Lemma 3.1

statements and ideas :

給定區間$[-M, M]$，單個 hidden layer 的 tanh 網路，

$\frac{s+1}{2}$，就能近似所有奇數次單項式，

ex: x, x^3, x^5 … x^s, s為奇數，

甚至能近似他們的導數

background,

對 Taylor series, $\tanh(y) = y - \frac{1}{3}y^3 + \frac{2}{15}y^5 - \cdots$

用中心差分法 (centered finite difference)

利用 tanh 是奇函數的特性

讓低次項互相底消 ($\sigma(-t) = -\sigma(t)$)

define a function f,

$$f_{p,h}(y) = \frac{1}{\sigma^{(p)}(0)\, h^p} \sum_{i=1}^{p} (-1)^i \binom{p}{i} \sigma\left(\left(\frac{p}{2} - i\right) h y\right)$$

例子一： $p=1$

$$f_{1,h}(y) = \frac{\tanh\left(\frac{h}{2} y\right) - \tanh\left(-\frac{h}{2} y\right)}{\sigma'(0)\, h}$$

$\sigma = \tanh \rightarrow \sigma'(0) = 1$

又 $\tanh\left(-\frac{h}{2} y\right) = -\tanh\left(\frac{h}{2} y\right)$

$$\therefore \quad f_{1,h}(y) = \frac{2\tanh\left(\frac{h}{2}y\right)}{h}$$

$$\tanh\left(\frac{h}{2}y\right) = \frac{h}{2}y - \frac{1}{3}\left(\frac{h}{2}y\right)^3 + O(h^5 y^5)$$

$$f_{1,h}(y) = y - \frac{h^2}{12}y^3 + O(h^4 y^5)$$

for $h \to 0$, $f_{1,h}(y) \to y$

例子二， $p=3$

$$f_{3,h}(y) = \frac{\tanh\left(\frac{3h}{2}y\right) - 3\tanh\left(\frac{h}{2}y\right) + 3\tanh\left(-\frac{h}{2}y\right) - \tanh\left(-\frac{3h}{2}y\right)}{\sigma^{(3)}(0)\, h^3}$$

$$\sigma^{(3)}(0) = -2 \quad , \quad \sigma(-t) = -\sigma(t)$$

$$f_{3,h}(y) = \frac{2\left[\tanh\left(\frac{3h}{2}y\right) - 3\tanh\left(\frac{h}{2}y\right)\right]}{-2h^3}$$

$$\sigma(hy) = \sigma(t) = t - \frac{1}{3}t^3 + \frac{2}{15}t^5 + O(t^9)$$

$$f_{3,h}(y) = \frac{-1}{h^3}\left[\sigma\left(\frac{3}{2}t\right) - 3\sigma\left(\frac{1}{2}t\right)\right]$$

一次项：

$$\frac{3}{2}t - 3\cdot\frac{1}{2}t = 0$$

三次项：

$$-\frac{1}{3}\left[\left(\frac{3}{2}t\right)^3 - 3\left(\frac{1}{2}t\right)^3\right] = -\frac{1}{3}\left[\frac{27}{8}t^3 - \frac{3}{8}t^3\right] = -t^3$$

$$f_{3,h}(y) = \frac{-t^3 + O(t^5)}{h^3} = y^3 + O(h^2y^5)$$

lemma 3-2.

statement and idea.

把偶數次方轉成奇數次方，補足 lemma 3-1

background.

recursive construction

example.

$$(y+\alpha)^3 = y^3 + 3\alpha y^2 + 3\alpha y + \alpha^3 \quad \text{—①}$$

$$(y-\alpha)^3 = y^3 - 3\alpha y^2 + 3\alpha^2 y - \alpha^3 \quad \text{—②}$$

$$(y+\alpha)^3 - (y-\alpha)^3 = 6\alpha y^2 + 2\alpha^3$$

$$y^2 = \frac{1}{6\alpha}\left[(y+\alpha)^3 - (y-\alpha)^3\right] - \frac{\alpha^2}{3}$$