Phase field evolution equation

The evolution equation for the phase field parameter d is given by Eqn.(41) in [1] as

$$\frac{g_c}{l}\left[d - l^2 \Delta d\right] = -g'(d)\Psi_0, \tag{1}$$

where $g(d)$ is the strain energy degradation function, g_c and l are constants, and Δ denotes the Laplacian operator. We rearrange Eqn.(1) and get

$$\Delta d - \frac{d}{l^2} = \frac{g'(d)\Psi_0}{g_c l}. \tag{2}$$

The "reference" strain energy density, Ψ_0, is defined in Eqn.(19) from [1] as

$$\Psi_0 = \frac{1}{2}\boldsymbol{\varepsilon} : \mathscr{C} : \boldsymbol{\varepsilon}, \tag{3}$$

where $\boldsymbol{\varepsilon}$ is the total infinitesimal strain tensor and $\mathscr{C}$ is the elastic stiffness tensor. Per Eqn.(23) in [1], the Cauchy stress $\boldsymbol{\sigma} = g(d)\mathscr{C} : \boldsymbol{\varepsilon}$ but by definition $\boldsymbol{\sigma} = \mathscr{C} : \boldsymbol{\varepsilon}_e$, where $\boldsymbol{\varepsilon}_e$ is the elastic strain. Therefore,

$$\boldsymbol{\varepsilon}_e = g(d)\boldsymbol{\varepsilon}.$$

Replacing $\boldsymbol{\varepsilon}$ with $\boldsymbol{\varepsilon}_e$ in Eqn.(3) we get

$$\Psi_0 = \frac{1}{2}\frac{1}{g(d)^2}\boldsymbol{\varepsilon}_e : \mathscr{C} : \boldsymbol{\varepsilon}_e. \tag{4}$$

Note that the strain energy density $\Psi = g(d)\Psi_0$ is *not* the typical definition of the strain energy density and has an additional multiplicative factor of $1/g(d)$ in Miehe's formulation. If we define an elastic compliance tensor as $\mathscr{S} = \mathscr{C}^{-1}$, then $\boldsymbol{\varepsilon}_e = \mathscr{S} : \boldsymbol{\sigma}$ and

$$\Psi_0 = \frac{1}{2}\frac{1}{g(d)^2}\boldsymbol{\sigma} : \mathscr{S} : \boldsymbol{\sigma}. \tag{5}$$

Substituting this into Eqn.(2), we get

$$\Delta d - \frac{d}{l^2} = \frac{1}{2 g_c l}\frac{g'(d)}{g(d)^2}\boldsymbol{\sigma} : \mathscr{S} : \boldsymbol{\sigma} \tag{6}$$

References

[1] Christian Miehe, Martina Hofacker, and Fabian Welschinger. A phase field model for rate-independent crack propagation: Robust algorithmic implementation based on operator splits. *Computer Methods in Applied Mechanics and Engineering*, 199(45):2765–2778, 2010.