HW-2.1

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Part-1

①: Gradient vector for $z = f(x,y) = ax+by+c$

Solution: $\nabla f(x,y) = \begin{bmatrix} \frac{df}{dx}(x,y) \\ \frac{df}{dy}(x,y) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$

②: Gredient vector for $f(x) = a_1x_1 + \cdots a_Nx_N + d$

Solution: $\nabla f(x) = \begin{bmatrix} \frac{df}{dx_1}(x) \\ \vdots \\ \frac{df}{dx_N}(x) \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix}$

③: Solution:

$z = A(x-x_0)^2 + B(y-y_0)^2 + C$

$f_x(x,y) = \left(\frac{d f(x,y)}{dx}\right)_y = A(x^2 - 2x_0 x + x_0^2) + B(y-y_0)^2 + C$

$= 2Ax - 2Ax_0$

$f_y(x,y) = \left(\frac{d f(x,y)}{dy}\right)_x = 2By - 2By_0$

④. Solution for.

x^T: $x = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}_{[3\times1]} \Rightarrow x^T = (3\ 1\ 4)_{[1\times3]}$, y^T: $y = [2\ 5\ 1]_{[1\times3]} \Rightarrow y^T = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}_{[3\times1]}$

$B^T = \begin{pmatrix} 3 & 5 & 1 \\ 5 & 2 & 4 \end{pmatrix}$, $x \cdot x = 9+1+16 = 26$

$x \cdot y^T = (3\ 1\ 4) \cdot (2\ 5\ 1) = 6+5+4 = 15$

$x \times y = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \times (2\ 5\ 1) = \begin{pmatrix} 6 & 15 & 3 \\ 2 & 5 & 1 \\ 8 & 20 & 4 \end{pmatrix}$. $y \times x = (2\ 5\ 1) \times \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = 6+5+4 = (15)$

$A \times x = \begin{bmatrix} 12+5+8 \\ 9+1+20 \\ 18+4+12 \end{bmatrix} = \begin{bmatrix} 25 \\ 30 \\ 34 \end{bmatrix}$. $A \times B = \begin{bmatrix} 12+25+2, & 20+10+8 \\ 9+5+5, & 15+2+20 \\ 18+20+3, & 30+8+12 \end{bmatrix} = \begin{bmatrix} 39, & 38 \\ 19, & 37 \\ 41, & 50 \end{bmatrix}$

B.reshape(1, 6) = [3, 5, 5, 2, 1, 4]

LLS – Single variable:

$$L(p) = L(m,b) = \sum_{i=1}^{N} (\hat{y_i} - m \cdot x_i - b)^2$$

$$\frac{\partial L(m,b)}{\partial m} = -2 \sum_{i=1}^{N} (y_i - m x_i - b) \cdot x_i$$

$$= -2 \sum_{i=1}^{N} x_i y_i + 2m \sum_{i=1}^{N} x_i^2 + 2b \sum_{i=1}^{N} x_i = 0$$

$$\frac{\partial L(m,b)}{\partial b} = -2 \sum_{i=1}^{N} (y_i - m x_i - b)$$

$$= -2 \sum_{i=1}^{N} y_i + 2m \cdot \sum_{i=1}^{n} x_i + 2nb = 0$$

$$\Rightarrow \sum y_i = m \cdot \sum x_i + n \cdot b$$

$$\Rightarrow \bar{y_i} = m \bar{x_i} + b \Rightarrow b = \bar{y_i} - m \bar{x_i}$$

Therefore: $-2 \sum_{i=1}^{N} x_i y_i + 2m \sum_{i=1}^{N} x_i^2 + 2b \sum_{i=1}^{N} x_i = 0$

$$\Rightarrow \sum_{i=1}^{N} x_i y_i = m \cdot \sum_{i=1}^{N} x_i^2 + b \cdot \sum_{i=1}^{N} x_i \Rightarrow \sum_{i=1}^{N} x_i y_i = m \sum_{i=1}^{N} x_i^2 + (\bar{y} - m\bar{x}) \sum_{i=1}^{N} x_i$$

$$\Rightarrow m = \frac{\sum (x_i y_i - \bar{y} x_i)}{\sum (x_i^2 - \bar{x} \cdot x_i)} = \frac{\sum x_i y_i - \frac{1}{n}(\sum x_i)(\sum y_i)}{\sum x^2 - \frac{1}{n} \sum x_i \cdot \sum x_i} = \frac{\sum (x_i - \bar{x}) \cdot (y_i - \bar{y})}{\sum (x_i - \bar{x})^2} = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)}$$

$$\therefore b = \bar{y} - \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)} \cdot \bar{X}$$

LLS → Multi-Variable:

Let $X = \{(x_1, y_1) \cdots (x_m, y_m)\}$ $y = x \cdot \beta + \beta_0$

Then SSR ⇒ $S(\beta) = \sum (y_i - x_i \cdot \beta - \beta_0)^2$

Then taking partial derivative:

$$\frac{\partial S(\beta)}{\partial \beta} = \sum 2(y_i - x_i \beta - \beta_0) \cdot (-x_i)$$

$$= \sum (-2) \cdot (x_i y_i + x_i^2 \beta - \beta_0 x_i)$$

$$= 2 \cdot \sum (x_i^2 \beta + \beta_0 x_i - x_i y_i)$$

$$\frac{\partial S(\beta)}{\partial \beta_0} = \sum 2(y_i - x_i \beta - \beta_0) \cdot (-1)$$

$$= \sum (-2) \cdot (y_i - x_i \beta - \beta_0)$$

$$= 2 \sum (x_i \beta + \beta_0 - y_i) = 2(m \beta \frac{\sum x_i}{m} + m \beta_0 - m \cdot \frac{\sum y_i}{m}) = 0$$

$\Rightarrow \beta_0 = \bar{y} - \beta \bar{x}$ (similar as single-Variable case)

Then we let $\frac{\partial S(\beta)}{\partial \beta} = 0 \Rightarrow 2 \sum [x_i^2 \beta + (\bar{y} - \beta \bar{x}) \cdot x_i - x_i y_i] = 0$

$$\beta = \frac{\sum x_i y_i - y \sum x_i}{\sum x_i^2 - \bar{x} \cdot \sum x_i} = \frac{\sum x_i y_i - \bar{y} \sum x_i - m\bar{y}\bar{x} + m\bar{y}\bar{x}}{\sum x_i^2 - 2\bar{x} \sum x_i + \bar{x} \sum x_i} = \frac{\sum (x_i y_i - \bar{y} x_i - y_i \bar{x} + \bar{y} \bar{x})}{\sum (x_i - \bar{x})^2}$$

$$= \frac{\sum (x_i - \bar{x}) \cdot (y_i - \bar{y})}{\sum (x_i - \bar{x})^2}$$

$$S(\beta) = \sum_{i=1}^{M} |y_i - \sum_{j=1}^{n} x_{ij} \cdot \beta_j|^2 = \| y - X \beta^T \|^2$$

$y - X\beta^T = 0 \Rightarrow X \cdot \beta^T = y$ $X^T X \beta^T = X^T y$

$\Rightarrow \beta^T = (X^T X)^{-1} X^T y$: solved.