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MIPS: An Efficient Global Parametrization Method

Kai Hormann and Günther Greiner

Abstract. The problem of parametrizing 3D data points is fundamental for many applications in computer-aided geometric design, e.g. surface fitting, texture mapping, and remeshing. We present a new method for constructing a global parametrization of a triangulated (topologically disk-like) surface over a planar region with minimal distortion. In contrast to many existing approaches which need the boundary of the parametrization to be fixed in advance, the boundary develops naturally with this new algorithm.

§1. Introduction

In general, a triangulated set of data points $P_i \in \mathbb{R}^3$ with triangles $T_j = \Delta(P_{j_0}, P_{j_1}, P_{j_2})$ and a 2-manifold domain $\Omega \subset \mathbb{R}^3$, over which the points are to be parametrized, are given. In most cases the domain is either planar ($\Omega \subset \mathbb{R}^2 \subset \mathbb{R}^3$) or a polygonal mesh with planar facets. The task is now to find parameter values $p_i \in \Omega$, one for each data point P_i, such that the topology of the point set is preserved, i.e., the triangles in the parameter domain $t_j = \Delta(p_{j_0}, p_{j_1}, p_{j_2})$ must not overlap.

After determining the parameter values, the interpolation problem can be written as follows: *find a function* $F : \Omega \to \mathbb{R}^3$ *with* $F(p_i) = P_i$ [5,8,12]. The simplest solution to this problem is the piecewise linear function that linearly maps each parameter triangle t_j to the corresponding surface triangle T_j (i.e., $F(t_j) = T_j$). This function is typically used in the case of texture mapping, where color information is defined in the parameter domain and mapped onto the 3D object to make it look more realistic [1,13]. The function F can also be used for remeshing the triangulated data points in order to get a mesh with regular connectivity, so that multiresolution analysis and subdivision techniques can be applied [3,10].

We only address the case of triangulated point sets that are topologically disk-like (i.e., having a boundary and no holes) and thus can be parametrized

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over a simply connected planar domain $\Omega \subset \mathbb{R}^2$. In Section 2 a summary of the previous work dedicated to this problem is given and the limitations of the existing methods are outlined. Our method to overcome these limitations will be explained in detail in Section 3. In Section 4 we show the advantages of the presented approach, giving some examples of surface approximation and remeshing with the new parametrizations. The paper concludes with a discussion of the drawbacks of the proposed technique, and suggestions for future investigations.

§2. Previous Work

While it is quite clear how to solve the *local* problem, i.e., parametrizing a set of points surrounding a reference point R, which can be done e.g. by an exponential mapping or by projection into an adequate tangent plane at R, the *global* problem is more complicated and has been addressed in several earlier papers.

Bennis et al. [1] propose a method based on differential geometry: they map isoparametric curves of the surface onto curves in the parameter domain such that the geodesic curvature at each point is preserved. The parametrization is then extended to both sides of that initial curve until some distortion threshold is reached. But this method as well as the one presented in [13] by Maillot et al. require the surface to be split into several independent regions, and therefore cannot be seen as a solution to the global problem.

Ma and Kruth [12] project the data points P_i onto a parametric base surface $S : \Omega \to \mathbb{R}^3$, and the parameter values of the projected points are taken as p_i. The approaches in [3,5,8,14] have the following strategy in common:

1) find a parametrization for the *boundary points*,
2) minimize an edge-based energy function

$$E = \frac{1}{2} \sum_{\{i,j\} \in \text{Edges}} c_{ij} \, \|p_i - p_j\|^2 \tag{1}$$

to determine the parametrization for the *inner points*.

The edge coefficients c_{ij} can be chosen in different ways. While Floater chooses them so that the geometric shape of the surface is preserved [5], Greiner and Hormann set $c_{ij} = \frac{1}{\|P_i - P_j\|^r}$ for some $r \geq 0$, as they want to minimize the energy of a network of springs [8]. Both methods are generalizations of well-known results for the parametrization of curves [4,6,11]. Furthermore, Taubin used the energy function (1) for smoothing polyhedral surfaces [14] and found $r = 1$ to produce good results.

A different method is introduced by Pinkall and Polthier in [14], and by Eck et al. in [3], where the *Dirichlet energy* of the piecewise linear function F^{-1} that maps the surface triangles T_j to the corresponding parameter triangles t_j is considered. It can be expressed as in (1) with $c_{ij} = \frac{1}{4}(\cot\alpha + \cot\beta)$, where α and β are the angles opposite to the edge $\overline{P_i P_j}$ in the two adjacent surface triangles.

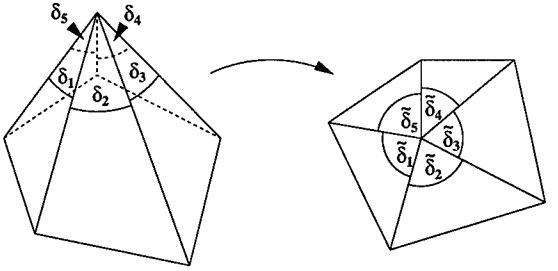


Fig. 1. A pyramid cannot be parametrized without distortions.

In all cases, minimizing (1) is equivalent to solving a non-singular sparse positive definite matrix system, that is (apart from Floater's method) even symmetric. Though this is a comparatively fast way to find a parametrization, it suffers from the fact that it is not clear how to choose the initial parametrization of the boundary points. Floater maps them to the boundary of the unit square using chord length parametrization, Greiner and Hormann project them into the plane that fits all boundary points best in the least square sense, and Eck et al. use parameter values lying on a circle. Note the importance of choosing a convex configuration for the boundary points, since triangle flipping may occur otherwise. Triangle flipping can also be caused by negative weights c_{ij}, which may happen with the method of [3,14] at sharp peaks.

These techniques seem to be rather arbitrary and do not take the geometry of the boundary points into account. In the next section we will introduce a parametrization method that yields parameter values not only for the inner points, but also for the boundary points. Since this method also generates parametrizations that are "as isometric as possible" (i.e., having minimal distortion), we will call them: **M**ost **I**sometric **P**arametrization**S** (MIPS).

§3. MIPS—Most Isometric Parametrizations

Let us briefly review the situation: we are given a set of triangulated data points $P_i \in \mathbb{R}^3$ with a boundary and no holes, and want to find a parametrization, i.e., a set of parameter values $p_i \in \mathbb{R}^2$ so that the topology is preserved. In order to define the quality of a parametrization, we consider the piecewise linear interpolation function $f : \mathbb{R}^3 \to \mathbb{R}^2$ that maps the data points to the corresponding parameter values, i.e., $f(P_i) = p_i$.

As the triangulated surface may be geometrically complex, this function will inevitably cause some deformation to the shape of the triangles. Consider e.g. the configuration in Fig. 1, which can only be parametrized without any deformations if the angles δ_i add up to 2π. In general, only for developable surfaces (e.g. planes, cylindrical and conical surfaces) an isometric parametrization without any distortion can be found. To keep the distortion as small as possible, we must somehow measure this deformation so that the best parametrization can be found in a minimization process.

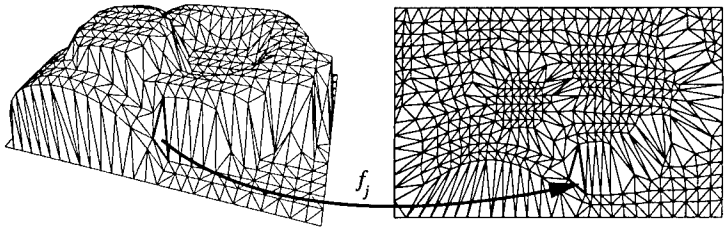


Fig. 2. An atomic linear map between surface and parameter triangle.

Clearly, f can be decomposed into *atomic linear maps* f_j (see Fig. 2) that map a surface triangle $T_j = \Delta(P_{j_0}, P_{j_1}, P_{j_2})$ to the corresponding parameter triangle $t_j = \Delta(p_{j_0}, p_{j_1}, p_{j_2})$. Thus it is sufficient to measure the distortion of linear maps: if E were such a deformation functional, the best parametrization could simply be found by minimizing $\sum_j E(f_j)$.

While Pinkall and Polthier in [14] and Eck et al. in [3] consider the Dirichlet energy $E_D(f) = \frac{1}{2}\int \|\nabla f\|^2$ as a measure of deformation, Maillot et al. propose the Green-Lagrange deformation tensor $\|I_f - \mathrm{Id}\|^2$ that describes the distance of the first fundamental form of f, $I_f = \nabla f^t \cdot \nabla f$, to the identity matrix in some 2×2-matrix norm [13].

An energy functional that measures the deformation of a linear function should have the following properties: it should be

1) unaffected by translations,
2) unaffected by orthogonal transformations,
3) unaffected by scalings,

since the shape of triangles is not changed by these operations. Furthermore, it is desirable to avoid degeneracies, so we need

4) a functional that punishes collapsing triangles very badly.

Notice that the Dirichlet energy meets the first and second condition but favors small parameter triangles, contradicting the other two conditions. Indeed, if the parameter values of the boundary points are not fixed, the minimum of that functional is the singular parametrization where all parameter values p_i collapse to one point. The Green-Lagrange deformation tensor also fails to meet the third and fourth condition and the second one is only fulfilled if the chosen matrix norm is invariant to orthogonal transformations.

Now, let $g(x) = Ax + b$ be an atomic linear map that maps a surface triangle T to a parameter triangle t. Note that by introducing a local coordinate system at T, this function can be seen as a mapping from $\mathbb{R}^2 \to \mathbb{R}^2$.

Because of the first condition, the constant part b of the function g should not be taken into account by the desired deformation functional. Remembering the singular value decomposition of a matrix $U^t A V = \Sigma = \begin{pmatrix} \sigma_1 & \\ & \sigma_2 \end{pmatrix}$, where $\sigma_1 \geq \sigma_2$ are the singular values of A and U and V are orthogonal matrices, the functional should further depend only on σ_1 and σ_2, thus fulfilling the second condition. As the singular values are the lengths of the semi-axes of

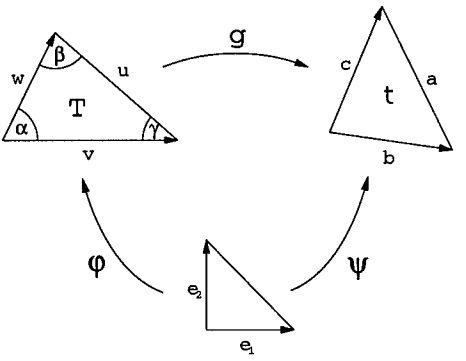


Fig. 3. Decomposition of a linear map g.

the ellipse $\{Ax : \|x\|_2 = 1\}$, the ratio σ_1/σ_2 seems to be a good measure of the deformation of g that also fulfills the third and fourth condition by punishing vanishing triangles with ∞. From Linear Algebra it is known [7] that this ratio is the 2-norm condition number of the matrix:

$$\kappa_2(A) = \|A\|_2\|A^{-1}\|_2 = \frac{\sigma_1}{\sigma_2}.$$

Since the 2-norm condition number of even a 2×2-matrix is rather costly in numerical computations, we decided to use the condition number based on the *Frobenius Norm* $\|\cdot\|_F$ instead, which still meets the four conditions and is much easier to handle:

$$\begin{aligned}\kappa_F(A) = \|A\|_F\|A^{-1}\|_F &= \sqrt{\sigma_1^2+\sigma_2^2}\sqrt{(\tfrac{1}{\sigma_1})^2+(\tfrac{1}{\sigma_2})^2} \\ &= \frac{\sigma_1^2+\sigma_2^2}{\sigma_1\sigma_2} = \frac{\sigma_1}{\sigma_2}+\frac{\sigma_2}{\sigma_1} = \kappa_2(A)+\frac{1}{\kappa_2(A)} \qquad (2) \\ &= \frac{\operatorname{trace}(A^tA)}{\det A}. \qquad (3)\end{aligned}$$

From (2) we can see how close κ_F and κ_2 are related and that it is no major difference whether we minimize the one or the other in order to get linear mappings with low distortion. We will now use (3) to get a representation of κ_F that is suitable for numerical computations.

If we decompose the linear function g according to Fig. 3, where $\{e_1, e_2\}$ is the canonical basis in $\mathbb{R}^2$, we have $g = \psi \circ \varphi^{-1}$. Further, we have $A = \partial g = \partial\psi\partial\varphi^{-1}$ and a little calculation (see [14]) yields

$$\begin{aligned}\kappa_F(g) := \kappa_F(A) = \frac{\operatorname{trace}(A^tA)}{\det A} &= \frac{\operatorname{trace}(\partial g^t\partial g)}{\det\partial\psi\det\partial\varphi^{-1}} = \frac{4E_D(g)}{\det\partial\psi} \qquad (4) \\ &= \frac{\cot\alpha|a|^2+\cot\beta|b|^2+\cot\gamma|c|^2}{\det\partial\psi}.\end{aligned}$$

From (4) and the observation that $\det\partial\psi$ is twice the area of the parameter triangle t, we can interpret the deformation energy κ_F as the Dirichlet energy per parameter area.

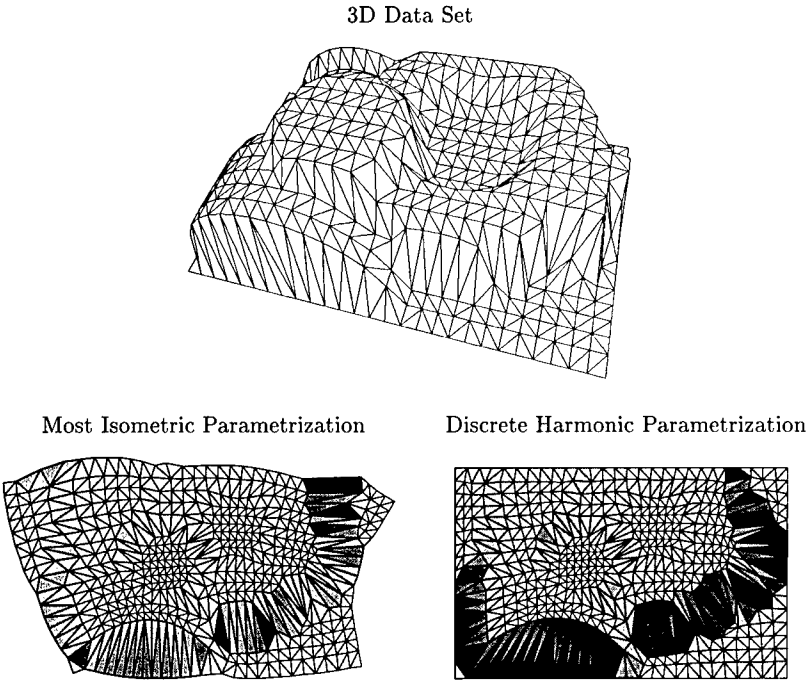


Fig. 4. Data set and gray-coded Dirichlet energy of different parametrizations.

Now, by minimizing the deformation functional $\kappa = \sum_j \kappa_F(f_j)$, we will get a set of parameter values $p_i \in \mathbb{R}^2$ that defines a parametrization with minimal distortion. Note that, as κ meets the four conditions from above, the minimum will only be unique up to movements and scalings. Anyway, this is not a drawback and can be fixed by retaining two arbitrarily chosen parameter values.

The main advantage of the proposed approach is that it is no longer necessary to fix the parameter values of the boundary points in advance. Instead, the boundary of the parametrization will develop most naturally in such a way that the deformation energy κ is minimized.

§4. Examples

We now illustrate the advantages of our new approach by showing some examples of surface approximation and texture mapping with different parametrizations.

In Fig. 4, a triangulated surface with 476 data points and 864 triangles can be seen. This data set has been parametrized with our new method and by minimizing the Dirichlet energy according to [3,14] which produced the best results of all the approaches mentioned in Sec. 2. Since the minimization of the Dirichlet energy can also be interpreted as a discrete harmonic mapping (see [3]), we will call the result Discrete Harmonic Parametrization.

Most Isometric Parametrization Discrete Harmonic Parametrization

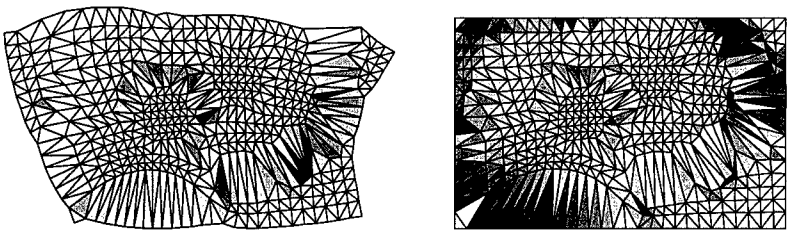


Fig. 5. Gray-coded κ deformation energy of different parametrizations.

We have encoded the amount of Dirichlet energy per triangle as gray tones (white color signifies low and black color denotes high energy). One can clearly see that the second method generates great deformations especially near the border of the parametrization which is due to the arbitrarily chosen parameter values for the boundary points. This effect is even more distinct if we look at the κ deformation energy per triangle, which has been gray-coded in Fig. 5.

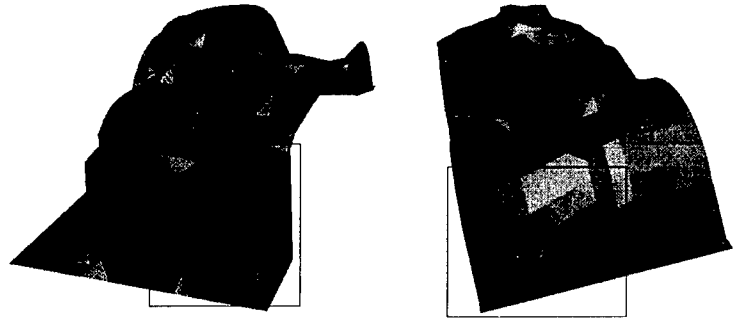
Fig. 6 shows how the deformations of these parametrizations affect an approximating surface. We have gray-coded the *mean curvature* of the surfaces in order to emphasize the fact that strong deformations in the parametrization of the data points cause the surface to wrinkle in these areas (dark color refers to high absolute values of the mean curvature).

Finally, Fig. 7 shows an example of remeshing. A base mesh of 4 triangles has been split six times by a regular 1-to-4-split, generating a triangle mesh with 16,384 triangles and regular connectivity, i.e., all vertices have valence 6 except for the one that refers to the central vertex of the base mesh and the boundary vertices. The Hoppe mannequin head has been remeshed with this semi-regular mesh using different parametrizations. The *Chord Length Parametrization*, used in the example to the right, refers to (1) with $c_{ij} = \frac{1}{\|P_i - P_j\|}$ (see [8] for details).

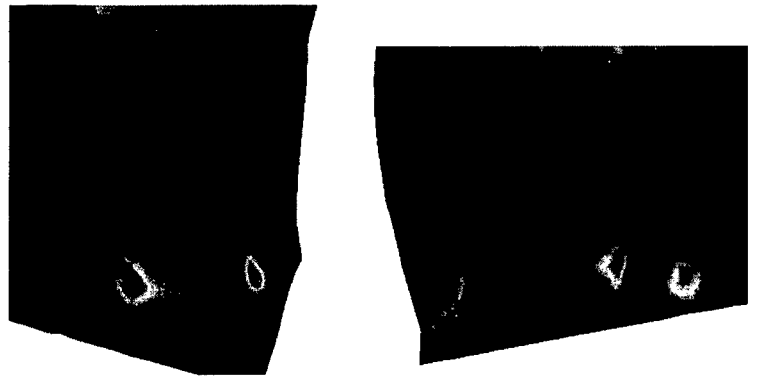
§5. Conclusion

We have presented a new method for constructing parametrizations of triangulated surfaces with a boundary. This parametrization can be used for surface approximation, texture mapping, and remeshing of the original mesh. The main advantage of our approach is that, in contrast to existing methods, the parametrization of the boundary data points is done in the same way as the parametrization of the inner points. Therefore we think that our approach is more natural than the other methods which set the parameter values at the boundary heuristically.

Often, the problem of parametrizing triangulated surfaces with holes occurs. So far, we have tacitly ignored this problem, but the proposed method is capable of dealing with these situations, in principle. Nevertheless, it might happen that overlapping parameter triangles will be generated at holes, which



Most Isometric Parametrization



Discrete Harmonic Parametrization

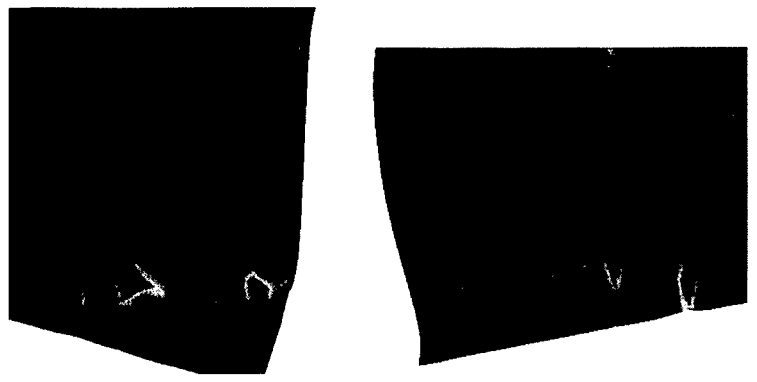


Fig. 6. Curvature plot of approximating surfaces with different parametrizations.

Most Isometric Parametrization Chord Length Parametrization

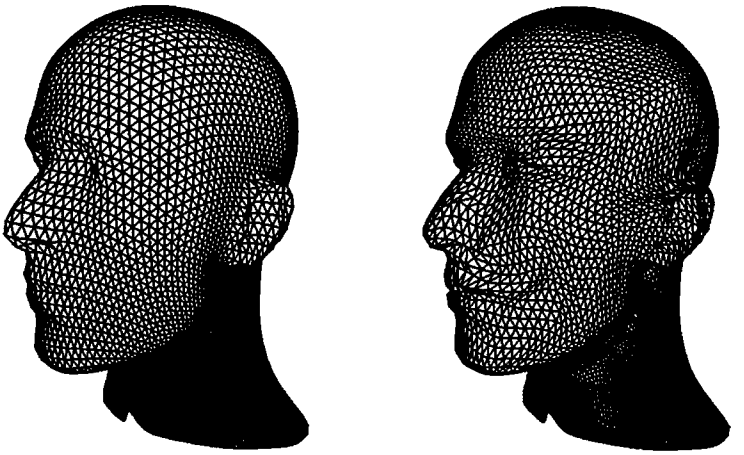


Fig. 7. Remeshing with different parametrizations.

can be fixed by triangulating the hole in a preprocessing step and removing the additional triangles afterwards.

The main drawback of our approach is that it requires the minimization of a rational quadratic function, while the other parametrization techniques only need to minimize a quadratic term which can be done by solving a sparse linear system of equations.

Our future work will therefore be concentrated on developing hierarchical methods for efficiently solving the problem. The concept of progressive meshes, introduced by Hoppe in [9] seems to be a good basis for such investigations.

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