Case Study: Geometric Invariance and Local Features | Jason Corso
Foundations of Computer Vision | University of Michigan | September 26, 2017

1 How can we formulate this notion of local uniqueness?

First, we know how to operate on spatial ranges. We could consider a window W and a binary spatial range operator combined with a domain operator. Consider a composed function. $E(T)$ for transform T:

$$E(T) = \sum_{(x,y)\in W} (\mathbf{I}(T(x,y)) - \mathbf{I}(x,y))^2$$

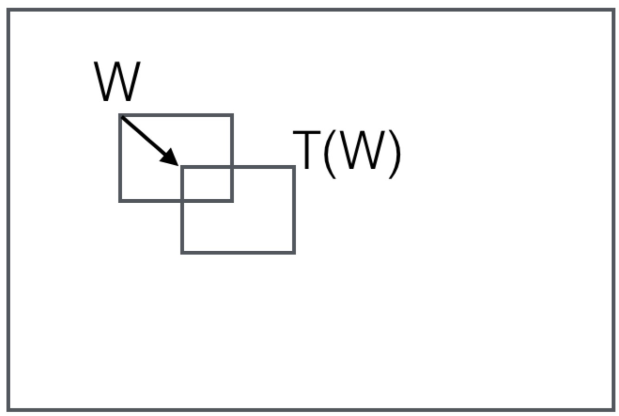

Figure 1: transform T

Note we could write out the correlation function, but it will not lead to a clean model of intensity corners.

What is T()?

- Translation: $\mathbf{I}(T_T(x,y)) = \mathbf{I}(x+u, y+v)$
- Rotation: $\mathbf{I}(T_R(x,y)) = \mathbf{I}(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$

Let's consider translation for reasons that will soon become clean.

$$\begin{aligned}\mathbf{I}(T_T(x,y)) &= \mathbf{I}(x+u, y+v)\\ &= \mathbf{I}(x,y) + \frac{\partial \mathbf{I}}{\partial x}(x,y)\cdot u + \frac{\partial \mathbf{I}}{\partial y}(x,y)\cdot v + \frac{\partial^2 \mathbf{I}}{\partial x^2}(x,y)\cdot u^2 + \frac{\partial^2 \mathbf{I}}{\partial y^2}(x,y)\cdot v^2\\ &+ higher\ order\ terms\end{aligned}$$

(Assume $\mathbf{I}$ is infinitely differentiable)

Since we are only looking in a small, local neighborhood of $\mathbf{I}(x,y)$, then a first-order approximation seems reasonable.

$$\mathbf{I}(x+u, y+v) \approx \mathbf{I}(x,y) + \frac{\partial \mathbf{I}}{\partial x}(x,y)\cdot u + \frac{\partial \mathbf{I}}{\partial y}(x,y)\cdot v$$

Back to our sum-of-squared differences objective function:

$$\begin{aligned}
E(T;u,v) &= \sum_{(x,y)\in W} (\mathbf{I}(x+u,y+v) - \mathbf{I}(x,y))^2 \\
&\approx \sum_{(x,y)\in W} (\cancel{\mathbf{I}(x,y)} + \frac{\partial \mathbf{I}}{\partial x}(x,y)\cdot u + \frac{\partial \mathbf{I}}{\partial y}(x,y)\cdot v - \cancel{\mathbf{I}(x,y)})^2 \\
&= \sum_{(x,y)\in W} (\frac{\partial \mathbf{I}}{\partial x}(x,y)\cdot u + \frac{\partial \mathbf{I}}{\partial y}(x,y)\cdot v)^2
\end{aligned}$$

We simplify notation: $\mathbf{I}_x(x,y) = \frac{\partial \mathbf{I}}{\partial x}(x,y),\ \mathbf{I}_y(x,y) = \frac{\partial \mathbf{I}}{\partial y}(x,y)$

Let's expand:

$$\begin{aligned}
E(T;u,v) &\approx \sum_{(x,y)\in W} (\mathbf{I}_x(x,y)u + \mathbf{I}_y(x,y)v)^2 \\
&= \sum_{(x,y)\in W} (\mathbf{I}_x(x,y)^2u^2 + \mathbf{I}_y(x,y)^2v^2 + 2\mathbf{I}_x(x,y)\mathbf{I}_y(x,y)uv) \\
&= \sum_{(x,y)\in W} \left(\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} \mathbf{I}_x(x,y)^2 & \mathbf{I}_x(x,y)\mathbf{I}_y(x,y) \\ \mathbf{I}_x(x,y)\mathbf{I}_y(x,y) & \mathbf{I}_y(x,y)^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \right)
\end{aligned}$$

We can verify that this is equivalent by carrying through the products. Next we need to push in the sum:

$$\begin{aligned}
E(T;u,v) &\approx \sum_{(x,y)\in W} (\mathbf{I}_x(x,y)u + \mathbf{I}_y(x,y)v)^2 \\
&= \sum_{(x,y)\in W} (\mathbf{I}_x(x,y)^2u^2 + \mathbf{I}_y(x,y)^2v^2 + 2\mathbf{I}_x(x,y)\mathbf{I}_y(x,y)uv) \\
&= \begin{bmatrix} u & v \end{bmatrix} \underbrace{\begin{bmatrix} \sum_{(x,y)\in W} \mathbf{I}_x(x,y)^2 & \sum_{(x,y)\in W} \mathbf{I}_x(x,y)\mathbf{I}_y(x,y) \\ \sum_{(x,y)\in W} \mathbf{I}_x(x,y)\mathbf{I}_y(x,y) & \sum_{(x,y)\in W} \mathbf{I}_y(x,y)^2 \end{bmatrix}}_{Structure\ Tensor} \begin{bmatrix} u \\ v \end{bmatrix}
\end{aligned}$$

Let $\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$ denote this Structure Tensor.

Intuitively, $\mathbf{H}$ captures local gradient changes of the image and structure of them.

2 How can we analyze the intrinsic structure of H?

Use Eigenvalues and eigenvectors.

$$\begin{aligned}
&\det\left[\mathbf{H} - \lambda\mathbf{I}\right] = \det\begin{bmatrix} h_{11}-\lambda & h_{12} \\ h_{21} & h_{22}-\lambda \end{bmatrix} = 0 \\
&\Rightarrow (h_{11}-\lambda)(h_{22}-\lambda) - h_{21}h_{12} = 0 \\
&\Rightarrow h_{11}h_{22} - \lambda h_{11} - \lambda h_{12} + \lambda^2 - h_{21}h_{12} = 0 \\
&\Rightarrow \lambda^2 - (h_{11}+h_{22})\lambda + h_{11}h_{22} - h_{21}h_{12} = 0
\end{aligned}$$

Solving: $\lambda^2 - (h_{11} + h_{22})\lambda + h_{11}h_{22} - h_{21}h_{12} = 0$
Recall Quadratic Equation/Formula:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\begin{aligned}\lambda_{\pm} &= \frac{1}{2}\left(h_{11} + h_{22} \pm \sqrt{-(h_{11} + h_{22})^2 - 4(h_{11}h_{22} - h_{21}h_{12})}\right) \\ &\overset{simplify}{=} \frac{1}{2}\left(h_{11} + h_{22} \pm \sqrt{(h_{11} - h_{22})^2 + 4h_{21}h_{12}}\right)\end{aligned}$$

Now we have two eigenvalues , that measure the degree of varration in structure. Let's look at what they mean.
Compute eigenvectors $\mathbf{x}_-$ and $\mathbf{x}_+$ by solving:

$$\left[\mathbf{H} - \lambda_- \mathbf{I}\right] \mathbf{x}_- = 0 \; and \; \left[\mathbf{H} - \lambda_+ \mathbf{I}\right] \mathbf{x}_+ = 0$$

$\mathbf{H}\mathbf{x}_+ = \lambda_+ \mathbf{x}_+$:
$\mathbf{x}_+$ is the direction of largest increase in $E(T)$,λ_+ is the magnitude of that increase
$\mathbf{H}\mathbf{x}_- = \lambda_- \mathbf{x}_-$:
$\mathbf{x}_-$ is the direction of smallest increase in $E(T)$,λ_- is the magnitude of that increase

Think about our line:

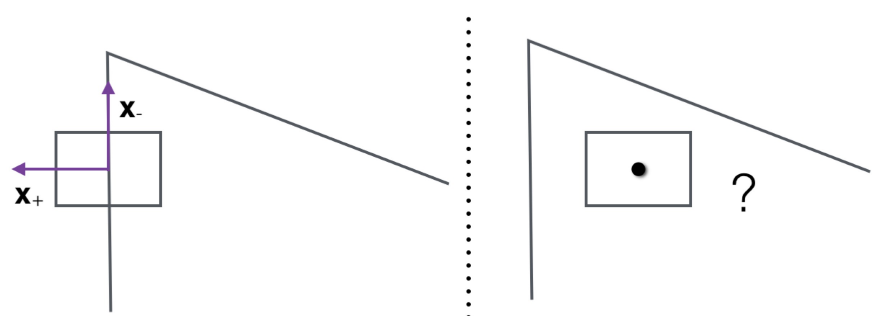

Figure 2: Looking for local uniqueness

3 Taylor Series Background

3.1 Single Variable

Let f be an infinitely differentiable function in some open interval around $x = a$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

3.2 Linear approximation

In an open interval around $x = a$, discard terms of order 2 and greater

$$f(x) \approx f(a) + f'(a)(x - a)$$

3.3 Multi-variable Taylor Series

Let f be an infinitely differentiable function in some open neighborhood around $(x, y) = (a, b)$

$$\begin{aligned} f(x,y) = & f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ & + \frac{1}{2}(f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2) \\ & + \ldots \end{aligned}$$

4 Rotation Invariance of Harris Structure Tensor

Intuition: the rotation varies the coordinate system but not the intrinsic variation of neighborhood of the point

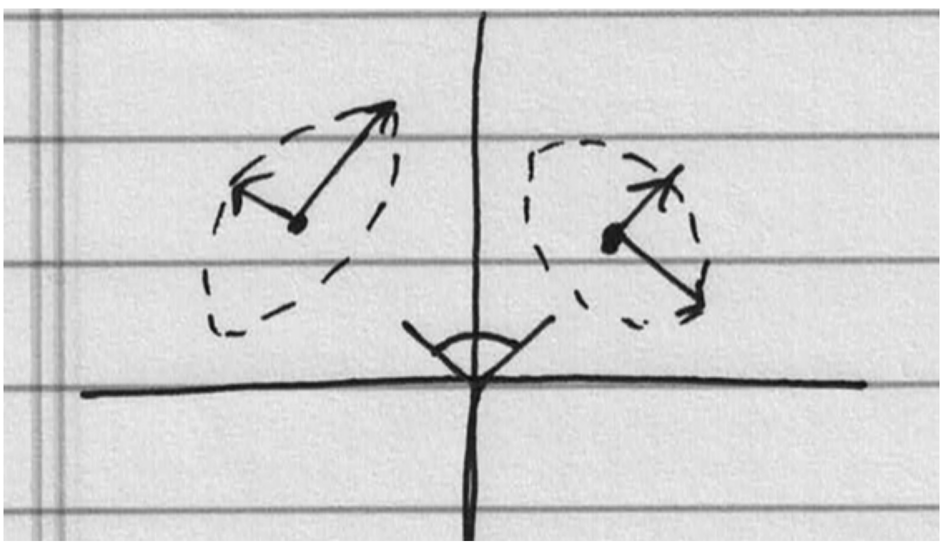

Figure 3: Rotation Invariance of Harris Structure Tensor

4.1 Rotated Structure Tensor

$$\underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} x+u \\ y+v \end{bmatrix} \Rightarrow \mathbf{R}\begin{bmatrix} x \\ y \end{bmatrix} + \mathbf{R}\begin{bmatrix} u \\ v \end{bmatrix}$$

$$
\begin{aligned}
&Let\ x' = x\cos\theta - y\sin\theta,\ y' = x\sin\theta + y\cos\theta \\
&\quad u' = u\cos\theta - v\sin\theta,\ v' = u\sin\theta + v\cos\theta \\
&\Rightarrow \sum_{x',y' \in \mathbf{W}'} [\mathbf{I}(x'+v', y'+v') - \mathbf{I}(x', y')] \\
&\Rightarrow \sum_{(x',y') \in \mathbf{W}'} \left(\begin{bmatrix} u' & v' \end{bmatrix} \begin{bmatrix} \mathbf{I}_{x'}(x',y')^2 & \mathbf{I}_{x'}(x',y')\mathbf{I}_{y'}(x',y') \\ \mathbf{I}_{x'}(x',y')\mathbf{I}_{y'}(x',y') & \mathbf{I}_{y'}(x',y')^2 \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} \right)
\end{aligned}
$$

4.2 Preservation of Eigenvalues

- Two $n \times n$ matrices A and B are said to be similar whenever there exists a nonsingular matrix S such that $S^{-1}AS = B$, $S^{-1}AS$ is called the similarity transform of A.
- Eigenvectors are a similarity transform
  A is diagonalizable if and only if A possesses a set of n linearly independent eigenvectors. Moreover, $S^{-1}AS = diag(\lambda_1, \lambda_2, \dots \lambda_n)$ if and only if the columns of S are the eigenvectors s_i corresponding to eigenvalues $\lambda_i\ for\ i = 1, 2, \dots, n$
- Similar matrices have the same characteristic polynomial and hence the same eigenvalues with the same multiplization
  Proof. Recall :

$$
\begin{aligned}
det(S^{-1} &= 1/det(S)) && (1) \\
det(AB) &= det(A)det(B) && (2) \\
det(A - \lambda\mathbf{I}) &= det(P^{-1}BP - \lambda\mathbf{I}) = det(P^{-1}(B - \lambda\mathbf{I})P) && (3) \\
&= det(P^{-1})det(B - \lambda\mathbf{I})det(P) = det(B - \lambda\mathbf{I}) && (4)
\end{aligned}
$$

  This implies that the eigenvalues of a matrix representation are invariant under a change of basis. They are independent of any coordinate representation.