The equiprobable model of Pascal

Key words: sample space, event, probability.

Definition. The set of all possible outcome of an even is called the sample space. If this set is finite, we can list up its elements as:

$$\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$$

Definition Any subset of Ω is called an event.

Examples

Flip of a coin $\Omega_1 = \{H, T\}$

Throw of a die $\Omega_2 = \{1,2,3,4,5,6\}$

Flip of two coins

$$\Omega_3 = \{(H,H),(H,T),(T,H),(T,T)\}$$

Flip of two die

$$\Omega_4 = \{(1,1),(1,2),\dots,(6,6)\}$$

Examples of events:

1) The outcome of the coin is H. $A_1 = \{H\}$
2) The outcome of the die is an even number $A_2 = \{2,4,6\}$
3) There is an odd number of tails $A_3 = \{(H,T),(T,H)\}$

The first die and the second die show the same number

$$A = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$$

Definition

The probability of an event A is defined by

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|}$$

where $|\cdot|$ denotes the number of elements.

Flip of a coin $\quad \Omega_1 = \{H, T\}$ $\qquad |\Omega_1| = 2$

Throw of a die $\quad \Omega_2 = \{1, 2, 3, 4, 5, 6\}$ $\qquad |\Omega_2| = 6$

Flip of two coins

$$\Omega_3 = \{(H,H), (H,T), (T,H), (T,T)\} \qquad |\Omega_3| = 4$$

Flip of two dice

$$\Omega_4 = \{(1,1), (1,2), \dots, (6,6)\} \qquad |\Omega_4| = 36$$

hence

$$\mathbb{P}(A_1) = 1/2$$

$$\mathbb{P}(A_2) = 1/2$$

$$\mathbb{P}(A_3) = 1/2$$

$$\mathbb{P}(A_4) = 6/36 = 1/6.$$

Example. A three-digit number is randomly chosen. Find the probability that the sum of its digits is an even number.

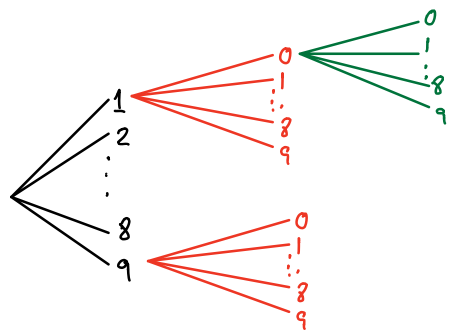
$$\Omega = \{100, 101, \dots, 999\}$$

$$|\Omega| = 999 - 100 + 1 = 900$$

how to describe A ?

$$\square \quad \square \quad \square \quad \square$$

Counting Principle I.



$\#$ vertices of the tree $= 9 \times 10 \times 10 = 900$

$$\mathbb{P}(A) = \frac{9 \times 10 \times 5}{9 \times 10 \times 10} = \frac{1}{2}$$

Definition For a positive integer n, $n!$ (read n factorial) is defined by:

$$n! = n \cdot (n-1) \cdot \dots \cdot 2 \cdot 1$$

We also define $0! = 1$.

Stirling's asymptotic formula for $n!$:

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

This means that

$$\lim_{n \to \infty} \frac{n!}{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n} = 1.$$

Counting Principle II

A set with n elements, has exactly $\binom{n}{k}$ subsets with k elements, where

$$\binom{n}{k} = \frac{n!}{(n-k)!\ k!}$$

and $n! = n \cdot (n-1) \cdot \dots 1$.

Example In how many ways can we form a football team from 15 football players?

Example A bowl contains 8 red an 9 blue marbles. 6 marbles are randomly taken out. Find the probability that 4 of them are red and two of them are blue.

$$|\Omega| = \binom{17}{6} \qquad |A| = \binom{8}{4} \times \binom{9}{2}$$

$$\mathbb{P}(A) = \frac{\binom{8}{4} \cdot \binom{9}{2}}{\binom{17}{6}}$$

Set theoretic notations:

A, B two events

$A \cup B$: outcomes which are in A or in B

$A \cap B$: outcomes which are in A and in B

A^c : outcomes that are not in A.

Ex

$A \cap B^c$:

$A^c \cap B^c$

We write $A \subset B$ if every outcome in A is in B, that is, when occurance of A implies that of B.

Axiomatic Probability

Ω : sample space , $\mathbb{P}$: $\underset{\Omega}{\text{all set of subsets of}} \longrightarrow [0,1]$

assigning to every set $A \subseteq \Omega$ its probability $0 \leq \mathbb{P}(A) \leq 1$, in such a way that

Axiom 1 • $\mathbb{P}(\Omega) = 1$

Axiom 2 • if $A \cap B = \emptyset$ then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.

Proposition In any probability space $(\Omega, \mathbb{P})$ we have

(a) $\mathbb{P}(\emptyset) = 0$

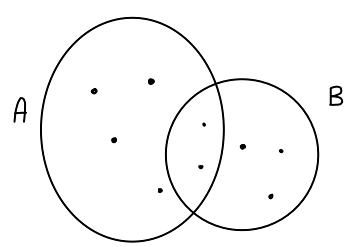
(b) For any event A, $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

(c) For any two events A, B, we have

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

For (c), also note that in the equiprobable case, it is equivalent to

$$|A \cup B| = |A| + |B| - |A \cap B|.$$



Example n students are in a party. What is the probability at least two of them are born on the same day? (ignore the leap years)

$$\Omega = \{(x_1, x_2, \dots, x_n) : 1 \leqslant x_i \leqslant 365\}$$

$$A = \{(x_1, x_2, \dots, x_n) : x_i = x_j \text{ for some } i \neq j\}$$

It is clear that:

$$|\Omega| = 365 \cdot \dots 365 = 365^n.$$

in order to compute $\mathbb{P}(A)$, we compute $\mathbb{P}(A^c)$.

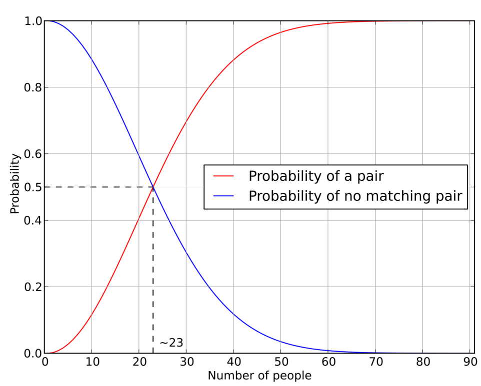
$$|A^c| =$$

hence

$$\mathbb{P}(A^c) = \frac{365 \cdot 364 \dots (365-n+1)}{365^n}$$

from here:

$$\mathbb{P}(A) = 1 - \frac{365 \cdot 364 \dots (365-n+1)}{365^n}$$



source: wikipedia

$n=2$: $$\mathbb{P}(A) = 1 - \frac{365 \cdot 364}{365 \cdot 365} = \frac{1}{365}.$$