Understanding Analysis, 2nd Edition: Stephen Abbott
Chapter 2 - Section 2.3

1. Let $x_n \geq 0$ for all $n \in \mathbf{N}$.
   (a) If $(x_n) \implies 0$, show that $(\sqrt{(x_n)} \implies 0$.
   (b) If $(x_n) \implies x$, show that $(\sqrt{(x_n)} \implies \sqrt{x}$.

Solution: Note that we *cannot* use either the algebraic or order limit theorem here, because they require that *both limits exist.* In this problem, the existence of $\lim \sqrt{(x_n)}$ is what we need to prove in this first place!

With that out of the way, we need to fall back to good ol' epsilon-delta proof.

(a) We need to prove that for any $\epsilon > 0$, there exists N such that $n \geq N \implies |\sqrt{x_n}| < \epsilon$. We can discard the abs sign since $x \geq 0$ and sqrt is non-negative.

We know $(x_n) \implies 0$ so we can pick N such that $x_n < \epsilon^2$ which'd make $\sqrt{x_n} < \epsilon$.

(b) We need N such that $|\sqrt{x_n} - \sqrt{x}| < \epsilon$.

We want $|x_n - x|$ to appear so let's rationalize : $|\frac{x_n - x}{\sqrt{x_n + \sqrt{x}}}| < \epsilon$.

As is common in epsilon-delta proof, we can use inequalities to discard terms and simplify expressions. In this case, we're given $x_n \geq 0$ for all n so we have $|\frac{x_n - x}{\sqrt{x_n + \sqrt{x}}}| \leq |\frac{x_n - x}{\sqrt{x}}| < \epsilon$.

Since we're given $(x_n) \implies x$ we can make $|x_n - x|$ as small as we want, so let's pick N such that $n \geq N \implies |x_n - x| < \epsilon\sqrt{x}$, so we have $\frac{x_n - x}{\sqrt{x}} < (\frac{\epsilon\sqrt{x}}{\sqrt{x}} = \epsilon)$, as needed.

Note that this is only valid if $x \neq 0$, but the case $x = 0$ has been handled in part (a).

2. Using only the definition of convergence, prove that if $(x_n) \implies 2$, then

   (a) $(\frac{2x_n - 1}{3}) \implies 1$;

   (b) $(1/x_n) \implies 1/2$.

> **Solution:**
>
> (a) We need $|\frac{2(x_n-2)}{3}| < \epsilon$. Pick $n \geq N$ such that $|x_n - 2| < \frac{3\epsilon}{2}$.
>
> (b) TODO.

3. **(Squeeze Theorem)**. Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbf{N}$ and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

> **Solution:** Just like problem 1, we can't use the algebraic or order limit theorem because it requires that $\lim y_n$ exist, which is what we're trying to prove.
>
> Intuitively, we can pick an N that works for both (x_n) and (z_n) simultaneously and it should work for (y_n) as well. Visualize it: either both (x_n) and (z_n) are to one side of l, or (x_n) is to the left and z_n to the right. Either case, (y_n) will also be in the ϵ neighborhood of l.
>
> To put it more formally, pick N that satisfies both x and z, then $|y - l| \leq max(|x - l|, |z - l|) < \epsilon$.
>
> The proof in Wikipedia is also nice.

4. Let $(a_n) \implies 0$ and use the Algebraic Limit Theorem to compute each of the following limits (assuming the fractions are always defined):

   (a) $\lim \frac{1+2a_n}{1+3a_n-4a_n^2}$.

   (b) $\lim \frac{(a_n+2)^2-4}{a_n}$.

   (c) $\lim \frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}$.

> **Solution:**
>
> (a) 1.

(b) Simplify to $\frac{a_n^2+4a_n}{a_n} = a_n + 4$, then the limit is 4.

(c) $\lim \frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}$.

(d) 2.

5. Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, .., x_n, y_n, ..)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

Solution: The problem statement doesn't say it, but we can actually prove that (z_n) converges to the same limit as (x_n) and (y_n).

Let's prove the "if" part first: if (x_n) and (y_n) converges to the same limit then (z_n) is convergent. Pick an N_{xy} that works for both (x_n) and (y_n), then let $N_z = 2N_{xy}$.

The other direction is similar. We want to prove that if (z_n) is convergent then (x_n) and (y_n) converge to the same limit. Pick an N_z for (z_n). This N_z immediately works for both (x_n) and (y_n), since it covers all the y_n where $n \geq floor(N_z/2)$ in (z_n). Similarly for (x_n).

6. (TODO again). The answer is -1, but try to redo it next time.

7. Give an example or state that the request is impossible by referencing the proper theorem(s):

   (a) sequences (x_n) and (y_n) which both diverge but whose sum $(x_n + y_n)$ converges.

   (b) sequences (x_n) and (y_n) where (x_n) converges, (y_n) diverges, and $(x_n + y_n)$ converges.

   (c) a convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges.

   (d) an unbounded sequence (a_n) and a convergent sequence (b_n) with $(a_n - b_n)$ bounded.

   (e) two sequences (a_n) and (b_n) where $(a_n b_n)$ and (a_n) converge but (b_n) does not.

Solution:

(a) $(x_n) = 1, -1, 1, -1, ...$ and $(y_n) = -1, 1, -1, 1, ...$ Then $(x_n + y_n) = 0, 0, 0, ...$

(b) This is impossible. Intuitively, if (x_n) converges, then for $(x_n + y_n)$ to also converge then x_n can only be "bumped" by something that's also convergent and not randomly by a divergent sequence.

Formally, we'll prove that if (x_n) and $(x_n + y_n)$ converge then (y_n) must be convergent (or in logic speak, let the statement that a sequence is convergent have a truth value "True", then we want to prove "a and not b and c" is false by proving that "(a and c) implies b").

Let $z_n = (x_n + y_n)$, then $\lim y_n = \lim z_n + -x_n = \lim(z_n) + -1 * \lim(x_n)$. So by the algebraic limit theorem, $\lim y_n$ exists.

(c) It's tempting to say that by the algebraic division theorem, this is impossible. But note that the theorem applies only when $\lim b_n \neq 0$.

Consider $(b_n) = 1, 1/2, 1/3, ...$ The limit is 0, but $(1/b_n) = 1, 2, 3, ...$ diverges.

(d) Impossible. By theorem 2.3.2, every convergent sequence is bounded, so (b_n) is bounded. Since $(a_n) = (a_n - b_n) + (b_n)$ and the sum of the bounds of the RHS must be a bound for (a_n) as well.

(e) This is the reverse of part (c). Let $(a_n) = 1, 1/2, 1/3, ...$ and $(b_n) = 1, 2, 3, ...$ Then $(a_n b_n) = 1, 1, 1, ...$

8. Let $(x_n) \implies x$ and let $p(x)$ be a polynomial.

   (a) Show $p(x_n) = p(x)$.

   (b) Find an example of a function $f(x)$ and a convergent sequence $(x_n) \implies x$ where the sequence $f(x_n)$ converges, but not to $f(x)$.

Solution:

(a) $p(x_n) = \sum_{i=i}^{m} a_i x_n^i$ for a polynomial of degree m. We can apply the algebraic limit theorem: $\lim a_i x_n^i = a_i \lim(x_n) \lim(x_n)$...(i times) $= a_i x^i$.

(b) Although the book hasn't touched the concept of continuity at this point, but we just need to make the f discontinuous at x. For example, let $(x_n) = 1/n$, this sequence converges to 0. Now we make $f(0)$ discontinuous. For example, $f(y) = 1$ if $y \neq 0$ else 0. f converges to 1, but $f(0) = 0$.

9. (a) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim(a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?

   (b) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b?

   (c) Use (a) to prove Theorem 2.3.3, part (iii), for the case when $a = 0$.

Solution:

(a) We need $|a_n b_n| < \epsilon$. Let B be a bound of (a_n), then $|a_n b_n| \leq |B||b_n|$. Since we have $\lim b_n = 0$, we make $|b_n| < \frac{\epsilon}{|B|}$.

We can't use the Algebraic Limit Theorem because it requires that we know that (a_n) is convergent.

(b) No. If we try the epsilon-delta style proof, we'll get $|a_n(b_n - L)| + |(a_n - 1)L|$. Unlike part (a), we can't just directly bound $a_n - 1$. We only know that a_n is bounded, but $a_n - 1$ may lie outside the bound.

For a concrete counterexample, let $(a_n) = 1, -1, 1, -1, ..$ and $(b_n) = 5, 5, 5, 5....$

(c) Prove the limit multiplication theorem in case $(a_n) \implies 0$. Well, we just need to note that convergent sequence is always bounded and directly apply part (a).