Understanding Analysis, 2nd Edition: Stephen Abbott
Chapter 1 - Section 1.2

1. Basically it's a bounded sequence.

2. Prove:

   - $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$.
   - $\lim \frac{2n^2}{n^3+3} = 0$.
   - $\lim \frac{\sin(n^2)}{\sqrt[3]{n}} = 0$.

Solution:

- This one is simple algebra. The final simplified form will be $\frac{3}{25n+20} < \epsilon \implies n > \frac{3}{25\epsilon} - \frac{4}{5}$.
- The point of this one is to teach us that not all epsilon-delta style problems need to be solved by relentless algebra.

  In this case, we find that the given expression cannot be factored any further. But we can note that $\frac{2n^2}{n^3+3} < \frac{2n^2}{n^3}$ so we want $\frac{2}{n} < \epsilon \implies n > \frac{\epsilon}{2}$.

  Basically, don't be afraid to discard constants and use inequalities to simplify expressions.
- Like the previous problem, we don't need to do algebra here. Since $\sin(x) \leq 1$ for all x, then $\frac{\sin(n^2)}{\sqrt[3]{n}} < \frac{1}{\sqrt[3]{n}} < \epsilon \implies n > \frac{1}{\epsilon^3}$.

3. Straightforward.

4. Give an example or state that the request is impossible.

   (a) A sequence with infinite number of ones that does not converge to one.

   (b) A sequence with an infinite number of ones that converges to a limit not equal to one.

   (c) A divergent sequence such that for every $n \in \mathbf{N}$ it is possible to find n consecutive ones somewhere in the sequence.

Solution:

(a) Let S = 1, 1e6, 1, 1e6, ... Pick any ϵ less than half the distance, then for any value L, its ϵ nghbrhd cannot contain both 1 and 1e6.

(b) Impossible. Let $L \neq 1$ be the limit and $d = |L - 1|$. Take $\epsilon = d/2$. Since there are infinitely many ones, there is a one that's outside $(L - \epsilon, L + \epsilon)$. This contradicts the fact that L is the limit.

(c) 1, 1e6, 1, 1, 1e6, 1, 1, 1, 1e6, ...

5. Easy.

6. Prove limit is unique.

Solution: Classic problem. The idea is to prove by contradiction. Suppose there are two limits L_1 and L_2 and let d be their difference. Pick $\epsilon < \frac{d}{2}$, then a number cannot be in the ϵ nghbrhd of both L_1 and L_2 simultaneously.

More formally, by definition of convergence, there exists N_1 such that $n \geq N_1 \implies |a_n - L_1| < \frac{d}{2}$. Similiarly, there exists N_2 such that $n \geq N_2 \implies |a_n - L_2| < \frac{d}{2}$. Let $N = max(N_1, N_2)$, then for any $n \geq N$, a_n satisfies both inequalities.

Now for the contradiction: if a point is simultaneously in the nghbrhd of both L_1 and L_2, then the sum of distance to that point from both L's must be $\geq d$. In other words, we have the triangle inequality $|L_1 - L_2| \leq |a_n - L_1| + |a_n - L_2|$. But the RHS are both less than $\frac{d}{2}$, so we have $d \leq |a_n - L_1| + |a_n - L_2| < d$, a contradiction.

7. See the book for the statement.

Solution:

(a) Eventually but not frequently.

(b) Eventually implies frequently, but not the other way around. So "eventually" is a stronger condition.

(c) "Eventually" can be used to describe convergence. Let S_ϵ be the set of all a_n such that $|a_n - L| < \epsilon$, then the limit of a sequence is L if the sequence is eventually in S_ϵ.

(d) Frequently but not eventually.

8. Doesn't look interesting, skip.