Elements of Probability

(3.1) Consider a discrete random variable X that with the PMF given by

$$p_X(x) = \begin{cases} k|x| & \text{if } x = -3, -1, 1, 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the value of k.
(b) Find $\mathbb{P}[|X| > 2]$.
(c) Find the PMF of the random variables $Y = X^2$ and $Z = X^3$.

Solution. (a) We have

$$1 = \sum_{x \in \{-3,-1,1,3\}} k|x| = 8k.$$

From here it follows that $k = \frac{1}{8}$.

(b) We have

$$\mathbb{P}[X > 2] = \mathbb{P}[X = \pm 3] = \frac{6}{8}.$$

(c) Note that $Y = X^2$ takes values $1, 9$, and

$$\mathbb{P}[Y = 1] = \mathbb{P}[X = 1] + \mathbb{P}[X = -1] = \frac{1}{8} + \frac{1}{8} = \frac{2}{8}.$$

Similarly,

$$\mathbb{P}[Y = 9] = \mathbb{P}[X = 3] + \mathbb{P}[X = -3] = \frac{3}{8} + \frac{3}{8} = \frac{6}{8}.$$

In a similar fashion, one can see that the probability mass function of Z is given by

$$p_Z(1) = p_Z(-1) = \frac{1}{8}, \quad p_Z(27) = p_Z(-27) = \frac{3}{8}.$$

(3.2) Theresa May is proposing a Brexit deal to the House of Commons which consists of 650 MPs.

(a) Assume first that each MP decides individually and independent of the rest of MPs to vote for the proposal with probability $p = 0.52$. Assuming that the prime minister needs more than half of the votes for passing the bill, find the probability that the bill is passed. You can use the special widget of *Wolfram's alpha* for doing the calculation.
(b) A group of 20 Pro-Brexit MPs have decided to vote against the bill. Compute the probability in part (a).
(c) Repeat part (a) and (b) for $p = 0.50$ and $p = 0.48$ and compare the results.

Solution. Let N denote the number of MPs who vote for the proposal. We are interested in $\mathbb{P}[N > 325]$. In part (a), N has binomial distribution $(n = 650, p = 0.52)$ and in part (b), we have $n = 630, p = 0,52$. Using Wolfram Alpha we have for part (a)

$$\mathbb{P}[N > 325] \approx 0.83$$

and for part (b) we have

$$\mathbb{P}[N > 325] \approx 0.56$$

Part (c) is similar, using $p = .5$ and $p = 0.48$ we have

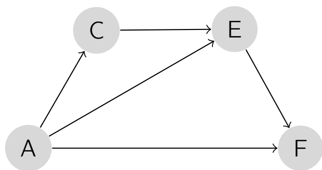
$$p = 0.50 \Rightarrow \mathbb{P}[N > 350] \approx 0.48$$

$$p = 0.48 \Rightarrow \mathbb{P}[N > 350] \approx 0.14$$

(3.3) A network connects computers A and F via intermediate nodes C, E as shown below. For each pair of directly connected nodes, there is a probability $p = 3/4$ that the connection from i to j is up. Assume that the link failures are independent events.

(a) Find the probability that the connection from A to F through at least one of the paths is up.

(b) Due to weather condition, connections AC, CE, EF, AE are simultaneously on or off, with probability $p = 3/4$. The connection AF which is not affected by weather is independency open with probability $p = 3/4$. Under this assumption, compute the probability that the connection from A to F through at least one of the paths is up, and compare the result to part (a).



Solution. Let us denote by C_1, C_2, and C_3 the events that the paths $ACEF$, AEF, and AF are up, respectively. Note that $ACEF$ is up, when all connections AC, CE and EF are up. Since the failures in roads are independent we have

$$\mathbb{P}[C_1] = p^3.$$

Similarly, we have

$$\mathbb{P}[C_2] = p^2, \qquad \mathbb{P}[C_3] = p.$$

Let us now consider $C_1 \cap C_2$. Note that $C_1 \cap C_2$ entails that all four paths AC, CE, EF, AE are up, and hence $\mathbb{P}[C_1 \cap C_2] = p^4$. Similarly, we obtain

$$\mathbb{P}[C_1 \cap C_3] = p^4, \qquad \mathbb{P}[C_2 \cap C_3] = p^3, \qquad \mathbb{P}[C_1 \cap C_2 \cap C_3] = p^5.$$

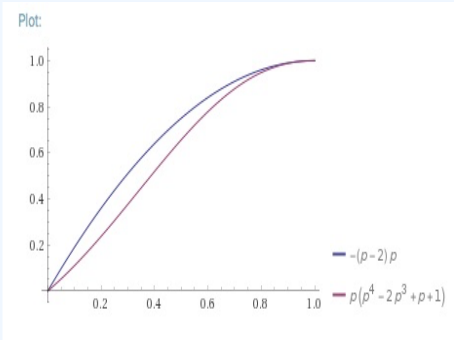
From here, we obtain

$$\mathbb{P}[Connect] = \mathbb{P}[C_1 \cup C_2 \cup C_3] = (p^3 + p^2 + p) - (p^4 + p^4 + p^3) + p^5 = p + p^2 - 2p^4 + p^5.$$

(b) Let C_1 denote the event that the road from A to F through C and E are open. Note that by the assumption in part (b), this event has probability p. Let C_2 denote the event that the path AF is open. This event has probability p too. Hence,

$$\mathbb{P}[Connect] = \mathbb{P}[C_1 \cup C_2] = 2p - p^2.$$

As the following graph shows the probability in part (b) is always higher than in part (a).



(3.4) Suppose X is a random variable with a geometric distribution with parameter p.

(a) Show that

$$\mathbb{P}[X > k] = (1-p)^k.$$

(b) Show that for all $n, k > 0$, we have

$$\mathbb{P}[X = n+k | X > k] = \mathbb{P}[X = n].$$

Solution. There are two different ways of computing $\mathbb{P}[X > k]$. One can proceed directly: we have

$$\mathbb{P}[X > k] = \sum_{j=k+1}^{\infty} \mathbb{P}[X = j] = \sum_{j=k+1}^{\infty} p(1-p)^{j-1} = p(1-p)^k \sum_{j=0}^{\infty} (1-p)^j = \frac{p(1-p)^k}{p} = (1-p)^k.$$

Alternatively, we can use the fact that the geometric distribution is the distribution of the first success in a sequence of Bernoulli trials (such as flipping a coin) where the probability of success in each round is p. Hence $X > k$ if and only if the first k rounds lead to failure which has probability $(1-p)^k$.

(b)

$$\begin{aligned} \mathbb{P}[X = n+k | X > k] &= \frac{\mathbb{P}[X = n+k \text{ and } X > k]}{\mathbb{P}[X > k]} = \frac{\mathbb{P}[X = n+k]}{\mathbb{P}[X > k]} \\ &= \frac{p(1-p)^{n+k-1}}{(1-p)^k} = p(1-p)^{n-1} = \mathbb{P}[X = n]. \end{aligned} \tag{6}$$

(3.5) Consider a coin which lands H with probability p and T with probability $1-p$. The coin is flipped until H shows up for the *second* time. Let N denote the number of required flips.

(a) For warm-up, show that $\mathbb{P}[N = 0] = \mathbb{P}[N = 1] = 0$, and $\mathbb{P}[N = 2] = p^2$.

(b) Show that $\mathbb{P}[N = 3] = 2p^2(1-p)$

(c) In general, show that the PMF of N is given by

$$\mathbb{P}[N = k] = (k-1)p^2(1-p)^{k-2}, \qquad k = 2, 3, \ldots$$

Hint: $N = k$ exactly when the k-th flip results in H and all but one of the previous $k-1$ flips result in T.

Solution. (a) First note that one requires at least two flips for the second H to show up, it is clear that $N = 0$ and $N = 1$ are impossible. Moreover, $N = 2$ if and only if the first two flips result in H, hence using the independence we have $\mathbb{P}[N = 2] = p^2$.

(b) Note that $N = 3$ occurs exactly when the outcome of the third flip is H (which happens with probability p) and the outcomes of the first two flips are HT or TH. The probability of the latter is $2p(1-p)$, hence

$$\mathbb{P}[N = 3] = 2p^2(1-p).$$

(c) The argument is similar to part (b). For $N = k$, one needs the outcome of the k-th flip to be H, and in the previous $k-1$ flips, there are exactly one H and $k-2$ tails. The probability of the latter is given by

$$\binom{k-1}{1} p(1-p)^{k-2}.$$

It follows that

$$\mathbb{P}[N = k] = (k-1)p^2(1-p)^{k-1}.$$

(3.6) (Bonus) Suppose that the probability of a coin landing heads is p, and that outcome of successive throws of the coin are independent. Let E denote the event that first HH appears before the first TT. Denote by X the outcome of the first throw.

(a) Show that

$$\mathbb{P}[E|X = H] = p + (1-p)\mathbb{P}[E|X = T].$$

(b) Find a similar formula for $\mathbb{P}[E|X = T]$.

(c) Use parts (a) and (b) to compute $\mathbb{P}[E]$.

Solution. (a) Suppose that $X = H$, that is the outcome of the first flip is H. Then either the next flip is H, which happens with probability p, or it is T. In the second case, which happens with probability $1-p$, we can pretend that the flips have started from the second one, that is, we can assume that $X = T$. This establishes the first equation.

(b) Suppose that $X = T$. Then for E to occur, one needs the next flip to be H (since otherwise TT has occurred before TT). The probability of this happening is clearly p. After this, we have a sequence starting with T, in which HH is supposed to occur before TT. Hence, we have

$$\mathbb{P}[E|X = T] = p\mathbb{P}[E|X = H].$$

Combining this with the equation

$$\mathbb{P}[E|X = H] = p + (1-p)\mathbb{P}[E|X = T].$$

from part (a), we obtain

$$\mathbb{P}[E|X = H] = \frac{p}{1-p+p^2}.$$
$$\mathbb{P}[E|X = T] = \frac{p^2}{1-p+p^2}.$$

Using conditioning we have

$$\mathbb{P}[E] = \mathbb{P}[E|X = H]\,\mathbb{P}[X = H] + \mathbb{P}[E|X = T]\,\mathbb{P}[X = T] = \frac{(2-p)p^2}{1-p+p^2}. \tag{7}$$