JTMS-12: Probability and Random Processes

Fall 2020

M. Bode

Lecture 9:
Functions of Random Variables

Lecture 9

Chapter 3: Functions of Random Variables

3.1 Functions of Random Variables

3.2 $Y = g(X)$

3.3 $Z = g(X,Y)$

3.4 $V = g(X,Y)$, W $= h(X,Y)$

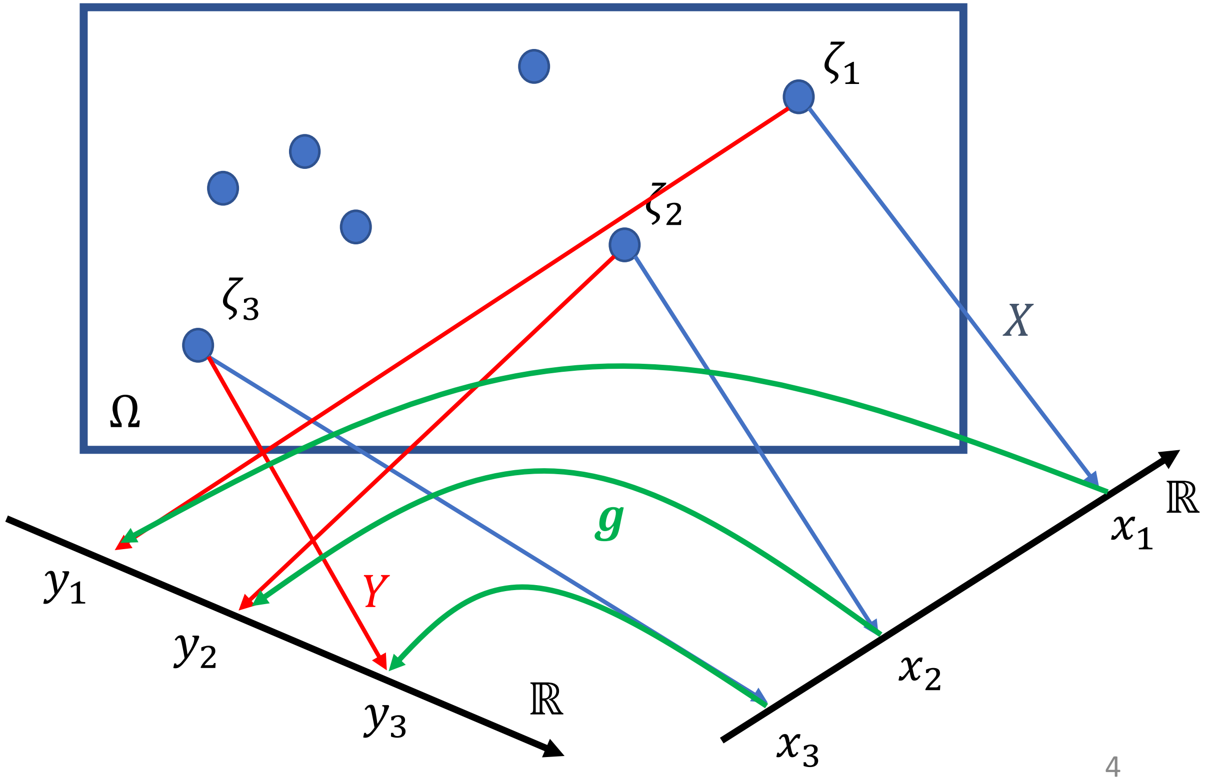
Functions of Random Variables

Idea: Map outcomes to (real) numbers.

The random variable $X\colon\ \Omega \rightarrow \mathbb{R}$ maps all outcomes from the sample description space to a real number.

Re-lable: $y = g(x)$

Re-interpret: $Y\colon\ \Omega \rightarrow \mathbb{R}\,,\ Y(\zeta) = g(X(\zeta))$



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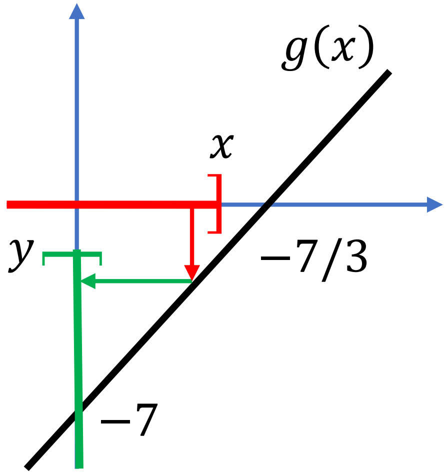
Functions of Random Variables

Example 1:

Consider a r.v. X that is uniformly distributed over (0,5), and

$$Y = 3X - 7 = g(X)$$

Find $F_Y(y)$ and $f_Y(y)$.



Standard approach …

$$F_Y(y) = P[Y \leq y]$$

$$= P[g(X) \leq y] = P[3X - 7 \leq y]$$

$$= P\left[X \leq \frac{y+7}{3}\right]$$

$$= F_X\left(\frac{y+7}{3}\right)$$

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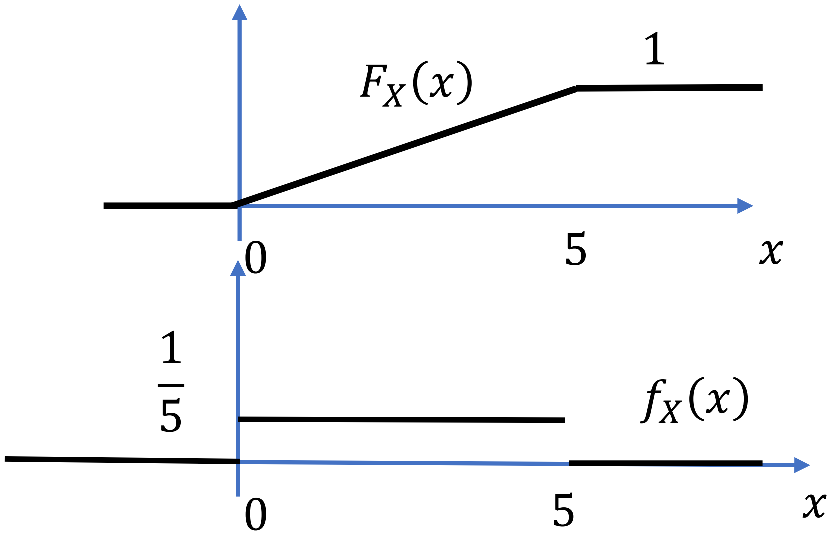


Functions of Random Variables

solve:

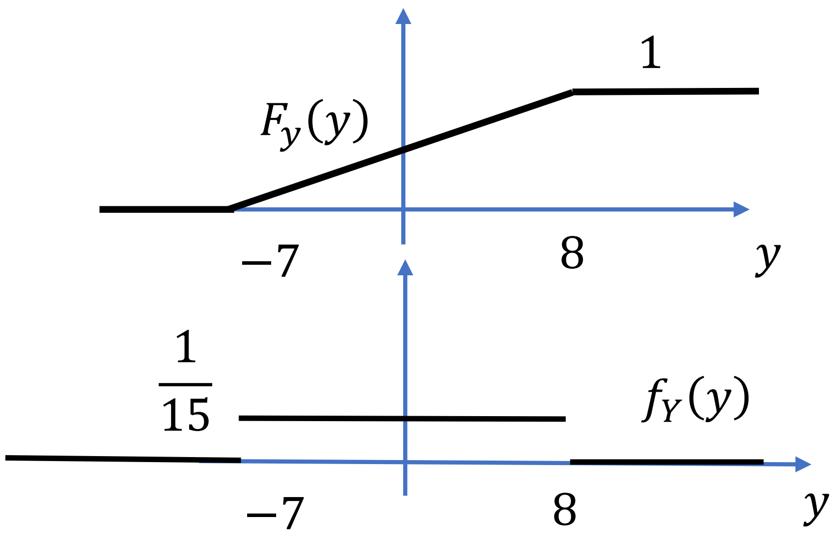
Consider a r.v. X that is uniformly distributed over (0,5), and

$$Y = 3X - 7 = g(X)$$

Find $F_Y(y)$ and $f_Y(y)$.

Standard approach … $F_Y(y) = F_X\left(\frac{y+7}{3}\right)$

$$\Rightarrow f_Y(y) = \frac{1}{3} f_X\left(\frac{y+7}{3}\right)$$



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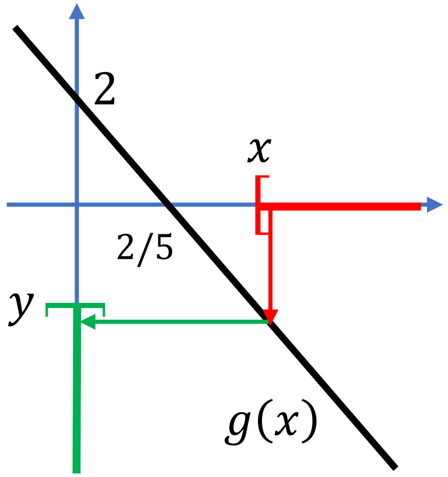
Example 2:

Consider a r.v. X that is uniformly distributed over (0,5), and

$$Y = -5X + 2 = g(X)$$

Find $F_Y(y)$ and $f_Y(y)$.



Standard approach …

$$F_Y(y) = P[Y \le y]$$

$$= P[g(X) \le y] = P[-5X + 2 \le y]$$

$$= P\left[X \ge \frac{y-2}{-5}\right] = 1 - P\left[X < \frac{y-2}{-5}\right]$$

For <u>continuous</u> rvs.:

$$F_Y(y) = 1 - F_X\left(\frac{y-2}{-5}\right)$$

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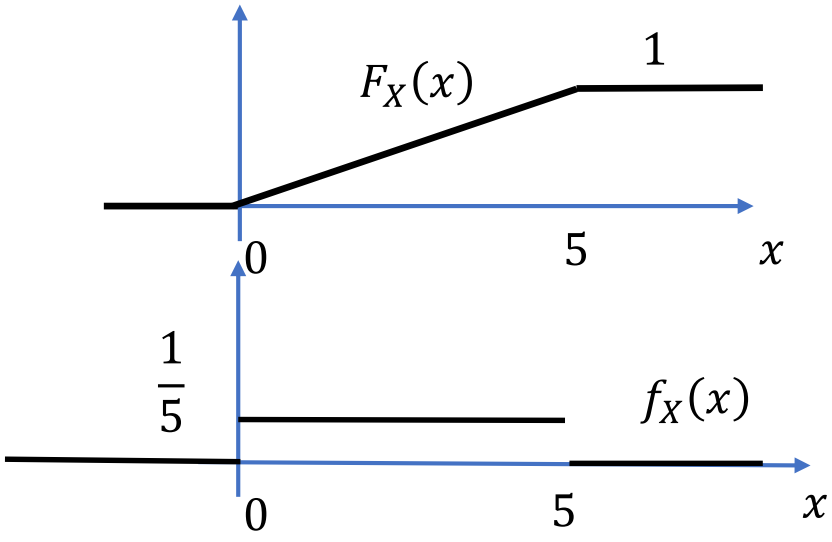


Functions of Random Variables

solve:

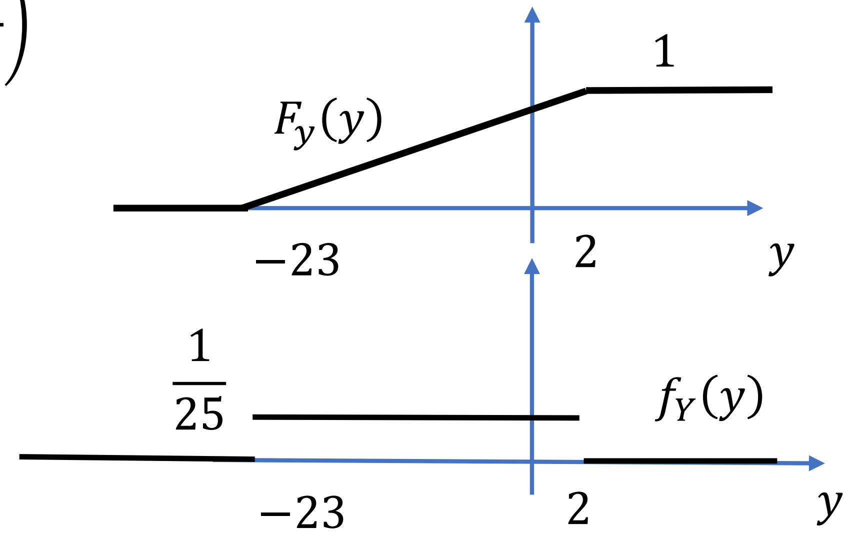
Consider a r.v. X that is uniformly distributed over (0,5), and

$$Y = -5X + 2 = g(X)$$

Find $F_Y(y)$ and $f_Y(y)$.

Standard approach … $F_Y(y) = 1 - F_X\left(\frac{y-2}{-5}\right)$

$$\Rightarrow f_Y(y) = \frac{1}{5} f_X\left(\frac{y-2}{-5}\right)$$



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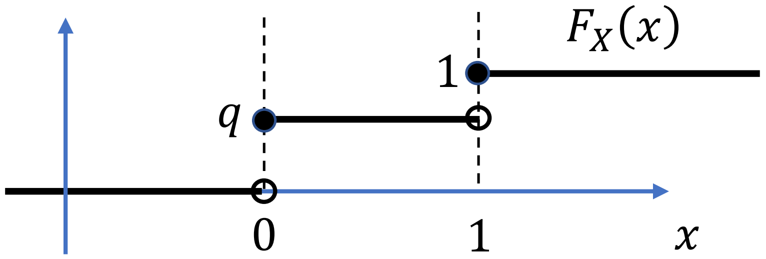


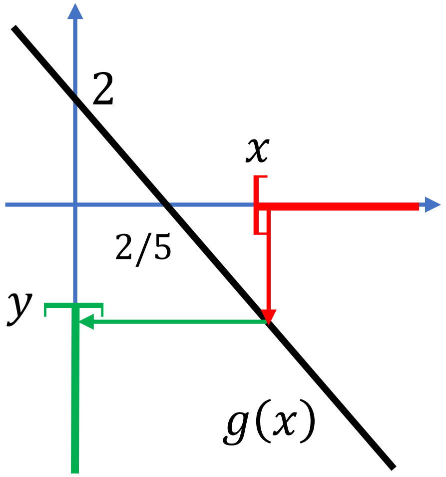
Functions of Random Variables

Example 3 (discrete):

Consider a ``fair coin´´ X, $P[X = 0] = q$, $P[X = 1] = p$

$$Y = -5X + 2 = g(X)$$

Find $F_Y(y)$ and $f_Y(y)$.



Standard approach …

$$F_Y(y) = P[Y \leq y]$$

$$= P[g(X) \leq y] = P[-5X + 2 \leq y] = P\left[X \geq \frac{y-2}{-5}\right]$$

$$= 1 - P\left[X < \frac{y-2}{-5}\right] - P\left[X = \frac{y-2}{-5}\right] + P\left[X = \frac{y-2}{-5}\right]$$

$$\Rightarrow F_Y(y) = 1 - F_X\left(\frac{y-2}{-5}\right) + P\left[X = \frac{y-2}{-5}\right]$$

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solve:

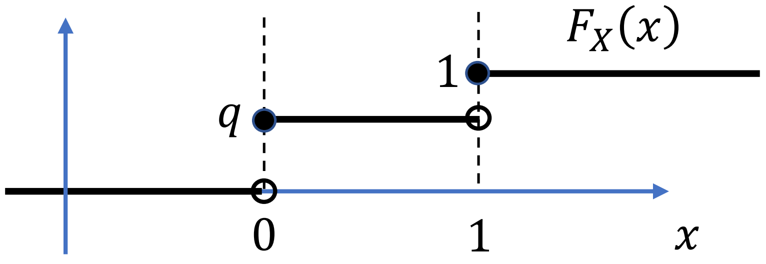
Consider a ``fair coin´´ X, $P[X=0]=q$, $P[X=1]=p$

$$Y = -5X + 2 = g(X)$$

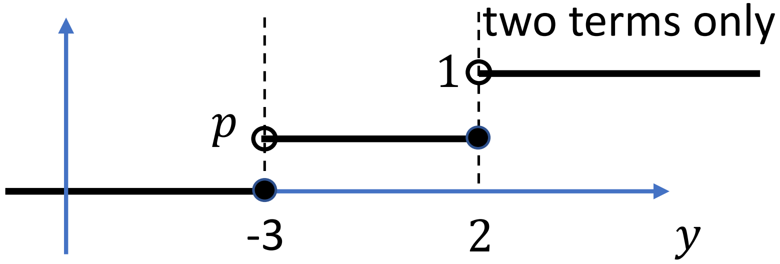
Find $F_Y(y)$ and $f_Y(y)$.

Standard approach …

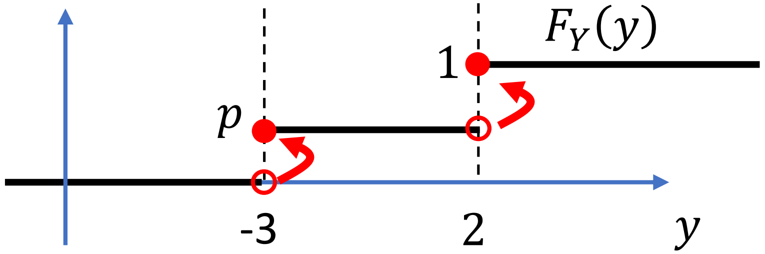
$$F_Y(y) = 1 - F_X\left(\frac{y-2}{-5}\right) + P\left[X = \frac{y-2}{-5}\right]$$



Incomplete: First two terms only



Complete $F_Y(y)$



The density $f_Y(y)$ is easy to guess, right?

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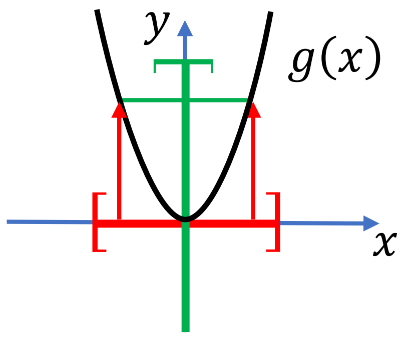
Example 4:

Consider a r.v. X, and

$$Y = X^2 = g(X)$$

Find $F_Y(y)$ and $f_Y(y)$.



Standard approach …

$$F_Y(y) = P[Y \leq y] = P[g(X) \leq y] = P[X^2 \leq y]$$

$\boxed{1)\ y \geq 0}$ $F_Y(y) = P[-\sqrt{y} \leq X \leq \sqrt{y}] = P[X \leq \sqrt{y}] - P[X < -\sqrt{y}] =$

$$\Rightarrow F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) + \underbrace{P[X = -\sqrt{y}]}_{\substack{=0 \\ for\ continous\ r.v.}}$$

$\boxed{2)\ y < 0}$

$$F_Y(y) = 0$$

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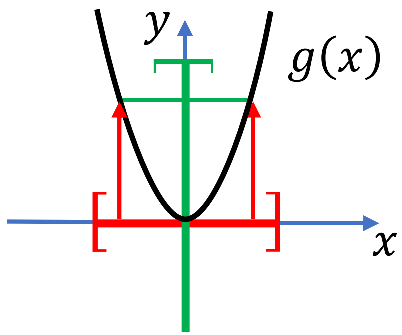
Functions of Random Variables

solve:

Consider a r.v. X, and

$$Y = X^2 = g(X)$$



Find $F_Y(y)$ and $f_Y(y)$.

Standard approach …

$$F_Y(y) = \begin{cases} F_X(\sqrt{y}) - F_X(-\sqrt{y}) + \underbrace{P[X = -\sqrt{y}]}_{\substack{=0 \\ for\ continous\ r.v.}} & ; y \geq 0 \\ 0 & ; y < 0 \end{cases}$$

Straight forward in continuous cases:

$$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}) = \ \ldots & ; y \geq 0 \\ 0 & ; y < 0 \end{cases}$$

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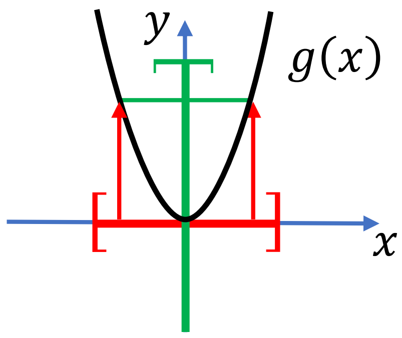
solve:

Consider a r.v. X, and

$$Y = X^2 = g(X)$$



$$\Rightarrow f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y})$$

Discuss:

a) Uniform over [-1,1]

b) Normal $\sim\mathcal{N}(0,1)$

c) Coin

Careful in discrete cases: May be easier to find the density directly.

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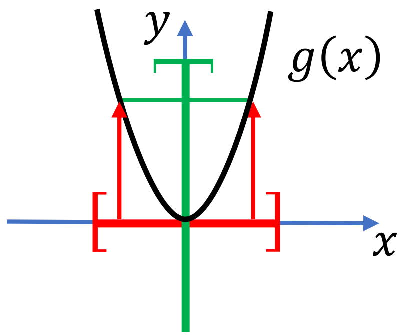

$$\Rightarrow f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y})$$

a) Uniform over [-1,1]

$$f_X(x) = \begin{cases} 0; x > 1 \\ \frac{1}{2}; -1 \le x \le 1 \\ 0; x < -1 \end{cases}$$

$$\Rightarrow f_Y(y) = \begin{cases} 0; y > 1 \\ \frac{1}{2\sqrt{y}}; 0 \le y \le 1 \\ 0; y < 0 \end{cases}$$

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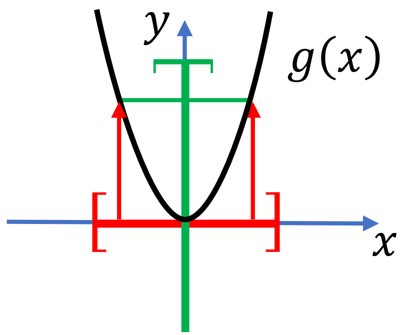

$$\Rightarrow f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y})$$

b) Normal: $X \sim \mathcal{N}(0,1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

$$\Rightarrow f_Y(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi y}} \exp\left(-\dfrac{y}{2}\right) \\ 0; y < 0 \end{cases}$$

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Consider a r.v. X, and

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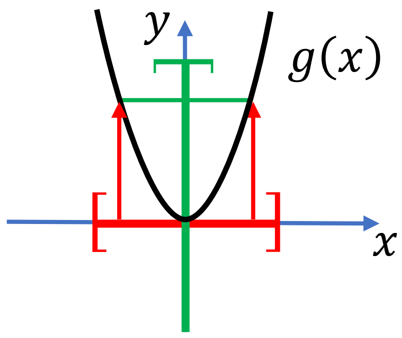

$$\Rightarrow f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y})$$

c) Coin

$$f_X(x) = q\delta(x) + p\delta(x-1)$$

$$\Rightarrow f_Y(y) = q\delta(y) + p\delta(y-1)$$

The End

Next time: Chp. 3