Understanding Analysis, 2nd Edition: Stephen Abbott
Chapter 2 - Section 2.6

1. Prove that every convergent sequence is a Cauchy sequence.

Solution:
Intuition: Since the sequence is convergent, let the limit be L. For any ϵ_a there exists an N_a that works.

Now for the Cauchy sequence, we need to make all elements eventually within ϵ_b. Choose $\epsilon_a \leq \epsilon b/2$. It should be easy to visualize that for all $a_n, n \geq N_a$, the largest distance an element can be from each other is one below and one above the limit L, both of which is below $\epsilon_a \leq \epsilon_b/2$.

More formally, for all $n, m \geq N_a$, we have:
$|a_n - a_m| \leq |a_n - L| + |a_m - L| < \epsilon_a + \epsilon_a \leq 2 * \frac{\epsilon_b}{2} = \epsilon_b$.

2. Give an example or argue that it's impossible.
   (a) A Cauchy sequence that is not monotone.
   (b) A Cauchy sequence with an unbounded subsequence.
   (c) A divergent monotone sequence with a Cauchy subsequence.
   (d) An unbounded sequence containing a subsequence that is Cauchy.

Solution: For all of these, we'll use the fact that over the reals, a Cauchy sequence is equivalent to a convergent sequence.

(a) {1, -1, 1/2, -1/2, 1/3, -1/3, ...} is not monotone and converges to 0.

(b) Impossible. Every convergent sequence is bounded and a subsequence of a bounded sequence is also bounded.

(c) Impossible. A divergent monotone sequence must be unbounded (because by MCT, a bounded monotone sequence must be convergent). Subsequence of an unbounded sequence must also be unbounded, but a Cauchy sequence must be bounded.

(d) $\{1, 1, 2, 1/2, 3, 1/3, 4, 1/4\}$. The subsequence $\{1, 1/2, 1/3, 1/4, ...\}$ converges to 0.

3. Give a direct argument that does not use the Cauchy criterion or the Algebraic Limit Theorem. If (x_n) and (y_n) are Cauchy

   (a) $(x_n + y_n)$ is Cauchy.

   (b) Do the same for $(x_n y_n)$.

Solution: Both proofs are reminiscent of the ones for the Algebraic Limit Theorem.

(a) We want to prove that given any $\epsilon > 0$, there is an N such that for all $n, m \geq N$, $|(a_n + b_n) - (a_m + b_m)| < \epsilon$.

Rearrange to $|(a_n - a_m) + (b_n - b_m)| \leq |a_n - a_m| + |b_n - b_m|$. Since (a_n) and (b_n) are Cauchy, there exist N_a, N_b such that both terms are less than $\frac{\epsilon}{2}$. Pick $N = max(N_1, N_2)$ then $|a_n - a_m| + |b_n - b_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

(b)

$$\begin{aligned}
|a_n b_n - a_m b_m| &= |a_n b_n - a_m b_n + a_m b_n - a_m b_m| \\
&\leq |(a_n - a_m) b_n| + |(b_n - b_m) a_m| \\
&= |a_n - a_m||b_n| + |b_n - b_m||a_m|
\end{aligned}$$

Now we use the fact that every Cauchy sequence is bounded, so there is an L_a such that $|a_n| \leq |L_a|$ for all n. Similarly for b_n and L_b. Let $L = max(L_a, L_b)$, then

$$|a_n - a_m||b_n| + |b_n - b_m||a_m| \leq |a_n - a_m||L| + |b_n - b_m||L|$$

There exist N_a and N_b such that both $|a_n - a_m|$ and $|b_n - b_m|$ are both less than $\frac{\epsilon}{2|L|}$. Pick $N = max(N_a, N_b)$, then

$$\begin{aligned}
|a_n - a_m||L| + |b_n - b_m||L| &< \frac{\epsilon}{2|L|}|L| + \frac{\epsilon}{2|L|}|L| \\
&= \epsilon
\end{aligned}$$

One more special case to handle: if $|L| = 0$ then both (a_n) and (b_n) are just the zero sequence.

4. Let a_n and b_n be Cauchy sequences. Decide whether each of the following sequences is a Cauchy sequence, justifying each conclusion.

   (a) $c_n = |a_n - b_n|$

   (b) $c_n = (-1)^n a_n$

   (c) $c_n = \lfloor a_n \rfloor$

Solution:

(a) Yes. We first prove that $|a_n|$ is Cauchy for any Cauchy sequence $|a_n|$. Use the fact that $|a-b| \geq ||a|-|b||$ for any real numbers a and b (the distance between two numbers is at least as great as the distance between their absolute values).

To prove $|a_n|$ is Cauchy we need to prove that for any $\epsilon > 0$, there exists N such that for all $n, m \geq N$, $||a_n| - |a_m|| < \epsilon$. Now use the inequality: $||a_n| - |a_m|| \leq |a_n - a_m|$ and the rest follows from the fact that a_n is Cauchy.

Back to the original problem: $a_n - b_n$ is Cauchy by the Algebraic Limit Theorem so $|a_n - b_n|$ is Cauchy as well.

(b) No. $\{1, 1, 1, ...\}$ is Cauchy but $\{1, -1, 1, -1, ...\}$ is not.

(c) No. We'll use the Cauchy criterion and talk about convergence instead.

- Counterexample 1: $\{1, 0.9, 1, 0.99, 1, 0.999, ...\}$ converges to 1 but its floor $\{1, 0, 1, 0, ...\}$ diverges.
- Counterexample 2: This one's more formulaic. Let $a_{2k-1} = 1$ and $a_{2k} = 1 - \frac{1}{k}$ for $k = (1, 2, ...)$. The sequence looks like $(1, 1-\frac{1}{1}, 1, 1-\frac{1}{2}, 1, 1-\frac{1}{3}, ...)$. The sequence converges to 1 but its floor $(1, 0, 1, 0, ...)$ diverges.

5. Consider the following (invented) definition: A sequence (s_n) is *pseudo-Cauchy* if, for all $\epsilon > 0$, there exist an N such that if $n \geq N$, then $|s_{n+1} - s_n| < \epsilon$. Decide which one of the following two propositions is actually true. Supply a proof or a counterexample.

   (a) Pseudo-Cauchy sequences are bounded.

   (b) If (x_n) and (y_n) are pseudo-Cauchy then $(x_n + y_n)$ is pseudo-Cauchy as well.

Solution:

(a) Not necessarily. One counterexample is $(s_n) = \sqrt{n}$.

(b) Yes. $|(x_n + y_n) - (x_{n+1} + y_{n+1})| = |(x_n - x_{n+1}) + (y_n - y_{n+1}| \leq |x_n - x_{n+1}| + |y_n - y_{n+1}|$. Make each term less than $\epsilon/2$.