Bifurcation and stability issues in gradient theories with softening

Ahmed Benallal, Jean-Jacques Marigo

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A. Benallal[1] and J.-J. Marigo[2]

[1] LMT-Cachan, 61 Avenue du Président Wilson, 94235 Cachan Cedex, France
[2] Laboratoire de Modélisation en Mécanique (UMR 7607), Université Paris VI, 4 Place Jussieu, Case 162, 75252 Paris Cedex 05

E-mail: `benallal@lmt.ens-cachan.fr,marigo@lmm.jussieu.fr`

Abstract. A bifurcation and stability analysis is carried out here for a bar made of a material obeying a gradient damage model with softening. We show that the associated initial boundary-value problem is ill-posed and one should expect mesh sensitivity in numerical solutions. However, in contrast to what happens for the underlying local damage model, the damage localization zone has a finite thickness and stability arguments can help in the selection of solutions.

1. Introduction

Generalized continua have been the subject of intensive studies for the last two decades. Two main streams, both stressing the importance of incorporating a material lengthscale in the constitutive behaviour led to this situation: in one hand, a great amount of experimental evidence shows definite size effect for various materials such as in torsion of thin copper wires [1], indentation hardness of metals and ceramics [2]; other phenomena are known in the literature whereby the smaller is the size, the stronger is the response, as for instance the strengthening of metals by a given volume fraction of hard particles which is greater for smaller than for larger ones [3] and the well known fact that fine grained metals are stronger than those with coarse grains [4]; in the other hand, strain softening materials where the numerical computations predict a dissipated energy that is strongly mesh-sensitive [5] for a local continuum and where the associated boundary-value problem is not mathematically well-posed. In this last context, the significance of including a material length scale to improve the description of failure by localization in soil and rock has been shown in [6] and various representations of the behaviour with an internal lengthscale have been introduced in the literature among which the micropolar continuum [7], the higher gradient theories [8] and nonlocal models written either in a gradient format [9, 10, 11] or in an integral one [12].

This paper is mainly concerned with *gradient models* with *softening* and its objective is twofold: to provide a bifurcation analysis and show in one hand that despite the introduction of the material lengthscale, the associated *nonlinear* initial boundary-value problem can still be ill-posed leading to the possibility of mesh-sensitivity of numerical computations; to carry out in the other hand a stability analysis and to show that in contrast to the underlying local model, one can provide a selection criterion for the solutions using stability considerations, in the spirit of the global variational approach proposed by [13] and [14].

Ill-posedness is well understood for linear boundary value problems in continuum mechanics and the general conditions for its occurrence are rigorously established [15]. For the mathematical point of view, this is due to the fact that the boundary-value problem may exhibit an infinite number of linearly independent solutions or to the fact that the solutions of this boundary-value problem do not depend continuously on the data. There is also the possibility for the continuous problem that in order to have a finite number of linearly independent solutions and that these solutions depend continuously on the data, one has to enforce an infinite number of linearly independent conditions over these data. These are the general three sources for ill-posedness and necessary and sufficient conditions are known in the case of a *linear boundary-value problem* for this ill-posedness to occur. These conditions are respectively the loss of ellipticity of the governing equations, the loss of the boundary complementing condition and eventually the loss of the interfacial boundary condition when the solid is heterogeneous. The first condition is a local condition that is dependent only on the rate-independent constitutive equations and implies the singularity of the material acoustic tensor. While also local in nature, the two other conditions involve the boundary or the interfacial conditions and imply a kind of compatibility between theses conditions and the constitutive behaviour. These two conditions may fail in the elliptic regime where they retain their importance.

For nonlinear problems, these conditions are not available and the usual approach is to analyse the associated rate boundary value problem. When this rate problem is nonlinear (as for rate-independent materials considered here), the analysis is rather made for the associated linear comparison solid. However the link between well-posedness for the linear comparison solid and that of the non linear problem is not well understood. Further, well-posedness of the rate boundary-value problem does not necessarily imply well posedness of the initial boundary-value problem and this may be true even when a material lengthscale is present.

Some of these facts are shown here through the simple example of one-dimensional gradient model where we provide a bifurcation and stability analysis. This is carried out for a bar of a finite length. The bifurcation analysis shows that one can construct an infinite number of linearly independent solutions implying the ill-posedness of the nonlinear initial boundary-value problem. The stability analysis allows to select among all these solutions the stable paths as potential candidates for the response of the bar. The paper is structured as follows. In section 2, the constitutive gradient damage relations are described with emphasis on the global variational approach. Section 3 contains the bifurcation analysis for the bar while section 4 summarizes the stability concept and its application.

2. Gradient damage model and variational approach

2.1. Constitutive setting

We consider a one-dimensional non local damage model in which the damage variable α is a scalar growing from 0 to infinity. The behavior of the material at a material point x is characterized by the state function W which depends on the local strain $u'(x)$ (u denoting the displacement and the prime denoting the spatial derivative), the local damage value $\alpha(x)$ and the gradient $\alpha'(x)$ of the damage field at x :

$$W(u', \alpha, \alpha') = \tfrac{1}{2}E(\alpha)u'^2 + w(\alpha) + \tfrac{1}{2}E_0\ell^2\alpha'^2 \tag{1}$$

where $E(\alpha)$ represents the Young modulus of the material at the damage state α and $w(\alpha)$ can be interpreted as the density of the dissipated energy by the material during a homogeneous damaging process (*i.e.* a process such that $\alpha' = 0$).

2.2. Variational approach in the case of the traction of a bar

In contrast to what is classically done, we formulate here the damage constitutive equations at the level of the whole structure. We consider a homogeneous bar whose natural reference configuration is the interval $(0, L)$. The bar is made of the non local damaging material characterized by the state function W given by (1). The end $x = 0$ of the bar is fixed, while the displacement of the end $x = L$ is prescribed by a hard device to a value increasing with time from 0 to infinity:

$$u_t(0) = 0, \qquad u_t(L) = t\varepsilon_1 L, \qquad t \geq 0, \tag{2}$$

where t denotes the time, u_t is the displacement field of the bar at time t and $\varepsilon_1 = \frac{\sigma_1}{E_0}$ is the elastic yield strain.

The damage evolution problem of the bar is obtained via an energetic variational formulation. We recall briefly the basic ingredients of a such variational formulation. Let $\mathcal{C}_t$ and $\mathcal{D}$ be respectively the affine space of kinematically admissible displacement fields at time t and the convex cone of admissible damage fields

$$\mathcal{C}_t = \{v \in H^1(0, L) \ : \ v(0) = 0, v(L) = t\varepsilon_1 L\}, \tag{3}$$

$$\mathcal{D} = \{\beta \in H^1(0, L) \ : \ \beta(x) \geq 0\}, \tag{4}$$

$H^1(0, L)$ denoting the usual Sobolev space of functions defined on $(0, L)$ which are square integrable and the first derivative of which is also square integrable. The linear space associated to $\mathcal{C}_t$ is $H^1_0(0, L) = \{v \in H^1(0, L) \ : v(0) = v(L) = 0\}$.

At any pair (u, α) admissible at time t, we associate *the total energy of the bar*

$$(u, \alpha) \in \mathcal{C}_t \times \mathcal{D} \mapsto \mathcal{P}(u, \alpha) = \int_0^L W(u'(x), \alpha(x), \alpha'(x)) \, dx. \tag{5}$$

The set of admissible displacement rates $\dot{u}$ (the dot denoting the time derivative) can be identified with $\mathcal{C}_1$, while the set of admissible damage rates $\dot{\alpha}$ can be identified with $\mathcal{D}$, the inequality $\dot{\alpha} \geq 0$ denoting the irreversibility of the damaging process.

By assuming that the bar is undamaged at time $t = 0$, the damage evolution problem can be read as:

$$\begin{aligned}&\text{For } t > 0, \text{ find } (u_t, \alpha_t) \in \mathcal{C}_t \times \mathcal{D} \text{ such that } (\dot{u}_t, \dot{\alpha}_t) \in \mathcal{C}_1 \times \mathcal{D} \\ &\text{and} \quad \mathcal{P}'(u_t, \alpha_t)(v - \dot{u}_t, \beta - \dot{\alpha}_t) \geq 0 \quad \forall (v, \beta) \in \mathcal{C}_1 \times \mathcal{D}\end{aligned} \tag{6}$$

with the initial condition

$$\alpha_0(x) = 0, \tag{7}$$

where $\mathcal{P}'(u, \alpha)(v, \beta)$ denotes the Gâteaux derivative of $\mathcal{P}$ at (u, α) in the direction (v, β), *i.e.*

$$\mathcal{P}'(u, \alpha)(v, \beta) = \int_0^L E(\alpha) u' v' dx + \int_0^L \left(\left(\tfrac{1}{2} E'(\alpha) u'^2 + w'(\alpha) \right) \beta + E_0 \ell^2 \alpha' \beta' \right) dx. \tag{8}$$

2.3. Nonlinear initial boundary-value problem for the bar

By inserting $\beta = \dot{\alpha}_t$ and $v = \dot{u}_t + w$, with $w \in H^1_0(0, L)$, into(6), we obtain the variational formulation of the equilibrium of the bar, *i.e.*

$$\int_0^L E(\alpha_t(x)) u'(x) w'(x) \, dx = 0 \qquad \forall w \in H^1_0(0, L), \tag{9}$$

from which we deduce that the stress must be uniform:

$$E(\alpha_t(x))u_t'(x) = \sigma_t, \quad \forall x \in (0, L). \tag{10}$$

By using the boundary condition (2), we obtain the relation between the stress σ_t and the damage field α_t

$$\varepsilon_t \equiv t\varepsilon_1 = \frac{\sigma_t}{L}\int_0^L \frac{dx}{E(\alpha_t(x))}, \tag{11}$$

ε_t representing the overall strain of the bar at time t.

By inserting (9)–(11) into (6) we obtain the variational inequality governing the damage field evolution:

$$\tfrac{1}{2}\sigma_t^2 \int_0^L \frac{E'(\alpha_t)}{E(\alpha_t)^2}\beta dx + \int_0^L w'(\alpha_t)\beta dx + E_0\ell^2 \int_0^L \alpha_t'\beta' dx \geq 0, \tag{12}$$

where the inequality must hold for all $\beta \in \mathcal{D}$ and becomes an equality when $\beta = \dot{\alpha}_t$. After an integration by parts and by using classical arguments of the calculus of variations, we obtain the set of local conditions satisfied by the damage field at any point of $(0, L)$

$$\text{Irreversibility Condition} : \dot{\alpha}_t \geq 0 \tag{13}$$

$$\text{Damage yield Criterion} : f = \tfrac{1}{2}\sigma_t^2 \frac{E'(\alpha_t)}{E(\alpha_t)^2} + w'(\alpha_t) - E_0\ell^2\alpha_t'' \geq 0 \tag{14}$$

$$\text{Alternative Condition} : \dot{\alpha}_t \left(\tfrac{1}{2}\sigma_t^2 \frac{E'(\alpha_t)}{E(\alpha_t)^2} + w'(\alpha_t) - E_0\ell^2\alpha_t'' \right) = 0 \tag{15}$$

with the boundary conditions

$$\alpha_t'(0) = \alpha_t'(L) = 0 \tag{16}$$

and the initial condition (7). Conditions (13), (14) and (15) are nothing else than the usual Kuhn-Tucker conditions $\dot{\alpha}_t \geq 0$, $f \geq 0$ and $f\dot{\alpha}_t = 0$. Note also that the global approach used here leads exactly to the same results obtained in [16] and [17].

2.4. Rate boundary-value problem

The properties of bifurcation, uniqueness or stability of the solutions of the damage evolution problem can be obtained by analyzing the rate damage problem, *i.e.* the problem governing at a given time t the damage rate $\dot{\alpha}_t$ by assuming that the damage state α_t is known. In its variational form, the damage rate problem is obtained by differentiating the damage evolution problem. Let us briefly recall how it is obtained.

Let α_t be an admissible damage field and let u_t be the associated displacement field giving the equilibrium of the bar at time t, see (10). The total energy of the bar is given by the functional $\alpha \mapsto \mathcal{E}_t(\alpha)$ defined on $\mathcal{D}$ by:

$$\mathcal{E}_t(\alpha) = \frac{E_0\ell^2}{2}\int_0^L \alpha'(x)^2 dx + \int_0^L w(\alpha(x))\, dx + \frac{t^2\varepsilon_1^2 L^2}{2\int_0^L \frac{dx}{E(\alpha(x))}}. \tag{17}$$

Its first directional derivative is the linear form defined on $H^1(0,L)$ by

$$\mathcal{E}_t'(\alpha)(\beta) = \int_0^L \left(E_0\ell^2\alpha'\beta' + w'(\alpha)\beta + \frac{t^2\varepsilon_1^2 L^2}{2\left(\int_0^L \frac{dx}{E(\alpha)}\right)^2}\frac{E'(\alpha)}{E(\alpha)^2}\beta \right) dx. \tag{18}$$

Thus, the variational inequation (12) is equivalent to $\mathcal{E}_t'(\alpha_t)(\beta) \geq 0$ for all $\beta \in \mathcal{D}$. Let α_t be a solution at time t. Differentiating once more, we obtain the damage rate problem

$$\begin{aligned}&\text{Find } \dot\alpha_t \in \mathcal{D} \text{ such that}\\ &\mathcal{E}_t''(\alpha_t)(\dot\alpha_t, \beta - \dot\alpha_t) + \dot{\mathcal{E}}_t'(\alpha_t)(\beta - \dot\alpha_t) \geq 0, \quad \forall \beta \in \mathcal{D}\end{aligned} \tag{19}$$

where the second directional derivative $\mathcal{E}_t''(\alpha_t)$ is a bilinear symmetric form defined on $H^1(0,L)^2$ and $\dot{\mathcal{E}}_t'(\alpha_t)$ denotes the partial derivative of $\mathcal{E}_t(\alpha)$ with respect to t at $\alpha = \alpha_t$ (it is a linear form defined on $H^1(0,L)$).

3. Bifurcation phenomena in gradient damage

To simplify the presentation and in order to obtain closed form solutions, we will only consider in the sequel the following particular case:

$$E(\alpha) = \frac{E_0}{(1+\alpha)^2}, \qquad w(\alpha) = \frac{\sigma_1^2}{E_0}\alpha, \qquad \alpha \geq 0, \tag{20}$$

E_0 denoting the initial Young modulus, σ_1 the elastic yield stress and ℓ the internal length of the material.

In this case, the damage yield criterion (14) and the relation (11) become

$$E_0^2\ell^2\alpha_t''(x) + \sigma_t^2(1+\alpha_t(x)) \leq \sigma_1^2, \quad \forall x \in (0,L), \tag{21}$$

$$\sigma_t = \frac{t\sigma_1 L}{\int_0^L (1+\alpha_t(x))^2\,dx}. \tag{22}$$

Moreover the second derivative reads as

$$\mathcal{E}_t''(\alpha_t)(\beta) = E_0\ell^2\int_0^L \beta'^2\,dx + \frac{4\sigma_t^3}{tE_0\sigma_1 L}\left(\int_0^L (1+\alpha_t)\beta\,dx\right)^2 - \frac{\sigma_t^2}{E_0}\int_0^L \beta^2\,dx, \tag{23}$$

where σ_t denotes the equilibrium stress associated to the homogeneous damage α_t, see (22), and

$$\dot{\mathcal{E}}_t'(\alpha_t)(\beta) = -\frac{2\sigma_t^2}{tE_0}\int_0^L (1+\alpha_t)\beta\,dx. \tag{24}$$

For $t \in [0,1)$, the response of the bar is elastic, the damage field remains at its initial value 0, the inequality in (21) is strict. At time $t = 1$, the inequality is an equality at every material point and the damage can evolve everywhere. The goal of this section is to prove that we can construct a continuum of solutions for the damage evolution problem for $t > 1$, when the length of the bar is sufficiently large. Let us first note that, in any interval where the damage yield criterion (21) is satisfied as an equality at time t (such points are called damaging points), the damage field is given by

$$\alpha_t(x) = \frac{\sigma_1^2}{\sigma_t^2} - 1 + a_t\sin\frac{\sigma x}{E_0\ell} + b_t\cos\frac{\sigma x}{E_0\ell} \tag{25}$$

where a_t and b_t are two time dependent scalars.

3.1. The homogeneous solution.

The homogeneous solution corresponds to the solution where the damage field is uniform at each time, $\alpha_t(x) = \alpha_t$. This solution always exists and we easily deduce from (21) and (22) that

$$\sigma_t = t^{-1/3}\sigma_1, \qquad \alpha_t = t^{2/3} - 1, \quad \forall t > 1. \tag{26}$$

In a diagram $\sigma - \varepsilon$, σ being the equilibrium stress and ε the overall strain of the bar ($\varepsilon_t = t\varepsilon_1$), the overall response of the bar corresponds to the descending branch in Figure 1, showing a *softening* behavior of the material.

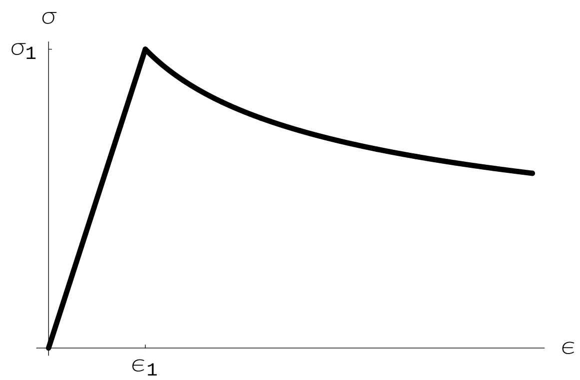

Figure 1. Homogeneous response of the bar: the segment line corresponds to the elastic response ($\alpha_t = 0$, $0 \le t < 1$), the decreasing branch corresponds to a spatially uniform damage growing with time ($\alpha_t(x) = t^{2/3} - 1$).

3.2. Uniqueness criterion

To see whether the solution can be unique, we can use the damage rate problem (19). Let α_t and σ_t be the homogeneous solution at time t of the evolution problem and given by (26). The rate variational problem admits the solution $\dot{\alpha}_t(x) = \frac{2}{3}t^{-1/3}$ for all $x \in (0, L)$ which corresponds to the rate of the homogeneous solution. This will be the unique solution provided $\mathcal{E}''_t(\alpha_t)$ is a definite positive quadratic form on $H^1(0, L)$. Here, the second derivative reads as

$$\mathcal{E}''_t(\alpha_t)(\beta) = E_0\ell^2 \int_0^L \beta'^2\,dx + \frac{4\sigma_t^2}{E_0 L}\left(\int_0^L \beta\,dx\right)^2 - \frac{\sigma_t^2}{E_0}\int_0^L \beta^2\,dx. \tag{27}$$

By introducing the Rayleigh ratio $\mathcal{R}_t$ defined on $H^1(0, L) \setminus \{0\}$ by

$$\mathcal{R}_t(\beta) = \frac{E_0\ell^2 \int_0^L \beta'^2\,dx + \frac{4\sigma_t^2}{E_0 L}\left(\int_0^L \beta\,dx\right)^2}{\frac{\sigma_t^2}{E_0}\int_0^L \beta^2\,dx}, \tag{28}$$

it immediately appears that the rate damage problem admits a unique solution if $\min_{H^1(0,L)\setminus\{0\}} \mathcal{R}_t > 1$. After some calculations, too long to be reported here, we obtain

$$\min_{H^1(0,L)\setminus\{0\}} \mathcal{R}_t = \min\left\{4, \left(\frac{\sigma_c}{\sigma_t}\right)^2\right\} \tag{29}$$

where

$$\sigma_c = \pi E_0 \frac{\ell}{L}. \tag{30}$$

Thus, we can conclude that

(i) *If* $\sigma_1 \leq \sigma_c$, *i.e.* if $L \leq \pi\ell/\varepsilon_1$, then the homogeneous solution is the unique solution of the damage evolution problem. This feature can be used for instance in an experimental setting to identify properly the homogeneous behaviour of the material.

(ii) *If* $\sigma_1 > \sigma_c$, *i.e.* if $L > \pi\ell/\varepsilon_1$, then the homogeneous damage rate $\dot{\alpha}_t = \frac{2}{3}t^{-1/3}$ is the unique solution of the rate damage problem provided that $\sigma_t < \sigma_c$, *i.e.* when $t > \frac{\sigma_1^3}{\sigma_c^3}$. Bifurcations are *eventually* possible from the homogeneous solution at any time in the interval $[1, \frac{\sigma_1^3}{\sigma_c^3}]$. When available, first bifurcation will occur at $t = 1$.

3.3. Examples of bifurcated branches at t=1

To construct non homogeneous solutions, we can investigate solutions where the equality in (21) holds only in a time-dependent part of the bar, i.e the bar elastically unloads with the initial stiffness E_0 in the rest part. Different scenarii may exist depending on the length of the bar.

For instance, one can assume that the equality holds in the interval $(0, D_t)$ and in this case we obtain the following solution

$$\alpha_t(x) = \begin{cases} 2\left(\frac{\sigma_1^2}{\sigma_t^2} - 1\right)\cos^2\frac{\pi x}{2D_t} & , \text{ if } 0 \leq x \leq D_t \\ 0 & , \text{ otherwise} \end{cases}, \tag{31}$$

where the width D_t is related to the stress equilibrium σ_t by

$$D_t = \pi\frac{E_0}{\sigma_t}\ell \tag{32}$$

and the overall response reads as

$$t = \frac{\sigma_t}{\sigma_1} + \frac{\sigma_c}{2\sigma_1}\left(3\frac{\sigma_1^4}{\sigma_t^4} - 2\frac{\sigma_1^2}{\sigma_t^2} - 1\right). \tag{33}$$

This half-sinusoidal damage field can appear provided that the bar is long enough so that $D_1 \leq L$, *i.e.* provided that $\sigma_c < \sigma_1$. The length D_1 is the size of the damaging zone just after $t = 1$. For $t > 1$, the irreversibility condition (13) is satisfied and the damage grows provided that $4\sigma_c > \sigma_1$. So, if

$$\frac{\varepsilon_1}{4} \leq \pi\frac{\ell}{L} \leq \varepsilon_1, \tag{34}$$

the solution is valid as long as $D_t \leq L$, *i.e.* for $t \in [1, t_c]$ with

$$t_c = \frac{\sigma_1^2}{\sigma_c^2}\left(3\frac{\sigma_1^2}{\sigma_c^2} - 2\right). \tag{35}$$

During this time interval, the damaging zone extends gradually to all the bar, cf. Figure 3.

<u>Remark</u>.

(i) One can construct symmetrically one half-sinusoidal damage field in the interval $(L-D_t, L)$. The global response is the same.

(ii) At $t = t_c$ the tip of the damaged zone reaches the end $x = L$ and can no more propagate. To continue this branch, we must consider solutions in which a part of the bar is in an unloading phase (the inequality is then strict in (21)). The details are not given here.

(iii) If the bar is too small, i.e. *if $L < \pi\ell/\varepsilon_1$, then the half-sinusoidal damage field cannot appear for lack of place. We recover the uniqueness property that we have obtained from the rate damage problem.*

(iv) If the bar is too long, i.e. *if $L > 4\pi\ell/\varepsilon_1$, then the global response $\sigma - \varepsilon$ has a* snap-back *near $(\varepsilon_1, \sigma_1)$,* i.e. *$\frac{d\sigma}{d\varepsilon}(\varepsilon_1-) > 0$, cf. Figure 2. So, since the overall strain $\varepsilon_t = t\varepsilon_1$ must increase, the stress must brutally decrease and the damage field must brutally increase. The response is no more continuous in time.*

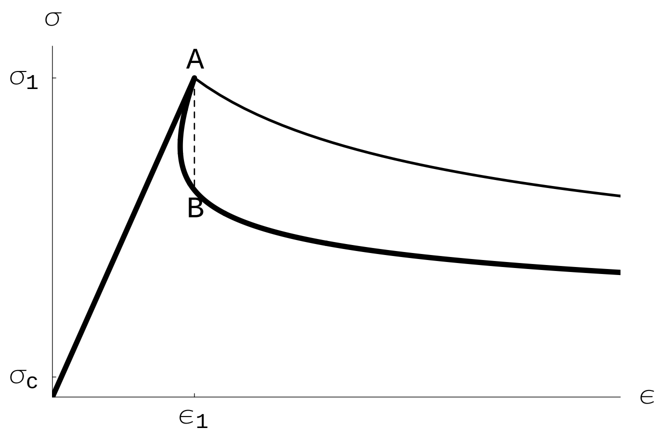

Figure 2. *Global response of the bar corresponding to the growing of one half-sinusoidal damage field (thick line) compared to the homogeneous response (thin line). The length is sufficiently large ($L > 4\pi\ell/\varepsilon_1$) and a snap-back is present at the bifurcation point A. Consequently, the initiation of damage will be discontinuous, the global response jumps from A to B at time $t = 1$.*

Other similar but qualitatively different solution is when the equality holds in the interval $(x_0 - D_t, x_0 + D_t)$ where D_t is a variable length and x_0 is a given point. We obtain the following solution

$$\alpha_t(x) = \begin{cases} 2\left(\frac{\sigma_1^2}{\sigma_t^2} - 1\right)\cos^2\frac{\pi(x-x_0)}{2D_t} & , \text{ if } |x - x_0| \le D_t \\ 0 & , \text{ otherwise} \end{cases}, \tag{36}$$

with D_t still given by (32), but the overall response reads now as

$$t = \frac{\sigma_t}{\sigma_1} + \frac{\sigma_c}{\sigma_1}\left(3\frac{\sigma_1^4}{\sigma_t^4} - 2\frac{\sigma_1^2}{\sigma_t^2} - 1\right). \tag{37}$$

This sinusoidal damage field can appear, centered at a given point x_0, provided that the bar is long enough. Let us consider the case where $x_0 = L/2$. The sinusoidal damage field can appear, centered at the middle of the bar, if $2D_1 \le L$, *i.e.* provided that $2\sigma_c \le \sigma_1$. The length $2D_1$ is the size of the damaging zone just after $t = 1$. For $t > 1$, the irreversibility condition (13) is satisfied and the damage grows provided that $8\sigma_c \ge \sigma_1$. So, if

$$\frac{\varepsilon_1}{4} \le 2\pi\frac{\ell}{L} \le \varepsilon_1, \tag{38}$$

the solution is valid as long as $2D_t \le L$, *i.e.* for $\sigma_t \in [2\sigma_c, \sigma_1]$. During this time interval, the damaging zone extends gradually to all the bar, cf. Figure 3.

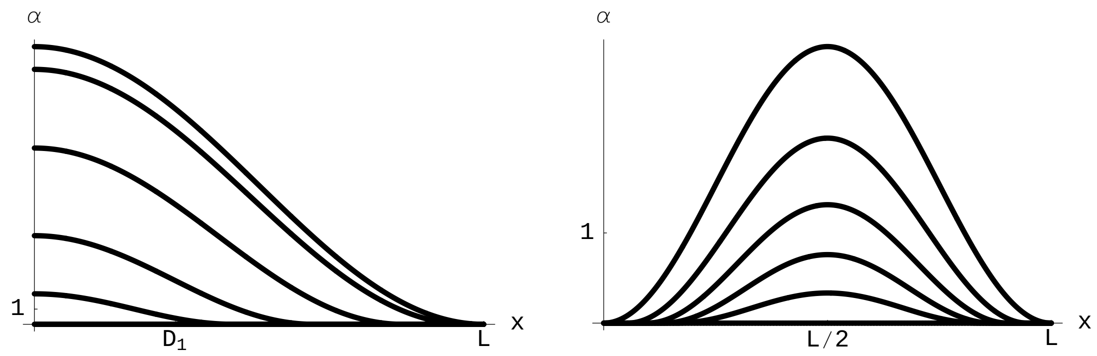

Figure 3. On the left : *Growing of one half-sinusoidal damage field starting at the end $x = 0$. The size of the damaged zone is equal to D_1 at $t = 1+$, then increases progressively with t and all the bar is damaged at $t = t_c$.*
On the right : *Growing of one sinusoidal damage field centered at the middle of the bar. The size of the damaging zone is equal to $2D_1$ at $t = 1+$, then increases progressively with t untill all the bar is damaging.*

If the bar is longer, one can construct solutions with n sinusoidal waves and in this case (33) is simply replaced by

$$t = \frac{\sigma_t}{\sigma_1} + n\frac{\sigma_c}{\sigma_1}\left(3\frac{\sigma_1^4}{\sigma_t^4} - 2\frac{\sigma_1^2}{\sigma_t^2} - 1\right). \tag{39}$$

The global responses corresponding to these non homogeneous solutions are plotted in Figure 4.

3.4. Bifurcation from the homogeneous branch.

We propose to construct in this subsection a continuum of solutions for the damage evolution problem by considering bifurcation branches from any point of the homogeneous one when the length of the bar is sufficiently large. We assume that

$$L > \pi\ell/\varepsilon_1 \tag{40}$$

and we proceed as follows

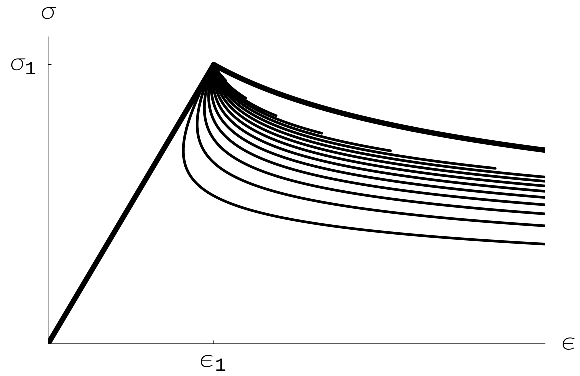

Figure 4. *In thin lines, the global response due to the growing of n-sinusoidal damage fields. The lowest curve corresponds to $n = 0.5$, the next to $n = 1$, ... The number of curves depends on the length of the bar: the longer is the bar, the larger is the number of curves. The curve n stops when the n-sinusoidal damage field covers all the bar.*

(i) Let $\alpha_b > 0$ be a given value of the damage variable. In the case of the homogeneous response, the corresponding time at which this damage state is reached, the corresponding overall strain and the corresponding equilibrium stress are given by

$$t_b = (1+\alpha_b)^{3/2}, \quad \varepsilon_b = (1+\alpha_b)^{3/2}\varepsilon_1, \quad \sigma_b = \frac{\sigma_1}{\sqrt{1+\alpha_b}}. \tag{41}$$

(ii) For $t > t_b$, we seek for a non homogeneous solution, the damage growing in the part $(0, D_t)$ of the bar while the damage remains at the value α_b in the remainder part (D_t, L) of the bar. By using (21)–(25) we find

$$\alpha_t(x) = \begin{cases} \alpha_b + 2\left(\frac{\sigma_1^2}{\sigma_t^2} - 1 - \alpha_b\right)\cos^2\frac{\pi x}{2D_t} & , \text{ if } 0 \le x \le D_t \\ \alpha_b & , \text{ otherwise} \end{cases}, \tag{42}$$

with

$$D_t = \pi\frac{E_0}{\sigma_t}\ell \tag{43}$$

and

$$t = (1+\alpha_b)^2\frac{\sigma_t}{\sigma_1} + \frac{\sigma_c}{2\sigma_1}\left(3\frac{\sigma_1^4}{\sigma_t^4} - 2(1+\alpha_b)\frac{\sigma_1^2}{\sigma_t^2} - (1+\alpha_b)^2\right). \tag{44}$$

The damage field corresponds to the growing of one half-sinusoid from the initial value α_b, cf. Figure 5. This solution is valid as long as $D_t \le L$, *i.e.* as long as $\sigma_t \ge \sigma_c$. So the value of the initial damage α_b must be chosen arbitrarily provided that $\sigma_b > \sigma_c$. Accordingly, by taking $\alpha_b \in [0, \frac{\sigma_1^2}{\sigma_c^2} - 1)$, we have obtained an infinite family of solutions indexed by α_b, cf. Figure 6.

Remark. *These results reinforce those we have obtained from the rate damage problem, since they prove that a bifurcation from the homogeneous branch is* really *possible at any time in the interval* $[1, \frac{\sigma_1^3}{\sigma_c^3}]$.

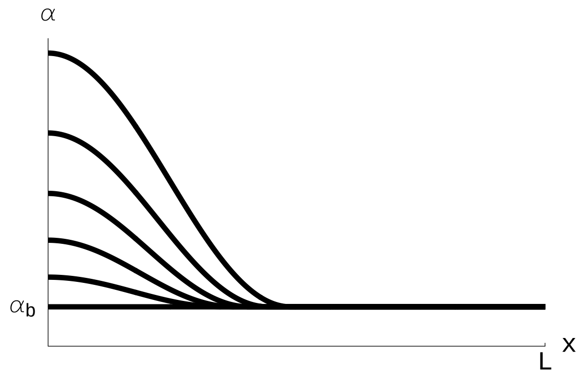

Figure 5. *Growing of one half-sinusoidal damage field from an initially homogeneous damage state α_b.*

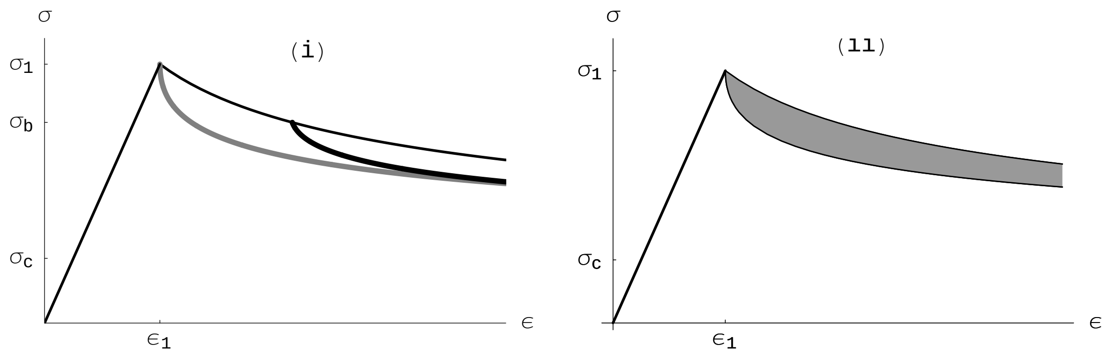

Figure 6. (i) *Thick black line:* the bifurcated branch; *Thin line:* the homogeneous branch; *Thick gray line:* the one half-sinusoidal damage branch.
(ii) *Gray area:* the continuum of possible global responses due to the possibility of a bifurcation at any point of the homogeneous branch.

4. Stability and selection of solutions

4.1. Stability criterion

The previous analysis shows that the damage evolution problem is ill-posed in the sense that it admits a continuum family of solutions . Therefore, the question of selection of the solutions raises in order to choose those solutions that can be really observed. We suggest here to answer this question by stability considerations and postulate that the solutions that are potentially feasible are the stable ones. And more precisely, those solutions that minimize the total energy. Following [14] and [13], we can use the concept of *unilateral local minimum* of the energy that we briefly introduce below. Let α_t be an admissible damage field and let u_t be the associated displacement field giving the equilibrium of the bar at time t, *i.e.*

$$E(\alpha_t(x))u_t'(x) = \sigma_t, \quad \forall x \in (0, L), \quad \text{with} \quad \sigma_t = \frac{t\varepsilon_1 L}{\int_0^L \frac{dx}{E(\alpha_t(x))}}. \tag{45}$$

The total energy of the bar is given by the functional $\alpha \mapsto \mathcal{E}_t(\alpha)$ defined on $\mathcal{D}$ by (17). We say that the bar is *stable* at time t in its damaged state α_t if and only if α_t is a unilateral local minimum of $\mathcal{E}_t$ on $\mathcal{D}$, *i.e.*

$$\text{(Ulm)} \quad \exists r > 0, \forall \beta \in \mathcal{D} : \|\beta\| = 1, \forall h \in [0, r], \quad \mathcal{E}_t(\alpha_t) \leq \mathcal{E}_t(\alpha_t + h\beta).$$

Let us note that we have only to compare the energy of α_t with the energy of the damage states which are accessible from α_t. This unilateral restriction is due to the irreversibility condition. This condition means that, if we can find in a neighborhood of α_t an accessible damage state with a smaller energy, then the state α_t is unstable and the bar will evolve spontaneously to some state with a smaller energy. The concept of neighborhood is defined by the choice of the norm $\|\cdot\|$. Here, the natural norm is the norm of $H^1(0, L)$.

4.2. Stability of the homogeneous states

For illustration, we will only study the stability of the homogeneous states of the bar, the analysis can be extended to the other solutions but is too long to be reported here. Let $\alpha_t = t^{2/3} - 1$ be the homogeneous damage state of the bar at the time $t > 1$, $h > 0$ and $\beta \in \mathcal{D}$. By developing $\mathcal{E}_t(\alpha_t + h\beta)$ with respect to h, we get

$$\mathcal{E}_t(\alpha_t + h\beta) = \mathcal{E}_t(\alpha_t) + h\mathcal{E}_t'(\alpha_t)(\beta) + \tfrac{1}{2}h^2\mathcal{E}_t''(\alpha_t)(\beta) + o(h^2), \tag{46}$$

where the primes denote directional derivatives. Since $\mathcal{E}_t'(\alpha_t)(\beta) = 0$, the stability condition consists in finding the sign of the second derivative in any positive direction β. In the particular case of the model (20), the second derivative is given by (23). By considering the Rayleigh ratio (28), it immediately appears that for the the homogeneous damage state α_t to be stable, it is necessary that $\min_{\mathcal{D}\setminus\{0\}} \mathcal{R}_t \geq 1$. It is sufficient however for this state to be stable that $\min_{\mathcal{D}\setminus\{0\}} \mathcal{R}_t > 1$. After some calculations which are not reproduced here, we obtain

$$\min_{\mathcal{D}\setminus\{0\}} \mathcal{R}_t = \min \left\{ 4, \left(\frac{4\sigma_c}{\sigma_t} \right)^{2/3} \right\} \tag{47}$$

and we can conclude that

(i) *If* $\sigma_1 \leq 4\sigma_c$, *i.e.* if $L \leq 4\pi\ell/\varepsilon_1$, then all the homogeneous damage states are stable.

(ii) *If* $\sigma_1 > 4\sigma_c$, *i.e.* if $L > 4\pi\ell/\varepsilon_1$, then only the homogeneous damage states at $t \geq \left(\frac{\sigma_1}{4\sigma_c}\right)^3$ are stable. Consequently, since the beginning of the homogeneous branch is not stable, the bar cannot be deformed uniformly and a non homogeneous solution will appear at $t = 1$.

Let us now compare the properties of stability and of bifurcation that we have obtained for the homogeneous solution. Since $\mathcal{D}$ is included in $H^1(0, L)$, we always have

$$4 \geq \min_{\mathcal{D}\setminus\{0\}} \mathcal{R}_t \geq \min_{H^1(0,L)\setminus\{0\}} \mathcal{R}_t$$

and uniqueness of the rate solution implies stability of the state. But the converse is not always true, a homogeneous state could be stable even if a bifurcation is possible at this point. Let us examine the different cases, α_t and σ_t are still given by (26).

Case	Stability	Bifurcation
$\sigma_t > 4\sigma_c$	No	Yes
$\sigma_c < \sigma_t < 4\sigma_c$	Yes	Yes
$\sigma_t < \sigma_c$	Yes	No

Table 1. Stability of an homogeneous state and possibility of bifurcation from this state following the value of the equilibrium stress.

Now, if we consider bars with different length, we obtain the following scenarii:

(i) *Small bars:* $L < \pi\ell/\varepsilon_1$. All the homogenous sates are stable and no bifurcation is possible. The homogeneous response is the unique solution of the damage evolution problem.

(ii) *Intermediate bars:* $\pi\ell/\varepsilon_1 < L < 4\pi\ell/\varepsilon_1$. All the homogenous sates are stable, but bifurcations are possible in the time interval $[1, \frac{\sigma_1^3}{\sigma_c^3}]$.

(iii) *Long bars:* $L > 4\pi\ell/\varepsilon_1$. Since the homogeneous states are unstable for $t \in [1, \frac{\sigma_1^3}{64\sigma_c^3})$, a non homogeneous solution appears at $t = 1$.

5. conclusions

A bifurcation and stability analysis was undertaken here for a simple gradient damage model in one-dimensional situation. The full nonlinear initial value problem was solved in closed form for a bar with a finite length. A uniqueness criterion was obtained as well as conditions for bifurcation. These are mainly dependent on the ratio between $\eta = \frac{L}{l}$ the length of the bar to the internal lengthscale involved in the model. The longer the bar (or the smaller the lengthscale), the more solutions are obtained. The localization zone, represented here by the damaged zone, has always a finite thickness. From the obtained results, the following conclusions are drawn:

For the linear comparison solid and as expected, we found a finite number of independent solutions and this number increases with the ratio η, leading to a densification of eigenvalues. It is likely that in the limit $\eta \to \infty$, features of the local underlying model are recovered and in particular ill-posedness.

For the real (nonlinear) solid, this ill-posedness occurs even for a finite ratio η as an infinte number of independent solutions can be constructed. Indeed, bifurcations are available from any homogeneous state of the bar. Note that this holds despite the well-posedness associated to the linear comparison solid.

Now in contrast to the underlying local model where all the damaged states are shown to be unstable, we have shown (though only for the homogeneous states) the existence of stable states and paths for the gradient model and we suggested that for the real nonlinear solid, these stable paths can be selected (among all the solutions) as potential responses of the bar. Another important issue is the numerical implementation of this selection criterion in a numerical analysis. We also underlined the possibility of stable states and paths beyond bifurcation.

Extension of the results to three-dimensional situations and their links with the Hill's general theory of uniqueness and stability [18] is quite appealing and is under investigation.

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