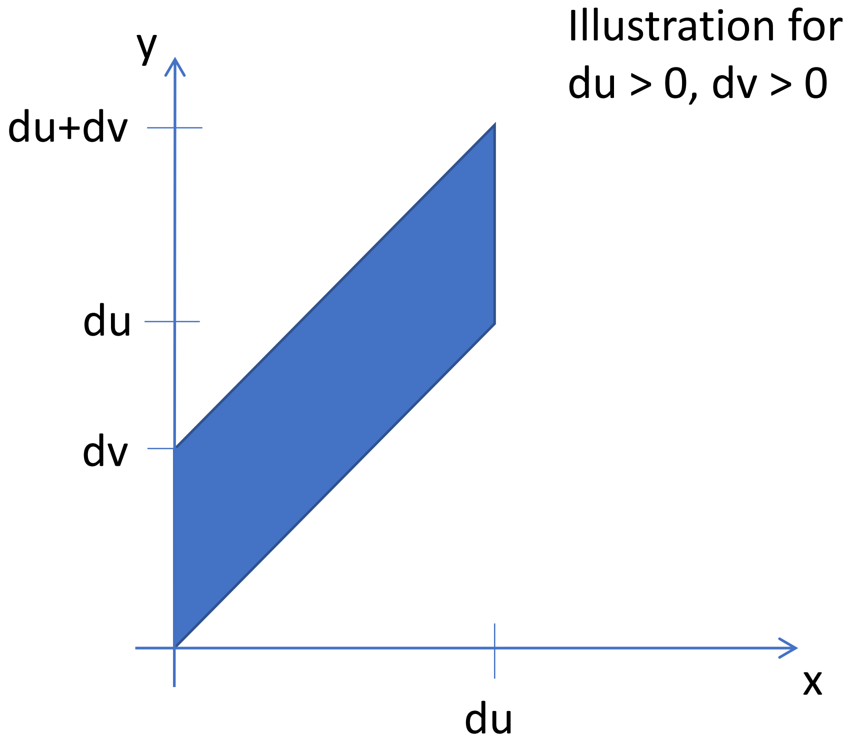
Functions of Random Variables

Determine $f_{UV}(u,v)$ … direct method



<u>Special case:</u>

Use $x = u$ and $y = u + v$

Start with a rectangle in the (u,v) – plane.

Consider the four points at positions:

$(u_i, v_i) = (0,0), (du, 0), (du, dv), (0, dv)$

The enclosed area is du·dv … or its absolute value.

Transform to the (x,y) - plane:

$(x_i, y_i) = (0,0), (du, du), (du, du + dv), (0, dv)$

with the same area, and the ratio is 1.

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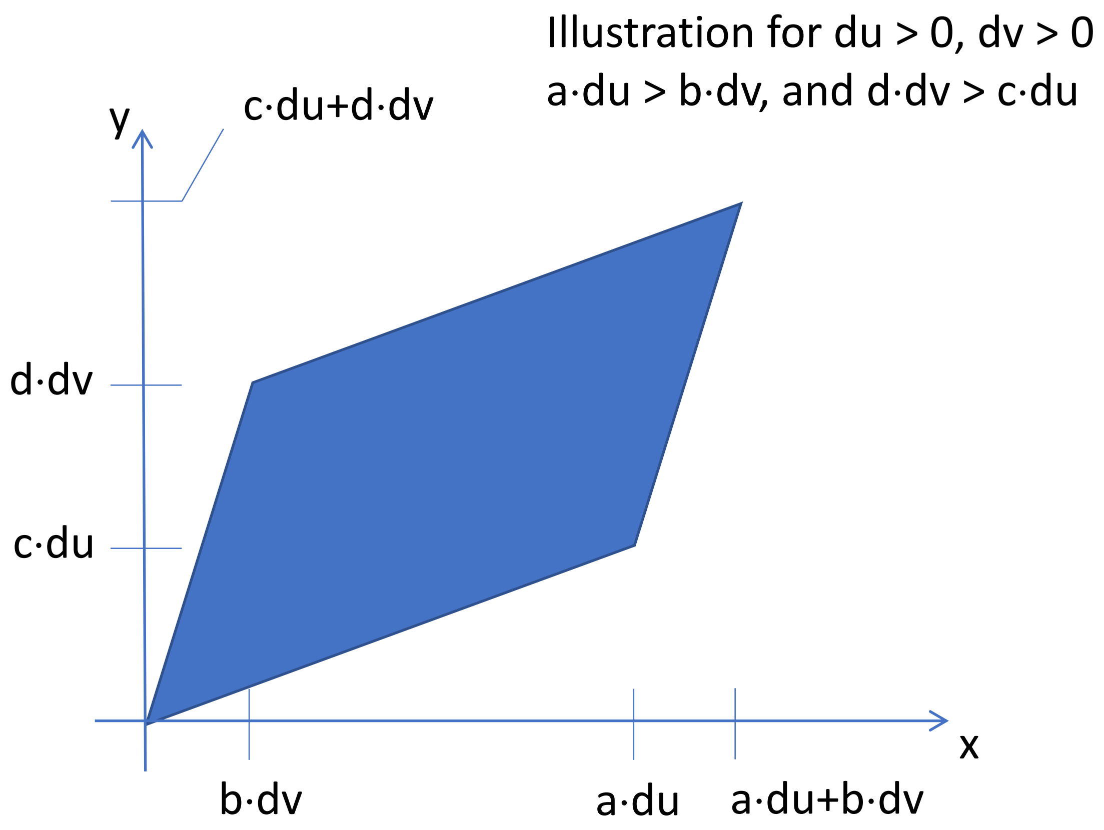

Functions of Random Variables

Determine $\boldsymbol{f_{UV}(u,v)}$ … direct method

<u>General case:</u> Use $x = au + bv$ and $y = cu + dv$

Again, start with a rectangle in the (u,v) – plane.

Consider the same four points as before:

$(u_i, v_i) = (0,0), (du, 0), (du, dv), (0, dv)$

The enclosed area is du·dv.

Now, transform to the (x,y) - plane:

$(x_i, y_i) = (0,0), (a \cdot du, c \cdot du),$

$(a \cdot du + b \cdot dv, c \cdot du + d \cdot dv), (b \cdot dv, d \cdot dv)$

Assume that a, b, c, d are all positive.

What's the area in this case ?

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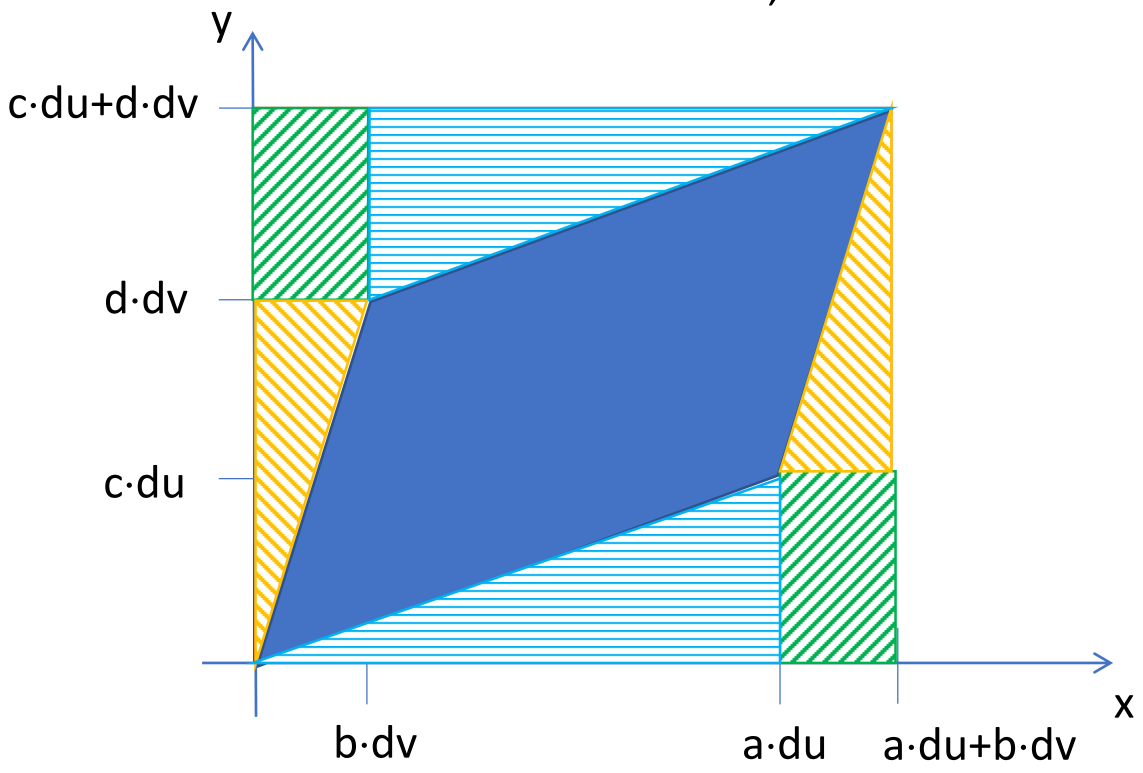
Functions of Random Variables

Determine $f_{UV}(u, v)$ … direct method

Illustration for du > 0, dv > 0
a·du > b·dv, and d·dv > c·du



<u>General case:</u> Use $x = au + bv$ and $y = cu + dv$

Area of the blue parallelogram:

$$(a \cdot du + b \cdot dv) \cdot (c \cdot du + d \cdot dv)$$

$$-2b \cdot dv \cdot c \cdot du - b \cdot dv \cdot d \cdot dv - a \cdot du \cdot c \cdot du$$

$$= (ad - bc) \cdot du \cdot dv = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \cdot du \cdot dv$$

So, here the ratio is

$$\frac{\Delta(x, y)}{\Delta(u, v)} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

If the determinant happens to be negative, we need to take its absolute value.

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Functions of Random Variables

Determine densities like $\boldsymbol{f_{UV}(u, v)}$ … direct method

Jacobian matrix perspective:

Use $x_i = \varphi_i(y_1, \dots, y_n)$ … locally, we have a linear mapping

$$\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{y})$$

$$\boldsymbol{x} + \boldsymbol{dx} = \boldsymbol{\varphi}(\boldsymbol{y} + \boldsymbol{dy}) \approx \boldsymbol{\varphi}(\boldsymbol{y}) + \boldsymbol{J}\ \boldsymbol{dy}$$

$$\boldsymbol{dx} = \mathbf{J}\ \mathbf{dy}$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial \varphi_1}{\partial y_1} & \cdots & \frac{\partial \varphi_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial \varphi_n}{\partial y_1} & \cdots & \frac{\partial \varphi_n}{\partial y_n} \end{bmatrix}$$

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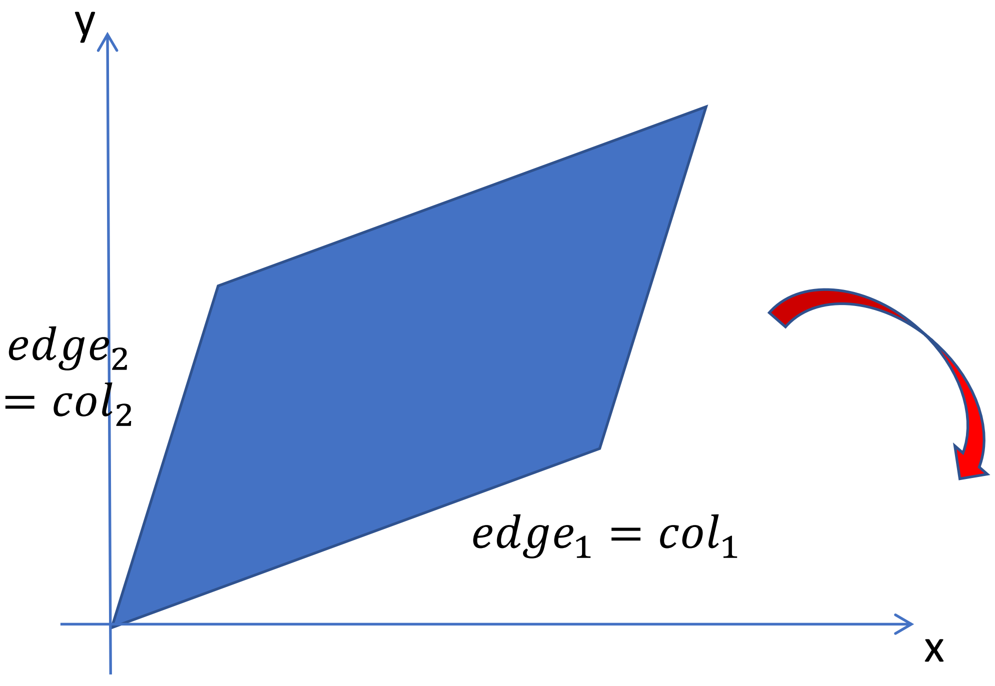
What did we do?

$$\overset{\text{base}}{} \quad \overset{\text{height}}{}$$

$$J = \begin{bmatrix} * & * \\ * & * \end{bmatrix} \xrightarrow{rotation} \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} = RJ$$

Notice: Area = $|\det(RJ)| = \underbrace{\det(R)}_{=1} |\det(J)| = |\det(J)|$



Functions of Random Variables

Determine densities like $f_{UV}(u, v)$ … direct method

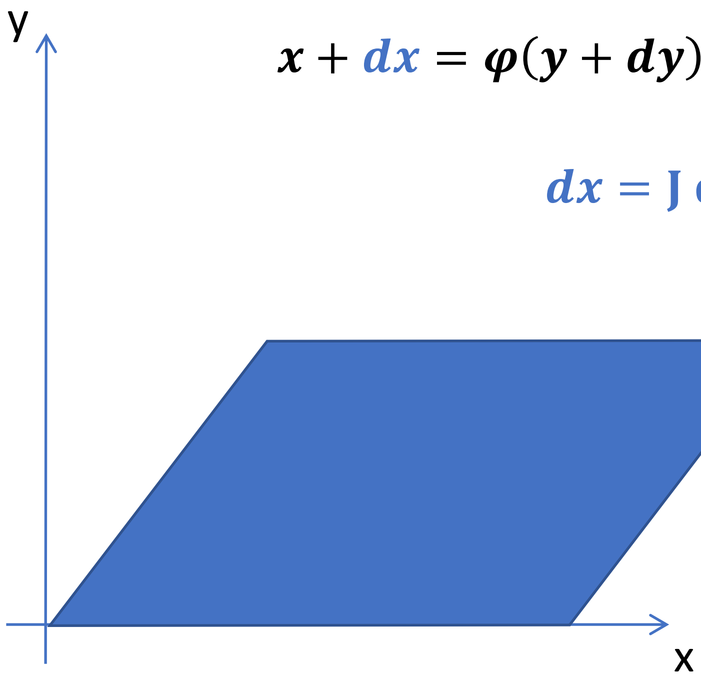
<u>Jacobian matrix perspective – different perspectives even:</u>

Use $x_i = \varphi_i(y_1, \dots, y_n)$ … locally, we have a linear mapping

$$\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{y})$$

$$\boldsymbol{x} + \boldsymbol{dx} = \boldsymbol{\varphi}(\boldsymbol{y} + \boldsymbol{dy}) \approx \boldsymbol{\varphi}(\boldsymbol{y}) + \boldsymbol{J}\,\boldsymbol{dy}$$

$$\boldsymbol{dx} = \mathbf{J}\ \mathbf{dy}$$



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Higher dimensions …

$$J = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \xrightarrow{rotation} \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \xrightarrow{rotation} \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$$

Edge 1 to axis 1
Edge 2 to plane (1,2) … keep angle with edge1 = axis 1
…

Area / Volume

$$= |\det(\ldots R_2\, R_1\, J)| = |\det(J)|$$

Functions of Random Variables

Determine densities like $f_{UV}(u, v)$ … direct method

Jacobian matrix perspective:

Use $x_i = \varphi_i(y_1, \ldots, y_n)$ … locally, we have a linear mapping

$$\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{y})$$

$$\boldsymbol{x} + \boldsymbol{dx} = \boldsymbol{\varphi}(\boldsymbol{y} + \boldsymbol{dy}) \approx \boldsymbol{\varphi}(\boldsymbol{y}) + \boldsymbol{J}\, \boldsymbol{dy}$$

$$\boldsymbol{dx} = \mathbf{J}\ \mathbf{dy}$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial \varphi_1}{\partial y_1} & \cdots & \frac{\partial \varphi_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial \varphi_n}{\partial y_1} & \cdots & \frac{\partial \varphi_n}{\partial y_n} \end{bmatrix}$$

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Higher dimensions …

Or via induction:

Just rotate the base ``area´´ away from the last axis.

$$\begin{bmatrix} * & \cdots & * & * \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ * & \cdots & * & * \end{bmatrix} \xrightarrow{rotation} \begin{bmatrix} * & \cdots & * & * \\ \vdots & & \vdots & \vdots \\ * & \cdots & * & \vdots \\ 0 & \cdots & 0 & * \end{bmatrix}$$

Intuition: Real 3-d box …

Reasoning: Base is (n-1)-dim. … rotate into the (n-1)-dim subspace spanned by axes 1, …, n-1
To see the rotation it takes, consider the normal vectors (orth. complements) of those two (n-1)-dim. subspaces, and the angle in between.

Functions of Random Variables

Determine densities like $\boldsymbol{f_{UV}(u,v)}$ … direct method

Jacobian matrix perspective:

Use $x_i = \varphi_i(y_1, \dots, y_n)$ … locally, we have a linear mapping

$$\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{y})$$

$$\boldsymbol{x} + \boldsymbol{dx} = \boldsymbol{\varphi}(\boldsymbol{y} + \boldsymbol{dy}) \approx \boldsymbol{\varphi}(\boldsymbol{y}) + \boldsymbol{J}\,\boldsymbol{dy}$$

$$\boldsymbol{dx} = \mathbf{J}\ \mathbf{dy}$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial \varphi_1}{\partial y_1} & \cdots & \frac{\partial \varphi_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial \varphi_n}{\partial y_1} & \cdots & \frac{\partial \varphi_n}{\partial y_n} \end{bmatrix}$$

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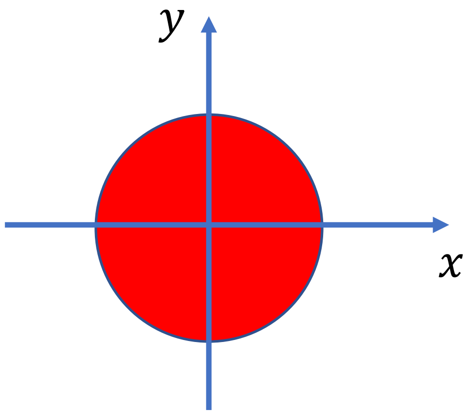

Functions of Random Variables

Determine densities like $f_{UV}(u, v)$ … direct method

<u>Example 1:</u>

Recall our earlier example:

… Change integration variables:

$$\begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix} \Rightarrow dxdy \to rdrd\theta$$

Jacobian matrix:

$$J = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \Rightarrow \det(J) = r(\cos^2\theta + \sin^2\theta) = r$$

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Jacobian matrix:

$$J = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \Rightarrow \det(J) = -\frac{1}{2} \Rightarrow |\det(J)| = \frac{1}{2}$$

Functions of Random Variables

Determine densities like $f_{UV}(u, v)$ … direct method

<u>Example 2:</u>

$$V = g(X, Y) = X + Y, \qquad W = h(X, Y) = X - Y$$

$$\Leftrightarrow \quad X = \frac{V + W}{2}, \qquad Y = \frac{V - W}{2}$$

➔

$$f_{VW}(v, w) = \frac{1}{2} f_{XY}\left(\frac{v + w}{2}, \frac{v - w}{2}\right)$$

<u>Mind:</u>

$$f_{VW}(v, w) = f_{XY}\big(x(v, w), y(v, w)\big) \frac{\Delta(x, y)}{\Delta(v, w)}$$

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Jacobian matrix:

$$\frac{\Delta(v,w)}{\Delta(x,y)} = \left|\det\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right)\right| = 2$$

Functions of Random Variables

Determine densities like $f_{UV}(u, v)$ … direct method

<u>Example 2 - Alternative Perspective:</u>

$$V \; = \; g(X,Y) = X + Y, \qquad W \; = h(X,Y) = X - Y$$

$$\Leftrightarrow \quad X \; = \frac{V+W}{2}\,, \qquad Y \; = \frac{V-W}{2}$$

$\Rightarrow$

$$f_{VW}(v,w) = \frac{1}{2} f_{XY}\left(\frac{v+w}{2}, \frac{v-w}{2}\right)$$

<u>**Mind:**</u>

$$\boldsymbol{f_{VW}(v,w)} = \boldsymbol{f_{XY}}\big(x(v,w), y(v,w)\big)\,\frac{\Delta(x,y)}{\Delta(v,w)}$$

The End

Next time: Recap Chp. 3
& Exam-type tasks