Elements of Probability

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Conditioning

Sometimes in order to computer the probability of an event A, it is easier to split the sample space tp

$$\Omega = B_1 \cup B_2 \cdots \cup B_n.$$

$$\begin{aligned}\mathbb{P}[A] &= \mathbb{P}[\cup_{i=1}^n (A \cap B_i)] \\ &= \sum_{i=1}^n \mathbb{P}[A \cap B_i] \\ &= \sum_{i=1}^n \mathbb{P}[A|B_i]\,\mathbb{P}[B_i].\end{aligned}$$

Theorem (Conditioning)

Let $\Omega = B_1 \cup B_2 \cdots \cup B_n$ be a partitioning of the sample space and A be an event. Then

$$\mathbb{P}[A] = \sum_{i=1}^n \mathbb{P}[A|B_i]\,\mathbb{P}[B_i].$$

Conditioning examples

Example

Alex has 5 coins in his pocket.Two are double-headed. one is double-tailed and the other two are normal.One of the coins is randomly chosen and flipped.

1. What is the probability that the outcome is heads?
2. He opens his eyes and sees that the outcome is heads.What is the probability that the flipped coin is double-headed?

B_{HH}, B_{TT}, B_{HT} :double-headed, double-tailed or normal. A: outcome is heads.

$$\mathbb{P}[A|B_{HH}] = 1, \quad \mathbb{P}[A|B_{TT}] = 0, \quad \mathbb{P}[A|B_{HT}] = \frac{1}{2}.$$

$$\begin{aligned}\mathbb{P}[A] &= \mathbb{P}[A|B_{HH}]\mathbb{P}[B_{HH}] + \mathbb{P}[A|B_{TT}]\mathbb{P}[B_{TT}] + \mathbb{P}[A|B_{HT}]\mathbb{P}[B_{HT}] \\ &= 1 \cdot \frac{2}{5} + 0 \cdot \frac{1}{5} + \frac{1}{2} \cdot \frac{2}{5} = \frac{3}{5}\end{aligned}$$

$\mathbb{P}[B_{HH}|A] = \frac{\mathbb{P}[A|B_{HH}]\mathbb{P}[B_{HH}]}{\mathbb{P}(A)} = \frac{\frac{2}{5}}{\frac{3}{5}} = \frac{2}{3}.$

Conditioning examples

Example

An urn contains r red and b blue balls. A ball is drawn from the urn and discarded.

1. What is the probability that the discarded ball is blue?
2. Without knowing the color of the first color, what is the probability that a second ball drawn is blue?

R_1: first ball red. B_1: first ball blue. B_2: the second ball blue.

$$\mathbb{P}\left[B_2\right] = \mathbb{P}\left[B_2|B_1\right]\mathbb{P}\left[B_1\right] + \mathbb{P}\left[B_2|R_1\right]\mathbb{P}\left[R_1\right]$$

$$\mathbb{P}\left[B_1\right] = \frac{b}{b+r}, \quad \mathbb{P}\left[R_1\right] = \frac{r}{b+r}$$

$$\mathbb{P}\left[B_2|B_1\right] = \frac{b-1}{b+r-1}, \quad \mathbb{P}\left[B_2|R_1\right] = \frac{b}{b+r-1}$$

$$\mathbb{P}\left[B_2\right] = \mathbb{P}\left[B_2|B_1\right]\mathbb{P}\left[B_1\right] + \mathbb{P}\left[B_2|R_1\right]\mathbb{P}\left[R_1\right] = \frac{b}{b+r}.$$

Bayes' theorem

Imagine a real-world situation in which an event A can be *caused* by one of events $B_1, \ldots, B_n$. We would like to compute the probability of the event B_i was the cause in light of the evidence that A has occurred.

Theorem (Bayes' Formula)
Let $\Omega = B_1 \cup B_2 \cup \cdots \cup B_n$ be a partitioning of the sample space Ω. Then we have

$$\mathbb{P}\left[B_i | A\right] = \frac{\mathbb{P}\left[A|B_i\right]\mathbb{P}\left[B_i\right]}{\sum_{j=1}^{n}\mathbb{P}\left[A|B_j\right]\mathbb{P}\left[B_j\right]}.$$

Example

Through a transmission channel two types of messages can be sent: $\mathbf{0}$ and $\mathbf{1}$. We assume that 40% of the time a $\mathbf{1}$ is transmitted. The probability that $\mathbf{0}$ is correctly received is 0.80 and the probability that a transmitted $\mathbf{1}$ is correctly received is 0.90. Determine

a) How often is a $\mathbf{0}$ being received?

b) Given that a $\mathbf{1}$ received, the probability that $\mathbf{1}$ was transmitted.

$T_{\mathbf{0}}$: $\mathbf{0}$ is transmitted. $T_{\mathbf{1}}$: $\mathbf{1}$ is transmitted.

$R_{\mathbf{0}}$: $\mathbf{0}$ is received. $R_{\mathbf{1}}$: $\mathbf{1}$ is received.

$$\mathbb{P}[T_{\mathbf{1}}] = \frac{4}{10}, \qquad \mathbb{P}[T_{\mathbf{0}}] = \frac{6}{10}$$

$$\mathbb{P}[R_{\mathbf{0}} | T_{\mathbf{0}}] = \frac{8}{10}, \quad \mathbb{P}[R_{\mathbf{1}} | T_{\mathbf{0}}] = \frac{2}{10}, \quad \mathbb{P}[R_{\mathbf{1}} | T_{\mathbf{1}}] = \frac{9}{10}, \quad \mathbb{P}[R_{\mathbf{0}} | T_{\mathbf{1}}] = \frac{1}{10}.$$

Bayes' theorem

$$\mathbb{P}[T_{\mathbf{1}}] = \frac{4}{10}, \qquad \mathbb{P}[T_{\mathbf{0}}] = \frac{6}{10}$$

$$\mathbb{P}[R_{\mathbf{0}}|T_{\mathbf{0}}] = \frac{8}{10}, \quad \mathbb{P}[R_{\mathbf{1}}|T_{\mathbf{0}}] = \frac{2}{10}, \quad \mathbb{P}[R_{\mathbf{1}}|T_{\mathbf{1}}] = \frac{9}{10}, \quad \mathbb{P}[R_{\mathbf{0}}|T_{\mathbf{1}}] = \frac{1}{10}.$$

$$\mathbb{P}[R_0] = \mathbb{P}[R_0|T_0]\,\mathbb{P}[T_0] + \mathbb{P}[R_0|T_1]\,\mathbb{P}[T_1] = \frac{8}{10}\cdot\frac{6}{10} + \frac{1}{10}\cdot\frac{4}{10} = \frac{52}{100}.$$

$$\mathbb{P}[T_1|R_1] = \frac{\mathbb{P}[R_1|T_1]\,\mathbb{P}[T_1]}{\mathbb{P}[R_1|T_1]\,\mathbb{P}[T_1] + \mathbb{P}[R_1|T_0]\,\mathbb{P}[T_0]} = \frac{\frac{9}{10}\cdot\frac{4}{10}}{\frac{9}{10}\cdot\frac{4}{10} + \frac{2}{10}\cdot\frac{6}{10}} = \frac{36}{48} = \frac{3}{4}.$$

Example

Suppose 5% of all cancers are malignant and suppose we have a test that is 90% accurate in determining malignancy. Suppose further that a test result has come back positive. Find the probability that the tumor is malignant.

M:the tumor is malignant. P the test is positive.

Assumption: $\mathbb{P}[P|M] = 0.90$ and $\mathbb{P}[P|M^c] = 0.10$.

$$\begin{aligned}\mathbb{P}[M|P] &= \frac{\mathbb{P}[P|M]\,\mathbb{P}[M]}{\mathbb{P}[P|M]\,\mathbb{P}[M] + \mathbb{P}[P|M^c]\,(\mathbb{P}[M^c]} \\ &= \frac{\frac{90}{100}\frac{5}{100}}{\frac{90}{100}\frac{5}{100} + \frac{10}{100}\frac{95}{100}} \\ &= \frac{450}{1400} = 0.32\end{aligned}$$

Independence

Recall from the previous section that for events A and B, the conditional probability of A given B is defined by

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}.$$

Definition

Events A and B are called independent when

$$\mathbb{P}[A|B] = \mathbb{P}[A],$$

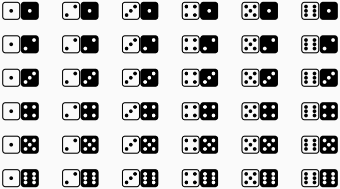
Equivalently when

$$\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B].$$

Examples of independence

A pair of dice are rolled. Consider the events

1. A: The first die's score is at most 3.
2. B: The second die's score is at least 5.
3. C: Sum of the scores of the two dice is equal to 6.



$$\mathbb{P}[A] = \frac{18}{36} = \frac{1}{2}.$$
$$\mathbb{P}[B] = \frac{12}{36} = \frac{1}{3}.$$
$$\mathbb{P}[C] = \frac{5}{36}.$$

$$\mathbb{P}[A \cap B] = \frac{6}{36} = \frac{1}{6} = \mathbb{P}[A]\,\mathbb{P}[B].$$
$$\mathbb{P}[A \cap C] = \frac{3}{36} \neq \frac{5}{72} = \mathbb{P}[A]\,\mathbb{P}[C].$$
$$\mathbb{P}[B \cap C] = \frac{1}{36} \neq \frac{5}{108} = \mathbb{P}[B]\,\mathbb{P}[C].$$

Make independence work for you

A biased coin turns up heads with probability 2/3 and tails with probability 1/3. How can this coin be used to a start a football match?

Throw the coin twice

$$\mathbb{P}[HT] = \frac{2}{3} \times \frac{1}{3} = \frac{2}{9}.$$

$$\mathbb{P}[TH] = \frac{1}{3} \times \frac{2}{3} = \frac{2}{9}.$$

$$\mathbb{P}[\text{ Undecided}] = 1 - \frac{4}{9} = \frac{5}{9} \approx 0.55$$

$$\mathbb{P}[\text{ Undecided after 5 repetitions}] \approx (0.55)^5 \approx 0.02.$$

Examples

Example

Werder Bremen football team wins each game with probability 20 percent and loses with probability 80 percent. What is the probability that they win exactly 4 games out of 34 games.

$$\mathbb{P}[E] = \binom{34}{4}\left(\frac{4}{10}\right)^{4}\left(\frac{6}{10}\right)^{30} \approx 0.09.$$