Understanding Analysis, 2nd Edition: Stephen Abbott
Chapter 1 - Section 1.6 - Cantor's Theorem

1. Show that $(0, 1)$ is uncountable if and only if R is uncountable.

Solution: First we prove the necessity: if $(0, 1)$ is uncountable then R is uncountable. This follows immediately from the fact that $(0, 1)$ is a subset of R. A superset of an uncountable set must be uncountable as well.

Now the sufficiency: if R is uncountable then $(0, 1)$ is uncountable. I initially wanted to prove this by contrapositive: if $(0, 1)$ is countable then R is countable. But this is tricky. The fact that $(0, 1)$ is a subset of R does not imply R is countable. Then I tried using the fact that the union of (infinitely many) countable sets is countable, by merging infinitely many intervals. But this requires proving that the intervals other than $(0, 1)$ are also countable.

The idea hit me to produce a bijection between $(0, 1)$ and R instead, and indeed such a bijection is possible by Exercise 1.5.4. For example, the tangent function maps $(\frac{-\pi}{2}, \frac{\pi}{2})$ to R. We need just need to map $(0, 1)$ to $(\frac{-\pi}{2}, \frac{\pi}{2})$ and compose it with tan. Find the line equation joining $(0, 1)$ to $(\frac{-\pi}{2}, \frac{\pi}{2})$ and we get $\pi x - \frac{\pi}{2}$, therefore $f(x) = tan(\pi x - \frac{\pi}{2})$ is a bijection between $(0, 1)$ and R. This proves both necessity and sufficiency at once.

2. (a) Explain why the real number $x = .b_1b_2b_3...$ cannot be $f(1)$.

   (b) Now explain why $x \neq f(2)$, and in general why $x \neq f(n)$ for any $n \in N$.

   (c) Point out the contradiction that arises from these observations and conclude that $(0, 1)$ is uncountable.

> **Solution:**
>
> (a) Because b_1 is already different from a_{11}.
>
> (b) In general, $b_n \neq a_{nn}$, so the decimal expansion differs in at least one digit.
>
> (c) If $(0, 1)$ were countable, then $x = f(n)$ for an n, since x is defined to be a real number $\in (0, 1)$. But from part (b), $x \neq f(n)$ for all n, a contradiction. Therefore $(0, 1)$ must be uncountable.

3. Supply rebuttals to the following complaints about the proof of Theorem 1.6.1 (the open interval $(0, 1)$ is uncountable).

   (a) Every rational number has a decimal expansion, so we could apply the same argument to show that the set of rational numbers between 0 and 1 is uncountable. However, because we know that every subset of Q must be uncountable, the proof of Theorem 1.6.1 must be flawed.

   (b) Some numbers have *two* different decimal representations. Specifically, any decimal expansion that terminates can also be written with repeating 9's. For instance, $1/2$ can be written as 0.5 or as 0.49999.... Doesn't this cause some problems?

> **Solution:**
>
> (a) Every rational number has a decimal expansion, but the converse is not true; not every decimal expansion represents a rational number. In the proof of Theorem 1.6.1, x may be irrational, so it can't be used to conclude that Q is uncountable.
>
> (b) First, it's worth noting that the *only way* a number can have two different decimal expansion is the one the problem statement says; one expansion that terminates, and another with repeating 9's. I claim this without proof, as the book also intentionally uses decimal representations without formal definitions.

The proof of Theorem 1.6.1 works fine because it uses only the digits 2 and 3, so the counterexample it constructs is a number with a unique decimal expansion. It cannot be the case that x is just another representation of the decimal expansion of a real number $f(n)$.

4. Let S be the set consisting of all sequences of 0's and 1's. Observe that S is not a particular sequence, but rather a large set whose elements are sequences, namely,

$$S = \{(a_1, a_2, a_3, ...) : a_n = 0 \text{ or } 1\}.$$

As an example, the sequence $(1, 0, 1, 0, 1, 0, ...)$ is an element of S, as is the sequence $(1, 1, 1, 1, 1, ...)$. Give a rigorous argument showing that S is uncountable.

Solution: The same diagonal argument works. We just replace the decimal digits 0-9 with 0 and 1. Consider the following table like in the book

$$\begin{array}{c|l}
\mathbb{N} & (0,1) \\
\hline
1 & f(1) = \ .\, \mathbf{a_{11}}\, a_{12}\, a_{13}\, a_{14}\, a_{15}\, a_{16}\, \cdots \\
2 & f(2) = \ .\, a_{21}\, \mathbf{a_{22}}\, a_{23}\, a_{24}\, a_{25}\, a_{26}\, \cdots \\
3 & f(3) = \ .\, a_{31}\, a_{32}\, \mathbf{a_{33}}\, a_{34}\, a_{35}\, a_{36}\, \cdots \\
4 & f(4) = \ .\, a_{41}\, a_{42}\, a_{43}\, \mathbf{a_{44}}\, a_{45}\, a_{46}\, \cdots \\
5 & f(5) = \ .\, a_{51}\, a_{52}\, a_{53}\, a_{54}\, \mathbf{a_{55}}\, a_{56}\, \cdots \\
6 & f(6) = \ .\, a_{61}\, a_{62}\, a_{63}\, a_{64}\, a_{65}\, \mathbf{a_{66}}\, \cdots \\
\vdots & \vdots
\end{array}$$

where $a_{mn} \in \{0, 1\}$. Construct a sequence b from the diagonal such that $b_n = 1$ if $a_{nn} = 0$ else $b_n = 0$. Exactly the same proof as Exercise 2 works here.

5. (a) Let $A = \{a, b, c\}$. List the eight elements of $P(A)$. (Do not forget that $\emptyset$ is considered a subset of every set).
   (b) If A is finite with n elements, show that $P(A)$ has 2^n elements.

Solution:

(a) $\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$.

(b) For a particular subset, an element is either chosen or not chosen. So there are two choices for each of the n elements, hence 2^n.