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T1. $A^2=B^2=C^2=I \quad [B,C]=iA$.

对于 $[B,C]=iA$ 即 $BC-CB=iA$，有.

$$BCB-CB^2=BCB-C=iAB \quad (1)$$

$$B^2C-BCB=C-BCB=iBA. \quad (2)$$

(1)+(2)得 $AB+BA=0$

同理有. $AC+CA=0$.

T2. $A^2=0$ $\cancel{AA^+=A}$ $AA^+ + A^+A = I$, $B=A^+A$

(a) $B^2 = A^+A A^+A = A^+A(1-A^+A^+) = A^+A - A^+AAA^+$

$= A^+A = B$

(b) 由 $B^2=B$，得 $B(B-1)=0$，故 $B=0,1$

∵ B本征态无简并

$B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

设 $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ 则 $\cancel{A^*A = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$

$\cancel{= \begin{pmatrix} |a|^2+|c|^2 & a^*b+c^*d \\ b^*a+d^*c & |b|^2+|d|^2 \end{pmatrix}}$

则 $A^*A = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} |a|^2+|c|^2 & a^*b+c^*d \\ b^*a+d^*c & |b|^2+|d|^2 \end{pmatrix}$

$= B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

∵ $|a|^2+|c|^2=0$ ∴ $a=c=0$

$A^2 = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & bd \\ 0 & d^2 \end{pmatrix} = 0$

解得 $d=0$

$AA^+ = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ b^+ & 0 \end{pmatrix} = \begin{pmatrix} |b|^2 & 0 \\ 0 & 0 \end{pmatrix}$

$A^+A = \begin{pmatrix} 0 & 0 \\ b^+ & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & |b|^2 \end{pmatrix}$

$AA^+ + A^+A = \begin{pmatrix} |b|^2 & 0 \\ 0 & |b|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$|b|^2=1$ $b=e^{ia}$ $A = \begin{pmatrix} 0 & e^{ia} \\ 0 & 0 \end{pmatrix}$ (a为实数)

T3. $\vec{A}\times\vec{B} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ \vec{i} & \vec{j} & \vec{k} \end{vmatrix} = (A_xB_y - A_yB_x)\vec{k} - (A_xB_z - A_zB_x)\vec{j}$

$+ (A_yB_z - A_zB_y)\vec{i}$

$$(\vec{A}\cdot\vec{\sigma})(\vec{B}\cdot\vec{\sigma}) = (A_x\vec{\sigma_x} + A_y\vec{\sigma_y} + A_z\vec{\sigma_z})(B_x\vec{\sigma_x} + B_y\vec{\sigma_y} + B_z\vec{\sigma_z})$$

$$= (A_xB_x + A_yB_y + A_zB_z) + \cancel{(A_xB_y + A_yB_x}$$

$$+ (A_xB_y - A_yB_x)\vec{\sigma_x}\vec{\sigma_y}$$

$$+ (A_xB_z - A_zB_x)\vec{\sigma_x}\vec{\sigma_z}$$

$$+ (A_yB_z - A_zB_y)\vec{\sigma_y}\vec{\sigma_z}$$

$$= \vec{A}\cdot\vec{B} + i[(A_xB_y - A_yB_x)\vec{\sigma_z} - (A_xB_z - A_zB_x)\vec{\sigma_y}$$

$$+ (A_yB_z - A_zB_y)\vec{\sigma_x}]$$

$$= \vec{A}\cdot\vec{B} + i(\vec{A}\times\vec{B})\cdot\vec{\sigma}.$$

T4. $U^{+}U=UU^{+}=1$

设 $U=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $U^{+}=\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$

$\because U^{+}U=1$ $\therefore \begin{cases} a^*a+c^*c=1 & (1) \\ a^*b+c^*d=0 & (2) \\ b^*b+d^*d=1 & (3) \\ \cancel{b^*c} \\ b^*a+d^*c=0 & (4) \end{cases}$

$\because UU^{+}=1$ $\therefore \begin{cases} aa^*+bb^*=1 & (5) \\ cc^*+dd^*=1 & (6) \\ ac^*+bd^*=0 & (7) \\ ca^*+db^*=0. & (8) \end{cases}$

又 $\det U=1$ $\therefore ad-bc=1.$ (9)

由 (1)、(3)、(5)、(6) 得 $bb^*=cc^*$，$aa^*=dd^*$.

又由 (2)(4)、(7)、(8)、得

$a=\cos w e^{i\alpha}$ $b=\sin w e^{i\beta}$ $c=\sin w e^{i\delta}$ $d=\cos w e^{i\gamma}$

$\cos^2 w e^{i(\alpha+\gamma)}-\sin^2 w e^{i(\beta+\delta)}=1$

即 $\cos^2 w e^{i(\alpha+\gamma)}-\sin^2 w e^{i(\beta+\delta)}=1.$

又$\because$ 这一方程成立当且仅当 $e^{i\delta}=e^{-i\beta}$

$e^{i(\alpha+\gamma)}=1$ $e^{i(\beta+\delta)}=-1$ 即 $e^{i\gamma}=e^{-i\alpha}$

故有 $U=\begin{pmatrix} \cos w e^{i\alpha} & \sin w e^{i\beta} \\ -\sin w e^{-i\beta} & \cos w e^{-i\alpha} \end{pmatrix}$

其中 α,β,w 为三个实参量.

T7. $\vec{\sigma}\cdot\vec{n}$的本征值为±1

$$\vec{\sigma}\cdot\vec{n} = \sigma_x \sin\theta\cos\varphi + \sigma_y \sin\theta\sin\varphi + \sigma_z\cos\theta$$

$$= \begin{pmatrix} \cos\theta & \sin\theta\cos\varphi - i\sin\theta\sin\varphi \\ \sin\theta\cos\varphi + i\sin\theta\sin\varphi & -\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix}$$

特征多项式 $\begin{vmatrix} \cos\theta-\lambda & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta-\lambda \end{vmatrix} = \lambda^2-1$

故本征态 $\lambda=\pm1$

$\lambda=1$ 对应的本征态 $|\uparrow\rangle_{\vec{n}} = \begin{pmatrix} \cos\frac{\theta}{2}e^{-\frac{i\varphi}{2}} \\ \sin\frac{\theta}{2}e^{\frac{i\varphi}{2}} \end{pmatrix}$

$\lambda=-1$ 对应的本征态 $|\downarrow\rangle_{\vec{n}} = \begin{pmatrix} -\sin\frac{\theta}{2}e^{-\frac{i\varphi}{2}} \\ \cos\frac{\theta}{2}e^{\frac{i\varphi}{2}} \end{pmatrix}$

z方向上 $|\uparrow\rangle_z = e^{-\frac{i\varphi}{2}}\begin{pmatrix}1\\0\end{pmatrix}$ $\quad |\downarrow\rangle_z = e^{\frac{i\varphi}{2}}\begin{pmatrix}0\\1\end{pmatrix}$

略去整体相因子

$\vec{\sigma}$在z方向上的投影的本征态

$$|\uparrow\rangle_z = \begin{pmatrix}1\\0\end{pmatrix} \quad |\downarrow\rangle_z = \begin{pmatrix}0\\1\end{pmatrix}$$

故测量自旋在$\vec{n}$方向投影，得到$\frac{\hbar}{2}$的概率为

$$|\langle\uparrow_{\vec{n}}|\uparrow\rangle|^2 = \cos^2\frac{\theta}{2}$$

得到$-\frac{\hbar}{2}$的概率为 $|\langle\downarrow_{\vec{n}}|\uparrow\rangle|^2 = \sin^2\frac{\theta}{2}$

T_{12}. 复合系统量子态 $|\psi\rangle = |\psi\rangle_A \otimes |\psi\rangle_B$

$$= |\uparrow\rangle \otimes \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle\right)$$

由 $|S=1, S_z=1\rangle = |\uparrow\uparrow\rangle$

$$|S=1, S_z=-1\rangle = |\downarrow\downarrow\rangle$$

$$|S=1, S_z=0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$|S=0, S_z=0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

得 $|\psi\rangle = \frac{1}{\sqrt{2}}|S=1, S_z=1\rangle + \frac{1}{2}|S=1, S_z=0\rangle$

$$+ \frac{1}{2}|S=0, S_z=0\rangle.$$