MIXED MODE CRACK PROPAGATION SIMULATION USING MATERIAL FORCES AND THE EXTENDED FINITE ELEMENT METHOD

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ABSTRACT

A new approach to simulate crack growth within an extended finite element framework is presented which incorporates the material force concept and enables a single step evaluation of crack state and crack growth direction based on a continuum mechanics approach. This represents an alternative to the common two step procedure that first evaluates the stress intensity factors followed by a determination of the crack growth direction using empirical criteria based on maximum tangential stress or maximum energy release rate. The proposed approach is presented and used for investigations of different crack and crack growth configurations which include mixed mode loading and crack interaction with other discontinuities. Empirical, experimental and additional numerical results, based on stress intensity factors, are compared, which confirm the applicability of the proposed approach and its capability to simulate crack growth.

KEYWORDS

extended finite element method, material forces, numerical crack growth simulation

INTRODUCTION

The usage of enhanced numerical simulation techniques like the extended finite element method (xFEM) increased significantly during the recent past. The fields of application of these methods cover a wide range but are especially attractive for simulations of cracked structures. The most important tasks herein are an evaluation of the severity of the crack and its growth direction. Well known and commonly used approaches to determine the severity of the crack state are for example energy integral formulations and the stress intensity factor concept. The crack propagation direction in two dimensions can then be determined with empirical formulations based on the in-plane stress intensity factors K_I and K_{II} which requires enhanced methods like for example the interaction integral technique when applying an energy integral formulation.

An alternative approach to evaluate cracked structures is the concept of material forces, which is also referred to as configurational force concept. The material force concept is based on Eshelby´s energy momentum tensor and describes the internal force that acts on the continuum in order to provide an optimum energy balance. In the presence of internal defects the internal energy equilibrium is disturbed and can be described by the material force. Since a discontinuity represents a significant defect within the continuum, this approach can also be used to evaluate cracks. In addition to that the material force provides an inherent directional information, following from its vector characteristics, that can be interpreted in terms of crack propagation and enables a one step evaluation of the above mentioned tasks which makes it very attractive for investigation of fracture mechanics and crack growth problems.

The present work introduces an approach to combine the extended finite element method (xFEM) and the material force concept in order to merge the benefits of both approaches. After the implementation, the applicability of the combined approach is investigated using different crack and crack growth scenarios.

ENERGY MOMENTUM TENSOR AND MATERIAL FORCES

Eshelby was the first to observe that the total force acting on any kind of defect within a continuum (e.g. singularity or inhomogeneity) can be evaluated by integration of a tensor quantity over a domain which encloses the defect [1]. This force was later referred to as *material force* or *configurational force* and the tensor quantity is referred to as *energy momentum tensor* which is also known as *Eshelby tensor*.

Energy momentum tensor

Following Eshelby in [1] the energy momentum tensor Σ_{ij} is given by the following equation, when limiting the observations to small deformations as done by Mueller et al. in [2].

$$\Sigma_{lj} = W\delta_{lj} - \frac{\partial W}{\partial u_{i,j}} u_{i,l} = W\delta_{lj} - \sigma_{ij} u_{i,l} \qquad \text{with} \quad \frac{\partial W}{\partial u_{i,j}} = \sigma_{ij} \tag{1}$$

In analogy to the physical stress equilibrium $\sigma_{ij} + f_i^{phy} = 0$, a configurational stress equilibrium can then be defined as

$$\Sigma_{lj,j} + f_j^{mat} = 0 \tag{2}$$

where f_j^{mat} represents material body forces which vanish for a homogeneous material and in the absence of physical body forces f_i^{phy}.

Material force acting on a defect

As already mentioned, the material force $\mathcal{M}_l$ at position X_l is determined by integrating the energy momentum tensor Σ_{lj} over a defined domain that includes the position of interest which in two dimensions becomes an integral over a surface Ω.

$$\mathcal{M}_l = \int_{\Omega_j} \Sigma_{lj} d\Omega_j \tag{3}$$

In case any kind of internal defect occurs within Ω_j, the resulting material force vector $\mathcal{M}_l$ takes non-zero values and its direction information can be interpreted in such a fashion that the material force vector $\mathcal{M}_l$ points into the direction where material points should be shifted in order to obtain an energetically optimized state of the continuum. This concept can directly be applied to fracture mechanics and the material force $\mathcal{M}_l$ from above can be used to evaluate cracked structures. In addition the material force inherently includes a directional information due to its vector properties which can be interpreted in terms of crack growth so that a crack will propagate in the opposite direction of $\mathcal{M}_l$ as already stated in [2].
Introducing a test function δ_u together with a standard FE interpolation, a weak formulation of the problem can be defined based on the configurational stress equilibrium in equation (2) which results in an expression for the determination of the material force at node I for a single element

$$\mathcal{M}_e^I = \int_{\Omega_e} \Sigma_{ij}\, N_{,j}^I \, d\Omega_e \tag{4}$$

The integral in (4) can readily be evaluated by numerical integration and the material force contribution from all elements n_e sharing node I defines the total material force $\mathcal{M}^I$ at node I.

$$\mathcal{M}^I = \sum_{e=1}^{n_e} \mathcal{M}_e^I \tag{5}$$

Investigating the vector plot of the resulting material forces in Figure 2a, additional material forces in the vicinity of the crack tip can be observed which are caused by an inaccurate approximation of the singular stress-strain field by the FE discretization in this region. This is an important aspect and the material force solution can be improved when these *spurious* material forces are included for the overall material force solution at the crack tip [3]. This improving effect represents the primary motivation and basic idea for the proposed approach to combine the material force concept and the extended finite element method (xFEM) which will be introduced in the following.

EXTENDED FINITE ELEMENT METHOD

The extended finite element method enables a mesh independent representation of arbitrary discontinuities by locally adding additional nodal degrees of freedom to the standard FE approximation. Some relevant aspects of the basic theory of the xFEM and its numerical implementation within the present work will shortly be highlighted in this section.

Review on theoretic aspects of the xFEM

The concept of a locally enriched finite element technique is based on the partition of unity property of finite element shape functions and was first introduced by Belytschko and Black [4] who use nodal enrichment functions for representation of the discontinuity. Moës et al. [5] improved this first approach by introducing a second type of enrichment function, represented by a step function, which enabled a fully mesh-independent representation of the crack. This technique was later referred to as extended finite element method (xFEM) and was investigated and improved by a large number of researchers.
Using the enriched finite element approximation given by [5] the displacement u^h at position x_i for an internal crack, like the one depicted in Figure 1a, is given by the following general form

$$u^h = \sum_{i=1}^{I} N_i u_i + \sum_{j=1}^{J} N_j a_j H(x_i) + \sum_{k=1}^{K_1} N_k \left(\sum_{\alpha=1}^{4} b_k^{\alpha,1} \Phi_{\alpha,1}(x_i) \right) + \sum_{k=1}^{K_2} N_k \left(\sum_{\alpha=1}^{4} b_k^{\alpha,2} \Phi_{\alpha,2}(x_i) \right) \quad (6)$$

The first term on the right hand side of (6) represents the standard finite element solution from all nodes I in the domain. J denotes the node set of all nodes that are enriched with step function enrichment and K_1 and K_2 are the sets containing the nodes at the crack tips to be enriched with singularity function enrichment (cp. Figure 1a). For cases where the crack tip is located near an element edge the standard crack tip enrichment can result in an unbalanced enrichment situation which negatively influences the results. An enhanced crack tip enrichment that enriches all nodes within a circle of radius r_{enr} around the crack tip with singularity function enrichment helps to overcome this problem and is also illustrated in Figure 1a. Following [5] the step function enrichment $H(x_i)$ is defined as the Heaviside step function whose sign depends on the orientation of point x_i with respect to the crack. Instead of using its position vector x_i the step function value can also be defined using the distance vector ξ_i representing the shortest normal distance to the crack which enables a convenient description of the crack using a level set approach.

$$H(x_i) = H(\xi_i) = \begin{cases} +1 & \text{for nodes "above" the crack line} \\ -1 & \text{for nodes "below" the crack line} \end{cases} \quad (7)$$

The singularity function enrichment $\Phi_\alpha(x_i)$ is defined in local polar crack tip coordinates r and θ and are defined by the asymptotic crack tip functions given in [5].

$$\Phi(x_i) = \Phi(r,\theta) = \left[\sqrt{r}\sin\left(\frac{\theta}{2}\right),\ \sqrt{r}\cos\left(\frac{\theta}{2}\right),\ \sqrt{r}\sin\left(\frac{\theta}{2}\right)\sin(\theta),\ \sqrt{r}\cos\left(\frac{\theta}{2}\right)\sin(\theta) \right] \quad (8)$$

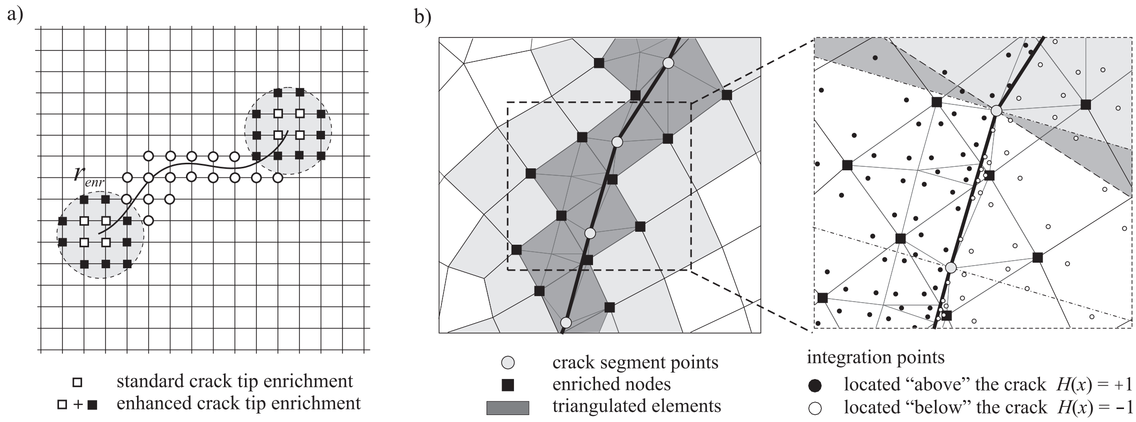


Fig. 1: crack mesh interaction and nodal enrichment for standard and enhanced crack tip enrichment ; b) exemplary crack section illustrating element subdivision and enrichment of nodes and integration points

The additional degrees of freedom a_j and b_k in (6) are only introduced locally for the identified sets of enriched nodes, either being defined by the Heaviside step function or the crack tip singularity function enrichment from above. This leads to a local enlargement of the element stiffness matrices for all elements containing one or more enriched nodes depending on the type of enrichment and number of enriched nodes.

Numerical implementation aspects

A numerical implementation of an extended finite element method requires special attention with respect to a variety of different aspects arising from the theoretical approach. Besides an adapted numerical integration scheme to ensure a correct integration of elements being intersected by the crack, this includes an algorithm to identify these interacting elements, as well as the correct enrichment identification and update of nodes and integration points. Even though a detailed survey on these issues can not be given here, Figure 1b gives a detail of a crack representation which illustrates the basic element discretization, the resulting element subdivision of triangulated elements, the enriched nodes and the step function enrichment of interacting integration points from the xFEM implementation for the present work.

INCORPORATION OF MATERIAL FORCE CONCEPT INTO XFEM FRAMEWORK

Domain approach - based on nodal material forces around the crack tip

An incorporation of the material force concept into an extended finite element framework faces the basic problem that the crack is represented independently of the FE discretization within the xFEM and there is no longer a crack tip node available for evaluation of the crack tip material force.

Recalling the investigations from above, an occurrence of additional material forces in the vicinity of the crack tip can be observed for a standard finite element solution (cp. Figure 2a) whose inclusion has a beneficial effect on the results and improves the material force solution at the crack tip. Transferring these observations to the extended finite element framework leads to the basic idea of the present work which aims at a combination of the material force concept and the xFEM. The approach assumes that the material force at the *virtual* crack tip within an xFEM can be determined by considering all nodal material forces $\mathcal{M}^I$ within a defined domain Ω_{tip}

$$\mathcal{M}_{\text{tip}}^{\text{domain}} = \sum_{I=1}^{n_{\text{domain}}} \mathcal{M}^I \tag{9}$$

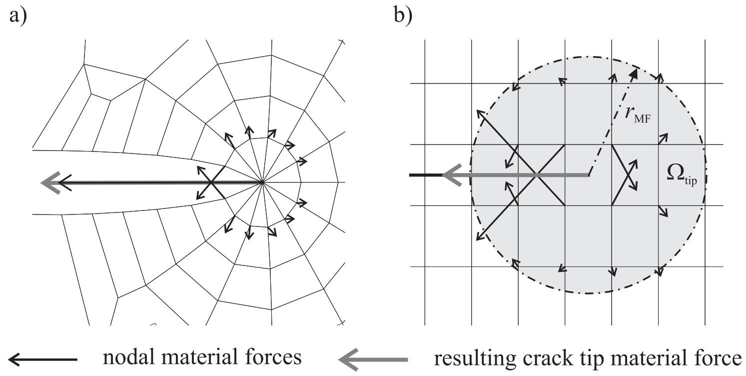


Fig. 2: nodal material forces around the crack tip and resulting crack tip material force: a) standard finite element method ; b) domain approach within xFEM

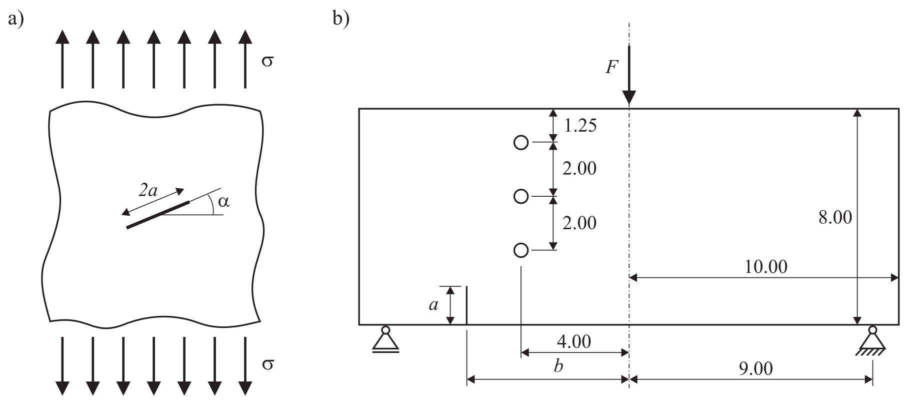
where $\mathcal{M}_{\text{tip}}^{\text{domain}}$ represents the resulting material force at the *virtual* crack tip. The size of Ω_{tip} is controlled by a radius r_{MF} which is illustrated by the shaded region in Figure 2b together with the nodal material forces and the resulting crack tip material force vector for an edge crack scenario. The radius r_{MF} is determined by the characteristic length $l^\star$ of the crack tip element, defined as the square root of crack tip element area, and a multiplication factor β_{MF}.

$$r_{\text{MF}} = l^\star \cdot \beta_{\text{MF}} = \sqrt{A_{\text{tip}}} \cdot \beta_{\text{MF}} \tag{10}$$

The multiplication factor β_{MF} controls the amount of nodal material forces contributing to $\mathcal{M}_{\text{tip}}^{\text{domain}}$ and is defined by the user. Investigating Figure 2b it can be seen that the material force contributions from the nodes of the crack tip element and the adjacent crack front element provide the major contributions to $\mathcal{M}_{\text{tip}}^{\text{domain}}$ while the remaining nodal material forces within Ω_{tip} are comparatively small and only refine the solution.

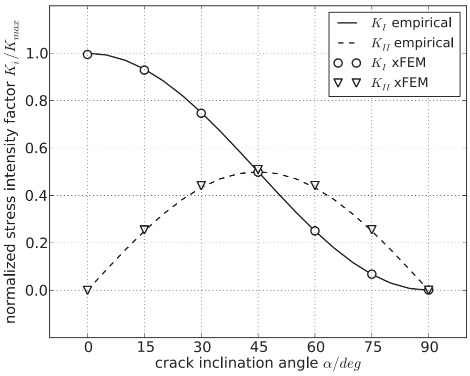
NUMERICAL EXAMPLES

In order to evaluate the proposed approach and enable a judgement on its capabilities to simulate crack growth, different crack and crack growth configurations are investigated. Exemplary simulation results are presented and compared with available empirical and experimental solutions as well as with additional numerical results.

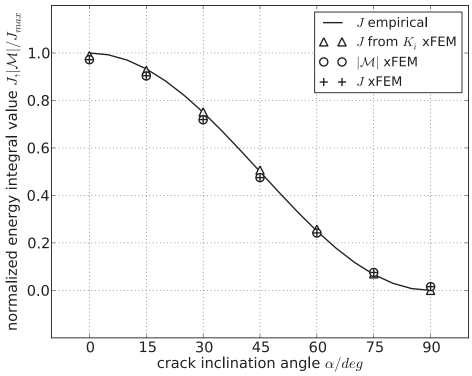


scenario	b	a
#1	5.0	1.5
#2	6.0	1.0

Fig. 3: investigated crack scenarios: a) inclined center crack ; b) three point bending specimen from [6]



(a) stress intensity factor (SIF) solution



(b) J-integral and material force solution

Fig. 4: simulation result summary for the crack scenario of an inclined center crack in an infinite width plate and comparison with the empirical solution from [7]

Inclined center crack scenario

A mixed mode crack scenario is used for basic evaluation of the new approach. The scenario is defined by a center crack in an infinite plate under uniaxially load as illustrated in Figure 3a. Different mixed mode crack loadings can be simulated by increasing the inclination angle α of the crack. The empirical stress intensity factors for this scenario can be found in [7].

$$K_I = \sigma\sqrt{\pi a}\cos^2(\alpha) \quad ; \quad K_{II} \; = \sigma\sqrt{\pi a}\sin(\alpha)\cos(\alpha) \tag{11}$$

The stress intensity factor, J-integral and material force solution is determined for varying crack inclination angles and the results are given in Figure 4. The stress intensity factor solution in Figure 4a which is evaluated using the interaction integral technique from Yau et al. [7] shows very good accordance with the empirical solution for all crack inclination angles with very small errors of less than 2%. The J-integral and material force solutions in Figure 4b also show good accordance for all investigated crack inclination angles with small errors which proves the applicability of the proposed approach for crack state evaluation.

Crack propagation investigations

In order to investigate the capabilities of the proposed method to simulate crack growth, including the crack path, two different crack growth configurations of a plate under three-point bending loading from [6] are considered which are illustrated in Figure 3b. The plate exhibits three eccentric holes and the initial crack is located at an off center position near the holes which is defined by the parameters a and b that are given in the table in Fig. 3b. The off-center loading in combination with an additional interaction of crack and holes results in mixed mode crack growth scenarios with increased complexity. The crack growth increment is fixed to a constant value of da = 0.15 and all simulations are performed using the xFEM. In the first case the crack state and crack propagation direction are determined using the direction information that is provided by the material force vector $\mathcal{M}$ resulting from the proposed approach which will be referred to as *MF-based* in the following. In the second case the crack path is determined using the in-plane stress intensity factors determined using the interaction integral technique from [7] and equation (12) which is taken from [8] and determines the crack deflection angle based on empirical assumptions. This second approach will be referred to as *SIF-based* due to its direct dependence on the stress intensity factors.

$$\varphi = -\arccos\left(\frac{3K_{II}^2 + K_I\sqrt{K_I^2 + 8K_{II}^2}}{K_I^2 + 9K_{II}^2}\right) \tag{12}$$

The influence of the evaluation radius r_{MF}, which defines the domain that is used for determination of the resulting material force at the virtual crack tip, is studied by varying the multiplication factor β_{MF} in (10). The usage of the β_{MF} values 1.5, 2.25 and 3.0 results in a constant enlargement of Ω_{tip} and increases the amount of considered nodal material forces for each case. Simulation results for the first scenario are given in Figure 5 enabling a comparison of the crack path predictions from both numerical approaches on the bottom side, including a comparison with the experimentally determined crack path from [6] which is illustrated on the right hand side. As can be seen in the detail views on the bottom side of Figure 5 the element discretization remains fixed for all simulations and no re-meshing is required during the crack growth simulation. The resulting crack paths for the *MF-based* and the *SIF-based* approach are given as dashed and solid lines respectively in the plots for increasing evaluation radii r_{MF}. Within this context the *SIF-based* solution is determined using a fixed evaluation radius of $r_{\mathsf{SIF}} < da$ in order to ensure an evaluation of the interaction integral over a straight crack segment. The accordance

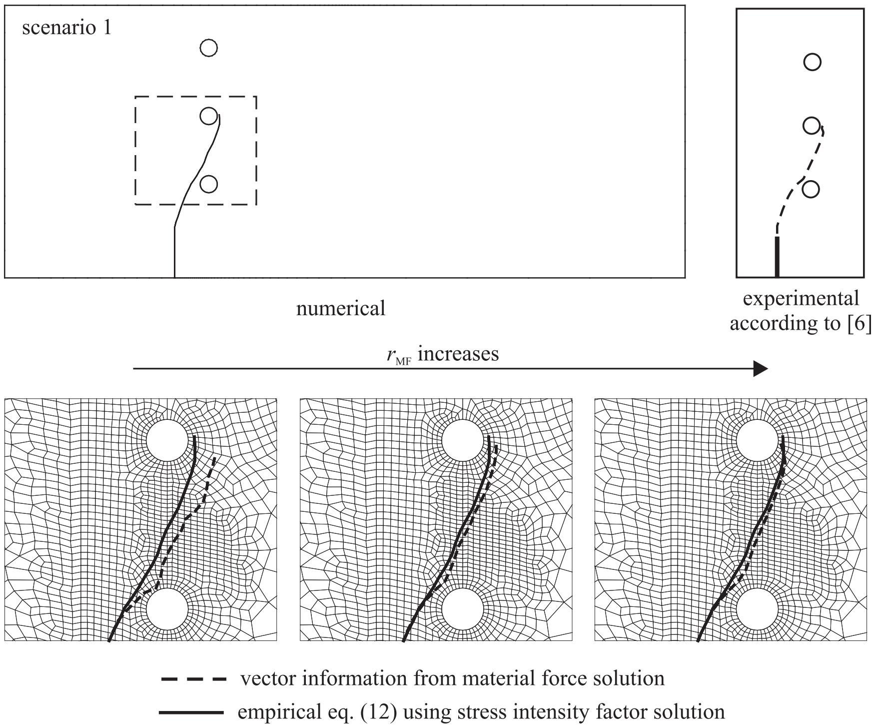


Fig. 5: result comparison for crack scenario 1 of the Bittencourt example from Figure 3b

between the numerical crack path solutions improves with increasing r_{MF} and provides a good representation of the experimentally determined crack path for the largest evaluation radius. In addition to that an increase of r_{MF} leads to an alignment of the *MF-based* and *SIF-based* crack path solutions. It should be mentioned, however, that this improving effect on the material force solution is certainly limited and an immoderate increase of the evaluation radius will eventually lead to a deterioration of the results when r_{MF} for example encloses another defect or singularity. A closer investigation of this effect is beyond the scope of the present work and will be subject of future investigations.

SUMMARY

The motivation of this work was to merge the benefits of the material force concept and the extended finite element method in order to efficiently investigate cracks and crack growth in fracture mechanics problems.

In this context the mesh independent crack representation within an extended finite element framework offers an attractive and efficient way to simulate cracks and their growth without any need for re-meshing operations. The vectorial material force concept represents a generalization of the scalar J-integral and provides an alternative way to evaluate the crack state including an inherent information on the crack growth direction based on a continuum mechanic approach.
An approach was presented which incorporates the material force concept within an extended finite element framework that is based on observations with respect to additional material forces in the vicinity of the crack tip and leads to a domain evaluation of the material force at the virtual crack tip position within the xFEM. The proposed approach was then used for simulation of different crack configurations and crack growth scenarios under mixed mode loading. The results were compared with an additional numerical solution, which uses the stress intensity factor solution and an empirical expressions for determination of the crack growth direction, as well as with empirical formulations and experimental results when available. This investigation also included a study on the influence of simulation parameters on the quality of resulting crack path solutions. The good accordance among the different simulations as well as with respect to empirical and experimental results proves the applicability of the proposed approach and confirms its capability to efficiently simulate crack growth processes.

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