Random variables

When we use probability theorem in real life, our sample space can be viewed as the set of all possible scenarios.

Example: Modeling the stock market. Each point ω in the sample space can be viewed as a possible state of the world at some point in the future.

We are typically *not* interested in ω itself, but rather in quantities that depend on ω.

A typical example is the price of a stock S, which depends on the state of the world ω, and hence can be viewed as a function on the sample space Ω.

More generally, we are interested in assigning a numerical quantity to an outcome $\omega \in \Omega$ of the experiment that captures one particular aspect. This leads to the following definition.

Random variables: definition

Definition

Consider a probability space with the sample space Ω. A function

$$X : \Omega \to \mathbb{R}$$

is called a real valued *random variable*. Similarly, a function $X : \Omega \to \mathbb{R}^n$ is called a vector-valued random variable.

Example

Suppose that the flipping of a coin can result in heads with probability p and in tails with probability $1 - p$, This coin is tossed n times. For each outcome ω consider:

$$X_1(\omega) = \{\text{first head}\},$$
$$X_2(\omega) = \{\text{first tail}\},$$
$$X_3(\omega) = \{\text{total number of H}\},$$
$$X_4(\omega) = \{\text{total number of T}\}.X_5(\omega) \qquad = \{\text{total number of HH}\}.$$

Definition

A random variable X is called **discrete** if it takes a finite or countable number of values. The **probability mass function** of X is the function defined by

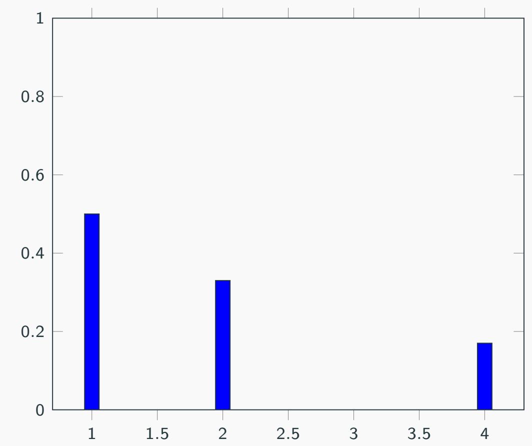
$$p(x) = \mathbb{P}[X = x].$$

Example

Suppose X takes values $1, 2, 4$ with probabilities:

$$\mathbb{P}[X = 1] = \frac{1}{2}, \quad \mathbb{P}[X = 2] = \frac{1}{3}, \quad \mathbb{P}[X = 4] = \frac{1}{6}.$$

x	1	2	4
$\mathbb{P}[X = x]$	$1/2$	$1/3$	$1/6$



Bernoulli random variables

The simplest discrete random variables are Bernoulli random variables.

Definition

A random variable X is called the *Bernoulli* random variable with parameter p if it only takes values 0 and 1, and

$$\mathbb{P}[X = 1] = p, \qquad \mathbb{P}[X = 0] = 1 - p.$$

A Bernoulli random variable X tells us whether something happened or not. The probability of happening $\mathbb{P}[X = 1]$ is called the parameter of X.

Example

A die is rolled. Let X be the random variable that tells us whether the outcome is larger than 4 or not. X has parameter $p = 2/6$.

⚀ ⚁ ⚂ ⚃ $\longrightarrow$ 0

⚄ ⚅ $\longrightarrow$ 1

Computations with discrete random variables

Example

Suppose X is a random variable taking values $0, 1, -1$ with

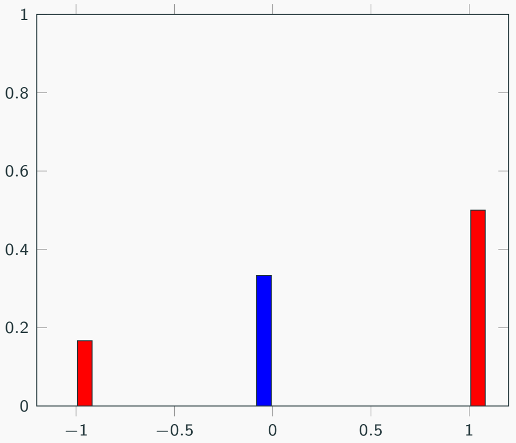
$$\mathbb{P}[X = 1] = \frac{1}{2}, \quad \mathbb{P}[X = 0] = \frac{1}{3}, \quad \mathbb{P}[X = -1] = \frac{1}{6}/$$

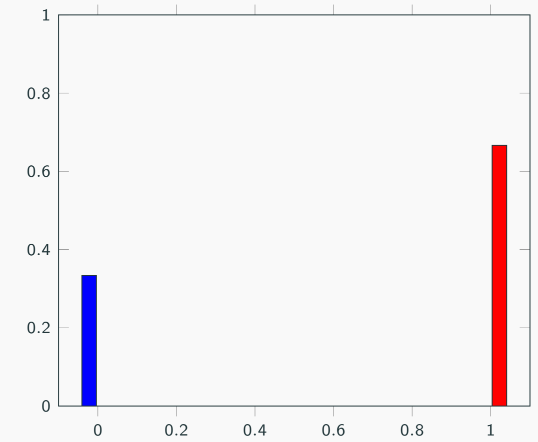
1. Compute $\mathbb{P}[X \geq 0]$.
2. Compute $\mathbb{P}[X \neq 0]$.
3. Find the probability mass function for $Y = X^2$.

$$\mathbb{P}[X \geq 0] = \mathbb{P}[X = 0] + \mathbb{P}[X = 1] = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

$$\mathbb{P}[X \neq 0] = \mathbb{P}[X = -1] + \mathbb{P}[X = 1] = \frac{1}{6} + \frac{1}{2} = \frac{2}{3}.$$

Computations with discrete random variables





y	0	1
$\mathbb{P}[Y = y]$	$1/3$	$2/3$

Binomial distribution

Consider a coin that comes up heads with probability p. The coin is thrown n times. Suppose that the outcomes of different rounds are independent.

Suppose $n = 2$: Then

$$\begin{aligned}
\mathbb{P}[HH] &= \mathbb{P}[\text{ first} H]\,\mathbb{P}[\text{ second} H] = p^2. \\
\mathbb{P}[HT] &= \mathbb{P}[\text{ first} H]\,\mathbb{P}[\text{ second} T] = p(1-p). \\
\mathbb{P}[TH] &= \mathbb{P}[\text{ first} T]\,\mathbb{P}[\text{ second} H] = (1-p)p. \\
\mathbb{P}[TT] &= \mathbb{P}[\text{ first} T]\,\mathbb{P}[\text{ second} T] = (1-p)^2.
\end{aligned} \tag{1}$$

$$\begin{array}{cccc}
\text{HH} & \longrightarrow & 2 & p^2 \\
\text{HT TH} & \longrightarrow & 1 & 2p(1-p) \\
\text{TT} & \longrightarrow & 0 & (1-p)^2
\end{array}$$

Suppose $n = 3$. Then the number of heads could be $0, 1, 2, 3$

$$\begin{array}{ccc c}
\text{HHH} & \longrightarrow & 3 & p^3 \\
\text{HHT HTH THH} & \longrightarrow & 2 & 3p^2(1-p) \\
\text{HTT THT TTH} & \longrightarrow & 1 & 3p(1-p)^2 \\
\text{TTT} & \longrightarrow & 0 & (1-p)^3
\end{array}$$

Definition

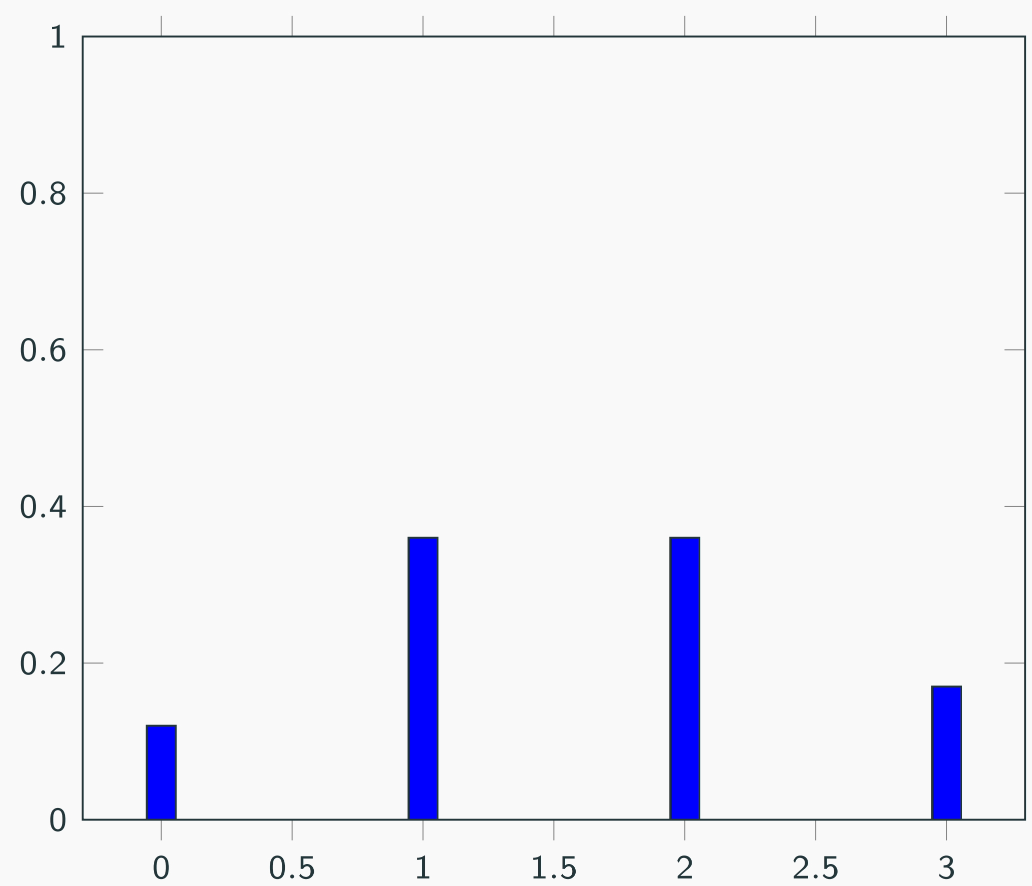
A random variable X has the Binomial distribution with parameters (n, p) if,

$$\mathbb{P}[X = k] = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{if } 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}$$

Binomial distribution bar charts

Suppose $n = 3$ and $p = 1/2$. Then the values X can attain are $0, 1, 2, 3$. We have

x	0	1	2	3
$\mathbb{P}[X = x]$	1/8	3/8	3/8	1/8



Binomial distribution bar charts

Suppose $n = 3$ and $p = 2/3$. Then the values X can attain are $0, 1, 2, 3$. We have

x	0	1	2	3
$\mathbb{P}[X = x]$	$1/27$	$6/27$	$12/27$	$8/27$