Lecture 3 (Review)

Ω: sample space , $A \subseteq \Omega$ event

$$0 \leq \mathbb{P}(A) \leq 1$$

Axioms of probability:

- $\mathbb{P}(\Omega) = 1$
- If A, B are disjoint (i.e. $A \cap B = \emptyset$) then

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$$

Consequences:

I: $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$ (complement rule)

II. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$ (union rule)

Applications

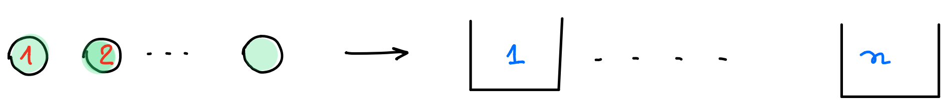
Example Suppose n balls are randomly placed into n boxes.
Find the probability that the first box contains at least one ball.

n balls in n boxes : $|\Omega| = n^n$

A: first box contains at least one ball

A^c = first box is empty.

$$|A^c| = ?$$



$$\mathbb{P}(A^c) = \frac{(n-1)^n}{n^n} = \left(1 - \frac{1}{n}\right)^n$$

$$\mathbb{P}(A) = 1 - \left(1 - \frac{1}{n}\right)^n.$$

Lecture 3

The inclusion and Exclusion Principle

Theorem

(Union of disjoint event)

(a) Suppose $A_1, \ldots, A_n$ are n events which are mutually disjoint,

i.e. $A_i \cap A_j = \emptyset$ for $i \neq j$. Then

$$\mathbb{P}(A_1 \cup \cdots \cup A_n) = \mathbb{P}(A_1) + \cdots + \mathbb{P}(A_n).$$

(union of non-disjoint events)

(b) Suppose $A_1, \ldots, A_n$ are n events. Then

$$\mathbb{P}(A_1 \cup \cdots \cup A_n) \leq \mathbb{P}(A_1) + \cdots + \mathbb{P}(A_n).$$

Proof.

For (a):

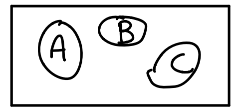
$$\begin{aligned}\mathbb{P}(A_1 \cup A_2 \cup A_3) &= \mathbb{P}(A_1 \cup (A_2 \cup A_3)) \\ &\leq \mathbb{P}(A_1) + \mathbb{P}(A_2 \cup A_3) \\ &\leq \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3).\end{aligned}$$

$$\begin{aligned}\mathbb{P}(A_1 \cup A_2 \cup A_3) &= \mathbb{P}(A_1 \cup (A_2 \cup A_3)) \\ &\leq \mathbb{P}(A_1) + \mathbb{P}(A_2 \cup A_3) \\ &\leq \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3).\end{aligned}$$

Example If $\mathbb{P}(A)=0.02$, $\mathbb{P}(B)=0.03$, $\mathbb{P}(C)=0.05$

- If we know that $A\cap B=A\cap C=B\cap C=\emptyset$

Then $\mathbb{P}(A\cup B\cup C)=0.02+0.03+0.05=0.10$



- If we don't know, all we can say is

$$0.05\leq \mathbb{P}(A\cup B\cup C)\leq 0.10$$

Union of non-disjoint events II

$$\mathbb{P}(A_1\cup A_2)=\mathbb{P}(A_1)+\mathbb{P}(A_2)-\mathbb{P}(A_1\cap A_2).$$

Let $n=3$.

$$\mathbb{P}(A_1\cup A_2\cup A_3)=\mathbb{P}(A_1\cup(A_2\cup A_3))$$

$$=\mathbb{P}(A_1)+\mathbb{P}(A_2\cup A_3)-\mathbb{P}(A_1\cap(A_2\cup A_3))$$

$$\mathbb{P}(A_2\cup A_3)=\mathbb{P}(A_2)+\mathbb{P}(A_3)-\mathbb{P}(A_2\cap A_3)$$

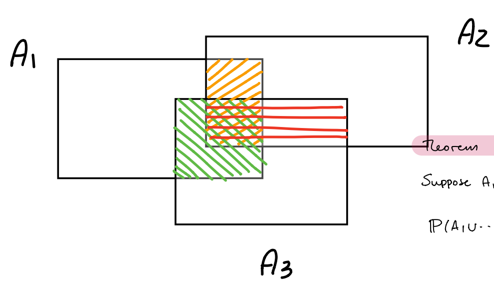
$$\mathbb{P}(A_1\cap(A_2\cup A_3))=\mathbb{P}((A_1\cap A_2)\cup(A_1\cap A_3))$$

$$=\mathbb{P}(A_1\cap A_2)+\mathbb{P}(A_1\cap A_3)-\mathbb{P}(A_1\cap A_2\cap A_3)$$

So

$$\mathbb{P}(A_1\cup A_2\cup A_3)=\mathbb{P}(A_1)+\mathbb{P}(A_2)+\mathbb{P}(A_3)-\mathbb{P}(A_2\cap A_3)$$

$$-\mathbb{P}(A_1\cap A_2)-\mathbb{P}(A_1\cap A_3)+\mathbb{P}(A_1\cap A_2\cap A_3).$$



Theorem (Principle of inclusion and exclusion)

Suppose $A_1, A_2, \dots, A_n$ are events in a prob. space. Then

$$\mathbb{P}(A_1\cup\cdots\cup A_n)=\sum_{i=1}^{n}\mathbb{P}(A_i)-\sum_{i<j}\mathbb{P}(A_i\cap A_j)+\sum_{i<j<k}\mathbb{P}(A_i\cap A_j\cap A_k)-\cdots+(-1)^{n+1}\mathbb{P}(A_1\cap\cdots\cap A_n)$$

$$\mathbb{P}(A_1 \cup A_2 \cup A_3 \cup A_4) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) + \mathbb{P}(A_4)$$
$$- \big(\mathbb{P}(A_1 \cap A_2) + \mathbb{P}(A_1 \cap A_3) + \mathbb{P}(A_1 \cap A_4) + \mathbb{P}(A_2 \cap A_3) + \mathbb{P}(A_2 \cap A_4) + \mathbb{P}(A_3 \cap A_4) \big)$$
$$+ \mathbb{P}(A_1 \cap A_2 \cap A_3) + \mathbb{P}(A_1 \cap A_2 \cap A_3) + \mathbb{P}(A_1 \cap A_3 \cap A_4) + \mathbb{P}(A_2 \cap A_3 \cap A_4)$$
$$- \mathbb{P}(A_1 \cap A_2 \cap A_3 \cap A_4)$$

Example A 3-digit number is randomly chosen. Find the probability of the event that at least one of its digits is even.

$$\Omega = \{100, 101, \dots, 999\}$$
$$A = \{ n = \boxed{a}\boxed{b}\boxed{c} : a \text{ or } b \text{ or } c \text{ is even} \}$$

$A = A_1 \cup A_2 \cup A_3$.

$A_1 =$ first digit (from the left) is even
$A_2 =$ second " is even
$A_3 =$ third " " "

$\mathbb{P}(A_1) =$ ___ $\qquad \mathbb{P}(A_2) = \qquad \mathbb{P}(A_3) =$

$\mathbb{P}(A_1 \cap A_2) = \qquad \mathbb{P}(A_1 \cap A_3) = \qquad \mathbb{P}(A_2 \cap A_3) =$

$\mathbb{P}(A_1 \cap A_2 \cap A_3) =$

Alternative solution

$$\mathbb{P}(A^c) = \mathbb{P}(\text{all digits are odd}) = \frac{5 \times 5 \times 5}{900} = \frac{5}{36}$$

So $\mathbb{P}(A) = 1 - \frac{5}{36} = \frac{31}{36}$

Example An integer n is randomly chosen from the set $\{1, 2, 3, \ldots, 100\}$. Find the probability of the event that n is divisible by 2 or 3 or 5 ?

Let us denote

A : n is divisible by 2

B : n is divisible by 3

C : n is divisible by 5

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{50}{100}$$

$$\mathbb{P}(C) = \frac{33}{100}$$

$$\mathbb{P}(B) = \frac{}{|\Omega|} = \frac{20}{100}$$

$$\mathbb{P}(A \cap B) = \frac{16}{100}$$

$$\mathbb{P}(B \cap C) = \frac{6}{100}$$

$$\mathbb{P}(A \cap C) = \frac{10}{100}$$

$$\mathbb{P}(A \cap B \cap C) = \frac{3}{100}$$

$$\mathbb{P}(A \cup B \cup C) = \frac{74}{100}$$

Example

n letters (written to different people) are randomly placed into n envelopes. Find the probability that no letter is placed into the right envelope.

A : no letter is placed in the right envelope

n	$\mathbb{P}(A)$
1	0
2	$1/2$
3	$2/6 = 1/3$

$A^c =$ at least one letter is placed in the right envelope

$A_k =$ event that letter #k is placed in envelope k.

So $$A^c = A_1 \cup A_2 \cup \cdots \cup A_k.$$

$$\mathbb{P}(A_i) = \frac{1}{n}$$

$$\mathbb{P}(A_i \cap A_j) = \frac{1}{n(n-1)}$$

$$\mathbb{P}(A_i \cap A_j \cap A_k) = \frac{1}{n(n-1)(n-2)}$$

So

$$\mathbb{P}(A^c) = n \cdot \frac{1}{n} - \binom{n}{2} \cdot \frac{1}{n(n-1)} + \binom{n}{3} \frac{1}{n(n-1)(n-2)}$$

$$- \cdots + (-1)^{n+1} \binom{n}{n} \cdot \frac{1}{n!}$$

$$\mathbb{P}(A) = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \cdot \frac{1}{n!}$$