Analytical Solution for Bending of Coaxial Orthotropic Cylinders

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Abstract: A general analytical solution is obtained for the stresses and displacements of an elastic body consisting of an assembly of coaxial hollow circular cylinders made of orthotropic material, with or without a core, and subjected to bending, tensile and torsion loads. Two types of conditions are considered at the interfaces between cylinders: no slip and no friction. A numerical application is used to illustrate the theoretical results. Results show that there is no coupling between bending and tension-torsion and that there is no deviation in the bending curvature. It was finally found that some warping of the cross section develops under bending, meaning that the Bernoulli-Euler hypothesis would not strictly apply in the case of orthotropic cylinders.

INTRODUCTION

This work addresses the problem of obtaining an analytical solution for the stresses and displacements of an elastic body that consists of an assembly of several coaxial thick-walled circular hollow cylinders made of orthotropic material, with or without a core, and subjected to tensile, torsion and bending loads. The cylinders under study are homogeneous and have cylindrical anisotropy, a case characterized by the fact that elastic constants appearing in the constitutive equation are constant in a cylindrical coordinate system. This may be opposed to the case called rectilinear anisotropy where elastic constants are constant in a Cartesian coordinate system.

There are many applications for such bodies and for the solution sought here. Fiber reinforced composites play an important role in modern technology and are a direct application. Also, this solution may be used for many types of conductors where we find numerous layers of protection or insulation material, metallic wires and/or optic fibers, which are helically laid and may be modeled as orthotropic tubes.

Basic equations of the mathematical theory of anisotropic elasticity are available from several sources, most notably, from Lekhniskii (1949, 1981). It is his approach which will be used in the present work. For the case of cylindrical anisotropy, he has obtained the set of partial differential equations that represents the problem at hand but has only worked out the simpler case of one cylinder subjected to axisymmetric loading [a solution to that particular problem has also been given independently by Blouin and Cardou (1988)]. He has also solved the cylinder bending problem for a particular case of anisotropy; namely, orthotropy with material axes coinciding with the cylindrical coordinates. No solution appears to have been published to date in the open literature for the general case of cylindrical anisotropy. It should be noted however, that for the case of rectilinear anisotropy, which is a mathematically different problem, general beam bending

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has been studied by Lekhniskii (1942), and a general solution has recently been obtained by Martynovich and Martynovich (1992). Other related works deal with thin-walled laminated composite tubes considered either, as shell elements (Reis and Goldman 1987) or, as thin orthotropic layers (Hobbs and Raoof 1982). In particular, an exact solution has been given for laminated anisotropic cylindrical shells by Bhaskar and Varadan (1993) with layer constitutive equations similar to those being used here. However, overall geometry (circular arc shells vs. circular cylinders), boundary conditions and applied loads are quite different. This leads to a solution method (based on a plane strain assumption) and results which have little in common with those being presented here.

In this paper, a solution will first be obtained for one cylindrically orthotropic tube subjected to the loads shown in Fig. 1. Boundary conditions between two cylinders will then be studied to obtain a solution for multilayered systems. A numerical application will be used to illustrate the theoretical results.

DESCRIPTION OF PROBLEM

The system under study is shown in Fig. 1. It is a hollow cylinder, with internal and external radii a and b, made of homogeneous orthotropic material, and subjected to traction, twisting moment and bending moment. It will undergo global deformations: axial strain ε, rotation per unit length ϑ, and curvatures of the center line κ_x and κ_y. This cylinder has cylindrical anisotropy as defined by Lekhnitskii (1981). The axis of anisotropy is axis z, meaning that mechanical properties of the cylinder are axially symmetric.

For simplicity, the present analysis will be limited to material that is orthotropic, although it could be generalized for the case of general anisotropy. By definition, orthotropic material has orthogonal planes of symmetry whose intersections define the three principal directions of the material. In the present case, one of these principal directions is radial and the two others are called longitudinal and tangential. If an analogy is made with helically wound fibrous material, the longitudinal direction would correspond to the fibre axis. The angle between the longitudinal direction would correspond to the fibre axis. The angle between the longitudinal direction and the plane normal to the z-axis is called helix angle.

The following assumptions, hypotheses and simplifications are used: elastic body under small strains, constant loads along the z-axis, no shear load resultant, stresses and strains that are functions of r and θ only and are

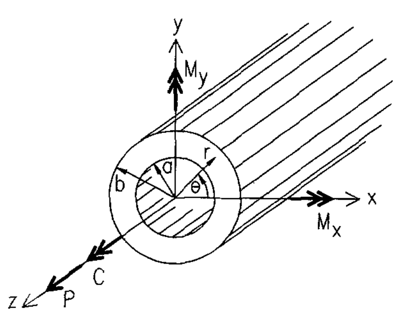

FIG. 1. Cylinder under Study and Applied Loads

independent of z. This implies constant curvature of the bent cylinder. The Bernoulli-Euler hypothesis is not used and results will in effect show warping and rotation of cross sections.

CONSTITUTIVE EQUATIONS

It is assumed that the elastic constants of the material are known in the local coordinate system of the material. This implies that nine independent constants in the directions of the three principal directions are known. A compliance matrix is formed with these constants and transformed for the different global coordinate system of the cylinder. This transformation is a rotation about radial axis, by an angle that is complement of the helix angle. More details are given by Lekhnitskii (1981). The obtained constitutive equation is in the form of (1), where the nonzero coefficients of the compliance matrix are identified

$$\begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ & C_{22} & C_{23} & C_{24} & 0 & 0 \\ & & C_{33} & C_{34} & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & \text{sym} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{\theta z} \\ \tau_{rz} \\ \tau_{r\theta} \end{Bmatrix} \qquad (1)$$

It may be emphasized that although 13 distinct constants appear in this last equation, they are derived from only nine independent constants.

Reduced elastic constants β_{ij}, which are defined by Lekhnitskii (1981) in terms of the elastic constants C_{ij} will also be used

$$\beta_{ij} = C_{ij} - \frac{C_{i3}C_{3j}}{C_{33}} \qquad (2)$$

where one may note that $\beta_{i3} = \beta_{3j} = 0$.

DISPLACEMENTS

Equations for the displacements u_r, u_θ, and w are obtained by integration of the equations of the strain tensor. Details of this procedure are given by Lekhnitskii [(1981) chapter 3], and results only are given here, for completeness. Lekhnitskii defines

$$D = C_{13}\sigma_r + C_{23}\sigma_\theta + C_{33}\sigma_z + C_{34}\tau_{\theta z} \qquad (3)$$

where D = a function of r and θ only, and uses this expression with (1) and (2), to rewrite the six equations of the strain tensor, which are then integrated. Results are obtained for D and for the displacements

$$D = \kappa_x r \sin\theta - \kappa_y r \cos\theta + \varepsilon \qquad (4)$$

$$u_r = -\frac{z^2}{2}(\kappa_x \sin\theta - \kappa_y \cos\theta) + U + u_r' \qquad (5a)$$

$$u_\theta = -\frac{z^2}{2}(\kappa_x \cos\theta + \kappa_y \sin\theta) + \vartheta rz + V + u_\theta' \qquad (5b)$$

$$w = z(\kappa_x r \sin\theta - \kappa_y r \cos\theta + \varepsilon) + W + w' \qquad (5c)$$

where ε, ϑ, and κ_i = global deformations of the cylinder, κ_i being defined as curvature in the plane perpendicular to direction i; u'_r, u'_θ, and w' = rigid-body displacements; and U, V, and W = functions of r and θ only, representing displacements caused by strains in axial position $z = 0$. Function W represents warping and rotation of the cross section, and any nonzero value for W indicates that the Bernoulli-Euler hypothesis is not valid. These functions are obtained from integration of

$$\frac{\partial U}{\partial r} = \beta_{11}\sigma_r + \beta_{12}\sigma_\theta + \beta_{14}\tau_{\theta z} + \frac{C_{13}}{C_{33}} D \tag{6a}$$

$$\frac{\partial V}{r\partial \theta} + \frac{U}{r} = \beta_{12}\sigma_r + \beta_{22}\sigma_\theta + \beta_{24}\tau_{\theta z} + \frac{C_{23}}{C_{33}} D \tag{6b}$$

$$\frac{\partial V}{\partial r} + \frac{\partial U}{r\partial \theta} - \frac{V}{r} = \beta_{56}\tau_{rz} + \beta_{66}\tau_{r\theta} \tag{6c}$$

$$\frac{\partial W}{\partial r} = \beta_{55}\tau_{rz} + \beta_{56}\tau_{r\theta} \tag{6d}$$

$$\frac{\partial W}{r\partial \theta} = \beta_{14}\sigma_r + \beta_{24}\sigma_\theta + \beta_{44}\tau_{\theta z} + \frac{C_{34}}{C_{33}} D - \vartheta r \tag{6e}$$

It may be noted that term $-\vartheta r$ from the last of these equations is not present in Lekhnitskii's (1981) equations (23.10). It is assumed that this omission is a mere transcription error since it is easily shown that this term is required to ensure consistency of the equations.

STRESS FUNCTIONS

Lekhnitskii's (1981) stress functions F and Φ are used to obtain the system of partial differential equations that represents the present problem. They are defined by

$$\sigma_r = \frac{\partial F}{r\partial r} + \frac{\partial^2 F}{r^2\partial \theta^2} \tag{7a}$$

$$\sigma_\theta = \frac{\partial^2 F}{\partial r^2} \tag{7b}$$

$$\tau_{r\theta} = \frac{\partial F}{r^2\partial \theta} - \frac{\partial^2 F}{r\partial r\partial \theta} \tag{7c}$$

$$\tau_{rz} = \frac{\partial \Phi}{r\partial \theta} \tag{7d}$$

$$\tau_{\theta z} = -\frac{\partial \Phi}{\partial r} \tag{7e}$$

The six strain-compatibility equations are expressed in terms of these stress functions. Some equations are then identically satisfied since stresses do not depend on z. The remaining equations give the system

$$L_4 F + L_3 \Phi = \frac{2}{r}\frac{C_{13} - C_{23}}{C_{33}}(\kappa_x \sin\theta - \kappa_y \cos\theta) \tag{8a}$$

$$L_3'F + L_2\Phi = \frac{C_{34}}{C_{33}}\left(2\kappa_x \sin\theta - 2\,\kappa_y \cos\theta + \frac{\varepsilon}{r}\right) - 2\vartheta \qquad (8b)$$

where the L_i = differential operators of order i. They are written as

$$L_4 = -\beta_{22}\frac{\partial^4}{\partial r^4} - \frac{2\beta_{12} + \beta_{66}}{r^2}\frac{\partial^4}{\partial r^2 \partial\theta^2} - \frac{\beta_{11}}{r^4}\frac{\partial^4}{\partial\theta^4} - \frac{2\beta_{22}}{r}\frac{\partial^3}{\partial r^3}$$

$$+ \frac{2\beta_{12} + \beta_{66}}{r^3}\frac{\partial^3}{\partial r \partial\theta^2} + \frac{\beta_{11}}{r^2}\frac{\partial^2}{\partial r^2} - \frac{2\beta_{11} + 2\beta_{12} + \beta_{66}}{r^4}\frac{\partial^2}{\partial\theta^2} - \frac{\beta_{11}}{r^3}\frac{\partial}{\partial r}$$

$$L_3 = \beta_{24}\frac{\partial^3}{\partial r^3} + \frac{\beta_{14} + \beta_{56}}{r^2}\frac{\partial^3}{\partial r \partial\theta^2} + \frac{\beta_{14} - 2\beta_{24}}{r}\frac{\partial^2}{\partial r^2}$$

$$L_3' = \beta_{24}\frac{\partial^3}{\partial r^3} + \frac{\beta_{14} + \beta_{56}}{r^2}\frac{\partial^3}{\partial r \partial\theta^2} + \frac{\beta_{14} + \beta_{24}}{r}\frac{\partial^2}{\partial r^2} - \frac{\beta_{14} + \beta_{56}}{r^3}\frac{\partial^2}{\partial\theta^2}$$

$$L_2 = -\beta_{44}\frac{\partial^2}{\partial r^2} - \frac{\beta_{55}}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{\beta_{44}}{r}\frac{\partial}{\partial r}$$

System (8) is due to Lekhnitskii (1949, 1981). Its solution was, however, obtained only for some particular cases. One of the objectives of this work is to present a more general solution of this system.

Separation of Variables

With the aim of separating variables to obtain a system of ordinary differential equations, a solution is sought in the form of

$$F = f_1(r)(\kappa_x \sin\theta - \kappa_y \cos\theta) + f_2(r) \qquad (9a)$$

$$\Phi = \varphi_1(r)(\kappa_x \sin\theta - \kappa_y \cos\theta) + \varphi_2(r) \qquad (9b)$$

By substituting (9) into system (8), two distinct systems of ordinary differential equations are obtained. The first of these systems is in terms of functions f_1 and φ_1, and represents a pure bending problem

$$\left(\beta_{22}\frac{d^4}{dr^4} + \frac{2\beta_{22}}{r}\frac{d^3}{dr^3} - \frac{\beta_{11} + 2\beta_{12} + \beta_{66}}{r^2}\frac{d^2}{dr^2} + \frac{\beta_{11} + 2\beta_{12} + \beta_{66}}{r^3}\frac{d}{dr}\right.$$

$$\left. - \frac{\beta_{11} + 2\beta_{12} + \beta_{66}}{r^4}\right) f_1 + \left(-\beta_{24}\frac{d^3}{dr^3} + \frac{\beta_{14} - 2\beta_{24}}{r}\frac{d^2}{dr^2}\right.$$

$$\left. + \frac{\beta_{14} + \beta_{56}}{r^2}\frac{d}{dr}\right)\varphi_1 = \frac{2}{r}\frac{C_{13} - C_{23}}{C_{33}} \qquad (10a)$$

$$\left(-\beta_{24}\frac{d^3}{dr^3} - \frac{\beta_{14} + \beta_{24}}{r}\frac{d^2}{dr^2} + \frac{\beta_{14} + \beta_{56}}{r^2}\frac{d}{dr} - \frac{\beta_{14} + \beta_{56}}{r^3}\right) f_1$$

$$+ \left(\beta_{44}\frac{d^2}{dr^2} + \frac{\beta_{44}}{r}\frac{d}{dr} - \frac{\beta_{55}}{r^2}\right)\varphi_1 = 2\frac{C_{34}}{C_{33}} \qquad (10b)$$

The second system is in terms of functions f_2 and φ_2. It represents an axisymmetric (tension-torsion) problem

$$\left(\beta_{22}\frac{d^4}{dr^4} + \frac{2\beta_{22}}{r}\frac{d^3}{dr^3} - \frac{\beta_{11}}{r^2}\frac{d^2}{dr^2} + \frac{\beta_{11}}{r^3}\frac{d}{dr}\right) f_2 - \left(\beta_{24}\frac{d^3}{dr^3} - \frac{\beta_{14} - 2\beta_{24}}{r}\frac{d^2}{dr^2}\right)\varphi_2 = 0 \quad (11a)$$

$$\left(-\beta_{24}\frac{d^3}{dr^3} - \frac{\beta_{14} + \beta_{24}}{r}\frac{d}{dr}\right) f_2 + \left(\beta_{44}\frac{d^2}{dr^2} + \frac{\beta_{44}}{r}\frac{d}{dr}\right)\varphi_2 = \frac{C_{34}}{C_{33}}\frac{\varepsilon}{r} - 2\vartheta \quad (11b)$$

General Solution of Pure Bending Problem

System (10) is of the Cauchy-Euler type. Solution of the homogeneous system will be of the form $f_1 = Kr^{m+1}$; $\varphi_1 = Kgr^m$, where K = an arbitrary constant. Substitution of this solution into (10b) gives

$$g = \frac{\beta_{24}m^3 + (\beta_{14} + \beta_{24})m^2 - \beta_{56}m}{\beta_{44}m^2 - \beta_{55}}$$

and the characteristic equation is obtained by substitution into (10a)

$$am^6 + bm^4 + cm^2 = 0$$

with

$$a = \beta_{22}\beta_{44} - \beta_{24}^2$$

$$b = \beta_{24}(2\beta_{14} + \beta_{24} + 2\beta_{56}) - \beta_{44}(\beta_{11} + 2\beta_{12} + \beta_{22} + \beta_{66}) - \beta_{22}\beta_{55} + \beta_{14}^2$$

$$c = \beta_{55}(\beta_{11} + 2\beta_{12} + \beta_{22} + \beta_{66}) - \beta_{56}^2$$

This characteristic equation has a double root for $m = 0$ and four roots given by

$$m_{1\cdots4} = \pm\sqrt{\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}} \quad (12)$$

It may be useful to note that for all numeric cases treated up to now, this last equation has given four real distinct roots.

An additional solution is needed for the double root $m = 0$. The solution corresponding to this root is $f_1 = Kr$; $\varphi_1 = 0$. Following a standard procedure for this type of problem, the additional solution will be given by $f_1 = Kr \ln r$; $\varphi_1 = gK$, which satisfies the system if $g = \beta_{56}/\beta_{55}$.

A particular solution for the nonhomogeneous system will be in the form $f_1 = (\mu_1/2)r^3$; $\varphi_1 = \mu_2 r^2$. Substitution of this solution in (10) gives a system of two linear equations for the two unknown constants μ_1 and μ_2. The solution is written in matrix form as

$$\begin{Bmatrix}\mu_1\\ \mu_2\end{Bmatrix} = \begin{bmatrix} -2\beta_{14} - 6\beta_{24} + \beta_{56} & 4\beta_{44} - \beta_{55} \\ -\beta_{11} - 2\beta_{12} + 3\beta_{22} - \beta_{66} & 2\beta_{14} - 2\beta_{24} + \beta_{56}\end{bmatrix}^{-1} \frac{1}{C_{33}}\begin{Bmatrix}2C_{34}\\ C_{13} - C_{23}\end{Bmatrix} \quad (13)$$

This completes the solution of the first system (10) of ordinary differential equations corresponding to the pure bending problem. The solution is rewritten in a manner that will permit simplification of further equations

$$f_1 = \sum_{i=1}^{4} \frac{K_i}{m_i} r^{m_i+1} + K_5 r + K_6 r \ln r + \frac{\mu_1}{2} r^3 \quad (14a)$$

$$\varphi_1 = \sum_{i=1}^{4} K_i g_i r^{m_i} + K_6 \frac{\beta_{56}}{\beta_{66}} + \mu_2 r^2 \quad (14b)$$

where K_i are six arbitrary constants; m_i are the four roots of the characteristic equation, given by (12); g_i are four constants obtained from

$$g_i = \frac{\beta_{24} m_i^2 + (\beta_{14} + \beta_{24}) m_i - \beta_{56}}{\beta_{44} m_i^2 - \beta_{55}}; \qquad i = 1 \text{ to } 4$$

μ_1 and μ_2 are constants given by the solution of (13).

General Solution of Axially Symmetric Problem

The general solution of system (11) of ordinary differential equations, although it is a little simpler, is obtained in a similar manner as the preceding one. It has been published by Lekhnitskii (1981). It is rewritten using the present notation

$$f_2 = \sum_{i=1}^{2} \frac{K_i'}{m_i' + 1} r^{m_i'+1} + K_3' + K_4' r + \frac{K_5'}{2} r^2 + \frac{\mu_3}{3} \vartheta r^3 \quad (15a)$$

$$\varphi_2 = \sum_{i=1}^{2} \frac{K_i' g_i'}{m_i'} r^{m_i'} + K_4' \frac{\beta_{11}}{\beta_{14}} \ln r + K_5' \frac{\beta_{14} + \beta_{24}}{\beta_{44}} r + K_6'$$

$$+ \frac{C_{34}}{C_{33}\beta_{44}} \varepsilon r + \frac{\mu_4}{2} \vartheta r^2 \quad (15b)$$

where K_i' are six arbitrary constants; and m_1' and m_2' = roots of the characteristic equation.

$$m_{1,2}' = \pm \sqrt{\frac{\beta_{11}\beta_{44} - \beta_{14}^2}{\beta_{22}\beta_{44} - \beta_{24}^2}}$$

g_1' and g_2' are constants given by

$$g_i' = \frac{\beta_{14} + \beta_{24} m_i'}{\beta_{44}}; \qquad i = 1, 2$$

μ_3 and μ_4 are constants obtained from the solution of

$$\begin{Bmatrix} \mu_3 \\ \mu_4 \end{Bmatrix} = \begin{bmatrix} \beta_{14} + 2\beta_{24} & -\beta_{44} \\ 4\beta_{22} - \beta_{11} & \beta_{14} - 2\beta_{24} \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

Complete General Solution

The complete general solution of system (8) for the stress functions is written using (9) with the preceding results

$$F = (\kappa_x \sin\theta - \kappa_y \cos\theta) \left(\sum_{i=1}^{4} \frac{K_i}{m_i} r^{m_i+1} + K_5 r + K_6 r \ln r + \frac{\mu_1}{2} r^3 \right)$$

$$+ \sum_{i=1}^{2} \frac{K_i'}{m_i' + 1} r^{m_i'+1} + K_3' + K_4' r + \frac{K_5'}{2} r^2 + \frac{\mu_3}{3} \vartheta r^3 \quad (16a)$$

$$\Phi = (\kappa_x \sin\theta - \kappa_y \cos\theta) \left(\sum_{i=1}^{4} K_i g_i r^{m_i} + K_6 \frac{\beta_{56}}{\beta_{66}} + \mu_2 r^2 \right) + \sum_{i=1}^{2} \frac{K_i' g_i'}{m_i'} r^{m_i'}$$

$$+ K_4' \frac{\beta_{11}}{\beta_{14}} \ln r + K_5' \frac{\beta_{14} + \beta_{24}}{\beta_{44}} r + K_6' + \frac{C_{34}}{C_{33}\beta_{44}} \varepsilon r + \frac{\mu_4}{2} \vartheta r^2 \quad (16b)$$

STRESSES AND DISPLACEMENTS

Expressions for stresses are obtained by using this last solution and its derivatives into (7), and σ_z is obtained from (3) and (4). Constants K_5, K_3', and K_6' may be set equal to zero since the corresponding stresses vanish during the derivation process. Equations for stresses are written here in their final form, omitting the terms containing constants K_6 and K_4' which, as will be seen, must equal zero, and using constant μ_5, which will be defined subsequently

$$\sigma_r = (\kappa_x \sin\theta - \kappa_y \cos\theta) \left(\sum_{i=1}^{4} K_i r^{m_i - 1} + \mu_1 r \right)$$

$$+ \sum_{i=1}^{2} K_i' r^{m_i' - 1} + \mu_3 \vartheta r + \mu_5 \varepsilon \quad (17a)$$

$$\sigma_\theta = (\kappa_x \sin\theta - \kappa_y \cos\theta) \left(\sum_{i=1}^{4} K_i (m_i + 1) r^{m_i - 1} + 3\mu_1 r \right)$$

$$+ \sum_{i=1}^{2} K_i' m_i' r^{m_i' - 1} + 2\mu_3 \vartheta r + \mu_5 \varepsilon \quad (17b)$$

$$\tau_{r\theta} = (\kappa_x \cos\theta + \kappa_y \sin\theta) \left(-\sum_{i=1}^{4} K_i r^{m_i - 1} - \mu_1 r \right) \quad (17c)$$

$$\tau_{rz} = (\kappa_x \cos\theta + \kappa_y \sin\theta) \left(\sum_{i=1}^{4} K_i g_i r^{m_i - 1} + \mu_2 r \right) \quad (17d)$$

$$\tau_{\theta z} = (\kappa_x \sin\theta - \kappa_y \cos\theta) \left(-\sum_{i=1}^{4} K_i g_i m_i r^{m_i - 1} - 2\mu_2 r \right)$$

$$- \sum_{i=1}^{2} K_i' g_i' r^{m_i' - 1} - \mu_4 \vartheta r - \left(\mu_5 \frac{\beta_{14} + \beta_{24}}{\beta_{44}} + \frac{C_{34}}{C_{33}\beta_{44}} \right) \varepsilon \quad (17e)$$

$$\sigma_z = \frac{1}{C_{33}} [\kappa_x r \sin\theta - \kappa_y r \cos\theta + \varepsilon - C_{13}\sigma_r - C_{23}\sigma_\theta - C_{34}\tau_{\theta z}] \quad (17f)$$

Displacements are given by (5), where U, V, and W, which are functions of r and θ only, are obtained from integration of (6). Eqs. (6*a*) and (6*b*), respectively, give expressions for U and V. To satisfy (6*c*) with the expressions thus obtained, constant K_6 must equal zero. Furthermore, in the integration process a nonperiodic function of θ appears, which must equal zero in order to avoid the possibility of multiple valued displacements. This requires $K_4' = 0$; $K_5' = \varepsilon\mu_5$; where μ_5 = a constant depending on material properties

$$\mu_5 = \frac{C_{34}(\beta_{24} - \beta_{14}) + \beta_{44}(C_{13} - C_{23})}{C_{33}[\beta_{14}^2 - \beta_{24}^2 + \beta_{44}(\beta_{22} - \beta_{11})]}$$

Function W is obtained from integration of (6d) and (6e). The two expressions thus obtained are equivalent only if constants K_6 and K_4' are set equal to zero. Functions U, V, and W are finally written as

$$U = (\kappa_x \sin\theta - \kappa_y \cos\theta)\left(\sum_{i=1}^{4} K_i U_i' r^{m_i} + U_5' r^2\right) + \sum_{i=1}^{2} K_i' U_i'' r^{m_i} + u_3''\vartheta r^2 + U_4''\varepsilon r \quad (18a)$$

$$V = (\kappa_x \cos\theta + \kappa_y \sin\theta)\left(\sum_{i=1}^{4} K_i V_i' r^{m_i} + V_5' r^2\right) \quad (18b)$$

$$W = (\kappa_x \cos\theta + \kappa_y \sin\theta)\left(\sum_{i=1}^{4} K_i W_i' r^{m_i} + W_5' r^2\right) \quad (18c)$$

where

$$U_i' = \frac{1}{m_i}[\beta_{11} + \beta_{12}(m_i + 1) - \beta_{14} g_i m_i]; \qquad i = 1 \text{ to } 4$$

$$U_5' = \frac{1}{2}\left[\mu_1(\beta_{11} + 3\beta_{12}) - 2\beta_{14}\mu_2 + \frac{C_{13}}{C_{33}}\right]$$

$$U_i'' = \frac{1}{m_i'}(\beta_{11} + \beta_{12} m_i' - \beta_{14} g_i'); \qquad i = 1, 2$$

$$U_3'' = \frac{1}{2}[\mu_3(\beta_{11} + 2\beta_{12}) - \mu_4\beta_{14}]$$

$$U_4'' = \frac{1}{C_{33}}\left(C_{13} - \frac{\beta_{14}}{\beta_{44}} C_{34}\right) + \mu_5\left(\beta_{11} + \beta_{12} - \beta_{14}\frac{\beta_{14} + \beta_{24}}{\beta_{44}}\right)$$

$$V_i' = \frac{1}{m_i}[\beta_{11} + \beta_{12} - \beta_{22} m_i(m_i + 1) - g_i m_i(\beta_{14} - \beta_{24} m_i)]; \qquad i = 1 \text{ to } 4$$

$$V_5' = \frac{1}{2}\left[\mu_1(\beta_{11} + \beta_{12} - 6\beta_{22}) - 2\mu_2(\beta_{14} - 2\beta_{24}) + \frac{C_{13} - 2C_{23}}{C_{33}}\right]$$

$$W_i' = \frac{1}{m_i}(\beta_{55} g_i - \beta_{56}); \qquad i = 1 \text{ to } 4$$

$$W_5' = \frac{1}{2}(\beta_{55}\mu_2 - \beta_{56}\mu_1)$$

BOUNDARY CONDITIONS

Conditions on the cylindrical surfaces and at the ends are now studied to solve for the unknown constants K_i and K_i', and the global deformations ε, ϑ, and κ. Rigid-body displacements will also be studied to insure compatibility of displacements in an assembly of several coaxial cylinders.

Rigid-Body Displacements

Rigid-body displacements u_r', u_θ', and w', appearing in (5), are written as

$$u_r' = z(-\omega_1 \sin\theta + \omega_2 \cos\theta) + u_0' \cos\theta + v_0' \sin\theta \tag{19a}$$

$$u_\theta' = z(-\omega_1 \cos\theta - \omega_2 \sin\theta) + \omega_3 r - u_0' \sin\theta + v_0' \cos\theta \tag{19b}$$

$$w' = r(\omega_1 \sin\theta - \omega_2 \cos\theta) + w_0' \tag{19c}$$

where u_0', v_0', w_0', ω_1, ω_2, and ω_3 = respectively, translations and rotations of the body with respect to the three coordinate axes. Although it is customary in this type of problem to assume that all the rigid-body displacements are equal to zero, it is easily shown that in this case, such an assumption would overdetermine the problem: there would be too many equations for the number of unknown constants, meaning an incompatibility in the radial and tangential displacements of the cylinders. Rotations ω_i and translation w_0' along the z-axis may still be set equal to zero, but a translation in the x-y plane is required to ensure compatibility of the bent cylinders. This is due to Poisson's ratio effect, which modifies the shape of a cross section in bending. It is assumed that these translations are proportional to the curvature and may be expressed as

$$u_0' = -\upsilon\kappa_y; \qquad v_0' = \upsilon\kappa_x,$$

where υ is a constant that is to be determined for each cylinder except the first one, where it may be set equal to zero, and indeed it will be seen that this permits to obtain a solution that ensures compatibility of the bent cylinders.

Conditions on Cylindrical Boundaries

Equations for the interface between cylinder n and cylinder $n + 1$ are now written. Radius at the interface is given by b. Previous equations apply to each separate cylinder with all the parameters indexed accordingly. The index applying to the cylinder is appended to the other indices and a separating comma is used when necessary (e.g. $K_{1,n}$, $C_{22,n}$).

Two cases will be considered: perfect bonding between cylinders (no slip) and no friction. With either case, sufficient prestress is assumed so that there is no loss of contact between cylinders. Under the no slip case, there is continuity of stresses σ_r, $\tau_{r\theta}$, and τ_{rz}, and of displacements u_r, u_θ, and w. Continuity of σ_r yields two equations

$$\sum_{i=1}^{4} K_{i,n} b^{m_{i,n}-1} - K_{i,n+1} b^{m_{i,n+1}-1} = (\mu_{1,n+1} - \mu_{1,n})b \tag{20}$$

$$\sum_{i=1}^{2} K'_{i,n}b^{m'_{i,n}-1} - K'_{i,n+1}b^{m'_{i,n+1}-1}$$

$$= (\mu_{3,n+1} - \mu_{3,n})\vartheta b + (\mu_{5,n+1} - \mu_{5,n})\varepsilon \quad (21)$$

Continuity of $\tau_{r\theta}$ gives a repetition of (20) while continuity of τ_{rz} is written as

$$\sum_{i=1}^{4} K_{i,n}g_{i,n}b^{m_{i,n}-1} - K_{i,n+1}g_{i,n+1}b^{m_{i,n+1}-1} = (\mu_{2,n+1} - \mu_{2,n})b \quad (22)$$

Continuity of displacement u_r yields the following two equations

$$\upsilon_n - \upsilon_{n+1} + \sum_{i=1}^{4} K_{i,n}U'_{i,n}b^{m_{i,n}} - K_{i,n+1}U'_{i,n+1}b^{m_{i,n+1}} = (U'_{5,n+1} - U'_{5,n})b^2 \quad (23)$$

$$\sum_{i=1}^{2} K'_{i,n}U''_{i,n}b^{m'_{i,n}} - K'_{i,n+1}U''_{i,n+1}b^{m'_{i,n+1}}$$

$$= (U''_{3,n+1} - U''_{3,n})\vartheta b^2 + (U''_{4,n+1} - U''_{4,n})\varepsilon b \quad (24)$$

and continuity of u_θ and w each gives one equation

$$\upsilon_n - \upsilon_{n+1} + \sum_{i=1}^{4} K_{i,n}V'_{i,n}b^{m_{i,n}} - K_{i,n+1}V'_{i,n+1}b^{m_{i,n+1}} = (V'_{5,n+1} - V'_{5,n})b^2 \quad (25)$$

$$\sum_{i=1}^{4} K_{i,n}W'_{i,n}b^{m_{i,n}} - K_{i,n+1}W'_{i,n+1}b^{m_{i,n+1}} = (W'_{5,n+1} - W'_{5,n})b^2 \quad (26)$$

In the no-friction case, longitudinal and tangential slip between cylinders is allowed. This means that there will generally be some discontinuities in the displacements u_θ and w. There is still continuity of σ_r and u_r, with the corresponding (20), (21), (23), and (24) seen previously, and the stresses $\tau_{r\theta}$ and τ_{rz} will have a zero value at the interface $r = b$, yielding the following four equations

$$\sum_{i=1}^{4} K_{i,n}b^{m_{i,n}-1} = -\mu_{1,n}b \quad (27)$$

$$\sum_{i=1}^{4} K_{i,n+1}b^{m_{i,n+1}-1} = -\mu_{1,n+1}b \quad (28)$$

$$\sum_{i=1}^{4} K_{i,n}g_{i,n}b^{m_{i,n}-1} = -\mu_{2,n}b \quad (29)$$

$$\sum_{i=1}^{4} K_{i,n+1}g_{i,n+1}b^{m_{i,n+1}-1} = -\mu_{2,n+1}b \quad (30)$$

It may be noted that (20) is now redundant considering (27) and (28), leaving seven independent equations overall.

When there is a core, the same equations still apply to the interface between the core (indexed 0) and the cylinder indexed 1. Often, the core would be isotropic and equations for the stresses and displacements then become much simpler. However, whether the core is isotropic or orthotropic, it is easily shown that, to insure stresses are defined at the center, there remains only 3 arbitrary constants, corresponding to positive m and m' values.

Finally, the case of a free surface is considered. This case occurs at the external face of the outer cylinder and at the interior face of the inner cylinder when there is no core. Stresses σ_r, $\tau_{r\theta}$, and τ_{rz} are given the value zero on the free surface. For the external face of the outer cylinder, (27) and (29) are used with the following one, which comes from the zero value of σ_r

$$\sum_{i=1}^{2} K'_{i,n} b^{m'_{i,n}-1} = -\mu_{3,n}\vartheta b - \mu_{5,n}\varepsilon \tag{31}$$

where radius b takes the value of the external radius of the outer cylinder. In the case of the internal face of the first cylinder, (27), (29), and

$$\sum_{i=1}^{2} K'_{i,n+1} b^{m'_{i,n+1}-1} = -\mu_{3,n+1}\vartheta b - \mu_{5,n+1}\varepsilon \tag{32}$$

are used, where b takes the value of the internal radius of the first cylinder.

These equations allow construction of a system of linear equations for the solution of the unknown constants. In the case of a set of N coaxial cylinders without a core, there are two free surfaces, yielding three equations each, plus $(N - 1)$ interfaces giving seven equations each, and the value υ_1 is set to zero. This gives $7N$ equations for the same number of arbitrary constants (K_1 to K_4, K'_1, K'_2, and υ for each cylinder). In the case of a set of N coaxial tubes with a core, there is one free surface for three equations plus N interfaces yielding seven equations each, and there are $7N$ unknown constants for the N cylinders plus three for the core.

The system of linear equations thus obtained may be divided into two distinct subsystems. The first one is made of (20), (22), (23), and (25)–(30) and contains constants K_1 to K_4 and υ. It may be directly solved to give the value of the aforementioned constants for each cylinder. The second subsystem is made of (21), (24), (31), and (32), and only contains constants K'_1 and K'_2. It may only be solved when the global deformations ε and ϑ are known. This system may be written in matrix form as

$$\mathbf{M1K'} = \mathbf{M2}\begin{Bmatrix}\varepsilon\\ \vartheta\end{Bmatrix}$$

where

$$K' = \langle K'_{1,0}\, K'_{1,1}\, K'_{2,1} \cdots K'_{1,N}\, K'_{2,N}\rangle^T$$

$K'_{1,0}$ is present only if there is a core. **M1** and **M2** = matrices formed from the terms of the equations of the system.

The solution is written in terms of ε and ϑ

$$\mathbf{K'} = \mathbf{M1}^{-1}\mathbf{M2}\begin{Bmatrix}\varepsilon\\ \vartheta\end{Bmatrix} \tag{33}$$

End Conditions

End conditions are given by

$$P = \sum_{n=0}^{N} \int_0^{2\pi} \int_{a_n}^{b_n} \sigma_z r \, d\theta \, dr \tag{34a}$$

$$C = \sum_{n=0}^{N} \int_0^{2\pi} \int_{a_n}^{b_n} \tau_{\theta z} r^2 \, d\theta \, dr \tag{34b}$$

$$M_x = \sum_{n=0}^{N} \int_0^{2\pi} \int_{a_n}^{b_n} \sigma_z r^2 \sin\theta \, d\theta \, dr \tag{34c}$$

$$M_y = -\sum_{n=0}^{N} \int_0^{2\pi} \int_{a_n}^{b_n} \sigma_z r^2 \cos\theta \, d\theta \, dr \tag{34d}$$

where P, C, M_x, and M_y = external forces and moments; n = index of a cylinder; a and b = internal and external radii of each cylinder ($a = 0$ for the core). Two similar relations could be written for shear load resultants V_x and V_y. It may however be checked, using (22) or (29) and (30), that these equations give values identically equal to zero for V_x and V_y. Integration of (34a) and (34b) gives

$$\begin{aligned} P = \sum_{n=0}^{N} \frac{2\pi}{C_{33,n}} \Bigg\{ & \sum_{i=1}^{2} K'_{i,n}[C_{13,n} + C_{23,n} m'_{i,n} - C_{34,n} g'_{i,n}] \frac{a_n^{m'_{i,n}+1} - b_n^{m'_{i,n}+1}}{m'_{i,n}+1} \\ & + [\mu_{3,n}(C_{13,n} + 2C_{23,n}) - \mu_{4,n} C_{34,n}]\vartheta \frac{a_n^3 - b_n^3}{3} \\ & + \left[\mu_{5,n} \left(C_{13,n} + C_{23,n} - C_{34,n} \frac{\beta_{34,n} + \beta_{24,n}}{\beta_{44,n}} \right) \right. \\ & \left. - \frac{C_{34,n}^2}{C_{33,n}\beta_{44,n}} - 1 \right] \varepsilon \frac{a_n^2 - b_n^2}{2} \Bigg\} \end{aligned} \tag{35}$$

$$\begin{aligned} C = \sum_{n=0}^{N} 2\pi \Bigg\{ & \sum_{i=1}^{2} K'_{i,n} g'_{i,n} \frac{a_n^{m'_{i,n}+2} - b_n^{m'_{i,n}+2}}{m_{i,n}+2} + \mu_{4,n}\vartheta \frac{a_n^4 - b_n^4}{4} \\ & + \left[\mu_{5,n} \frac{\beta_{14,n} + \beta_{24,n}}{\beta_{44,n}} + \frac{C_{34,n}}{C_{33,n}\beta_{44,n}} \right] \varepsilon \frac{a_n^3 - b_n^3}{3} \Bigg\} \end{aligned} \tag{36}$$

which are written in matrix form

$$\begin{Bmatrix} P \\ C \end{Bmatrix} = \mathbf{M3K}' + \mathbf{M4} \begin{Bmatrix} \varepsilon \\ \vartheta \end{Bmatrix}$$

Matrices **M3** and **M4** are formed from the terms of (35) and (36). Column vector **K**$'$ is given by (33), and substitution yields

$$\begin{Bmatrix} P \\ C \end{Bmatrix} = \mathbf{B} \begin{Bmatrix} \varepsilon \\ \vartheta \end{Bmatrix}; \qquad \mathbf{B} = \mathbf{M4} + \mathbf{M3M1}^{-1}\mathbf{M2} \tag{37}$$

where **B** = rigidity matrix. For axially symmetric loads (tension and torsion), its terms are

$$\mathbf{B} = \begin{bmatrix} (EA) & B_{12} \\ B_{21} & (GJ) \end{bmatrix}$$

It should be noted that matrix **B** thus obtained is symmetric, meaning that coupling terms B_{12} and B_{21} are equal.

From integration of (34*c*) and (34*d*), identical equations are obtained that give the bending rigidity of the assembly of cylinders

$$M_x = (EI)\kappa_x$$

$$M_y = (EI)\kappa_y$$

$$(EI) = \sum_{n=1}^{N} \frac{\pi}{C_{33,n}} \left\{ \sum_{i=1}^{4} K_{i,n}[C_{13,n} + C_{23,n}(m_{i,n} + 1) - C_{34,n}g_{i,n}m_{i,n}] \frac{a_n^{m_{i,n}+2} - b_n^{m_{i,n}+2}}{m_{i,n}+2} + [\mu_{1,n}(C_{13,n} + 3C_{23,n}) - 2\mu_{2,n}C_{34,n} - 1] \frac{a_n^4 - b_n^4}{4} \right\} \quad (38)$$

It may be noted that there is no coupling between M_x and κ_y or between M_y and κ_x, and that curvature caused by a bending moment occurs in a plane perpendicular to the axis of the applied moment. Results obtained also show that there is no coupling between axially symmetric loads and deformations (P, C, ε, ϑ), and bending loads and deformations (M, κ). Global rigidity of the assembly of cylinders is finally expressed as

$$\begin{Bmatrix} P \\ C \\ M \end{Bmatrix} = \begin{bmatrix} (EA) & B_{12} & 0 \\ B_{21} & (GJ) & 0 \\ 0 & 0 & (EI) \end{bmatrix} \begin{Bmatrix} \varepsilon \\ \vartheta \\ \kappa \end{Bmatrix} \quad (39)$$

thus permitting to obtain global deformations of the assembly when loads are given.

The theoretical results obtained here for stresses and displacements are mathematically equivalent to those obtained by Blouin and Cardou (1988) when only tensile and torsion loads are considered. They are also equivalent to those obtained by Lekhnitskii (1981) for a particular case of the present problem where one of the principal directions of the material coincides with the z axis of the cylinder (i.e. the helix angle is 90°), this cylinder being under an axial force and a bending moment. Also, numerical results obtained with the present equations are very close to those obtained by Étienne (1991) with the Finite Element Method for bending and axially symmetric loads.

NUMERICAL APPLICATION

Some numerical results are now given for a simple example assembly of two concentric cylinders without a core, whose characteristics are given in Tables 1 and 2. The C_{ij}-values given are those of (1) after transformation of the compliance matrix for helix angle. The rigidity of the assembly (i.e. the elements of the rigidity matrix in (39)) is given in Table 3 for the two cases (no slip and no friction) of conditions on the interface. It is interesting to note in these results that the type of condition on the interface has influence only on the bending rigidity. This is a consequence of the fact that tensile and torsion loads do not induce slip.

TABLE 1. Geometrical Properties of Cylinders

Geometrical property (1)	Cylinder 1 (2)	Cylinder 2 (3)
a (mm)	2	8
b (mm)	8	14
Helix angle (degrees)	105	65

TABLE 2. Material Properties of Cylinders

Material property (1)	Cylinder 1 (mm²/N) (2)	Cylinder 2 (mm²/N) (3)
c_{11}	1.052632E-004	1.052632E-004
c_{12}	−6.456815E-006	−6.691803E-006
c_{13}	−8.280027E-006	−8.045040E-006
c_{14}	1.052632E-006	−1.612725E-006
c_{22}	1.263265E-004	1.359339E-004
c_{23}	−2.463235E-005	−4.513900E-005
c_{24}	−2.831390E-005	2.255642E-005
c_{33}	4.176170E-005	7.316760E-005
c_{34}	7.713743E-005	−9.735841E-005
c_{44}	3.391176E-004	2.570911E-004
c_{55}	4.832532E-004	4.553485E-004
c_{56}	−6.250000E-005	9.575556E-005
c_{66}	2.667468E-004	2.946515E-004

TABLE 3. Stiffness Coefficients of Assembly of Cylinders

(1)	Unit (2)	No slip (3)	No friction (4)
EA	MN	19.8807	19.8807
B_{12}	kN·m	43.0542	43.0542
GJ	N·m²	483.3007	483.3007
EI	N·m²	707.7323	498.4763

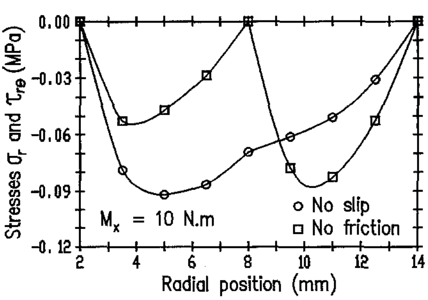

FIG. 2. Radial Stress σ_r (at θ = 90°) and Shear Stress $\tau_{r\theta}$ (at θ = 180°) of Example Assembly

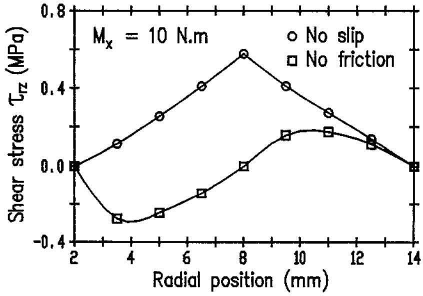

FIG. 3. Shear Stress τ_{rz} (at θ = 0°) of Example Assembly

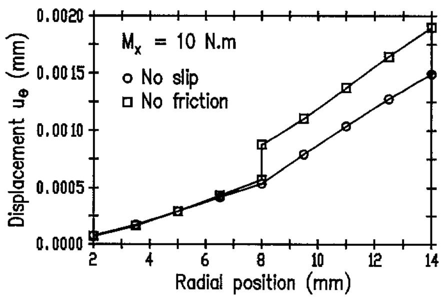

FIG. 4. Tangential Displacement (at θ = 0° and z = 0) of Example Assembly

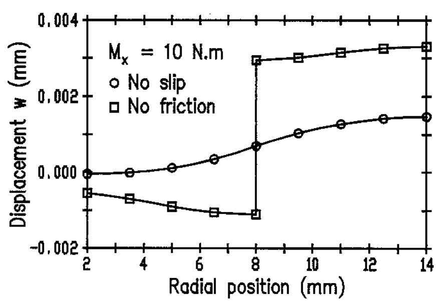

FIG. 5. Axial Displacement (at θ = 0° and z = 0) of Example Assembly

Results for stresses and displacements due to an axial load are not given here since such results have been obtained by Blouin and Cardou (1989). Figs. 2–5 give some of the stresses and displacements caused by a 10 N·m M_x bending moment applied to the previously described assembly of cylinders, and for the two cases of conditions on the interface between the cylinders. These stresses and displacements are sine or cosine functions of the angular position. From (33), (38), and (39), it may be seen that under a M_x bending moment, κ_y, ε, ϑ and the K_i' are all equal to zero, leaving only the sin θ terms for σ_r, σ_θ, σ_z, $\tau_{\theta z}$ and u_r, and the cos θ terms for $\tau_{r\theta}$, τ_{rz}, u_θ, and w in (17) and (18). Thus, Figs. 2–5 were drawn for the angular position of maximum stress or displacement. It may also be noted in (17) that σ_r has the same value (under a pure bending moment) as $\tau_{r\theta}$ except for the sign and phase differences; they have thus been combined in Figs. 2–5.

It is interesting to observe the results obtained at the radial position of 8 mm, corresponding to the interface between the two cylinders.

Results on Fig. 2 for σ_r show that there is no pressure exerted between the cylinders under the no friction condition. Under the no slip condition, there would be a positive radial stress at an angular position of 270°, indicating a tendency for the two cylinders to separate, which has to be counterbalanced either by a sufficient axial prestress or by the bonding between cylinders.

The slips that occur under no friction condition are well seen in Figs. 4 and 5, being represented by the discontinuities of u_θ and w curves. Longitudinal slip is more important than tangential slip by an order of magnitude. Maximum slip occurs on the neutral plane.

The friction required to prevent slipping is seen in Figs. 2 and 3 with the no slip curves of $\tau_{r\theta}$ and τ_{rz}. Total friction required would be the vectorial sum of the two shear stresses. If there is no bonding between cylinders, this friction may only be produced from a negative radial stress. Since σ_r is zero on the neutral plane, stresses caused by bending cannot help to develop friction required to prevent slipping. A sufficient traction on the cylinders could however produce the required radial stress.

An important result is shown by the w curves in Fig. 5. They represent warping of the cross section. This result indicates that the Bernoulli-Euler hypothesis may not be rigorously applicable for orthotropic cylinders under bending. The effect is slightly less important in the no slip condition because material longitudinal principal directions being in opposite directions, the two cylinders counteract each other.

CONCLUSIONS

The primary objective of this work was to derive equations describing the behaviour of a system of coaxial orthotropic cylinders under bending, tensile and torsion loads. To meet this objective, Lekhnitskii's (1981) stress functions F and Φ were used to form a system of two partial differential equations in terms of r and θ, which were analytically solved to yield equations for stresses and displacements of one orthotropic cylinder. The behaviour of an assembly of several cylinders was obtained using adequate boundary conditions at interfaces between cylinders for two limiting cases, namely no slip and no friction. To reach this solution, simplifications were made: cylinders must have cylindrical anisotropy, and stresses and strains were assumed constant along the axis of the cylinder, meaning that only constant curvature may be modeled and that there may not be any shear load resultant.

Theoretical results obtained show that, if there are no end effects, there will be no coupling between bending and tension-torsion, meaning that a cylinder under bending will not elongate or rotate, and vice versa. It was also found that the curvature caused by a bending moment occurs in a plane perpendicular to the axis of the applied moment, without deviation.

Numerical results obtained for a simple application have permitted to evaluate slip of one cylinder with respect to the other when there is no friction, and to evaluate the amount of friction required to prevent slipping. It was also shown that, under the no slip case, a positive radial stress develops at some angular positions on the interface between the cylinders, and it has to be counterbalanced either by a sufficient axial prestress or by bonding between cylinders to prevent loss of contact. It was finally found that some warping of the cross section develops under bending, meaning that the

Bernouli-Euler hypothesis would not strictly apply in the case of these orthotropic cylinders.

ACKNOWLEDGMENTS

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APPENDIX II. NOTATION

The following symbols are used in this paper:

a = internal radius of cylinder;
$\mathbf{B}$, B_{ij} = rigidity matrix, its elements;
b = external radius of cylinder, also interface radius;
C = twisting moment;
$\mathbf{C}$, C_{ij} = compliance matrix, its elements;
D = function of r and Θ;
(EA) = axial rigidity of assembly of cylinders;
(EI) = bending rigidity of assembly of cylinders;
F, f = stress functions;
(GJ) = twisting rigidity of assembly of cylinders;
g, g' = constants;
K, K' = integration constants;
M, M_x, M_y = bending moment;
m, m' = roots of characteristic equations;

N = number of layers (cylinders);
P = axial load (traction);
r = radial coordinate;
U = radial displacement at $z = 0$;
u_r = radial displacement;
u_r' = radial rigid-body displacement;
u_θ = tangential displacement;
u_θ' = tangential rigid-body displacement;
V = tangential displacement at $z = 0$;
W = axial displacement at $z = 0$;
w = axial displacement;
w' = axial rigid-body displacement;
z = axial coordinate;
β_{ij} = reduced elastic constants;
γ = shear deformation;
ε = deformations;
θ = tangential coordinate;
ϑ = unit rotation of assembly of cylinders;
κ, κ_x, κ_y = curvatures of assembly of cylinders;
$\mu_{1\cdots5}$ = constants;
σ = stress;
τ = shear stress;
υ = rigid-body displacement; and
Φ, φ = stress functions.