Approximation of Functionals Depending on Jumps by Elliptic Functionals via Γ-Convergence

LUIGI AMBROSIO
II Universitá di Roma "Tor Vergata"

AND

VINCENZO MARIA TORTORELLI
Scuola Normale Superiore, Pisa

Abstract

We show how it is possible to approximate the Mumford-Shah (see [29]) image segmentation functional

$$\mathcal{G}(u, K) = \int_{\Omega \setminus K} [|\nabla u|^2 + \beta(u-g)^2]\, dx + \alpha \mathcal{H}^{n-1}(K),$$

$$u \in W^{1,2}(\Omega \setminus K),\ K \subset \Omega \quad \text{closed in} \quad \Omega$$

by elliptic functionals defined on Sobolev spaces. The heuristic idea is to consider functionals $\mathcal{G}_h(u, z)$ with z ranging between 0 and 1 and related to the set K. The minimizing z_h are near to 1 in a neighborhood of the set K, and far from the neighborhood they are very small. The neighborhood shrinks as $h \to +\infty$. For a similar approach to the problem compare Kulkarni; see [25]. The approximation of $\mathcal{G}_h$ to $\mathcal{G}$ takes place in a variational sense, the De Giorgi Γ-convergence.

Introduction

In recent years new interest has grown toward problems of calculus of variations which have among the unknowns a hypersurface $K \subset \mathbf{R}^n$. These problems, suggested by mathematical physics (see [18], [16]) and by other areas of applied mathematics (see [24], [29]) involve the minimization of functionals of the type

$$(1) \qquad \mathcal{F}(u, K) = \int_{\Omega \setminus K} f(x, u, \nabla u)\, dx + \int_K \varphi(x, u^+, u^-, \nu)\, d\mathcal{H}^{n-1}(x),$$

where $\Omega \subset \mathbf{R}^n$ is an open bounded set, $\mathcal{H}^{n-1}$ is the Hausdorff $(n-1)$-dimensional measure, K varies in a class of sufficiently regular closed sets of Ω, u varies in $W^{1,1}(\Omega \setminus K)$, ν is normal to K, and u^+, u^- are the traces of u on the opposite sides of K.

In this paper we are concerned with the functional

$$(2) \qquad \begin{gathered} \mathcal{G}(u, K) = \int_{\Omega \setminus K} [|\nabla u|^2 + \beta(u-g)^2]\, dx + \alpha \mathcal{H}^{n-1}(K), \\ u \in W^{1,2}(\Omega \setminus K),\ K \subset \Omega \quad \text{closed in} \quad \Omega \end{gathered}$$

Communications on Pure and Applied Mathematics, Vol. XLIII, 999–1036 (1990)
 CCC 0010-3640/90/080999-38$04.00

where α, $\beta > 0$ are fixed parameters, and $g \in L^\infty(\Omega)$. In the case $n = 2$ this functional has been suggested by Mumford-Shah (see [29]) for a variational approach to image segmentation. The function g in (2) represents the image of a group of objects given by a camera, with discontinuities along the edges of the objects. By minimizing the functional (2) one tries to distinguish the discontinuities due to the edges and shadows from the discontinuities due to noise and small irregularities. The functional penalizes large sets K, and outside the set K the function u is required to be close to g and $W^{1,2}$. In the papers [29], [30] Mumford-Shah conjecture that $\mathcal{G}$ has minimizers, and they study their behavior under some a priori regularity assumptions. Recently, the Mumford-Shah conjecture concerning existence of minimizers has been proved for general integers n by De Giorgi-Carriero-Leaci; see [17]. In the case $n = 2$ a different proof has been discovered by Dal Maso-Morel-Solimini; see [13]. The idea in common to both proofs is to use a weak formulation of the problem by setting

$$G(u) = \int_\Omega [|\nabla u|^2 + \beta(u-g)^2]\, dx + \alpha \mathcal{H}^{n-1}(S_u), \tag{3}$$

where S_u is the discontinuity set of u in an approximate sense, and u varies in a special class of functions of bounded variation, denoted by $SBV(\Omega)$. This class consists of all functions of bounded variation such that the distributional derivative is absolutely continuous with respect to Lebesgue measure plus an $(n-1)$-dimensional measure. Since (see [2], [3], [17])

$$\mathcal{H}^{n-1}(K) < +\infty, \quad u \in W^{1,1}(\Omega \backslash K) \cap L^\infty(\Omega \backslash K) \Rightarrow$$
$$u \in SBV(\Omega), \mathcal{H}^{n-1}(S_u \backslash K) = 0,$$

the domain $SBV(\Omega)$ contains the original domain of $\mathcal{G}$. By means of general compactness and lower semicontinuity theorems in $SBV(\Omega)$ (see [1], [2], [3]) it can be shown that G has minimizers in $SBV(\Omega)$. The regularity theorems in [13], [17] show that

$$u \in C^1(\Omega \backslash K), \qquad \mathcal{H}^{n-1}(\overline{S_u} \cap \Omega \backslash S_u) = 0,$$

for any minimizer u so that, by setting $K = \overline{S_u} \cap \Omega$, we recover a minimum of $\mathcal{G}$.

The problem of finding effective algorithms for computing the minimizers of G is still widely discussed; see [7], [20], [22], [23], [24], [35]. In this paper we suggest an approach to this problem by approximating in a variational sense, defined by Γ-convergence (see, for instance, [5], [12], [14], [15]), the functional G by elliptic functionals. The outcome of recent numerical simulations, made by R. March in Pisa, has been very encouraging. Anyway, independently of the possible computational applications of this result, it is interesting to notice that Γ-limits of elliptic but not equicoercive functionals may be completely different functionals, containing in our case a penalisation $\alpha\mathcal{H}^{n-1}(K)$. This phenomenon was discovered in the first years of development of Γ-convergence by Modica-Mortola; see [27]. Recently, the abstract Modica-Mortola result has been very successfully applied to explain interface phenomena of a class of fluids; see [26].

Our approximating elliptic functionals are formally defined by

$$G_h(u, z) = \int_\Omega \left[(|\nabla u|^2 + |\nabla z|^2)(1 - z^2)^{2h} + \frac{1}{4}(\alpha^2 h^2) z^2 \right] dx \tag{4}$$
$$+ \beta \int_\Omega |u - g|^2,$$

and the approximation takes place as $h \to +\infty$. In (4) the variable $z \in [0, 1]$ plays the role of control variable on the gradient of u, and it depends on the jump set K. We recall that Γ-convergence of G_h to G implies

$$(u_h, z_h) \text{ minimizes } G_h, \quad u_h \to u \quad \Rightarrow \quad u \text{ minimizes } G. \tag{5}$$

Since $\inf G_h \leqq \beta \|g^2\|_\infty \operatorname{meas}(\Omega)$, it is easily seen that the functions z_h in (5) converge to 0 in $L^2(\Omega)$. On the other hand, if $u \notin W^{1,2}(\Omega)$, the coefficients $c_h = (1 - z_h^2)^{h+1}$ cannot be controlled from below with a strictly positive constant, hence z_h converges to 1 somewhere. We shall in general prove that for any sequence (u_h, z_h) such that

$$\lim_{h \to +\infty} G_h(u_h, z_h) = G(u) < +\infty$$

the coefficients $c_h \to 1$ almost everywhere and the measures

$$\mu_h(B) = \int_B |\nabla c_h| \, dx = 2(h+1) \int_B z_h (1 - z_h^2)^h |\nabla z_h| \, dx \tag{6}$$

weakly converge to the measure $2\mathscr{H}^{n-1} \llcorner S_u$, the restriction of $2\mathscr{H}^{n-1}$ to S_u (see Remark 4.3).

The proof of Γ-convergence consists of two steps (see [5], [12], [14]). The first step is the proof of inequality

$$\liminf_{h \to +\infty} G_h(u_h, z_h) \geqq G(u) \tag{7}$$

for every sequence $u_h \to u$, $z_h \to 0$. This is done in Section 4, first in the case $n = 1$ and then by a slicing argument. The second step is the construction of sequences $u_h \to u$, $z_h \to 0$ such that

$$\limsup_{h \to +\infty} G_h(u_h, z_h) \leqq G(u). \tag{8}$$

We construct the sequences first for all functions u whose jump set S_u satisfies a very mild regularity condition (see [19], Section 3.2.37):

$$\lim_{\rho \to 0^+} \frac{\text{meas}(\{x \in \Omega : \text{dist}(x, S_u) < \rho\})}{2\rho} = \mathscr{H}^{n-1}(S_u). \tag{9}$$

The sequences in the general case are then found by a diagonal argument by approximating each function u such that $G(u) < +\infty$ by functions u_k satisfying (9) in such a way that $G(u_k) \to G(u)$. We prove the approximation property by taking as u_k functions minimizing

$$\int_\Omega |\nabla v|^2 \, dx + \mathscr{H}^{n-1}(S_v) + k \int_\Omega |v - u|^2 \, dx,$$

and we use a regularity assumption on Ω fulfilled by rectangles and C^2 domains.

1. Notations and Statement of the Main Results

We begin by fixing some basic notions. By $\Omega \subset \mathbf{R}^n$ we denote a generic bounded open set. By $\mathscr{H}^{n-1}$, $\mathcal{L}^n$, we denote respectively the Hausdorff $(n-1)$-dimensional measure in $\mathbf{R}^n$ and the Lebesgue n-dimensional measure in $\mathbf{R}^n$. We also set

$$\mathbf{S}^{n-1} = \{x \in \mathbf{R}^n : |x| = 1\} \qquad B_\rho(x) = \{y \in \mathbf{R}^n : |y - x| < \rho\},$$

and

$$\text{meas}(B) = \mathcal{L}^n(B).$$

Let μ, σ be measures, and let f be the density of σ with respect to μ; we use the standard notation

$$f = \frac{\sigma}{\mu}$$

for the density, and

$$f \cdot \mu$$

for the absolutely continuous part of σ. We denote by $\mathbf{B}(\Omega)$ the σ-algebra of Borel sets in Ω, and we set

$$\mu \llcorner B(C) = \mu(B \cap C) \; \forall C \in \mathbf{B}(\Omega)$$

whenever $B \in \mathbf{B}(\Omega)$ and μ is a measure. The following space of functions will be often mentioned in this paper:

$$\mathcal{B}(\Omega) = \{u: \Omega \to \mathbf{R} : u \quad \text{is a Borel function}\}. \tag{1.1}$$

The space $\mathcal{B}(\Omega)$ is endowed with the convergence in measure; that is, a sequence u_h converges to u if and only if

$$\lim_{h \to +\infty} \int_K \frac{|u_h - u|}{1 + |u_h - u|} \, dx = 0$$

for every compact set $K \subset \Omega$. This space can be endowed with a distance which induces the convergence in measure.

Given a bounded function $u \in \mathcal{B}(\Omega)$, we denote by S_u the complement of the Lebesgue set of u, i.e.,

$$S_u = \left\{x \in \Omega : \nexists\, z \in \mathbf{R} \quad \text{such that} \quad \lim_{\rho \to 0^+} \rho^{-n} \int_{B_\rho(x)} |u - z| \, dx = 0\right\}. \tag{1.2}$$

In case u is not bounded, we set

$$S_u = \bigcup_{N=1}^{\infty} S_{N \wedge u \vee -N}. \tag{1.3}$$

We also define approximate upper and lower limits $u^+(x)$, $u^-(x)$ by

$$u^+(x) = \inf\left\{t \in [-\infty, +\infty] : \lim_{\rho \to 0^+} \frac{\operatorname{meas}(\{y \in B_\rho(x) : u(y) > t\})}{\rho^n} = 0\right\}, \tag{1.4}$$

$$u^-(x) = \sup\left\{t \in [-\infty, +\infty] : \lim_{\rho \to 0^+} \frac{\operatorname{meas}(\{y \in B_\rho(x) : u(y) < t\})}{\rho^n} = 0\right\}. \tag{1.5}$$

It can be easily seen that

$$S_u = \{x \in \Omega : u^-(x) < u^+(x)\}.$$

Let $BV(\Omega)$ be the space of functions $u \in L^1(\Omega)$ such that the distributional derivative can be represented by a measure with finite total variation $Du : \mathbf{B}(\Omega) \to \mathbf{R}^n$. The functions $u \in BV(\Omega)$ are called functions of bounded variation. For the main properties of this class of functions we refer to [19], [21], [36], [37]. We shall recall in Section 2 the properties useful for our purposes. We are interested in a "special" class of functions of bounded variation, in order to exclude functions like the Cantor-Vitali function. Following the same approach of [2], [16], we denote by $SBV(\Omega)$ the space of functions $u \in BV(\Omega)$ such that Du is absolutely continuous with respect to $\mathcal{L}^n + \mathscr{H}^{n-1} \llcorner S_u$. We also set

$$(1.6) \qquad GSBV(\Omega) = \{u \in \mathcal{B}(\Omega) : N \wedge u \vee -N \in SBV(\Omega)\ \forall N \in \mathbf{N}\}.$$

It can be shown (see Section 2) that to each function $u \in GSBV(\Omega)$ there corresponds a Borel function $\nabla u : \Omega \to \mathbf{R}^n$ such that $\nabla u = Du/\mathcal{L}^n$ almost everywhere if $u \in SBV(\Omega)$ and

$$(1.7) \qquad \nabla u = \nabla(N \wedge u \vee -N) \quad \text{a.e. on} \quad \{x \in \Omega : |u| \leq N\}\ \forall N \in \mathbf{N}$$

(we shall see in Section 2 that this function can be interpreted as a differential in an approximate sense). The functional we are interested in is defined by

$$(1.8) \qquad F(u) = \int_\Omega |\nabla u|^2\, dx + \alpha \mathscr{H}^{n-1}(S_u) \qquad u \in GSBV(\Omega),$$

where $\alpha > 0$ is a fixed parameter. In [2], by using compactness and semicontinuity theorems in $GSBV(\Omega)$, it is shown that the problem

$$(\mathcal{P}) \quad \min\left\{F(u) + \beta \int_\Omega |u - g|^\gamma\, dx : u \in GSBV(\Omega)\right\} \qquad \beta, \gamma > 0, g \in \mathcal{B}(\Omega)$$

has at least one solution. Moreover, any solution belongs to $SBV(\Omega)$ if $g \in L^\infty(\Omega)$.

We want to approximate F by means of elliptic functionals defined on Sobolev spaces. Formally, our functionals F_h are defined by

$$(1.9) \qquad F_h(u, z) = \int_\Omega \left[(|\nabla u|^2 + |\nabla z|^2)(1 - z^2)^{2h} + \frac{1}{4}(\alpha^2 h^2) z^2\right] dx.$$

Here, $z \in [0, 1]$ is an extra variable which controls the gradient of u. The formal expression (1.9) certainly makes sense if $u, z \in W^{1,2}(\Omega)$. However, due to a lack of coercivity of F_h in the Sobolev spaces, it is necessary to enlarge the domain in order to get compactness of the minimizing sequences of the functional

$$F_h(u, z) + \beta \int_\Omega |u - g|^\gamma\, dx.$$

In Section 3 we find a natural domain $\mathcal{D}_{h,n}(\Omega)$ such that

$$W^{1,2}(\Omega) \times \{z \in W^{1,2}(\Omega) : 0 \leq z \leq 1 \text{ a.e.}\} \subset \mathcal{D}_{h,n}(\Omega)$$
$$\subset \mathcal{B}(\Omega) \times \{z \in \mathcal{B}(\Omega) : 0 \leq z \leq 1 \text{ a.e.}\}$$

and we show that (1.9) makes sense for $(u, z) \in \mathcal{D}_{h,n}(\Omega)$. Moreover, the problem

$$(\mathcal{P}_h) \quad \min\left\{F_h(u, z) + \beta \int_\Omega |u - g|^\gamma \, dx : (u, z) \in \mathcal{D}_{h,n}(\Omega)\right\}$$

$$\beta, \gamma > 0, g \in \mathcal{B}(\Omega)$$

has at least one solution. The approximation of F_h to F takes place in a variational sense, the De Giorgi Γ-convergence (see Section 2). We need, as we said in the introduction, the following "reflection" condition on $\partial\Omega$:

($\mathcal{R}$) *there exists a neighborhood U of $\partial\Omega$ and a one to one map $\varphi : U \cap \Omega \to U \setminus \bar{\Omega}$ Lipschitz continuous with its inverse and such that*

$$\lim_{y \to x} \varphi(y) = x \quad \forall x \in \partial\Omega.$$

We also assume for sake of simplicity that Ω is a Lipschitz domain.

We can now state the main result of this paper.

THEOREM 1.1. *Assuming condition ($\mathcal{R}$), let $\mathcal{F}_h : \mathcal{B}(\Omega) \times \mathcal{B}(\Omega) \to [0, +\infty]$ be defined by*

$$(1.10) \qquad \mathcal{F}_h(u, z) = \begin{cases} F_h(u, z) & \text{if} \quad (u, z) \in \mathcal{D}_{h,n}(\Omega); \\ +\infty & \text{otherwise}. \end{cases}$$

Then, the functionals $\mathcal{F}_h$ $\Gamma(\mathcal{B}(\Omega) \times \mathcal{B}(\Omega))$-converge to the functional

$$(1.11) \qquad \mathcal{F}(u, z) = \begin{cases} F(u) & \text{if} \quad u \in GSBV(\Omega), z = 0; \\ +\infty & \text{otherwise}. \end{cases}$$

Moreover, for any choice of $\beta, \gamma > 0$, $g \in L^\infty(\Omega)$, all the problems $(\mathcal{P}_h)$ have solutions $\bar{w}_h = (\bar{u}_h, \bar{z}_h) \in \mathcal{D}_{h,n}(\Omega)$, the set $\{\bar{u}_h\}_{h \in \mathbf{N}}$ is compact in $\mathcal{B}(\Omega)$ and we have the implication

$$(1.12) \qquad (\bar{u}_{h_k}, \bar{z}_{h_k}) \text{ solves } (\mathcal{P}_{h_k}), \quad \bar{u}_{h_k} \to \bar{u} \quad \Rightarrow \quad \bar{u} \text{ solves } (\mathcal{P}).$$

We remark that the sequences $\bar{z}_{h_k}$ in (1.12) strongly converge to 0 in $L^2(\Omega)$. However, as we remarked in the introduction, the total variations of $(1 - \bar{z}_{h_k}^2)^{h_k+1}$ weakly converge as measures to $2\mathcal{H}^{n-1} \llcorner S_u$ (Remark 4.3), so that the minimizing set S_u can be identified by means of $\bar{z}_{h_k}$. If we replace $\mathcal{D}_{h,n}(\Omega)$ by

$$W^{1,2}(\Omega) \times \{z \in W^{1,2}(\Omega) : 0 \leqq z \leqq 1\}$$

in (1.10), then $\Gamma(\mathcal{B}(\Omega) \times \mathcal{B}(\Omega))$-convergence still holds (Remark 5.4), but the problem of minimizing

$$\mathcal{F}_h(u, z) + \beta \int_\Omega |u - g|^\gamma \, dx$$

could not have solution. We shall see in Section 3 that for any sequence $(w_k) = (u_k, z_k) \subset \mathcal{D}_{h,n}(\Omega)$ such that $F_h(w_k)$ is bounded there exists $w \in \mathcal{D}_{h,n}(\Omega)$ and a subsequence w_{h_k} such that $z_{k_p} \to z$ almost everywhere on Ω and $u_{k_p} \to u$ almost everywhere on $\{x \in \Omega : z(x) < 1\}$. Since F_h does not control ∇u on the set $\{z = 1\}$, it is easily seen that the result is sharp. We have chosen the integrals

$$\beta \int_\Omega |u - g|^\gamma \, dx \tag{1.13}$$

as lower semicontinuous perturbations of the functionals F_h, because this ensures existence of solutions of the approximate problems. Moreover, the same can actually be shown, with minor changes in our proof, for a wide class of perturbations including (1.13).

2. Γ-Convergence, Functions of Bounded Variation

We begin by recalling the definition and the main properties of Γ-convergence (see, for instance, [5], [12], [14], [15]). Let (E, d) be a separable metric space, and let $f_h : E \to [0, +\infty]$ be functions. We say that f_h $\Gamma(E)$-converges to $f : E \to [0, +\infty]$ if the following two conditions

$$\forall x_h \to x \quad \text{one has} \quad \liminf_{h \to +\infty} f_h(x_h) \geqq f(x) \tag{2.1}$$

and

$$\exists x_h \to x \quad \text{such that} \quad \limsup_{h \to +\infty} f_h(x_h) \leqq f(x) \tag{2.2}$$

are fulfilled for every $x \in E$. The Γ-limit, if it exists, is unique and lower semicontinuous. Moreover, every sequence f_h admits a subsequence which Γ-converges.

PROPOSITION 2.1. *Assume that f_h $\Gamma(E)$-converges to f. Then, the following statements hold.*

(i) *$f_h + g$ $\Gamma(E)$-converges to $f + g$ for every continuous function $g : E \to \mathbf{R}$.*
(ii) *Let $t_h \downarrow 0$. Then, every cluster point of the sequence of sets*

$$\{x \in E : f_h(x) \leqq \inf_E f_h + t_h\}$$

minimizes f.

(iii) *Assume that the functions f_h are lower semicontinuous and for every $t \in [0, +\infty[$ there exists a compact set $K_t \subset E$ with*

$$\{x \in E : f_h(x) \leqq t\} \subset K_t \quad \forall h \in \mathbf{N}.$$

Then, the functions f_h have minimizers in E, and any sequence x_h of minimizers of f_h admits subsequences converging to some minimizer of f.

The first statement in Proposition 2.1 tells us that Γ-convergence is stable under continuous perturbations. The second statement means that limits of almost minimizers are minimizers. Finally, the last one ensures existence and compactness of the sequence of minimizers.

Now we recall some basic results from the theory of functions of bounded variation. Let $u \in \mathcal{B}(\Omega)$, and assume that $z = u^+(x) = u^-(x) \in \mathbf{R}$. Then, we say that $p \in \mathbf{R}^n$ is the approximate differential of u at x if $v^+(x) = 0$, where

$$v(y) = \frac{|u(y) - z - \langle p, y - x\rangle|}{|y - x|} \qquad \forall y \in \mathbf{R}^n \setminus \{x\}.$$

Of course, differentiability implies approximate differentiability. The approximate differential, if it exists, is unique, and we denote it by $\nabla u(x)$. Moreover, the domain of ∇u is a Borel set, and ∇u is a Borel function (see, for instance, [2]). The main property of approximate differentials is the following:

$$\exists \nabla u(x), \quad \lim_{\rho \to 0^+} \frac{\operatorname{meas}(\{y \in B_\rho(x) : u(y) \neq v(y)\})}{\rho^n} = 0 \quad \Longrightarrow \quad \exists \nabla v(x), \quad \nabla u(x) = \nabla v(x) \tag{2.3}$$

for every pair of functions $u, v \in \mathcal{B}(\Omega)$. In particular,

$$\nabla u(x) = 0 \quad \text{a.e. on} \quad \{y \in \Omega : u(y) = c\} \tag{2.4}$$

for any $c \in \mathbf{R}$. We also have that $\varphi \circ v$ is approximately differentiable at x provided ∇v exists at x and $\varphi \in C^1$ in a neighborhood of $v^+(x) = v^-(x)$. We remark that, although Lipschitz functions are differentiable almost everywhere, there exist examples of functions $u \in W^{1,1}$ which are not differentiable in sets of positive measure; see [34]. However, Calderon-Zygmund [9] proved that BV functions, and in particular $W^{1,1}$ functions, are approximately differentiable almost everywhere, and the approximate differential is almost everywhere equal to $Du/\mathcal{L}^n$. By (2.3) and the Calderon-Zygmund theorem we immediately get:

PROPOSITION 2.2. *Any function $u \in GSBV(\Omega)$ is approximately differentiable almost everywhere.*

By (2.3), the approximate differential satisfies the condition (1.7).

Given a Lipschitz function $\tau : \Omega \to \mathbf{R}$, the following co-area formula

$$\int_{-\infty}^{+\infty} \int_{\{\tau = t\}} \theta(x)\, d\mathscr{H}^{n-1}(x)\, dt = \int_{\Omega} \theta(x) |\nabla\tau|\, dx \tag{2.5}$$

holds (see [19], Section 3.2.12) for every non-negative Borel function θ. In particular, applying (2.5) with $\theta = 1$, $\tau(x) = s' \wedge \operatorname{dist}(x, A) \vee s$, and recalling that (see [19], Section 3.2.34)

$$|\nabla \operatorname{dist}(\cdot, A)| = 1 \quad \text{a.e. on} \quad \mathbf{R}^n \setminus \bar{A},$$

we get

(2.6)

$$\operatorname{meas}(\{x \in \Omega : s < \operatorname{dist}(x, A) < s'\}) = \int_s^{s'} \mathscr{H}^{n-1}(\{x \in \Omega : \operatorname{dist}(x, A) = t\})\, dt$$

whenever $0 < s < s'$, $A \subset \mathbf{R}^n$. We have also

$$\operatorname{meas}(\{x \in \Omega : \operatorname{dist}(x, A) = t\}) = 0 \qquad \forall t > 0. \tag{2.7}$$

In the particular case $n = 1$, the co-area formula

$$\int_{-\infty}^{+\infty} \sum_{x \in \tau^{-1}(t)} \theta(x)\, dt = \int_{\Omega} \theta(x) |\nabla\tau|\, dx \qquad \theta \in \mathscr{B}(\Omega), \theta \geqq 0 \tag{2.8}$$

holds even for functions $\tau \in W^{1,1}(\Omega)$. In fact, there exists (see [19], Section 3.1.16) an increasing sequence of compact set $K_k \subset \Omega$ such that

$$E = \Omega \setminus \bigcup_{k=1}^{\infty} K_k$$

is negligible, and u on K_k is the restriction of a Lipschitz function τ_k. Since (2.3), (2.5) yield

$$\begin{aligned}\int_{-\infty}^{+\infty} \int_{\{\tau = t\} \cap K_k} \theta(x)\, d\mathscr{H}^0(x)\, dt &= \int_{-\infty}^{+\infty} \int_{\{\tau_k = t\} \cap K_k} \theta(x)\, d\mathscr{H}^0(x)\, dt \\ &= \int_{K_k} \theta(x) |\nabla\tau_k|\, dx = \int_{K_k} \theta(x) |\nabla\tau|\, dx,\end{aligned}$$

and since $\operatorname{meas}(u(E)) = 0$, by letting $k \to +\infty$ we get (2.8).

Now, we need to recall some technical results concerning the one-dimensional sections of functions of bounded variation. Let $\nu \in \mathbf{S}^{n-1}$ be a fixed direction. We set

$$(2.9)\qquad \begin{cases} \pi_\nu = \{x \in \mathbf{R}^n : \langle x, \nu \rangle = 0\}; \\ \Omega_x = \{t \in \mathbf{R} : x + t\nu \in \Omega\} \quad (x \in \pi_\nu); \\ \Omega_\nu = \{x \in \pi_\nu : \Omega_x \neq \emptyset\}. \end{cases}$$

The sets Ω_x are the one-dimensional slices of Ω indexed by $x \in \pi_\nu$, and Ω_ν is the projection of Ω on π_ν. We also consider the restrictions $u_x \in \mathscr{B}(\Omega_x)$ of the function u, defined by

$$(2.10)\qquad u_x(t) = u(x + t\nu) \qquad (x \in \Omega_\nu).$$

In [1] the following structure theorem is proved (we consider for simplicity functions $u \in SBV(\Omega) \cap L^\infty(\Omega)$).

THEOREM 2.3. *Let $u \in L^\infty(\Omega)$ be a function such that*

$$(\mathrm{i})\qquad u_x \in SBV(\Omega_x) \quad \text{for} \quad \mathscr{H}^{n-1} - \text{a.e. } x \in \Omega_\nu,$$

and

$$(\mathrm{ii})\qquad \int_{\Omega_\nu} \left[\int_{\Omega_x} |\nabla u_x| \, dt + \mathscr{H}^0(S_{u_x}) \right] d\mathscr{H}^{n-1}(x) < +\infty,$$

for any choice of $\nu \in \mathbf{S}^{n-1}$. Then, $u \in SBV(\Omega)$ and $\mathscr{H}^{n-1}(S_u) < +\infty$. Conversely, let $u \in SBV(\Omega) \cap L^\infty(\Omega)$ such that $\mathscr{H}^{n-1}(S_u) < +\infty$. Then, the conditions (i), (ii) *are satisfied for every $\nu \in \mathbf{S}^{n-1}$. We also have*

$$(\mathrm{iii})\qquad \langle \nabla u(x + t\nu), \nu \rangle = \nabla u_x(t) \quad \text{for} \quad \text{a.e. } t \in \Omega_x$$

for $\mathscr{H}^{n-1}$-almost every $x \in \Omega_\nu$, and there exists a Borel function $\nu_u : S_u \to \mathbf{S}^{n-1}$ depending only on S_u such that

$$(\mathrm{iv})\qquad \int_{S_u} |\langle \nu_u, \nu \rangle| \, d\mathscr{H}^{n-1} = \int_{\Omega_\nu} \mathscr{H}^0(S_{u_x}) \, d\mathscr{H}^{n-1}(x).$$

By using Theorem 2.3, we can recover the structure of functions $u \in SBV(\Omega)$ by the structure of the one-dimensional sections u_x. In particular, we recover the approximate differential by the (classical) differentials of the functions u_x, and the jump set S_u by the jump sets S_{u_x} of u_x. The function ν_u in (iv) is normal to S_u in an approximate sense (see [19], Section 3.2.16). Note that (iv) yields $\mathscr{H}^0(S_{u_x}) < +\infty$ for $\mathscr{H}^{n-1}$-almost every $x \in \Omega_\nu$, so that u_x is piecewise absolutely continuous for $\mathscr{H}^{n-1}$-almost every x.

Similar results also hold for $GSBV(\Omega)$, without the restriction $\mathscr{H}^{n-1}(S_u) < +\infty$. A particular case is the following well-known result (see [1], [19]): if $u \in W^{1,1}(\Omega)$, then $\mathscr{H}^{n-1}(S_u) = 0$ and

$$u_x \in W^{1,1}(\Omega_x), \quad \langle \nabla u_x(x + t\nu), \nu \rangle = \nabla u_x(t) \quad \text{a.e. in} \quad \Omega_x \tag{2.11}$$

for $\mathscr{H}^{n-1}$-almost every $x \in \Omega_\nu$, for any choice of the direction ν. Conversely, if $u \in L^1(\Omega)$ satisfies (2.11) and

$$\int_{\Omega_\nu} \int_{\Omega_x} |\nabla u_x| \, dt \, d\mathscr{H}^{n-1}(x) < +\infty \tag{2.12}$$

for any choice of the direction ν, then $u \in W^{1,1}(\Omega)$.

Finally, we prove a lemma which will be often useful in the sequel.

LEMMA 2.4. *Let $v : \Omega \to \mathbf{R}$ be a bounded continuous function, and let $a \in \mathbf{R}$. Then,*

$$v \in W^{1,1}(\Omega \setminus \{v = a\}) \quad \Rightarrow \quad v \in W^{1,1}(\Omega).$$

Proof: By the above mentioned slicing properties of $W^{1,1}$ functions, it is not restrictive to assume $n = 1$. Let $C = \{x \in \Omega : v(x) = a\}$. By the differentiability properties of absolutely continuous functions and by (2.4) we obtain that the approximate differential ∇v exists almost everywhere in Ω and $\nabla v \in L^1(\Omega)$, $\nabla v = 0$ almost everywhere on C. Let $A_i = (s_i, t_i)$ be the connected components of $\Omega \setminus C$, let $x_0 \in C$ be a fixed point, $b \in \Omega \setminus C$. If $b \geqq x_0$, we need only to show that

$$v(b) = \int_{\Omega \cap [x_0, b]} \nabla v \, dt + a.$$

Let j be such that $b \in (s_j, t_j)$; since $s_i, t_i \in C$ we get

$$\int_{\Omega \cap [x_0, b]} \nabla v \, dt = \int_{\Omega \cap [x_0, b] \setminus C} \nabla v \, dt = \sum_{\substack{i : t_i < b \\ s_i \geqq x_0}} \int_{s_i}^{t_i} \nabla v \, dt + \int_{s_j}^{b} \nabla v \, dt = -a + v(b).$$

The proof in the case $b \leqq x_0$ is similar.

3. The Approximating Functionals F_h

In this section we find a domain suitable for coercivity and lower semicontinuity of the functionals F_h formally defined by

$$F_h(u, z) = \int_\Omega [|\nabla u|^2 + |\nabla z|^2](1 - z^2)^{2h} + \frac{1}{4}(\alpha^2 h^2) z^2 \, dx. \tag{3.1}$$

In equation (3.1) the set $\Omega \subset \mathbf{R}^n$ is open and bounded, and $h \geqq 1$ is a fixed integer. This formula certainly makes sense if u, z belong to the Sobolev space $W^{1,2}(\Omega)$. However, the coefficient $(1 - z^2)^{2h}$ multiplying the gradients jeopardizes coercivity.

We shall identify a natural domain $\mathcal{D}_{h,n}(\Omega)$ of the functional F_h, and we prove compactness and lower semicontinuity properties which ensure existence of minimizers of the functionals

$$F_h(u, z) + \beta \int_\Omega |u - g|^\gamma \, dx \qquad \beta, \gamma > 0, g \in \mathcal{B}(\Omega). \tag{3.2}$$

Finally, we prove equicoercivity of the functionals F_h, according to Proposition 2.1 (iii).

We often set $w = (u, z)$, and we always tacitly assume that $0 \leqq z \leqq 1$ almost everywhere. To discover the finiteness domain of the functional F_h we are led to consider suitable changes of variables. We set

$$\varphi(t) = \int_0^t (1 - s^2)^h \, ds, \qquad \psi(s, t) = s(1 - t^2)^{h+1}. \tag{3.3}$$

For functions $w \in W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ the chain rule easily yields

$$F_h(w) \geqq \int_\Omega |\nabla(\varphi \circ z)|^2 \, dx, \tag{3.4}$$

and, whenever u is bounded, we get

$$F_h(w) \geqq c \int_\Omega |\nabla(\psi(u, z))|^2 \, dx, \tag{3.5}$$

for a constant c depending only on $\|u\|_\infty$. Hence, it is natural to define

$$\begin{aligned} \mathcal{D}_{h,n} = \{(u, z) \in \mathcal{B}(\Omega) \times \mathcal{B}(\Omega) : \varphi \circ z \in W^{1,2}(\Omega), \\ \psi(N \wedge u \vee -N, z) \in W^{1,2}(\Omega) \ \forall N \in \mathbf{N}\}. \end{aligned} \tag{3.6}$$

Since $\varphi \in C^1([0, 1))$, it is easily seen using Proposition 2.2 that any function $w =$

$(u, z) \in \mathcal{D}_{h,n}$ is approximately differentiable almost everywhere in the set $\{x \in \Omega : z(x) < 1\}$. Thus, expression (3.1) certainly makes sense for $w \in \mathcal{D}_{h,n}$ provided the product $|\nabla w|^2(1 - z^2)^{2h}$ is set by definition equal to 0 in the set $\{x \in \Omega : z(x) = 1\}$.

In the particular case $n = 1$, the functions $w \in \mathcal{D}_{h,n}$ such that $F_h(w) < +\infty$ can be easily characterized.

PROPOSITION 3.1. *Let $w = (u, z) \in \mathcal{B}(\Omega) \times \mathcal{B}(\Omega)$. Then $w \in \mathcal{D}_{h,1}$, $F_h(w) < +\infty$ if and only if z is continuous, $w \in W^{1,2}_{\mathrm{loc}}(\{z < 1\})$ and*

$$\int_\Omega |\nabla w|^2(1 - z^2)^{2h}\, dt < +\infty. \tag{3.7}$$

Proof: Assume that $w \in \mathcal{D}_{h,1}$, $F_h(w) < +\infty$. Then, since φ is strictly increasing and continuous, and $\varphi \circ z$ is continuous, also z is continuous. Let $A \Subset \{x \in \Omega : z(x) < 1\}$ be an open set. There exists $\gamma \in (0, 1)$ such that $z(x) \leqq \gamma$ on A. Since $\varphi^{-1} \in C^1([0, \varphi(\gamma)])$ and

$$z = \varphi^{-1} \circ \varphi \circ z, \qquad u = \frac{\psi(u, z)}{(1 - z^2)^{h+1}}$$

we obtain that $w \in W^{1,2}(A)$. The proof of the opposite implication is an easy consequence of Lemma 2.4.

Now we prove a compactness theorem for the sublevels of the functional in (3.2).

THEOREM 3.2. *Let $\beta, \gamma, C > 0$, $g \in \mathcal{B}(\Omega)$, and let*

$$\mathcal{K} = \left\{ w \in \mathcal{D}_{h,n} : F_h(w) + \beta \int_\Omega |u - g|^\gamma\, dx \leqq C \right\}.$$

Then, given any sequence $(w_k) \subset \mathcal{K}$ there exists a subsequence (w_{k_p}) and $w \in \mathcal{D}_{h,n}$ such that z_{k_p} converges almost everywhere to z, and u_{k_p} converges almost everywhere to u in the set $\{x \in \Omega : z(x) < 1\}$. Moveover, we can assume that $u(x) = g(x)$ on $\{x \in \Omega : z(x) = 1\}$.

Proof: By the inequality (3.4) we obtain that the family

$$\mathcal{P} = \{\varphi \circ z : w = (u, z) \in \mathcal{K}\}$$

is bounded in $W^{1,2}(\Omega)$. Similarly, (3.5) yields that the family

$$\mathcal{G}_N = \{\psi(N \wedge u \vee -N, z) : w = (u, z) \in \mathcal{K}\}$$

is bounded in $W^{1,2}(\Omega)$ for any $N \in \mathbf{N}$. Given a sequence $(w_k) \subset \mathcal{H}$, we can assume up to subsequences that there exists a Borel function $z : \Omega \to [0, 1]$ such that

$$\varphi \circ z_k \to \varphi \circ z \in W^{1,2}(\Omega)$$

almost everywhere in Ω, and there exist Borel functions $v_N \in \mathcal{B}(\Omega)$ such that

$$\psi(N \wedge u_k \vee -N, z_k) = (N \wedge u_k \vee -N)(1 - z_k^2)^{h+1} \to v_N \in W^{1,2}(\Omega)$$

almost everywhere in Ω for any $N \in \mathbf{N}$. This easily implies that z_k converges almost everywhere to z and u_k converges almost everywhere to $u(x) \in [-\infty, +\infty]$ in the set $\{x \in \Omega : z(x) < 1\}$. We set by definition $u(x) = g(x)$ if $z(x) = 1$, and we claim that $w = (u, v) \in \mathcal{D}_{h,n}$. In fact, the conditions listed in (3.6) are fulfilled by construction. We need only to verify that $u(x) \in \mathbf{R}$ almost everywhere. Let

$$g(x, t) = \begin{cases} \beta |t - g(x)|^{\gamma} & \text{if} \quad t \in \mathbf{R}; \\ +\infty & \text{if} \quad t = \pm\infty. \end{cases}$$

The function $g(x, t)$ is lower semicontinuous in t, therefore the Fatou lemma yields

$$\int_{\{z<1\}} g(x, u) \leqq \liminf_{k \to +\infty} \int_{\{z<1\}} g(x, u_k)\, dx \leqq C < +\infty,$$

so that $u(x)$ is finite almost everywhere.

Now we prove the lower semicontinuity of the functional in (3.2). We first need a slicing theorem for functions in $\mathcal{D}_{h,n}$.

THEOREM 3.3. *Let $w \in \mathcal{D}_{h,n}(\Omega)$, and let $\nu \in \mathbf{S}^{n-1}$. Then, recalling notations* (2.9), (2.10), *the functions w_x belong to $\mathcal{D}_{h,1}(\Omega_x)$ and*

$$\nabla u_x(t) = \langle \nabla u(x + t\nu), \nu \rangle,$$

$$\nabla z_x(t) = \langle \nabla z(x + t\nu), \nu \rangle \quad \text{a.e. on} \quad \Omega_x \setminus \{t : z(x + t\nu) = 1\},$$

for $\mathcal{H}^{n-1}$-almost every $x \in \Omega_\nu$.

Proof: It is a straightforward consequence of (2.11), (2.3), and (3.6).

THEOREM 3.4. *Let $(w_k) = (u_k, z_k) \subset \mathcal{D}_{h,n}(\Omega)$, $w = (u, z) \in \mathcal{D}_{h,n}(\Omega)$ be such that*

$$\begin{cases} u_k \to u & \text{almost everywhere on } \{x \in \Omega : z(x) < 1\}; \\ z_k \to z & \text{almost everywhere on } \Omega. \end{cases}$$

Then

$$F_h(w) \leqq \liminf_{k \to +\infty} F_h(w_k).$$

Moreover, if $u(x) = g(x)$ *on* $\{x \in \Omega : z(x) = 1\}$, *then*

$$F(w) + \beta \int_\Omega |u - g|^\gamma \, dx \leqq \liminf_{k \to +\infty} F(w_k) + \beta \int_\Omega |u_k - g|^\gamma \, dx.$$

Proof: It is not restrictive to assume that $F_h(w_k)$ is bounded, hence by (3.4) the sequence $\varphi \circ z_k$ is bounded in $W^{1,2}(\Omega)$ and weakly converges to $\varphi \circ z$. The functional F_h can be represented as follows:

$$(3.8) \quad F_h(w) = \int_\Omega |\nabla u^2|(1 - z^2)^{2h} + \int_\Omega |\nabla(\varphi \circ z)|^2 \, dx + \frac{1}{4}(\alpha^2 h^2) \int_\Omega z^2 \, dx.$$

The second and the third term in the right member of (3.8) are trivially lower semicontinuous. In order to prove the lower semicontinuity of the first term, we first consider the case $n = 1$. In this case the weak convergence in $W^{1,2}(\Omega)$ of $\varphi \circ z_k$ yields uniform convergence of $\varphi \circ z_k$, therefore z_k uniformly converges to z. Let $A \Subset \{x \in \Omega : z(x) < 1\}$ be an open set. Since for k large enough $A \Subset \{x \in \Omega : z_k(x) < 1\}$ as well, the functions $u_k \in W^{1,2}(A)$ and they are converging in $L^1(A)$, a well-known lower semicontinuity theorem in Sobolev spaces yields (see, for instance, [32], Theorem 2, page 144)

$$\int_A |\nabla u|^2(1 - z^2)^{2h} \, dx \leqq \liminf_{k \to +\infty} \int_A |\nabla u_k|^2(1 - z_k^2)^{2h} \, dx$$

$$\leqq \liminf_{k \to +\infty} \int_\Omega |\nabla u_k|^2(1 - z_k^2)^{2h} \, dx.$$

We get the desired inequality by letting $A \uparrow \{x \in \Omega : z(x) < 1\}$.

Now we briefly describe how, by using the slicing Theorem 3.3, it is possible to get the lower semicontinuity inequality in the general case $n \geqq 1$. The same argument will be described more carefully in Section 4. By using the slicing Theorem 3.3 and the Fatou lemma we get

$$\int_A |\langle \nabla u, \nu \rangle|^2(1 - z^2)^{2h} \, dx \leqq \liminf_{k \to +\infty} \int_A |\langle \nabla u_k, \nu \rangle|^2(1 - z_k^2)^{2h} \, dx$$

$$\leqq \liminf_{k \to +\infty} \int_A |\nabla u_k|^2(1 - z_k^2)^{2h} \, dx$$

for every open set $A \subset \Omega$ and every $\nu \in \mathbf{S}^{n-1}$. By taking the supremum in the lattice of measures (see (4.8)) we conclude the proof of the first inequality. The second inequality is a straightforward consequence of the Fatou lemma.

LEMMA 3.5. *Let $z : \Omega \to [0, 1]$ be a function such that $\varphi \circ z \in W^{1,2}(\Omega)$. Then*

(i) $z_\gamma = z \wedge \gamma \in W^{1,2}(\Omega)\ \forall \gamma < 1, \quad c = (1 - z^2)^{h+1} \in W^{1,2}(\Omega)$;

(ii) $$\lim_{\gamma \uparrow 1} F_h(u, z_\gamma) = F_h(u, z) \qquad \forall u \in W^{1,2}(\Omega).$$

Proof: (i) Since $\varphi^{-1} \in C^1([0, \varphi(1))$,the function $z_\gamma = \varphi^{-1}((\varphi \circ z) \wedge \varphi(\gamma))$ belongs to $W^{1,2}(\Omega)$. Since by (2.3) z is approximately differentiable almost everywhere and $|(1 - z^2)^h \nabla z| = |\nabla(\varphi \circ z)| \in L^2(\Omega)$, it can be easily seen that the set

$$c_\gamma = (1 - z_\gamma^2)^{h+1} \qquad 0 \leqq \gamma < 1$$

is bounded in $W^{1,2}(\Omega)$, hence $c \in W^{1,2}(\Omega)$.

(ii) Trivial.

We do not know whether for any functions $w \in \mathcal{D}_{h,n}(\Omega)$ it is possible to find functions $w_k \in W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ such that $F_h(w_k) \to F_h(w)$. Now we prove the equicoercivity of the functionals (3.2).

THEOREM 3.6. *Let $(w_h) = (u_h, z_h) \subset \mathcal{D}_{h,n}(\Omega)$ be a sequence such that*

$$F_h(w_h) + \beta \int_\Omega |u_h - g|^\gamma \, dx \leqq C < +\infty$$

for some constants $\beta, \gamma, C > 0$ and $g \in \mathcal{B}(\Omega)$. Then, there exists a subsequence u_{h_k} and $u \in \mathcal{B}(\Omega)$ such that $(u_{h_k}, z_{h_k}) \to (u, 0)$ in $\mathcal{B}(\Omega) \times \mathcal{B}(\Omega)$.

Proof: Since

$$F_h(w_h) \geqq \frac{1}{4}(\alpha^2 h^2) \int_\Omega z_h^2 \, dx,$$

the sequence z_h strongly converges to 0 in $L^2(\Omega)$. Let us consider

$$c_h = (1 - z_h^2)^{h+1}, \qquad \psi_{h,N} = (N \wedge u_h \vee -N)(1 - z_h^2)^{h+1}.$$

Since Lemma 3.5 and (3.6) yield $c_h, \psi_h \in W^{1,2}(\Omega)$ and

$$\int_\Omega |\nabla c_h| \, dx \leqq 2(h+1) \int_\Omega z_h(1-z_h^2)^h |\nabla z_h| \, dx \leqq 4F_h(w_h),$$

$$\int_\Omega |\nabla \psi_{h,N}|^2 \, dx \leqq c(N) F_h(w_h),$$

there is no loss of generality if we assume that z_h converges to 0, c_h converges to $c \in BV(\Omega)$ (see [21]), and $\psi_{h,N}$ converges to $\psi_N : \Omega \to [-N, N]$ almost everywhere in Ω for every natural number $N \in \mathbf{N}$. Since

$$\int_\Omega \liminf_{h \to +\infty} h^2 z_h^2 \, dx \leqq \liminf_{h \to +\infty} \int_\Omega h^2 z_h^2 \, dx < +\infty,$$

we have

$$\liminf_{h \to +\infty} h^2 z_h^2 < +\infty$$

almost everywhere in Ω, so that $c = 1$ almost everywhere. In particular

$$N \wedge u_h \vee -N = \psi_{h,N}/c_h \to \psi_N$$

almost everywhere, and this gives a pointwise limit $u(x) \in [-\infty, +\infty]$ of $u_h(x)$ outside a negligible set. By the same argument of Theorem 3.2 we obtain that $u(x) \in \mathbf{R}$ almost everywhere.

4. The Inequality for the Lower Γ-Limit

In this section we prove the inequality:

$$\liminf_{h \to +\infty} F_h(u_h, z_h) \geqq F(u)$$

for all sequences $w_h = (u_h, z_h) \in \mathcal{D}_{h,n}(\Omega)$ such that $(u_h, z_h) \to (u, 0) = w$ in $\mathcal{B}(\Omega) \times \mathcal{B}(\Omega)$. In the following, it will be convenient for us also to consider functionals depending on the domain of integration Ω. This leads to the following definition: we set

$$\begin{aligned} F_-(w, \Omega) = F_-((u,0), \Omega) = \inf\{ & \liminf_{h \to +\infty} F_h(w_h, \Omega) : w_h \in \mathcal{D}_{h,n}(\Omega), \\ & w_h \to (u,0) \text{ in } \mathcal{B}(\Omega) \times \mathcal{B}(\Omega)\}, \end{aligned} \tag{4.1}$$

where $F_h(w, \Omega)$ are the functionals studied in Section 3 (we emphasize the dependence of F_h on the domain Ω). Then, the goal of this section can be restated as follows:

PROPOSITION 4.1. *For all $\Omega \subset \mathbf{R}^n$ open and all $u \in \mathcal{B}(\Omega)$ we have*

$$F_-((u, 0), \Omega) \geqq F(u, \Omega), \tag{4.2}$$

where

$$F(u, \Omega) = \begin{cases} \int_\Omega |\nabla u|^2 \, dx + \alpha \mathscr{H}^{n-1}(S_u) & \text{if } u \in GSBV(\Omega); \\ +\infty & \text{otherwise}. \end{cases} \tag{4.3}$$

For notational simplicity, we set $\alpha = 1$. We underscore some facts that are often used in what follows. First, we notice that F_- is increasing in the set variable and

$$F(u, \Omega) = \sup\{F(u, A) : A \text{ open}, A \Subset \Omega\},$$

so that there is no loss of generality if we assume Ω bounded. Moreover, (1.3), (1.7), (2.3) yield

$$\begin{gathered} F(u, \Omega) = \lim_{N \to +\infty} F(N \wedge u \vee -N, \Omega) \\ F_-((u, 0), \Omega) \geqq F_-((N \wedge u \vee -N, 0), \Omega) \qquad \forall N \in \mathbf{N}, \end{gathered} \tag{4.4}$$

so that we can also assume u to be a bounded function. Therefore, in the following we always tacitly assume the functions and the open sets to be bounded. Finally we remark that F_- is not only monotone increasing in the set variable but also superadditive, i.e.,

$$F_-(w, A) + F_-(w, B) \leqq F_-(w, A \cup B) \quad \text{whenever} \quad A \cap B = \emptyset.$$

To prove Proposition 4.1 we reduce the statement to the case $n = 1$ by a slicing argument, and then we prove the inequality in this case by using essentially Lemma 4.2 below. To avoid confusion, when $n = 1$ we denote the approximating functionals by G_h, and their lower Γ-limit (defined by (4.1)) by G_-. We also recall that by Proposition 3.1 we have

$$w = (u, z) \in \mathcal{D}_{h,1}(\Omega) \quad \Rightarrow \quad z \quad \text{continuous}, \quad u, z \in W^{1,2}_{\text{loc}}(\Omega \setminus \{z = 1\})$$

and

$$G_h(w, \Omega) = \int_{\Omega \setminus \{z=1\}} [|\nabla u|^2 + |\nabla z|^2](1 - z^2)^{2h} + \frac{1}{4}(\alpha^2 h^2) z^2 \, dx.$$

Similarly, the functional corresponding to $F(u, \Omega)$ in (4.3) with $n = 1$ will be denoted by G.

LEMMA 4.2. *Let $w = (u, 0)$ be defined on an open ball $B_\eta(x)$ of $x \in \mathbf{R}$. Then,*

(i)
$$G_-(w, B_\rho(x)) \geqq \int_{B_\rho(x)} |\nabla u|^2 \, dt$$

for all $\rho \in (0, \eta)$ such that $u \in W^{1,2}(B_\rho(x))$. On the other hand, if $u \notin W^{1,2}(B_\rho(x))$ for any $\rho \in (0, \eta)$, then

(ii)
$$G_-(w, B_\rho(x)) \geqq 1 \quad \forall \rho \in (0, \eta).$$

In particular, if I is an open bounded interval and $G_-(w, I) < +\infty$, then there is a set $J \subset I$ whose cardinality is less than $G_-(w, I)$ such that $u \in W^{1,2}(I \backslash J)$. Thus $u \in SBV(I)$ and

$$\int_I |\nabla u|^2 \, dx + \mathscr{H}^0(S_u \cap I) \leqq G_-(w, I).$$

Before proving this central lemma we show how it allows the reduction of Proposition 4.1 to the case $n = 1$. Since the slices Ω_x in (2.9) are generic open sets in $\mathbf{R}$, we need the following lemma.

LEMMA 4.3. *If $G_-(w, I) \geqq G(w, I)$ for any bounded open interval I, then the same inequality holds for any bounded open set $I \subset \mathbf{R}$.*

Proof: Let I be an open bounded subset of $\mathbf{R}$, and let $\{I_k\}_{k \in \mathbf{N}}$ be the set of its connected components (possibly $I_k = \varnothing$ for large k). Since G_- is increasing and superadditive, we have

$$G_-(w, I) \geqq G_-\left(w, \bigcup_{k=1}^{p} I_k\right) \geqq \sum_{k=1}^{p} G_-(w, I_k) \geqq \sum_{k=1}^{p} G(w, I_k) = G\left(w, \bigcup_{k=1}^{p} I_k\right)$$

for any $p > 0$. Since $G(w, \cdot)$ is the trace of a Borel measure, the statement is proved by letting $p \to +\infty$.

Proof of Proposition 4.1: Let Ω be a bounded open subset of $\mathbf{R}^n$, and let $w = (u, 0)$ with u bounded function. We assume that $F_-(w, \Omega) < +\infty$. We recall notations (2.9): π_ν is the hyperplane orthogonal to $\nu \in \mathbf{S}^{n-1}$, $\Omega_\nu \subset \pi_\nu$ is the projection of Ω on π_ν, and Ω_x, w_x are the one-dimensional slices of Ω and w indexed by $x \in \Omega_\nu$. We remember that for every $\nu \in S^{n-1}$ we have (Theorem 3.3)

$$w \in \mathcal{D}_{h,n}(\Omega) \quad \Rightarrow \quad w_x \in \mathcal{D}_{h,1}(\Omega_x),$$

and

$$\nabla u_x(t) = \langle \nabla u(x + t\nu), \nu \rangle,$$

$$\nabla v_x(t) = \langle \nabla v(x + t\nu), \nu \rangle \quad \text{a.e. on} \quad \Omega_x \setminus \{t : z(x + t\nu) = 1\},$$

for $\mathcal{H}^{n-1}$-almost all $x \in \Omega_\nu$. This is of fundamental importance. In fact, given $\nu \in S^{n-1}$, A open subset of Ω and a sequence $(w_h) \subset \mathcal{D}_{h,n}(A)$ such that

$$\lim_{h \to +\infty} F_h(w_h, A) = F_-(w, A),$$

by the Fatou lemma we infer

$$(4.5) \qquad \int_{A_\nu} \liminf_{h \to +\infty} G_h(w_{hx}, A_x)\, d\mathcal{H}^{n-1}(x) \leqq \liminf_{h \to +\infty} F_h(w_h, A) \leqq F_-(w, A).$$

Since w_{hx} converges to w_x for $\mathcal{H}^{n-1}$-almost every $x \in \Omega_\nu$, and since $F_-(w, \Omega) < +\infty$ it follows that $G_-(w_x, \Omega_x)$ is finite for $\mathcal{H}^{n-1}$-almost every $x \in \Omega_\nu$. Lemma 4.2 and Lemma 4.3 yield $u_x \in SBV(\Omega_x) \cap L^\infty(\Omega_x)$ for $\mathcal{H}^{n-1}$-almost all $x \in \Omega_\nu$ and

$$(4.6) \qquad \int_{A_x} |\nabla u_x|^2\, dt + \mathcal{H}^0(S_{u_x} \cap A_x) = G(w_x, A_x) \leqq G_-(w_x, A_x).$$

By integrating (4.6) on A_ν, using (4.5) and Theorem 3.3 we get

$$(4.7) \qquad \begin{aligned} &\int_A |\langle \nabla u, \nu \rangle|^2\, dx + \int_{S_u \cap A} |\langle \nu_u, \nu \rangle\, d\mathcal{H}^{n-1} \\ &\qquad = \int_{A_\nu} \left[\int_{A_x} |\nabla u_x|^2\, dt + \mathcal{H}^0(S_{u_x} \cap A_x) \right] dx \leqq F_-(w, A). \end{aligned}$$

Setting $A = \Omega$ in the inequality above and using the Hölder inequality we obtain $u \in SBV(\Omega)$ because ν is arbitrary. As A varies in the open subsets of Ω one has that $F(u, A)$ is the trace of a Radon measure. Moreover, it is the least upper bound in the lattice of Radon measures of the integrals in (4.7):

$$F(u, \cdot) = \sup_{\nu \in S^{n-1}} \{ |\langle \nabla u, \nu \rangle|^2 \cdot \mathcal{L}^n + |\langle \nu_u, \nu \rangle| \cdot \mathcal{H}^{n-1} \llcorner S_u \}.$$

More explicitly we have

$$(4.8)\qquad F(u,\Omega)=\sup\left\{\sum_{i=1}^{+\infty}\int_{A_i}|\langle\nabla u,\nu_i\rangle|^2\,dx+\int_{S_u\cap A_i}|\langle\nu_u,\nu_i\rangle|\,d\mathscr{H}^{n-1}\right\},$$

where $\{A_i\}_{i\in\mathbf{N}}$ is any sequence of pairwise disjoint open subsets of A and $\nu_i\in\mathbf{S}^{n-1}$ for each $i\in\mathbf{N}$. Since by (4.7) each term of each summation in (4.8) is dominated by $F_-(w,A_i)$, by superadditivity and monotonicity of F_- we get our assertion.

Proof of Lemma 4.2: We assume that $G_-(w,B_\rho(x))<+\infty$. Let us first consider the case $u\in W^{1,2}(B_\rho(x))$. Up to subsequences, we can also suppose that

$$\exists\lim_{h\to+\infty}G_h(w_h,B_\rho(x))$$

$$=G_-(w,B_\rho(x))<+\infty,\qquad \exists\lim_{h\to+\infty}\int_{B_\rho(x)}|\nabla u_n|^2(1-z_h^2)^{2h}\,dx,$$

and

$$\lim_{h\to+\infty}w_h(y)=(u(y),0)\quad\text{a.e. on}\quad B_\rho(x).$$

We recall the definition of G_h in this case:

$$G_h(w_h,B_\rho(x))=\int_{B_\rho(x)}\left[|\nabla u_h|^2(1-z_h^2)^{2h}+|\nabla z_h|^2(1-z_h^2)^{2h}+\frac{1}{4}h^2z_h^2\right]dy.$$

Under these hypotheses we will prove that

$$(4.9)\qquad \liminf_{h\to+\infty}\int_{B_\rho(x)}[|\nabla u_h|^2(1-z_h^2)^{2h}]\,dy\geqq\int_{B_\rho(x)}|\nabla u|^2\,dy.$$

By the same argument of Theorem 3.6, the coefficients $c_h=(1-z_h^2)^{h+1}\in W^{1,2}(B_\rho(x))$ converge almost everywhere to 1. However, we see that ∇c_h cannot be uniformly controlled in any $L^p(B_\rho(x))$ with $p>1$, hence uniform convergence of c_h is not ensured and this does not allow the direct application of a semicontinuity argument. Our choice is to prove, by studying the limit behavior of the areas where the c_h are near to 1, that for any $\delta<1$ there exists an open set $J\subset B_\rho(x)$ of full measure (i.e., meas$(J)=2\rho$) such that

$$K\subset\{y\in B_\rho(x):c_h(y)\geqq\delta\}\quad\text{for}\quad h\quad\text{large enough},$$

for any compact set $K \subset J$. We fix $\sigma \in (\delta, 1)$. The requirement of full measure will be directly achieved by the almost sure convergence of c_h to 1 by the condition

$$B_\rho(x) \setminus J \subset \bigcup_{m \in \mathbf{N}} \bigcap_{h \geq m} (\{ y \in B_\rho(x) : c_h(y) \leqq \sigma \}) \tag{4.10}$$

up to a finite set. We denote the closed superlevel sets $\{ y \in B_\rho(x) : c_h(y) \geqq t \}$ by A_h^t. The suggestion is that if each superlevel set were an interval then their limits would also be intervals. Hence we look for a uniform control of the number of connected components of the superlevel sets, by using the co-area formula (2.8) and boundedness in $W^{1,1}(B_\rho(x))$ of c_h. Then we consider the Kuratowski limit of these components. By using (2.8) with $\theta = 1$ we get

$$+\infty > M \geqq \int_{B_\rho(x)} \left[|\nabla z_h|^2 (1 - z_h^2)^{2h} + \frac{1}{4} h^2 z_h^2 \right] dy$$

$$\geqq \frac{1}{4} \int_{B_\rho(x)} |\nabla c_h| \, dy \geqq \frac{1}{4} \int_\delta^\sigma \mathscr{H}^0(\{ y : c_h(y) = t \}) \, dt.$$

Hence there exist real numbers $\delta_h \in (\delta, \sigma)$ such that

$$\frac{4M}{\sigma - \delta} \geqq \mathscr{H}^0(\{ y : c_h(y) = \delta_h \}),$$

so that

$$A_h^{\delta_h} \quad \text{has at most} \quad \frac{4M}{\sigma - \delta} + 1 \quad \text{connected components.}$$

Let us rename $A_h^{\delta_h}$ as A_h. Then there is a number k depending on $\sigma - \delta$ and M, such that for each h

$$A_h = \bigcup_{i=1}^k I_h^i, \quad \forall i < j \quad \max I_h^i < \min I_h^j, \quad \forall i \; I_h^i \quad \text{is a closed interval or} \quad \varnothing.$$

Up to subsequences, for each i let I_∞^i be the Kuratowski limit of the I_h^i. We have

$$I_\infty^i \quad \text{connected and closed,} \quad \forall i < j \quad \max I_\infty^i \leqq \min I_\infty^j.$$

We set

$$J = \bigcup_{i=1}^k \operatorname{int}(I_\infty^i).$$

Since

$$B_\rho(x)\setminus A_h \subset \{y \in B_\rho(x) : c_h(y) \leqq \sigma\},$$

the set J satisfies condition (4.10) (the exceptional set contains at most k points). Moreover, since J has at most k connected components, every compact subset of J is contained for h large enough in A_h^δ, therefore

$$\liminf_{h\to+\infty} \int_{B_\rho(x)} [|\nabla u_h|^2(1 - z_h^2)^{2h}]\, dy$$
$$\geqq \liminf_{h\to+\infty} \int_K [|\nabla u_h|^2(1 - z_h^2)^{2h}]\, dy \geqq \delta^2 \int_K |\nabla u|^2\, dy,$$

for every open $K \Subset J$. By letting $K \uparrow J$ and $\delta \uparrow 1$, we prove (4.9).

Now, let us assume that $u \notin W^{1,2}(B_\rho(x))$ for any $\rho < \eta$ and let us show inequality (ii). We fix $\rho \in (0, \eta)$. Possibly extracting a subsequence, we assume that

$$\exists \lim_{h\to+\infty} G_h(w_h, B_\rho(x)) = G_-(w, B_\rho(x)) < +\infty.$$

We will prove the inequality

$$\liminf_{h\to+\infty} \int_{B_\rho(x)} hz_h(1 - z_h^2)^h |\nabla z_h|\, dy \geqq 1. \tag{4.11}$$

Choosing $\sigma < \rho$, there exists a sequence $\{y_h\}_{h\in\mathbf{N}} \subset B_\sigma(x)$ such that

$$\lim_{h\to+\infty}(1 - z_h^2(y_h))^{2h} = 0,$$

for otherwise one would get

$$\limsup_{h\to+\infty} \inf_{y\in B_\sigma(x)} (1 - z_h^2(y))^{2h} > 0,$$

and then $u \in W^{1,2}(B_\sigma(x))$. By the Fatou lemma we get

$$\int_{B_\rho(x)} \liminf_{h\to+\infty} h^2 z_h^2\, dy \leqq \liminf_{h\to+\infty} \int_{B_\rho(x)} h^2 z_h^2\, dy < +\infty,$$

so that, possibly extracting a sequence, we can find $x' \in B_\rho(x)$, $x' < x - \sigma$ such that $h^2 z_h^2(x') \to 0$. Repeating the same argument we find, up to subsequences, $x'' \in B_\rho(x)$, $x'' > x + \sigma$ such that $h^2 z_h^2(x'') \to 0$. In particular, we get

$$\lim_{h\to+\infty} (1 - z_h^2(x'))^{2h} = \lim_{h\to+\infty} (1 - z_h^2(x''))^{2h} = 1.$$

Finally, recalling that by Lemma 3.5 the functions $c_h = (1 - z_h^2)^{h+1}$ belong to $W^{1,2}(B_\rho(x))$ we get

$$\begin{aligned}
\lim_{h \to +\infty} G_h(w_h, B_\rho(x)) &\geqq \limsup_{h \to +\infty} \int_{B_\rho(x)} \left[|\nabla z_h|^2 (1 - z_h^2)^{2h} + \frac{1}{4} h^2 z_h^2 \right] dy \\
&\geqq \limsup_{h \to +\infty} \int_{B_\rho(x)} [h z_h |\nabla z_h| (1 - z_h^2)^h] \, dy \\
&\geqq \limsup_{h \to +\infty} \left\{ \left| \int_{x'}^{y_h} [h z_h \nabla z_h (1 - z_h^2)^h] \, dy \right| + \left| \int_{y_h}^{x''} [h z_h \nabla z_h (1 - z_h^2)^h] \, dy \right| \right\} \\
&\geqq \limsup_{h \to +\infty} \left\{ \left| \int_{z_h(x')}^{z_h(y_h)} hs(1 - s^2)^h \, ds \right| + \left| \int_{z_h(y_h)}^{z_h(x'')} hs(1 - s^2)^h \, ds \right| \right\} \\
&\geqq \frac{1}{2} \limsup_{h \to +\infty} \frac{h}{h+1} [(1 - z_h^2(x'))^{h+1} - (1 - z_h^2(y_h))^{h+1} \\
&\qquad + (1 - z_h^2(x''))^{h+1} - (1 - z_h^2(y_h))^{h+1}] = 1.
\end{aligned}$$

The last statements of the Lemma are a straightforward consequence of (i), (ii), and the superadditivity of G_-.

Remark 4.3. Now we can briefly describe the proof of weak convergence of measures

$$\mu_h(B) = \int_B |\nabla c_h| \, dx = 2(h+1) \int_B z_h (1 - z_h^2)^h |\nabla z_h| \, dx$$

to the measure $2\mathscr{H}^{n-1} \llcorner S_u$, under the assumption $F_h(u_h, z_h) \to F(u) < +\infty$. In fact, the inequality

$$\liminf_{h \to +\infty} \mu_h(A) \geqq 2\mathscr{H}^{n-1}(A \cap S_n) \qquad A \subset \Omega \text{ open.} \tag{4.12}$$

in the case $n = 1$ follows by (4.11) and in the general case by the slicing argument of the proof of Proposition 4.1. In order to show weak convergence, we need only to show that (see, for instance, [6])

$$\limsup_{h \to +\infty} \mu_h(\Omega) \leqq 2\mathscr{H}^{n-1}(S_u). \tag{4.13}$$

By (4.9) and the usual slicing argument we get

$$\liminf_{h\to+\infty}\int_\Omega |\nabla u_h|^2(1-z_h^2)^{2h}\,dx \geqq \int_\Omega |\nabla u|^2\,dx. \tag{4.14}$$

Moreover, the assumption $F_h(u_h, z_h) \to F(u)$ yields

$$\limsup_{h\to+\infty}\int_\Omega |\nabla u_h|^2(1-z_h^2)^{2h}\,dx + \frac{1}{2}\mu_h(\Omega) \leqq \int_\Omega |\nabla u|^2\,dx + \mathscr{H}^{n-1}(S_u). \tag{4.15}$$

Inequalities (4.14), (4.15), combined with inequality (4.12) with $A = \Omega$ yield that $\mu_h(\Omega) \to 2\mathscr{H}^{n-1}(S_u)$ and

$$\lim_{h\to+\infty}\int_\Omega |\nabla u_h|^2(1-z_h^2)^{2h}\,dx = \int_\Omega |\nabla u|^2\,dx.$$

5. The Inequality for the Upper Γ-Limit

This section is devoted to the proof of the second inequality which defines Γ-convergence (see (2.2)) and the proof of Theorem 1.1. Therefore we aim to construct sequences $(w_h) = (u_h, z_h) \subset \mathscr{D}_{h,n}(\Omega)$ converging to $(u, 0)$ in $\mathscr{B}(\Omega) \times \mathscr{B}(\Omega)$ and such that

$$\limsup_{h\to+\infty} F_h(w_h) \leqq F(u). \tag{5.1}$$

As in Section 4 we assume for simplicity $\alpha = 1$. We begin with the following key proposition.

PROPOSITION 5.1. *Assume that Ω is a bounded open set and*

$$u \in SBV(\Omega) \cap L^\infty(\Omega).$$

Then, there exists a sequence $(w_h) = (u_h, z_h) \subset \mathscr{D}_{h,n}(\Omega)$ such that

(5.2)

$$\limsup_{h\to+\infty} F_h(w_h) \leqq \int_\Omega |\nabla u|^2\,dx + \limsup_{\rho\to 0^+}\frac{\text{meas}(\{x\in\Omega : \text{dist}(x, S_u) < \rho\})}{2\rho}.$$

Moreover, the sequence can be chosen in such a way that $u_h, z_h \in W^{1,2}(\Omega)$.

Proof: We denote by $A_\delta \subset \Omega$ the set

$$\{x \in \Omega : \text{dist}(x, A) < \delta\}$$

whenever $A \subset \Omega$, $\delta > 0$. We can assume that

$$|\nabla u| \in L^2(\Omega), \quad L = \limsup_{\rho\to 0^+}\frac{\text{meas}(\{x\in\Omega : \text{dist}(x, S_u) < \rho\})}{2\rho} < +\infty. \tag{5.3}$$

If $u \in W^{1,2}(\Omega)$ it is sufficient to take $u_h = u$ and $z_h = 0$. On the other hand, if $u \notin W^{1,2}(\Omega)$, then the coefficients $(1 - z_h^2)^{2h}$ have to be infinitesimal near S_u, because

$$\lim_{h \to +\infty} \int_\Omega |\nabla u_h|^2 \, dx = +\infty$$

for any sequence $(u_h) \subset W^{1,2}(\Omega)$ converging to u in $\mathcal{B}(\Omega)$. We restrict our choice to the functions w_h such that

$$u_h = u \quad \text{on} \quad \Omega \setminus (S_u)_{b_h}, \tag{5.4}$$

and

$$z_h = \begin{cases} 1 & \text{on} \quad (S_u)_{b_h}; \\ \eta_h & \text{on} \quad \Omega \setminus (S_u)_{b_h + a_h}, \end{cases} \qquad \eta_h \leqq z_h \leqq 1, \tag{5.5}$$

where a_h, b_h, η_h are strictly positive and

$$\lim_{h \to +\infty} a_h = \lim_{h \to +\infty} b_h = \lim_{h \to +\infty} \eta_h = 0. \tag{5.6}$$

We choose b_h in such a way that

$$\lim_{h \to +\infty} h^2 b_h = 0, \tag{5.7}$$

so that assumption (5.3) yields

$$\lim_{h \to +\infty} h^2 \mathrm{meas}(S_u)_{b_h} = 0. \tag{5.8}$$

We also choose, for reasons which will be clear later,

$$\eta_h = \frac{1}{h} \sqrt{\int_0^1 (1 - s^2)^h \, ds}. \tag{5.9}$$

We remark that since the distributional derivative Du is absolutely continuous with respect to $\mathcal{L}^n + \mathcal{H}^{n-1} \llcorner S_u$, u belongs to $W^{1,2}(\Omega \setminus \overline{S_u})$. Hence, given $\psi_h \in C_0^\infty(\mathbf{R}^n)$ such that

$$0 \leqq \psi_h \leqq 1, \quad \psi_h \equiv 1 \quad \text{on} \quad (S_u)_{b_h/2}, \quad \psi_h \equiv 0 \quad \text{on} \quad \Omega \setminus (S_u)_{b_h},$$

the functions u_h defined by

$$u_h = (1 - \psi_h) u$$

satisfy (5.4), belong to $W^{1,2}(\Omega)$, and converge to u in $\mathcal{B}(\Omega)$. To choose in the appropriate way the sequence a_h, let us try to estimate from above $F_h(w_h)$ under assumptions (5.4), (5.5), (5.6), (5.7). Defining

$$A_h(z) = \int_{(S_u)_{b_h+a_h} \setminus (S_u)_{b_h}} \left[|\nabla z|^2 (1 - z^2)^{2h} + \frac{1}{4} h^2 z^2 \right] dx,$$

we get (recall also (2.6))

$$\begin{aligned} (5.10) \quad F_h(w_h) = &\int_{\Omega \setminus (S_u)_{b_h+a_h}} |\nabla u|^2 (1 - \eta_h^2)^{2h}\, dx \\ &+ \int_{(S_u)_{b_h+a_h} \setminus (S_u)_{b_h}} |\nabla u|^2\, (1 - z_h^2)^{2h}\, dx \\ &+ \frac{1}{4} h^2 \eta_h^2 \operatorname{meas}(\Omega \setminus (S_u)_{b_h+a_h}) + \frac{1}{4} h^2 \operatorname{meas}((S_u)_{b_h}) + A_h(z_h). \end{aligned}$$

Under the assumption (5.6), since (5.3) implies that $\Omega \cap \overline{S_u}$ is negligible and $(1 - \eta_h^2)^{2h} \to 1$, the first term converges to

$$\int_\Omega |\nabla u|^2\, dx.$$

We want to show that a suitable choice of a_h, z_h ensures that the sum of the remaining terms is less than L up to infinitesimals. In fact, the second term and the fourth term in (5.10) are infinitesimal by (5.3), (5.8). The third term can be estimated with $h^2 \eta_h^2 \operatorname{meas}(\Omega)$ which is infinitesimal by (5.9).

The term $A_h(z_h)$ represents the cost of the transition between η_h and 1 in the tubular neighborhood of S_u. We must then choose z_h such that $(u_h, z_h) \in \mathcal{D}_{h,n}(\Omega)$ and

$$(5.11) \qquad \limsup_{h \to +\infty} A_h(z_h) \leqq L.$$

The idea is to set $z_h = \tilde{z}_h \circ \tau$, where by definition $\tau(x) = \operatorname{dist}(x, S_u)$, to restrict ourselves, by using the co-area formula and chain rules, to locally Lipschitz functions $\tilde{z}_h$ depending on one real variable only. For such z_h one has the following reduction of the term $A_h(z_h)$:

$$\begin{aligned} A_h(z_h) &= \int_{b_h}^{b_h+a_h} \left\{ \int_{\{y:\tau(y)=t\}} \left[|\nabla z_h(y)|^2 (1 - z_h^2(y))^{2h} + \frac{1}{4} h^2 z_h^2(y) \right] d\mathcal{H}^{n-1}(y) \right\} dt \\ &= \int_{b_h}^{b_h+a_h} \left\{ \int_{\{y:\tau(y)=t\}} \left[|\nabla z_h(y)|^2 (1 - \tilde{z}_h^2(t))^{2h} + \frac{1}{4} h^2 z_h^2(t) \right] d\mathcal{H}^{n-1}(y) \right\} dt. \end{aligned}$$

By using the equality $|\nabla\tau| = 1$ almost everywhere (see [19], Section 3.2.34) and the chain rule for compositions for Lipschitz functions we get

$$(5.12)\quad A_h(z_h)$$
$$= \int_{b_h}^{b_h+a_h}\left[|\nabla\tilde{z}_h(t)|^2(1-\tilde{z}_h^2(t))^{2h} + \frac{1}{4}h^2\tilde{z}_h^2(t)\right]\mathscr{H}^{n-1}(\{y:\tau(y)=t\})\,dt.$$

Now, to get the best $(\tilde{z}_h, a_h)$ following [27], one would choose the solutions of the problems

$$\min\left\{\int_{b_h}^{b_h+a_h}\left[|\nabla\theta(t)|^2(1-\theta^2(t))^{2h} + \frac{1}{4}h^2\theta^2(t)\right]\mathscr{H}(t)\,dt : a_h > 0,\right.$$
$$\left.\theta(b_h) = 1, \quad \theta(b_h + a_h) = 0\right\},$$

where $\mathscr{H}(t) = \mathscr{H}^{n-1}(\{y:\tau(y)=t\})$. Modica and Mortola remark that problems of the following type

$$\min\left\{\int_0^p [|\nabla\theta(t)|^2 + f^2(\theta(t))]\,dt : \theta(0),\quad \theta(p)\quad \text{assigned}\right\}$$

have computable minimum values, by reducing via the Cauchy-Schwartz inequality $a^2 + b^2 \geqq 2ab$ the problems to

$$\min\left\{2\int_0^p |\nabla\theta(t)|\,|f|(\theta(t))\,dt : \theta(0),\quad \theta(p)\quad \text{assigned}\right\}$$

and then by executing a change of the variable of integration they find that the infimum is

$$2\int_{\theta(0)}^{\theta(p)} |f|(s)\,ds$$

and the solutions (θ, p) are given by

$$\nabla\theta(t) = |f|(\theta(t)),\quad \theta(0),\quad \theta(p)\quad \text{assigned}.$$

In our case, the multiplier $\mathscr{H}(t)$ does not allow an explicit calculation of the minimum values. This difficulty will be bypassed by using integration by parts. We define θ_h as the solution of the differential equation

$$\nabla\theta = \frac{h\theta}{2(1-\theta^2)^h}, \qquad \theta(0) = \eta_h, \tag{5.13}$$

where η_h is defined in (5.9). Then, a_h is the maximal existence interval length of the solution. A standard argument shows that

$$a_h = 2\int_{\eta_h}^1 \frac{(1-s^2)^h}{hs}\,ds \leqq \frac{2}{h\eta_h}\int_0^1 (1-s^2)^h\,ds = 2\sqrt{\int_0^1 (1-s^2)^h\,ds} \downarrow 0,$$

so that a_h fulfill (5.6). We set

$$\tilde{z}_h(t) = \theta_h(b_h + a_h - t) \qquad t \in [b_h, b_h + a_h],$$

and

$$\mathcal{A}(t) = \operatorname{meas}(\{x \in \Omega : \operatorname{dist}(x, S_u) < t\}) \qquad t > 0.$$

Since $\theta_h(a_h) = 1$, the function $\tilde{z}_h$ is strictly decreasing and ranges between η_h and 1. We also recall (see (2.6)) that $\mathcal{A} \in W^{1,1}_{\mathrm{loc}}((0, +\infty))$ and $\nabla\mathcal{A} = \mathscr{H}$ almost everywhere. Now, by using (5.12) and (5.13), we evaluate $A_h(z_h)$. We denote in the sequel by $o(h)$ terms infinitesimal as $h \to +\infty$.

$$A_h(z_h) = \int_{b_h}^{b_h+a_h} \left[|\nabla\tilde{z}_h(t)|^2(1-\tilde{z}_h^2(t))^{2h} + \frac{1}{4}h^2\tilde{z}_h^2(t)\right]\mathscr{H}(t)\,dt$$

$$= \frac{1}{2}h^2 \int_{b_h}^{b_h+a_h} \tilde{z}_h^2\mathscr{H}(t)\,dt.$$

Integrating by parts we obtain

$$A_h(z_h) = \frac{1}{2}(h^2\eta_h^2)\mathcal{A}(b_h + a_h) - \frac{1}{2}h^2\mathcal{A}(b_h) - h^2\int_{b_h}^{b_h+a_h} \tilde{z}_h(\nabla\tilde{z}_h)\mathcal{A}(t)\,dt$$

$$= -h^2\int_{b_h}^{b_h+a_h} \tilde{z}_h(\nabla\tilde{z}_h)\mathcal{A}(t)\,dt + o(h).$$

Now we use assumption (5.3), which gives the existence of a sequence $\omega_h \downarrow 0$ such that

$$\mathcal{A}(t) \leqq 2t(L + \omega_h) \qquad \forall t \in [0, b_h + a_h],$$

and we find

$$A_h(z_h) \leqq -2(L + \omega_h)h^2 \int_{b_h}^{b_h+a_h} \tilde{z}_h(\nabla\tilde{z}_h)t\,dt + o(h).$$

A new integration by parts yields

$$A_h(z_h) \leqq (L + \omega_h)\left[h^2 b_h - h^2\eta_h^2(b_h + a_h) + h^2 \int_{b_h}^{b_h + a_h} \tilde{z}_h^2 \, dt \right] + o(h)$$

$$= (L + \omega_h)h^2 \int_{b_h}^{b_h + a_h} \tilde{z}_h^2 \, dt + o(h).$$

Finally, by using the definition of $\tilde{z}_h$ and (5.13) we get

$$A_h(z_h) \leqq 2(L + \omega_h)h \int_{b_h}^{b_h + a_h} \tilde{z}_h(1 - z_h^2)^h \nabla\tilde{z}_h \, dt + o(h)$$

$$= (L + \omega_h) \frac{h}{h+1} (1 - \eta_h^2)^{h+1} + o(h),$$

and (5.11) is proved. Now, in order to show that $w_h = (u_h, z_h) \in \mathcal{D}_{h,n}(\Omega)$ we need only to verify that $\varphi \circ z_h \in W^{1,2}(\Omega)$ with $\varphi(t)$ defined in (3.3). By construction $\tilde{z}_h \in C^1([\eta_h, 1))$, so that

$$z_h, \varphi \circ z_h \in W^{1,2}_{\mathrm{loc}}(\Omega \setminus \{x : \operatorname{dist}(x, S_u) = b_h\}).$$

Since $F_h(u_h, z_h) < +\infty$, it is easy to see that $\nabla(\varphi \circ z_h) \in L^2(\Omega)$, therefore Lemma 2.4 yields $\varphi \circ z_h \in W^{1,2}(\Omega)$.

Finally, the last statement of the Proposition is a straightforward consequence of Lemma 3.5 (ii).

The limits

$$\mathcal{M}^*(S_u) = \limsup_{\rho \to 0^+} \frac{\operatorname{meas}(\{x \in \Omega : \operatorname{dist}(x, S_u) < \rho\})}{2\rho},$$

$$\mathcal{M}_*(S_u) = \liminf_{\rho \to 0^+} \frac{\operatorname{meas}(\{x \in \Omega : \operatorname{dist}(x, S_u) < \rho\})}{2\rho}$$

are known in the literature as Minkowski $(n-1)$-dimensional upper and lower content respectively. It can be shown (see [9], Sections 3.2.37 and 3.2.39) that

$$\mathcal{M}_*(S_u) \geqq \mathcal{H}^{n-1}(S_u)$$

for any function $u \in BV(\Omega)$. The inequality $\mathcal{M}^*(S_u) \leqq \mathcal{H}^{n-1}(S_u)$ is also true under very mild regularity assumptions on S_u, for instance

$$\mathcal{H}^{n-1}(\overline{S_u} \setminus S_u) = 0, \qquad \overline{S_u} \subset f(K),$$

with $K \Subset R^{n-1}$ and f Lipschitz. Proposition 5.1 shows the possibility of constructing sequences w_h satisfying (5.1) for all functions $u \in SBV(\Omega) \cap L^\infty(\Omega)$ whose jump set S_u satisfies the condition

$$\mathcal{M}^*(S_u) = \mathcal{H}^{n-1}(S_u) = \mathcal{M}_*(S_u). \tag{5.14}$$

In the next proposition we consider the problem of finding the sequences for general functions $u \in GSBV(\Omega)$. We denote by $\mathcal{F}(\Omega)$ the class of functions $u \in GSBV(\Omega)$ such that either $F(u) = +\infty$ or there exists a sequence $(u_k) \subset SBV(\Omega) \cap L^\infty(\Omega)$ converging to u in $\mathcal{B}(\Omega)$, such that $F(u_k) \to F(u)$ and

$$\lim_{k \to +\infty} \mathcal{M}^*(S_{u_k}) - \mathcal{H}^{n-1}(S_{u_k}) = 0$$

for all $k \in \mathbf{N}$.

PROPOSITION 5.2. *For any function $u \in \mathcal{F}(\Omega)$ it is possible to find sequences $u_h, z_h \in W^{1,2}(\Omega)$ such that $w_h = (u_h, z_h) \in \mathcal{D}_{h,n}(\Omega)$, $w_h \to (u, 0)$ in $\mathcal{B}(\Omega) \times \mathcal{B}(\Omega)$ and*

$$\limsup_{h \to +\infty} F_h(w_h) \leqq F(u).$$

Proof: It follows by Proposition 5.1 by using a diagonal argument.

Now we see how the extent of $\mathcal{F}(\Omega) \subset GSBV(\Omega)$ depends on the regularity of the minimizers of the Mumford-Shah functional.

PROPOSITION 5.3. (i) *Let $u \in SBV(\Omega) \cap L^\infty(\Omega)$ be a minimizer of the functional*

$$\int_\Omega |\nabla u|^2\, dx + \mathcal{H}^{n-1}(S_u) + \beta \int_\Omega |u - g|^2\, dx \tag{5.15}$$

for some $g \in L^\infty(\Omega)$. Then,

$$\lim_{\rho \to 0^+} \frac{|\{x \in \Omega : \operatorname{dist}(x, K) < \rho\}}{2\rho} = \mathcal{H}^{n-1}(K)$$

for any compact set $K \subset \overline{S_u} \cap \Omega$.

(ii) *If Ω satisfies the condition $(\mathcal{R})$, then $\mathcal{F}(\Omega) = GSBV(\Omega)$.*

Proof: (i) The idea of the proof is to approximate K by sets S such that $\mathcal{M}^*(S) = \mathcal{H}^{n-1}(S)$ and then estimate $\mathcal{M}^*(K \setminus S)$. In order to do this we need the following density estimate:

$$\text{(5.16)} \quad \mathscr{H}^{n-1}(S_u \cap B_\rho(x)) \geqq \beta\rho^{n-1} \quad \forall x \in K, \quad \rho \in]0, \operatorname{dist}(x, \partial\Omega)[,$$

for a suitable constant $\beta > 0$. This estimate has been proved in the two-dimensional case by Dal Maso-Morel-Solimini; see [13]. In the general case the same estimate can be proved by the methods of [17] (see also [11], Proposition 1.2 and Remark 3.13). By the general properties of jump sets of functions of bounded variation (see [19], Sections 3.2.39 and 4.5.9), K can be covered, up to $\mathscr{H}^{n-1}$-small sets, with disjoint compact subsets of C^1 hypersurfaces. Moreover, by [19], Section 3.2.39, it follows that any such set B satisfies

$$\mathcal{M}^*(B) = \mathscr{H}^{n-1}(B) = \mathcal{M}_*(B).$$

In particular, the inequality $\mathcal{M}_*(K) \geqq \mathscr{H}^{n-1}(K)$ is satisfied and for any $\delta > 0$ it is possible to find a compact set $S \subset K$ such that $\mathscr{H}^{n-1}(K \setminus S) < \delta$ and $\mathcal{M}^*(S) = \mathscr{H}^{n-1}(S)$. Let ω_n be the measure of the unit ball in $\mathbf{R}^n$ and

$$c(n) = \sup\{p : \exists x_1, \cdots, x_p \in B_1(0) \quad \text{such that} \quad |x_i - x_j| \geqq 1 \ \forall i \neq j\}.$$

Let $\lambda > 0$ be fixed, and let us denote by A_ρ the open ρ-neighborhood of $A \subset \mathbf{R}^n$. For any $\rho > 0$ we have

$$K_\rho \subset S_{(1+2\lambda)\rho} \cup (K \setminus S_{\lambda\rho})_\rho.$$

We can cover the compact set $E = K \setminus S_{\lambda\rho}$ with a finite number of balls $B_{\lambda\rho}(x_1)$, $\cdots$, $B_{\lambda\rho}(x_p)$ centered in E such that $|x_i - x_j| \geqq \lambda\rho$ whenever $i \neq j$. Since any point belongs to at most $c(n)$ balls, by integrating the density estimate we infer

$$c(n)\mathscr{H}^{n-1}(S_u \cap E_{\lambda\rho}) \geqq p\beta(\lambda\rho)^{n-1}.$$

Since

$$E_\rho \subset \bigcup_{i=1}^{p} B_{(1+2\lambda)\rho}(x_i)$$

we get

$$|E_\rho| \leqq \frac{c(n)\omega_n}{\lambda^{n-1}\beta}(1 + 2\lambda)^n \rho \mathscr{H}^{n-1}(S_u \cap E_{\lambda\rho}),$$

so that, since $E_{\lambda\rho} \subset K_{\lambda\rho} \setminus S$, we find

$$\frac{|K_\rho|}{2\rho} \leqq \frac{|S_{(1+2\lambda)\rho}|}{2\rho} + \frac{c(n)\omega_n}{2\lambda^{n-1}\beta}(1 + 2\lambda)^n \mathscr{H}^{n-1}(S_u \cap K_{\lambda\rho} \setminus S).$$

By letting $\rho \downarrow$ we find

$$\mathcal{M}^*(K) \leqq (1 + 2\lambda)\mathcal{H}^{n-1}(K) + \delta \frac{c(n)\omega_n}{2\lambda^{n-1}\beta}(1 + 2\lambda)^n.$$

The inequality follows by letting first $\delta \downarrow 0$ then $\lambda \downarrow 0$.

(ii) A diagonal argument shows that

$$(u_k) \subset \mathcal{F}(\Omega), \quad F(u_k) \to F(u) < +\infty, \quad u_k \to u \text{ in } \mathcal{B}(\Omega) \quad \Rightarrow \quad u \in \mathcal{F}(\Omega).$$

Since (1.3), (1.7), (2.3) yield

$$F(u) = \lim_{N \to +\infty} F(N \wedge u \vee -N),$$

it is sufficient to show that $\mathcal{F}(\Omega) \supset SBV(\Omega) \cap L^\infty(\Omega)$. Let $u \in SBV(\Omega) \cap L^\infty(\Omega)$ be a fixed function and let us extend u to $\Omega' = \Omega \cup U$ by setting

$$u(x) = u(\varphi^{-1}(x)) \qquad \forall x \in U \backslash \bar{\Omega}.$$

Then (see, for instance, [36], [37]) $u \in SBV(\Omega') \cap L^\infty(\Omega')$ and

$$\mathcal{H}^{n-1}(S_u \cap \partial\Omega) = 0 \tag{5.17}$$

because φ is the identity in $\partial\Omega$. We now consider solutions $u_k \in SBV(\Omega') \cap L^\infty(\Omega')$ of the minimum problem

$$\min\left\{\int_{\Omega'} |\nabla v|^2\, dx + \mathcal{H}^{n-1}(S_v) + k\int_{\Omega'} |v - u|^2\, dx : v \in SBV(\Omega')\right\}.$$

Clearly $u_k \to u$ and (i) yield

$$\limsup_{\rho \to 0^+} \frac{|\{x \in \Omega : \operatorname{dist}(x, S_{u_k}) < \rho\}|}{2\rho} \leqq \mathcal{H}^{n-1}(\bar{\Omega} \cap \overline{S_{u_k}}) = \mathcal{H}^{n-1}(\Omega \cap S_{u_k})$$

for any $k \in \mathbf{N}$. Let μ_k be the measures

$$\mu_k(B) = \int_B |\nabla u_k|^2\, dx + \mathcal{H}^{n-1}(B \cap S_{u_k})$$

and let

$$\mu(B) = \int_B |\nabla u|^2\, dx + \mathcal{H}^{n-1}(B \cap S_u).$$

The inequality

$$\limsup_{k\to+\infty} \mu_k(\Omega') \leqq \mu(\Omega')$$

follows by the definition of u_k, and the inequality

$$\liminf_{k\to+\infty} \mu_k(A) \geqq \mu(A) \qquad A \subset \Omega' \text{ open}$$

follows by the lower semicontinuity of F in SBV (see [1], [2], [3]). In particular, the measures μ_k weakly converge to μ (see [6]) and $\mu_k(B) \to \mu(B)$ for any set B such that $\mu(\partial B) = 0$. It follows by (5.17) that $\mu_k(\Omega) \to \mu(\Omega)$ and $\mu_k(\partial\Omega) \to 0$, so that $F(u_k) \to F(u)$ and $u \in \mathcal{F}(\Omega)$.

Finally, we can show Theorem 1.1, which summarizes all the results of this paper.

Proof of Theorem 1.1: By the definition of Γ-convergence, we have to check inequalities (2.1), (2.2). We begin with (2.1). Let $(w_h) = (u_h, z_h) \subset \mathcal{B}(\Omega) \times \mathcal{B}(\Omega)$ be converging to $w = (u, z)$. There is no loss of generality if we assume $w_h \in \mathcal{D}_{h,n}(\Omega)$ and $z = 0$. Then, the inequality is given by Proposition 4.1. On the other hand, inequality (2.2) follows by Proposition 5.2 and Proposition 5.3(ii). The problems $(\mathcal{P}_h)$ of Section 1 have solutions $\bar{w}_h = (\bar{u}_h, \bar{z}_h) \in \mathcal{D}_{h,n}(\Omega)$ because of the compactness of minimizing sequences (Theorem 3.2) and lower semicontinuity of the functional (Theorem 3.4). The set of solutions is compact because of the equicoercivity of the functionals (it suffices to take $C = \beta \|g\|_\infty^\gamma \operatorname{meas}(\Omega)$ in Theorem 3.6). Finally we prove the implication (1.12). Assume that $\bar{u}_{h_k}$ converges to $\bar{u}$ in $\mathcal{B}(\Omega)$. We have to show

$$F(u) + \beta \int_\Omega |u - g|^\gamma \, dx \geqq F(\bar{u}) + \beta \int_\Omega |\bar{u} - g|^\gamma \, dx \tag{5.18}$$

for any $u \in GSBV(\Omega)$. Since $g \in L^\infty(\Omega)$, and since

$$F(u) \geqq F(N \wedge u \vee -N) \qquad \forall N \in \mathbf{N},$$

we need to check (5.18) only for functions $u \in SBV(\Omega) \cap L^\infty(\Omega)$ such that $\|u\|_\infty \leqq \|g\|_\infty$. Let $(w_h) = (u_h, z_h) \subset \mathcal{D}_{h,n}(\Omega)$ be such that $w_h \to (u, 0)$ and

$$F(u) = \lim_{h\to+\infty} F_h(w_h).$$

A truncation argument shows that we can assume $u_h \in L^\infty(\Omega)$ and $\|u_h\|_\infty \leqq \|g\|_\infty$. Since

$$\int_\Omega |u-g|^\gamma \, dx = \lim_{h\to+\infty} \int_\Omega |u_h - g|^\gamma \, dx,$$

we get

$$F(u) + \beta \int_\Omega |u-g|^\gamma \, dx = \lim_{h\to+\infty} F_h(w_h) + \beta \int_\Omega |u_h - g|^\gamma \, dx$$

$$\geqq \liminf_{k\to+\infty} F_{h_k}(\bar{w}_{h_k}) + \beta \int_\Omega |\bar{u}_{h_k} - g|^\gamma \, dx \geqq F(\bar{u}) + \beta \int_\Omega |\bar{u} - g|^\gamma \, dx,$$

and (5.18) is proved.

Remark 5.4. If we replace the space $\mathcal{D}_{h,n}(\Omega)$ in (1.10) by

$$W^{1,2}(\Omega) \times \{z \in W^{1,2}(\Omega) : 0 \leqq z \leqq 1\},$$

then Γ-convergence still holds. In fact, the lim inf inequality (2.2) is trivially true. The lim sup inequality is also true because in Proposition 5.2 we have chosen the functions u_h in $W^{1,2}(\Omega)$ and z_h such that $\varphi(z_h) \in W^{1,2}(\Omega)$; see Lemma 3.5(ii).

Added in Proof. There is a large freedom in choosing the approximating functions. For instance, a different choice could be

$$H_\epsilon(u,z) = \int_\Omega \left[\epsilon |\nabla z|^2 + z^2 |\nabla u|^2 + \frac{\alpha^2 (z-1)^2}{4\epsilon} \right] dx.$$

The proof of Γ-convergence is the same, with minor changes. Functionals formally very similar to H_ϵ (with u unitary vector) have recently been suggested by Ericksen as a new mathematical model for the statics of liquid crystals.

Acknowledgments. We thank E. De Giorgi for having suggested to us the approximation problem and for many other useful suggestions. We also thank R. March (C.N.R.-I.E.I., Pisa) for his work on the numerical treatment of this subject.

The second author was supported by an Istituto Nazionale di Alta Matematica "Francesco Severi" research fellowship.

Bibliography

[1] Ambrosio, L., *A compactness theorem for a special class of functions of bounded variations*, Boll. Un. Mat. It. 3-B, 1989, pp. 857–881.

[2] Ambrosio, L., *Existence theory for a new class of variational problems*, to appear in Arch. Rat. Mech. Anal.

[3] Ambrosio, L., *Variational problems in SBV*, Acta Appl. Math. 17, 1989, pp. 1–40.
[4] Ambrosio, L., and Dal Maso, G., *A general chain rule for distributional derivatives*, Proc. AMS. 108, 1990, pp. 691–702.
[5] Attouch, H., *Variational Convergence for Functions and Operators*, Pitman, Boston, 1984.
[6] Bergström, H., *Weak Convergence of Measures*, Academic Press, New York, 1982.
[7] Blake, A., and Zisserman, A., *Localising discontinuities using weak continuity constraints*, Pattern Recog. Lett. 6, 1987, pp. 51–59.
[8] Brezis, H., Coron, J. M., and Lieb, E. H., *Harmonic maps with defects*, Comm. Math. Phys. 107, 1986, pp. 649–705.
[9] Calderon, A. P., and Zygmund, A., *On the differentiability of functions which are of bounded variation in Tonelli's sense*, Rev. Unit. Mat. Arg. 20, 1960, pp. 102–121.
[10] Carriero, M., Leaci, A., Pallera, D., and Pascali, E., *Euler conditions for a minimum problem with free discontinuity surfaces*, preprint, 1988.
[11] Carriero, M., and Leaci, A., *Existence theory for a Dirichlet problem with free discontinuity set*, to appear in Nonl. Anal.
[12] Dal Maso, G., and Modica, L., *A general theory of variational functionals*, in *Topics in Functional Analysis* 1980–81, Scuola Normale Sup., Pisa, 1981.
[13] Dal Maso, G., Morel, J. M., and Solimini, S., *A variational method in image segmentation: Existence and approximation results*, to appear in Acta Math.
[14] De Giorgi, E., and Franzoni, T., *Su un tipo di convergenza variazionale*, Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fit. Mat. Natur. 58, 1975, pp. 842–850.
[15] De Giorgi, E., and Franzoni, T., *Su un tipo di convergenza variazionale*, Rend. Sem. Mat. Brescia 3, 1979, pp. 63–101.
[16] De Giorgi, E., and Ambrosio, L., *Un nuovo tipo di funzionale del Calcolo delle Variazioni*, Att. Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat. 2–8 (82), 1988.
[17] De Giorgi, E., Carriero, M., and Leaci, A., *Existence theorem for a minimum problem with free discontinuity set*, Arch. Rat. Mech. Anal. 108, 1989, pp. 195–218.
[18] Ericksen, J. L., *Equilibrium theory of liquid crystals*, Advances in Liquid Crystals 2, Academic Press, 1976, pp. 233–298.
[19] Federer, H., *Geometric Measure Theory*, Springer-Verlag, Berlin, 1969.
[20] Geman, D., and Geman, S., *Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images*, IEEE Trans. Pattern Anal. Mach. Intell. 6, 1984, pp. 721–741.
[21] Giusti, E., *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser, Boston, 1984.
[22] March, R., *A regularization model for stereo vision with controlled continuity*, C.N.R.-I.E.I. Internal Report B4-07, Pisa, 1989.
[23] Marr., D., and Hildreth, E., *Theory of edge detection*, Proc. Roy. Soc. London. B204, 1980, pp. 301–328.
[24] Marroquin, J., Mitter, S., and Poggio, T., *Probabilistic solutions of ill posed problems in computational vision*, J. Amer. Stat. Assoc. 82, 1987, p. 397.
[25] Mitter, S., Richardson, T., and Kulkarni, S. R., *An existence theorem and lattice approximation for a variational problem*, Preprint Cent. For Intell. Control Systems, M.I.T., to appear in Proc. of the Workshop on Signal Processing, Inst. for Math. and Appl., University of Minnesota.
[26] Modica, L., *The gradient theory of phase transitions and the minimal interface criterion*, Arch. Rat. Mech. Anal. 98, 1987, pp. 123–142.
[27] Modica, L., and Mortola, S., *Un esempio di Γ-convergence*, Boll. Un. Mat. Ital. 14-B, 1977, pp. 285–299.
[28] Morgan, F., *Geometric Measure Theory: A Beginner's Guide*, Academic Press, New York, 1988.
[29] Mumford, D., and Shah, J., *Boundary detection by minimizing functionals*, Proc. IEEE Conf. on Computer Vision and Pattern Recognition, San Francisco, 1985.
[30] Mumford, D., and Shah, J., *Optimal approximations by piecewise smooth functions and associated variational problems*, Comm. Pure Appl. Math. 42, 1989, pp. 577–685.
[31] Rosenfeld, A., and Kak, A. C., *Digital Picture Processing*, Academic Press, New York, 1976.

[32] Serrin, J., *On the definition and properties of certain variational integrals*, Trans. Amer. Mat. Soc. 101, 1961, pp. 139–167.
[33] Simon, L., *Lectures on Geometric Measure Theory*, Proc. Centre for Math. Anal. 3, Australian National University, 1983.
[34] Stein, E., *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.
[35] Terzopoulos, D., *Regularization of inverse visual problems involving discontinuities*, IEEE Trans. Pattern Anal. Mach. Intell. 8, 1988, pp. 413–424.
[36] Vol'pert, A. I., and Huhjaev, S. I., *Analysis in classes of discontinuous functions and equations of mathematical physics*, Martinus Nijhoff, Dordrecht, 1985.
[37] Vol'pert, A. I., *The spaces BV and quasilinear equations*, Math. USSR Sb. 17, 1972, pp. 255–267.

Received July, 1989.