Euler Elastic Problem

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1 Governing Equation Derivation

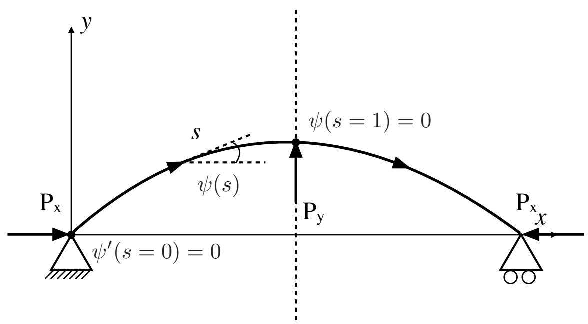


Figure 1: Simply supported beam under general load

Consider a simply supported beam with total length $s = 2$. The tangent angle of the beam is $\psi(s)$. Due to symmetry, we only consider the left half part. The total energy as a functional of $\psi(s)$ is

$$\Pi = \frac{E}{2}\int_0^1 I(s)\kappa^2 ds - P_x\Delta_x - P_y\Delta_y \tag{1}$$

where

$$\begin{aligned} \kappa &= \frac{d\psi(s)}{ds} \\ \Delta_x &= 1 - \int_0^1 \cos\psi\, ds \\ \Delta_y &= \int_0^1 \sin\psi\, ds \end{aligned} \tag{2}$$

The admissible space of $\psi(s)$ is that

$$\mathcal{V} = \{\psi(s) \in C^1([0,1],\mathcal{R})\} \tag{3}$$

Take derivation of functional (1), we got

$$\begin{aligned} \delta\Pi &= E\int_0^1 I(s)\frac{d\psi}{ds}\frac{d\delta\psi}{ds}ds - P_x\int_0^1 \sin\psi\,\delta\psi\, ds - P_y\int_0^1 \cos\psi\,\delta\psi\, ds \\ &= EI(s)\frac{d\psi}{ds}\delta\psi\bigg|_{s=0}^{s=1} - \int_0^1 \left(E\left(I(s)\frac{d\psi}{ds}\right)' + P_x\sin\psi + P_y\cos\psi\right)\delta\psi\, ds \end{aligned} \tag{4}$$

Since the boundary conditions can be $\psi(s) = 0$ or $\left.\frac{d\psi}{ds}\right|_s = 0$ on $s = 0, 1$, the first term in eqn (4) always vanishes. Thus $\delta\Pi = 0$ for arbitrary $\delta\psi$ gives out the governing equation

$$E\left(I(s)\frac{d\psi}{ds}\right)' + P_x \sin\psi + P_y \cos\psi = 0 \tag{5}$$

Here we use arc length coordinate in material configuration, so the in-extensibility constraint is implicitly satisfied. The governing equation (5) is solved numerically in Mathematica under different boundary conditions.

2 Simply supported beam with a Vertical Point Load at the middle point (half)

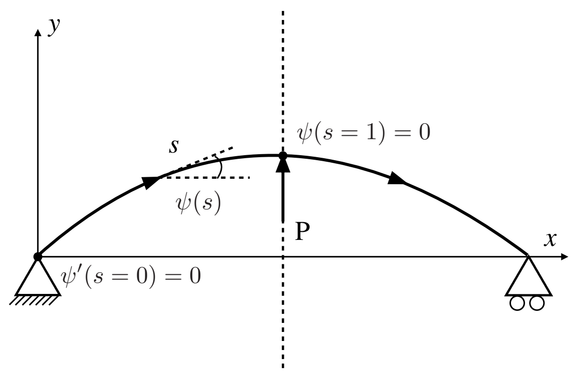


Figure 2: Simply supported beam under vertical load

The boundary conditions are $\left.\frac{d\psi}{ds}\right|_{s=0} = 0,\ \psi(1) = 0$, let $P_x = 0,\ P_y = 1,\ I(s) = 1 + s$, the deformed configuration is

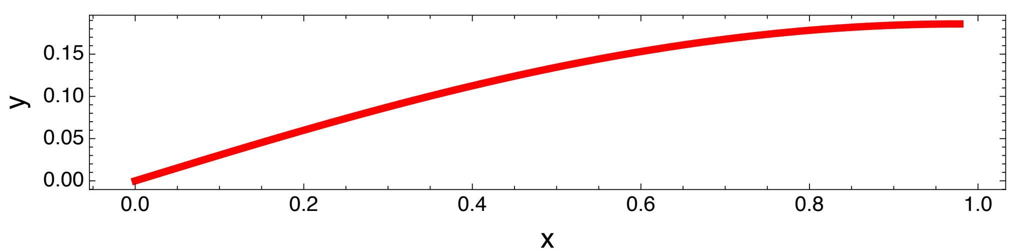


Figure 3: Simply supported beam under vertical load

3 Fixed-fixed beam with a Vertical Point Load at the middle point (half)

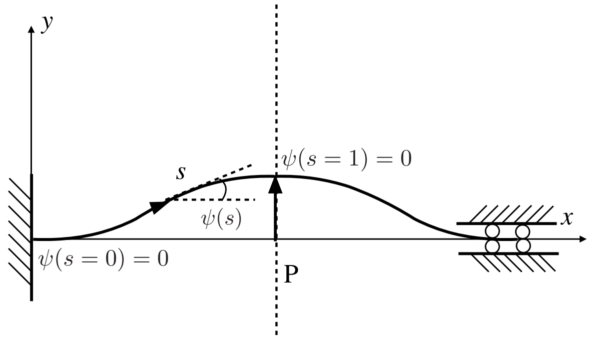


Figure 4: Simply supported beam under vertical load

The boundary conditions are $\psi(0)=0,\ \psi(1)=0$, let $P_x=0,\ P_y=4,\ I(s)=1$, the deformed configuration is

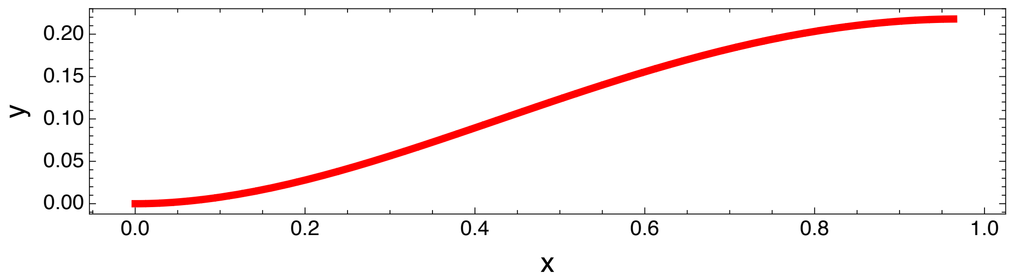


Figure 5: Simply supported beam under vertical load

4 Simply supported beam with Axial Load at two ends

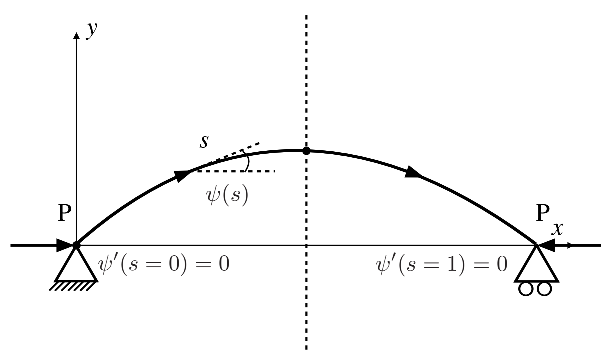


Figure 6: Simply supported beam with Axial Load at two ends

The boundary conditions are $\left.\frac{d\psi}{ds}\right|_{s=0}=0,\ \left.\frac{d\psi}{ds}\right|_{s=1}=0$. Let $P_y=0,\ I(s)=1$. Then this is an eigenvalue problem. The solution can be not unique for certain P_x. Here we use Fourier series to approximate the solution

$$\psi(s)=a_1\cos(\pi s)+a_2\cos(2\pi s)+a_3\cos(3\pi s)+a_4\cos(4\pi s)+a_5\cos(5\pi s)+a_6\cos(6\pi s) \tag{6}$$

which has already satisfied the boundary conditions.

Then we find the coefficients by minimize the total energy functional and get the approximated solution $\psi(s)$. The curve of load w.r.t w (the amplitude of $\psi(s)$) is plotted in the following figure

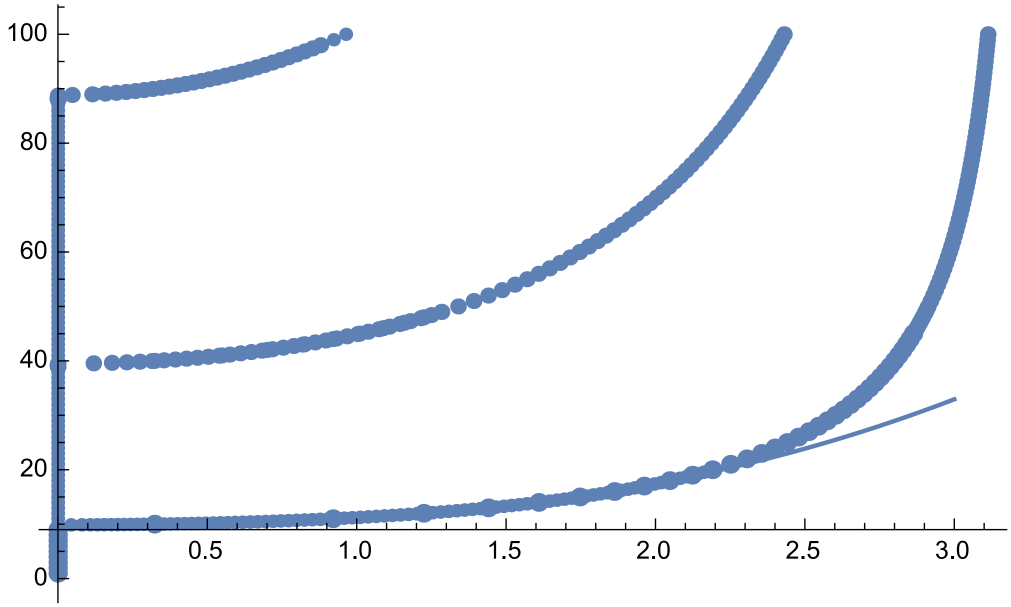
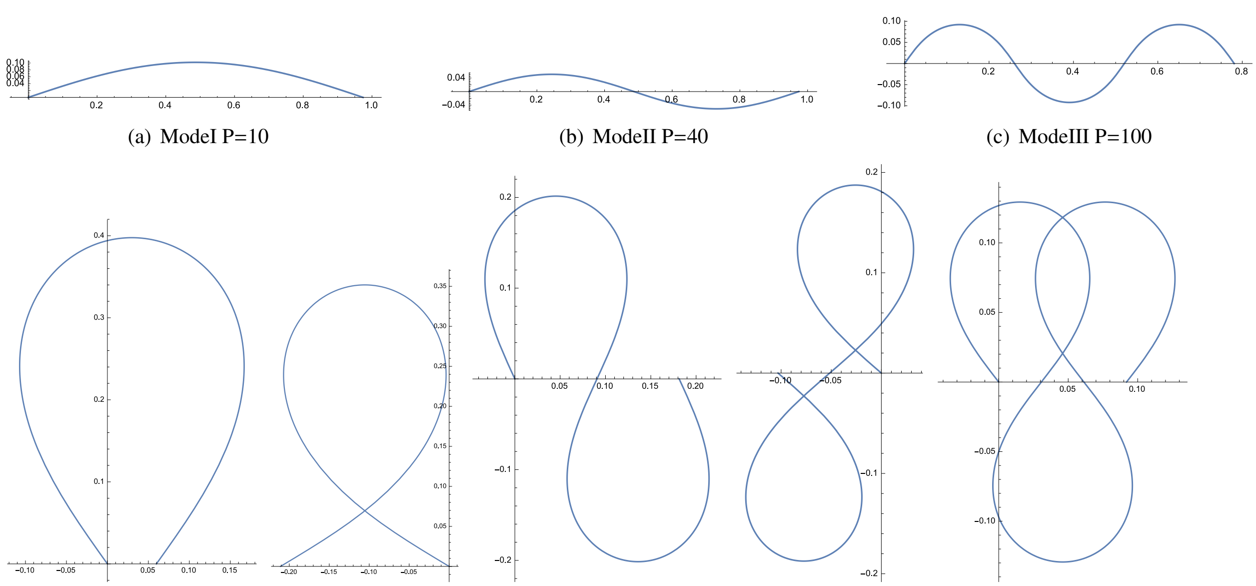


Figure 7: Load vs w curve



(a) ModelI P=10 (b) ModelI P=40 (c) ModelII P=100

(d) ModelI P=20 (e) ModelI P=30 (f) ModelII P=70 (g) ModelII P=100 (h) ModelIII P=180

Figure 8: Some deformed configurations.

Here I am not able to get the analytical expression of Load vs deflection around the point on the load axis. Assume that $P - P_{cr} = Cw^d$. we got a fitting function as

$$P - 9.9701 = 1.11316w^{2.75538} \tag{7}$$