Understanding Analysis, 2nd Edition: Stephen Abbott
Chapter 2 - Section 2.7

1. Proving the Alternating Series Test amounts to showing that the sequence of partial sums

$$s_n = a_1 - a_2 + a_3 - \ldots \pm a_n$$

converges.

   (a) Prove it by showing that (s_n) is a Cauchy sequence.

   (b) Supply another proof using the Nested Interval Property.

   (c) Consider the subsequences (s_{2n}) and s_{2n+1}, and show how the Monotone Converge Theorem leads to a third proof.

Solution:

(a) Intuition: look back at the condition of the Alternating Series Test and visualize the graph of (s_n).

It's a zigzag (from the alternating signs) where if we start from an index i, then the **absolute value** of all the suffix sum starting from i is $\leq a_i$ (from the fact that (a_n) is non-decreasing and $a_n \geq 0$). Formally, for any $i < j$, we have:

$$|s_j - s_i| = |\sum_{k=i+1}^{j} (-1)^{k+1} a_k| \leq a_{i+1}$$

*this can be formally proved by induction, but I'm too lazy to write it.
note that this applies regardless of the parity of i. I was initially confused by this. The important point is the **absolute value sign.

This means that to fulfill the Cauchy criterion, we only need to find N such that for all $n \geq N$, $a_n < \epsilon$, because we know the (absolute) suffix sum starting from a_N will all be $\leq a_N$. In other words, if we can find such an N, then we can set $i = N$, then $a_{i+1} \leq a_i < \epsilon$ implies $|s_j - s_i| \leq a_{i+1} < \epsilon$ for all $i < j$.

To find such an N, we use condition (ii) of the Alternating Series Test: $(a_n) \implies 0$, so for any $\epsilon > 0$ there exist N such that $n \geq N$, $|a_n| < \epsilon$.

(b) TODO

(c) Like part (a), we can look at the zig-zag pattern of (s_n). Then we can prove that (s_{2n+1}) is monotone non-increasing and bounded below by 0, while (s_{2n}) is monotone non-decreasing and bounded above by a_1. By MCT both sequences converge. It remains to prove that they converge to the same limit.

Let L be the limit of (s_{2n}). Then there exists an N_1 such that for all $n \geq N_1$, $|s_{2n} - L| < \epsilon$.

To prove that (s_{2n+1}) also converges to L, look at the zig-zag pattern again. The idea is we want to pick N such that, from the previous even s_{2n}, we don't rise too far as to be outside the ϵ-range of L. To do this, we know that $(a_n) \to 0$, so there exists N_2 such that for all $n \geq N_2$, $|a_n| < \frac{\epsilon}{2}$. Let N_1 such that $|s_{2n} - L| < \frac{\epsilon}{2}$. Now pick $N = max(N_1, N_2)$, then for any $\epsilon > 0$, we have

$$|s_{2n+1} - L| = |s_{2n} - L + a_{2n+1}| \leq |s_{2n} - L| + |a_{2n+1}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

UPDATE: there's a much smarter solution. Let (s_{2n}) converge to L. Then to prove (s_{2n+1}) also converges to L

$$\lim s_{2n+1} = \lim(s_{2n} + a_{n+1}) = \lim(s_{2n}) + \lim a_{2n+1} = S + 0 = S$$

2. Converges or diverges:

   (a) $\sum_{n=1}^{\infty} \frac{1}{2^n + n}$.

   (b) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$.

   (c) $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \frac{7}{12} + ...$

   (d) $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + ...$

   (e) $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + ...$

Solution:

(a) We know that $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent geometric series (because $r = \frac{1}{2}$ satisfies $|r| < 1$). By the Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{2^n + n}$ converges.

I initially proved this by comparing with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ instead, which the book has proved to converge in example 2.4.4, page 57. This works, but comparing with 2^n feels more natural.

(b) This is what the book means in page 73 about combining the Comparison Test with the Absolute Convergence Test to handle series that contain some negative terms.

Example 2.4.4 has proved that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. We have $|\sin(n)| \leq 1$ for all n, so $|\frac{\sin(n)}{n^2}| \leq \frac{1}{n^2}$ By the Comparison Test, the series $\sum_{n=1}^{\infty} |\frac{\sin(n)}{n^2}|$ converges and by the Absolute Convergence Test, $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ also converges.

(c) The series is $\sum_{n=2}^{\infty} (-1)^n \frac{n}{2n-2}$. The sequence $\frac{n}{2n-2}$ converges to $\frac{1}{2}$. To see this, divide both nume and denom by n to get $\frac{1}{2-\frac{2}{n}}$ and apply the Algebraic Limit Theorem. Or intuitively, for large n the term $\frac{n}{2n}$ will eventually dominate, so we can "discard" the "- 2" term in the denom.

(d) (TODO again. Got help from MSE).

I was initially trying to apply the convergence tests, which didn't work.

Since the sign pattern repeat every 3 terms, we can try grouping by 3: $(1+\frac{1}{2}-\frac{1}{3})+(\frac{1}{4}+\frac{1}{5}-\frac{1}{6})+(\frac{1}{7}+\frac{1}{8}-\frac{1}{9})+....$ This is *less* than $(1)+(\frac{1}{4})+(\frac{1}{7})+... = \sum_{k=1}^{\infty} \frac{1}{3k-2}$, and the last series diverges, because

$$\sum_{k=1}^{\infty} \frac{1}{3k-2} = \frac{1}{3} \sum_{k=1}^{\infty} \frac{1}{k-\frac{2}{3}} \geq \frac{1}{3} \sum_{k=1}^{\infty} \frac{1}{k}$$

And the Harmonic Series diverges.

(e) Group by two:
$$(1 - \frac{1}{2^2}) + (\frac{1}{3} - \frac{1}{4^2}) + (\frac{1}{5} - \frac{1}{6^2}) + ...$$

And we have:

$$(1 - \frac{1}{2^2}) + (\frac{1}{3} - \frac{1}{4^2}) + (\frac{1}{5} - \frac{1}{6^2}) + ... \geq \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + ... = \sum_{k=1}^{\infty} \frac{1}{2k}$$

So the series diverges, because we can use the Comparison Test on the last inequality, and we have $\sum_{k=1}^{\infty} \frac{1}{2k} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k}$ and the Harmonic Series diverges.

3. (a) Provide the details for the proof of the Comparison Test (Theorem 2.7.4) using the Cauchy Criterion for Series.

   (b) Give another proof using the Monotone Convergence Theorem.

Solution: Recall the condition for the Comparison Tests: $0 \leq a_k \leq b_k$.

- As the book says in page 73, this follows immediately from the inequality $|a_{m+1} + a_{m+2} + ... + a_n| \leq |b_{m+1} + b_{m+2} + ... + b_n|$.

  - If (b_k) converges, then (a_k) converges.
    We have $|a_{m+1} + a_{m+2} + ... + a_n| \leq |b_{m+1} + b_{m+2} + ... + b_n| < \epsilon$.

  - If (a_k) diverges, then (b_k) diverges.
    $\epsilon < |a_{m+1} + a_{m+2} + ... + a_n| \leq |b_{m+1} + b_{m+2} + ... + b_n|$

- We'll prove that if (b_k) converges then (a_k) converges. Let $s_n = \sum_{i=1}^{n} b_i$ be the sequence of partial sums of (b_k). Since $(b_k) \geq 0$ and convergent, by MCT it's bounded. Since $(a_k) \leq (b_k)$, the same bound works for (a_k). The partial sums of (a_k) is also non-decreasing, so by MCT (a_k) is also convergent.

  The proof for divergence is similar.

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4. Give an example of each or explain why the request is impossible, referencing the proper theorem(s).

   (a) Two series $\sum x_n$ and $\sum y_n$ that both diverge but where $\sum x_n y_n$ converges.

   (b) A convergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n y_n$ diverges.

   (c) Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum (x_n + y_n)$ both converge but $\sum y_n$ diverges.

   (d) A sequence (x_n) satisfying $0 \leq x_n \leq 1/n$ where $\sum (-1)^n x_n$ diverges.

Solution:

(a) Let $\sum x_n$ and $\sum y_n$ both be the harmonic series, then $\sum x_n y_n = \sum_{i=1}^{\infty} \frac{1}{p^2}$ which has been proven to converge in Example 2.4.4

(b) Let $\sum x_n$ be the alternating harmonic series and $(y_n) = 1, -1, 1, -1, ...$ Then $\sum x_n y_n$ is the harmonic series, which diverges.

Remark: I initially tried to prove that this was possible, by considering the sequence of partial sums and multiplying that with (y_n). But the problem becomes "must the product of a convergent sequence and a bounded sequence be convergent"? This shouldn't be true, since the Algebraic Limit Theorem states that *both* sequences must be convergent, not only bounded.

I then tried to think of a bounded non-convergent sequence, and $1, -1, 1, -1, ...$ came to mind. But seeing the alternating signs instantly reminded me of the alternating harmonic series, and the problem was done.

I guess the moral of the story is: $1, -1, 1, -1, ...$ is always a useful case to consider for any problem that's related to "bounded but not necessarily convergent" sequence.

(c) Impossible. By the Algebraic Limit Theorem for Series, $\sum y_n = \sum(x_n + y_n) + (-1)(\sum_{x_n})$. Since the RHS converges, $\sum y_n$ must converge as well.

(d) Let (x_n) be 0 if n is odd or $\frac{1}{n}$ otherwise. Then the series (counting from $n = 1$) $0 + \frac{1}{2} - 0 + \frac{1}{4} - 0 + \frac{1}{6}...$ diverges.

Moral of the story: the Alternating Series Test requires that the sequence be monotone.

5. Now that we have proved the basic facts about geometric series, supply a proof for Corollary 2.4.7.

Solution: The corollary is: the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. Note that p is not necessarily an integer.

The fact about geometric series that we're going to use is Example 2.7.5: a geometric series with ratio r converges iff $|r| < 1$.

We first prove the "only if" direction: if $p > 1$ then the series converges. This looks easier than the "if" direction because here we're given the assumption $p > 1$.

We use the Cauchy Condensation Test to turn this into a geometric series. The test states that the series converges iff $\frac{1}{1^p} + 2\frac{1}{2^p} + 4\frac{1}{4^p} + 8\frac{1}{8^p} + ... = \sum_{n=1}^{\infty} 2^n \frac{1}{2^{np}} = 2^{n(1-p)}$ converges. This is a geometric series with ratio $r = 2^{1-p}$. If $p > 1$ then $|r| < 1$, so the condensed series converges, hence the original series also converges.

Next, the "if" direction: if $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges then $p > 1$.

First, we discard the possibility that $p \leq 0$, since in that case the terms of series does not even converge to 0. Now assume $p > 0$, then the series satisfies the precondition for the Cauchy Condensation Test (the series must be decreasing and ≥ 0).

Note that the Condensation Test is both a necessity and sufficiency test, so we can claim that if $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges then $\sum_{n=1}^{\infty} 2^n \frac{1}{2^{np}}$ must converge as well. Then we use the same fact as the "only if" direction; the series converges iff $r = 2^{(1-p)} < 1$, and this condition is satisfied iff $1 - p < 0 \implies p > 1$.

By the Cauchy Condensation Test, $\sum \frac{1}{n^p}$ converges iff $\sum 2^n (\frac{1}{2^n})^p$ converges. But notice that $\sum 2^n (\frac{1}{2^n})^p = \sum (\frac{1}{2^{p-1}})^n$.

By the Geometric Series Test, this series converges iff $|\frac{1}{2^{p-1}}| < 1$. Solving for p we find that p must satisfy $p > 1$.

6. Let's say that a series *subverges* if the sequence of partial sums contains a subsequence that converges. Decide which of the following statements are valid.

   (a) If (a_n) is bounded, then $\sum a_n$ subverges.

   (b) All convergent series are subvergent.

   (c) If $\sum |a_n|$ subverges, then $\sum a_n$ subverges as well.

   (d) If $\sum a_n$ subverges, then (a_n) has a convergent subsequence.

Solution:

   (a) False. Take $(a_n) = 1, 1, 1, 1, ...$ The sequence of partial sums $1, 2, 3, 4, ...$ diverges and none of its subsequences converge.

   (b) Yes. A convergent series must have its sequence of partial sums converge as well, and any subsequence of a convergent sequence must converge to the same limit (Theorem 2.5.2).

   (c) Yes. The proof is very similar to the proof of the Absolute Convergence Test.

   Let (s_{n_m}) be the subsequence of the partial sums of $\sum |a_n|$ that converges. By the Cauchy Criterion, there exists $M \in \mathbf{N}$ such that for all $p, q \geq M$, $|a_{n_{p+1}}| + ... + |a_{n_q}| < \epsilon$. By the triangle inequality, $|a_{n_{p+1}} + ... + a_{n_q}| \leq |a_{n_{p+1}}| + ... + |a_{n_q}| < \epsilon$, so the same subsequence indices in $\sum a_n$ converges as well.

(d) I initially tried to prove that this was possible, using the inequality from part (c). But the fact that $|a_{n_{p+1}} + ... + a_{n_q}| < \epsilon$ does not tell us anything about whether any individual $|a_n|$ is less than ϵ. The only thing it guarantees is that we can get convergence if we sum possibly multiple terms of a_n each time.

This gives an idea of a counterexample: construct a subvergent series where the subvergence only shows every block of n terms ($n > 1$). So something like

$$1 - 1 + 1 - 1 + 1...$$

won't work. But

$$6 + (-1) + (-2) + (-3) + 60 + (-10) + (-20) + (-30) + 600 + (-100) + ...$$

works. The partial sums are:

$$6, 5, 3, 0, 60, 50, 30, 0, 600, 500, ...$$

so there's a subsequence of $0, 0, 0, ...$ But the sequence:

$$6, (-1), (-2), (-3), 60, (-10), (-20), ...$$

clearly diverges.

Remark: I guess the moral of the problem is to show the hierarchy (from strong to weak) between convergence, subsequence convergence, and boundedness.

7. (a) Show that if $a_n > 0$ and $\lim(na_n) = l$ with $l \neq 0$, then the series $\sum a_n$ diverges.

(b) Assume $a_n > 0$ and $\lim(n^2 a_n)$ exists. Show that $\sum a_n$ converges.

Solution:

(a) First, note that the n in $\lim(na_n)$ is *not* a constant! It's instead the indices: $(na_n) = a_1, 2a_2, 3a_3, ...$

———————— INCORRECT SOLUTION FOR POSTERITY ————————

My first intuition is that we will probably prove that (a_n) doesn't converge to 0, since that's the easiest criteria to prove divergence.

Next, I tried to play around with the sequence (na_n) and see what I conclusions I can make about (a_n). We are told $l \neq 0$, so I thought (a_n) can't converge to

0. I initially tried to use the Algebraic Limit Theorem: $lim(n) * lim(a_n) = l$, but it's not applicable here because $lim(n)$ diverges (the theorem requires that both limits are known).

But it's still not too difficult to prove that $lim(a_n) \neq 0$. We prove it by contradiction. If $lim(a_n) = 0$, then $lim(na_n) = 0$, because we can always pick N such that $n \geq N \implies |a_n| < \frac{\epsilon}{n} \implies |na_n| < \epsilon$, contradicting the assumption that $lim(na_n) = l$ with $l \neq 0$.

We've proved that $lim(a_n) \neq 0$, but to prove that $\sum a_n$ diverges we still need to prove that (a_n) is not the zero sequence. This is where the assumption $a_n > 0$ comes into play.

ALL THIS IS INCORRECT. The epsilon-delta proof is wrong because $\frac{\epsilon}{n}$ is not a fixed quantity. Concrete counterexample: the harmonic series converges to 0 but $(n\frac{1}{n})$ converges to 1. So yes, (a_n) can converge to 0 even if (na_n) doesn't.

———————— CORRECT SOLUTION ————————

First, an intuition, although this uses a concept that the book doesn't introduce (big O).

Note that $a_n > 0$ implies $l > 0$.

The growth rate of the sequence (n) is linear, so for (na_n) to converge, (a_n) must decrease at a rate of *at least* $O(\frac{1}{n})$. For example, if $(a_n) = \frac{2}{n}$ then $lim(na_n) = 2$ and if $(a_n) = \frac{1}{2n}$ then $lim(na_n) = \frac{1}{2}$. They're all $O(\frac{1}{n})$, up to a constant factor, where the constant factor is the limit.

Of course, (a_n) can decrease *faster* than $O(\frac{1}{n})$ e.g. $(a_n) = \frac{1}{n^2}$. But if (a_n) decreases any faster then $O(\frac{1}{n})$, (na_n) will eventually converge to 0, and this is where the assumption $l \neq 0$ comes. It ensures that (a_n) is exactly $O(\frac{1}{n})$, so $\sum(a_n)$ is a constant multiple of the harmonic series, which diverges.

Now the formal solution. (na_n) converges means there is an N such that eventually $|na_n - l| < \epsilon$ for all $n \geq N$. The idea is that we want to use this inequality to get a lower bound for (a_n), and the lower bound should be a *positive* constant multiple < 1 of the harmonic series.

To do this, plug a plausible number to the ϵ. If we put $\epsilon = 1$, then $|na_n - l| < 1 \implies na_n > l - 1 \implies a_n > (l-1)\frac{1}{n}$. This bound is too lax because $(l-1)$ could be negative e.g. the bound doesn't say that (a_n) can't be the

zero sequence. Instead, l itself is a constant, so we can pick $\epsilon = \frac{l}{2}$ to derive $(a_n) > (\frac{l}{2})(\frac{1}{n})$. Since $l > 0$, $\frac{l}{2}$ is positive.

So we've proved that there exists N such that for all $n \geq N$, (a_n) eventually becomes a constant multiple of the harmonic series. We've also satisfied all the conditions for the Comparison Test: $0 \leq (a_n) < (\frac{l}{2})(\frac{1}{n})$ and we know that the RHS diverges, so $\sum(a_n)$ diverges.

(b) This is very similar to part (a) with the following differences:

- We want lower bound instead of upper bound.

- There's no $l \neq 0$ condition, so l can be zero (but still can't be negative, since $a_n > 0$). This means we need a different ϵ than part (a), since $\frac{l}{2}$ could be zero.

Pick $\epsilon = 1$ to get $a_n < (l+1)\frac{1}{n^2}$ then use the Comparison Test. Picking any $\epsilon > 0$ will also work.

8. True or counterexample.

(a) If $\sum a_n$ converges absolutely then $\sum a_n^2$ converges absolutely.

(b) If $\sum a_n$ converges and (b_n) converges then $\sum a_n b_n$ converges.

(c) If $\sum a_n$ converges conditionally, then $\sum n^2 a_n$ diverges.

Solution:

(a) True. INCORRECT PROOF: Let $b_n = |a_n|$. Then by the Algebraic Limit Theorem, $\sum b_n^2 = \sum |a_n^2|$ converges. It's incorrect because the Alegbraic Limit Theorem for series doesn't cover multiplications.

Here's a correct proof from my own attempt. If $\sum a_n^2$ converges absolutely, then there exists N such that for all $m, n \geq N$, $|a_m^2 + ... + a_n^2| < \epsilon$. Now we use the inequality $(x^2 + y^2) \leq (|x| + |y|)^2$ to get $|a_m^2 + ... + a_n^2| \leq (|a_m| + ... + |a_n|)^2 < \epsilon$. Since $\sum a_n$ converges absolutely, then we can make $(|a_m| + .. |a_n|)^2 < \sqrt{\epsilon}$, completing the proof.

Here's a smarter proof from the internet. Since $\sum |a_n|$ converges absolutely then the sequence $|a_n|$ must converge to 0, so there exists N such that for all $n \geq N$, $|a_n| < 1$. For those n, $|a_n^2| < |a_n|$, so we can apply the Comparison Test to conclude that $|a_n^2|$ converges absolutely.

(b) I was trying to find a counterexample, failed, and then tried to prove the statement is true as follows.

If $\sum a_n b_n$ converges then there exists N such that for all $m, n \geq N$, $|a_m b_m + ... + a_n b_n| < \epsilon$. I initially thought we can control b_n to make it less than 1, so that each $a_i b_i \leq a_i$ and apply the inequality $|a_m b_m + ... + a_n b_n| < |a_1 + ... + a_n| < \epsilon$. But this is wrong because $a_i b_i \leq a_i$ only follows if a_i is non-negative.

This suggests that a counterexample can be found by alternating the signs of a_n. Let $a_n = \frac{(-1)^n}{\sqrt{n}}$ and b_n be the same. $\sum a_n$ converges by the Alternating Series Test. (b_n) converges to 0. But $\sum a_n b_n = \frac{1}{n}$ which diverges.

(c) Be careful with the choice of a_n. I initially tried to use $(a_n) = \frac{(-1)^n}{n^3}$, but the problem states that $\sum a_n$ converges *conditionally*, whereas my example converges *absolutely*.

Like in problem 7(a), intuitively this should be true. For if $\sum n^2 a_n$ converges, then a_n must decrease at a rate of at least $O(\frac{1}{n^2})$, but that growth rate converges absolutely.

Now we want to find a bound for a_n. Let's see what we have. $\sum a_n$ converges conditionally gives $\lim(a_n) = 0$, but that's not too useful. $\sum n^2 a_n$ diverges doesn't give any bounds either. Instead we can show the contrapositive. If $\sum n^2 a_n$ converges then $\lim(n^2 a_n) = 0$ so we can bound $|a_n| < \frac{\epsilon}{n^2}$. Note that even if some a_n are negatives, we know that any sequence of the form $\frac{c}{n^2}$ for some constant c converges absolutely, so a_n must converge absolutely.

Remarks:

- From part (b), when given the assumption that a sequence converges and asked to find a counterexample, the a *conditionally* convergent sequence is often useful.

- I took way too long on part (c), even though it's quite similar to problem 7(a). Basically, once we get the intuition that a sequence must have a bounded growth rate, immediately expand all definitions to try to find a more precise bound.

9. **Ratio Test.** Given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, the Ratio Test states that if (a_n) satisfies
$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1.$$
then the series converges absolutely.

(a) Let r' satisfy $r < r' < 1$. Explain why there exists an N such that $n \geq N$ implies $|a_{n+1}| \leq |a_n| r'$.

(b) Why does $|a_N| \sum (r')^n$ converge?

(c) Now, show that $\sum |a_n|$ converges, and conclude that $\sum a_n$ converges.

Solution:

(a) It should be intuitive to see why this should be true. Given, say, $r = \frac{1}{4}$, the condition of the Ratio Test says that eventually, every element is a quarter of its previous element. Now given, say $r' = \frac{1}{2}$, then of course this means every element is smaller than half of its previous, since $\frac{1}{4} < \frac{1}{2}$.

Formally, there exists an N such that $n \geq N$ implies
$$\left| \frac{a_{n+1}}{a_n} - r \right| < \epsilon \implies \left| \frac{a_{n+1}}{a_n} \right| < \epsilon + r$$
$$\implies |a_{n+1}| < (\epsilon + r)|a_n|$$

By definition, for any ϵ there exists such an N, so $(\epsilon + r)$ is effectively just a constant. Let $r' = \epsilon + r$.

UPDATE: an easier way to think of the formal proof is this. Since $r < r'$, simply choose epsilon that's small enough as to stay below r' e.g. $\epsilon = |r - r'|$, then apply the definition of limit.

(b) Note that a_N is a constant; it's the N we've chosen in part (a). This is simply a geometric series with $0 < r' < 1$, so it converges.

(c) Part (a) is basically saying that there is an N such that from N onwards, $|a_N|$ is a *peak* i.e. $|a_n| \leq |a_N|$ for all $n \geq N$. More precisely, we know that for all such n, $|a_{n+1}| < r'|a_n|$ (we can extend it by induction).

$\sum_{k=0}^{N-1} |a_n|$ only forms a finite sum, so it doesn't affect converge. For the tail $n \geq N$, we can use the Comparison Test to compare $\sum |a_n|$ with $|a_N| \sum (r')^n$ as for all $n \geq N$, we have $|a_n| \leq |a_N|(r')^n$. We know from part (b) that the latter series converges, so $\sum |a_n|$ converges and by the Absolute Convergence Test, $\sum a_n$ converges as well.

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10. **Infinite Products.** Review exercise 2.4.10 about infinite products and then answer the following questions:

   (a) Does $\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{9}{8} \cdot \frac{17}{16}...$ converge?

   (b) The infinite product $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{9}{10}...$ certainly converges. (Why?) Does it converge to zero?

   (c) In 1655, John Wallis famously derived the formula
   $$\left(\frac{2 \cdot 2}{1 \cdot 3}\right) \left(\frac{4 \cdot 4}{3 \cdot 5}\right) \left(\frac{6 \cdot 6}{5 \cdot 7}\right) \left(\frac{8 \cdot 8}{7 \cdot 9}\right) ... = \frac{\pi}{2}$$

   Show that the left side of this identity at least converges to something.

Solution: For reference, Exercise 2.4.10 part (b) claims that the product

$$\prod_{n=1}^{\infty}(1 + a_n) = (1 + a_1)(1 + a_2)(1 + a_3)..., \quad \text{where } a_n \geq 0$$

converges if and only if $\sum_{n=1}^{\infty} a_n$ converges.

   (a) The product is $\prod_{n=0}^{\infty} a_n = \prod_{n=0}^{\infty}(\frac{2^n+1}{2^n}) = \prod_{n=0}^{\infty}(1 + \frac{1}{2^n})$. Let $(b_n) = \frac{1}{2^n}$. b_n is positive for all n and $\sum b_n$ converges (it's a geometric series with $r = \frac{1}{2}$). So we can apply Exercise 2.4.10 part (b) to conclude that $\prod a_n$ converges.

   (b) Each term is strictly positive and < 1, so the sequence of partial products is monotonically decreasing and bounded (by 0 and $\frac{1}{2}$). By MCT, the infinite product converges. TODO: don't know yet how to prove it doesn't converge to zero.

   (c) The infinite product is $\prod_{n=1}^{\infty} \frac{(2n)(2n)}{(2n-1)(2n+1)} = \frac{4n^2-1+1}{4n^2-1} =$

$$\prod_{n=1}^{\infty} \frac{(2n)(2n)}{(2n-1)(2n+1)} = \frac{4n^2}{4n^2 - 1}$$
$$= \frac{4n^2 - 1 + 1}{4n^2 - 1}$$
$$= 1 + \frac{1}{4n^2 - 1}$$

   $\frac{1}{4n^2-1}$ is strictly positive for all $n > 0$. By Comparison Test with $\sum \frac{1}{n^2}$, $\sum \frac{1}{4n^2-1}$ converges. Now we can apply Exercise 2.4.10 to conclude that the infinite product converges.

11. Find examples of two series $\sum a_n$ and $\sum b_n$ both of which diverge but for which $\sum min(a_n, b_n)$ converges. To make it more challenging, produce examples where (a_n) and (b_n) are strictly positive and decreasing.

Solution: First thought: from a convergent sequence, modify its odd and even indices to diverge for (a_n) and (b_n) respectively. From here it's easy to produce a concrete example: $(a_n) = 1$ if n odd else $\frac{1}{n^2}$. Similarly for (b_n) but with even indices. Then $\sum min(a_n, b_n) = \sum \frac{1}{n^2}$ which converges.

UPDATE: even simpler example, $(a_n) = 1, 0, 1, 0, ...$ and $(b_n) = 0, 1, 0, 1, ...$.

The challenge is more difficult, and I can't get the odd / even pattern to work.

TODO: do it again. Got help from MSE this time.

Okay, here's the intuition. First, let's see why changing just the odd indices cannot work. We initially have a strictly positive, decreasing convergent series. Since it's convergent, the sequence converges to zero. We want to change the odd indices so that the whole sequence diverges, while still maintaining the decreasing property.

But as we get further and further into the sequence, the maximum value we can put in the odd indices becomes smaller because we need to maintain the decreasing property, until eventually we can only increase them by such small amounts that we can't make the sequence diverge.

Take the convergent series $\sum \frac{1}{n^2}$ as an example. For an odd index n, the value we can change a_n to is bounded by $\frac{1}{(n-1)^2}$ and $\frac{1}{(n+1)^2}$ which gets decreasingly smaller.

This suggests two things: 1) we need to change more than one indices at a time and 2) we'll need to change increasingly longer subarray as the sequence goes on.

And now the answer. We're going to make $min(a_n, b_n) = \frac{1}{n^2}$. First, $a_1 = b_1 = 1$. Next, we're going to keep a_n flat at $\frac{1}{4}$, until it sums up to 1.

$$a_n = 1, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$$
$$b_n = 1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}$$

Next we alternate. Keep b_n flat at $\frac{1}{36}$ until it sums up to 1

$$a_n = 1, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{36}, \frac{1}{49}, ..., \frac{1}{42^2}$$
$$b_n = 1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \frac{1}{36}, \frac{1}{36}, ..., \frac{1}{42^2}$$

Then keep a_n flat at $\frac{1}{42^2}$, and so on.

12. **(Summation-by-parts).** Let (x_n) and (y_n) be sequences, let $s_n = x_1 + x_2 + \dots + x_n$ and set $s_0 = 0$. Use the observation that $x_j = s_j - s_{j-1}$ to verify the formula

$$\sum_{j=m}^{n} x_j y_j = s_n y_{n+1} - s_{m-1} y_m + \sum_{j=m}^{n} s_j (y_j - y_{j+1})$$

Solution:

$$\sum_{j=m}^{n} x_j y_j = \sum_{j=m}^{n} (s_j - s_{j-1}) y_j$$

$$= \sum_{j=m}^{n} s_j y_j - \sum_{j=m}^{n} s_{j-1} y_j$$

$$= (s_m y_m + \dots + s_n y_n) - (s_{m-1} y_m + \dots + s_{n-1} y_n)$$

$$= s_n y_n + s_{m-1} y_m + \sum_{j=m}^{n-1} s_j y_j - \sum_{j=m+1}^{n} s_{j-1} y_j$$

$$= s_n y_n + s_{m-1} y_m + \sum_{j=m}^{n-1} s_j y_j - \sum_{j=m}^{n-1} s_j y_{j+1}$$

$$= s_n y_n + s_{m-1} y_m + \sum_{j=m}^{n-1} s_j (y_j - y_{j+1})$$

This is *almost* the same as the formula in the problem statement, with slightly diffent indexing. To get the exact same formula:

$$s_n y_n + s_{m-1} y_m + \sum_{j=m}^{n-1} s_j (y_j - y_{j+1})$$

$$= s_n y_n + s_{m-1} y_m + \sum_{j=m}^{n-1} s_j (y_j - y_{j+1}) + s_n (y_{n+1} - y_n) - s_n (y_{n+1} - y_n)$$

$$= s_n y_{n+1} - s_{m-1} y_m + \sum_{j=m}^{n} s_j (y_j - y_{j+1})$$

13. **(Abel's Test).** Abel's Test for convergence states that if the series $\sum_{k=1}^{\infty} x_k$ converges, and if (y_k) is a sequence satisfying

$$y_1 \geq y_2 \geq \ldots \geq 0$$

then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

(a) Use Exercise 2.7.12 to show that

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k(y_k - y_{k+1}),$$

where $s_n = x_1 + x_2 + \ldots + x_n$

(b) Use the Comparison Test to argue that $\sum_{k=1}^{\infty} s_k(y_k - y_{k+1})$ converges absolutely, and show how this leads directly to a proof of Abel's Test.

Solution:

(a) Using the formula from Exercise 2.7.12, we get

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} - s_0 y_1 + \sum_{k=1}^{n} s_k(y_k - y_{k+1})$$

and we have $s_0 = 0$ so the term $-s_0 y_1$ disappears.

(b) Since the problem is asking us to use the Comparison Test, intuitively we want to produce an upper bound of $\sum_{k=1}^{\infty} s_k(y_k - y_{k+1})$.

s_k is bounded, by the fact that $\sum_{k=1}^{\infty} x_k$ converges. Let M be an upper bound. Then we have

$$\sum_{k=1}^{\infty} |s_k|(y_k - y_{k+1}) \leq M \sum_{k=1}^{\infty} (y_k - y_{k+1})$$

Note that $(y_k - y_{k+1})$ is always positive, so we don't need absolute signs. Now if we can prove that $\sum_{k=1}^{\infty}(y_k - y_{k+1})$ converges, then by the Algebraic Limit Theorem $M \sum_{k=1}^{\infty}(y_k - y_{k+1})$ converges absolutely and by the Comparison Test $\sum_{k=1}^{\infty} s_k(y_k - y_{k+1})$ converges absolutely as well.

Let's look at the sequence of partial sums: $\sum_{k=1}^{n}(y_k - y_{k+1})$.

$$(y_1 - y_2), (y_1 - y_3), (y_1 - y_4), \ldots$$

This sequence is non-decreasing (because $y_n \geq y_{n+1}$) and bounded above by y_1, so it converges by the Monotone Convergence Theorem.

Finally, we show how this proves Abel's Test. From part (a)

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k(y_k - y_{k+1})$$

We want to prove that the sequence of this partial sum converges as $n \to \infty$.

Because $\sum x_k$ converges, the sequence (s_n) converges, and since (y_n) converges, it follows that $(s_n y_{n+1})$ converges. We've proved that $\sum_{k=1}^{\infty} s_k(y_k - y_{k+1})$ converges absolutely.

Therefore, the right-hand side converges as $n \to \infty$, hence the sequence of partial sums $(\sum_{k=1}^{n} x_k y_k)$ converges.

14. **(Dirichlet's Test).** Dirichlet's Test for convergence states that if the partial sums of $\sum_{k=1}^{\infty} x_k$ are bounded (but not necessarily convergent), and if (y_k) is a sequence satisfying $y_1 \geq y_2 \geq y_3 \geq ... \geq 0$ with $\lim y_k = 0$, then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

  (a) Point out how the hypothesis of Dirichlet's Test differs from that of Abel's Test in Exercise 2.7.13, but show that essentially the same strategy can be used to provide a proof.

  (b) Show how the Alternating Series Test (Theorem 2.7.7) can be derived as a special case of Dirichlet's Test.

Solution:

  (a) The difference is that the partial sums $\sum_{k=1}^{n} x_k$ are bounded but $\sum_{k=1}^{\infty}$ may not be convergent. We now also have (y_k) converges to 0, unlike Abel's Test where we only know that (y_k) converges (without actually knowing the limit).

  Essentially the same strategy can be used to prove Dirichlet's Test. First, the summation-by-parts formula from the previous problem part (a)

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k(y_k - y_{k+1}),$$

  is still valid.

  Next we use the same strategy as part (b) to prove that $\sum_{k=1}^{\infty} s_k(y_k - y_{k+1})$ converges absolutely. Note that the properties we needed in the previous problem to prove this fact were:

- s_k is bounded. Which is still true in this problem by the fact that $\sum_{k=1}^{n} x_k$ is bounded.

- The sequence of partial sums $\sum_{k=1}^{n}(y_k - y_{k+1})$ converges. Which is still true in this problem because (y_n) is still non-increasing and bounded.

So the fact that $\sum_{k=1}^{\infty} s_k(y_k - y_{k+1})$ converges absolutely still holds.

The last part is the only one with a little difference. We need to show that the sequence of partial sums

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k(y_k - y_{k+1})$$

converges as $n \to \infty$.

Unlike Abel's Test, we can't argue that (s_n) converges, because we no longer have the assumption that $\sum_{k=1}^{\infty} x_k$ converges. But we still know that (s_n) is bounded. We also have $(y_n) \to 0$, which implies $(s_n y_{n+1}) \to 0$. The rest is the same as Abel's Test.

(b) We restate the Alternating Series Test here for convenience. Let (a_n) be a sequence satisfying

- $a_1 \geq a_2 \geq a_3 \geq ... \geq a_n \geq a_{n+1} \geq ...$ and

- $(a_n) \to 0$

Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converges.

To get the alternating series from Dirichlet's Test, let $(y_n) = (a_n)$ and $(x_n) = (-1)^{n+1}$.

(y_n) still satisfies the condition for Dirichlet's Test. Note that the conditions of the Alternating Series Test implies $a_n \geq 0$ (which is required by Dirichlet). The partial sums of $x_n = (-1)^{n+1}$

$$s_1 = 1, s_2 = 0, s_3 = 1, s_4 = 0, ...$$

so $0 \leq s_n \leq 1$ i.e. (s_n) is bounded, as required by Dirichlet's test.