Lecture 9

Definition A random variable X is said to have a Poisson random variable with parameter λ if it takes values $k = 0, 1, 2, \ldots$ and moreover its PMF is given by

$$p_X(k) = \mathbb{P}[X=k] = e^{-\lambda} \cdot \frac{\lambda^k}{k!} \qquad k = 0, 1, 2, \ldots$$

Note

$$\sum_{k=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} \cdot e^{\lambda} = 1$$

Where does the Poisson random variable come from?

Consider binomial random variables with parameter n, p with

n large
p small

such that

$$\boxed{np = \lambda}$$

Let us consider

$$\mathbb{P}[X=0] = \binom{n}{0} p^0 (1-p)^n = \left(1 - \frac{\lambda}{n}\right)^n \approx e^{-\lambda}$$

where we used the formula

$$\boxed{\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x}$$

$$\mathbb{P}[X=1] = \binom{n}{1} p^1 (1-p)^{n-1} = n \cdot \frac{\lambda}{n} \left(1 - \frac{\lambda}{n}\right)^{n-1}$$

$$= \lambda \left(1-\frac{\lambda}{n}\right)^{n} \cdot \left(1-\frac{\lambda}{n}\right)^{-1}$$

$$\approx \lambda \cdot e^{-\lambda} \cdot 1$$

$$= e^{-\lambda} \cdot \lambda$$

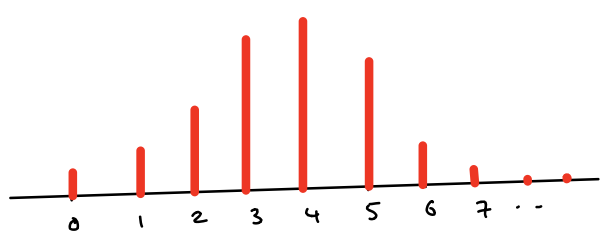
More generally,

$$\mathbb{P}[X=k] = \binom{n}{k} p^k (1-p)^{n-k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$\approx \frac{n^k}{k!} \cdot \frac{\lambda^k}{n^k} \left(1-\frac{\lambda}{n}\right)^n \cdot \left(1-\frac{\lambda}{n}\right)^{-k}$$

$$\approx \frac{1}{k!} \lambda^k \cdot e^{-\lambda}$$

$$= \frac{\lambda^k}{k!} e^{-\lambda}.$$



<u>Example</u> The number of daily car accidents in a city is a Poisson random variable with $\lambda = 2$. What is the probability that on a given day there are at most 3 accidents?

$$\mathbb{P}(X \le 3) = \mathbb{P}(X=0) + \mathbb{P}(X=1) + \mathbb{P}(X=2) + \mathbb{P}(X=3)$$

$$= e^{-2}\left(\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!}\right) = e^{-2} \cdot \frac{19}{3} = 0.85$$

Expectation and Variance

Suppose X is a discrete random variable taking values $x_1, x_2, \ldots, x_k$ with the PMF $P_X(x)$. Then the expected value or mean of X is defined by

$$\mathbb{E}[X] = \sum_{i=1}^{k} x_i P_X(x_i) = x_1 P_X(x_1) + \cdots + x_k P_X(x_k).$$

If X takes infinitely many values $x_1, x_2, \ldots$, we define similarly,

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} x_i P_X(x_i)$$

Example Suppose X is a random variable with

$$P_X(k) = \begin{cases} \lambda \cdot 2^k & k = 0,1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

Values

x	0	1	2	3
$P_X(x)$	λ	2λ	4λ	8λ
	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{8}{15}$

...

Find $\mathbb{E}[X]$.

<u>Solution</u> Since X is a random variable

$$P_X(0) + P_X(1) + P_X(2) + P_X(3) = 1$$

$$\lambda + 2\lambda + 4\lambda + 8\lambda = 1 \Rightarrow \lambda = \frac{1}{15}$$

Hence:

$$\mathbb{E}[X] = 0 \cdot P_X(0) + 1 \cdot P_X(1) + 2 \cdot P_X(2) + 3 \cdot P_X(3)$$

$$= 0 \cdot \frac{1}{15} + 1 \cdot \frac{2}{15} + 2 \cdot \frac{4}{15} + 3 \cdot \frac{8}{15}$$

$$= 0 + \frac{2}{15} + \frac{8}{15} + \frac{24}{15} = \frac{34}{15}.$$

Expected value of important random variables

- X Bernoulli with parameter p

X	0	1
	$1-p$	p

$$\mathbb{E}[X] = p \cdot 1 + (1-p) \cdot 0 = p$$

- X Binomial with parameters n, p

$$\mathbb{E}[X] = \sum_{k=0}^{n} \binom{n}{k} p^k \cdot (1-p)^{n-k} \cdot k = n \cdot p$$

- X Geometric with parameter p

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} p(1-p)^{k-1} \cdot k = \frac{1}{p}$$

- X Poisson with parameter λ : $\mathbb{E}[X] = \lambda$

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^k}{(k-1)!}$$

$$= \lambda \sum_{l=0}^{\infty} e^{-\lambda} \frac{\lambda^l}{l!} = \lambda .$$

Consider a lottery with outcomes $x_1, x_2, \ldots, x_n$ which are realized with probabilities $p_1, \ldots, p_n$. What is the value of this lottery?

<u>Example</u> Consider tossing a coin which comes up H with prob p and T with probability $1-p$.

Outcome	H	T
Payoff	1	0
Prob	1/2	1/2

what is the value of this game?

Outcome	H	T
Payoff	1	0
Prob	p	$1-p$

value = p.

In general, the value of a lottery with possible outcomes

$$x_1, x_2, \ldots, x_n$$

which can be realized with probabilities $p_1, \ldots, p_n$

is given by $E[\text{Payoff}] = x_1 \cdot p_1 + x_2 \cdot p_2 + \cdots + x_n p_n$

In other words

gamble with outcome $\begin{array}{c|c|c|c} x_1 & x_2 & \cdots & x_n \\ \hline p_1 & p_2 & & p_n \end{array}$ $\sim$ a sure amount of $x_1 p_1 + \cdots + x_n p_n$

What about

$\begin{array}{c|c|c} & 0€ & 10€ \\ \hline & 1/2 & 1/2 \end{array}$ vs. $\begin{array}{c|c} & 5€ \\ \hline & 1 \end{array}$

$\begin{array}{c|c|c} & 0€ & 1000,000€ \\ \hline & 1/2 & 1/2 \end{array}$ $\begin{array}{c|c} & 500,000 \\ \hline & 1 \end{array}$

even $\begin{array}{c|c} & 450,000 \\ \hline & 1 \end{array}$

Example (Bernoulli, 1738)

A fair coin is flipped until a heads comes up. Suppose that the payoff is given by the following table

N	1	2	3	4	5	6	...
Payoff	2€	4€	8€	16€	32€	64€	...
Probability	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	...

How much will you pay to enter this game?

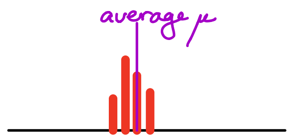
$$\mathbb{E}[\text{Payoff}] = 2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} + 8 \cdot \frac{1}{8} + 16 \cdot \frac{1}{16} + \cdots = \infty$$

Variance of a random variable

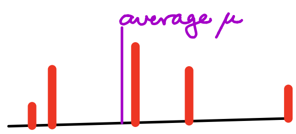
Consider a random variable X with $\mathbb{E}[X]=\mu$.

Wanted: a measure of how spread out X is:

<u>Example</u>



low variance



high variance

How to measure variance?

First attempt: take $Y = X - \mathbb{E}[X]$, and consider $\mathbb{E}[Y]$.

BUT: $\mathbb{E}[Y] = \mathbb{E}[X - \mu_X] = \mathbb{E}[X] - \mu_X = \mu_X - \mu_X = 0$

not surprising: due to cancellations.

Second attempt: take $Y = (X-\mu_X)^2$ and consider $\mathbb{E}[Y]$.

Definition The variance of a random variable X is defined by

$$\mathrm{Var}[X] = \mathbb{E}[(X-\mu)^2]$$

where $\mu = \mathbb{E}[X]$.

<u>Example</u> Suppose X is a discrete random variable with the PMF given by

x	1	2	5
$p_X(x)$	$1/5$	$2/5$	$2/5$

Find $\mathbb{E}[X]$ and $\mathrm{Var}[X]$

Find $\mathbb{E}[X]$ and $Var[X]$.

$$\mathbb{E}[X] = 1 \cdot \frac{1}{5} + 2 \cdot \frac{2}{5} + 5 \cdot \frac{2}{5} = 3$$

x	-2	-1	2
$P_Y(x)$	$1/5$	$2/5$	$2/5$

$$Var[X] = (-2)^2 \cdot \frac{1}{5} + (-1)^2 \cdot \frac{2}{5} + 2 \cdot \frac{2}{5}$$
$$= \frac{4}{5} + \frac{2}{5} + \frac{8}{5} = \frac{14}{5}$$

Example. Suppose

$$\mathbb{E}[X] = p$$
$$Var[X] = \mathbb{E}[(X-p)^2] = (1-p)^2 p + p^2(1-p)$$
$$= p(1-p)$$

Theorem Suppose X is a random variable. Then

$$Var[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

Proof

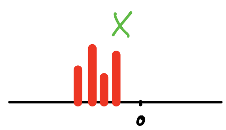
$$Var[X] = \mathbb{E}[(X-\mu_X)^2]$$
$$= \mathbb{E}[X^2 - 2\mu_X X + \mu_X^2]$$
$$= \mathbb{E}[X^2] - 2\mu_X \mathbb{E}[X] + \mu_X^2$$
$$= \mathbb{E}[X^2] - 2\mu_X^2 + \mu_X^2$$
$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

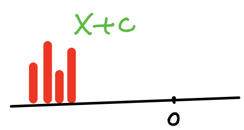
Theorem Suppose X is a random variable. Then:

- $\mathrm{Var}[X] \geqslant 0$
- $\mathrm{Var}[X+c] = \mathrm{Var}[X]$
- $\mathrm{Var}[cX] = c^2\,\mathrm{Var}[X]$

Proof

$$\begin{aligned}\mathrm{Var}[X+c] &= \mathbb{E}[(X+c)^2] - (\mathbb{E}[X]+c)^2 \\ &= \mathbb{E}[X^2+2cX+c^2] - \mathbb{E}[X]^2 - 2c\mathbb{E}[X] - c^2 \\ &= \mathbb{E}[X^2] - \mathbb{E}[X]^2\end{aligned}$$





Useful formulas

distribution	parameters	mean	variance
Bernoulli	p	p	$p(1-p)$
Binomial	n, p	np	$np(1-p)$
geometric	p	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson	λ	λ	λ