3D COMPUTER VISION

Single-View Geometry

C2: Representation of a 3D Moving Scene

3D Euclidean Space

- Inner product: $\langle u, v\rangle = u^T v \in \mathbb{R}^3$
- Cross product: $u \times v = -v \times u \quad \in \mathbb{R}^3$
  $u \times (\alpha v + \beta w) = \alpha u \times v + \beta u + w$
- Skew-symmetric: $u \times v = \widehat{u} v$

$$\widehat{u} v = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

$$[u]_\times = \widehat{u} = -\widehat{u}^T,\ \widehat{u} u = 0, x^T \widehat{u} x = 0$$

when $\|u\| = 1$ then $\widehat{u}\,\widehat{u}^T\,\widehat{u} = \widehat{u}$

- Orthogonality: $\langle u \times v, v\rangle = \langle u \times v, u\rangle = 0$

Rotational Motion

- Rotational motion: $R_{cw} = R_{wc}^{-1} = R_{wc}^T$
- Special Orthogonal group:

$$SO(3) \doteq \{R \in \mathbb{R}^{3\times3} | R^T R = I, det(R) = +1\}$$

- Exponential Map: $so(3) \doteq \{\widehat{\omega} \in \mathbb{R}^{3\times3} | \omega \in \mathbb{R}^3\}$

$$\exp\colon so(3) \to SO(3); \quad \widehat{\omega} \mapsto e^{\widehat{\omega}}$$

$$R = e^{\widehat{\omega}} \quad \widehat{\omega} = log(R)$$

  rotating around some fixed axis ω by a certain angle $\|\omega\|$
- Rodrigues' Formula:

$$e^{\widehat{\omega}} = I + \frac{\widehat{\omega}}{\|\omega\|}\sin(\|\omega\|) + \frac{\widehat{\omega}^2}{\|\omega\|^2}(1 - \cos(\|\omega\|))$$

Rigid-body Motion

- Special Euclidean group:

$$SE(3) \doteq \{g = (R, T) | R \in SO(3), T \in \mathbb{R}^3\}$$

- Motion Composition: $g(t_3, t_1) = g(t_3, t_2) g(t_2, t_1)$

$$g_{13} = \begin{bmatrix} R_{12}R_{23} & R_{12}T_{23} + T_{12} \\ 0 & 1 \end{bmatrix}$$

- Motion inverse: $g(t_2, t_1)^{-1} = g(t_1, t_2)$

$$g^{-1} = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^T & -R^T T \\ 0 & 1 \end{bmatrix} \in SE(3)$$

- Exponential Map:

$$se(3) \doteq \left\{ \widehat{\xi} = \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix} \middle| \widehat{\omega} \in so(3), v \in \mathbb{R}^3 \right\} \subset \mathbb{R}^{4\times4}$$

$$\exp\colon \mathrm{se}(3) \to \mathrm{SE}(3); \quad \widehat{\xi} \mapsto e^{\widehat{\xi}}$$

$$R = e^{\widehat{\xi}} \quad \widehat{\xi} = log(R)$$

Coordinate and velocity transformation

- $X(t) = R(t)X_0 + T(t)$ $\quad \dot{X}(t) = \widehat{\omega} X(t) + v(t)$
- Adjoint map $\quad \widehat{\omega} \mapsto R\widehat{\omega}R^T$

Other!!

- Euler Angles: $R = R_z(\alpha) R_y(\beta) R_x(\gamma)$

$$\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

- A Unit Quaternion:

$$\mathbf{q}^{-1} = e^{-\frac{\theta}{2}(u_x i + u_y j + u_z k)} = \cos\frac{\theta}{2} - (u_x \mathbf{i} + u_y \mathbf{j} + u_z \mathbf{k}) \sin\frac{\theta}{2}$$

  A rotation about the unit vector u by an angle θ
- Homogeneous coordinates $\quad (x, y) \leftrightarrow (\lambda x, \lambda y, \lambda), \quad \forall \lambda \neq 0$
- Homogeneous Linear Least Squares problem: $\quad Ax = 0$

- Line equation $\quad ax + by + c = 0$

$$\mathbf{x}^\top l = 0, \quad \text{where} \quad l = (a, b, c)^\top$$

- Conic equation $\quad ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$

$$\mathbf{x}^\top C \mathbf{x} = 0, \quad \text{where} \quad C = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

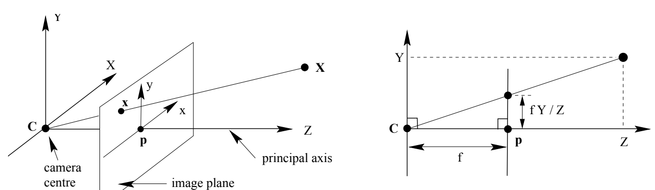
- Parallel lines $\quad l = (a, b, c), l' = (a, b, c')$
  $l \times l' \sim (b, -a, 0)$
- Duality $\quad p = l_1 \times l_2, \ \ l = p_1 \times p_2$
- Ideal points $\quad p_\infty = (x_1, x_2, 0)$
- Line at infinity $\quad l_\infty = (0, 0, 1)$
- Plane at infinity $\quad \pi_\infty = (0, 0, 0, 1)$

Table 2.1. Rotation and rigid-body motion in 3D space.

	Rotation SO(3)	Rigid-body motion SE(3)
Matrix rep	$R: \begin{cases} R^T R = I \\ det(R) = 1 \end{cases}$	$g = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}$
Coord (3D)	$X = RX_0$	$X = RX_0 + T$
Inverse	$R^{-1} = R^T$	$g^{-1} = \begin{bmatrix} R^T & -R^T T \\ 0 & 1 \end{bmatrix}$
Composition	$R_{ik} = R_{ij} R_{jk}$	$g_{ik} = g_{ij} g_{jk}$
Exp. rep	$R = exp(\widehat{\omega})$	$g = exp(\widehat{\xi})$
Velocity	$\dot{X} = \widehat{\omega} X$	$\dot{X} = \widehat{\omega} X + v$
Adjoint map	$\widehat{\omega} \mapsto R\widehat{\omega}R^T$	$\widehat{\xi} \mapsto g\widehat{\xi}g^{-1}$

C3: Image Formation

Camera Model



- World → Camera: $\quad X_c = RX_w + T$
- Camera → Image Plane: $\quad x = -f\frac{X}{Z}, y = -f\frac{Y}{Z}$
- Ideal Pinhole Camera $\quad \pi : \mathbb{R}^3 \mapsto \mathbb{R}^2; \quad X \mapsto x$
- 3D homogeneous coordinates:

$$\lambda \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{K_f} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\Pi_0} \underbrace{\begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \\ 1 \end{bmatrix}}_{X_c = g_0 X_w}$$

- Image Plane → Pixel: $\quad x' = K_s x$
- 3D homogeneous coordinates:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & s_\theta & o_x \\ 0 & s_y & o_y \\ 0 & 0 & 1 \end{bmatrix}}_{K_s} \underbrace{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}_{\text{Metric}}$$

  where s_θ= skew factor, (o_x, o_y)= principal point in terms of pixel dimensions (center offsets), f= focal length, (s_x, s_y)= number of pixels per unit distance in image coordinates (scaling factors).
- Camera calibration(intrinsic): $K = K_s K_f$ (5 DOF)
- Camera model: $\quad X' = K\Pi_0 g X$
- Projection matrix: $\quad \Pi = K\Pi_0 g = [KR, KT]$

$$\lambda x' = \Pi X_0 \ (\lambda\text{: projective dept})$$

C4: Image Primitives and Correspondence

Small baseline: feature tracking and optical flow

$$I_1(x) = I_2(h(x)) = I_2(x + \Delta x),\ \Delta X \doteq \mathrm{u} dt$$

image velocity: $\quad u = -G^{-1} b$

$$G = \begin{bmatrix} \sum I_x^2 & \sum I_x I_y \\ \sum I_x I_y & \sum I_y^2 \end{bmatrix},\ b = \begin{bmatrix} \sum I_x I_t \\ \sum I_y I_t \end{bmatrix}$$

- Rank(G) = 0 blank wall problem (flat area)
- Rank(G) = 1 aperture problem (line)
- Rank(G) = 2 enough texture, good feature candidates

Similarity measure: Sum of Squared Differences (SSD)

Large baseline: Feature matching

$$h(\tilde{x}) = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \tilde{x} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

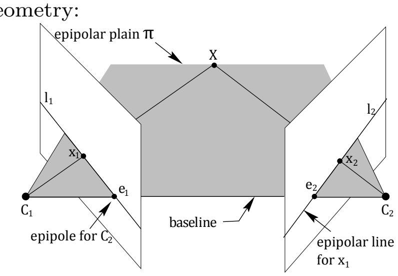
Similarity measure: Normalized cross-correlation (NCC)

Point feature selection: Harris: $C(x) = det(G) + \kappa.trace^2(G)$

Two-View Geometry

C5: Reconstruction from 2 Calibrated Views

Epipolar geometry:



Pose Recovery from the Essential Matrix:

- Decompose(SVD): $E = U\Sigma V^T$
  where $\Sigma = diag\{\sigma, \sigma, 0\}$ and $U, V \in SO(3)$
- Calculate two pairs of camera matrices:

$$(\widehat{T}_1, R_1) = (UR_Z(+\tfrac{\pi}{2})\Sigma U^T, UR_Z(+\tfrac{\pi}{2})V^T)$$

$$(\widehat{T}_2, R_2) = (UR_Z(-\tfrac{\pi}{2})\Sigma U^T, UR_Z(-\tfrac{\pi}{2})V^T)$$

The eight-point linear algorithm:

- Calculate Kronecker product for each pair: $a = x_1 \otimes x_2$

$$a \doteq [x_1 x_2, x_1 y_2, x_1 z_2, y_1 x_2, y_1 y_2, y_1 z_2, z_1 x_2, z_1 y_2, z_1 z_2]^T$$

- From a set of n point matches, we obtain a set of linear equations of the form

$$\chi \doteq [a^1, a^2, ..., a^n]^T$$

- Solve linear equation: $\chi E^s = 0$
  Where E^s is stacked vector of E

$$E^s \doteq [e_{11}, e_{21}, e_{31}, e_{12}, e_{22}, e_{32}, e_{13}, e_{23}, e_{33}]^T$$

Planar scenes and homography $\widehat{e_2}H = F$

$\widehat{e_2}T = 0$??slide-section5: p144

$$\underbrace{\left[\widehat{x_2^j} R x_1^j, \widehat{x_2^j} T\right]}_{M^j} \begin{bmatrix}\lambda_1^j\\ \gamma\end{bmatrix}$$

- Depth cues (parallax) can only be recovered when T is nonzero.
- Any pair of images of an arbitrary scene captured by a purely rotating camera is related by a planar homography.
- **Parallax:** All points on the reference plane are aligned. Points outside it are offset, relative to their distance from the reference plane.
- Warping the **silhouettes** of an object from image plane to a plane in the scene using a planar homography is equivalent to projecting the visual hull of the object onto the plane.

Table 5.3+ my Modifications

	Epipolar cons	**Homography**
Image point	$x_2 \sim Rx_1 + T$ $X_2 = RX_1 + T$	$x_2 \sim Hx_1$ $X_2 = HX_1$
Geometry cons	$x_2^T E x_1 = 0$	$\widehat{x_2} H x_1 = 0$
Epipolar lines	$l_1 \sim E^T x_2$ $l_2 \sim E x_1$	$l_2 \sim \widehat{x_2} H x_1$ $l_1 \sim H^T l_2$
Matrices	$E = \widehat{T}R$	$H = R + \frac{1}{d} T N^T$
Map	point $\rightarrow$ line	point $\rightarrow$ point
Relation	$\exists v \in \mathbb{R}^3, H = \widehat{T}^T E + T v^T$ $E = \widehat{T} H \qquad H^T E + E^T H = 0$	
Continuous motion	$x^T \widehat{\omega}\widehat{v} x + u^T \widehat{v} x = 0$	$\widehat{x}(\widehat{\omega} + \frac{1}{d} v N^T) X = \widehat{u} X$
Matrices	$E = \begin{bmatrix}\frac{1}{2}(\widehat{\omega}\widehat{v} + \widehat{v}\widehat{\omega}) \\ \widehat{v}\end{bmatrix}$	$H = w + \ vNT$
Linear Algo	8 points	4 points
Decomposition	1 possible solution (5 DOF)	2 possible solutions (8 DOF)

C6: Reconstruction from 2 Uncalibrated Views

$F = U \Sigma V^T$

where $\Sigma = diag\{\sigma_1, \sigma_2, 0\}$, $det(F) = 0$

Normalization of 8-Point:

Transform image to $[-1, 1] \times [-1, 1]$

1. Compute centroid (c_1, c_2) and shift origin to centroid.
2. Compute mean distance $(\bar{d})$ and scale to $s = \sqrt{2}/\bar{d}$
3. Transform the image coordinates according to $\hat{x}_1 = T_1 x_1$ and $\hat{x}_2 = T_2 x_2$

$$T = \begin{bmatrix} s & 0 & -s.c_1 \\ 0 & s & -s.c_2 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow F = T_2^T \hat{F} T_1$$

Table 6.4 **Calibrated vs Uncalibrated Epipolar**

	Calibrated	**Uncalibrated**
Image point	$x \sim RX + T$	$x' = Kx \sim R'X' + T'$
Cam (motion)	$g = (R, T)$	$g' = (KRK^{-1}, KT)$
Geometry cons	$x_2^T E x_1 = 0$	${x_2'}^T F x_1' = 0$
Matrices	$E = \widehat{T}R$ $= K^T F K$	$F = \widehat{T'} R'$ $= K^{-T} E K^{-1}$ [1] $= \widehat{T'} K R K^{-1}$
Epipoles	$E e_1 = 0$ $e_2^T E = 0$	$e_1 = K R^T T, F e_1 = 0,$ $e_2 = KT = T', e_2^T F = 0$
Epipolar lines	$l_1 \sim E^T x_2$ $l_2 \sim E x_1$	$l_1 \sim F^T x_2'$ $l_2 \sim F x_1'$, $l = \widehat{e} x$
Decomposition	$E \rightarrow [R, T]$ 5(3+2) DOF	$F \rightarrow [\widehat{T'}^T F, T']$ 8(9-1) DOF
Reconstruction	$Euclidean : X_e$	$Projective : X_p = H X_e$

Table 6.5. (not complete) Geometric Stratification

	Euclidean	**Affine**	**Projective**
Struc	$X_e = g_e X$ $= H_a^{-1} X_a$	$X_a = H_a X_e$ $= H_p^{-1} X_p$	$X_p = H_p X_a$
Trans	$g_e = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}$	$H_a = \begin{bmatrix} K & 0 \\ 0 & 1 \end{bmatrix}$	$H_p = \begin{bmatrix} I & 0 \\ -v^T v_4^{-1} & v_4^{-1} \end{bmatrix}$
Proj	$\Pi_e = [KR, KT]$	$\Pi_a = \Pi_e H_a^{-1}$	$\Pi_p = \Pi_a H_p^{-1}$

C10: Symmetry

- Symmetric structures: There exist several vantage points from which they appear identical (Equivalent Views).
- Fundamental Types of Symmetry:
  - Rotational symmetry: Obtained by rotating the board about its normal.
  - Reflective symmetry.
  - Translational symmetry.

$$x \sim \Pi_0 g_0 X \Rightarrow g(x) \sim \Pi_0 g_0 g X$$

- g_0: Initial pose of 3D point p
- $g_0 g$: Virtual camera vantage point
- $g' = g_0 g g_0^{-1}$: relative transformation between the original image and the equivalent view.
- $x = (g_0 g g_0^{-1})(g_0 X)$: Coordinate of 2D point, relative to the virtual camera coordinate.

$$g' = \begin{cases} R' = R_0 R R_0 \in O(3) \\ T' = (I - R')T_0 + R_0 T \in \mathbb{R}^3 \end{cases}$$

Symmetry-Based Reconstruction:

1. Two pairs of symmetric image points.
2. Recover essential matrix (or homography)
3. Decompose E (or H) to obtain $\{R', T', N\}$
4. Solve Lyapunov equation $R' R_0 - R_0 R = 0$, to obtain R_0 & T_0.

In reflection symmetry, we have $R' = I - 2T'(T')^T$, if $|T'| = 1$ so $E = \widehat{T}R = \widehat{T} \rightarrow$ To recover, only two pairs of symmetric points are needed (3 DoF for T').

* Alignment of Two Symmetric Objects in One Image $\Rightarrow$ calculate Relative pose, intersection line

$\alpha = \frac{d_2}{d_1} = \frac{N_1^T x}{N_2^T x} \qquad g_2 \leftarrow [R_2, \alpha T_2], g_{21} = g_2 g_1^{-1}$

* Alignment of Two Images through the Same Symmetric Cell $\Rightarrow$ calculate scale factor

No use of the homography between cells $\Rightarrow$ baseline independent

C11: Building of a 3D Model from Images

Pipeline: 1-detection. 2-matching. 3-epipolar geometry (F-matrix). 4-Two-view reconstruction. 5-Incrementally addition of more views (Bundle Adjustment). 6-projective reconstruction + Euclidean Upgrade. 7-Auto-calibration. 8-Epipolar Rectificatino+Dense stereo matching. 9-Structure Triangulation+Texture mapping.

Feature correspondence

1. Feature tracking (narrow baseline): Interframe motion
2. Feature matching (wide baseline): Detect features independently in each image

SSD: $\min_d E(d) \doteq \sum_{\tilde{x} \in W(x)} [I_2(\tilde{x} + d) - I_1(\tilde{x})] \rightarrow$ see: C4 image velocity

normalized cross-correlation (NCC):

$$NCC(A, d, x) = \frac{\sum_{\tilde{x} \in W(x)} (I_1(\tilde{x}) - \bar{I}_1)(I_2(A\tilde{x} + d) - \bar{I}_2)}{\sqrt{\sum_{\tilde{x} \in W(x)} (I_1(\tilde{x}) - \bar{I}_1)^2 \sum_{\tilde{x} \in W(x)} (I_2(A\tilde{x} + d) - \bar{I}_2)^2}}$$

where $NCC(A, d, x) \in [-1, 1]$ 1=most similar $NCC > \tau$

$\bar{I}_1 = \frac{1}{N} \sum_{\tilde{x} \in W(x)} I_1(\tilde{x})$, $\bar{I}_2 = \frac{1}{N} \sum_{\tilde{x} \in W(x)} I_2(\tilde{x})$

Sampson distance: $d^j \doteq \frac{({x_2^j}^T F x_1^j)^2}{\|\widehat{e}_3 F x_1^j\|^2 + \|{x_2^j}^T F \widehat{e}_3\|^2}$

Rectification

we are looking for $H_1, H_2 \in \mathbb{R}^{3\times 3}$ that satisfy:

$$H_1 e_1 \sim [1, 0, 0]^T,\ H_2 e_2 \sim [1, 0, 0]^T$$

Rectification Makes the epipolar lines in parallel.

- **Find** H_2
  1. Translates the image center $[O_x, O_y, 1]^T$ to the origin $[0, 0, 1]^T$.
  2. Rotates around the z-axis for the epipole to lie on x-axis $\alpha = atan(-y_e/x_e)$
  3. Transforms the epipole from x-axis to infinity
  4. $H_2 = G G_R G_T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/x_e & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & O_x \\ 0 & 1 & O_y \\ 0 & 0 & 1 \end{bmatrix}$
- **Find** H_1
  1. $H_1 = H_2 H$, $H = (\widehat{T'})^T F + T' v^T$ since $v \in \mathbb{R}^3$ can be arbitrary.
  2. Choose v in such a way that the distance between x_2' and $H x_1'$ for previously matched feature points is minimized.

$$\min_v \sum_{j=1}^n \| \widehat{{x_2'}^j} ((\widehat{T'})^T F + T' v^T) {x_1'}^j \|$$

[1]From MaSKS Lemma 5.4, we have the identity $K^{-T} \widehat{T} K^{-1} = \widehat{T'}$ when $det(K) = +1$

$$H = H_p H_a H_e = \underbrace{\begin{bmatrix} I & 0 \\ v^T & 1 \end{bmatrix}}_{\text{Projective}} \underbrace{\begin{bmatrix} K & 0 \\ 0^T & 1 \end{bmatrix}}_{\text{Affine}} \underbrace{\begin{bmatrix} R & T \\ 0^T & 1 \end{bmatrix}}_{\text{Euclidean}}$$

Taylor series:

$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots .$

$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

It can be shown that the distance d from point $P(x_0, y_0)$ to the line $ax + by + c = 0$ is equal to:
$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2+b^2}}$

References:

[1] Y. Ma, S. Soatto, J. Kosetska, and S. Sastry, *An invitation to 3D computer vision.* Springer-Verlag, New York, 2004. `MaSKS`

[2] Hartley, R. I. & Zisserman, A. second (Ed.) *Multiple view geometry in computer vision.* Cambridge University Press, 2004

[3] Prof. Shohreh Kasaei, *Advance vision course notes*, spring 2014.

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Tables 1 Some usefull tables

Transformations

Group	**2D Matrix**	**DOF**	**Group**	**2D Matrix**	**DOF**
Translation	$\begin{bmatrix}1 & 0 & t_x\\ 0 & 1 & t_y\\ 0 & 0 & 1\end{bmatrix}$	2D=2, 3D=3	Rotation	$\begin{bmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}$	2D=1, 3D=3
Scale	$\begin{bmatrix}S_x & 0 & 0\\ 0 & S_y & 0\\ 0 & 0 & 1\end{bmatrix}$	2D=2, 3D=3	Sheer	$\begin{bmatrix}0 & Sh_x & 0\\ Sh_y & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$	2D=2, 3D=3
Euclidean (Rigid)	$\begin{bmatrix}r_{11} & r_{12} & t_x\\ r_{21} & r_{22} & t_y\\ 0 & 0 & 1\end{bmatrix}$	2D=3, 3D=6	Similarity (metric)	$\begin{bmatrix}sr_{11} & sr_{12} & t_x\\ sr_{21} & sr_{22} & t_y\\ 0 & 0 & 1\end{bmatrix}$	2D=4, 3D=7
Affine	$\begin{bmatrix}a_{11} & a_{12} & t_x\\ a_{21} & a_{22} & t_y\\ 0 & 0 & 1\end{bmatrix}$	2D=6,3D=12	Projective	$\begin{bmatrix}p_{11} & p_{12} & p_{13}\\ p_{21} & p_{22} & p_{23}\\ p_{31} & p_{32} & p_{33}\end{bmatrix}$	2D=8,3D=15

Table 2.1. Rotation and rigid-body motion in 3-D space.

	Rotation SO(3)	Rigid-body motion SE(3)
Matrix representation	$R : \begin{cases} R^T R = I \\ det(R) = 1 \end{cases}$	$g = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}$
Coordinates (3-D)	$X = RX_0$	$X = RX_0 + T$
Inverse	$R^{-1} = R^T$	$g^{-1} = \begin{bmatrix} R^T & -R^T T \\ 0 & 1 \end{bmatrix}$
Composition	$R_{ik} = R_{ij} R_{jk}$	$g_{ik} = g_{ij} g_{jk}$
Exp. representation	$R = exp(\widehat{\omega})$	$g = exp(\widehat{\xi})$
Velocity	$\dot{X} = \widehat{\omega} X$	$\dot{X} = \widehat{\omega} X + v$
Adjoint map	$\widehat{\omega} \mapsto R \widehat{\omega} R^T$	$\widehat{\xi} \mapsto g \widehat{\xi} g^{-1}$

Table 5.3+ my Modifications

	Epipolar constraint	**(Planar) Homography Geometry**
Image point	$x_2 \sim Rx_1 + T$, $X_2 = RX_1 + T$	$x_2 \sim Hx_1$, $X_2 = HX_1$
Geometry constraint	$x_2^T E x_1 = 0$	$\widehat{x_2} H x_1 = 0$
Epipolar lines	$l_1 \sim E^T x_2, l_2 \sim Ex_1$	$l_2 \sim \widehat{x_2} H x_1, l_1 \sim H^T l_2$
Matrices	$E = \widehat{T} R$	$H = R + \frac{1}{d} T N^T$
Map	point $\to$ line	point $\to$ point
Relation	$\exists v \in \mathbb{R}^3, H = \widehat{T}^T E + T v^T$	$E = \widehat{T} H$ $\quad H^T E + E^T H = 0$
Continuous motion	$x^T \widehat{\omega} \widehat{v} x + u^T \widehat{v} x = 0$	$\widehat{x}(\widehat{\omega} + \frac{1}{d} v N^T) X = \widehat{u} X$
Matrices	$E = \begin{bmatrix} \frac{1}{2}(\widehat{\omega}\widehat{v} + \widehat{v}\widehat{\omega}) \\ \widehat{v} \end{bmatrix}$	$H = w + \ vNT$
Linear Algorithms	8 points	4 points
Decomposition	1 possible solution (5 DOF)	2 possible solutions (8 DOF)

Table 6.4 **Calibrated vs Uncalibrated Epipolar**

	Calibrated Case	**Uncalibrated Case**
Image point	$x \sim RX + T$	$x' = Kx \sim R'X' + T'$
Camera (motion)	$g = (R, T)$	$g' = (KRK^{-1}, KT)$
Geometry constraint	$x_2^T E x_1 = 0$	${x'_2}^T F x'_1 = 0$
Matrices	$E = \widehat{T} R = K^T F K$	$F = \widehat{T'} R' = K^{-T} E K^{-1} = \widehat{T'} K R K^{-1}$ †
Epipoles	$Ee_1 = 0, e_2^T E = 0$	$e_1 = KR^T T, Fe_1 = 0,$ $e_2 = KT = T', e_2^T F = 0$
Epipolar lines	$l_1 \sim E^T x_2, l_2 \sim Ex_1$	$l_1 \sim F^T x'_2, l_2 \sim Fx'_1$, $l = \widehat{e} x$
Decomposition	$E \to [R, T]$, 5(3+2) DOF	$F \to [\widehat{T'}^T F, T']$, 8(9-1) DOF
Reconstruction	$Euclidean : X_e$	$Projective : X_p = HX_e$

Table 6.5. (not complete) Geometric Stratification

	Euclidean	**Affine**	**Projective**
Structure	$X_e = g_e X = H_a^{-1} X_a$	$X_a = H_a X_e = H_p^{-1} X_p$	$X_p = H_p X_a$
Transformation	$g_e = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}$	$H_a = \begin{bmatrix} K & 0 \\ 0 & 1 \end{bmatrix}$	$H_p = \begin{bmatrix} I & 0 \\ -v^T v_4^{-1} & v_4^{-1} \end{bmatrix}$
Projection	$\Pi_e = [KR, KT]$	$\Pi_a = \Pi_e H_a^{-1} = [KRK^{-1}, KT]$	$\Pi_p = \Pi_a H_p^{-1} = [KRK^{-1} + KTv^T, v_4 KT]$

Summary of (Auto)Calibration Methods slide 6-103

$P = [\widehat{e_2} F | e_2]$