Mixed formulation for elasticity

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1 Displacement formulation of Elasticity

In genral we solve the equations

$$\sigma_{ij,j} = \rho \ddot{u}_i, \tag{1a}$$
$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij}, \tag{1b}$$
$$\epsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{1c}$$

Stress in terms of displacements read

$$\sigma_{ij} = \mu\left(u_{i,j} + u_{j,i}\right) + \lambda u_{k,k}\delta_{ij} \tag{2}$$

We have the traction from (2) to be

$$\sigma_{ij}n_j = \mu\left(u_{i,j} + u_{j,i}\right)n_j + \lambda u_{k,k}\delta_{ij}n_j, \tag{3}$$
$$t_i = \mu\left(u_{i,j} + u_{j,i}\right)n_j + \lambda u_{k,k}n_i \tag{4}$$

Combining all the above equations we get

$$\sigma_{ij,j} = \mu\left(u_{i,jj} + u_{j,ij}\right) + \lambda u_{k,kj}\delta_{ij}, \tag{5a}$$
$$\sigma_{ij,j} = \mu\left(u_{i,jj} + u_{j,ij}\right) + \lambda u_{k,kj}\delta_{ij}, \tag{5b}$$
$$= \mu u_{i,jj} + \mu u_{k,ki} + \lambda u_{k,ki}, \tag{5c}$$
$$= \mu u_{i,jj} + \left(\mu + \lambda\right)u_{k,ki} \tag{5d}$$

or in co-ordiante free form

$$\{\operatorname{div}\boldsymbol{\sigma}\} = \mu\nabla^2\boldsymbol{u} + \left(\lambda + \mu\right)\nabla\left(\nabla\cdot\boldsymbol{u}\right) \tag{6}$$

Thus, the governing equation becomes,

$$\mu u_{i,jj} + (\mu + \lambda)\, u_{k,ki} + f_i = \rho \ddot{u}_i, \tag{7}$$

$$u_i(x_i) = \hat{u}_i \quad (x_i) \in \Gamma_{g_i} \tag{8}$$

$$\mu\,(u_{i,j} + u_{j,i})\, n_j + \lambda u_{k,k} n_i = \hat{t}_i \quad on\,(x_i) \in \Gamma_{h_i} \tag{9}$$

1.1 Finite Element Formulation

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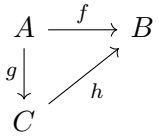
2 Mixed formulation

$$\mu\,(u_{i,jj} + u_{j,ij}) - \frac{2\mu}{3} u_{k,ki} - p_{,i} = \rho \ddot{u}_i, \tag{10a}$$

$$p + K u_{i,i} = 0 \tag{10b}$$

The equations

3 Diagrams



4 Tables and Figures

Variable	Value	Unit
Speed of light c	3.00×10^{8}	$\mathrm{m\,s^{-1}}$
Planck constant h	6.626×10^{-34}	$\mathrm{J\,s}$

Table 1: Constants.

A Extra Macros

Add any further macro definitions here.