Understanding Analysis, 2nd Edition: Stephen Abbott
Chapter 1 - Section 1.6 - Cantor's Theorem

1. Show that $(0,1)$ is uncountable if and only if R is uncountable.

Solution: First we prove the necessity: if $(0,1)$ is uncountable then R is uncountable. This follows immediately from the fact that $(0,1)$ is a subset of R. A superset of an uncountable set must be uncountable as well.

Now the sufficiency: if R is uncountable then $(0,1)$ is uncountable. I initially wanted to prove this by contrapositive: if $(0,1)$ is countable then R is countable. But this is tricky. The fact that $(0,1)$ is a subset of R does not imply R is countable. Then I tried using the fact that the union of (infinitely many) countable sets is countable, by merging infinitely many intervals. But this requires proving that the intervals other than $(0,1)$ are also countable.

The idea hit me to produce a bijection between $(0,1)$ and R instead, and indeed such a bijection is possible by Exercise 1.5.4. For example, the tangent function maps $(\frac{-\pi}{2},\frac{\pi}{2})$ to R. We need just need to map $(0,1)$ to $(\frac{-\pi}{2},\frac{\pi}{2})$ and compose it with tan. Find the line equation joining $(0,1)$ to $(\frac{-\pi}{2},\frac{\pi}{2})$ and we get $\pi x - \frac{\pi}{2}$, therefore $f(x) = tan(\pi x - \frac{\pi}{2})$ is a bijection between $(0,1)$ and R. This proves both necessity and sufficiency at once.

2. (a) Explain why the real number $x = .b_1b_2b_3...$ cannot be $f(1)$.
   (b) Now explain why $x \neq f(2)$, and in general why $x \neq f(n)$ for any $n \in N$.
   (c) Point out the contradiction that arises from these observations and conclude that $(0, 1)$ is uncountable.

Solution:

(a) Because b_1 is already different from a_{11}.

(b) In general, $b_n \neq a_{nn}$, so the decimal expansion differs in at least one digit.

(c) If $(0, 1)$ were countable, then $x = f(n)$ for an n, since x is defined to be a real number $\in (0, 1)$. But from part (b), $x \neq f(n)$ for all n, a contradiction. Therefore $(0, 1)$ must be uncountable.

3. Supply rebuttals to the following complaints about the proof of Theorem 1.6.1 (the open interval $(0, 1)$ is uncountable).
   (a) Every rational number has a decimal expansion, so we could apply the same argument to show that the set of rational numbers between 0 and 1 is uncountable. However, because we know that every subset of Q must be uncountable, the proof of Theorem 1.6.1 must be flawed.
   (b) Some numbers have *two* different decimal representations. Specifically, any decimal expansion that terminates can also be written with repeating 9's. For instance, 1/2 can be written as 0.5 or as 0.49999.... Doesn't this cause some problems?

Solution:

(a) Every rational number has a decimal expansion, but the converse is not true; not every decimal expansion represents a rational number. In the proof of Theorem 1.6.1, x may be irrational, so it can't be used to conclude that Q is uncountable.

(b) First, it's worth noting that the *only way* a number can have two different decimal expansion is the one the problem statement says; one expansion that terminates, and another with repeating 9's. I claim this without proof, as the book also intentionally uses decimal representations without formal definitions.

The proof of Theorem 1.6.1 works fine because it uses only the digits 2 and 3, so the counterexample it constructs is a number with a unique decimal expansion. It cannot be the case that x is just another representation of the decimal expansion of a real number $f(n)$.

4. Let S be the set consisting of all sequences of 0's and 1's. Observe that S is not a particular sequence, but rather a large set whose elements are sequences, namely,

$$S = \{(a_1, a_2, a_3, ...) : a_n = 0 \text{ or } 1\}.$$

As an example, the sequence $(1, 0, 1, 0, 1, 0, ...)$ is an element of S, as is the sequence $(1, 1, 1, 1, 1, ...)$. Give a rigorous argument showing that S is uncountable.

Solution: The same diagonal argument works. We just replace the decimal digits 0-9 with 0 and 1. Consider the following table like in the book

$$\begin{array}{c|l}
\mathbb{N} & (0,1) \\
\hline
1 & f(1) = \ .\, \mathbf{a_{11}}\, a_{12}\, a_{13}\, a_{14}\, a_{15}\, a_{16}\, \cdots \\
2 & f(2) = \ .\, a_{21}\, \mathbf{a_{22}}\, a_{23}\, a_{24}\, a_{25}\, a_{26}\, \cdots \\
3 & f(3) = \ .\, a_{31}\, a_{32}\, \mathbf{a_{33}}\, a_{34}\, a_{35}\, a_{36}\, \cdots \\
4 & f(4) = \ .\, a_{41}\, a_{42}\, a_{43}\, \mathbf{a_{44}}\, a_{45}\, a_{46}\, \cdots \\
5 & f(5) = \ .\, a_{51}\, a_{52}\, a_{53}\, a_{54}\, \mathbf{a_{55}}\, a_{56}\, \cdots \\
6 & f(6) = \ .\, a_{61}\, a_{62}\, a_{63}\, a_{64}\, a_{65}\, \mathbf{a_{66}}\, \cdots \\
\vdots & \vdots
\end{array}$$

where $a_{mn} \in \{0, 1\}$. Construct a sequence b from the diagonal such that $b_n = 1$ if $a_{nn} = 0$ else $b_n = 0$. Exactly the same proof as Exercise 2 works here.

5. (a) Let $A = \{a, b, c\}$. List the eight elements of $P(A)$. (Do not forget that $\emptyset$ is considered a subset of every set).

   (b) If A is finite with n elements, show that $P(A)$ has 2^n elements.

Solution:

(a) $\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$.

(b) For a particular subset, an element is either chosen or not chosen. So there are two choices for each of the n elements, hence 2^n.

6. (a) Using the particular set $A = \{a, b, c\}$, exhibit two different 1-1 mappings from A to $P(A)$.

   (b) Letting $C = \{1, 2, 3, 4\}$, produce an example of a 1-1 map $g : C \to P(C)$.

   (c) Explain why in parts (a) and (b), it's impossible to construct mappings that are *onto*.

Solution: Note that the elements of $P(A)$ are *sets*. So for example in part (a), $f(b) = b$ is not a valid mapping. It should instead be $f(b) = \{b\}$.

(a) $f(a) = \{a\}, f(b) = \{b\}, f(c) = \{c\}$ or $f(a) = \{b\}, f(b) = \{c\}, f(c) = \{a\}$.

(b) $g(1) = \{1\}, g(2) = \{2\}, g(3) = \{3\}, g(4) = \{4\}$.

(c) Because $|P(A)| > |A|$, at least for any nonempty finite set A. It's also true for infinite sets, but that's the next exercise.

7. Return to the particular functions constructed in Exercise 1.6.6 and construct the subset B that results using the preceding rule (see the book). In each case, note that B is not in the range of the function used.

Solution: We'll just do it for part (a) of Exercise 1.6.6. When the mapping is

$$f(a) = f(a) = \{a\}, f(b) = \{b\}, f(c) = \{c\}$$

then $B = \emptyset$, and this is outside the range of f.

When the mapping is

$$f(a) = \{b\}, f(b) = \{c\}, f(c) = \{a\}$$

then $B = \{a, b, c\}$, and this is again outside the range of f.

Also note that this is pretty much the diagonal argument. For a set $A = \{a_1, a_2, a_3, ...\}$ and say, $f(a_1) = \{a_1, a_5\}$, then draw the table with $f(a_1) = \{a_1, a'_2, a'_3, a'_4, a_5, a'_6, ...\}$ where an accent means the element is not part of $f(a_1)$, but we list it anyway so the table stays rectangular. Now go through the diagonal and construct B by taking accented elements.

Too lazy to draw the table here. Just see the illustration in Wikipedia's page about Cantor's diagonal argument.

8. (a) First, show that the case $a' \in B$ leads to a contradiction.

   (b) Now, finish the argument by showing that the case $a' \notin B$ is equally unacceptable.

Solution:

(a) Our assumptions are $B = f(a')$ and $a' \in B$.

From the second assumption, this must mean that $a' \notin f(a')$, because that's how we constructed B. But if we have $a' \in B$ and $a' \notin f(a')$, then $B \neq f(a')$, contradicting our first assumption.

(b) Our assumptions are $B = f(a')$ and $a' \notin B$.

From the second assumption, this must mean that $a' \in f(a')$, because that's how we constructed B. But if we have $a' \notin B$ and $a' \in f(a')$, then $B \neq f(a')$, contradicting our first assumption.