2023.05.06

T1. $E = \frac{h^2}{2m_e\lambda^2} = \frac{(hc)^2}{2m_ec^2\lambda^2} = \frac{(1.24\times10^{-6}\ eV\cdot m)^2}{2\times0.511\ MeV\times(5.5\times10^{-7}\ m)^2}$

$= 4.974\times10^{-6}\ eV$

T2. $\lambda = \frac{h}{\sqrt{2m_eE}} = \frac{hc}{\sqrt{2m_ec^2E}} = \frac{1.24\times10^{-6}\ eV\cdot m}{\sqrt{2\times0.511\ MeV\times50\ keV}}$

$= 5.486\times10^{-12}\ m$

T3. $\iint [\psi(x,y,t=0)]^2\,dx\,dy = \iint (x^2+y^2)\exp\{-2(x^2+y^2)\}\,dx\,dy$

$= \iint r^2 e^{-2r^2}\, r\,dr\,d\theta = \frac{\pi}{4}$

$p(x,y,t=0) = \frac{4}{\pi}[\psi(x,y,t=0)]^2$

$= \frac{4}{\pi}(x^2+y^2)\exp\{-2(x^2+y^2)\}$

T4. (1) $\int_{-\infty}^{+\infty}|\psi(x)|^2dx=\int_{-\infty}^{+\infty}e^{-2|x|}dx=1$

故归一化后的波函数为 $\psi(x)=e^{-|x|}\ (x\in R)$

(2) $\phi(p)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty}e^{-\frac{i}{\hbar}px}\psi(x)dx$

$=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty}e^{-\frac{ipx}{\hbar}}\cdot e^{-|x|}dx$

$=\frac{1}{\sqrt{2\pi\hbar}}\cdot\frac{1}{1+\frac{p^2}{\hbar^2}}$

T5. $\psi(x)=e^{-\frac{x^2}{6^2}}$

$\int_{-\infty}^{+\infty}|\psi(x)|^2dx=2\int_0^{\infty}e^{-\frac{2x^2}{6^2}}dx=\sqrt{\frac{\pi}{2}}6$.

$\psi(x)=(\sqrt{\frac{\pi}{2}}6)^{-1}\cdot e^{-\frac{x^2}{6^2}}$

$\langle\hat{x}\rangle=\int_{-\infty}^{+\infty}\psi^*(x)x\psi(x)dx=0$

$\langle\hat{p}\rangle=\int_{-\infty}^{+\infty}\psi^*(x)(-i\hbar\frac{\partial}{\partial x})\psi(x)dx=0$.

$\langle\hat{T}\rangle=\int_{-\infty}^{+\infty}\psi^*(x)(-\frac{\hbar}{2m}\frac{\partial^2}{\partial z^2})\psi(x)dx$

$=\int_{-\infty}^{+\infty}(\sqrt{\frac{\pi}{2}}6)^{-1}e^{-\frac{2x^2}{6^2}}(-\hbar^2)(-\frac{2}{6^2}+\frac{4x^2}{6^4})dx$

$=\frac{1}{2m}\frac{\hbar^2}{6^2}$

T6. $\psi(x)=\begin{cases}1 & ,|x|<1\\ 0 & ,|x|\geq 1\end{cases}$

$\int_{-\infty}^{+\infty}|\psi^2(x)|\,dx=2$ 归一化后 $\psi(x)=\begin{cases}\frac{\sqrt{2}}{2} & ,|x|<1\\ 0 & ,|x|\geq 1\end{cases}$

$$\langle x\rangle=\int_{-\infty}^{+\infty}\psi(x)\,x\,\psi(x)\,dx=0.$$

$$\langle x^2\rangle=\int_{-\infty}^{+\infty}\psi(x)\,x^2\psi^*(x)\,dx=\frac{1}{3}$$

$$\langle p\rangle=-i\hbar\,\psi(x)\frac{d}{dx}\psi(x)\,dx=0$$

$$\langle p^2\rangle=-\hbar^2\int_{-\infty}^{+\infty}\frac{d^2}{dx^2}\psi(x)\,dx=-\frac{\hbar^2}{2}\int_{-\infty}^{+\infty}\psi(x)\frac{d}{dx}[\delta(x+1)-\delta(x-1)]\,dx$$

$$=\frac{\hbar^2}{2}\int_{-\infty}^{+\infty}\psi'(x)[\delta(x+1)-\delta(x-1)]\,dx=\hbar^2\delta(0)$$

$\Delta x=\sqrt{\langle x^2\rangle-\langle x\rangle^2}=\frac{\sqrt{3}}{3}$ $\Delta p=\sqrt{\langle p^2\rangle-\langle p\rangle^2}=h\sqrt{\delta(0)}$ $\Delta x\Delta p=\hbar\sqrt{\frac{\delta(0)}{3}}>\frac{\hbar}{2}$

~~解~~

~~T10~~. 设此一维无限深势阱宽度为 a，

T11. 则在此一维无限深势阱宽度中粒子的能量本征函数为

$$\varphi_n=\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$$

$$\langle x\rangle=\int_0^a \varphi_n x\varphi_n\,dx=\frac{a}{2}$$

$$\langle x^2\rangle=\int_0^a \varphi_n x^2\varphi_n\,dx=\frac{a^2}{3}-\frac{a^2}{2n^2\pi^2}$$

$$\langle p\rangle=\int_0^a \varphi_n\left(-i\hbar\frac{\partial}{\partial x}\right)\varphi_n\,dx=0$$

$$\langle p^2\rangle=\int_0^a \varphi_n\left(-\hbar^2\frac{\partial^2}{\partial x^2}\right)\varphi_n\,dx=\left(\frac{\hbar n\pi}{a}\right)^2$$

$$\Delta x=\sqrt{\langle x^2\rangle-\langle x\rangle^2}=\frac{a}{\sqrt{12}}\sqrt{1-\frac{6}{n^2\pi^2}}$$

$$\Delta p=\sqrt{\langle p^2\rangle-\langle p\rangle^2}=\frac{\hbar n\pi}{a}$$

$$\Delta x\Delta p=\frac{\hbar}{\sqrt{12}}\sqrt{n^2\pi^2-6}\geq\frac{\hbar}{2}$$

第1个定态最接近上述不等式极限.

T_{12}. (1) $\psi(x)$

$(1^2+|i|^2)|c|^2=1$

$c=\frac{\sqrt{2}}{2}$

$\psi(x)=\frac{\sqrt{2}}{2}\phi_1(x)+\frac{\sqrt{2}}{2}i\phi_2(x)$

(2) $\psi(x,t)=\frac{\sqrt{2}}{2}[\phi_1(x)e^{-\frac{iE_1 x}{\hbar}}+i\phi_2(x)e^{-\frac{iE_2 x}{\hbar}}]$

代入 $\phi_n(x)=\sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a})$

$E_n=\frac{n^2\pi^2\hbar^2}{2ma^2}$

$|\psi(x,t)|^2=\frac{1}{2}[\psi_1(x)e^{-\frac{iE_1 x}{\hbar}}+i\psi_2(x)e^{-\frac{iE_2 x}{\hbar}}]^2$

$=\frac{1}{a}[\sin^2(\frac{\pi}{a}x)+\sin^2(\frac{2\pi x}{a})+i[\sin(\frac{\pi}{a}x)\sin(\frac{2\pi}{a}x)\cos(\frac{E_2-E_1}{\hbar}t)]$

(3) 在 $[0,a]$ 上的一维无限深势阱

$\phi_1(x)=\sqrt{\frac{2}{a}}\sin(\frac{\pi}{a}x)$ $\phi_2=\sqrt{\frac{2}{a}}\sin(\frac{2\pi}{a}x)$

$\langle x\rangle=\frac{a}{2}-\frac{16a}{9\pi^2}\sin 3\omega t$ $\langle x^2\rangle=(\frac{1}{3}-\frac{5}{16\pi^2})a^2-\frac{16a^2}{9\pi^2}\sin 3\omega t$

$\langle p\rangle=-\frac{8\hbar}{3a}\cos 3\omega t$ $\langle p^2\rangle=\frac{5\pi^2\hbar^2}{2a^2}$. 其中 $\omega=\frac{E_1}{\hbar}=\frac{\pi^2\hbar}{2ma^2}$

(4) 粒子处于 E_1, E_2 的概率均为 $\frac{1}{2}$，系统的能量平均值为

$\langle H\rangle=\frac{1}{2}(E_1+E_2)=\frac{5\pi^2\hbar^2}{4ma^2}$.

①证明:

T13. 势能

$$V(x)=\begin{cases}0, & x\in(0,a)\\ \infty, & 其它\end{cases}$$

由势能的形式得 $\psi=0$, $x\notin(0,a)$

当 $x\in(0,a)$ 时, 设定态波函数的通解为 $\psi(x)=A\sin(kx+d)$

其中 $k=\sqrt{\frac{2mE}{\hbar}}$, A、d为待定系数

由 $\begin{cases}A\sin(k\cdot\frac{a}{2}+d)=0\\ A\sin(k\cdot(-\frac{a}{2})+d)=0\end{cases}$ 得 $\begin{cases}a=\frac{n_1\pi}{k} & n_1=0,1,2,\dots\\ d=\frac{n_2\pi}{2} & n_2=0,1,2,\dots\end{cases}$

$\begin{cases}k=\frac{n\pi}{a}\\ k=\sqrt{\frac{2mE}{\hbar}}\end{cases}$ 则 $E=\frac{n^2\pi^2\hbar}{2ma^2}$

$$\varphi_n(x)=\begin{cases}\sin(k_n x+\frac{n\pi}{2}) & |x|\le\frac{a}{2},\ k_n=\frac{n\pi}{a}\\ 0 & ,|x|>\frac{a}{2}\end{cases}$$

$$\int_{-\infty}^{+\infty}|\varphi_n(x)|^2dx=\sqrt{\frac{a}{2}}$$

$$\varphi_n(x)=\begin{cases}\sqrt{\frac{2}{a}}\sin(k_n x+\frac{n\pi}{2}) & ,|x|\le\frac{a}{2},\ k_n=\frac{n\pi}{a}\\ 0 & ,|x|>\frac{a}{2}\end{cases}$$

② $\psi(x,0) = A\sin^3\frac{\pi x}{a}$

$$= \frac{A}{4}\cdot\left[3\sin\frac{\pi x}{a} - \sin\frac{3\pi x}{a}\right]$$

$$= \frac{A}{4}\cdot\sqrt{\frac{a}{2}}\left[3\cdot\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a} - \sqrt{\frac{2}{a}}\sin\frac{3\pi x}{a}\right]$$

$$= \frac{A}{4}\sqrt{\frac{a}{2}}\cdot\left[3\phi(x) - \phi_3(x)\right]$$

$$[3^2+(-1)^2]|C|^2=1$$

$$C=\frac{\sqrt{10}}{10}$$

$$A=\frac{4}{5a}\sqrt{5a}$$

$t>0$时 $\psi(x,t) = \frac{\sqrt{10}}{10}\left[3\cdot\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a}\cdot\exp\left(-\frac{i}{\hbar}E_1 t\right) + \sqrt{\frac{2}{a}}\sin\frac{3\pi x}{a}\cdot\exp\left(-\frac{i}{\hbar}E_3 t\right)\right]$

其中 $E_n = \frac{\pi^2\hbar^2}{2ma^2}n^2 \quad (n=1,2,3,\cdots)$

$t<0$时 $\psi(x,t) = \psi(x,0)$

$t>0$时 $\psi(x,t) = \frac{\sqrt{10}}{10}\left[3\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a}\exp\left(-\frac{i\pi^2\hbar^2}{2ma^2}t\right) + \sqrt{\frac{2}{a}}\sin\frac{3\pi x}{a}\exp\left(-\frac{9i\pi^2\hbar}{2ma^2}t\right)\right]$

$$\langle x\rangle = \int_0^a |\psi(x,t)|^2 x\,dx = \frac{a}{2}$$

$$\langle p\rangle = \int_0^a \psi(x)\left(-i\hbar\frac{\partial}{\partial x}\right)\psi(x)\,dx = 0$$

T14. $[0,a]$上无限深势阱的本征波函数为

$\varphi_n(x)=\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$

粒子的初始波函数为 $\psi(x,0)=\begin{cases}\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a}, & 0\le x\le a\\ 0, & x\le 0,\ x\ge a\end{cases}$

突变后：势阱的能量本征态 $E_n=\frac{\pi^2\hbar^2}{2m(2a)^2}n^2\ (n=1,2,3,\cdots)$

本征态 $\tilde{\varphi}_n(x)=\sqrt{\frac{2}{2a}}\sin\frac{n\pi x}{2a}$

$$\psi(x,0)=\sum_{n}^{\infty}C_n\tilde{\varphi}_n(x)$$

$$C_n=\int_0^{2a}\tilde{\varphi}_n(x)\psi(x,0)\,dx$$

$$=\int_0^{2a}\sqrt{\frac{1}{a}}\sin\frac{n\pi x}{2a}\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a}dx$$

$$=\frac{\sqrt{2}}{2a}\int_0^{a}\left[\cos\left(\frac{(n-2)\pi x}{2a}\right)-\cos\frac{(n+2)\pi x}{2a}\right]dx$$