Understanding Analysis, 2nd Edition: Stephen Abbott
Chapter 2 - Section 2.4

1. (a) Prove that the sequence define by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

converges.

(b) Now that we know $\lim_{x_n}$ exists, explain why $\lim_{x_{n+1}}$ must also exist and equal the same value.

(c) Take the limit of each side of the recursive equation in part (a) to explicitly compute $\lim_{x_n}$.

Solution:

(a) The sequence is decreasing (TODO: prove this formally) and bounded by 3 and 0, so by the MCT it converges.

(b) The limit is the same because shifting a sequence doesn't change its limit. More formally, by using the formal definition of convergence, any N that works for (x_n) works for (x_{n+1}) as well by setting $N_2 = N - 1$.

(c) Here, (x_n) and (x_{n+1}) are sequences, not variables. We have

$$\lim(x_{n+1}) = \lim(\frac{1}{4 - (x_n)})$$

By the algebraic limit theorem

$$\lim(x_{n+1}) = \frac{1}{4 - \lim(x_n)}$$

Let $\lim(x_n) = L$. By part (a) and (b)

$$L = \frac{1}{4 - L}$$

Solve the quadratic to get $L = 2 - \sqrt{3} \vee L = 2 + \sqrt{3}$. Because $x_1 = 3$, the sequence is decreasing and $L = 2 - sqrt3$.

2. (a) No, because the sequence doesn't converge. (b) Yes.

3. (a) Show that

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}$$

converges and find the limit.

(b) Does the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, ...$$

converge? If so, find the limit.

Solution:

(a) The sequence is $x_n = \sqrt{2 + x_{n-1}}$ with $x_1 = \sqrt{2}$. It's upper bounded by 2 as $x < 2 \implies \sqrt{2+x} < (\sqrt{4} = 2)$.

To prove the sequence is increasing, either get the quadratic $x < \sqrt{2+x} \implies x^2 - x - 2 < 0$ and note that the starting value x_1 is at the increasing side of the parabola.

Or use induction. The claim is true for x_1 and x_2. Now suppose it's true up to x_{n-1}. Now we need to prove $x_{n+1} > x_n$. Write it as $\sqrt{2 + x_{n-1}} > \sqrt{2 + x_{n-2}}$. By the induction hypothesis, we know $x_{n-1} > x_{n-2}$, so $\sqrt{2 + x_{n-1}} > \sqrt{2 + x_{n-2}}$ is true.

We've proven that the sequence is increasing and is bounded by $\sqrt{2}$ and 2, so by MCT the limit exist. Find it just like problem 1. Let $\lim(x_n) = L$, then $\lim(x_n) = \lim\sqrt{2 + (x_{n-1})} \implies L = \sqrt{2 + L}$ so $L = 2$ or $L = -1$. As the sequence is increasing and starts from $\sqrt{2}$, $L = 2$.

Note that the algebraic limit for sqrt has been proved at problem 2.3.1

(b) This is very similar to part (a). Prove the sequence is increasing, bounded by 2 (because $x < 2 \implies \sqrt{2x} < 2$), and the limit is also 2.

4. Skip.

5. **Calculating Square Roots**. Let $x_1 = 2$ and define

$$x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n})$$

.

(a) Show that x_n^2 is ≥ 2, and then use this to prove that $x_n - x_{n+1} \geq 0$. Conclude that $\lim x_n = \sqrt{2}$.

(b) Modify the sequence (x_n) so that it converges to $\sqrt{c}$.

Solution:

(a) The claim is true for x_1. Now by induction, if $x_n^2 \geq 2$, then the minimum of $x_{n+1}^2 = \frac{1}{4}(x_n^2 + 4 + \frac{4}{{x_n}^2})$ is attained by substituting $x_n^2 = 2 \implies \frac{1}{4}(2+4+2) = \frac{8}{4} = 2$.

To prove the sequence is decreasing, we use induction again. $x_1 \geq x_2 = \frac{3}{2}$. Now we need to prove $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n}) \leq x_n$. Do algebra to get $x_n^2 - 2 < 0 \implies x_n^2 \geq 2$, which by part (a) is true.

To confirm the limit, just do the same thing as problem 3.

(b) This is known as Heron's method. The modified sequence is $x_{n+1} = \frac{1}{2}(x_n + \frac{c}{x_n})$ with $x_1 = c$. The convergence proof follows part (a) very closely.

First we show by induction that $x_n^2 \geq c$ for all n. The claim is true for x_1. Now by induction, if $x_n^2 \geq c$, then the minimum of $x_{n+1}^2 = \frac{1}{4}(x_n^2 + 2c + \frac{c^2}{{x_n}^2})$ is attained by substituting $x_n^2 = c \implies \frac{1}{4}(c + 2c + c) = \frac{4c}{4} = c$.

Important: The solution manual seems to sweep the monotonicity proof under the rug by just saying it's similar to part (a). But it's not. This new sequence is **not** strictly decreasing. For example, plug $c = 0.1$, then the first two terms are increasing.

We claim that the sequence is eventually non-increasing starting from x_1. The trick is to use the AM-GM inequality.

First, note that x_n is simply the arithmetic mean of x_{n-1} and $\frac{c}{x_{n-1}}$. The geometric mean of these two values is $\sqrt{c}$, so by AM-GM we have $x_n \leq \sqrt{c}$. If we apply this to $n = 1$, then we know x_1 is always $\geq \sqrt{c}$, regardless of whether x_0 starts at a number larger than $\sqrt{c}$ (as in $c = 2$) or less (as in $c = 0.1$).

Now we can use induction like part(a) to prove that the sequence in non-increasing from x_1 onwards. Do algebra to get $x_n - x_{n+1} = x_n^2 - c$. We've proved that for $n \geq 1$, $x_n \geq \sqrt{c} \implies x_n^2 \geq c$, so $x_n - x_{n+1} = x_n^2 - c \geq 0$, so the sequence is non-increasing for $n \geq 1$.

6. TODO

7. TODO

8. TODO

9. TODO

10. **(Infinite Products.)** A close relative of the infinite series is the *infinite product*

$$\prod_{n=1}^{\infty} b_n = b_1 b_2 b_3 ...$$

which is understood in terms of its sequence of *partial products*

$$p_m = \prod_{n=1}^{m} b_m = b_1 b_2 b_3 ... b_m$$

Consider the special class of infinite products of the formal

$$\prod_{n=1}^{\infty} (1 + a_n) = (1 + a_1)(1 + a_2)(1 + a_3)..., \quad \text{where } a_n \geq 0.$$

(a) Find an explicit formula for the sequence of partial products in the case where $a_n = 1/n$ and decide whether the sequence converges. Write out the first few terms in the sequence of partial products in the case where $a_n 1/n^2$ and make a conjecture about the convergence of this sequence.

(b) Show, in general, that the sequence of partial product converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. (The inequality $1 + x \leq 3^x$ for positive x will be useful in one direction.)

Solution:

(a) For the case $a_n = 1/n$, we have the first few terms in the sequence of partial products:

$$\begin{aligned} p_1 &= 2 \\ p_2 = 2 \cdot \frac{3}{2} &= 3 \\ p_3 = 2 \cdot \frac{3}{2} \cdot \frac{4}{3} &= 4 \\ p_4 = 2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} &= 5 \end{aligned}$$

It's a telescoping product with $p_n = n + 1$, so the sequence of partial products diverges, hence the infinite product diverges.

For the case $a_n = 1/n^2$, plot the first hundred terms in Desmos and it looks like it converges?

(b) ———— Necessity ————

First we prove the necessity. If $\prod_{n=1}^{\infty}(1+a_n) = (1+a_1)(1+a_2)(1+a_3)...$ where $a_n \geq 0$ converges, then $\sum a_n$ converges.

Let's take a look at, say, p_3. It's $(1+a_1)(1+a_2)(1+a_3) = 1+(a_1+a_2+a_3)+(a_1a_2+a_1a_3+a_1a_2a_3)$.

In general, the m'th partial product p_m can be written as

$$(1+a_1)...(1+a_m) = 1 + (a_1 + ... + a_m) + c_m$$

where c_m is the sum of all products of two or more distinct a_k's. Given the hypothesis that $a_n \geq 0$ for all n, $c_m \geq 0$.

Given that the infinite product converges, then the sequence $(p_n) = 1 + (a_1 + ... + a_n) + c_n$ is bounded. It's also monotone non-decreasing, since all the terms are non-negative so

$$p_n \leq p_{n+1} = p_n \cdot (1 + a_{n+1})$$

and we also have

$$(a_1 + ... + a_n) \leq p_n$$

So boundedness of (p_n) implies boundedness of $\sum a_n$.

This implies $(a_1 + .. + a_n)$ i.e. the sequence of partial sums is also monotone and bounded, hence it converges.

——————— Sufficiency ———————

For the sufficiency direction, we want to prove that if $\sum a_n$ converges then $\prod_{i=1}^{\infty}(1+a_n)$ converges.

Because $a_n \geq 0$ for all n, $(1+a_n) \geq 1$ and the sequence of partial products $(p_m) = \prod_{n=1}^{m}(1+a_n)$ is monotone non-decreasing. If we can prove that it's bounded, then we can use to Monotone Convergence Theorem to conclude that it converges.

We use the inequality provided by the problem hint to get $\prod_{n=1}^{m}(1+a_n) \leq \prod_{n=1}^{m} 3^{a_n} = 3^{\sum_{n=1}^{m} a_n}$. By the hypothesis that $\sum a_n$ converges (hence bounded), $3^{\sum a_n}$ is bounded, so $\prod_{n=1}^{m}(1+a_n)$ is bounded as well.