# Lecture 6

Review:

conditional probability:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Example (Testing for a rare disease)

Alex is tested for a rare disease which afflicts 1% of the population. Assume that the accuracy of the test is 95% in the sense that if the patient has the disease the test turns positive with probability 95% and if the patient does not have the disease, the test turns negative with prob. 95%. Suppose that the test has turned positive. What is the prob. that Alex has this desease.

D: Alex has the desease

P: the test result is positive

$\mathbb{P}(D) = 1/100$, $\mathbb{P}(P|D) = \frac{95}{100}$, $\mathbb{P}(P^c|D^c) = \frac{95}{100}$.

$\mathbb{P}(D|P) = ?$

$$\mathbb{P}(D|P) = \frac{\mathbb{P}(P|D)\ \mathbb{P}(D)}{\mathbb{P}(P|D)\,\mathbb{P}(D) + \mathbb{P}(P|D^c)\,\mathbb{P}(D^c)}$$

$$= \frac{0.95 \times 0.01}{0.95 \times 0.01 + 0.05 \times 0.99}$$

$$\cong 0.16$$

The occurance of $B$ does not change the probability of $A$

when $\mathbb{P}(A|B) = \mathbb{P}(A) \Leftrightarrow \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)} = \mathbb{P}(A) \Leftrightarrow \mathbb{P}(A\cap B) = \mathbb{P}(A)\,\mathbb{P}(B).$

**Definition** Two events $A, B$ are called independent, when

$$\mathbb{P}(A\cap B) = \mathbb{P}(A)\,\mathbb{P}(B).$$

<u>Example</u> A coin is flipped 5 times. Consider the events:

$A$: the first flip results in heads

$B$: # of heads is even. Are $A, B$ independent?

$A = \{H****\}$ $\qquad \mathbb{P}(A) = 1/2$

$B = B_0 \cup B_2 \cup B_4$ $\qquad B_i$ : $i$ heads

$\mathbb{P}(B_0) = \frac{1}{2^5}$, $\mathbb{P}(B_2) = \frac{\binom{5}{2}}{2^5} = \frac{10}{32}$, $\mathbb{P}(B_4) = \frac{\binom{5}{4}}{2^5} = \frac{5}{32} \Rightarrow \mathbb{P}(B) = \frac{1}{2}$

$\mathbb{P}(A\cap B) = \frac{1}{4}$ so $A, B$ are independent.

<u>Remark</u> Independence is different from disjointness. In fact two independent events with positive probability are <u>never</u> disjoint!

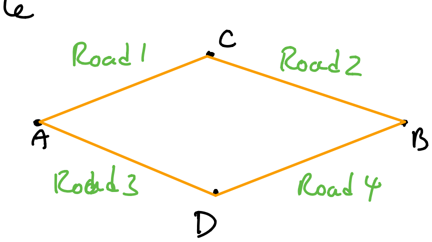
**Definition** Suppose $A_1, A_2, \ldots, A_n$ are $n$ events. We say that these events are independent if for any $1 \le i_1 < i_2 < \cdots < i_k \le n$

$$\mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k}) = \mathbb{P}(A_{i_1}) \cdots \mathbb{P}(A_{i_k}).$$

<u>Example</u> $A, B, C$ are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B), \quad \mathbb{P}(A \cap C) = \mathbb{P}(A)\,\mathbb{P}(C)$$
$$\mathbb{P}(B \cap C) = \mathbb{P}(B)\,\mathbb{P}(C), \quad \mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(C)$$

<u>Example</u> A parcle is supposed to be transported from point A to point B. This can be done via points C or D. Assume that each road is accessible with probability 90%. What is the probability that the parcel can be delivered, assuming that the accessibility of roads are independent.




$A_i$ = road $i$ is accessible.

$$A = (A_1 \cap A_2) \cup (A_3 \cap A_4)$$

$$\mathbb{P}(A) = \mathbb{P}(A_1 \cap A_2) + \mathbb{P}(A_3 \cap A_4) - \mathbb{P}(A_1 \cap A_2 \cap A_3 \cap A_4)$$
$$= 0.81 + 0.81 - 0.61 \cong 0.96$$