Notes on derivation of governing equation of Elastica

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1 Unit tangent vector and normal vector

We assume that the position vector of a point is given by

$$\mathbf{r}(s) = x_1(s)\hat{\mathbf{e}}_1 + x_2(s)\hat{\mathbf{e}}_2. \tag{1}$$

1.1 Unit tangent vector

According to the definition of tangent vector (refer to Frenet-Serret formulas), we have

$$\mathbf{T}(s) = \frac{d\mathbf{r}(s)}{ds} = x_1'(s)\hat{\mathbf{e}}_1 + x_2'(s)\hat{\mathbf{e}}_2. \tag{2}$$ {eq:rprime}

The norm of $\mathbf{T}(s)$ is

$$||\mathbf{T}(s)|| = \left(x_1'(s)^2 + x_2'(s)^2\right)^{1/2}. \tag{3}$$

From in-extensibility of the beam, we have

$$\left(x_1'(s)^2 + x_2'(s)^2\right)^{1/2} = 1, \tag{4}$$ {eq:in-extens

which means $||\mathbf{T}(s)|| = 1$. Thus we have the unit tangent vector

$$\hat{\mathbf{T}}(s) = \frac{\mathbf{T}(s)}{||\mathbf{T}(s)||} = \mathbf{T}(s) = \mathbf{r}'(s). \tag{5}$$ {eq:TUnit}

1.2 Unit normal vector

The unit normal vector (refer to Frenet-Serret formulas) is defined as

$$\begin{aligned}
\hat{\mathbf{N}}(s) &= \frac{d\mathbf{T}(s)}{ds}\left\|\frac{d\mathbf{T}(s)}{ds}\right\|^{-1} && (6)\\
&= \frac{x_1''(s)\hat{\mathbf{e}}_1 + x_2''(s)\hat{\mathbf{e}}_2}{(x_1''^2 + x_2''^2)^{1/2}}. && (7)
\end{aligned}$$

We take derivative of the in-extensibility condition (4) and get

$$x_1'x_1'' + x_2'x_2'' = 0. \tag{8}$$

Assuming that $x_1' = \cos(\theta)$ and $x_2' = \sin(\theta)$, we have

$$x_1''(s) = -\sin(\theta(s))\theta'(s), \tag{9}$$

$$x_2''(s) = \cos(\theta(s))\theta'(s), \tag{10}$$

$$x_1''(s)^2 + x_2''(s)^2 = \theta'(s)^2. \tag{11}$$

Therefore, the unit normal vector can be expressed as function of $\theta(s)$ as

$$\hat{\mathbf{N}}(s) = \operatorname{sign}\left[\theta'(s)\right]\left(-\sin(\theta(s))\hat{\mathbf{e}}_1 + \cos(\theta(s))\hat{\mathbf{e}}_2\right). \tag{12}$$ {eq:NUnit}

2 Traction vector

The traction vector $\mathbf{t}(\boldsymbol{\xi}(s))$ on position $\boldsymbol{\xi}(s)$ of the beam can be returned as:

$$\mathbf{t}(\boldsymbol{\xi}(s)) = t_1(\boldsymbol{\xi}(s))\hat{\mathbf{T}}(s) + t_2(\boldsymbol{\xi}(s))\hat{\mathbf{N}}(s). \tag{13}$$

where $\boldsymbol{\xi}(s) = \xi\hat{\mathbf{T}}(s) + \eta\hat{\mathbf{N}}(s) + \zeta\hat{\mathbf{B}}(s)$ and

$$\hat{\mathbf{B}}(s) = \hat{\mathbf{T}}(s) \times \hat{\mathbf{N}}(s) = \operatorname{sign}\left[\theta'(s)\right]\hat{\mathbf{e}}_3, \tag{14}$$

is called the binormal unit vector, $t_1(\boldsymbol{\xi}(s))$ and $t_2(\boldsymbol{\xi}(s))$ are the tangent and normal components of $\mathbf{t}(\boldsymbol{\xi}(s))$, respectively.

We take integration on $\mathbf{t}(\boldsymbol{\xi}(s))$ over cross section $\mathcal{A}(s)$

$$\int_{\mathcal{A}(s)} \mathbf{t}(\boldsymbol{\xi}(s))\, dA = \hat{\mathbf{T}}(s)\int_{\mathcal{A}(s)} t_1(\boldsymbol{\xi}(s))\, dA + \hat{\mathbf{N}}(s)\int_{\mathcal{A}(s)} t_2(\boldsymbol{\xi}(s))\, dA \tag{15}$$

$$= V(s)\hat{\mathbf{N}}(s) + T(s)\hat{\mathbf{T}}(s), \tag{16}$$

where $V(s) := \int_{\mathcal{A}(s)} t_2(\boldsymbol{\xi}(s))\, dA$, $T(s) := \int_{\mathcal{A}(s)} t_1(\boldsymbol{\xi}(s))\, dA$.

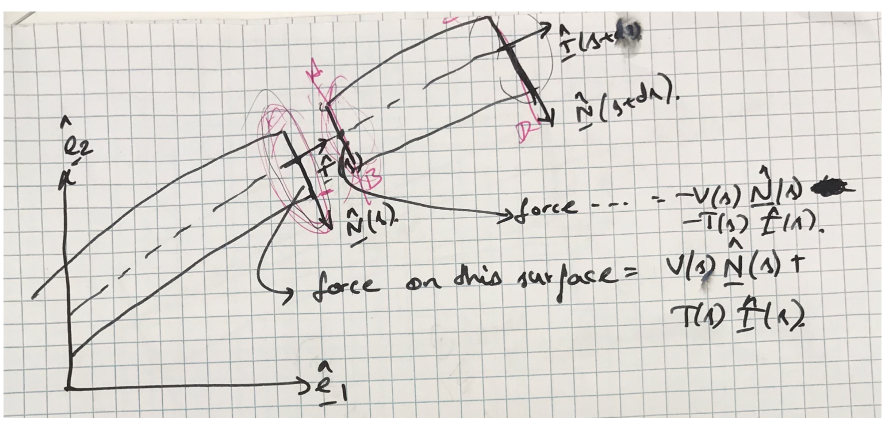

Figure 1: Schematic of the beam {fig:schemati

The tractions acting on the surface A-B (see Fig. 1) is

$$-\mathbf{t}(\boldsymbol{\xi}(s)) = \boldsymbol{\sigma}(s)\left(-\hat{\mathbf{T}}(s)\right) = -t_1(\boldsymbol{\xi}(s))\hat{\mathbf{T}}(s) - t_2(\boldsymbol{\xi}(s))\hat{\mathbf{N}}(s). \tag{17}$$

3 Moment on a cross section

We compute the moment $-\mathbf{M}(s)$ over the cross section A-B (see Fig. 1) at s is given by

$$
\begin{aligned}
-\mathbf{M}(s) &= \int_{\mathcal{A}(s)} \boldsymbol{\xi}(s) \times (-\mathbf{t}(\boldsymbol{\xi}(s)))\ dA && (18)\\
&= \int_{\mathcal{A}(s)} \left(\xi \hat{\mathbf{T}}(s) + \eta \hat{\mathbf{N}}(s) + \zeta \hat{\mathbf{B}}(s)\right) \times \left(-t_1(\boldsymbol{\xi}(s))\hat{\mathbf{T}}(s) - t_2(\boldsymbol{\xi}(s))\hat{\mathbf{N}}(s)\right)\ dA && (19)\\
&= \int_{\mathcal{A}(s)} \left(-\xi t_2 \hat{\mathbf{B}}(s) + \eta t_1(\boldsymbol{\xi}(s))\hat{\mathbf{B}}(s) - \zeta t_1(\boldsymbol{\xi}(s))\hat{\mathbf{N}}(s) + \zeta t_2(\boldsymbol{\xi}(s))\mathbf{t}(s)\right)\ dA && (20)\\
&= \int_{\mathcal{A}(s)} \eta t_1(\boldsymbol{\xi}(s))\hat{\mathbf{B}}(s)\, dA, && (21)
\end{aligned}
$$

{eq:MomentExp

noting that in the local coordinate system, over cross section, $\xi = 0$. Because of symmetry, $\int_{\mathcal{A}(s)} \zeta t_1(\boldsymbol{\xi}(s))\hat{\mathbf{N}}(s)\, dA = \int_{\mathcal{A}(s)} \zeta t_2(\boldsymbol{\xi}(s))\hat{\mathbf{T}}(s)\, dA = 0$.

We can write Eqn. (21) as

$$
\begin{aligned}
\mathbf{M}(s) &= M(s)\hat{\mathbf{B}}(s), && (22)\\
M(s) &= -\int_{\mathcal{A}(s)} \eta t_1(\boldsymbol{\xi}(s))\, dA. && (23)
\end{aligned}
$$

{eq:MomentSca

To build the link between moment and kinematic variables, we assume that the material of the beam is described by Hooke's Law. Therefore,

$$
t_1(\boldsymbol{\xi}(s)) = E\epsilon_1(\boldsymbol{\xi}(s)), \tag{24}
$$

{eq:hooke}

where E is Young's modulus and $\epsilon_1(\boldsymbol{\xi}(s))$ is the normal strain on the cross section at s. We will derive $\epsilon_1(\boldsymbol{\xi}(s))$ in polar coordinate system, as shown in Fig. 2.

The length of a fiber at $\eta\hat{\mathbf{N}}(s)$ is $(\rho(s) - \eta)$, where $\rho(s)$ is the local radius of curvature. The normal strain of this fiber is given by

$$
\epsilon_1(\boldsymbol{\xi}(s)) = \frac{(\rho(s) - \eta)d\theta - \rho(s)d\theta}{\rho(s)d\theta} = -\frac{\eta}{\rho(s)}. \tag{25}
$$

We also know that $1/\rho(s) = \kappa(s) = ||d\mathbf{T}(s)/ds|| = \theta'(s)\,\mathrm{sign}\,[\theta'(s)]$. Thus

$$
\epsilon_1(\boldsymbol{\xi}(s)) = -\eta\theta'(s)\,\mathrm{sign}\,[\theta'(s)]\,. \tag{26}
$$

{eq:normalstr

Substituting Eqns. (24) and (26) into Eqn. (23), we have that

$$
\begin{aligned}
M(s) &= -\int_{\mathcal{A}(s)} \eta E\epsilon_1(\boldsymbol{\xi}(s))\, dA, && (27)\\
&= E\theta'(s)\,\mathrm{sign}\,[\theta'(s)] \int_{\mathcal{A}(s)} \eta^2\, dA, && (28)\\
&= EI\theta'(s)\,\mathrm{sign}\,[\theta'(s)]\,, && (29)
\end{aligned}
$$

{eq:MomentFin

where $I = \int_{\mathcal{A}(s)} \eta^2\, dA$ is the second moment of inertia of the cross section.

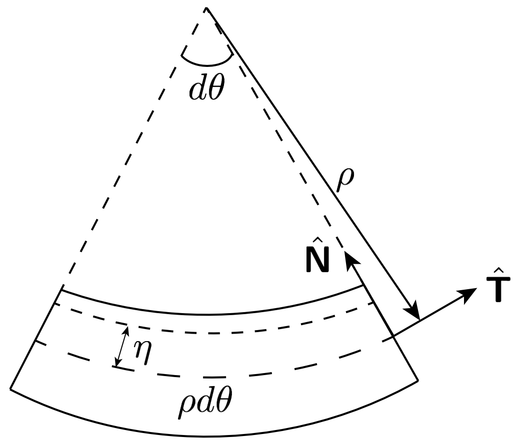

Figure 2: Schematic of a infinitesimal section of a beam. {fig:polar}

4 Balance of force

From force balance, we have equilibrium equation

$$\mathbf{P} + V(s)\hat{\mathbf{N}} + T(s)\hat{\mathbf{T}} = \mathbf{0}, \tag{30}$$ {eq:balance}

where $\mathbf{P} = P_1\hat{\mathbf{e}}_1 + P_2\hat{\mathbf{e}}_2$ is the external applied force on beam's cross section.

We project the force balance equation (30) in the direction of $\hat{\mathbf{N}}$ to obtain

$$\mathbf{P} \cdot \hat{\mathbf{N}} + V(s) = 0. \tag{31}$$ {eq:PV}

Recall that from Eqn. (12), we can simplify Eqn. (31) as

$$\operatorname{sign}\left[\theta'(s)\right]\left(-\sin(\theta(s))P_1 + \cos(\theta(s))P_2\right) + V(s) = 0. \tag{32}$$ {eq:ForceBala

5 Balance of moment

We apply balance of moment on the infinitesimal section of beam A-B-D-C (see Fig. 1),

$$-\mathbf{M}(s) + \int_{\mathcal{A}(s+ds)} \Delta\mathbf{r} \times \mathbf{t}(\boldsymbol{\xi}(s+ds))\, dA + \int_{\mathcal{A}(s+ds)} \boldsymbol{\xi} \times \mathbf{t}(\boldsymbol{\xi}(s+ds))\, dA = 0, \tag{33}$$ {eq:Balanceof

where

$$\begin{aligned}
\int_{\mathcal{A}(s+ds)} \Delta\mathbf{r} \times \mathbf{t}(\boldsymbol{\xi}(s+ds))\, ds &= \Delta\mathbf{r} \times \int_{\mathcal{A}(s+ds)} \mathbf{t}(\boldsymbol{\xi}(s+ds))\, ds && (34)\\
&= \Delta\mathbf{r} \times \left(V(s+ds)\hat{\mathbf{N}}(s+ds) + T(s+ds)\hat{\mathbf{N}}(s+ds)\right) && (35)\\
&= \left(\mathbf{r}'(s)ds + o(ds)\right) \times \Big(\left(V(s) + V'(s)ds + o(ds)\right)\Big(\hat{\mathbf{N}}(s) + \hat{\mathbf{N}}'(s)ds + o(ds)\Big)\Big) && (36)\\
&+ \left(\mathbf{r}'(s)ds + o(ds)\right) \times \Big(\left(T(s) + T'(s)ds + o(ds)\right)\Big(\hat{\mathbf{T}}(s) + \hat{\mathbf{T}}'(s)ds + o(ds)\Big)\Big) && (37)\\
&= \mathbf{r}'(s)ds \times \big(V(s)\hat{\mathbf{N}}(s) + T(s)\hat{\mathbf{T}}(s) + o(ds)\big) + o(ds), && (38)
\end{aligned}$$ {eq:Moment2}

and

$$\int_{\mathcal{A}(s+ds)} \boldsymbol{\xi} \times \mathbf{t}(\boldsymbol{\xi}(s+ds))\, dA = \mathbf{M}(s+ds) \tag{39}$$

$$= \mathbf{M}(s) + \mathbf{M}'(s)ds + o(ds). \tag{40}$$

{eq:Moment3}

Recall that from Eqn. (2) and (5), we have

$$\mathbf{r}'(s) = \hat{\mathbf{T}}(s). \tag{41}$$

Combing Eqns. (33), (38) and (40), we have

$$-\mathbf{M}(s) + \hat{\mathbf{T}}(s) \times \left(V(s)\hat{\mathbf{N}}(s) + T(s)\hat{\mathbf{T}}(s) + o(ds)\right) + \mathbf{M}(s) + \mathbf{M}'(s)ds + o(ds) \tag{42}$$

$$=V(s)\hat{\mathbf{B}}(s) + o(ds) + \mathbf{M}'(s)ds + o(ds) \tag{43}$$

$$= \left(V(s)\hat{\mathbf{B}}(s) + \mathbf{M}'(s)\right) ds + o(ds) \tag{44}$$

$$=0. \tag{45}$$

Thus we have

$$V(s)\hat{\mathbf{B}}(s) = -\mathbf{M}'(s), \tag{46}$$

which is equivalently (assuming the sign of $\theta'(s)$ does not change, i.e., $\hat{\mathbf{B}}(s)$ remain the same)

$$V(s) = -M'(s). \tag{47}$$

{eq:MomentBal

6 Final governing equation

Combing force balance (32) and moment balance (47), we have

$$M'(s) + \text{sign}\left[\theta'(s)\right] \left(P_1 \sin(\theta) - P_2 \cos(\theta)\right) = 0. \tag{48}$$

From Eqn. (29), we have

$$EI\theta''(s) + \left(P_1 \sin(\theta) - P_2 \cos(\theta)\right) = 0. \tag{49}$$