Understanding Analysis, 2nd Edition: Stephen Abbott
Chapter 1 - Section 1.2

1. Prove that $\sqrt{3}$ is irrational. Does the same proof work for $\sqrt{6}$ and $\sqrt{4}$.

> **Solution:** Rather than a proof by contradiction like in the book, a direct proof is more illuminating, although we need to cheat a bit by using unique prime factorization.
>
> Let $(\frac{p}{q})^2 = 3$. Since the LHS is a square, it must have an even number of prime factors, while the RHS has an ood number of factors, qed.
>
> This makes it immediately obvious that the proof works for any integer with an odd number of factors and conversely, it doesn't work for an integer with an even number of factors.

2. Show that there is no rational number satisfying $2^r = 3$.

> **Solution:** There is no solution to $2^p = 3^q$ because they have different parities.
>
> More generally, to check whether $a^n = b^m$ has a solution, we can look at the prime factorizations. First, the set of prime factors must match. Next, if we have p^x in a and p^y in b, then we have $xn = ym \implies \frac{x}{y} = \frac{m}{n}$. This ratio must agree for all prime factors.

3. See the book.

> **Solution:**
>
> (a) False, as in Example 1.2.2.
>
> (b) True. Note the crucial difference with part (a). In part (a) there's always the "next" set A_{m+1} that doesn't contain m. In this part, because A_1 is finite, then either $A_{i+1} = A_i$ in which case both contains m, or $A_{i+1} \subset A_i$ in which case the size of intersections decreases by one and since A_1 is finite and A_n not empty, this process must eventually end.

(c) False. Let $A = \{1\}, B = \{2\}$ and $C = \{3\}$.

(d) True.

(e) True.

4. Produce an infinite collection of sets $A_1, A_2, \ldots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$ and $\cup_{i=1}^{\infty} A_i = \mathbf{N}$.

Solution: First thought when I tried to visualize "crossing out integers" is the prime sieve e.g. like Sieve of Eratosthenes. So the sets would be $A_1 = \{2, 4, 6, 8, \ldots\}, A_2 = \{3, 9, \ldots\}$. But there's also 1. If the first set contains multiples of 1, then we'd only have a single set instead of infinite collection of sets.

It took me way too long to realize this but, we can just put 1 in any of the sets e.g. in the set that starts with 3, since 1 doesn't appear anywhere else.. Now I feel stupid.

There are many other excellent constructions. Another one I liked (and should have thought of, since it feels more "algorithmic") is to first divide by parity, and then divide the even numbers into two sets: one divisible by only 2 and the other divisible by 4, and so on..

A general way to decompose infinite sets is written here.

5. (**De Morgan's Laws**). Let A and B be subsets of $\mathbf{R}$.

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.

(b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.

(c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

Solution: Mostly straightforward. Too lazy to type.

6. Prove a bunch of absolute value stuffs. For all of these, the fastest way to gain an "intuitive" proof is to simply visualize them in the number line. Too lazy to type the full algebra.

7. Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over A; that is, $f(A) = \{f(x) : x \in A\}$.

   (a) Let $f(x) = x^2$. If $A = [0, 2]$ and $B = [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

   (b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

   (c) Show that, for an arbitrary function $g : \mathbf{R} \implies \mathbf{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbf{R}$.

   (d) Form an prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g.

Solution:

   (a) $f(A) = [0, 4]$ and $f(B) = [1, 16]$.

   $f(A \cap B) = f([1, 2]) = [1, 4]$ and $f(A) \cap f(B) = [1, 4]$ so they're equal.

   $f(A \cup B) = f([0, 4]) = [0, 16]$ and $f(A) \cup f(B) = [0, 16]$ so they're equal.

   (b) $A = [-4, -3]$ and $B = [3, 4]$. Then $f(A \cap B) = \emptyset$ but $f(A) \cap f(B) = f(A) = f(B) = [9, 16]$.

   (c) If $x \in A \cap B$, then $g(x) \in g(A)$ (because $x \in A$) and similarly $g(x) \in g(B)$, so $g(x) \in g(A) \cap g(B)$.

   (d) $g(A \cup B) = g(A) \cup g(B)$. This is not difficult to prove. Too lazy to type.

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8. Give an example of each or state that the request is impossible:

   (a) $f : \mathbf{N} \to \mathbf{N}$ that is injective but not surjective.

   (b) $f : \mathbf{N} \to \mathbf{N}$ that is surjective but not injective.

   (c) $f : \mathbf{N} \to \mathbf{Z}$ that is injective and surjective.

Solution:

   (a) $f(n) = 2n$. Odd numbers are not in the image of f.

   (b) $f(n) = floor(n/2)$. Yes, I'm including 0 as a natural number here, sue me.

(c) Split **N** by parity and **Z** by sign (with 0 going to the negative subset). We'll map even **N** to positive **Z** and odd **N** to non-positive **Z**. Define:

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ -\left\lceil \frac{n}{2} \right\rceil, & \text{if } n \text{ is odd} \end{cases}$$

9. Skip. Looks similar to problem 7.

10. Which are true and provide counterexamples if not.

    (a) Two real numbers satisfy $a < b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

    (b) Two real numbers satisfy $a < b$ if $a < b + \epsilon$ for every $\epsilon > 0$.

    (c) Two real numbers satisfy $a \leq b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

Solution:

   (a) False. We can have $a = b$.

   (b) False. We can have $a = b$.

   (c) True.

11. I'm kinda tired of the rest. Maybe some other day...