Understanding Analysis, 2nd Edition: Stephen Abbott
Chapter 3 - Section 3.2 - Open and Closed Sets

1. (a) Where in the proof of Theorem 3.2.3 part (ii) does the assumption that the collection of open sets be *finite* get used?

   (b) Give an example of a countable collection of open sets $\{O_1, O_2, O_3, ...\}$ whose intersection $\cap_{n=1}^{\infty} O_n$ is closed, not empty, and not all of R.

Solution:

(a) The part where we want to take the minimum ϵ-neighborhood $\epsilon = min\{\epsilon_1, \epsilon_2, ..., \epsilon_n\} > 0$. A minimum does not necessarily exist in an infinite set.

To be even more precise though, what we're lacking is a positive lower bound. Had there been a positive infimum (but not minimum), we could use that as ϵ (note that the proof is still valid even if we replace minimum with infimum. It's just that taking minimum is simpler for a finite set). However, the infimum may be zero, as can be seen in the next example.

(b) Let $O_n = (-\frac{1}{n}, \frac{1}{n})$. Both endpoints converge to 0, so $\cap_{n=1}^{\infty} O_n = \{0\}$ which is closed, not empty, and not all of R, as required.

2. Let $A = \{(-1)^n + \frac{2}{n} : n = 1, 2, 3, ...\}$ and $B = \{x \in Q : 0 < x < 1\}$. Answer the following questions for each set:

   (a) What are the limit points?

   (b) Is the set open? Closed?

   (c) Does the set contain any isolated points?

   (d) Find the closure of the set.