JTMS-12: Probability and Random Processes

Fall 2020

M. Bode

Lecture 11

Chapter 3: Functions of Random Variables

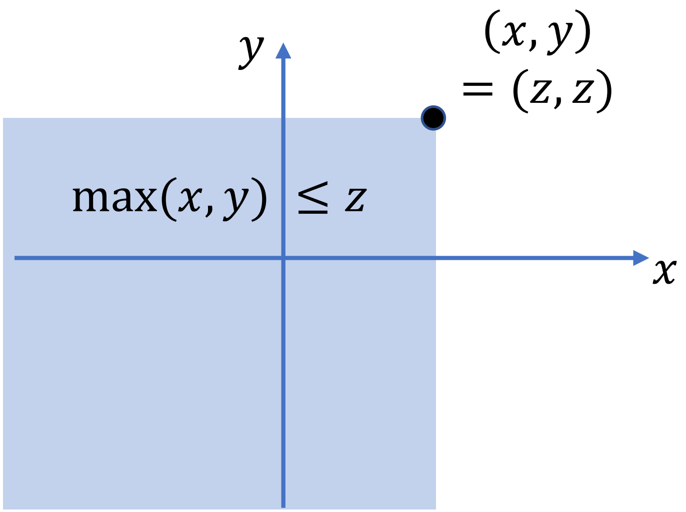
3.1 Functions of Random Variables

3.2 $Y = g(X)$

3.3 $Z = g(X,Y)$

3.4 $V = g(X,Y)$, $W = h(X,Y)$

Functions of Random Variables



Example 3:

Consider independent continuous r.v.s X, Y, and

$$Z = g(X,Y) = \max(X,Y)$$

Find $F_Z(z)$ and $f_Z(z)$.

Standard approach …

$$F_Z(z) = P[Z \le z] = P[g(X,Y) \le z] = P[\max(X,Y) \le z]$$

$$= P[X \le z, Y \le z]$$

$$= P[X \le z]P[Y \le z] = F_X(z)F_Y(z)$$

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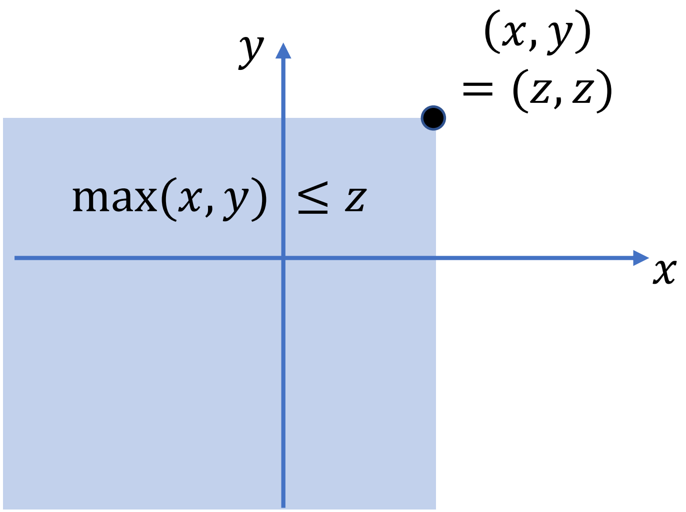
Functions of Random Variables



solve:

Consider independent

continuous r.v.s X, Y, and

$$Z = g(X,Y) = \max(X,Y)$$

Find $F_Z(z)$ and $f_Z(z)$.

Standard approach …

$$F_Z(z) = F_X(z)F_Y(z)$$

$$\Rightarrow f_Z(z) = \frac{d}{dz}F_Z(z) = f_X(z)F_Y(z) + F_X(z)f_Y(z)$$

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Challenge:

Consider independent

continuous r.v.s X, Y, and

$$Z = g(X,Y) = \min(X,Y)$$

Find $F_Z(z)$ and $f_Z(z)$.

Standard approach …

$$F_Z(z) = P[Z \leq z] = P[g(X,Y) \leq z] = P[\min(X,Y) \leq z]$$

?

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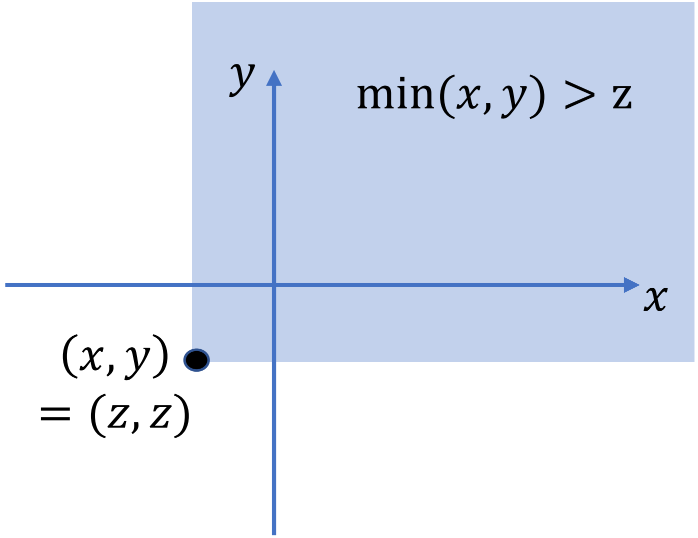

solve:

Consider independent

continuous r.v.s X, Y, and

$Z = g(X, Y) = \min(X, Y)$

Find $F_Z(z)$ and $f_Z(z)$.

Standard approach …

$$F_Z(z) = P[Z \leq z] = P[g(X, Y) \leq z] = P[\min(X, Y) \leq z]$$

$$= 1 - P[\min(X, Y) > z] = 1 - P[X > z, Y > z]$$

$$= 1 - (1 - P[X \leq z])(1 - P[Y \leq z])$$

$$= 1 - \big(1 - F_X(z)\big)\big(1 - F_Y(z)\big)$$

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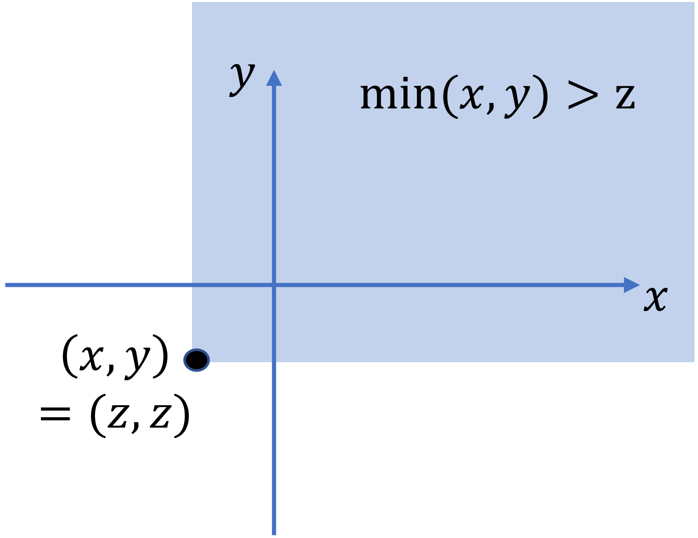

solve:

Consider independent

continuous r.v.s X, Y, and

$$Z = g(X,Y) = \min(X,Y)$$

Find $F_Z(z)$ and $f_Z(z)$.

Standard approach …

$$F_Z(z) = 1 - \big(1 - F_X(z)\big)\big(1 - F_Y(z)\big)$$

$$\Rightarrow f_Z(z) = \frac{d}{dz} F_Z(z) = f_X(z)\big(1 - F_Y(z)\big) + \big(1 - F_X(z)\big) f_Y(z)$$

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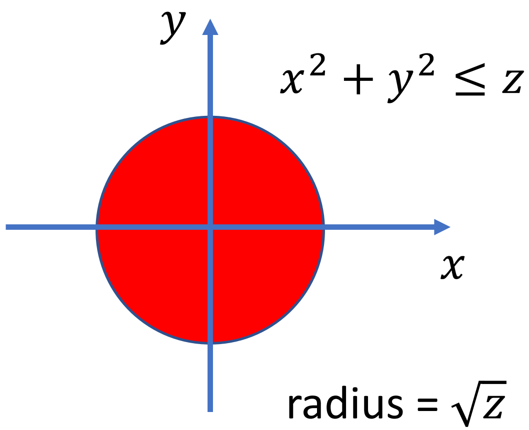
Functions of Random Variables

$\boxed{z \geq 0}$



Example 4:

Consider independent

r.v.s $X, Y \sim \mathcal{N}(0, \sigma^2)$ and

$Z = g(X,Y) = X^2 + Y^2$ (*power, energy*)

Find $F_Z(z)$ and $f_Z(z)$.

Standard approach …

$\boxed{z \geq 0}$

$$F_Z(z) = P[Z \leq z] = P[g(X,Y) \leq z] = P[X^2 + Y^2 \leq z]$$

$$= \iint_{\sqrt{z}-circle} \frac{1}{2\pi \cdot \sigma^2} \exp\left[-\frac{x^2 + y^2}{2\sigma^2}\right] dxdy$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\sqrt{z}} \frac{r}{\sigma^2} \exp\left[-\frac{r^2}{2\sigma^2}\right] drd\theta = -\exp\left[-\frac{r^2}{2\sigma^2}\right]\Bigg|_0^{\sqrt{z}}$$

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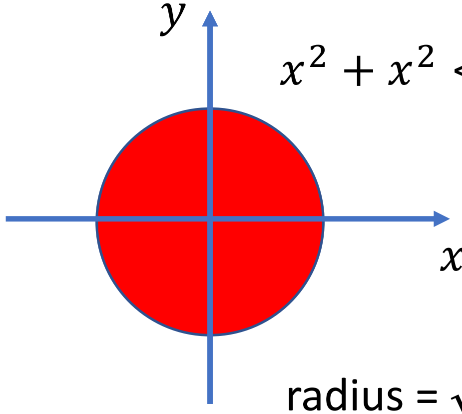
Functions of Random Variables

$\boxed{z \geq 0}$



solve:

Consider independent

r.v.s $X, Y \sim \mathcal{N}(0, \sigma^2)$ and

$Z = g(X, Y) = X^2 + Y^2$ (*power, energy*)

Find $F_Z(z)$ and $f_Z(z)$.

Standard approach …

$$F_Z(z) = 1 - \exp\left[-\frac{z}{2\sigma^2}\right]$$

$\boxed{z \geq 0}$

$$\Rightarrow f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{1}{2\sigma^2} \exp\left[-\frac{z}{2\sigma^2}\right]$$

Normal → exponential

Discuss ``density in the center´´ …

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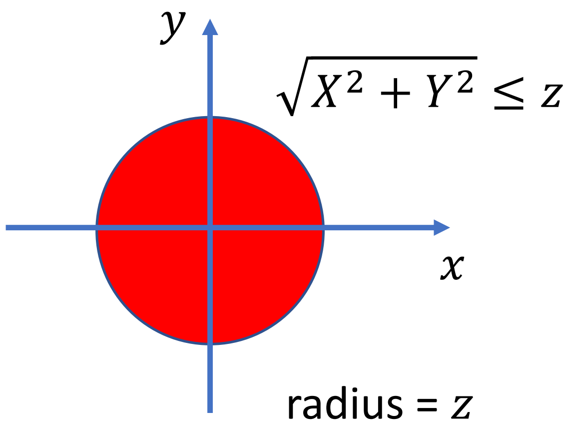
$\boxed{z \geq 0}$



Example 5 (COMPARE!):

Consider independent

r.v.s $X, Y \sim \mathcal{N}(0, \sigma^2)$ and

$Z = g(X, Y) = \sqrt{X^2 + Y^2}$ (*amplitude*)

Find $F_Z(z)$ and $f_Z(z)$.

Standard approach …

$$F_Z(z) = P[Z \leq z] = P[g(X, Y) \leq z] = P\left[\sqrt{X^2 + Y^2} \leq z\right]$$

$\boxed{z \geq 0}$

$$= \iint_{z-circle} \frac{1}{2\pi \cdot \sigma^2} \exp\left[-\frac{x^2 + y^2}{2\sigma^2}\right] dxdy$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^z \frac{r}{\sigma^2} \exp\left[-\frac{r^2}{2\sigma^2}\right] drd\theta = -\exp\left[-\frac{r^2}{2\sigma^2}\right]\Bigg|_0^z$$

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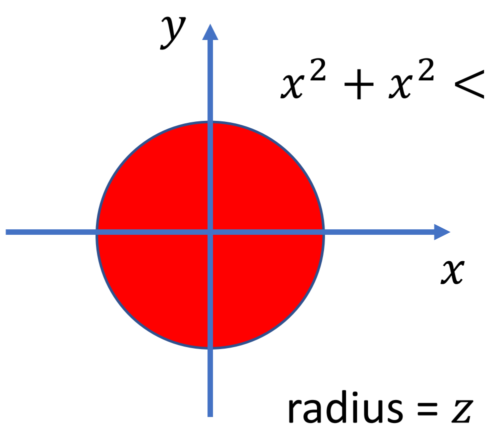
$\boxed{z \geq 0}$



$x^2 + x^2 < z$

radius = z

solve:

Consider independent

r.v.s $X, Y \sim \mathcal{N}(0, \sigma^2)$ and

$Z = g(X, Y) = \sqrt{X^2 + Y^2}$ (*amplitude*)

Find $F_Z(z)$ and $f_Z(z)$.

Standard approach …

$$F_Z(z) = 1 - \exp\left[-\frac{z^2}{2\sigma^2}\right]$$

$\boxed{z \geq 0}$

$$\Rightarrow f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{z}{\sigma^2} \exp\left[-\frac{z^2}{2\sigma^2}\right]$$

Discuss ``density in the center´´ …

Normal → Rayleigh

Two Functions of Two Random Variables

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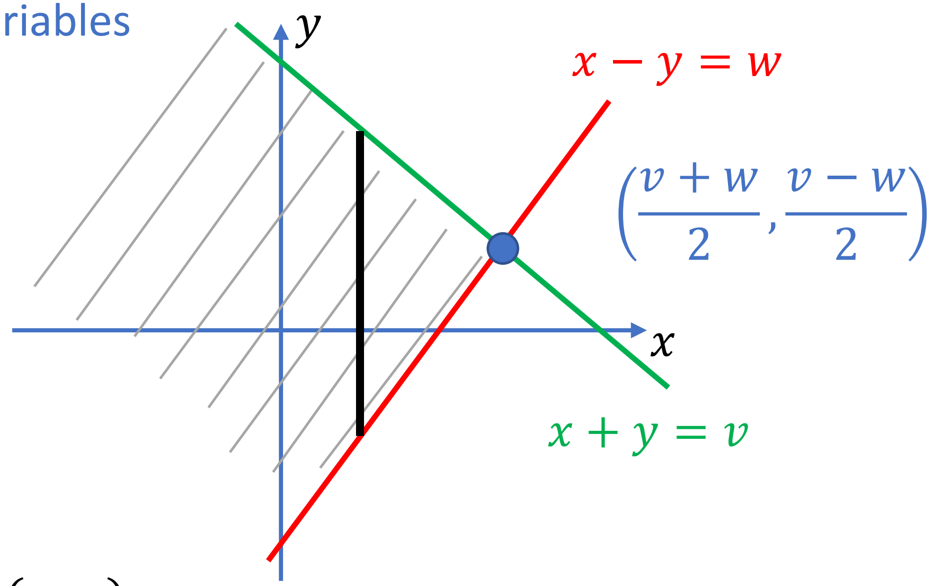
Functions of Random Variables

Example:

Consider continuous r.v.s X, Y, and

$$V \;=\; g(X,Y) = X + Y$$
$$W \;=\; h(X,Y) = X - Y$$

Find $F_{VW}(v,w)$ and $f_{VW}(v,w)$.



Standard approach …

$$F_{VW}(v,w) = P[V \le v, W \le w] = P[g(X,Y) \le v, h(X,Y) \le w]$$

$$= P[X + Y \le v, X - Y \le w] = \iint_{\substack{shaded \\ area}} f_{XY}(x,y)dxdy$$

$$= \int_{-\infty}^{\frac{v+w}{2}} \left(\int_{x-w}^{v-x} f_{XY}(x,y)dy \right) dx$$

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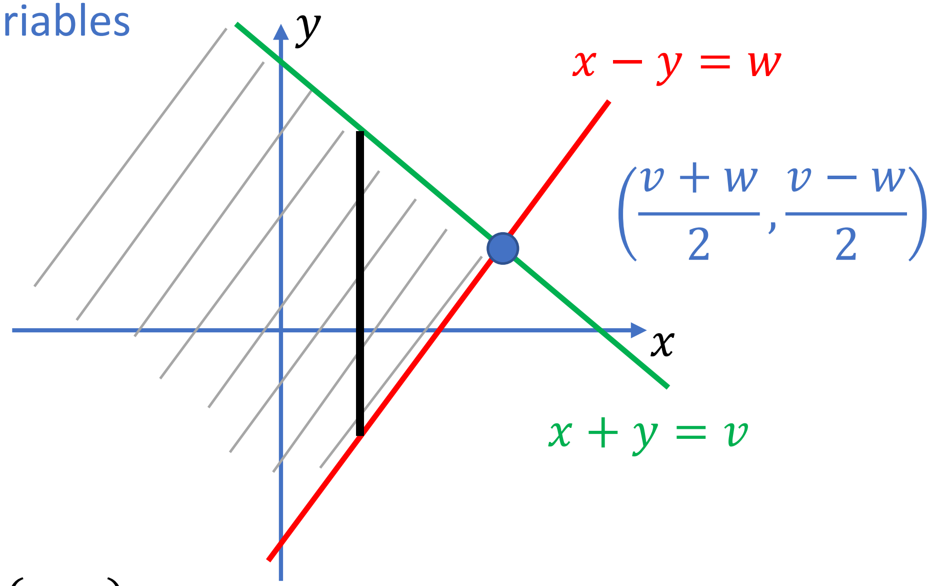

solve:

Consider continuous r.v.s X, Y, and

$$V = g(X, Y) = X + Y$$
$$W = h(X, Y) = X - Y$$

Find $F_{VW}(v, w)$ and $f_{VW}(v, w)$.

Standard approach …

$$F_{VW}(v, w) = \int_{-\infty}^{\frac{v+w}{2}} \left(\int_{x-w}^{v-x} f_{XY}(x, y)dy \right) dx$$

$$\Rightarrow f_{VW}(v, w) = \frac{\partial^2}{\partial v \partial w} F_{VW}(v, w) = \frac{1}{2} f_{XY}\left(\frac{v+w}{2}, \frac{v-w}{2}\right)$$

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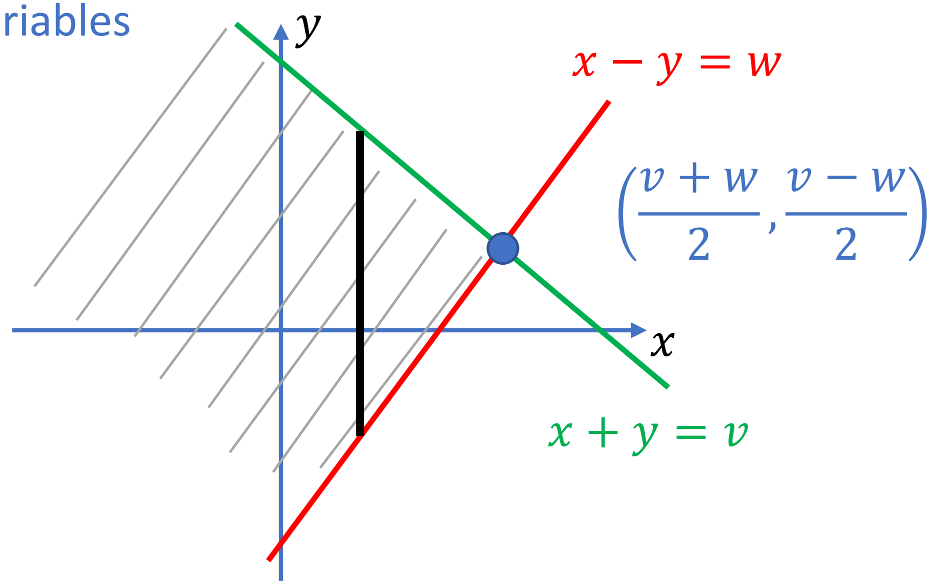

solve:

Consider continuous r.v.s X, Y, and

$$V = g(X, Y) = X + Y$$
$$W = h(X, Y) = X - Y$$

Details …

$$f_{VW}(v, w) = \frac{\partial^2}{\partial v \partial w} \int_{-\infty}^{\frac{v+w}{2}} \left(\int_{x-w}^{v-x} f_{XY}(x, y) dy \right) dx$$

$$= \frac{\partial}{\partial v} \left[\frac{1}{2} \int_{\frac{v-w}{2}}^{\frac{v-w}{2}} f_{XY}\left(\frac{v+w}{2}, y\right) dy + \int_{-\infty}^{\frac{v+w}{2}} f_{XY}(x, x-w) dx \right] = \frac{1}{2} f_{XY}\left(\frac{v+w}{2}, \frac{v-w}{2}\right)$$

The End

Next time: Cont. Chp. 3