Understanding Analysis, 2nd Edition: Stephen Abbott
Chapter 1 - Section 1.2

1. Doesn't look interesting. Skip for now.

2. Give an example or state that it's impossible.

   (a) A set B with inf $B \geq$ sup B.

   (b) A finite set that contains its infimum but not its supremum.

   (c) A bounded subset of $\mathbf{Q}$ that contains its supremum but not its infimum.

Solution:

(a) A single-element set has its infimum equal supremum.

(b) Slightly tricky. The half-open interval [0, 1) of real numbers is *not* a valid answer, because it's not a finite set.

The answer is it's impossible. A finite set always has its minimum and maximum as infimum and supremum respectively.

(c) (0, 1].

3. (a) Let A be nonempty and bounded below, and define $B = \{b \in \mathbf{R} :$ b is a lower bound for A$\}$. Show that sup B = inf A.

   (b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

Solution:

(a) First we need to assert that sup B actually exists. By the axiom completeness, any set bounded above has a supremum, and any element of A is an upper bound, so sup B exists.

Next we prove that sup B = inf A. Any real number less than sup B clearly cannot be the infimum, since it wouldn't be the *greatest* lower bound. There is also no lower bound greater than sup B, since by definition sup B is greatest of all lower bounds. This completes the proof.

(b) Because (a) already shows that the infimum exists (if the set is bounded below) as the supremum of another set, and it also gives the construction of such a set (by taking the set of all lower bounds).

4. Let $A_1, A_2, ...,$ be a collection of nonempty sets, each of which is bounded above.
   (a) Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup(\cup_{k=1}^{n} A_k)$.
   (b) Consider $\sup(\cup_{k=1}^{\infty} A_k)$. Does the formula in (a) extend to the infinite case?

Solution:

(a) max(sup A_1, sup A_2). max(sup A_1, ..., sup A_n). Too lazy to write the formal proof, but it should be intuitively clear.

(b) No, because the resulting union of supremums might not be bounded above. The simplest example is $A_i = i$.

5. As in example example 1.3.7, let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $c \in \mathbf{R}$. This time define the set $cA = \{ca : a \in A\}$.
   (a) If $c \geq 0$, show that $\sup(cA) = c \sup(A)$.
   (b) Postulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

Solution:

(a) c sup A is an upper bound because $a \leq \sup A$ for all a and that implies $ca \leq c \sup A$ for all a.

It is the *least* upper bound. Suppose x is any upper bound for cA. Then $x \leq ca \implies x/c \leq a$. This means x/c is an upper bound of A. But if it's an upper bound then $x/c \geq \sup A \implies x \geq c \sup A$.

(b) $c \inf A$. Too lazy to write.

6. Boring. Skip.

7. Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $a = \sup A$.

Solution: This is basically saying a is the maximum of the set.

8. Compute, without proofs, the suprema and infima (if they exist) of the following sets:
   (a) $\{m/n : m, n \in \mathbf{N} \text{ with } m < n\}$.
   (b) $\{(-1)^m/n : m, n \in \mathbf{N}\}$.
   (c) $\{n/(3n+1) : n \in \mathbf{N}\}$.
   (d) $\{m/(m+n) : m, n \in \mathbf{N}\}$.

Solution: Note that we don't include 0 in $\mathbf{N}$.

(a) Supremum: 1/2. Infimum: 0 (set $m = 1$ and n arbitrarily big).

(b) Supremum: 1. Infimum: -1.

(c) Supremum: 1/3. Infimum: 1/4.

(d) Supremum: 1/2. Infimum: 0.

9. (a) If sup $a <$ sup B, show that there exists an element $b \in B$ that is an upper bound for A.
   (b) Give an example to show that this is not always the case if we only assume sup $A \leq$ sup B.

Solution:

(a) The idea is that there must be an element of b between sup A and sup B.

Let $d =$ sup $B-$ sup A. Since sup B is the *least* upper bound, then there must be an element $b \in B$ such that $b >$ sup $B - d \implies b >$ sup A.

(b) Let $A = [0, 1]$ and $B = [0, 1)$.

10. **Cut Property**. The Cut Property of the real numbers is the following:
If A and B are nonempty, disjoint sets with $A \cup B = \mathbf{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists $c \in \mathbf{R}$ such that $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

   (a) Use the AoC to prove the Cut Property.

   (b) Show that the implication goes the other way; that is, assume $\mathbf{R}$ possesses the Cut Property and let E be a nonempty set that is bounded above. Prove sup E exists.

   (c) The punchline of parts (a) and (b) is that the Cut Property could be used in place of AoC as the fundamental axiom that distinguishes the real numbers from the rationals. To drive this point home, give a concrete example showing that the Cut Property is not a valid statement when $\mathbf{R}$ is replaced by $\mathbf{Q}$.

Solution:

(a) Any element in b is an upper bound of A so by AoC, A has a supremum (conversely, by exercise 3, B has an infimum). Take $c = \sup A$ (or $c = \inf B$).

(b) Since E be bounded above, let B be the set of all upper bounds i.e. $B = \{b \in \mathbf{R} : \text{ b is an upper bound of E}\}$. But this is yet enough to invoke the cut property, since it's not necessarily true that $E \cup B = \mathbf{R}$.

Let $A = \{a \in \mathbf{R} : a < e \text{ for some } e \in E\}$. It's easy to verify that combining A and E does not change the upper bounds of E, so let $E = A \cup E$. Now $E \cup B = \mathbf{R}$. Invoke the Cut Property: the c defined in the property is the supremum.

(c) The idea is that the "cut" point should be an irrational number. So let A be all rationals less than $\sqrt{2}$ and B all rationals greater than $\sqrt{2}$. Then $A \cup B = \mathbf{Q}$ since the only missing point is not a rational number.

Next, suppose there is a c such that $x \leq c$ whenever $x \in A$. Then c must be greater than $\sqrt{2}$, for if $c < \sqrt{2}$ then $c \in A$ but if $c \in A$ then we can always find a rational greater than c but less than $\sqrt{2}$. A similar argument shows that if c satisfies $x \leq c$ for all $x \in A$ then it cannot satisfy $x \geq c$ for all $x \in B$.

11. Decide if the following statements are true or false. Give a short proof if true, or a counterexample otherwise.

   (a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then $\sup A \leq \sup B$.

   (b) If $\sup A < \inf B$ for sets A and B, then there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.

   (c) If there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

Solution:

(a) True. $\sup B$ is an upper bound of A, so the *least* upper bound of A must, by definition, be less than or equal to $\sup B$.

(b) True. The idea is that there must be a gap between $\sup A$ and $\inf B$ and we can take any number in that gap. For example, take $c = (\sup A + \inf B)/2$.

(c) No. Take $A = [1, 2)$ and $B = (2, 3]$.