A Model of Anisotropic Damage by Mesocrack Growth; Unilateral Effect

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ABSTRACT: A three-dimensional model of anisotropic damage by mesocrack growth is first described in its basic version, employing a second-order tensorial damage variable. The model – concerning rate-independent, small strain, isothermal behaviour – allows to take into account residual effects due to damage and reduces any system of mesocracks to three equivalent orthogonal sets. This first version is then extended to account for elastic moduli recovery due to crack closure. Micromechanical considerations impose to employ a fourth-order crack-related tensor when the mesocracks are constrained against opening. Unlike some models which do not avoid (or rectify a posteriori) discontinuities of the stress-strain response, the approach herein ensures a priori the stress continuity and allows to express a convenient macroscopic opening-closure criterion. Nevertheless, the new formulation maintains the orthotropy of the effective properties (instead of an eventual, more general form of anisotropy). Finally, it appears that the extended version does not introduce additional material constants compared to the basic version. The model is tested by simulating the behaviour of Fontainebleau sandstone.

KEY WORDS: damage, brittle materials, mesocracks, anisotropy, orthotropy, residual effects, damage evolution, crack closure, damage deactivation, fourth-order crack tensor, stress continuity, closure criterion.

1. INTRODUCTION

PROGRESSIVE DEGRADATION OF mechanical properties of brittle (e.g., rock-like) materials is due to generation and growth of mesocracks. This deterioration phenomenon controls macroscopic events like induced anisotropy, residual (irreversible) strain/stress effects, dilatancy, etc. The existing mesocracks can be opened (active) or closed (inactive) along a loading path, leading physically to an elastic moduli recovery phenomenon, i.e., bilinear elasticity. The

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concern here is in associating directionality of the defects and related anisotropy with description of the deactivation phenomenon.

In spite of existing attempts – some of them providing practical criteria as, for example, the work by Chaboche (1993) – further investigations are needed to deal thoroughly with the problems of continuity of the underlying response in a rigorous way and to relate macroscopic criteria to mesostructural mechanisms. Concerning the state of the art, the following remarks can be put forward:

- Some existing theories (cf. the models evaluated by Chaboche, 1992a) do not ensure symmetry of the elastic compliance (or stiffness) tensor and/or continuity of the stress-strain response.
- Micromechanical models (Andrieux et al., 1986; Gambarotta and Lagomarsino, 1993), in spite of their ingenuity, employ quantities that are not adapted to an efficient structural analysis.
- Certain theories have to do with quite specific forms of damage (e.g., Ladevèze et al., 1994, for a class of composites) and fail to describe a more general degradation.
- Some authors propose modifications a posteriori of their model: modification of the stiffness (or compliance) tensor expressed in the principal damage tensor directions (Chaboche, 1993); "smoothing" of the Heaviside function in order to avoid discontinuities (Hansen and Schreyer, 1995).

This paper aims to introduce a new unilateral condition in the model of anisotropic damage by mesocrack growth proposed by Dragon et al. (1994) referring to the framework of Continuum Damage Mechanics and based on proper thermodynamic considerations. In Section 2, the main features of the model in its simplified "basic" version are pointed out. In Section 3, this basic version is enlarged in order to take into account the unilateral effect. This new approach is built on micromechanical considerations, but allows a formulation which remains convenient to handle at the macro-scale.

2. A MODEL OF ANISOTROPIC DAMAGE BY MESOCRACK GROWTH

Generation and growth of mesocracks is the unique dissipative mechanism considered in this model; the behaviour of the mesocracked material is assumed to be rate-independent, isothermal, and restrained to small strain.

2.1 Damage Variable

In order to account for orientation of mesodefects, a second-order tensor is chosen as a single damage internal variable: it is built upon the dyadic product $\mathbf{n}^i \otimes \mathbf{n}^i$ ($\mathbf{n}^i$ is a unit normal to the set i of the parallel mesocracks) and on the dimensionless scalar function $d^i(S)$, proportional to the extent of decohesion surface produced by mesocracking in the representative volume element:

$$\mathbf{D} = \sum_i d^i(S)\mathbf{n}^i \otimes \mathbf{n}^i \tag{1}$$

This microstructural insight is sustained by macroscopic interpretation relevant to the choice of second-order tensorial density: as a symmetric second-order tensor, **D** can be expressed in its principal axes according to spectral decomposition:

$$\mathbf{D} = \sum_{k=1}^{3} D_k \boldsymbol{\nu}^k \otimes \boldsymbol{\nu}^k \tag{2}$$

This means that any system of cracks, i.e., any collection of sets included in summation (1), is fully equivalent to three mutually perpendicular families of parallel mesocracks. This equivalence will be exploited furthermore in the expression of **D**-influenced strain energy leading to effective moduli representing a form of induced orthotropy with axes coinciding with the principal axes of **D**.

2.2 Strain Energy Function and Elastic-Damage Response

The model postulates the existence of a thermodynamic potential, in particular, the free energy w per unit volume, function of the arguments strain ϵ and damage **D**. w is an isotropic invariant of ϵ and **D**, i.e., can be regarded as a combination of independent invariants of ϵ and **D** including the simultaneous ones. w is restrained to depend linearly on **D**, corresponding to the hypothesis of non-interacting cracks (Kachanov, 1992) and is at most quadratic in ϵ, thus assuming linear elasticity for a fixed **D**. The presence of a linear term of ϵ allows to take into account residual (irreversible) effects (e.g., residual stress for $\epsilon = 0$) due to damage only, without reference to plasticity (Figure 1).

According to these assumptions, Dragon et al. (1994) propose the following expression of w:

$$w = \frac{1}{2}\lambda(\mathrm{tr}\,\epsilon)^2 + \mu\,\mathrm{tr}\,(\epsilon\cdot\epsilon) + g\,\mathrm{tr}(\epsilon\cdot\mathbf{D}) + \alpha\,\mathrm{tr}\,\epsilon\,\mathrm{tr}\,(\epsilon\cdot\mathbf{D})$$
$$+ 2\beta\,\mathrm{tr}\,(\epsilon\cdot\epsilon\cdot\mathbf{D}) \tag{3}$$

where λ and μ are the Lamé constants, α and β are supplementary constants related to moduli degradation due to damage, and g is a constant governing residual effects. For the discussion concerning the residual strain/stress effects due to

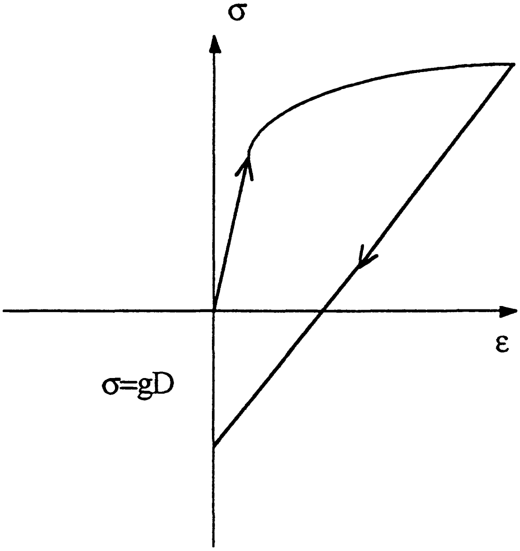

Figure 1. *Residual stress for uniaxial tension.*

damage, see for example, Dragon et al. (1994); the corresponding postulate in the framework of damage modelling was advanced relatively early (Dragon, 1976). For a more recent discussion of the subject, see for example, Najar (1994).

The differentiation of the free energy leads to the equations of state defining, respectively, the elastic behaviour and the thermodynamic affinity related to **D**:

$$\sigma = \frac{\partial w}{\partial \epsilon} = \lambda(\mathrm{tr}\ \epsilon)\mathbf{I} + 2\mu\epsilon + g\mathbf{D} + \alpha\ [\mathrm{tr}\ (\epsilon \cdot \mathbf{D})\mathbf{I} + (\mathrm{tr}\ \epsilon)\mathbf{D}]$$

$$+ 2\beta(\epsilon \cdot \mathbf{D} + \mathbf{D} \cdot \epsilon) \tag{4}$$

$$\mathbf{F}^{D} = -\frac{\partial w}{\partial \mathbf{D}} = -g\epsilon - \alpha\ (\mathrm{tr}\ \epsilon)\epsilon - 2\beta(\epsilon \cdot \epsilon) \tag{5}$$

The force $\mathbf{F}^D$ may be divided into two parts:

$$\mathbf{F}^{D} = \mathbf{F}^{D1} + \mathbf{F}^{D2};\quad \mathbf{F}^{D1} = -g\epsilon;\quad \mathbf{F}^{D2} = -\alpha\ (\mathrm{tr}\ \epsilon)\epsilon - 2\beta(\epsilon \cdot \epsilon) \tag{6}$$

where $\mathbf{F}^{D1}$ is the "locked" damage energy release rate, linked to residual effects, and $\mathbf{F}^{D2}$ is the damage energy release rate relative to modified reversible effects. Anticipating the modelling of damage growth, an additional partition of $\mathbf{F}^{D1}$ is introduced:

$$\mathbf{F}^{D1} = \mathbf{F}^{D1+} + \mathbf{F}^{D1-} = -g\epsilon^{+} - g(\epsilon - \epsilon^{+}); \quad \epsilon^{+} = \mathbf{P}^{+}:\epsilon \tag{7}$$

$\mathbf{P}^{+}$ is the fourth-order positive projection tensor selecting the positive eigenvalues of ϵ. $\mathbf{P}^{+}$ was first mentioned by Ortiz (1985), then explicitly given in a more complex form by Ju (1989). Lubarda et al. (1994) proposed a simplified equivalent expression employed, for example, by Hansen and Schreyer (1995) as follows:

$$\mathrm{P}^{+}_{ijkl} = Q^{+}_{ik}Q^{+}_{jl} \quad \text{with} \quad \mathbf{Q}^{+} = \sum_{n=1}^{3} H(\epsilon_n)\mathbf{q}^{n} \otimes \mathbf{q}^{n} \tag{8}$$

H is the Heaviside function, ϵ_n the nth eigenvalue, $\mathbf{q}^n$ the nth eigenvector of ϵ.

2.3 Damage Criterion and Evolution

The convex reversibility domain limited by the damage criterion $f = 0$ is written in $\mathbf{F}^D$ space. It is assumed that, concerning the form of the damage threshold in reference to the brittle splitting-like damage, the $\mathbf{F}^{D1+}$ term of the driving force $\mathbf{F}^D$ plays a determining role:

$$\begin{aligned} f(\mathbf{F}^D,\mathbf{D}) &= f(\mathbf{F}^D - \mathbf{F}^{D2} - \mathbf{F}^{D1-},\mathbf{D}) \\ &= \sqrt{\frac{1}{2}\,\mathrm{tr}\,[(\mathbf{F}^D - \mathbf{F}^{D2} - \mathbf{F}^{D1-})\cdot(\mathbf{F}^D - \mathbf{F}^{D2} - \mathbf{F}^{D1-})]} \\ &\quad + B\,\mathrm{tr}\,[(\mathbf{F}^D - \mathbf{F}^{D2} - \mathbf{F}^{D1-})\cdot\mathbf{D}] - (C_0 + C_1\,\mathrm{tr}\,\mathbf{D}) \leq 0 \end{aligned} \tag{9}$$

C_0 is the initial damage limit while the constants C_1 and B are linked to the variation of the domain $f \leq 0$ with the progression of damage. The aspect of the initial ($\mathbf{D} = \mathbf{0}$) damage threshold has been drawn for $B = 0$ in Figure 2 in the principal stress space corresponding to axisymmetric loading. The model parameters have been identified for Fontainebleau sandstone (cf. Table 1). Note in particular the dissymmetry between the compression and tension thresholds.

The damage growth rule is assumed to be rate-independent and when expressed in the framework of a standard scheme, it is found to reflect predominant

Table 1. Parameters identified on Fontainebleau sandstone.

Parameters	λ [MPa]	μ [MPa]	α [MPa]	β [MPa]	g [MPa]	C_0 [MPa]	C_1 [MPa]	B [1]
Sandstone	26,250	17,500	1900	−20,400	−110	0.001	0.55	0

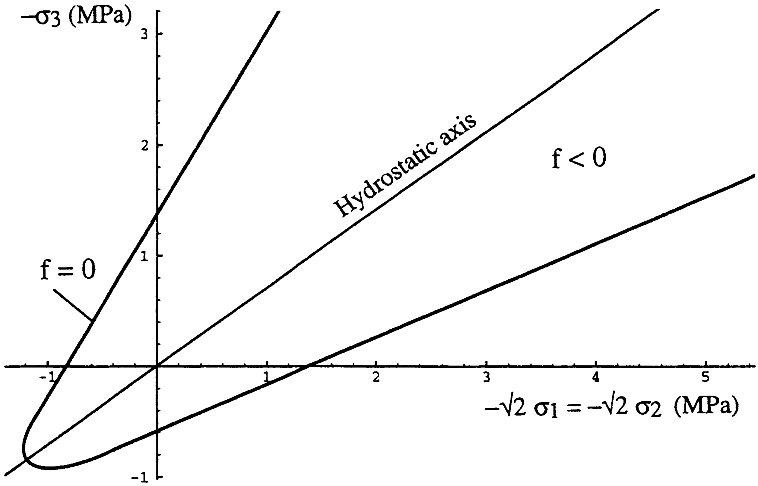

Figure 2. *Initial simulated damage locus in stress space corresponding to axisymmetric loading.*

features of brittle cracking as notably the orientation of mesocracks tending to follow the ϵ^+-directions for proportional loading segments. We assume thus $\dot{\mathbf{D}}$-normality with respect to f, according to Hill's maximum dissipation principle in the relevant $\mathbf{F}^D$-space:

$$\dot{\mathbf{D}} = \Lambda \frac{\partial f(\mathbf{F}^D - \mathbf{F}^{D2} - \mathbf{F}^{D1-}, \mathbf{D})}{\partial \mathbf{F}^D}, \quad \Lambda \geq 0 \tag{10a}$$

$$\dot{\mathbf{D}} = \begin{cases} 0 & \text{if } f < 0 \text{ or } f = 0, \ \dot{f} < 0 \\ \Lambda\left[\dfrac{\epsilon^+}{\sqrt{2 \operatorname{tr}(\epsilon^+ \cdot \epsilon^+)}} + B\mathbf{D}\right] & \text{if } f = 0 \text{ and } \dot{f} = 0 \end{cases} \tag{10b}$$

The directional term of Equation (10b) includes two terms: one term proportional to positive strain, the other relating $\dot{\mathbf{D}}$ to the actual value of $\mathbf{D}$, through the intervention of the damage-drag constant B.

To summarize, the proposed model employs eight parameters only: λ, μ, α, β, g, C_0, C_1, B. Seven of them can be easily determined from axisymmetric triaxial compression tests with unloading, see Dragon et al. (1994). λ and μ are conventionally calculated from E_0 and ν_0 (initial Young modulus and Poisson's ratio) in the elastic response range without damage growth if the material can be considered initially as undamaged or damaged with no initial anisotropy. C_0 and C_1 are determined from the nonlinear portion of experimental curves. The identification of α and β exploits the unloading portions. The point at which the unloading is performed should correspond to pronounced oriented damage, but has to be reasonably far from the maximum of the stress-strain curve to avoid interference with eventual bifurcation phenomena. To determine B, one should resort to an off-axis loading experiment for a pre-damaged specimen.

In order to examine the predictive ability of the model, Cormery (1994) and Pham (1994) have carried out a number of simulations that were compared to experimental data. We propose here (Figures 3a and 3b) the comparison between experiment (Ikogou, 1990) and simulation for the behaviour of Fontainebleau sandstone submitted to axisymmetric triaxial compression with a confining pressure P_c of 30 MPa. The set of parameters given in Table 1 was determined from axisymmetric tests with confining pressure of 10 and 20 MPa. The dilatancy effect resulting from pronounced damage is evidenced in Figure 3b.

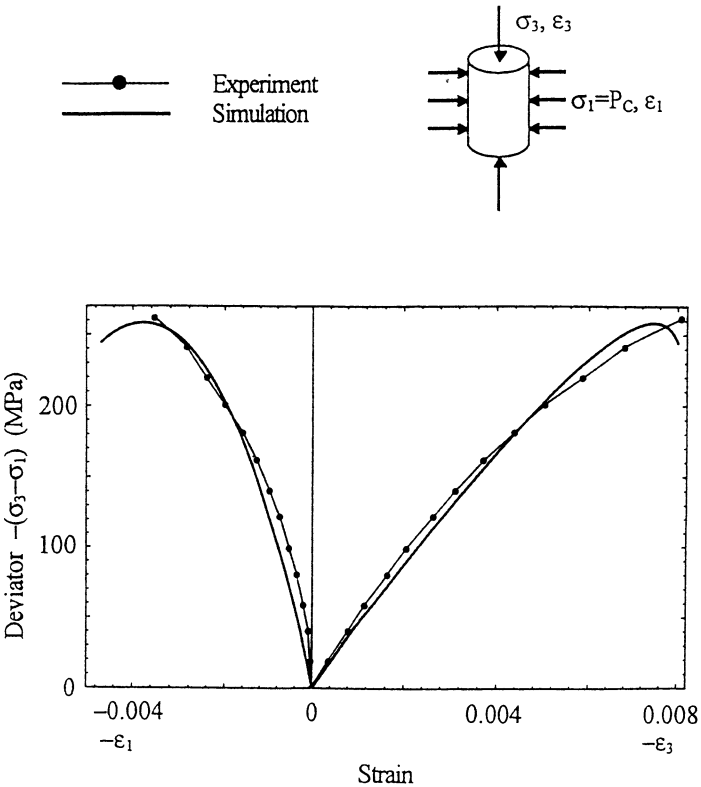

Figure 3a. Axisymmetric triaxial compression test—longitudinal and transversal response.

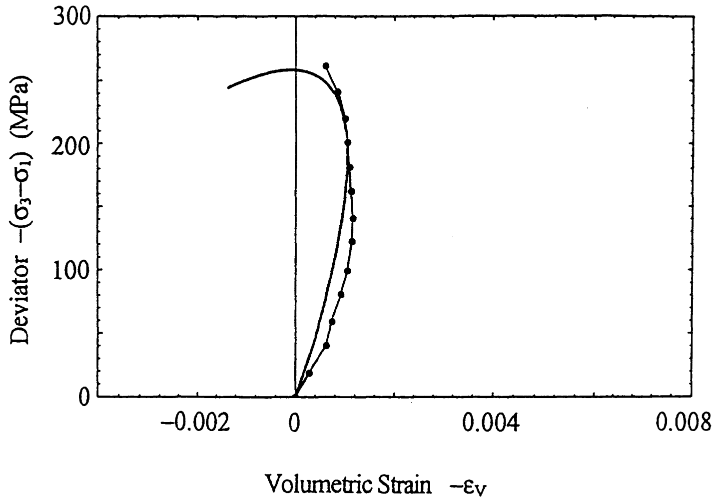

Figure 3b. *Axisymmetric triaxial compression test—volumetric strain.*

3. INTRODUCTION OF A UNILATERAL CONDITION

In this paper, the authors propose a new unilateral condition allowing recovery of moduli in relation to damage deactivation (mesocrack closure). The purpose is to complete the "basic" version of the model summarized in Section 2. The approach presented here is stimulated by some micromechanical arguments, especially by those by Kachanov (1992) concerning some remarkable facts about crack networks in the elastic medium.

3.1 Unilateral Condition

Consider a representative volume element (RVE) V of a material considered initially isotropic and homogeneous (constant initial compliance tensor $\mathbf{S}^0$), including an array of mesocracks. Each mesocrack i (or set of equioriented similar mesocracks) is characterized by its surface $\mathbf{S}^i$, its normal $\mathbf{n}^i$, and its crack displacement discontinuity vector $\mathbf{b}^i$. Classically, the expression for global strain on RVE giving explicitly the contribution of multiple cracks is:

$$\epsilon = \mathbf{S}^0:\sigma + \frac{1}{2V}\sum_i \int_{\mathbf{S}^i} (\mathbf{b} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{b})^i dS \qquad (11)$$

In the following, the cracks are assumed to be flat (i.e., $\mathbf{n}$ remains constant over the crack) and quasi-circular (radius a^i) so that:

$$\epsilon = \mathbf{S}^0{:}\boldsymbol{\sigma} + \frac{1}{2V}\sum_i (\langle\mathbf{b}\rangle \otimes \mathbf{n} + \mathbf{n} \otimes \langle\mathbf{b}\rangle)^i S^i \tag{12}$$

where $\langle\mathbf{b}^i\rangle$ is $\mathbf{b}$ averaged over the mesocrack set i. In this way, the global strain is composed of two terms: the first represents the matrix strain without damage, the second the specific contribution of the mesocracks. Relation (12) leads to the expression of the free enthalpy of the mesocracked solid:

$$u = \frac{1}{2}\boldsymbol{\sigma}{:}\mathbf{S}^0{:}\boldsymbol{\sigma} + \frac{1}{2V}\sum_i (\mathbf{n} \cdot \boldsymbol{\sigma} \cdot \langle\mathbf{b}\rangle)^i S^i \tag{13}$$

Kachanov (1992) gives the form of $\langle\mathbf{b}^i\rangle$ in a three-dimensional case for a crack embedded in an infinite body with stress $\boldsymbol{\sigma}$ at infinity (hypothesis of noninteracting cracks): in the most general case (mesocracks allowed to open), $\langle\mathbf{b}\rangle$ comprises normal and shear components:

$$\langle\mathbf{b}^i\rangle = \frac{16(1-\nu_0^2)}{3\pi} \cdot \frac{a^i}{E_0}(\mathbf{n}^i \cdot \boldsymbol{\sigma} \cdot \mathbf{n}^i)\mathbf{n}^i$$

$$+ \frac{16(1-\nu_0^2)}{3\pi(1-\nu_0/2)} \cdot \frac{a^i}{E_0}[\mathbf{n}^i \cdot \sigma - (\mathbf{n}^i \cdot \sigma \cdot \mathbf{n}^i)\mathbf{n}^i] \tag{14}$$

where E_0 and ν_0 are, respectively, the Young modulus and the Poisson's ratio of the matrix. Then:

$$u = u_0 + \Delta u = \frac{1}{2}\boldsymbol{\sigma}{:}\mathbf{S}^0{:}\boldsymbol{\sigma} + \frac{8(1-\nu_0^2)}{3(1-\nu_0/2)E_0}$$

$$\cdot \left[(\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}){:}\mathbf{D} - \frac{\nu_0}{2}\boldsymbol{\sigma}{:}\frac{1}{V}\sum_i (a^3\mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n})^i{:}\boldsymbol{\sigma}\right] \tag{15}$$

The presence of a fourth-order tensor causes deviation from orthotropy, but Kachanov (1980) proves that this deviation is small due to the multiplier $\nu_0/2$ preceding the second term in square brackets and, consequently, the influence of the fourth-order term can be neglected. Thus, for mesocracks allowed to open, a second-order crack-density variable, like $\mathbf{D}$, constitutes a sufficient approximation to describe the global behaviour of the mesocracked solid. This result (presence of a fourth-order term) is also obtained by methods considering mesocracks as three-dimensional defects ("void inclusions") whose thickness tends to zero

(Krajcinovic, 1989). Here, the direct approach (mesocracks directly seen as surface defects) seems to be more appropriate.

When the mesocracks are constrained against opening, implying thus unilateral contact, but are allowed to slide (without friction), the only possible displacement discontinuity occurs in the mesocrack plane; $\langle \mathbf{b} \rangle$ has a single shear component:

$$\langle \mathbf{b}^i \rangle \propto \mathbf{n}^i \cdot \boldsymbol{\sigma} - (\mathbf{n}^i \cdot \boldsymbol{\sigma} \cdot \mathbf{n}^i)\mathbf{n}^i \tag{16}$$

and the variation of free enthalpy becomes proportional to the following expression:

$$\Delta u \propto \left[(\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}):\mathbf{D} - \boldsymbol{\sigma}: \frac{1}{V} \sum_i (a^3\mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n})^i : \boldsymbol{\sigma} \right] \tag{17}$$

As a consequence of the disappearance of the multiplier $\nu_0/2$, the fourth-order term can no longer be neglected and orthotropy is violated. One concludes that the modelling of cracked solids, whereas mesocracks are constrained against opening, must taken into account a fourth-order crack-density term beside the second-order **D**-term. Other authors also mention the use of fourth-order tensors in large physical contexts such as elastoviscoplasticity (Chaboche, 1979) or creep (Chaboche, 1984).

These micromechanical preliminaries constitute the starting point for a reformulation of the model described in Section 2 to account for the unilateral effect. It first appears necessary to define a new fourth-order damage parameter. According to the considerations above, a rigorous fourth-order extension of **D** should be:

$$\overline{\mathbf{D}} = \sum_i d^i(S)\mathbf{n}^i \otimes \mathbf{n}^i \otimes \mathbf{n}^i \otimes \mathbf{n}^i \tag{18}$$

whose spectral decomposition can be written as:

$$\overline{\mathbf{D}} = \sum_{k=1}^{6} \overline{\mathrm{D}}_k \cdot \mathbf{A}^k \otimes \mathbf{A}^k \tag{19}$$

with $\overline{\mathrm{D}}_k$ the kth eigenvalue, $\mathbf{A}^k$ the kth second-order eigentensor. Such a decomposition encounters two main drawbacks:

1. It is difficult to confer a clear physical meaning on the eigentensor $\mathbf{A}^k$, unlike

Equation (2) where $\boldsymbol{\nu}^k$ appears as the normal to an equivalent set of mesocracks.
2. There is no trivial link between $\boldsymbol{\nu}^k$ and $\mathbf{A}^k$ so that the "elementary damage sets" in Equations (2) and (19) do not coincide and a formulation involving both decompositions would not be coherent.

This is why a simplified way is advanced consisting of harmonizing a fourth-order tensor in connection with Equation (2) in order to preserve the compatibility between both damage parameters:

$$\hat{\mathbf{D}} = \sum_{k=1}^{3} D_k \boldsymbol{\nu}^k \otimes \boldsymbol{\nu}^k \otimes \boldsymbol{\nu}^k \otimes \boldsymbol{\nu}^k \tag{20}$$

Note that $\hat{\mathbf{D}}$ appears as an opening-closure control parameter rather than as a new damage variable. Nevertheless, this new formulation keeps the macroscopic equivalence of any crack distribution of three orthogonal sets and relative orthotropy; it no longer allows a general form of anisotropy. This constitutes the main limitation relative to the introduction of $\hat{\mathbf{D}}$.

3.2 Decomposition of the Strain Free Energy and Elastic-Damage Response

According to the micromechanical preliminaries, the crack closure imposes the emergence of a fourth-order damage term in the expression of the free enthalpy increment Δu due to mesocracks [the terms $o(. . .)$ mean that the quantity in parenthesis is negligible with respect to the other terms]:

$$\begin{aligned} \text{open cracks:}\quad & u(\boldsymbol{\sigma},\mathbf{D}) = u_0(\boldsymbol{\sigma}) + \Delta u(\boldsymbol{\sigma},\mathbf{D}) + o(\boldsymbol{\sigma},\hat{\mathbf{D}}) \\ \text{closed cracks:}\quad & u(\boldsymbol{\sigma},\mathbf{D},\hat{\mathbf{D}}) = u_0(\boldsymbol{\sigma}) + \Delta u(\boldsymbol{\sigma},\mathbf{D},\hat{\mathbf{D}}) \end{aligned} \tag{21}$$

A dual representation, i.e., strain energy w (see Section 2), is obtained with the help of a Legendre transform ($\mathbf{D}$ and $\hat{\mathbf{D}}$ are here considered as passive parameters). w then takes two different forms depending on whether mesocracks are open (state 1) or closed (state 2):

$$\begin{aligned} \text{state 1 (open cracks):}\quad & w^1 = w^1(\boldsymbol{\epsilon},\mathbf{D}) = w^0(\boldsymbol{\epsilon}) + \Delta w^1(\boldsymbol{\epsilon},\mathbf{D}) + o(\boldsymbol{\epsilon},\hat{\mathbf{D}}) \\ \text{state 2 (closed cracks):}\quad & w^2 = w^2(\boldsymbol{\epsilon},\mathbf{D},\hat{\mathbf{D}}) = w^0(\boldsymbol{\epsilon}) + \Delta w^{2\prime}(\boldsymbol{\epsilon},\mathbf{D}) \\ & \qquad + \Delta w^{2\prime\prime}(\boldsymbol{\epsilon},\hat{\mathbf{D}}) \end{aligned} \tag{22}$$

The term in $\hat{\mathbf{D}}$ influences the elastic moduli by entering $\Delta w^{2\prime\prime}$ by simultaneous invariants quadratic in ϵ and linear in $\hat{\mathbf{D}}$. The research of such simultaneous invariants (Spencer, 1971) is largely simplified by the symmetry of ϵ and the particular form of $\hat{\mathbf{D}}$ [Equation (20)], i.e., $\hat{\mathbf{D}}$ being symmetric with respect to all rearrangements of its indices. As a result, there are only five independent simultaneous invariants:

$$\epsilon_{ii}\epsilon_{jj}\hat{D}_{kkll}, \quad \epsilon_{ik}\epsilon_{ik}\hat{D}_{jjll}, \quad \epsilon_{ii}\hat{D}_{jjkl}\epsilon_{kl}, \quad \epsilon_{kj}\epsilon_{jl}\hat{D}_{iikl}, \quad \epsilon_{ij}\hat{D}_{ijkl}\epsilon_{kl} \tag{23}$$

Since all the $\boldsymbol{\nu}^k$ are unit vectors, only one invariant (i.e., $\epsilon:\hat{\mathbf{D}}:\epsilon$) is truly pertinent to $\hat{\mathbf{D}}$; the four others can be expressed as invariants of ϵ and $\mathbf{D}$. So $\hat{\mathbf{D}}$ will enter Δw^2 through $\epsilon:\hat{\mathbf{D}}:\epsilon$. The free energy can be written in its two different forms:

- state 1: open mesocracks:

$$w^1 = \frac{1}{2}\lambda\,(\mathrm{tr}\,\epsilon)^2 + \mu\,\mathrm{tr}\,(\epsilon\cdot\epsilon) + g\,\mathrm{tr}\,(\epsilon\cdot\mathbf{D})$$
$$+ \alpha_1\,\mathrm{tr}\,\epsilon\,\mathrm{tr}\,(\epsilon\cdot\mathbf{D}) + 2\beta_1\,\mathrm{tr}\,(\epsilon\cdot\epsilon\cdot\mathbf{D}) + o(\epsilon:\hat{\mathbf{D}}:\epsilon) \tag{24}$$

- state 2: closed mesocracks:

$$w^2 = \frac{1}{2}\lambda\,(\mathrm{tr}\,\epsilon)^2 + \mu\,\mathrm{tr}\,(\epsilon\cdot\epsilon) + g\,\mathrm{tr}\,(\epsilon\cdot\mathbf{D})$$
$$+ \underbrace{\alpha_2\,\mathrm{tr}\,\epsilon\,\mathrm{tr}\,(\epsilon\cdot\mathbf{D}) + 2\beta_2\,\mathrm{tr}\,(\epsilon\cdot\epsilon\cdot\mathbf{D})}_{\Delta w'} + \underbrace{\gamma\epsilon:\hat{\mathbf{D}}:\epsilon}_{\Delta w''} \tag{25}$$

In the introduction of this paper, the emphasis was put on the difficulties in ensuring continuity of stress-strain response when the unilateral condition takes place. Whereas some authors rectify a posteriori the discontinuity generated by the deactivation (Hansen and Schreyer, 1995), the model given here postulates the continuity of stress between the states 1 and 2 a priori, i.e.:

$$\boldsymbol{\sigma}^1 = \boldsymbol{\sigma}^2 \quad \text{at closure-opening point} \tag{26}$$

Considering the stiffness discontinuity tensor $[[\mathbf{C}]] = (\partial^2 w^2/\partial\epsilon\partial\epsilon) - (\partial^2 w^1/\partial\epsilon\partial\epsilon)$, Equation (26) can be written as the equation of a surface S of the form in the strain space delimiting two linear elastic subdomains:

$$h(\epsilon) = [[\mathbf{C}]]:\epsilon = 0 \tag{27}$$

Remark: the condition [Equation (26)] of stress continuity is sufficient to impose the continuity of w; indeed, w is continuous if the stress jump is normal to the surface S separating states 1 and 2 (Wesolowski, 1969). The condition on the stress discontinuity is stronger here: there is no jump.

Since S separates the six-dimensional strain space into two six-dimensional subspaces, S has to be five-dimensional (i.e., dim Ker $[[\mathbf{C}]] = 5$) and $[[\mathbf{C}]]$ is of rank one (dim $-$ Im $[[\mathbf{C}]] = 1$, Curnier et al., 1995). It is then sufficient that all the determinants of second order of $[[\mathbf{C}]]$ be equal to zero (Wesolowski, 1969). The relevant calculations give rise to the following sufficient conditions:

$$\begin{cases} \alpha_1 = \alpha_2 = \alpha \\ \beta_1 = \beta_2 = \beta \end{cases} \tag{28a}$$

$$[[\mathbf{C}]] = 2\gamma\hat{\mathbf{D}} = 2\gamma\sum_{k=1}^{3} D_k \boldsymbol{\nu}^k \otimes \boldsymbol{\nu}^k \otimes \boldsymbol{\nu}^k \otimes \boldsymbol{\nu}^k \tag{28b}$$

Each equivalent set of cracks associated with $\boldsymbol{\nu}^k$, $k = 1,2,3$ can be considered independent. Let us consider in a first time a single set ($k = 1$). Equation (27) leads to:

$$h(\boldsymbol{\epsilon}) = \boldsymbol{\nu} \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{\nu} = 0 \tag{29}$$

which is the opening-closure criterion of the set of normal $\boldsymbol{\nu}$. A rigorous argument allowed us to find this criterion which was earlier postulated by Chaboche (1992b) in accordance with some micromechanical considerations (Andrieux et al. 1986).

Relation (30) may be extended to each equivalent set:

$$h^k(\boldsymbol{\epsilon}) = \boldsymbol{\nu}^k \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{\nu}^k = 0, \quad k = 1,2,3 \tag{30}$$

Thus, the equivalent damage set of normal $\boldsymbol{\nu}^k$ becomes inactive (resp. active) when the normal strain $\boldsymbol{\nu}^k \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{\nu}^k$ becomes negative (resp. positive); at this threshold, when passing from positive to negative $h^k(\boldsymbol{\epsilon})$, the term $\boldsymbol{\epsilon}{:}\hat{\mathbf{D}}{:}\boldsymbol{\epsilon}$ with the constant γ appears in the expression of w^2 so that there is recovery of stiffness in the direction normal to the closed cracks. In a convenient reference system related to the crack network (axis 1 normal to the cracks, axis 2 and 3 in the crack plane), this stiffness recovery condition can be written:

$$\frac{\partial^2 \Delta w'}{\partial \epsilon_{11} \partial \epsilon_{11}} + \frac{\partial^2 \Delta w''}{\partial \epsilon_{11} \partial \epsilon_{11}} = 0 \tag{31}$$

Equation (31) means that the damage induced loss of normal stiffness obtained by differentiation of the $\Delta w'$-term in Equation (25) is recovered through the supplementary term $\Delta w''$.

After calculation, Equation (31) leads to the simple relation:

$$\gamma = -\alpha - 2\beta \tag{32}$$

This result constitutes a major advantage: γ is not a new constant to be identified, it is calculated from the parameters α and β of the basic version.

Collecting Equations (24), (25), (28), (30), and (32), one obtains the set of modified equations taking into account the restitution of elastic moduli due to crack closure:

$$w = \frac{1}{2}\lambda(\mathrm{tr}\,\boldsymbol{\epsilon})^2 + \mu\,\mathrm{tr}(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}) + g\,\mathrm{tr}(\boldsymbol{\epsilon}\cdot\mathbf{D}) + \alpha\,\mathrm{tr}\,\boldsymbol{\epsilon}\,\mathrm{tr}(\boldsymbol{\epsilon}\cdot\mathbf{D}) + 2\beta\,\mathrm{tr}(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}\cdot\mathbf{D})$$

$$-(\alpha + 2\beta)\boldsymbol{\epsilon}:\left[\sum_{k=1}^{3} H(-\boldsymbol{\nu}^k\cdot\boldsymbol{\epsilon}\cdot\boldsymbol{\nu}^k)D_k\boldsymbol{\nu}^k\otimes\boldsymbol{\nu}^k\otimes\boldsymbol{\nu}^k\otimes\boldsymbol{\nu}^k\right]:\boldsymbol{\epsilon} \tag{33}$$

$$\boldsymbol{\sigma} = \frac{\partial w}{\partial\boldsymbol{\epsilon}} = \lambda(\mathrm{tr}\,\boldsymbol{\epsilon})\mathbf{I} + 2\mu\boldsymbol{\epsilon} + g\mathbf{D} + \alpha\,[\mathrm{tr}(\boldsymbol{\epsilon}\cdot\mathbf{D})\mathbf{I} + (\mathrm{tr}\,\boldsymbol{\epsilon})\mathbf{D}]$$

$$+ 2\beta(\boldsymbol{\epsilon}\cdot\mathbf{D} + \mathbf{D}\cdot\boldsymbol{\epsilon}) - 2(\alpha + 2\beta)\sum_{k=1}^{3} H(-\boldsymbol{\nu}^k\cdot\boldsymbol{\epsilon}\cdot\boldsymbol{\nu}^k)D_k(\boldsymbol{\nu}^k\cdot\boldsymbol{\epsilon}\cdot\boldsymbol{\nu}^k)\boldsymbol{\nu}^k\otimes\boldsymbol{\nu}^k \tag{34}$$

$$\mathbf{F}^D = -\frac{\partial w}{\partial\mathbf{D}} = -g\boldsymbol{\epsilon} - \alpha(\mathrm{tr}\,\boldsymbol{\epsilon})\boldsymbol{\epsilon} - 2\beta(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}) + (\alpha + 2\beta)\sum_{k=1}^{3} H(-\overset{\nu}{\boldsymbol{\nu}^k}\cdot\overset{\nu}{\boldsymbol{\epsilon}}\cdot\boldsymbol{\nu}^k)$$

$$\cdot(\boldsymbol{\nu}^k\cdot\boldsymbol{\epsilon}\cdot\boldsymbol{\nu}^k)^2\boldsymbol{\nu}^k\otimes\boldsymbol{\nu}^k \tag{35}$$

The foregoing enlarged version differs from the basic one by a term containing a Heaviside function H which separates two linear elastic domains. Note that although H is naturally discontinuous, w, $\boldsymbol{\sigma}$, and $\mathbf{F}^D$ remain continuous since the discontinuity of H takes place when the multiplier $\boldsymbol{\nu}^k\cdot\boldsymbol{\epsilon}\cdot\boldsymbol{\nu}^k$ [entering Equations (33), (34), and (35)] becomes zero.

Remark 1: the expression [Equation (35)] of the thermodynamic force $\mathbf{F}^D$ related to damage is valid for a given (fixed) configuration of principal directions of $\mathbf{D}$. In relation to a computational scheme relative to the model, a $\mathbf{D}$-

nonproportional path in the above sense has to be considered as a collection of **D**-principal-triad constant segments.

Remark 2: the relations above mention no evolution equation for $\hat{\mathbf{D}}$. Indeed, $\hat{\mathbf{D}}$ is not a new damage variable itself, since its expression is directly built upon **D**. In a computational scheme, Equation (10b) allows to actualize **D** and then $\hat{\mathbf{D}}$ through Equation (20) at each step of the calculation. Thus, $\hat{\mathbf{D}}$ does not require an evolution law of its own, Equation (10b) is sufficient.

Remark 3: the additional $\hat{\mathbf{D}}$-term allows to describe a "normal" stiffness recovery due to crack closure. Nevertheless, it fails to incorporate a "shear" damage deactivation due to frictional sliding of the mesocrack lips. The version herein considers sliding without friction (case of lubricated cracks, for example).

3.3 Examples

A series of simulations was carried out in order to prove the ability of the enlarged anisotropic damage model to describe the moduli recovery phenomenon due to crack closure. One of the simplest tests (in theory!) to bring out the unilateral effect consists in a uniaxial tension-compression loading. The material constants employed in the simulation are those identified on Fontainebleau sandstone (Table 1). One considers a homogeneous axisymmetric loading as follows (Figure 4).

- Step 1: tension along axis 3; short elastic response followed by degradation of the moduli due to generation of a set of parallel mesocracks of normal 3.

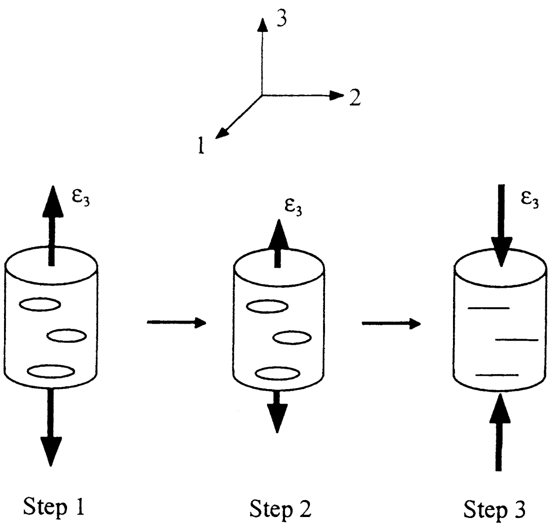

Figure 4. *Steps of the homogeneous axisymmetric simulation.*

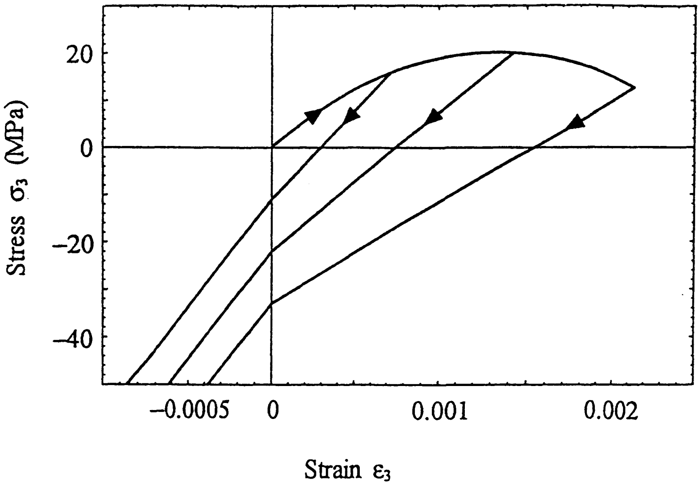

***Figure 5.** Stress-strain response for a tension-compression simulation.*

- Step 2: elastic unloading down to crack closure which occurs at relative normal strain reduced to zero (i.e., $\epsilon_3 = 0$).
- Step 3: compression after closure; elastic response with restored moduli.

Figure 5 corresponds to the stress-strain response of the simulation described above for three different damage states (i.e., three different unloading phases). The unloading portions include two parts: before the closure (i.e., $\epsilon_3 > 0$), the decreasing slopes (longitudinal Young modulus E_3) show the damage effect on the axial modulus (E_3 is, respectively, equal to 82%, 65%, and 47% of the initial Young modulus); after closure (i.e., $\epsilon_3 < 0$), whatever the damage level is, the curves are practically parallel (slopes equal to about 98% of the initial Young modulus), indicating genuine damage deactivation.

A less common test illustrating the moduli restoration consists in an axisymmetric triaxial compression with lateral overloading (Figure 6).

- Step 1: hydrostatic loading ($\sigma_3 = \sigma_1 = P_c$); elastic response.
- Step 2: longitudinal loading up to σ_{3MAX}; σ_1 remains constant; generation of an axisymmetric mesocrack family.
- Step 3: lateral loading whereas σ_3 remains equal to σ_{3MAX}; for a sufficiently pronounced value of σ_1, the mesocrack closure occurs and initial moduli are recovered.

Figure 7 shows the simulated response during the third step for a confining pressure P_c of 30 MPa. The strain origin ϵ_1^0 considered here coincides with the lateral strain value at the end of step 2. Three successive simulations were carried

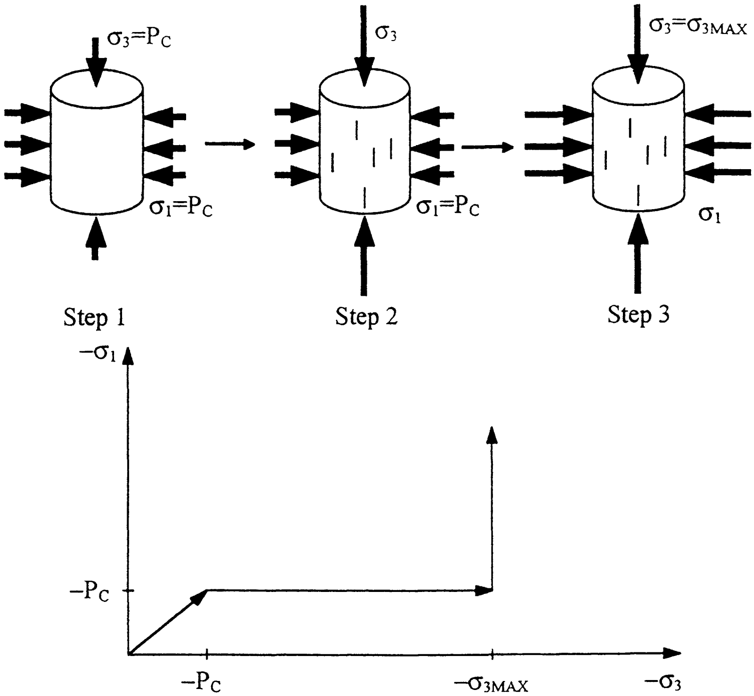

__Figure 6.__ Axisymmetric triaxial compression with lateral overloading.

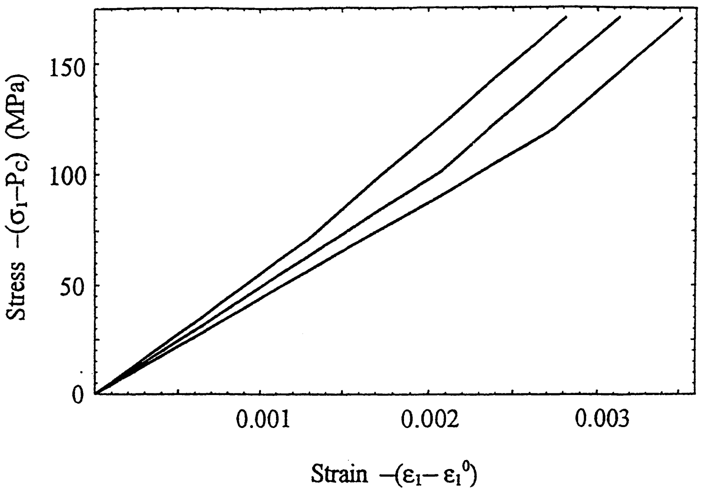

__Figure 7.__ Stress-strain response for triaxial compression with lateral overloading (Step 3).

out for different values of σ_{3MAX}, i.e., different damage level characterized by a reduction of the curve slope. Like in the tension-compression simulation, Figure 7 also exhibits a stiffness recovery of about 98% compared to the initial Young modulus.

4. CONCLUSION

The objective of this work was to associate a particular damage mechanism (oriented mesocracking) and a damage deactivation phenomenon in a consistent model able to predict complex brittle/quasi-ductile behaviour of a class of solids. The paper puts forward a new theoretical solution for bilinear orthotropic elasticity of a damaged (mesocracked) solid. The approach presented is rigorous in the way it is built on micromechanical considerations, but leads at the same time to convenient three-dimensional modelling. The unilateral effect is introduced with the help of a damage fourth-order crack-closure control parameter keeping the orthotropy generated by the formulation of the basic version. The corresponding crack-closure criterion, accessible at the macro-scale, is emerging from the modelling and not postulated a priori; moreover, no further information is needed since the enlarged version employs no additional constants compared to the basic one. This allows to ensure the continuity of the stress-strain response (no need to rectify a posteriori the discontinuities encountered). Beside the normal stiffness recovery, it would now be pertinent to account for another dissipative mechanism, namely friction on the closed mesocrack lips, inducing a partial restitution of the shear stiffness.

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