Understanding Analysis, 2nd Edition: Stephen Abbott
Chapter 1 - Section 1.6 - Cantor's Theorem

1. Show that $(0, 1)$ is uncountable if and only if R is uncountable.

> **Solution:** First we prove the necessity: if $(0, 1)$ is uncountable then R is uncountable. This follows immediately from the fact that $(0, 1)$ is a subset of R. A superset of an uncountable set must be uncountable as well.
>
> Now the sufficiency: if R is uncountable then $(0, 1)$ is uncountable. I initially wanted to prove this by contrapositive: if $(0, 1)$ is countable then R is countable. But this is tricky. The fact that $(0, 1)$ is a subset of R does not imply R is countable. Then I tried using the fact that the union of (infinitely many) countable sets is countable, by merging infinitely many intervals. But this requires proving that the intervals other than $(0, 1)$ are also countable.
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> The idea hit me to produce a bijection between $(0, 1)$ and R instead, and indeed such a bijection is possible by Exercise 1.5.4. For example, the tangent function maps $(\frac{-\pi}{2}, \frac{\pi}{2})$ to R. We need just need to map $(0, 1)$ to $(\frac{-\pi}{2}, \frac{\pi}{2})$ and compose it with tan. Find the line equation joining $(0, 1)$ to $(\frac{-\pi}{2}, \frac{\pi}{2})$ and we get $\pi x - \frac{\pi}{2}$, therefore $f(x) = tan(\pi x - \frac{\pi}{2})$ is a bijection between $(0, 1)$ and R. This proves both necessity and sufficiency at once.

2. (a) Explain why the real number $x = .b_1b_2b_3...$ cannot be $f(1)$.

   (b) Now explain why $x \neq f(2)$, and in general why $x \neq f(n)$ for any $n \in N$.

   (c) Point out the contradiction that arises from these observations and conclude that $(0,1)$ is uncountable.

Solution:

(a) Because b_1 is already different from a_{11}.

(b) In general, $b_n \neq a_{nn}$, so the decimal expansion differs in at least one digit.

(c) If $(0,1)$ were countable, then $x = f(n)$ for an n, since x is defined to be a real number $\in (0,1)$. But from part (b), $x \neq f(n)$ for all n, a contradiction. Therefore $(0,1)$ must be uncountable.

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3. Supply rebuttals to the following complaints about the proof of Theorem 1.6.1 (the open interval $(0,1)$ is uncountable).

   (a) Every rational number has a decimal expansion, so we could apply the same argument to show that the set of rational numbers between 0 and 1 is uncountable. However, because we know that every subset of Q must be uncountable, the proof of Theorem 1.6.1 must be flawed.

   (b) Some numbers have *two* different decimal representations. Specifically, any decimal expansion that terminates can also be written with repeating 9's. For instance, $1/2$ can be written as 0.5 or as 0.49999.... Doesn't this cause some problems?

Solution:

(a) Every rational number has a decimal expansion, but the converse is not true; not every decimal expansion represents a rational number. In the proof of Theorem 1.6.1, x may be irrational, so it can't be used to conclude that Q is uncountable.

(b) First, it's worth noting that the *only way* a number can have two different decimal expansion is the one the problem statement says; one expansion that terminates, and another with repeating 9's. I claim this without proof, as the book also intentionally uses decimal representations without formal definitions.

The proof of Theorem 1.6.1 works fine because it uses only the digits 2 and 3, so the counterexample it constructs is a number with a unique decimal expansion. It cannot be the case that x is just another representation of the decimal expansion of a real number $f(n)$.