COLLEGE OF ENGINEERING
ELECTRICAL ENGINEERING & COMPUTER SCIENCE
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Domain Operations

EECS 442 Computer Vision

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Content

- Geometric Primitives for Domain Operations
  - Points, Lines in 2D
- Domain Operations
  - Transformations in 2D

Geometric Primitives

- 2D points: pixel coordinates

$$\mathbf{x} = \begin{bmatrix} x & y \end{bmatrix}^{\mathsf{T}} \in \mathbb{R}^2$$

$x \in \mathbb{R}^2$

$\underset{2\times1}{} = \underset{2\times2}{A}\,\underset{2\times1}{x} + \underset{2\times1}{b}$

$\hat{A} = \begin{bmatrix} A & b \\ 0\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$

$\hat{A}\tilde{x}$

- Using homogeneous coordinates
  - Vectors differing by scale are equivalent.

$$\tilde{\mathbf{x}} = \begin{bmatrix} \tilde{x} & \tilde{y} & \tilde{w} \end{bmatrix}^{\mathsf{T}} \in \mathbb{P}^2$$

$$\tilde{\mathbf{x}} = \tilde{w} \begin{bmatrix} x & y & 1 \end{bmatrix}^{\mathsf{T}} = \tilde{w}\bar{\mathbf{x}}$$

$$\bar{\mathbf{x}} = \begin{bmatrix} x & y & 1 \end{bmatrix}$$

augmented vector

2D Projective Space

$$\mathbb{P}^2 = \mathbb{R}^3 - \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^{\mathsf{T}}$$

  - When the last element $\tilde{w} = 0$, call it an *ideal point.*

Geometric Primitives

- 2D lines with homogeneous coordinates

$$\tilde{\boldsymbol{l}} = \begin{bmatrix} a & b & c \end{bmatrix}^{\mathsf{T}}$$

$$\bar{\mathbf{x}}^{\mathsf{T}}\tilde{\boldsymbol{l}} = ax + by + c = 0$$

point $\bar{x}$ is on line $\tilde{l}$ when this is satisfied.

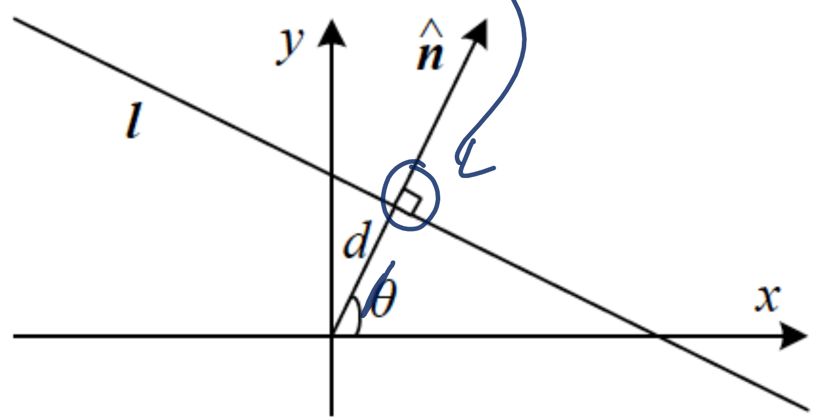
- Normalized coordinates

normal vector

$$\boldsymbol{l} = \begin{bmatrix} \hat{n}_x & \hat{n}_y & d \end{bmatrix}^{\mathsf{T}} = \begin{bmatrix} \hat{\mathbf{n}}^{\mathsf{T}} d \end{bmatrix}^{\mathsf{T}} \quad \text{s.t.} \quad \|\hat{\mathbf{n}}\| = 1$$

- Polar coordinates

$$\boldsymbol{l} = (\theta, d)$$
$$= \begin{bmatrix} \cos\theta & \sin\theta & d \end{bmatrix}$$



The image on this slides is sourced from the Szeliski book.

Geometric Primitives

- Intersection of two lines

$$\tilde{\mathbf{x}} = \tilde{\boldsymbol{l}}_1 \times \tilde{\boldsymbol{l}}_2$$

- Line connecting two points

$$\tilde{\boldsymbol{l}} = \tilde{\mathbf{x}}_1 \times \tilde{\mathbf{x}}_2$$

$$\begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Geometric Transformations

- 2D translation

Identity matrix

$$\mathbf{x}' = \begin{bmatrix} \mathcal{I} & \mathbf{t} \end{bmatrix} \bar{\mathbf{x}}$$

2×2 $2,3$ 3×1

$\bar{x}$ 3×1 $\tilde{x}$ 3 w

$$\bar{\mathbf{x}}' = \begin{bmatrix} \mathcal{I} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \bar{\mathbf{x}}$$

3×1

- 2D rotation and translation

3DoF – 2D rigid body or Euclidean transformation

$$\mathbf{x}' = \begin{bmatrix} \mathcal{R} & \mathbf{t} \end{bmatrix} \bar{\mathbf{x}} \qquad \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotation matrix

$$\mathcal{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\mathcal{R}\mathcal{R}^\mathsf{T} = \mathcal{I}$$

$$|\mathcal{R}| = 1$$

$-1 \rightarrow$ Reflection

$$\bar{\mathbf{x}}' = \begin{bmatrix} \mathcal{R} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \bar{\mathbf{x}}$$

3×3

$$\bar{x}' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \bar{x}\cos\theta - \bar{y}\sin\theta \\ x\sin\theta + \cos\theta \\ 1 \end{bmatrix} + t$$

Geometric Transformations

4DOF

- 2D scaled rotation or similarity transform

s t R

$$\bar{\mathbf{x}}' = \begin{bmatrix} s\mathcal{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \bar{\mathbf{x}} = \begin{bmatrix} a & -b & t_x \\ b & a & t_y \\ 0 & 0 & 1 \end{bmatrix} \bar{\mathbf{x}}$$

$a = s \cdot \cos\theta$

$b = s \cdot \sin\theta$

$$\begin{bmatrix} ax - by + t_x \\ bx + ay + t_y \\ 1 \end{bmatrix}$$

  - Constraint $a^2 + b^2 = 1$ is not enforced.

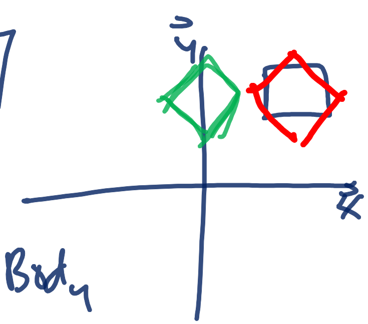
6DoF

- 2D affine transformation

$$\bar{\mathbf{x}}' = \mathcal{A}\bar{\mathbf{x}} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \bar{\mathbf{x}}$$

$c \equiv t_x$

$f \equiv t_y$



Rigid Body

Geometric Transformations

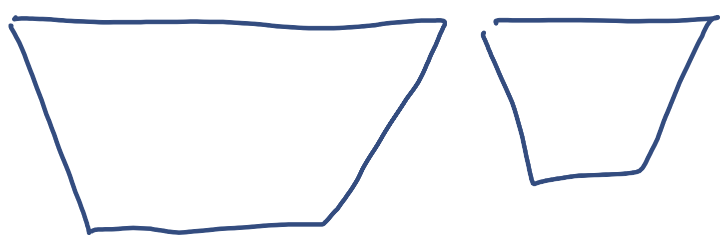
- 2D projective, also called the homography

$$\tilde{\mathbf{x}}' = \tilde{\mathcal{H}}\tilde{\mathbf{x}} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 1 \end{bmatrix} \tilde{\mathbf{x}}$$

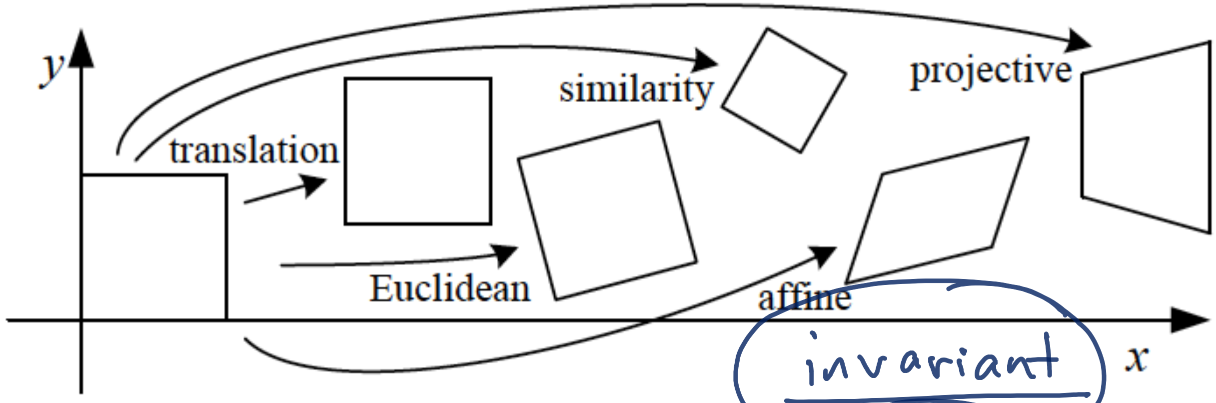
$$\begin{bmatrix} ax + by + c \\ dx + ey + f \\ gx + hy \end{bmatrix}$$

- Projective matrix $\tilde{\mathcal{H}}$ is defined up to scale.
- Inhomogeneous results are computed after homogeneous operation.

$$\bar{x} = \begin{bmatrix} \frac{ax+by+c}{gx+hy} \\ \cdot \end{bmatrix}$$



Hierarchy of 2D Planar Transformations



Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} I \mid t \end{bmatrix}_{2\times 3}$	2	orientation	
rigid (Euclidean)	$\begin{bmatrix} R \mid t \end{bmatrix}_{2\times 3}$	3	lengths	
similarity	$\begin{bmatrix} sR \mid t \end{bmatrix}_{2\times 3}$	4	angles	
affine	$\begin{bmatrix} A \end{bmatrix}_{2\times 3}$	6	parallelism	
projective	$\begin{bmatrix} \tilde{H} \end{bmatrix}_{3\times 3}$	8	straight lines	

$p_1, p_2 \in \mathbb{R}^2$

$D(p_1, p_2)$

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Def Rot. Mx $R(\theta)$

$q_1 = R p_1, \quad q_2 = R p_2$

$D(q_1, q_2)$

$$D(p_1, p_2) = D(q_1, q_2)$$

Projective Geometry

- These geometry basics are but the surface of an area important to computer vision called **projective geometry**.

	Euclidean	similarity	affine	projective
Transformations				
rotation	X	X	X	X
translation	X	X	X	X
uniform scaling		X	X	X
nonuniform scaling			X	X
shear			X	X
perspective projection				X
composition of projections				X
Invariants				
length	X			
angle	X	X		
ratio of lengths	X	X		
parallelism	X	X	X	
incidence	X	X	X	X
cross ratio	X	X	X	X

- Further reading: “An Introduction to Projective Geometry” by Stan Birchfield.