The distribution function and other probabilities

Definition
The distribution function of the random variable X is given by

$$F_X(t) = \mathbb{P}[X \le t] = \int_{-\infty}^{t} f_X(x)\, dx.$$

1. $$\mathbb{P}[X \ge t] = 1 - F_X(t).$$
2. $$\mathbb{P}[t_1 \le X \le t_2] = F_X(t_2) - F_X(t_1).$$

Relation between the probability density function and the distribution

Theorem
The relation between the probability density function and the distribution function is given by

1. $F_X(t)$ *can be obtained from the density function by* *integration*:
$$F_X(t) = \int_{-\infty}^{t} f_X(x)\, dx.$$
2. $f_X(t)$ *can be obtained from* $F_X(t)$ *by* *differentiation*:
$$f_X(x) = F_X'(x).$$

The uniform distribution

The simplest continuous distribution is the uniform distribution.

Definition

A random variable X has uniform distribution over the interval $[a, b]$, if its probability density function is given by

$$f_X(t) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq t \leq b \\ 0 & \text{otherwise} \end{cases}$$

When we say, choose a random number between a and b, we always talk about uniform distribution.

The special case of the uniform random variable

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(t) = \begin{cases} 0 & \text{if } t \leq a \\ \frac{t-a}{b-a} & \text{if } a \leq t \leq b \\ 1 & t \geq b. \end{cases}$$

Example

Suppose X has uniform distribution over the interval $[1, 3]$, Find the probability of the following events: (a) $1 \leq X \leq 2$. (b) $X \geq 2$. (c) $1 \leq X \leq 4$.

The distribution function is given by

$$F_X(t) = \begin{cases} 0 & \text{if } t \leq 1 \\ \frac{t-1}{2} & \text{if } 1 \leq t \leq 3 \\ 1 & t \geq 3. \end{cases}$$

Hence we have

$$\mathbb{P}\left[1 \leq X \leq 2\right] = \frac{1}{2}.$$

$$\mathbb{P}\left[X \geq 2\right] = 1 - F_X(2) = 1 - \frac{1}{2} = \frac{1}{2}.$$

$$\mathbb{P}\left[1 \leq X \leq 4\right] = F_X(4) - F_X(1) = 1 - 0 = 1.$$

Example

Suppose that the distribution of a random variable X is given by

$$F_X(t) = \begin{cases} 1 - e^{-t^2} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

Find the probability density function of X.

$$f_X(t) = F_X^{'}(t) = \begin{cases} 2te^{-t^2} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

Transformation of random variables

Example

Suppose X has uniform distribution over the interval $[0,2]$ and set $Y = X^3$. What is the distribution function and the probability density function of Y?

First we compute $F_Y(t)$.

$$F_Y(t) = \mathbb{P}[Y \leq t] = \mathbb{P}\left[X^3 \leq t\right] = \mathbb{P}\left[X \leq t^{1/3}\right].$$

$$F_Y(t) = \begin{cases} 1 & \text{if } t \leq 0 \\ \frac{1}{2}t^{1/3} & \text{if } 0 \leq t \leq 8 \\ 1 & \text{if } t \geq 8. \end{cases}$$

$$f_Y(t) = \begin{cases} \frac{1}{6}t^{-2/3} & \text{if } 0 < t < 8 \\ 0 & \text{otherwise} \end{cases}$$

Transformation of random variables

Suppose X has uniform distribution over the interval $[-1,1]$ and set $Y = X^2$. What is the distribution function and the probability density function of Y?

Since X takes values in $[-1,1]$, we know that $Y = X^2$ takes values in $[0,1]$.

For $0 < t < 1$ we have

$$F_Y(t) = \mathbb{P}[Y \leq t] = \mathbb{P}\left[X^2 \leq t\right] = \mathbb{P}\left[-\sqrt{t} \leq X \leq \sqrt{t}\right] = \int_{-\sqrt{t}}^{\sqrt{t}} \frac{1}{2} dx = \sqrt{t}.$$

Hence we have

$$f_Y(t) = \begin{cases} \frac{1}{2}t^{-1/2} & \text{if } 0 < t < 1 \\ 0 & \text{if } t < 0 \text{ or } t > 1. \end{cases}$$

Other prominent continuous random variables

Definition

A random variable X is said to have exponential distribution with parameter λ when its density function is given by

$$f_X(t) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The distribution function is thus given by for values of $t > 0$:

$$F_Y(t) = \int_0^t \lambda e^{-\lambda x} \, dx = 1 - e^{-\lambda t}.$$

In particular, we have

$$\mathbb{P}[X > t] = 1 - F_X(t) = e^{-\lambda t}.$$

Exponential random variables do not capture memory

$$\mathbb{P}[X > s + t | X > s] = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = \mathbb{P}[X > t].$$

Situations where exponential distributions may arise:

- Time between two car accidents.
- The number of words between two typos in a book.
- Lifetime of items that do not age.

Normal random variables

Normal or Gaussian random variables are some of the most important examples of continuous random variables. They arise naturally in *the central limit theorem.*

Definition

A continuous random variable X is said to have Gaussian or normal distribution with parameters (μ, σ^2), written $X \sim N(\mu, \sigma^2)$, if the probability density function of X is given by

$$f_X(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

A random variable with normal distribution with parameters $\mu = 0$ and $\sigma = 1$ is called a *standard normal distribution* or a standard Gaussian.

Probability density functions of normal random variables