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Some observations on localisation in non-local and gradient damage models

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ABSTRACT. – Classical continuum descriptions of material degradation may cease to be mathematically meaningful in case of softening-induced localisation of deformation. Several enhancements of conventional models have been proposed to remove this deficiency. The properties of two of these so-called regularisation methods, the non-local and the gradient approach, are examined and compared in a continuum damage context. It is shown that the enhanced models allow for the propagation of waves in the softening zone, in contrast to conventional damage models. For both types of enhancement wave propagation becomes dispersive. The behaviour under quasi-static loading conditions is studied numerically. Finite element simulations of a one-dimensional problem yield quite similar results.

1. Introduction

Many engineering materials show a more or less gradual decrease of their load-carrying capacity when deformed beyond a certain limit. This so-called softening behaviour acts as a precursor to complete rupture. Conclusive experimental evidence indicates that softening is a structural effect rather than an intrinsic material property (Read and Hegemier, 1984). Indeed, the mechanisms responsible for the softening behaviour can be accounted for in a natural fashion in micro-mechanical material models (*e.g.* Vosbeek, 1994; Vervuurt *et al.*, 1995). Nevertheless, numerical analyses based on these models are not suited for large scale simulations since they demand a tremendous computational effort.

An alternative is the description of softening at a continuum level. However, the straightforward application of strain softening in classical plasticity or continuum damage theory normally leads to erroneous results. Numerical analyses show an extreme sensitivity with respect to the fineness and direction of the spatial discretisation that is employed. Upon refinement of the discretisation, convergence is observed to a solution in which deformation is localised in an infinitely narrow band. This response is unacceptable from a physical point of view because it does not allow for any energy dissipation in

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the fracture process (Pietruszczak and Mróz, 1981; Bažant *et al.*, 1984; Bažant and Belytschko, 1985).

The underlying mathematical cause of the mesh dependence is that the partial differential equations which govern the deformation process locally lose hyperbolicity in dynamics and ellipticity in quasi-static analyses, rendering the initial-boundary value problem ill-posed (Triantafyllidis and Aifantis, 1986; Lasry and Belytschko, 1988; de Borst *et al.*, 1993). If the continuum context is to be preserved, the mathematical model must be enriched by additional terms to prevent this change of type and to allow for a correct description of localisation of deformation. A number of approaches can be distinguished here (*see* de Borst *et al.*, 1993 for a review), which share the property that an internal length scale is introduced in the material description, which regularises the localisation process.

Two of the regularisation techniques, namely the non-local and gradient approaches, have been connected already at an early stage (Beran and McCoy, 1970; Bažant, 1984; Lasry and Belytschko, 1988; Mühlhaus and Aifantis, 1991). The fact that gradient regularisation has been applied primarily in plasticity models (Aifantis, 1984; Schreyer and Chen, 1986; Mühlhaus and Aifantis, 1991; de Borst and Mühlhaus, 1992), whereas non-local theory has found its application almost exclusively in damage mechanics (Pijaudier-Cabot and Bažant, 1987; Bažant and Pijaudier-Cabot, 1988; Pijaudier-Cabot and Huerta, 1991), may have obscured this relation somewhat. A profound knowledge of the intrinsic similarities and differences between the two regularisation methods does not seem to exist. This paper is an attempt to partly fill this gap. To this end, the behaviour of a recently developed gradient-damage model for quasi-brittle fracture (Peerlings *et al.*, 1995, 1996) is examined in a one-dimensional context for both dynamics and statics and is compared with that of the underlying non-local formulation.

After recalling briefly the relevant constitutive relations, it is demonstrated that in a one-dimensional setting the gradient enhancement prevents the change of type of the governing equations. Then, the gradient formulation is compared to the non-local model with regard to wave propagation. An analytical analysis of the dispersive character of the wave propagation has been carried out for this purpose. The performance of both methods under quasi-static loading conditions is studied numerically. Finally, some conclusions are drawn on the basis of the presented results.

2. Constitutive relations

In order not to obscure the essentials of the discussion with the complexity accompanying a three-dimensional analysis, this study is limited to the one-dimensional case. An infinitely long cylindrical bar is considered, so that the delicate issue of non-standard boundary conditions need not be addressed (Mühlhaus and Aifantis, 1991; Peerlings *et al.*, 1995, 1996). The classical constitutive equation for quasi-brittle damage (*e.g.* Kachanov, 1986; Krajcinovic and Lemaitre, 1987) then reduces to

$$\sigma = (1 - D)E\varepsilon, \tag{1}$$

with σ and ε denoting the axial stress and strain, respectively, and E being Young's modulus of the undamaged material. The damage variable D is zero for the initial, undamaged material, while $D = 1$ characterises the complete loss of material coherence. In the standard, local model, damage evolution is driven by the deformation through the history variable κ, which represents the severest deformation the material has experienced. For the one-dimensional tensile case, κ is determined by the initial threshold κ_i and the Kuhn-Tucker conditions

$$\dot{\kappa} \geq 0, \qquad \varepsilon - \kappa \leq 0, \qquad \dot{\kappa}(\varepsilon - \kappa) = 0, \tag{2}$$

where a superimposed dot designates a time derivative. In this contribution, the relation between D and κ has been chosen as

$$D = \begin{cases} \dfrac{\kappa_c}{\kappa}\dfrac{\kappa - \kappa_i}{\kappa_c - \kappa_i} & \text{if } \kappa_i < \kappa \leq \kappa_c, \\ 1 & \text{if } \kappa > \kappa_c, \end{cases} \tag{3}$$

which yields a linear softening post-peak path in the stress-strain diagram (*Fig.* 1).

The key notion to non-local damage models is that damage evolution in a material point, say x, depends on the deformation of not only this point, but also of the surrounding volume. This is achieved by replacing (2) by

$$\dot{\kappa} \geq 0, \qquad \bar{\varepsilon} - \kappa \leq 0, \qquad \dot{\kappa}(\bar{\varepsilon} - \kappa) = 0, \tag{4}$$

where $\bar{\varepsilon}$ is a weighted average of the strain in the bar according to

$$\bar{\varepsilon}(x) = \int_{-\infty}^{\infty} g(\xi)\varepsilon(x + \xi)d\xi. \tag{5}$$

An appropriate weight function $g(\xi)$, with ξ the distance from point x, is the Gaussian function

$$g(\xi) = \frac{1}{\sqrt{2\pi}l}e^{-\frac{\xi^2}{2l^2}}. \tag{6}$$

Its bell shape reflects the gradual decrease of the influence of the strain on damage evolution with increasing distance. As shown in Pijaudier-Cabot and Bažant (1987); Bažant and Pijaudier-Cabot (1988) by a one-dimensional numerical example, and by Pijaudier-Cabot and Benallal (1993) in a more general, theoretical context, the non-local model is capable of properly capturing strain and damage localisation. The width of the localisation band is proportional to the internal length l which is introduced in the constitutive description through the weight function (6).

Two versions of gradient enhancement of the local damage model are considered here. Both can be derived from the non-local formulation, as has been demonstrated for the three-dimensional case in Peerlings *et al.* (1995, 1996). Substitution of a Taylor expansion of the local strain into (5) and use of the weight function (6) give

$$\bar{\varepsilon} = \varepsilon + \frac{1}{2} l^2 \frac{\partial^2 \varepsilon}{\partial x^2} + \frac{1}{8} l^4 \frac{\partial^4 \varepsilon}{\partial x^4} + \ldots \tag{7}$$

Thus, neglecting higher-order terms, the integral definition (5) of the non-local strain can be replaced by the differential approximation (*cf.* Lasry and Belytschko, 1988; de Borst and Mühlhaus, 1991)

$$\bar{\varepsilon} = \varepsilon + \frac{1}{2} l^2 \frac{\partial^2 \varepsilon}{\partial x^2}. \tag{8}$$

Definition (8) is less suitable in a finite element framework, because the explicit dependence on the second-order strain gradient necessitates a C^1-continuous interpolation of the displacement (Lasry and Belytschko, 1988). To avoid this severe restriction, a second-gradient formulation has been developed in which the gradient term is incorporated in a more implicit form. Differentiating (8) twice with respect to x and substitution of the result in (7) yields the expression

$$\bar{\varepsilon} - \frac{1}{2} l^2 \frac{\partial^2 \bar{\varepsilon}}{\partial x^2} = \varepsilon - \frac{1}{8} l^4 \frac{\partial^4 \varepsilon}{\partial x^4} + \ldots \tag{9}$$

Comparing (7) and (9), it can be concluded that an approximation of the same order is introduced if

$$\bar{\varepsilon} - \frac{1}{2} l^2 \frac{\partial^2 \bar{\varepsilon}}{\partial x^2} = \varepsilon \tag{10}$$

is employed as the definition of $\bar{\varepsilon}$ instead of (8). If (10) is treated as an additional differential equation for the variable $\bar{\varepsilon}$, complementary to the equations of motion, an efficient numerical solution algorithm can be constructed, for which C^0-continuity of the interpolation polynomials suffices (Peerlings *et al.*, 1995, 1996).

It is emphasised that an internal length scale, which is essential for the proper description of damage localisation, is present in both gradient formulations. If the parameter l which characterises this length scale is set to zero, the original, local model is recovered.

3. Change of type of the governing equations

The change of type of the partial differential equations which govern the response of conventional strain softening continuum models is illustrated most conveniently for the dynamic loading case. Wave propagation phenomena are described mathematically by a set of hyperbolic partial differential equations with appropriate initial and boundary conditions. As will be demonstrated next, this is also true for the local damage model as long as softening does not occur. When a material point enters the softening regime, however, the type of the governing equations in this point changes from hyperbolic – via parabolic in the limiting case – to elliptic. It is emphasised that this change of type is a local phenomenon; the type of non-linear partial differential equations may vary from point to point. With the emergence of a zone in which the equations are elliptic in an otherwise hyperbolic initial-boundary value problem, the mathematical description becomes ill-posed, since the initial and boundary conditions that are meaningful for the hyperbolic problem usually are inappropriate for the elliptical part. In enhanced continua this change of type should not occur.

The general equation of motion for a cylindrical bar under dynamic loading reads

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial v}{\partial t}, \tag{11}$$

where ρ denotes the density and v is the velocity. For the local model and for evolving damage, so that $\kappa = \varepsilon$, substitution of the constitutive relation (1) in this equation and application of the chain rule yield

$$\left(1 - D - \varepsilon \frac{\partial D}{\partial \kappa}\right) \frac{\partial \varepsilon}{\partial x} - \frac{1}{c_e^2} \frac{\partial v}{\partial t} = 0, \tag{12}$$

with $c_e = \sqrt{E/\rho}$ the wave velocity in a linear elastic bar. One should notice that the factor preceding the strain derivative equals the tangential stiffness on the softening path of the stress-strain curve divided by Young's modulus E. For the linear softening damage law (3) it is a negative constant (*Fig.* 1), given by

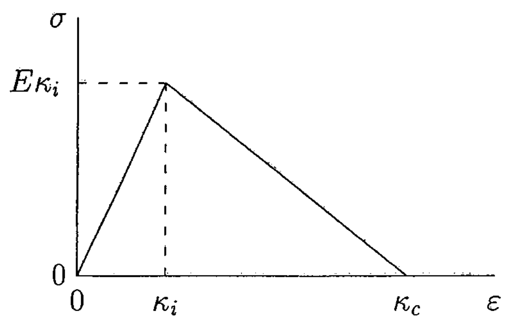


Fig. 1. – Stress-strain diagram for the local damage model.

$$\frac{1}{E} \frac{\partial \sigma}{\partial \varepsilon} = 1 - D - \varepsilon \frac{\partial D}{\partial \kappa} = -\frac{\kappa_i}{\kappa_c - \kappa_i}. \tag{13}$$

Collecting the field variables v and ε in the column

$$\mathbf{u} = \begin{bmatrix} v \\ \varepsilon \end{bmatrix} \tag{14}$$

and combining (12) with the kinematic relation

$$\frac{\partial \varepsilon}{\partial t} - \frac{\partial v}{\partial x} = 0, \tag{15}$$

the equations of motion can be written in the general form of a system of quasi-linear partial differential equations

$$\mathbf{P}(\mathbf{u})\frac{\partial \mathbf{u}}{\partial t} + \mathbf{Q}(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x} = \mathbf{r}(\mathbf{u}), \tag{16}$$

in which

$$\mathbf{P}(\mathbf{u}) = \begin{bmatrix} -\frac{1}{c_e^2} & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{Q}(\mathbf{u}) = \begin{bmatrix} 0 & 1 - D - \varepsilon\frac{\partial D}{\partial \kappa} \\ -1 & 0 \end{bmatrix}, \quad \mathbf{r}(\mathbf{u}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{17}$$

Notice that the system is in fact linear for the specific damage law (3) because of the constant softening modulus (*see* Eq. (13)). The set of partial differential Eqs. (16) is hyperbolic if the generalised eigenproblem

$$\mathbf{w}^T(\mathbf{P} - \lambda\mathbf{Q}) = \mathbf{0}^T \tag{18}$$

has n real eigenvalues λ and n linearly independent left eigenvectors $\mathbf{w}$, where $n = 2$ denotes the number of equations in the system (*see* for instance Courant and Hilbert, 1953; Whitham, 1974). The eigenvalues lead to the so-called characteristic directions in the x-t plane, along which the system may be simplified with regard to an analytical solution:

$$\frac{dx}{dt} = \frac{1}{\lambda}. \tag{19}$$

The solution of (18) is straightforward, yielding:

$$\lambda = \frac{\pm 1}{c_e\sqrt{1 - D - \varepsilon\frac{\partial D}{\partial \kappa}}}. \tag{20}$$

Since both eigenvalues are imaginary (*cf.* expression (13)), the system (16) must be classified as elliptic. In areas where damage is not growing and where it has not yet reached the critical value $D = 1$, the term $\varepsilon\partial D/\partial\kappa$ in (20) vanishes, and since $1 - D > 0$ we have two distinct real eigenvalues. Two distinct eigenvalues are always accompanied by two independent eigenvectors, so that we can conclude that the system is hyperbolic outside the process zone, leaving the total initial value problem ill-posed. For $D = 1$ the system becomes parabolic, but in this case, which represents complete fracture of the bar, the analysis is no longer physically meaningful.

For the gradient model with the explicit appearance of the second derivative of the strain (8), Eq. (12) is replaced by

$$(1 - D)\frac{\partial \varepsilon}{\partial x} - \varepsilon\frac{\partial D}{\partial \kappa}\frac{\partial \bar{\varepsilon}}{\partial x} - \frac{1}{c_e^2}\frac{\partial v}{\partial t} = 0. \tag{21}$$

If definition (8) is split as

$$\frac{1}{2}l^2\frac{\partial p}{\partial x} = \bar{\varepsilon} - \varepsilon, \qquad \frac{\partial \varepsilon}{\partial x} = p, \tag{22}$$

and $\mathbf{u}$ is defined as

$$\mathbf{u} = \begin{bmatrix} v \\ \varepsilon \\ \bar{\varepsilon} \\ p \end{bmatrix}, \tag{23}$$

the governing set of partial differential Eqs. (15), (21) and (22) can also be written in the form (16) with

(24)

$$\mathbf{P}(\mathbf{u}) = \begin{bmatrix} -\frac{1}{c_e^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{Q}(\mathbf{u}) = \begin{bmatrix} 0 & 1-D & -\varepsilon\frac{\partial D}{\partial \kappa} & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}l^2 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{r}(\mathbf{u}) = \begin{bmatrix} 0 \\ 0 \\ \bar{\varepsilon} - \varepsilon \\ p \end{bmatrix}.$$

If the eigenproblem (18) is solved for this system, a quadruple eigenvalue $\lambda = 0$ is found. However, four linearly independent eigenvectors can be associated with this value, so that the set of equations is nevertheless hyperbolic. It is straightforward to verify that it is hyperbolic also for non-evolving damage.

The modified gradient formulation according to (10) can be written as a system of first-order equations by setting $q = \partial\bar{\varepsilon}/\partial x$,

$$\mathbf{u} = \begin{bmatrix} v \\ \varepsilon \\ \bar{\varepsilon} \\ q \end{bmatrix}, \tag{25}$$

and

(26)

$$\mathbf{P}(\mathbf{u}) = \begin{bmatrix} -\frac{1}{c_e^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{Q}(\mathbf{u}) = \begin{bmatrix} 0 & 1-D & -\varepsilon\frac{\partial D}{\partial \kappa} & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}l^2 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{r}(\mathbf{u}) = \begin{bmatrix} 0 \\ 0 \\ \bar{\varepsilon} - \varepsilon \\ q \end{bmatrix}.$$

The solution of the corresponding eigenproblem yields the eigenvalues

$$\lambda = \frac{\pm 1}{c_e\sqrt{1-D}} \qquad \text{and} \qquad \lambda = 0. \tag{27}$$

The first pair of eigenvalues is real because $1-D$ is non-negative. Since two independent eigenvectors can be constructed for the double eigenvalue $\lambda = 0$, the problem remains hyperbolic also for this formulation.

Apart from the preservation of hyperbolicity in both gradient damage models, another interesting property can be deduced from the above analysis. For the formulation according to (10), two families of characteristics exist:

$$\frac{dx}{dt} = \pm c_e \sqrt{1-D} \qquad \text{and} \qquad dt = 0. \tag{28}$$

The first pair corresponds to the wave velocity of the damaged material and reflects the propagation of waves. The second, double characteristic, which is not encountered in a linear elastic material, is related to the "averaging" procedure. However, the model with the explicit gradient dependence yields a quadruple characteristic $dt = 0$. This description therefore does not incorporate the delay effect that is typical for wave propagation problems. More attention will be paid to this subject in the next section.

4. Wave propagation

A considerable amount of insight in the properties of softening material models can be gained from an analysis of wave propagation. In conventional continua, the change of type of the governing partial differential equations prohibits the propagation of loading waves in the softening zone. Deformation is trapped in an infinitely narrow band, in which the strain can grow unboundedly (Read and Hegemier, 1984; Bažant and Belytschko, 1985). Enhanced continuum descriptions, for which the governing equations remain hyperbolic also in the softening regime, allow for waves propagating in a softening zone. Wave propagation is dispersive for these models, that is, the velocity of a harmonic wave depends on its wave number and the shape of an arbitrary pulse is altered during its propagation. In fact, this property is essential for the proper description of localisation under dynamic loading, since it enables the emergence of a standing wave with a non-zero width, which can develop into a zone of localised deformation (Sluys, 1992; de Borst *et al.*, 1993; Sluys and de Borst, 1994).

The equations of motion (16) are generally highly non-linear. In order to arrive at a closed form solution, the usual assumption of a linear comparison solid is made (Hill, 1958; Lasry and Belytschko, 1988; Huerta and Pijaudier-Cabot, 1994). This means that loading is assumed for an infinitesimally small perturbation $\delta\mathbf{u} = \mathbf{u} - \mathbf{u}_0$. The equations of motion (16) are linearised around a homogeneous equilibrium state defined by the strain field ε_0, the corresponding damage D_0 and the stress σ_0, yielding the following linear system:

$$\mathbf{P}(\mathbf{u}_0)\frac{\partial(\delta\mathbf{u})}{\partial t} + \mathbf{Q}(\mathbf{u}_0)\frac{\partial(\delta\mathbf{u})}{\partial x} - \left(\frac{\partial\mathbf{r}}{\partial\mathbf{u}}\right)_0 \delta\mathbf{u} = \mathbf{0}. \tag{29}$$

A solution is sought in the form of the single harmonic wave

$$\delta\mathbf{u} = \hat{\mathbf{u}}e^{ik(x-ct)} \tag{30}$$

with wave number k and corresponding velocity c. Substitution of (30) into (29) shows that a non-trivial solution of this form exists if

$$\det\left(-ikc\,\mathbf{P}(\mathbf{u}_0)+ik\,\mathbf{Q}(\mathbf{u}_0)-\left(\frac{\partial\mathbf{r}}{\partial\mathbf{u}}\right)_0\right)=0. \tag{31}$$

For the local damage model, this equation reduces to the dispersion relation

$$\frac{k^2c^2}{c_e^2}-k^2\left(1-D_0-\varepsilon_0\left(\frac{\partial D}{\partial\kappa}\right)_0\right)=0, \tag{32}$$

which, apart from the trivial solution $k=0$, yields

$$c=c_e\sqrt{1-D_0-\varepsilon_0\left(\frac{\partial D}{\partial\kappa}\right)_0} \tag{33}$$

for the phase velocity of the harmonic wave (30). According to relation (13) this wave velocity is imaginary for all wave numbers k, so that loading waves cannot propagate in the softening regime. Also, substitution of (33) into (30) shows that the perturbation $\delta\mathbf{u}$ can grow unboundedly.

The wave propagation characteristics of the gradient damage models can be derived along the same lines. Application of expressions (23) and (24) in Eq. (31) yields for the model with explicit gradient dependence the non-trivial solution

$$c=c_e\sqrt{1-D_0-\varepsilon_0\left(\frac{\partial D}{\partial\kappa}\right)_0\left(1-\frac{1}{2}l^2k^2\right)}. \tag{34}$$

For this model, the wave velocity depends of the wave number k and is therefore dispersive. It is real for wave numbers $k\geq k_c$ with

$$k_c=\frac{1}{l}\sqrt{2\left(1-\frac{1-D_0}{\varepsilon_0\left(\frac{\partial D}{\partial\kappa}\right)_0}\right)}. \tag{35}$$

For the formulation according to (10), in which the gradient term is included in a more implicit fashion, elaboration of (31) gives

$$c=c_e\sqrt{1-D_0-\frac{\varepsilon_0\left(\frac{\partial D}{\partial\kappa}\right)_0}{1+\frac{1}{2}l^2k^2}}, \tag{36}$$

which is real for wave numbers greater than the critical value

$$k_c = \frac{1}{l}\sqrt{2\left(\frac{\varepsilon_0\left(\frac{\partial D}{\partial \kappa}\right)_0}{1 - D_0} - 1\right)}. \tag{37}$$

Wave propagation in a non-local damage model which is only slightly different from the one adopted in Section 2 has been analysed by Huerta and Pijaudier-Cabot (1994) under the same assumptions as introduced here. For the model according to (5) the phase velocity can be derived as

$$c = c_e\sqrt{1 - D_0 - \varepsilon_0\left(\frac{\partial D}{\partial \kappa}\right)_0 e^{-\frac{1}{2}k^2 l^2}}, \tag{38}$$

so that the wave speed is also dispersive in the non-local theory. The critical wave number k_c is

$$k_c = \frac{1}{l}\sqrt{2\ln\left(\frac{\varepsilon_0\left(\frac{\partial D}{\partial \kappa}\right)_0}{1 - D_0}\right)}. \tag{39}$$

The wave velocities according to (34), (36) and (38) have been plotted in Figure 2 at the peak stress ($\varepsilon_0 = \kappa_i$). Young's modulus was set to E = 20,000 MPa and the density was chosen such that a value of c_e = 1000 m/s is obtained for the elastic wave velocity. The damage model parameters were set to κ_i = 0.0001 and κ_c = 0.0125, respectively, while a value of $l = \sqrt{2}$ mm was used for the internal length parameter. For small wave numbers, the three regularisation methods appear to yield almost identical propagation properties. Below the critical wave number k_c, which at the peak stress is virtually equal for all three models, the wave speed is imaginary. In theory, the perturbation is unbounded for these values. This effect is introduced by limiting the analysis to a uniformly deformed linear comparison solid. In a "real" softening solid, damage can only be progressive in a zone of width π/k_c, while it remains constant in the surrounding material. The large wave lengths associated to wave numbers smaller than k_c cannot exist in such a softening zone,

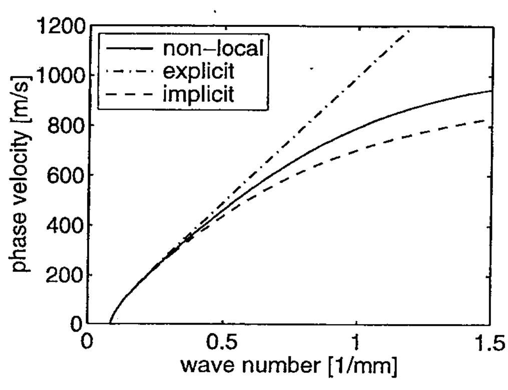


Fig. 2. – Wave velocities for the non-local and gradient models.

so that the response remains bounded (Lasry and Belytschko, 1988; Sluys and de Borst, 1994). For higher wave numbers, the omission of higher-order terms in deriving the gradient models from the non-local formulation becomes apparent. Higher-order terms then play a more important role and the curves are somewhat divergent.

A striking discrepancy is observed between the two gradient models. While the implicit formulation (10) and the non-local model exhibit a horizontal asymptote equal to the elastic wave speed c_e = 1000 m/s as the wave number approaches infinity, the formulation with the explicit relation (8) is not bounded. As a consequence, infinitely short waves can propagate with an infinite speed in the explicit model. This behaviour, which is in accordance with the four-fold characteristic $dt = 0$ of the non-linear problem as derived in the previous section, is unacceptable from a physical point of view. The explicit formulation therefore seems less suitable for dynamic problems, but it probably does not pose problems in static analyses, where the response is always immediate.

In contrast with plasticity-based softening models, where the dispersion properties remain constant for the complete softening branch in case of linear softening (Sluys, 1992; Sluys and de Borst, 1994), the dispersion diagram depends on the strain level ε_0 for the damage models. This is illustrated for the implicit formulation in Figure 3, in which the wave speed has been plotted versus the wave number for ε_0 = 0.0001, 0.0005 and 0.0025. Since damage evolution can be regarded as a stiffness degrading mechanism, wave propagation becomes slower for a higher damage level. Indeed, for the implicit gradient model the wave velocity is bounded by the elastic wave speed in the damaged material $c_e\sqrt{1-D_0}$. The non-local and the explicit gradient model both exhibit similar trends, although the velocity remains unbounded in the explicit model.

Figure 3 also shows that the critical wave number, below which the phase velocity is imaginary and waves cannot propagate, increases for an increasing damage level. Thus, the critical wave length, which sets the width of the localisation band (Sluys, 1992; Sluys and de Borst, 1994), decreases during the material degradation process. This observation is in accordance with the narrowing of the damage evolution zone in a static analysis (Peerlings *et al.*, 1995, 1996; *see* also the next section). Figure 4 shows the critical wave length $2\pi/k_c$, with k_c according to (35), (37) and (39), as a function of the uniform strain ε_0 for all three localisation limiters. For relatively small strains the deviations between the curves are negligible, but at higher strain levels considerable differences are observed.

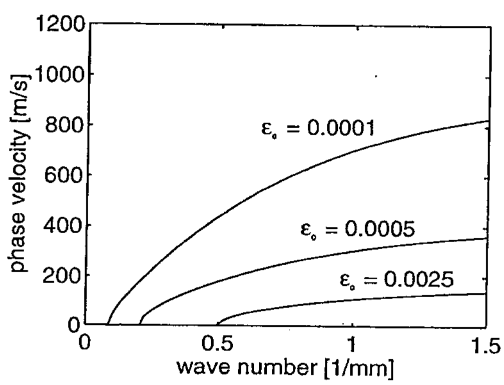


Fig. 3. – Influence of the uniform strain level ε_0 on the wave velocity; implicit gradient model.

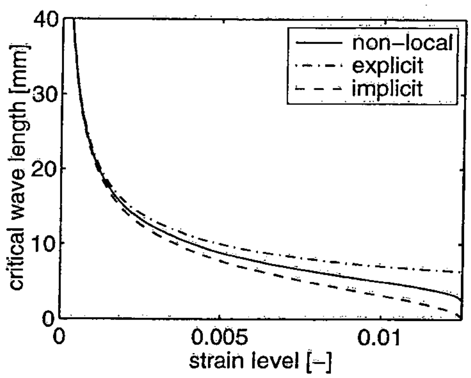


Fig. 4. – Critical wave length versus strain level.

These must be attributed to the higher-order terms which are present in the non-local model and have been neglected in constructing both gradient formulations. The non-local and the implicit gradient damage models predict a narrowing of the damage zone until zero width. Physically, this corresponds with a line crack, and a natural transition from a damaged zone into a line crack is obtained. In contrast, the explicit gradient formulation results in a finite width of the damage process zone also for complete loss of coherence, namely $\frac{\pi}{2}\sqrt{2}l$, which precludes a gradual transition into a line crack.

5. Quasi-static loading

Complications quite similar to those encountered in dynamic localisation problems may also occur in quasi-static problems. It is therefore useful to elucidate the similarities and specific features of gradient-enhanced and non-local models also for this situation. For this purpose, finite element analyses of a simple, one-dimensional bar problem have been performed with both the non-local and the implicit gradient damage model. As indicated in Section 2, the explicit model is less suitable with respect to a finite element implementation; it is therefore left out of consideration here.

Since the bar must have a finite length L, proper boundary conditions must be specified for the "averaging" procedure. For the non-local approach, the integration interval in (5) is limited to the bar length. The non-local strain $\bar{\varepsilon}$ is therefore redefined according to

$$(40) \qquad \bar{\varepsilon}(x) = \frac{1}{V_e}\int_{-x}^{L-x} g(\xi)\varepsilon(x+\xi)A(\xi)d\xi, \qquad \text{with } V_e = \int_{-x}^{L-x} g(\xi)A(\xi)d\xi,$$

with A the cross sectional area of the bar (Bažant and Jirásek, 1994). The gradient dependent formulation is completed by the natural boundary condition

$$(41) \qquad \frac{\partial\bar{\varepsilon}}{\partial x} = 0$$

at both ends of the bar (Peerlings *et al.*, 1995, 1996). With these assumptions, both models yield exactly the same response as the local model if the deformation is kept homogeneous.

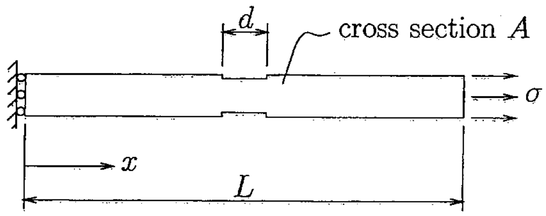
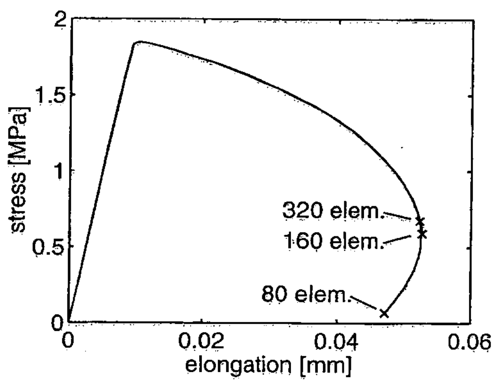
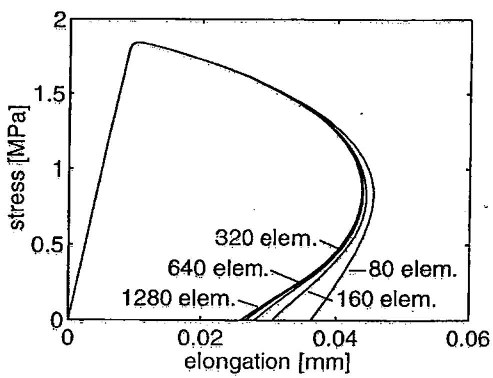


Fig. 5. – Bar problem configuration.

The bar, depicted in Figure 5, is subjected to a uniaxial, tensile load applied by an indirect displacement control procedure (Crisfield, 1980; de Borst, 1986). The length L of the bar has been set to 100 mm. A 10% reduction of its cross sectional area A has been applied in the middle d = 10 mm in order to trigger localisation of damage in this area. The interpolation of the displacement was taken quadratic for both regularisation methods; for the additional interpolation of $\bar{\varepsilon}$ in the gradient approach linear polynomials have been used. A consistent, full Newton-Raphson procedure has been applied for the gradient model; reference is made to Peerlings *et al.* (1995, 1996) for a detailed discussion of the solution procedure. Utilisation of a consistent tangent operator for the non-local model would result in an unfavourably structured matrix, the assembly of which is rather cumbersome (Pijaudier-Cabot and Huerta, 1991). A secant stiffness was adopted for this reason.



(a)



(b)

Fig.6. – Load-elongation curves computed with different meshes: (a) non-local model and (b) gradient model.

Figure 6 shows the load-deflection curves obtained for both regularisation techniques with increasing levels of mesh refinement. The markers in the curves for the non-local model (Figure 6(*a*)) indicate the loss of convergence of the incremental-iterative procedure for the respective meshes. This divergence is caused by the use of a secant stiffness matrix, which renders the indirect displacement control procedure less stable than with a consistent tangent operator, particularly for snap-back behaviour. Both regularisation techniques result in physically meaningful load-displacement curves with a finite energy dissipation upon refinement of the discretisation. For the non-local model a coarser mesh appears to suffice than for the gradient dependent formulation, although the comparison is obscured somewhat by the fact that the non-local softening path could not be followed until complete fracture.

As can be observed from Figure 7, in which the 320 element load-displacement curves of both models have been plotted, the responses of the enhanced continua agree quite well in a qualitative sense. The predicted tensile strengths are practically equal, but the gradient-dependent formulation exhibits a somewhat more brittle post-peak behaviour than the non-local model. Figure 8 presents the strain and damage profiles for elongations of 0.01, 0.02, 0.03 and 0.04 mm. For relatively small displacements, the strain distributions of the non-local and gradient dependent model match almost perfectly. For larger elongations the deformation (Figure 8(*a*)) and damage (Figure 8(*b*)) are slightly more localised in the gradient model than in the non-local continuum, which explains the more brittle softening branch in Figure 7. Both the gradual narrowing of the process zone (Figure 8(*a*)) and the increasing deviation between the two models for increasing deformations are consistent with the evolution of the critical wave lengths as derived in Section 4. The increasing deviation is again due to the higher-order derivatives that are present in the non-local model and absent in the gradient formulation.

6. Conclusion

The analyses presented in this contribution indicate that the responses obtained from non-local and two different second-order gradient enhanced continua are at least qualitatively very similar. The deviations are not negligible in a quantitative sense, but the gradient dependent formulations comprise all features of the non-local model that are essential for the proper description of localisation phenomena. Under dynamic loading

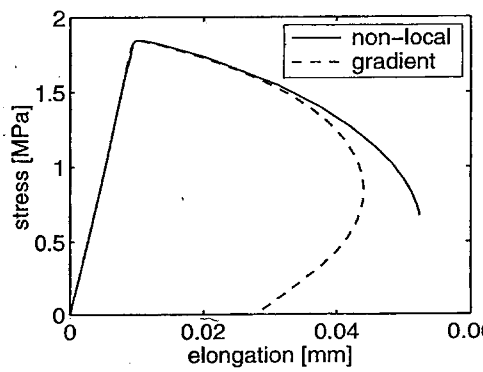


Fig. 7. – Comparison of load-elongation curves for the non-local and gradient dependent models.

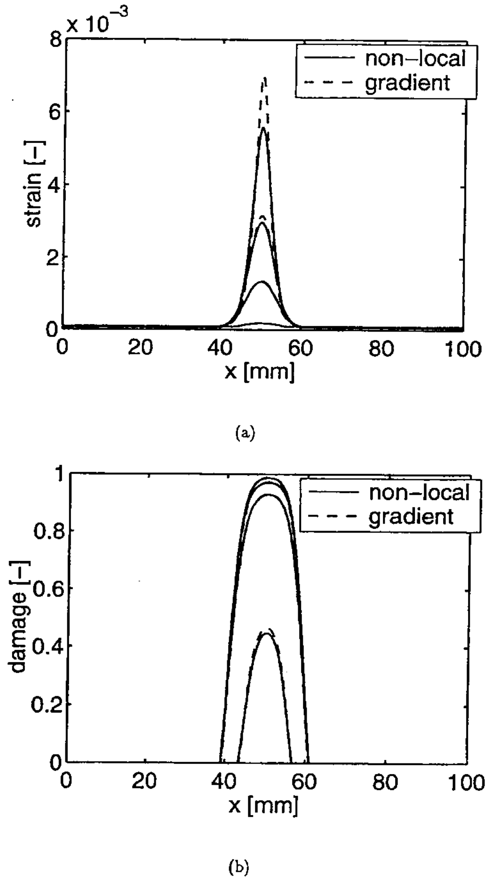


Fig. 8. – Strain (a) and damage (b) evolution for the non-local and gradient models.

conditions, the hyperbolicity of the equations of motion is preserved and the wave velocity is real and dispersive in the softening zone. The width of the deformation band in dynamics as well as statics is governed by the internal length scale introduced in both types of higher-order continua (non-local and gradient dependent).

Two versions of gradient enhancement of damage models have been considered.

The form indicated as implicit yields a somewhat more localised deformation than the non-local model. On the basis of a dispersion analysis the opposite trend is expected for the explicit formulation. However, the latter suffers from the theoretical shortcoming that it results in an infinite wave speed for loading waves in the softening region and has the computational disadvantage that it demands a C^1-continuous finite element interpolation. The fact that C^0-continuity of the interpolation suffices for the implicit model allows the construction of a stable and efficient numerical solution algorithm and thus renders this form an attractive alternative to non-local damage models.

The effect of the gradient enhancement on the response of the damage model is fundamentally different from the effect is has in a plasticity framework. Whereas gradient damage seems to be both conceptually and computationally simpler than gradient plasticity (*see* also de Borst *et al.*, 1995), the mathematical implications of the introduction of gradient terms are more far-reaching in damage theory. The multiplicative character of the constitutive equation causes the structural response to be non-linear, even if a linear softening law is applied. Also, the governing set of partial differential equations in the damage evolution zone does not remain linear as in plasticity (Sluys *et al.*, 1993), but becomes quasi-linear. The multiplicative character is also responsible for the gradual narrowing of the process zone, which seems more appropriate for fracture problems than the constant localisation width found in gradient plasticity (de Borst and Mühlhaus, 1992).

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