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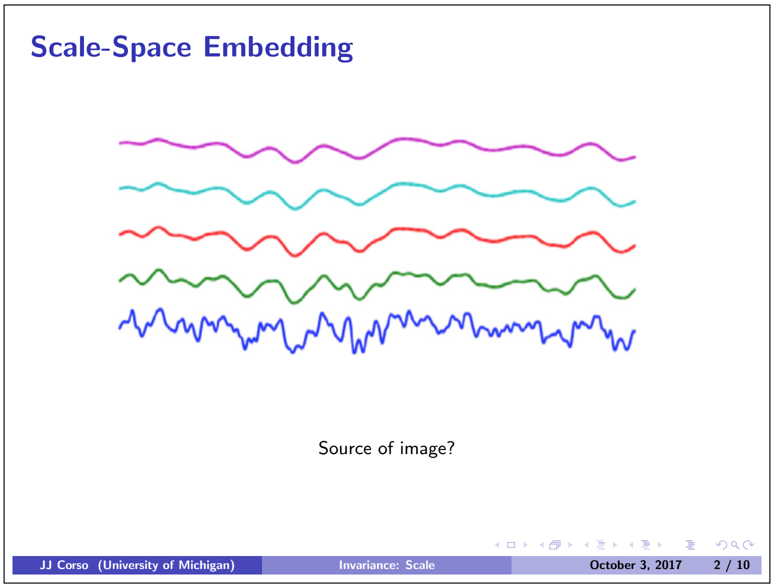
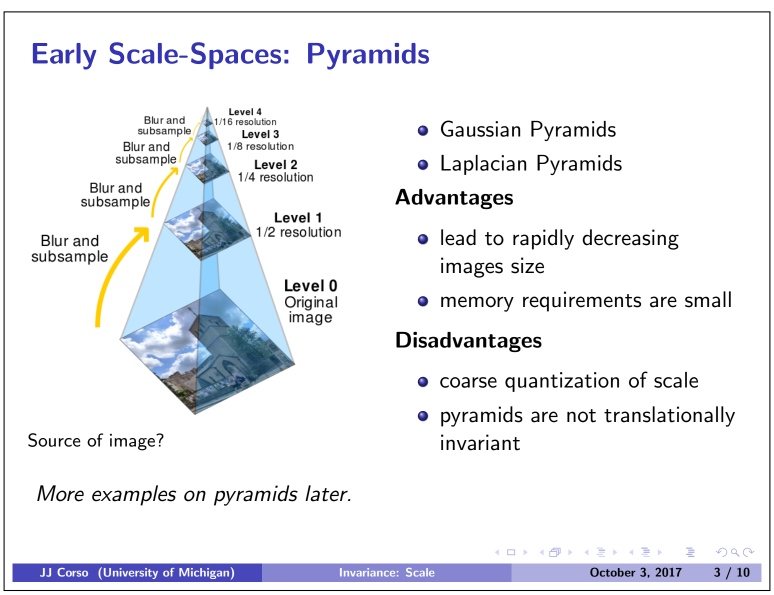
1 Scale Background

Slides begin with motivating examples of scale and scale-selection.

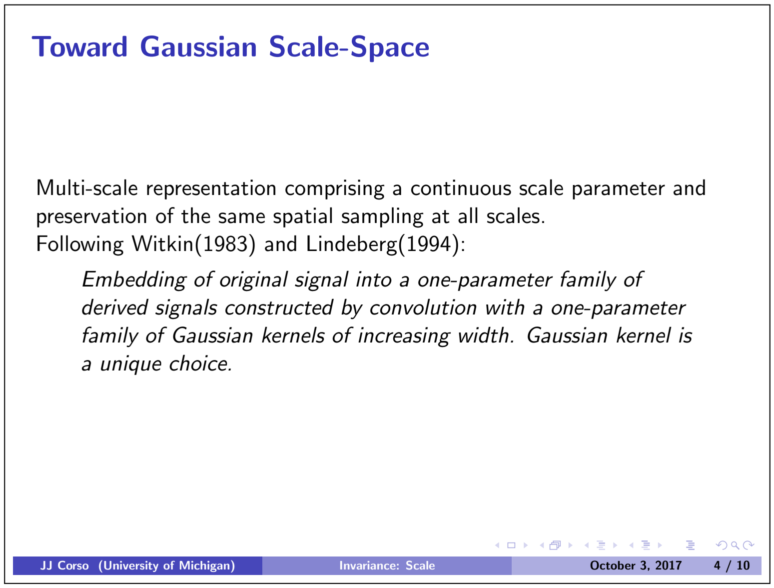
Loosely, scale-space is an embedding of a given signal into a one-parameter family of derived signals, where the family is parameterized by a scale parameter such that fine-scale structures are successively suppressed when the scale parameter is increased.



Goal: Scale-specific structures only occur at their respective scales in the scale-space.



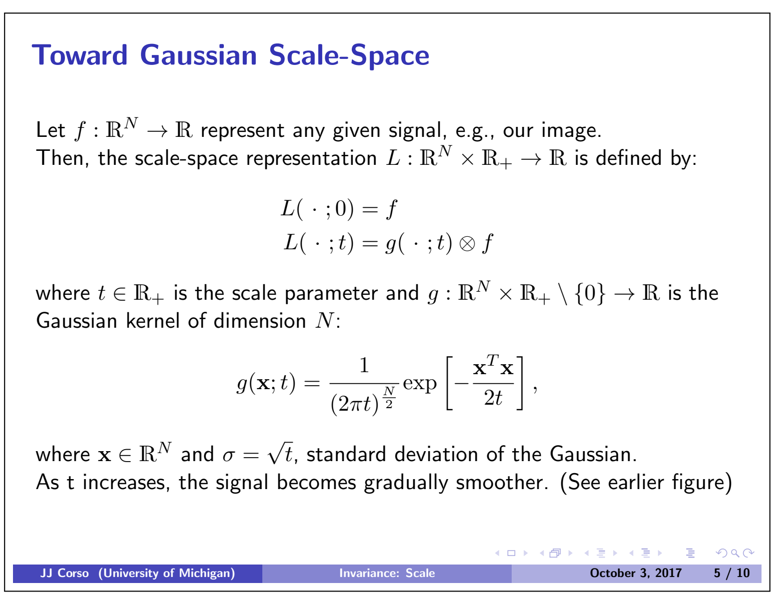
2 Scale-Space Representation



Multi-scale representation comprising a continuous scale parameter and preservation of the same spatial sampling at all scales.

Following Witkin(1983) and Lindeberg(1994):

> Embedding of original signal into a one-parameter family of derived signals constructed by convolution with a one-parameter family of Gaussian kernels of increasing width. Gaussian kernel is a unique choice.



Definition 1. Let $f : \mathbb{R}^N \to \mathbb{R}$ represent any given signal, e.g., our image.

Then, the scale-space representation $L : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$ is defined by:

$$L(\,\cdot\,; 0) = f \tag{1}$$

$$L(\,\cdot\,; t) = g(\,\cdot\,; t) \otimes f \tag{2}$$

where $t \in \mathbb{R}_+$ is the scale parameter and $g : \mathbb{R}^N \times \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}$ is the Gaussian kernel of dimension N:

$$g(\mathbf{x}; t) = \frac{1}{(2\pi t)^{\frac{N}{2}}} \exp\left[-\frac{\mathbf{x}^T\mathbf{x}}{2t}\right], \tag{3}$$

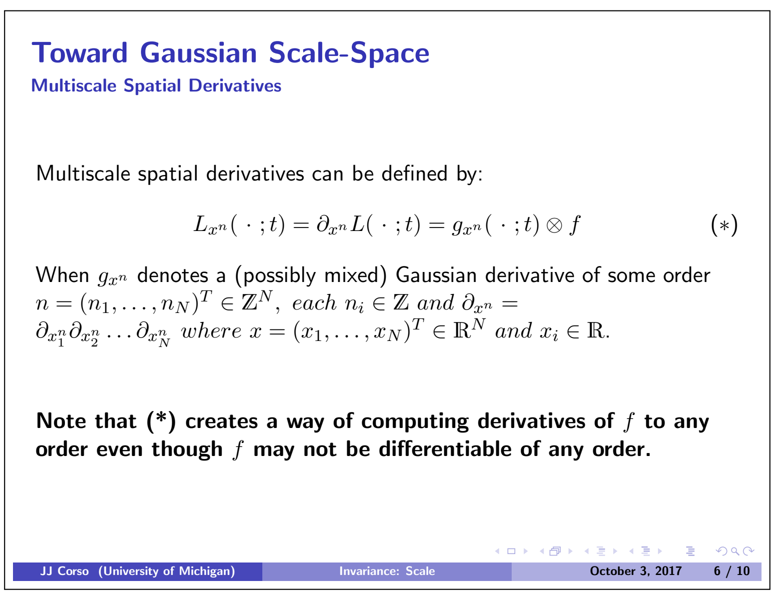
where $\mathbf{x} \in \mathbb{R}^N$ and $\sigma = \sqrt{t}$, standard deviation of the Gaussian.

Remark 1.1. As t increases, the signal becomes gradually smoother. (See earlier figure)

The scale-space family L can equivalently be defined as the solution to the diffusion equation:

$$\partial_t L = \frac{1}{2}\nabla^T\nabla L = \frac{1}{2}\sum_{i=1}^{N} \partial_{x_i}^2 L$$

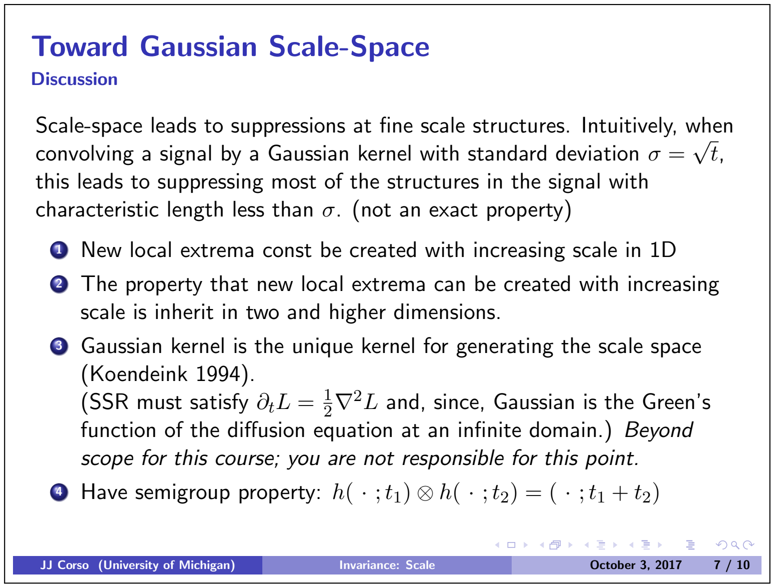
with the initial condition $L(\,\cdot\,; 0) = f$ This equation explains how a heat distribution L evolves in time in a homogeneous medium with uniform conductivity.



Definition 2. Multiscale spatial derivatives can be defined by:

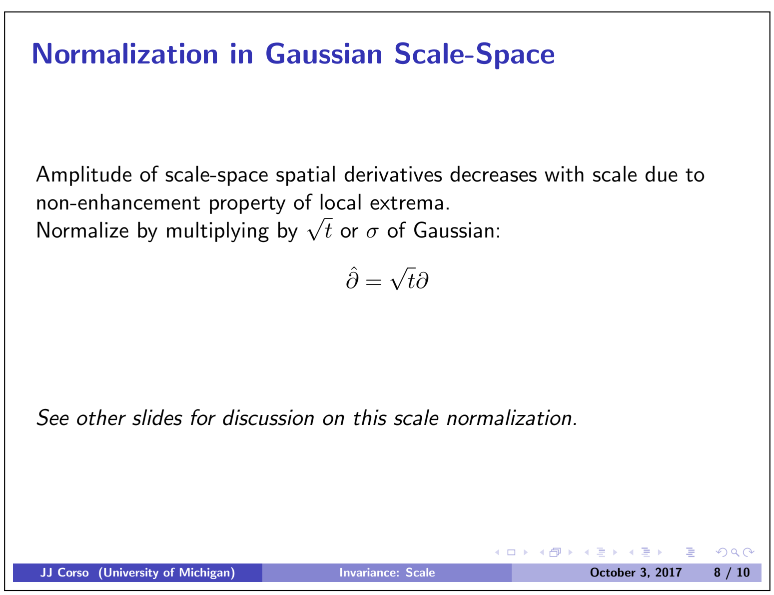
$$L_{x^n}(\,\cdot\,; t) = \partial_{x^n} L(\,\cdot\,; t) = g_{x^n}(\,\cdot\,; t) \otimes f \tag{*}$$

When g_{x^n} denotes a (possibly mixed) Gaussian derivative of some order $n = (n_1, \ldots, n_N)^T \in \mathbb{Z}^N$, *each* $n_i \in \mathbb{Z}$ *and* $\partial_{x^n} = \partial_{x_1^n}\partial_{x_2^n}\ldots\partial_{x_N^n}$ *where* $x = (x_1, \ldots, x_N)^T \in \mathbb{R}^N$ *and* $x_i \in \mathbb{R}$.
Note that $(*)$ creates a way of computing derivatives of f to any order even though f may not be differentiable of any order.



Remark 2.1. Scale-space leads to suppressions at fine scale structures. Intuitively, when convolving a signal by a Gaussian kernel with standard deviation $\sigma = \sqrt{t}$, this leads to suppressing most of the structures in the signal with characteristic length less than σ. (not an exact property)

1. New local extrema const be created with increasing scale in 1D
2. The property that new local extrema can be created with increasing scale in inherit in two and higher dimensions.
3. Gaussian kernel is the unique kernel for generating the scale space (Koendeink 1994). SSR must satisfy $\partial_t L = \frac{1}{2}\nabla^2 L$ and, since, Gaussian is the Green's function of the diffusion equation at an infinite domain.
4. Have semigroup property: $h(\,\cdot\,;t_1) \otimes h(\,\cdot\,;t_2) = (\,\cdot\,;t_1+t_2)$



Remark 2.2. Amplitude of scale-space spatial derivatives decreases with scale due to non-enhancement property of local extrema.

Normalize by multiplying by $\sqrt{t}$ or σ of Gaussian:

$$\hat{\partial} = \sqrt{t}\partial$$

See other slides for discussion on this scale normalization.

3 Example in scale selection: Normalization

Consider a sinusoidal input signal of frequency ω_0:

$$f(x) = \sin \omega_0 x$$

Solution of Diffusion Equation for scale space:

$$L(x;t) = e^{-\frac{\omega_0^2 t}{2}} \sin \omega_0 x$$

Amplitude of scale-space $L_{\max}$, and amplitude of m^{th} order derivative

$$L_{x^m,\max}(t) = \omega_0^m e^{-\omega_0^2 t/2}$$

$\gamma - normalized$ derivative:

$$\partial_{\zeta,\gamma-norm} = t^{\gamma/2}\partial_x,\ \zeta = \frac{x}{t^{\gamma/2}}$$

Amplitude of an m^{th} order normalized derivative as a function of scale is given by:

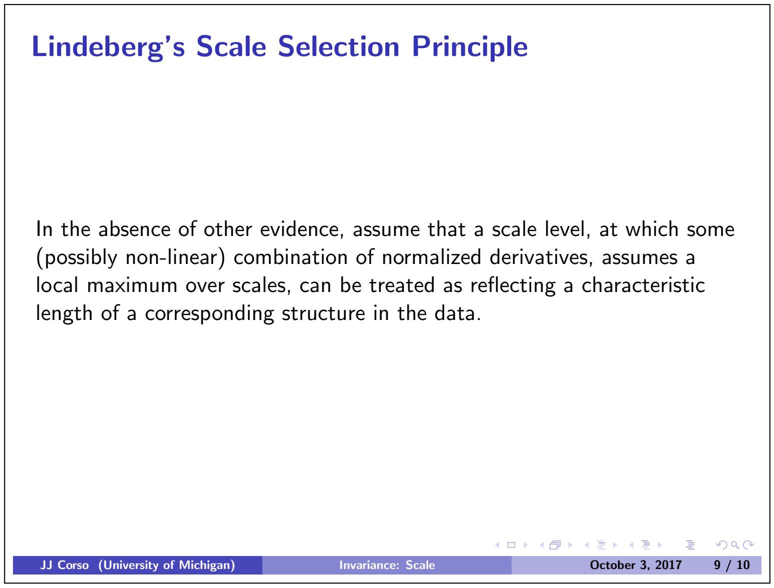
$$L_{\zeta^m,\max}(t) = t^{m\gamma/2} \cdot \omega_0^m \cdot e^{-\omega_0^2 t/2},\ unique\ max\ at:\ t_{\max,L\zeta m} = \frac{\gamma m}{\omega_0^2}$$

Scale $\sigma = \sqrt{t}$, wavelength $\lambda_0 = \frac{2\pi}{\omega_0}$, then scale at which amplitude of $\gamma - norm$ derivative assumes its maximum over scale is proportional to the wavelength λ_0 of the signal

$$\sigma_{\max,L\zeta m} = \frac{\sqrt{\gamma m}}{2\pi}\lambda_0$$

4 Scale Selection Principle

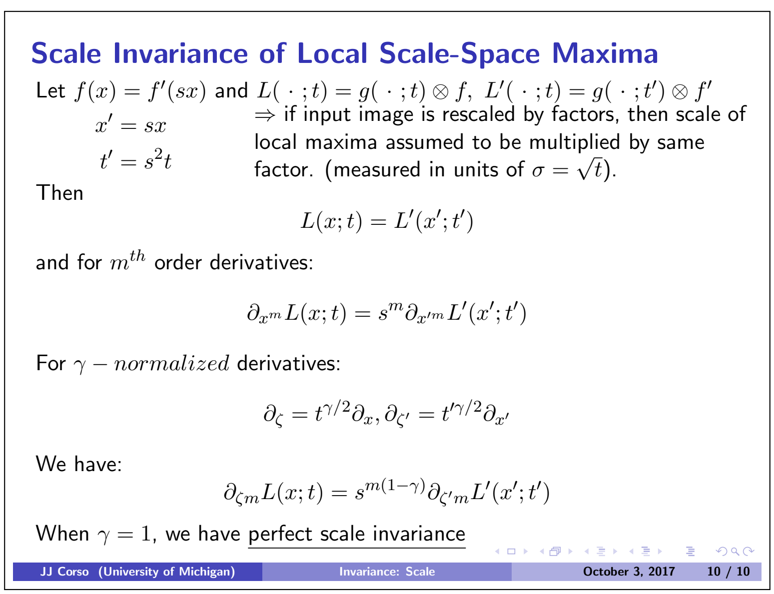
With the $\gamma - normalized$ derivative, we achieve a one-to-one correspondence between the matching response at the Gaussian derivative kernels and the wavelength of the signal.



4.1 Lindeberg's Principle of automatic scale selection

In the absence of other evidence, assume that a scale level, at which some (possibly non-linear) combination of normalized derivatives, assumes a local maximum over scales, can be treated as reflecting a characteristic length of a corresponding structure in the data.

5 Scale Invariance of Local Scale-Space Maxima



Consider $f(x) = f'(sx)$ and $L(\,\cdot\,;t) = g(\,\cdot\,;t) \otimes f,\ L'(\,\cdot\,;t) = g(\,\cdot\,;t') \otimes f'$

$$x' = sx, t' = s^2 t$$

$\Rightarrow$ if input image is rescaled by factors, then scale of local maxima assumed to be multiplied by same factor.(measured in units of $\sigma = \sqrt{t}$). Then

$$L(x'; t) = L'(x'; t')$$

and for m^{th} order derivatives:

$$\partial_{x^m} L(x; t) = s^m \partial_{x'^m} L'(x'; t')$$

For $\gamma - normalized$ derivatives:

$$\partial_\zeta = t^{\gamma/2} \partial_x, \partial_{\zeta'} = t'^{\gamma/2} \partial_{x'}$$

We have:

$$\partial_{\zeta m} L(x; t) = s^{m(1-\gamma)} \partial_{\zeta' m} L'(x'; t')$$

When $\gamma = 1$, we have perfect scale invariance

6 Scale Invariant Natural Image Statistics

Add examples; Invariance of gradient statistics.