Phase field evolution equation

1 Governing equation

The evolution equation for the phase field parameter d is given by Eqn.(41) in [1] as

$$\frac{g_c}{l}\left[d - l^2 \Delta d\right] = -g'(d)\Psi_0, \tag{1}$$

where $g(d)$ is the strain energy degradation function, g_c and l are constants, and Δ denotes the Laplacian operator. We rearrange Eqn.(1) and get

$$\Delta d - \frac{d}{l^2} = \frac{g'(d)\Psi_0}{g_c l}. \tag{2}$$

The "reference" strain energy density, Ψ_0, is defined in Eqn.(19) from [1] as

$$\Psi_0 = \frac{1}{2}\boldsymbol{\varepsilon} : \mathscr{C} : \boldsymbol{\varepsilon}, \tag{3}$$

where $\boldsymbol{\varepsilon}$ is the total infinitesimal strain tensor and $\mathscr{C}$ is the elastic stiffness tensor. Per Eqn.(23) in [1], the Cauchy stress $\boldsymbol{\sigma} = g(d)\mathscr{C} : \boldsymbol{\varepsilon}$ but by definition $\boldsymbol{\sigma} = \mathscr{C} : \boldsymbol{\varepsilon}_e$, where $\boldsymbol{\varepsilon}_e$ is the elastic strain. Therefore,

$$\boldsymbol{\varepsilon}_e = g(d)\boldsymbol{\varepsilon}.$$

Replacing $\boldsymbol{\varepsilon}$ with $\boldsymbol{\varepsilon}_e$ in Eqn.(3) we get

$$\Psi_0 = \frac{1}{2}\frac{1}{g(d)^2}\boldsymbol{\varepsilon}_e : \mathscr{C} : \boldsymbol{\varepsilon}_e. \tag{4}$$

Note that the strain energy density $\Psi = g(d)\Psi_0$ is *not* the typical definition of the strain energy density and has an additional multiplicative factor of $1/g(d)$ in Miehe's formulation. If we define an elastic compliance tensor as $\mathscr{S} = \mathscr{C}^{-1}$, then $\boldsymbol{\varepsilon}_e = \mathscr{S} : \boldsymbol{\sigma}$ and

$$\Psi_0 = \frac{1}{2}\frac{1}{g(d)^2}\boldsymbol{\sigma} : \mathscr{S} : \boldsymbol{\sigma}. \tag{5}$$

Substituting this into Eqn.(2), we get

$$\Delta d - \frac{d}{l^2} = \frac{1}{2g_c l}\frac{g'(d)}{g(d)^2}\boldsymbol{\sigma} : \mathscr{S} : \boldsymbol{\sigma} \tag{6}$$

2 Broadening phenomenon?

2.1 Displacement controlled

Consider a semi-infinite long strip in x direction that is subjected to an homogeneously applied total strain ε_0 in y direction, as shown in Fig. 1. Then the governing equation

$$\Delta d - \frac{d}{l^2} = \frac{g'(d)}{2g_c l}\boldsymbol{\varepsilon} : \mathscr{C} : \boldsymbol{\varepsilon}. \tag{7}$$

with $g(d) = (1-d)^2$ becomes

$$d'' - \frac{d}{l^2} = -\frac{E\varepsilon_0^2}{g_c l}(1-d) \tag{8}$$

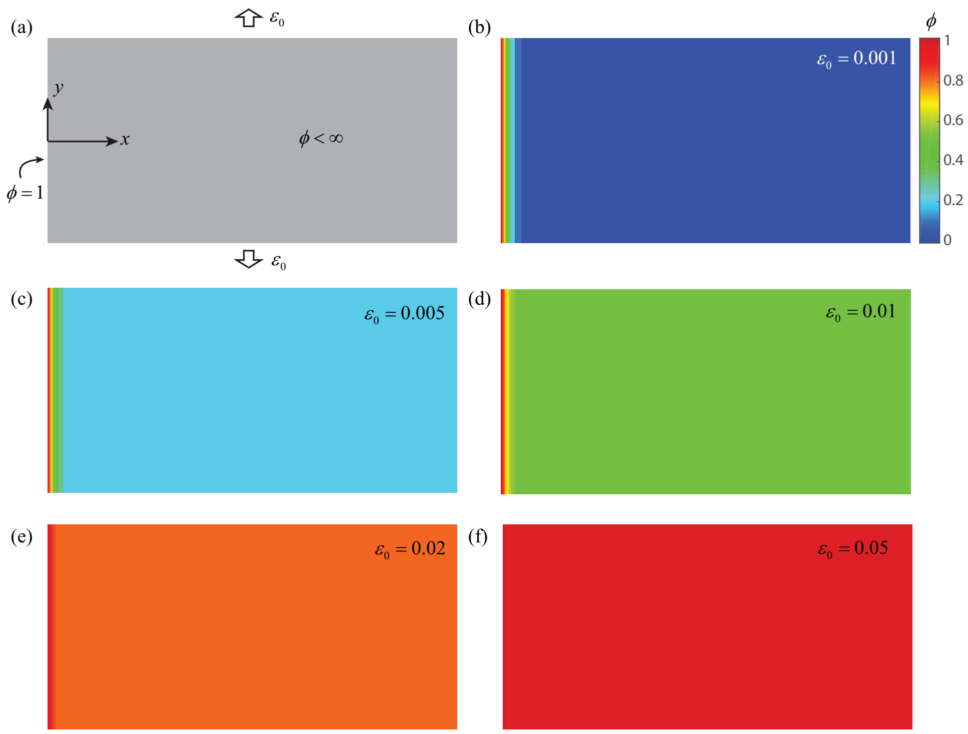

Figure 1: (a) A semi-infinite long strip subjected to an applied total strain ε_0. (b-f) The phase field ϕ at different strain levels

Let $\beta = \frac{E\varepsilon_0^2}{g_c l}$, $\alpha = \beta + \frac{1}{l^2}$. The general solution of

$$d'' - \alpha d + \beta = 0 \tag{9}$$

is

$$d(x) = \frac{\beta}{\alpha} + C_1 e^{-\sqrt{\alpha}x} + C_2 e^{\sqrt{\alpha}x} \tag{10}$$

According to the boundary conditions

$$d(0) = 1 \quad \text{and} \quad |d(x)| < \infty \quad \text{for} \quad x \in [0, \infty], \tag{11}$$

we determine the constants to be $C_1 = \frac{1}{1+\beta l^2}$ and $C_2 = 0$. Thus

$$d(x) = \frac{\beta}{\alpha} + \frac{1}{1+\beta l^2} e^{-\sqrt{\alpha}x}. \tag{12}$$

As an example, let $E = 5.0 \times 10^{-5}$, $g_c = 0.5$, $l = 1.0 \times 10^{-2}$. The plots of $d(x)$ at different strain levels are shown in Fig. **??**. We still need to check the equilibrium of stress to be satisfied. We note that

$$\sigma = g(d)\mathscr{C} : \varepsilon \tag{13}$$

with components $\sigma_{xx} = 0$, $\sigma_{xy} = 0$, and $\sigma_{yy} = (1 - d(x))^2 E\varepsilon_0$. Then the equilibrium follows by substituting the stress into the equation $\nabla \cdot \sigma = 0$.

2.2 Load controlled

Instead of applying displacement, we prescribe traction σ_0 at the top and button boundary. According to equilibrium equation, the only non-zero stress is $\sigma_{yy} = \sigma_0$. Then the governing equation of $d(x)$ is

$$d'' - \frac{d}{l^2} = -\frac{\sigma^2}{g_c l} \frac{g'(d)}{g(d)^2} = -\frac{\sigma^2}{g_c l E} \frac{1}{(1-d)^3}, \tag{14}$$

or written as

$$d'' - \alpha d + \beta (1-d)^{-3} = 0 \tag{15}$$

with $\alpha = \frac{1}{l^2}$ and $\beta = \frac{\sigma^2}{g_c l E}$. The boundary conditions are Eq. 11.

References

[1] Christian Miehe, Martina Hofacker, and Fabian Welschinger. A phase field model for rate-independent crack propagation: Robust algorithmic implementation based on operator splits. *Computer Methods in Applied Mechanics and Engineering*, 199(45):2765–2778, 2010.