Limiting Distribution of the Complex Roots of Random Polynomials

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Abstract

In this thesis, we follow the proof of a general result of Kabluchko and Zaporozhets in [8] about the limiting distribution of the complex roots of a (finite or infinite) random polynomial indexed by a natural number n, as $n \to \infty$, when the coefficients of the polynomial satisfy certain conditions. We briefly discuss a recent and related result by Bloom and Dauvergne in [1], which gives a stronger sense of convergence for polynomials of degree n. Next, we investigate the limiting distributions for a class of polynomials considered by Schehr and Majumdar in [11], in which the asymptotics for the expected number of real roots exhibit phase transitions along a parameter α. We provide a conjecture for the limiting distributions of this class, using a heuristic argument based on the methods in the result of Kabluchko and Zaporozhets, and find that, if the conjecture is correct, phase transitions occur in the limiting distributions, as well.

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Contents

1 Introduction

Random polynomials are (finite or infinite) polynomials whose coefficients are random variables. Two main topics of interest in the field of random polynomials are the following: 1) in the case of finite polynomials, the expected number of real roots as the degree tends toward infinity and 2) characterizing the distribution of complex roots. One well-studied random polynomial is the Kac polynomial, given by

$$K_n(z) = \sum_{k=0}^{n} z^k \xi_k, \quad n = 1, 2, \ldots \tag{1}$$

where the the ξ_k are independent and identically distributed standard Gaussian (complex-valued) random variables. The expected number of real roots of K_n is asymptotically $\frac{2}{\pi} \log n$, a result which can be proven by an explicit formula for counting the real roots [2]. In fact, this formula can be generalized to random sums of any differentiable functions and to multivariate normal distributions with any covariance matrix. In particular, let P_n be a random polynomial of the form

$$P_n(z) = \sum_{k=0}^{n} f_{k,n}(z)\xi_k \tag{2}$$

where $f_{0,n}, \ldots, f_{n,n}$ are differentiable functions and $(\xi_0, \ldots, \xi_n)$ is drawn from a multivariate normal distribution with covariance matrix Σ and zero mean. Let $f(t) = (f_{0,n}(t), \ldots, f_{n,n}(t))^T$ be a column vector of the functions. Then the expected number of real roots of P_n on any measurable set E, denoted $\mathbb{E}_E[N_n]$ or simply $\mathbb{E}[N_n]$ if $E = \mathbb{R}$, is given by

$$\mathbb{E}_E[N_n] = \int_E \frac{1}{\pi} \left(\frac{\partial^2}{\partial x \partial y} \left(\log f(x)^T \Sigma f(y) \right) |_{y=x=t} \right)^{1/2} dt \tag{3}$$

[2]. Depending on the functions $f_{k,n}$, the above formula, and more specifically the sum $f(x)^T \Sigma f(y)$, may or may not be easy to compute. In the case of the Kac polynomial K_n in (1), the sum of a finite geometric series yields

$$f(x)^T \Sigma f(y) = \frac{1 - (xy)^{n+1}}{1 - xy}$$

which simplifies the expression above to

$$\mathbb{E}_E[N_n] = \frac{1}{\pi}\int_E \sqrt{\frac{1}{(t^2-1)^2} - \frac{(n+1)^2 t^{2n}}{(t^{2n+2}-1)^2}}dt$$

from which it can be deduced, with some effort, that $\mathbb{E}[N_n] \sim \frac{2}{\pi}\log n$ [2]. In the case of the polynomial whose coefficients are i.i.d. Gaussians with variances $\binom{n}{k}$ (that is, take $f_{k,n}(z) = \sqrt{\binom{n}{k}}z^k$ in (2)), the formula is even simpler, with

$$f(x)^T \Sigma f(y) = (1+xy)^n$$

so that the integral calculation becomes

$$\mathbb{E}[N_n] = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\sqrt{n}}{1+t^2}dt = \sqrt{n}$$

[2]. Naturally, though, these computations, either for calculating $\mathbb{E}[N_n]$ or for calculating asymptotics, can become unwieldy when the variances do not result in a nice sum $f(x)^T\Sigma f(y)$.

In addition to finding a formula for the density of the expected number of real roots, as is given by the integrand in the formula (3) for $\mathbb{E}[N_n]$, there is an explicit formula for the density of the complex roots for certain classes of random functions using the argument principal [12]. Using the form $P_n(z)$ in (2) in which the ξ_k are i.i.d. standard Gaussians and the $f_{k,n}(z)$ are entire functions which are real-valued on the real line, a density can be computed in terms of the following sums:

$$A_0(z) = \sum_{k=0}^n f_{k,n}(z)^2 \qquad B_0(z) = \sum_{k=0}^n |f_{k,n}(z)|^2$$

$$A_1(z) = \sum_{k=0}^n f_{k,n}(z)^2 f'_{k,n}(z)^2 \qquad B_1(z) = \sum_{k=0}^n \overline{f_{k,n}(z)} f'_{k,n}(z)$$

$$A_2(z) = \sum_{k=0}^n f'_{k,n}(z)^2 \qquad B_2(z) = \sum_{k=0}^n |f'_{k,n}(z)|^2$$

[12]. In some cases, such as in the Weyl polynomial

$$W_n(z) = \sum_{k=0}^{n} \frac{1}{\sqrt{k!}} \xi_k \tag{4}$$

where the ξ_k are standard Gaussians, the above sums are easy to compute as $n \to \infty$ [12]. As before, the ease of this computation will very much depend on the $f_{k,n}(z)$.

Recently, the focus has been not on finding exact formulas for the expected density of real and complex roots, but instead on the limiting distribution of (all) roots as $n \to \infty$ in the case of random polynomials (we will later define “limiting distribution” more precisely). Some classic results include that the distribution of the complex roots of the Kac polynomial is uniform on the unit circle in the limit as $n \to \infty$ [7]. The limiting distribution of the Weyl polynomial, on the other hand, is uniform in the disc of radius $\sqrt{n}$ in the limit (Figure 1 in [8]). These results have been subject to increasing generalization by, for instance, showing that they hold for a broad class of distributions, rather than assuming the ξ_k are Gaussians. For instance, Ibragimov and Zaporozhets proved that the limiting distribution of the complex zeros of a polynomial of the form $\xi_0 + \xi_1 z + \cdots \xi_n z^n$, where the ξ_i are i.i.d., is uniform on the unit circle *if and only if* the i.i.d. coefficients have finite logarithmic moment [7]. In these results, the strategy does *not* involve calculating a density for fixed n, but rather observing the limiting behavior of $\frac{1}{n} \log |P_n(z)|$, whose relationship to counting the zeros of P_n we will study later. Along with increasing distributional generalizations, results about the limiting distribution of roots as $n \to \infty$ have been extended to infinite polynomials (recall that in P_n the dependence on n is not just in the number of terms in the sum, but also in the coefficients $f_{k,n}$) [8].

In this thesis, we rewrite a general result of Kabluchko and Zaporozhets (Theorem 2.8 in [8]) on the limiting distribution of the complex zeros of (finite or infinite) random polynomials whose coefficients satisfy certain conditions. First, we develop the relevant preliminaries from analysis and probability, then give insight into how the conditions were reverse engineered, and finally follow their technical proof. Then, we consider this result in the context of a related result by Bloom and Dauvergne (Theorem 6.5 in [1]) that has so far only been stated for finite polynomials but which gives a stronger sense of convergence. We next consider an interesting class of polynomials recently studied by Schehr and Majumdar in [11] in which the asymptotics for the expected number of real roots exhibit phase transitions. We investigate the limiting behavior of the complex zeros, which to our knowledge has yet to be characterized, with the idea that the limiting distributions will likewise exhibit phase transitions. We show that while the general result of Kabluchko and Zaporozhets does not apply to this class, the methods in the proof can be adapted. We present our conjecture for the limiting distribution along with a heuristic argument based on these methods supporting it, and note how this could be formalized in the future.

2 General Result of Kabluchko and Zaporozhets [8]

Let $\xi_0, \xi_1, \ldots$ be a sequence of independent and identically distributed complex-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We will consider random polynomials of the form

$$P_n(z;\omega) = \sum_{k=0}^{\infty} f_{k,n} z^k \xi_k \tag{5}$$

where $\omega \in \Omega$ is the event producing $\xi_0, \xi_1, \ldots$ and the $f_{k,n}$ are deterministic complex coefficients. Sometimes we may suppress ω and simply write $P_n(z)$, but will include ω when emphasizing the randomness of P_n. Often, we may be interested in a *finite* random polynomial of the form

$$\sum_{k=0}^{n} \xi_k f_{k,n} z^k$$

in which case we still have a polynomial of the form (5), since we may use the fact that $f_{k,n}$ is a function of k and n, and therefore set $f_{k,n} = 0$ for $k > n$.

In this section, we present a general result of Kabluchko and Zaporozhets (Theorem 2.8 in [8]) describing the limiting distribution of the complex roots of P_n as $n \to \infty$, in which the

ξ_k satisfy a moment condition (without any additional constraint on the distribution from which the ξ_k are drawn) and the $f_{k,n}$ are deterministic coefficients that can be approximated by a continuous function satisfying a few additional conditions. Further, all limiting distributions described in this result will be rotationally invariant. Importantly, we note a limitation of this result is that it will not necessarily describe the limiting distribution on all of $\mathbb{C}$, but will apply to a disc of radius $R \in (0, \infty]$, where R is determined by the decay of the coefficients $f_{k,n}$; the choice of R will ensure almost sure convergence of P_n on $\mathbb{D}_R$, and will also be used to bound the tail end of P_n in our later computations.

2.1 Preliminaries: Convergence of Random Measures

In this subsection, we describe precisely what we mean by the limiting distribution of the complex roots of P_n. We will write μ_n for the measure on $\mathbb{C}$ counting with multiplicity the complex zeros of P_n. In other words, for any Lebesgue measurable subset A of $\mathbb{C}$, $\mu_n(A)$ is the number of zeros (counted with multiplicity) of P_n contained in A. Sometimes, we will write $\mu_n(\omega)$ instead of μ_n to emphasize that μ_n is a random measure. We will be interested in the limiting behavior of the sequence of random measures $\frac{1}{n}\mu_n$ as $n \to \infty$. In order to describe the sense in which this limit is taken below, we introduce some notation and definitions, all of which are conventions that can be found in Chapter II of [6] and Chapter 9 of [5].

Let $C_c^\infty(X)$ be the space of all infinitely differentiable and compactly supported functions on X.

Definition 1. *The set of distribution functions, or generalized functions, on U, denoted $\mathcal{D}'(U)$, is the set of all continuous linear functionals on $C_c^\infty(U)$. In other words, if $F \in \mathcal{D}'(U)$, then*

$$F : C_c^\infty(U) \to \mathbb{R}$$

and F is linear and continuous. We refer to $C_c^\infty(U)$ as the space of test functions. If $\phi \in C_c^\infty(U)$, sometimes we will write $\langle F, \phi \rangle$ or $\int F\phi$ for $F(\phi)$.

We will endow $\mathcal{D}'(U)$ with the topology of pointwise convergence, that is, the weak* topology. In other words, for a sequence $F_1, F_2, \cdots \in \mathcal{D}'(U)$, we say that

$$F_n \to F$$

if for every test function $\phi \in C_c^\infty(U)$,

$$F_n(\phi) \to F(\phi)$$

in the usual sense of convergence of a sequence of real numbers.

Identifying Measures with Distributions: If ν is a measure, then we will use the convention of associating ν with the distribution F_ν defined by

$$F_\nu(\phi) = \int \phi d\nu.$$

We may make a slight abuse of notation when the context is clear and write ν for the distribution F_ν.

Identifying Functions with Distributions: If f is a continuous function on U, then we will use the convention of associating f with the distribution F_f defined by

$$F_f(\phi) = \int_U \phi(u) f(u) du.$$

We may make a slight abuse of notation when the context is clear and write f for the distribution F_f.

Definition 2. *Take $U = \mathbb{C}$ in Definition 1. Let $z_0 \in \mathbb{C}$. The Dirac delta distribution at z_0, or the point mass at z_0, is the distribution $\delta_{z_0} \in \mathcal{D}'(U)$ defined by*

$$\delta_{z_0}(\phi) = \phi(z_0).$$

Convergence of Measures: Using the identification of measures with distributions above, and the definition of the Dirac delta distribution, the counting measure μ_n is the sum of the point masses at the roots of P_n (counted with multiplicity). More specifically, if we define μ_n on some subset D of $\mathbb{C}$, and let $m(z)$ denote the multiplicity of the root at $z \in D$, we have that

$$\mu_n = \sum_{z \in \mathbb{D}: P_n(z)=0} m(z)\delta_z.$$

Now we can define convergence of a sequence of measures. We will say that a sequence of measures ν_n on $D \subset \mathbb{C}$ converges to ν if the corresponding distribution functions converge in the weak* topology, that is, if

$$\int_D \phi d\nu_n \to \int_D \phi d\nu$$

for every $\phi \in C_c^\infty(D)$.

We will be interested in the sequence of measures $\nu_n = \frac{1}{n}\mu_n$. The reason for scaling by $\frac{1}{n}$ is that when P_n is a polynomial of degree n, we have that $\frac{1}{n}\mu_n(\mathbb{C}) = 1$, so that $\frac{1}{n}\mu_n$ is a probability measure.

Convergence of Random Measures in Probability: We are now in a position to define a notion of convergence of random measures, which is the notion of convergence used in the result of Kabluchko and Zaporozhets. If $\nu_n = \nu_n(\omega)$ is a random measure, then we say that ν_n converges to ν in probability if

$$\int_D \phi d\nu_n \to \int_D \phi d\nu \qquad \text{in probability}$$

for every $\phi \in C_c^\infty(U)$.

Remark 1. *Let $\mathcal{M}(D)$ be the space of all locally finite measures on D. We may think of a sequence of random measures as defined above as a sequence of $\mathcal{M}(D)$-valued random* *variables. Convergence in the vague topology on $\mathcal{M}(D)$ is precisely $\int \phi d\nu_n \to \int \phi d\nu$ for every ϕ. So convergence in probability of the random measures ν_n in the weak* topology is equivalent to convergence in probability of the corresponding $\mathcal{M}(D)$-valued random variables in the vague topology (Section 1.2 in [8]).*

2.2 Preliminaries: Counting the Complex Roots of $P_n(z)$

Definition 3. *Let $\mathcal{D}'(U)$ be a distribution on U. The Laplacian of $F \in \mathcal{D}'(U)$, denoted ΔF, is the distribution in $\mathcal{D}'(U)$ defined by*

$$\langle \Delta F, \phi \rangle := \langle F, \Delta \phi \rangle$$

(Chapter 9 in [5]).

With this definition in hand, we will now be able to find an explicit formula for μ_n, the measure counting the zeros of P_n, in terms of P_n. We will then be able to work exclusively with this formula in finding the limiting distribution of $\frac{1}{n}\mu_n$.

Proposition 1. *(Theorem 3.3.2 in [6]) Let δ_{z_0} denote the point mass at $z_0 \in \mathbb{C}$ as above. Then*

$$\delta_{z_0} = \frac{1}{2\pi} \Delta \log |z - z_0|$$

Remark 2. *Here, we identify the function $\log |z - z_0|$ with the corresponding distribution in $\mathcal{D}'(\mathbb{C})$, and view Δ as mapping from $\mathcal{D}'(\mathbb{C})$ to $\mathcal{D}'(\mathbb{C})$ as in Definition 3.*

Proposition 2. *Let P_n be a (possibly infinite) polynomial and μ_n the measure on $\mathbb{C}$ counting the zeros of P_n, as above. Then*

$$\mu_n = \frac{1}{2\pi} \Delta \log |P_n(z)|.$$

Remark 3. *Here, as in the previous proposition, we identify the function $\log |P_n(z)|$ with the corresponding distribution in $\mathcal{D}'(\mathbb{C})$, and view Δ as mapping from $\mathcal{D}'(\mathbb{C})$ to $\mathcal{D}'(\mathbb{C})$. Further, we view μ_n as the distribution corresponding to the measure μ_n. In other words, we think* *of* $\frac{1}{2\pi}\Delta \log|P_n(z)|$ *as the density of the measure* μ_n*.*

Proof of Proposition 2. Let $D \subseteq \mathbb{C}$ and let $m(z_i)$ be the multiplicity of the zero $z_i \in D$ of $P_n(z)$. For some constant $c \in \mathbb{C}$, we can write

$$|P_n(z)| = \left| c \prod_{z_i \in D: P_n(z_i)=0} (z - z_i)^{m(z_i)} \right|$$

$$\iff \frac{1}{2\pi}\Delta \log|G_n(z)| = \frac{1}{2\pi} \sum_{z_i \in D: P_n(z_i)=0} m(z_i)\Delta \log|z - z_i|$$

$$\iff \frac{1}{2\pi}\Delta \log|G_n(z)| = \sum_{z_i \in D: P_n(z_i)=0} \delta_{z_i} = \mu_n$$

where we have used Proposition 1 and the definition of μ_n. □

2.3 Steps for Proving Convergence of Measures

Now that we have an explicit formula for μ_n in terms of $P_n(z)$, following [8], we can outline a set of possible steps for showing that $\frac{1}{n}\mu_n$ converges in probability to some distribution independent of n. We begin with a result that allows us to deduce convergence in probability of a sequence of random variables from almost sure convergence of its subsubsequences:

Lemma 1. *(Lemma 3.2 in [9]) Let* $X_1, X_2, \ldots$ *be a sequence of random variables in a metric space. Then* $X_n \to X$ *in probability if and only if every subsequence* X_{n_k} *has a subsequence* $X_{n_{k_l}}$ *such that* $X_{n_{k_l}} \to X$ *almost surely.*

First, we consider the sequence

$$p_n(z;\omega) = \frac{1}{n}\log|P_n(z;\omega)|.$$

In particular, $\frac{1}{2\pi}\Delta p_n = \frac{1}{n}\mu_n$ by Proposition 2. Under the conditions placed on the ξ_k and the $f_{k,n}$ in the general result we present, we will be able to study the limiting behavior of the sequence of random variables p_n and show that this behavior is unchanged under the application of the Laplacian, so that the limiting behavior of the p_n determines the limiting behavior of the $\frac{1}{n}\mu_n$. Suppose the following conditions are satisfied:

Condition 1. *Let $D \subseteq \mathbb{C}$. There is a subsequence p_{n_k} of p_n, where $n_k \geqslant i_k$ for some strictly increasing sequence i_k of indices, such that for all $z \in D$,*

$$p_{n_k}(z) \to \mathcal{L}(\log|z|) \quad \textit{almost surely as } k \to \infty$$

where $\mathcal{L} : D \to \mathbb{R}$ only depends on $|z|$.

Condition 2. *Let $D \subseteq \mathbb{C}$ and let p_{n_k} be any subsequence of p_n such that for Lebesgue-a.e. $z \in D$, whenever*

$$p_{n_k}(z) \to \mathcal{L}(\log|z|) \quad \textit{almost surely as } k \to \infty$$

we also have that

$$\Delta p_{n_k}(z) \to \Delta\mathcal{L}(\log|z|) \quad \textit{almost surely as } k \to \infty$$

so that in particular

$$\frac{1}{n_k}\mu_{n_k} \to \frac{1}{2\pi}\Delta\mathcal{L}(\log|z|) \quad \textit{almost surely as } k \to \infty.$$

By Lemma 1, if conditions 1 and 2 are satisfied, then $\frac{1}{n}\mu_n \to \frac{1}{2\pi}\Delta\mathcal{L}(\log|z|)$ in probability. The next two conditions together imply condition 1.

Condition 3. *Let $D \subseteq \mathbb{C}$. For every $\varepsilon > 0$, there exists an almost surely finite random variable $M(\varepsilon)$ uniform in z such that for sufficiently large n,*

$$|P_n(z)| \leqslant M(\varepsilon)e^{n(\mathcal{L}(\log|z|)+\varepsilon)} \tag{6}$$

where $\mathcal{L} : D \to \mathbb{R}$ only depends on $|z|$, and not on n or ε.

Condition 4. *Let $D \subseteq \mathbb{C}$. For every $\varepsilon > 0$, there exists a constant $C = C(\varepsilon, z)$, possibly* *depending on ε and z, such that*

$$\mathbb{P}\left(|P_n(z)| < e^{n(\mathcal{L}(\log|z|)-\varepsilon)}\right) \leqslant \frac{C}{g(n)}$$

where g is a function strictly increasing in n and $g(n) \to \infty$ as $n \to \infty$.

Suppose conditions 3 and 4 are satisfied. Let i_k be an increasing sequence of indices such that $\sum_k \frac{1}{g(i_k)} < \infty$ (such a sequence of i_k must exist since $g(n) \to \infty$). Now let n_k be an increasing sequence of indices satisfying $n_k \geqslant i_k$. Define A_{n_k} to be the event that $|P_{n_k}(z)| < e^{n_k(\mathcal{L}(\log|z|)-\varepsilon)}$. Then

$$\sum_k \mathbb{P}(A_{n_k}) \leqslant \sum_k \frac{C}{g(n_k)} \leqslant \sum_k \frac{C}{g(i_k)} < \infty.$$

By Borel-Cantelli, we have that for sufficiently large k,

$$-\varepsilon \leqslant \frac{1}{n_k}\log|P_{n_k}(z)| - \mathcal{L}(\log|z|) \qquad \text{almost surely.}$$

Further, condition 3 implies that for sufficiently large n,

$$\frac{1}{n}\log|P_n(z)| - \mathcal{L}(\log|z|) \leqslant 2\varepsilon \qquad \text{almost surely.}$$

Together the two inequalities imply that $p_{n_k} \to \mathcal{L}(\log|z|)$ almost surely as $k \to \infty$, so condition 1 is satisfied.

Remark 4. *The following observations are in order:*

- *In condition 3, we are simply bounding $|P_n(z)|$ above, and the only time that the randomness of the ξ_k's is used in this bound is in showing the existence of the almost surely finite random variable $M(\varepsilon)$ (later, the existence of $M(\varepsilon)$ will be ensured by the moment condition that we place on the ξ_k's).*
- *A useful observation is the factor of $e^{\varepsilon n}$ in the upper bound; in our later computations, we will see that this factor allows us to be a bit naive in our estimates, since we can*

afford to lose a factor of $e^{\varepsilon n}$.

- *We only need the upper bound to hold for a sequence n_k satisfying $n_k \geqslant i_k$, where i_k is determined by the function g in condition 4. In practice, however, it is not necessarily easier to relax condition 3 by only requiring that the bound hold for a subsequence n_k satisfying $n_k \geqslant i_k$, since i_k depends on g, which in turn does not necessarily have an immediate relationship to the upper bound.*

- *Condition 4 is the most obvious place in which the randomness of the ξ_k's is used. Specifically, it says that the values of $P_n(z)$ do not concentrate inside the circle of radius $e^{n(\mathcal{L}(\log|z|)-\varepsilon)}$. We will later introduce "anti-concentration inequalities" that will help us achieve this bound.*

2.4 Results from Probability

The following lemmas are the only results about randomness that are needed in the proof of the general result of Kabluchko and Zaporozhets. The first lemma uses the fact that if a sequence of i.i.d. random variables has a finite logarithmic moment, then it can be bounded above by a sequence of exponentials.

Lemma 2. *(Lemma 4.4 in [8]) Let $\xi_0, \xi_1, \ldots$ be a sequence of independent, identically distributed random variables and let $\varepsilon > 0$ be a fixed positive number. Define $S := \sup_{k=0,1,\ldots} \frac{|\xi_k|}{e^{\varepsilon k}}$. Then S is almost surely finite if and only if $\mathbb{E}[\log(1+|\xi_0|)] < \infty$.*

Proof of Lemma 2. For a non-negative random variable X, we have that

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X \geqslant x)dx.$$

Using lower and upper Riemann sums, we therefore have

$$\sum_{k=1}^{\infty} \mathbb{P}(X \geqslant k) \leqslant \mathbb{E}[X] \leqslant \sum_{k=0}^{\infty} \mathbb{P}(X \geqslant k).$$

Take $X = \frac{1}{\varepsilon}\log(1+|\xi_0|)$. Then by the above bounds for $\mathbb{E}[X]$,

$$\varepsilon \sum_{k=1}^{\infty} \mathbb{P}(|\xi_0| \geqslant e^{\varepsilon k} - 1) \leqslant \mathbb{E}[\log(1+|\xi_0|)] \leqslant \varepsilon \sum_{k=0}^{\infty} \mathbb{P}(|\xi_0| \geqslant e^{\varepsilon k} - 1)$$

If $\varepsilon \sum_{k=1}^{\infty} \mathbb{P}(|\xi_0| \geqslant e^{\varepsilon k} - 1) \leqslant \infty$, let A_k be the event that $|\xi_k| \geqslant e^{\varepsilon k} - 1$. Since by hypothesis $\sum_{k=1}^{\infty} \mathbb{P}(A_k) < \infty$, by Borel-Cantelli almost surely only finitely many of the A_k occur. So there exists K such that $|\xi_k| \leqslant e^{\varepsilon k} - 1$ for all $k \geqslant K$, and so $\frac{|\xi_k|}{e^{\varepsilon k}} \leqslant 1$. Then S is almost surely finite by accounting for the finitely many $k \leqslant K$. The other direction is similar. $\square$

In particular, this result will allow us to absorb the ξ_k in $P_n(z)$ into the factor of $M(\varepsilon)e^{n\varepsilon}$ in condition 3.

Let us introduce some terminology to discuss anti-concentration results, which we will use in order to prove condition 4.

Let X be a random variable in $\mathbb{R}$. Let $\mathbb{D}_r(x)$ be the d-dimensional sphere centered at $x \in \mathbb{R}^d$ of radius r. The concentration function of X with respect to spheres in $\mathbb{R}^d$ of radius r, denoted $Q_d(X; r)$, is the least upper bound on the probability that X belongs to any such sphere (over all possible centers)[4]. That is,

$$Q_d(X; r) = \sup_{x \in \mathbb{R}^d} \mathbb{P}(X \in \mathbb{D}_r(x)).$$

It will be useful to observe that if X and Y are independent random variables, then

$$Q_d(X + Y; r) \leqslant \min\{Q_d(X; r), Q_d(Y; r)\} \tag{7}$$

[4]. Note that in order to apply results that use this concentration function to our random polynomials, we will identify a complex-valued random variable X with its corresponding random vector in $\mathbb{R}^2$. We begin with the following lemma, extending a result of Littlewood and Offord, by Erdos. We will not use this inequality in the proof, but we introduce it as a

preface to a stronger inequality.

Lemma 3. *(See [3]) Let $\xi_1, \xi_2, \ldots$ be Rademacher random variables, that is, $\mathbb{P}(\xi_1 = 1) = \mathbb{P}(\xi_1 = -1) = \frac{1}{2}$. Let $x_1, x_2, \ldots, x_n \in \mathbb{C}$ satisfying $|x_i| \geqslant 1$. Define $S_n = \sum_{k=1}^{n} \xi_k x_k$. Then*

$$Q_2(S_n; r) \leqslant \frac{c(r)}{n^{1/2}}$$

where $c(r)$ is a constant only depending on r. In particular,

$$\mathbb{P}\left(|S_n| \leqslant r\right) \leqslant \frac{c(r)}{n^{1/2}}.$$

Recall from condition 4 that we seek a bound of the form

$$\mathbb{P}(|P_n(z)| \leqslant e^{n(\mathcal{L}(\log|z|)-\varepsilon)}) \leqslant \frac{C}{g(n)}.$$

We cannot apply Lemma 3 to P_n directly, since P_n may be infinite. However, suppose we write $P_n(z) = P_n^{(1)}(z) + P_n^{(2)}(z)$ where $P_n^{(1)}(z)$ is the sum of $\Theta(g(n)^2)$ terms from $P_n(z)$ and $P_n^{(2)}(z)$ is the sum of the remaining terms. By Proposition 7,

$$\mathcal{Q}(P_n(z); e^{n(\mathcal{L}(\log|z|)-\varepsilon)}) \leqslant \mathcal{Q}(P_n^{(1)}(z); e^{n(\mathcal{L}(\log|z|)-\varepsilon)})$$

and

$$|P_n^{(1)}(z)| \leqslant e^{n(\mathcal{L}(\log|z|)-\varepsilon)} \iff |P_n^{(1)}(z)| e^{-n(\mathcal{L}(\log|z|)-\varepsilon/2)} \leqslant e^{-n\varepsilon/2}$$

Note that $e^{-n\varepsilon/2} = \mathcal{O}(1)$. Now define $\alpha_{k,n} = f_{k,n} z^k e^{-n(\mathcal{L}(\log|z|)-\varepsilon/2)}$. If we could find $\Theta(g(n)^2)$ terms $\alpha_{k,n}$ such that $|\alpha_{k,n}| \geqslant 1$, and if we set $P_n^{(1)}(z)$ to be the sum of these terms, then by Lemma 3,

$$\mathbb{P}(|P_n(z)| \leqslant e^{n(I(\log|z|)-\varepsilon)}) = \mathcal{O}\left(\frac{1}{g(n)}\right)$$

which is what we seek.

Let $h(n) = g(n)^2$ be any function such that $h(n) \to \infty$ and h is strictly increasing in n. In summary, when the ξ_k are Rademacher random variables, it will suffice to demonstrate $\Theta(h(n))$ terms $f_{k,n}$ such that $|f_{k,n}z^k| \geqslant e^{n(\mathcal{L}(\log|z|)-\varepsilon/2)}$. Importantly, we have shown that we can reduce condition 4 to finding a lower bound on $f_{k,n}z^k$ for a finite number of terms. While our general result will have $h(n) = n$, it would suffice to find $h(n) = \log n$ terms, for example, with this lower bound, or something that approaches positive infinity even more slowly. Loosely speaking, then, we simply need to find enough $f_{k,n}$ such that $f_{k,n}z^k$ is large enough, while not needing to consider the remaining $f_{k,n}$. This will contrast with our approach to demonstrating condition 3, which will require bounding all of the terms of P_n.

The next result generalizes Lemma 3 to any random variables that are independent; with this result, we will be able to use the same exact strategy of finding $\Theta(h(n))$ terms $f_{k,n}z^k$ that are large enough.

Lemma 4 (Kolmogorov-Rogozin Inequality: Corollary 1 in [4]; Theorem 4.5 in [8])**.** *Let $X_1 \ldots, X_n$ be independent (not necessarily identical) random vectors in $\mathbb{R}^d$. Define $S_n = \sum_{k=1}^n X_k$. Then*

$$Q_d(S_n; r) \leqslant c(d) \left(\sum_{k=1}^{n} (1 - Q_d(X_k; r)) \right)^{-1/2}$$

where $c(d)$ is a constant depending on d. In particular, if in addition, the X_k are identically distributed and non-degenerate,

$$Q_d(S_n; r) \leqslant c(d, r) n^{-1/2}$$

where $c(d, r)$ is a constant depending on d and r.

Note that the second inequality follows from taking $1 - Q_d(X_k; r)$ to be the same constant for all k (by the identical condition), which is greater than 0 (by the non-degeneracy condition). We will arrive at the same conclusion as in the application of Lemma 3. For, again, we have that

$$\mathbb{P}\left(|P_n^{(1)}(z) \leqslant e^{n(\mathcal{L}(\log|z|)-\varepsilon)}\right) \leqslant Q_2(|P_n^{(1)}(z)|e^{-n(\mathcal{L}(\log|z|)-\varepsilon/2)}; e^{-n\varepsilon/2})$$

In applying Lemma 4 to the right-hand side, we need to find an upper bound for

$$1 - Q_2(f_{k,n} z^k e^{-n(\mathcal{L}(\log|z|)-\varepsilon/2)} \xi_k; e^{-n\varepsilon/2}).$$

It suffices to use

$$1 - Q_2(\xi_k; e^{-n\varepsilon/2})$$

when

$$|\alpha_{k,n}| = |f_{k,n} z^k e^{-n(\mathcal{L}(\log|z|)-\varepsilon/2)}| \geqslant 1$$

which is precisely the condition we sought before.

2.5 Reverse Engineering the Argument

In this section, we will give a brief sketch of how the conditions stated in the general result of Kabluchko and Zaporozhets were reverse engineered, and in the next section we precisely state these conditions and the general result, and complete the proof, using the methods outlined in the previous sections, and also filling in some of the technical details omitted from the paper. We established in the previous section that we want to find a function $\mathcal{L}$, independent of n, such that for all $\varepsilon > 0$, the following two conditions simultaneously hold:

Criterion 1. $|P_n(z)| \leqslant M(\varepsilon) e^{n(\mathcal{L}(\log|z|)+\varepsilon)}$ *where* $M(\varepsilon)$ *is almost surely finite.*

Criterion 2. *There exists a strictly increasing function* h *on* $\mathbb{N}$ *where* $h(n) \to \infty$ *such that* $|f_{k,n} z^k| \geqslant e^{n(\mathcal{L}(\log|z|)-\varepsilon)}$ *for* $\Theta(h(n))$ *values of* k*.*

To address criterion 1, suppose first that P_n is a finite polynomial with cn terms for some constant $c > 0$. Fix $\varepsilon > 0$. There exists an almost surely finite random variable $M(\varepsilon)$ such that $|\xi_k| \leqslant M(\varepsilon) e^{\varepsilon k/c} \leqslant M(\varepsilon) e^{\varepsilon n}$ by Lemma 2 (thus note that the following bound would be unaffected if the ξ_k were deterministic coefficients growing subexponentially, but we will require the randomness in proving the lower bound in condition 4). Thus, we seek an $\mathcal{L}$ for which

$$\sum_{k=0}^{cn} |f_{k,n} z^k| \leqslant e^{n(\mathcal{L}(\log |z|)+\varepsilon)}.$$

Again taking advantage of the fact that we may lose a factor of $e^{\varepsilon n}$, it suffices to find $\mathcal{L}$ for which

$$\max_{k=0,\ldots,cn} |f_{k,n} z^k| = e^{n(\mathcal{L}(\log |z|+\varepsilon)}.$$

The use of $=$ here instead of $\leqslant$ is done in anticipation of satisfying criterion 2 listed above (a $\leqslant$ inequality would still satisfy the criterion 1). That is, in anticipation of satisfying criterion 2, we seek a tight upper bound on $|f_{k,n} z^k|$. Rewriting the above equation, we seek

$$\max_{k=0,\ldots,cn} \log |f_{k,n}|^{1/n} + \frac{k}{n} \log |z| = \mathcal{L}(\log |z|) + \varepsilon.$$

The challenge is that $\mathcal{L}$ should not depend on n. Other properties are desired as well, such as sufficient smoothness of $\mathcal{L}$ in order for condition 2 to be satisfied. The strategy will be to approximate $f_{k,n}^{1/n}$ with a continuous function f (if possible), so that we can essentially replace $f_{k,n}^{1/n}$ with $f(\frac{k}{n})$ in the equation above. Importantly, the approximation of $f_{k,n}^{1/n}$ must only depend on the ratio $\frac{k}{n}$. We would then be able to rewrite the above condition as:

$$\max_{t\in[0,c]} \log f(t) + t \log |z| = \mathcal{L}(\log |z|)$$

and, crucially, $\mathcal{L}$ would not depend on n. Also note that continuity guarantees that the maximum exists on the interval $[0, c]$.

By requiring the upper bound on $\left|f_{k,n} z^k\right|$ to be tight and f to be continuous, we have set ourselves up to satisfy criterion 2, the lower bound on $|f_{k,n} z^k|$ for $\Theta(h(n))$ values of k, with little effort. Define

$$t^* = \arg \max_{t\in[0,c]} \log f(t) + t \log |z|$$

so that by definition

$$\log f(t^*) + t^* \log |z| = \mathcal{L}(\log |z|).$$

This is the point at which we directly exploit the continuity of f. Specifically, we know that there must exist an interval $T \ni t^*$ such that for all $t \in T$,

$$\log f(t) + t \log |z| \geqslant \mathcal{L}(\log |z|) - \varepsilon.$$

While T may be small in length, for large enough n, we can certainly find $\Theta(n)$ values of k for which $\frac{k}{n} \in T$. Now using the fact that $f(\frac{k}{n})$ approximates $f_{k,n}^{1/n}$, we know that for $\Theta(n)$ values of k,

$$\log f_{k,n}^{1/n} + \frac{k}{n} \log |z| \geqslant \mathcal{L}(\log |z|) - \varepsilon$$

$$\iff |f_{k,n} z^k| \geqslant e^{n(\mathcal{L}(\log |z|) - \varepsilon)}$$

as desired.

Above, we assumed that P_n was a finite polynomial with $\Theta(n)$ terms. It turns out that it will not be too difficult to extend the argument to infinite polynomials, since as k grows large, we will be able to exploit the fact that the terms of an infinite polynomial must decay to 0 as $k \to \infty$ in order for $P_n(z)$ to converge at z. The decay we require of the terms $f_{k,n} z^k \xi_k$ will determine the radius R such that the upper and lower bounds (the above two conditions) hold for $z \in \mathbb{D}_R$.

Fixing $\varepsilon > 0$, we want to write $P_n(z)$ as

$$P_n(z) = \sum_{k=0}^{A(\varepsilon)n} f_{k,n} z^k \xi_k + \sum_{A(\varepsilon)n}^{\infty} f_{k,n} z^k \xi_k$$

where $A(\varepsilon)$ depends only on ε (and not on n). For the left-hand sum, we will use the same argument as the one above on the finite sum from $k = 0$ to $k = cn$. For the right-hand sum, we would like $A(\varepsilon)$ to have been chosen large enough so that the right-hand side is bounded above by $M(\varepsilon) e^{n(\mathcal{L} \log |z|) + \varepsilon)}$. We will again use the subexponential property of the ξ_k although this time we will have infinitely many; we will absorb $e^{\varepsilon k}$ into a geometric series with terms $\left(f_{k,n}^{1/k} z e^{\varepsilon}\right)^k$ which will converge if $|z| < \left|f_{k,n}^{-1/k}\right| e^{-\varepsilon}$. This suggests that we would like to choose R as a lower bound on $f_{k,n}^{-1/k}$ that is independent of n. In other words, we would like

$$R \leqslant \liminf_{n,k/n\to\infty} f_{k,n}^{-1/k}.$$

2.6 Statement of General Result of Kabluchko and Zaporozhets

Theorem 1. *(Theorem 2.8 in [8]) Let $\xi_0, \xi_1, \ldots$ be non-degenerate, independent and identically distributed complex-valued random variables with finite logarithmic moment, i.e., $\mathbb{E}[\log(1 + |\xi_0|)] < \infty$. For $n \in \mathbb{N}$, define the infinite random polynomial $P_n : \mathbb{C} \to \mathbb{C}$ by*

$$P_n(z) = \sum_{k=0}^{\infty} f_{k,n} z^k \xi_k$$

where $f_{k,n} \in \mathbb{C}$ are deterministic coefficients. Let $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ be a function and $\beta \in (0, \infty]$ a positive (extended) real number satisfying the following conditions:

1. ***Positivity:*** *$f(t) > 0$ for $0 \leqslant t < \beta$ and $f(t) = 0$ when $t > \beta$.*

2. ***Continuity:*** *f is continuous on $[0, \beta)$ and left-continuous at β.*

3. ***Coefficient Approximation:*** *For any constant $c > 0$,*

$$\lim_{n\to\infty} \sup_{k\in[0,cn]} \left| |f_{k,n}|^{1/n} - f\left(\frac{k}{n}\right) \right| = 0.$$

4. ***Coefficient Decay:*** *Define $R := \liminf_{t\to\infty} f(t)^{-1/t}$.*

   - *Holding n fixed, $R \leqslant \liminf_{k\to\infty} |f_{k,n}|^{-1/k}$.*

   - *Additionally, $R \leqslant \liminf_{n,k/n\to\infty} |f_{k,n}|^{-1/k}$.*

Define $\mathcal{L} : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ by

$$\mathcal{L}(s) := \sup_{t\geqslant 0}(st + \log f(t)).$$

Define μ to be the rotationally invariant measure supported on $\mathbb{D}_R$ defined by

$$\mu(\mathbb{D}_r) = \mathcal{L}'(\log r), \quad r \in (0, R).$$

Let μ_n be the measure counting the zeros of P_n. Then

$$\frac{1}{n}\mu_n \to \mu \qquad \textit{in probability.}$$

Remark 5. *In condition 4, $\liminf_{n,k/n\to\infty} |f_{k,n}|^{-1/k} \geqslant R$ means the following: For all $\varepsilon > 0$, there exists a constant $N = N(\varepsilon)$ such that for all $n \geqslant N$ and all $k/n \geqslant N$, $|f_{k,n}|^{-1/k} \geqslant R - \varepsilon$.*

Before proving the theorem, we need to show that $\mathcal{L}'$ is well-defined, and that the series P_n converges almost surely for $|z| < R$. Both will follow from the coefficient decay condition.

Proposition 3. *$\mathcal{L}(s) < \infty$ when $s < \log R$ and $\mathcal{L}(s) = \infty$ when $s > \log R$.*

Proof of Proposition 3. Fix any $s \in \mathbb{R}$. When $s < \log R$, there exists t_0 such that for all $t \geqslant t_0$, $f(t)^{-1/t} \geqslant e^s$, which implies that $\log f(t) + st \leqslant 0$ for $t \geqslant t_0$. By the positivity and continuity conditions, $st + \log f(t)$ is bounded above on $[0, t_0]$. Therefore $st + \log f(t)$ is bounded above by some finite constant that does not depend on t. Also, $\mathcal{L}(s) \geqslant \log f(0) > -\infty$. So $\mathcal{L}(s) = \sup_{t\geqslant 0} st + \log f(t)$ is finite. When $s > \log R$, there exists $\alpha > 0$ such that $e^s > e^\alpha R$. Since $R = \liminf_{t\to\infty} f(t)^{-1/t}$, there exists a sequence of indices $t_n \to \infty$ such that $e^s > f(t_n)^{-1/t_n}$. In turn $st_n + \log f(t_n) > \alpha t_n$, and since $\alpha t_n \to \infty$, $\mathcal{L}(s) = +\infty$.

□

Proposition 4. *$\mathcal{L}$ is a convex function.*

Proof of Proposition 4. Let $\lambda \in [0, 1]$ and $s_1, s_2 \in \mathbb{R}$. Then

$$\mathcal{L}(\lambda s_1 + (1-\lambda)s_2) = \sup_{t\geqslant 0}(\lambda s_1 t + (1-\lambda)s_2 t + \lambda \log f(t) + (1-\lambda)\log f(t))$$

$$\leqslant \sup_{t\geqslant 0} \lambda s_1 t + \lambda \log f(t) + \sup_{t\geqslant 0}(1-\lambda)s_2 t + (1-\lambda)\log f(t)$$

$$\leqslant \lambda \sup_{t\geqslant 0}(s_1 t + \log f(t)) + (1-\lambda)\sup_{t\geqslant 0}(s_2 t + \log f(t))$$

$$= \lambda\mathcal{L}(s_1) + (1-\lambda)\mathcal{L}(s_2)$$

□

Together Propositions 3 and 4 imply that $\mathcal{L}'$ is well defined on $(-\infty, \log R)$.

Proposition 5. *The series $P_n(z)$ converges for $|z| < R$.*

Proof of Proposition 5. Fix $\varepsilon > 0$. Suppose first that $R = \infty$. Then $\lim_{k\to\infty} |f_{k,n}|^{-1/k} = \infty$. For sufficiently large k, $|f_{k,n}|^{-1/k} \geqslant \frac{1}{\varepsilon}$. Then $|P_n(z)| \leqslant M(\varepsilon)\sum_{k=0}^{\infty}(e^{\varepsilon}\varepsilon|z|)^k$ for some almost surely finite random variable $M(\varepsilon)$, which converges when $|z| < \frac{1}{e^{\varepsilon}\varepsilon}$. Taking $\varepsilon \to 0$, we have that P_n converges on all of $\mathbb{C}$.

Now suppose $R < \infty$. Since $R \leqslant \liminf_{k\to\infty} |f_{k,n}|^{-1/k}$, we have that for sufficiently large k, $|f_{k,n}| < (R-\varepsilon)^{-k}$. Then $|P_n(z)| \leqslant M(\varepsilon)\sum_{k=0}^{\infty}\left(\frac{|z|e^{\varepsilon}}{R-\varepsilon}\right)^k$, which converges when $|z| < \frac{R-\varepsilon}{e^{\varepsilon}}$. Taking $\varepsilon \to 0$, we have that P_n converges on $\mathbb{D}_R$. □

Concluding the proof of Theorem 1: First, we will show conditions 3 and 4 are satisfied, which implies condition 1. Then we will show that condition 1 implies condition 2, and so we will have

$$\frac{1}{n}\mu_n \to \mu \quad \text{in probability}$$

where the density of μ is given by $\frac{1}{2\pi}\Delta\mathcal{L}(\log|z|)$. Note that since the density is a function of $|z|$, μ is rotationally invariant. We will therefore be able to specify the measure μ by $\mu(\mathbb{D}_r)$ for $r \in (0, R)$. In polar coordinates, the Laplacian of a function f is given by:

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}.$$

Taking $f = \mathcal{L}(\log|z|)$, the second summand disappears because f does not depend on θ, so we have that for any $r' \in (0, R)$,

$$\mu(\mathbb{D}_{r'}) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{r'}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\mathcal{L}(\log r)\right) r dr d\theta$$

$$= \int_0^{r'} \frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} \mathcal{L}(\log r) \right) dr$$

$$= \mathcal{L}'(\log r')$$

as desired. It also follows that if $\mathcal{L}''$ exists, then the density ρ of μ is given by

$$\rho(z) = \frac{\mathcal{L}''(\log |z|)}{2\pi |z|^2}.$$

2.7 Proof of Conditions 3 and 4

Proposition 6. *(Section 4.3 in [8]) Let $\varepsilon > 0$. If $R < \infty$, we take $z \in \mathbb{D}_{e^{-2\varepsilon}R}\backslash\{0\}$. If $R = \infty$, we take $z \in \mathbb{D}_{1/\varepsilon}\backslash\{0\}$. Then there exists a positive integer $N = N(\varepsilon)$ and an almost surely finite random variable $M(\varepsilon)$, both of which do not depend on z, such that for all $n \geqslant N(\varepsilon)$,*

$$|P_n(z)| \leqslant M(\varepsilon) e^{n(\mathcal{L}(\log |z|)+\varepsilon)}.$$

Note that by having $M = M(\varepsilon)$ and $N = N(\varepsilon)$ hold uniformly in z, we may take $\epsilon \to 0$ so that the above bound holds for all $z \in D_R\backslash\{0\}$.

Proof of Proposition 6. We first identify $A = A(\varepsilon)$ such that we can split the sum $P_n(z)$ at $k = An$. On the tail end, we will use the decay of the coefficients $f_{k,n}$, while on the finite sum we will use the coefficient approximation condition, which will directly apply for any $A > 0$. Thus, $A = A(\varepsilon)$ can be chosen solely in view of the tail end of the series satisfying the upper bound.

By the decay condition, there exists $A = A(\varepsilon)$ such that for all $n \geqslant A$ and for all $k \geqslant An$, $|f_{k,n}|^{-1/k} \geqslant e^{-\varepsilon}R$ when $R < \infty$ and $|f_{k,n}|^{-1/k} \geqslant e^{\varepsilon}$ when $R = \infty$. By our choice of z, we then have that for all $k \geqslant An$ and for all $n \geqslant A$, $|f_{k,n}| \leqslant \left(e^{-\varepsilon}|z|e^{2\varepsilon}\right)^{-k}$ when $R < \infty$ and $|f_{k,n}| \leqslant e^{-\varepsilon k}$ when $R = \infty$.

For $R < \infty$, there exist almost surely finite random variables $M'(\varepsilon)$ and $M''(\varepsilon)$ such that

$$\left|\sum_{k=An}^{\infty} f_{k,n} z^k \xi_k\right| \leqslant M'(\varepsilon) \sum_{k=An}^{\infty} e^{\varepsilon k/2} |f_{k,n}||z|^k \leqslant M'(\varepsilon) \sum_{k=An}^{\infty} e^{-\varepsilon k/2} = M''(\varepsilon) e^{-A'n}$$

where in the last equality we have used the sum for a geometric series, absorbed a constant depending only on ε into $M'(\varepsilon)$ to get $M''(\varepsilon)$, and taken $A' = \frac{\varepsilon}{2} A$.

For $R = \infty$, there exist almost surely finite random variables $M'(\varepsilon)$ and $M''(\varepsilon)$ such that

$$\left|\sum_{k=An}^{\infty} f_{k,n} z^k \xi_k\right| \leqslant M'(\varepsilon) \sum_{k=An}^{\infty} e^{\frac{\varepsilon k}{2}} e^{-\varepsilon k} \varepsilon^{-k} = M'(\varepsilon) \sum_{k=An}^{\infty} \left(\frac{e^{-\varepsilon/2}}{\varepsilon}\right)^k = M''(\varepsilon) e^{-A'n}$$

where again we have used the sum of a geometric series in the last equality and taken $A' = \frac{\varepsilon}{2} A$.

Finally, assume $A > 0$ was chosen large enough so that $A' > -\log f(0)$ and thus $-A' < \mathcal{L}(\log|z|)$. Then in both cases of R,

$$\left|\sum_{k=An}^{\infty} f_{k,n} z^k \xi_k\right| \leqslant M''(\varepsilon) e^{-A'n} \leqslant M''(\varepsilon) e^{n\mathcal{L}(\log|z|)}.$$

Now we handle the finite part of the sum. By the coefficient approximation condition, for sufficiently large n,

$$|f_{k,n}| < (f(k/n) + \delta)^n \quad \text{for all } k \in [0, An].$$

We then repeat the strategy in the heuristic argument of bounding the finite sum by bounding the largest term via the continuous function f, and then taking advantage of the fact that we are allowed to lose a factor of $e^{\varepsilon n}$ in using this bound for all An terms. There exists an almost surely finite random variable $S(\varepsilon)$ and a constant $c(\varepsilon)$ depending only on ε such that for sufficiently large n,

$$\left|\sum_{k=0}^{An} |f_{k,n} z^k \xi_k\right| \leqslant \sum_{k=0}^{An} S(\varepsilon) e^{\varepsilon k/A} \left(|z|^{k/n} f(k/n) + |z|^{k/n} \delta\right)^n$$

$$\leqslant S(\varepsilon)e^{\varepsilon n}(An+1)\left(e^{\mathcal{L}(\log|z|)}+c(\varepsilon)\delta\right)^n$$

$$\leqslant S(\varepsilon)e^{2\varepsilon n}\left(e^{\mathcal{L}(\log|z|)}+c(\varepsilon)\delta\right)^n$$

where the bound $|z|^{k/n} < c(\varepsilon)$ comes from the fact that $|z|$ is bounded as a function of ε and $k/n \in [0, A]$ where A depends only on ε. Further, assume we have chosen δ small enough so that

$$\delta c(\varepsilon) \leqslant \varepsilon e^{-A'} \leqslant \varepsilon e^{\mathcal{L}(\log|z|)}$$

(note that δ will depend only on ε). Continuing the bound,

$$\left|\sum_{k=0}^{An} |f_{k,n} z^k \xi_k\right| \leqslant S(\varepsilon)e^{2\varepsilon n}e^{n\mathcal{L}(\log|z|)}(1+\varepsilon)^n \leqslant S(\varepsilon)e^{n(\mathcal{L}(\log|z|)+3\varepsilon)}$$

where we have used the fact that $(1+\varepsilon)^n \leqslant e^{\varepsilon n}$. Combining the finite sum and the tail end of the series gives the desired bound:

$$\left|\sum_{k=0}^{An} f_{k,n} z^k \xi_k\right| \leqslant M(\varepsilon)e^{n(\mathcal{L}(\log|z|)+3\varepsilon)}$$

where $M(\varepsilon) = M''(\varepsilon) + S(\varepsilon)$ is an almost surely finite random variable. ☐

Proposition 7. *(See also Section 4.4 in [8]) Fix $\varepsilon > 0$ and $z \in \mathbb{D}_R\backslash\{0\}$. There exist $\Theta(n)$ values of k such that*

$$|f_{k,n} z^k| \geqslant e^{n(\mathcal{L}(\log|z|)-\varepsilon)}$$

By the argument in Section 2.4, Proposition 7 implies condition 4, as desired.

Note that we do not require the $\Theta(n)$ values of k for which the above bound holds to be independent of z. This is because we only need $p_{n_k} \to \mathcal{L}(\log|z|)$ in condition 1 to converge pointwise. We required the constants in Proposition 6 to hold uniformly in z because the bound on $|P_n(z)|$ given in this proposition only applied for z in a disc whose radius was a function of R and ε (here, our bound applies for any $z \in \mathbb{D}_R\backslash\{0\}$, with no dependence on ε).

Proof of Proposition 7. Since $\mathcal{L}(\log|z|)$ is chosen to be a tight upper bound on $t\log|z|$ +

$\log f(t)$, there exists t' such that

$$t' \log |z| + \log f(t') > \mathcal{L}(\log |z| - \varepsilon)$$

where $t' \in [0, \beta]$ because $\mathcal{L}(\log |z|)$ is finite for $|z| < R$ and $\log f(t) = -\infty$ for $t > \beta$. By the continuity condition, there exists a finite (open or closed) interval of nonzero length $T \ni t'$, possibly with its right endpoint as β, such that

$$t \log |z| + \log f(t) > \mathcal{L}(\log |z|) - \varepsilon \quad \text{for all } t \in T.$$

We pause to note that this is the most salient use of continuity in the proof.
Now fixing n, we consider all possible values of k such that $\frac{k}{n} \in T$. For large enough n, there are certainly $\Theta(n)$ values of k for which $\frac{k}{n} \in T$. We rewrite the above inequality as:

$$\frac{k}{n} \log |z| + \log f(k/n) > \mathcal{L}(\log |z|) - \varepsilon.$$

Further, since T is a finite interval, each such k belongs to $[0, cn]$ for some constant c. We may now apply the coefficient approximation condition and conclude that for large enough n,

$$|f_{k,n} z^k| \geqslant e^{n(\mathcal{L}(\log |z|) - 2\varepsilon)}$$

for $\Theta(n)$ values of k, where we have accumualated an additional factor of $e^{-\varepsilon n}$ in using the approximation. □

It is instructive to observe where each of the conditions in the statement of the result is used in the upper and lower bounds (Propositions 6 and 7), in hopes of using tools from this proof to derive the limiting distributions of random polynomials even when they fail to satisfy these conditions, which we will explore in a later section.

- The **decay condition** is used in showing $\mathcal{L}$ is well-defined on $(-\infty, \log R)$ (uses the definition of R); the series converges almost surely when $|z| < R$ (uses the first bullet); and in bounding the tail end of the series (uses the second bullet)

- The **approximation condition** is a means of finding $\mathcal{L}$ which *does not depend on* n.

- Proposition 7 is the most important place in which the **continuity condition** is used in the proof of the general result, and moreover one of the motivations behind including this condition. In Proposition 6, we never appeal directly to the fact that f is continuous; in fact, we could imagine that in finding an upper bound on a finite polynomial $P_n(z)$, we could apply the same strategy of finding the maximum term of $P_n(z)$, but doing so without maximizing via a continuous function that approximates the coefficients $f_{k,n}^{1/n}$ *as a function of the ratio* k/n.

- There are weaker conditions that would imply Proposition 7 that do not require comparing the coefficients $f_{k,n}$ to a continuous function f. Here is one possibility. Let $\alpha_{k,n} = f_{k,n}z^k$ and let k^* be an index satisfying

$$|\alpha_{k^*,n}| > e^{n(\mathcal{L}(\log|z|)-\varepsilon)}.$$

Ultimately we need to show that $\alpha_{k,n}$ does not decay too quickly near k^*. For instance, it would suffice to show that there exists a constant $0 < c < 1$ such that for all $k \in [k^*, k^* + \varepsilon' n]$ for some ε', $|\alpha_{k+1}| \geqslant c|\alpha_k|$. Then

$$|\alpha_{k^*+\varepsilon' n,n}| \geqslant c^{\varepsilon' n}|\alpha_{k^*,n}| \geqslant e^{\varepsilon' n \log c}e^{n(\mathcal{L}(log|z|)-\varepsilon)} \geqslant e^{n(\mathcal{L}(\log|z|)-\varepsilon'')}$$

It would even suffice to take $k \in [k^*, k^* + \varepsilon' h(n)]$ where $h(n) \to \infty$ is strictly increasing and grows more slowly than n, such as $h(n) = \log n$ or $h(n) = \sqrt{n}$. This is yet another instance in which we can take advantage of the ability to lose a factor of $e^{\varepsilon n}$ in our estimates.

2.8 Condition 2 Holds (see also Proposition 4.3 in [8])

Recall that we have defined $p_n = \frac{1}{n}\log|P_n(z)|$. Now let p_{n_k} be a subsequence of p_n such that for Lebesgue-a.e. $z \in \mathbb{D}_R \backslash \{0\}$,

$$p_{n_k} \to \mathcal{L}(\log|z|) \quad \text{almost surely as } k \to \infty. \tag{8}$$

We want to show that

$$\Delta p_{n_k} \to \Delta\mathcal{L}(\log|z|) \quad \text{almost surely as } k \to \infty \text{ in } \mathcal{D}'(\mathbb{D}_R) \tag{9}$$

where (9) is in the sense of distributions. First we will show that (8) holds in $\mathcal{D}'(\mathbb{D}_R)$, and since the Laplacian is a continuous operator, we will then be able to conclude that (9) holds in $\mathcal{D}'(\mathbb{D}_R)$. First, we observe that there exists a set $E_0 \subset \Omega$ of measure zero such that for all $\omega \notin E_0$,

$$\lim_{k\to\infty} p_{n_k}(z;\omega) \to \mathcal{L}(\log|z|) \quad \text{for Lebesgue-a.e. } z \in \mathbb{D}_R. \tag{10}$$

This follows from the (8) and an application of Fubini's theorem (we have switched the order of z and ω given in (8), since in (8) Lebesgue-a.e. z is fixed and the limit holds for $\mathbb{P}$-a.e. ω, whereas in (10) $\mathbb{P}$-a.e. ω is fixed and the limit holds for Lebesgue-a.e. z). Now we would like to show that there exists a measurable set $E \subset \Omega$ of measure zero, such that for $\omega \notin E$, and for all $\phi \in C_c^\infty(\mathbb{D}_R)$,

$$\lim_{k\to\infty} \int_{\mathbb{D}_R} \phi(z) p_{n_k}(z;\omega)dz = \int_{\mathbb{D}_R} \phi(z)\mathcal{L}(\log|z|)dz$$

since this is the definition of convergence in $\mathcal{D}'(\mathbb{D}_R)$. In other words, we would like to justify the interchange of the limit and the integral and then appeal to the almost sure limit (10). Recall that ϕ is compactly supported, and let $K \subset \mathbb{D}_R$ be the support of ϕ. Equivalently, we need to show that

$$\lim_{k\to\infty} \int_K \phi(z) p_{n_k}(z;\omega)dz = \int_K \phi(z)\mathcal{L}(\log|z|)dz.$$

By the bounded convergence theorem, it will suffice to show that $\phi(z)p_{n_k}(z)$ is uniformly bounded on K. Clearly ϕ is uniformly bounded on K, since ϕ is continuous and K is compact. To prove that $p_{n_k}(z)$ is uniformly bounded on K, recall from Proposition 6 that

$$|P_n(z)| \leqslant M(\varepsilon)e^{n(\mathcal{L}(\log|z|)+\varepsilon)}$$

$$p_n(z) = \frac{1}{n}\log|P_n(z)| \leqslant \frac{1}{n}\log M(\varepsilon) + \mathcal{L}(\log|z|) + \varepsilon$$

where $M(\varepsilon)$ is almost surely finite. Therefore, we need now only prove that $\mathcal{L}(\log|z|)$ is uniformly bounded on K. Since K is compact, there exists R' such that $0 < |z| \leqslant R' < R$. So we need to show that $\mathcal{L}$ is uniformly bounded for $s \in (-\infty, \log R']$. We employ essentially the same argument as in Proposition 3, but this time we need t_0 to hold uniformly. Note first that $\mathcal{L}$ is bounded below, since $\mathcal{L}(s) \geqslant \log f(0)$ for any $s < \log R$. Secondly, by the decay condition, since $\log R' < \log R$, there exists t_0 such that for all $t \geqslant t_0$, $f(t)^{-1/t} \geqslant e^{\log R'} \geqslant e^s$. In turn $st+\log f(t) \leqslant 0$, so that $\mathcal{L}(s) \leqslant \max\{0, \sup_{t\in[0,t_0]} st+\log f(t)\}$, which is finite because f is continuous and $[0, t_0]$ is compact. Since t_0 was chosen uniformly for all $s \leqslant \log R'$, we can conclude that almost surely p_{n_k} is uniformly bounded above on K. Therefore, there exists a set E of measure zero (containing E_0) such that for $\omega \notin E$,

$$\lim_{k\to\infty} p_{n_k}(z;\omega) = \mathcal{L}(\log|z|) \text{ in } \mathcal{D}'(\mathbb{D}_R).$$

Applying Δ to both sides (which can be done because Δ is continuous), we conclude that for every $\omega \notin E$,

$$\Delta p_{n_k}(z;\omega) \to \Delta\mathcal{L}(\log|z|) \text{ in } \mathcal{D}'(\mathbb{D}_R),$$

which is what we sought to prove. (We acknowledge Professor Miklos Racz for assisting with this argument.)

3 Related Result: Bloom and Dauvergne

In this section, we briefly discuss a recent related result by Bloom and Dauvergne (Theorem 6.5 in [1]), which gives conditions under which the normalized counting measures of *finite* polynomials with n terms converge to a limiting distribution *almost surely*. The conditions stated in the result of Kabluchko and Zaporozhets applied to polynomials of n terms are a special case of the conditions in Bloom and Dauvergne, therefore giving a stronger result for this particular case. While the overall methods used in the proof of this result are the same as the ones shown above, that is, finding upper and lower bounds for $\frac{1}{n}\log|P_n(z)|$, the main difference occurs in the lower bound. Recall that in proving the lower bound, it was sufficient to find a function g, increasing to ∞, and a subsubsequence n_k of $\mathbb{N}$ with indices large enough, so that the series $\sum_k \frac{1}{g(n_k)}$ converged. Such a function g was found through application of the Kolmogorov-Rogozin inequality. Importantly, to prove convergence in probability of the normalized counting measures, it was sufficient by Lemma 1 to find the almost surely converging *subsubsequence* $\frac{1}{n_k}\log|P_{n_k}(z)| = p_{n_k}(z)$ (condition 1), which was implied by the existence of the function g along with the upper bound, which held for all sufficiently large n (rather than merely for a specific subsubsequence, as in the lower bound). To prove almost sure convergence of the normalized counting measures, however, the lower bound will have to apply for all n sufficiently large, rather than only for a subsubsequence. This implies that a stronger anti-concentration result is needed. Bloom and Dauvergne use a small ball probability theorem of Nguyen and Vu [10]. Note that whereas the Kolmogorov-Rogozin inequality (Lemma 4) held for independent random variables, the small ball probability theorem will require that the random variables be identical in addition. Below, we state the version of the theorem used in Bloom and Dauvergne.

Theorem 2. *(Theorem 6.1 in [1]) Let* $0 < \varepsilon < 1$ *and* $C > 0$ *be constants, and* $r > 0$ *a parameter possibly depending on* n. *Let* $\xi_0, \ldots, \xi_n$ *be i.i.d. non-degenerate complex-valued random variables. Define* $S_n = \sum_{k=0}^n a_k\xi_k$, *where the coefficients* a_k *satisfy* $\sum_{k=0}^n |a_k|^2 = 1$. *Further, assume that*

$$\mathcal{Q}(S_n; r) \geqslant n^{-C}.$$

Then there exists a constant D, depending only on ξ_0 and ε, such that for any $n' \in (n^\varepsilon, n)$, at least $n - n'$ of the coefficients a_k can be covered by a union of $\max(\frac{Dn^C}{\sqrt{n'}}, 1)$ balls of radius r.

Unlike in the Kolmogorov-Rogozin inequality, the above anti-concentration result requires that the concentration function $\mathcal{Q}(S_n; r)$ be sufficiently large, and places a normalization constraint on the coefficients in S_n (whereas the Kolmogorov-Rogozin inequality does not consider coefficients). Through the following lemma, Bloom and Dauvergne show how the small ball probability theorem can be used to obtain almost sure convergence. In applying this lemma, they will take $a_{k,n} = f_{k,n} z^k$.

Lemma 5. *(Lemma 6.2 in [1]) Let $S_n = \sum_{k=0}^n a_{k,n}\xi_k$, where $\xi_0, \ldots, \xi_n$ are i.i.d. non-degenerate complex-valued random variables. Denote the norm of the vector $(a_{0,n}, \ldots, a_{n,n})$ by $||a_n|| = \left(\sum_{k=0}^n |a_{k,n}|^2\right)^{1/2}$, and suppose that*

$$\lim_{n\to\infty} \frac{1}{2n} \log ||a_n||^2 = A.$$

Let $w_{k,n} = a_{k,n}/||a_n||$. Suppose that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for n large enough, the set

$$\mathcal{W}_n = \{w_{k,n} : 0 \leqslant k \leqslant n\}$$

cannot be contained in a union of $n^{2/3+\delta}$ balls of radius $e^{-\varepsilon n}$. Then

$$\liminf_{n\to\infty} \frac{1}{n} \log |S_n| \geqslant A \quad \textit{almost surely.}$$

Observe that the main condition in this lemma is that the (normalized) coefficients cannot be too close together. The lemma is proved by a direct appliation of Borel-Cantelli, specifically, by showing that for every $\varepsilon > 0$,

$$\sum_{n=0}^{\infty} \mathcal{Q}\left(\sum_{k=0}^{n} w_{k,n}\xi_k; e^{-\varepsilon n}\right) < \infty$$

which is the same strategy as in condition 4, only the sum is taken over all n, rather than over indices from a subsequence. Theorem 2 is applied by taking $C = 1 + \delta/2$ and showing that the series above is finite by appealing to the condition that for n large enough, the $w_{k,n}$ are far enough apart.

Now we state the almost sure convergence result of Bloom and Dauvergne.

Theorem 3. *(Theorems 6.3 and 6.5 in [1]) Let $P_n(z) = \sum_{k=0}^{n} f_{k,n} z^k \xi_k$, where the $f_{k,n} \in \mathbb{C}$ are deterministic coefficients and the ξ_k are i.i.d. non-degenerate complex-valued random variables with finite logarithmic moment, i.e., $\mathbb{E}[\log(1 + |\xi_0|)] < \infty$. Suppose the coefficients $f_{k,n}$ satisfy the following conditions.*

1. *There is a continuous function $V : \mathbb{C} \to \mathbb{R}$ such that for all $z \in \mathbb{C}$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{k=0}^{n} |f_{k,n}||z|^k \right) = V(z)$$

   where the convergence is locally uniform. Additionally, $V(z)$ is subharmonic and $V(z) - \log|z|$ is bounded as $|z| \to \infty$.

2. *For Lebesgue-a.e. $z \in \mathbb{C}$, the following holds: For any $\varepsilon > 0$, there exists $N = N(\varepsilon, z)$ sufficiently large and $\delta = \delta(\varepsilon, z)$ such that for all $n \geqslant N$,*

$$\left| \{ k \in [0, n] : |f_{k,n}||z|^k \geqslant e^{n(V(z)-\varepsilon)} \} \right| \geqslant n^{2/3+\delta}.$$

Then for Lebesgue-a.e. z,

$$\lim_{n\to\infty} \frac{1}{n} \log |P_n(z)| = V(z) \quad \textit{almost surely}$$

and

$$\frac{1}{n}\mu_n \to \frac{1}{2\pi} \Delta V(z) \quad \textit{almost surely.}$$

An important difference between the conditions in Theorem 3 and the conditions in Theorem 1 is the use of the continuous function f to approximate the coefficients and the use of the function $\mathcal{L}$ in Theorem 1. In Theorem 1, the upper bound was achieved by maximizing the terms $|f_{k,n}z^k|$ via the continuous function f and the definition of $\mathcal{L}$. Further, by construction of $\mathcal{L}$ and the continuity of f, we were able to find enough terms $|f_{k,n}z^k|$ that are large enough for the lower bound. The first condition in Theorem 3 provides a potentially more general way of arriving at the upper bound, without explicitly going through functions like f and $\mathcal{L}$ to maximize specific terms. The second condition likewise provides a potentially more general way of arriving at the lower bound, but with the added requirement that the number of terms $|f_{k,n}z^k|$ that are large enough is at least $n^{2/3+\delta}$ (rather than, say, $\Theta(h(n))$ such terms). As a consequence, it can be shown that when the conditions in Theorem 1 are satisfied for a *finite polynomial with n terms*, the conditions in Theorem 3 are also satisfied, and $V(z)$ of course is equal to $\mathcal{L}(\log|z|)$ (Special Cases of Theorem 6.5 in [1]).

4 Constructing Limiting Distributions

A nice consequence of Theorem 1 is that, given a rotationally invariant measure μ (with a mild restriction), we can construct a random polynomial of the form $P_n(z) = \sum_{k=0}^{\infty} f_{k,n}z^k\xi_k$ such that $\frac{1}{n}\mu_n \to \mu$ in probability on some disc of radius $R > 0$. This result of Kabluchko and Zaporozhets is stated below.

Theorem 4. *(Theorem 2.9 in [8]) Let μ be a rotationally invariant measure on $\mathbb{C}$ such that*

- *Letting $R := \sup\{r > 0 : \mu(\mathbb{D}_r) < \infty\}$, $R \in (0, \infty]$,*

$$\mu(\mathbb{C}\backslash\mathbb{D}_R) = 0.$$

- *$\int_0^{R'} \frac{\mu(\mathbb{D}_r)}{r} dr < \infty$ for every $0 < R' < R$.*

Then there exists a random polynomial P_n such that $\frac{1}{n}\mu_n \to \mu$ in probability on some disc of radius R.

Note that whereas in Theorem 1, the measure μ was not necessarily completely described by R, that is, it was not necessarily the case that $\mu(\mathbb{C}\backslash\mathbb{D}_R) = 0$, Theorem 4 provides that μ

is completely determined by $\mu(\mathbb{D}_{R'})$ for $R' \in (0, R)$. Also note that if the second hypothesis holds for some R', then it holds for all R'. While we do not include a proof of this theorem, we will include how the polynomial P_n is constructed for purposes of the next subsection.

Given the hypotheses in Theorem 4, for all $s < \log R$, define

$$\mathcal{L}(s) := \int_{-\infty}^{s} \mu(\mathbb{D}_{e^r})dr$$

and define $u : [0, \infty) \to \mathbb{R}$ by

$$u(t) := \sup_{s \in \mathbb{R}}(st - \mathcal{L}(s)).$$

The polynomial satisfying Theorem 4 will be given by

$$f_{k,n} := e^{-nu(k/n)}$$

$$P_n(z) := \sum_{k=0}^{\infty} f_{k,n} z^k \xi_k$$

for any i.i.d. ξ_k satisfying $\mathbb{E}[\log(1 + |\xi_0|)] < \infty$.

4.1 Uniform Distribution in the Annulus

We mentioned in the introduction that the limiting distribution of the Kac polynomial K_n is uniform on the unit circle while the limiting distribution of the Weyl polynomial W_n is uniform on the disc of radius $\sqrt{n}$. A natural question, which does not appear to have been addressed in the literature, is finding a polynomial whose limiting distribution is uniform in an annulus. We address this question here, through a straightforward application of Theorem 4, and will later see the relevance of this example to other random polynomials we consider in the next section.

Let $1 \leqslant r_1 < r_2$. We claim the following polynomial is uniform in the annulus with inner and outer radii r_1 and r_2, respectively:

$$P_n^{(r_1,r_2)}(z) = \sum_{k=0}^{n} f_{k,n} z^k \xi_k$$

where the ξ_k are any i.i.d. random variables with finite logarithmic moment and

$$f_{k,n} = \left(\frac{k}{n}(r_2^2 - r_1^2) + r_1^2\right)^{-\left(\frac{r_1^2 n}{2(r_2^2-r_1^2)}+\frac{k}{2}\right)} \exp\left\{\frac{r_1^2 n}{2(r_2^2 - r_1^2)}\right\} \exp\left\{\frac{k}{2}\right\}.$$

One can verify that

$$u(t) = \left(\frac{t}{2} + \frac{r_1^2}{2(r_2^2 - r_1^2)}\right)\log(t(r_2^2 - r_1^2) + r_1^2) - \left(\frac{t}{2} + \frac{r_1^2}{2(r_2^2 - r_1^2)}\right), \quad t \in [0,1]$$

and $u(t) = \infty$ for $t > 1$. This means that

$$\mathcal{L}(s) = \frac{1}{r_2^2 - r_1^2}\left(\frac{e^{2s}}{2} - sr_1^2\right), \quad \log r_1 \leqslant s \leqslant \log r_2$$

and $\mathcal{L}(s) = 0$ when $s < \log r_1$ and $\mathcal{L}(s) = s$ when $s > \log r_2$. We then have that

$$\mathcal{L}'(\log r) = \mu(\mathbb{D}_r) = \frac{r^2 - r_1^2}{r_2^2 - r_1^2}, \quad r_1 \leqslant r \leqslant r_2$$

and $\mu(\mathbb{D}_r) = 0$ when $0 \leqslant r < r_1$ and $\mu(\mathbb{D}_r) = 1$ for $r > r_2$; μ describes the distribution that is uniform in the annulus between radii r_1 and r_2, as desired.

A useful observation is that the two exponential factors on the right in the definition of $f_{k,n}$ can essentially be ignored, since the first exponential factor is a constant multiplicative factor (it only depends on n) and the second exponential factor can be absorbed into z^k (by scaling z by $\sqrt{e}$). For simplicity, let's consider when $r_1 = 1$ and $r_2 = 2$. Then

$$P_n^{(1,2)}(z) = \sum_{k=0}^{n}\left(\frac{3k}{n} + 1\right)^{-\frac{3k+n}{6}} \exp\left\{\frac{3k+n}{6}\right\} z^k \xi_k.$$

Ignoring the exponential factor, which will rescale the distribution by $\sqrt{e}$, we are mainly interested in the term

$$\left(\frac{3k}{n}+1\right)^{-\frac{3k+n}{6}}.$$

Using the approximation $1+x \approx e^x$ for x small,

$$\left(\frac{3k}{n}+1\right)^{-\frac{3k+n}{6}} \approx \exp\left\{-\frac{3}{2}\frac{k^2}{n}\right\}\exp\left\{-\frac{k}{2}\right\}$$

where again the factor of $e^{-k/2}$ can essentially be ignored. This approximation will be related to a class of random polynomials we encounter in the next section.

5 An Interesting Class: Phase Transitions in Real Root Asymptotics and Limiting Distributions

We now turn to the class of random polynomials considered by Schehr and Majumdar in [11]:

$$P_n^\alpha(z) = \sum_{k=0}^{n} e^{-k^\alpha/2} z^k \xi_k$$

where $\alpha > 0$ is a parameter and the ξ_k are independent standard Gaussian random variables. They prove that as α varies, two phase transitions occur in the expected number of real roots, denoted $\mathbb{E}[N_n^\alpha]$, specifically at $\alpha = 1$ and $\alpha = 2$:

$$\mathbb{E}[N_n^\alpha] \sim \frac{2}{\pi}\log n \quad \text{when } 0 \leqslant \alpha \leqslant 1$$

$$\mathbb{E}[N_n^\alpha] \sim \frac{2}{\pi}\sqrt{\frac{\alpha-1}{\alpha}}\, n^{\alpha/2} \quad \text{when } 1 < \alpha < 2$$

$$\mathbb{E}[N_n^\alpha] \sim n \quad \text{when } \alpha > 2$$

where $f(n) \sim g(n)$ if $\lim_{n\to\infty} f(n)/g(n) = 1$. We are specifically interested in the first phase transition at $\alpha = 1$. When $0 \leqslant \alpha \leqslant 1$, the behavior of P_n^α, in terms of the number of real roots, mimics that of Kac's polynomials. For $\alpha = 1$, this is not surprising, since the variance of e^k can essentially be absorbed into z^k and just constitutes a rescaling of the Kac polynomial. Note also, in this regime $0 \leqslant \alpha \leqslant 1$, the asymptotic dose not at all depend on α. We see a phase transition occur at $\alpha = 1$, in which the number of real roots goes from being logarithmic to proportional to $n^{\alpha/2}$. It is useful to consider that the expected number of real roots for the Weyl polynomials, given by

$$W_n(z) = \frac{1}{\sqrt{k!}} z^k \xi_k,$$

is asymptotically $\frac{2}{\pi}\sqrt{n}$ [11]. The variance of $\frac{1}{k!}$ can be estimated by $e^{-k \log k}$, which interpolates between the variances of e^{-k} ($\alpha = 1$) and e^{-k^α} for $\alpha > 1$. Thus, the behavior of the Weyl polynomials is reflected in the behavior of e^{-k^α} when α approaches 1 from the right, and this is a reasonable result given that, in variance, the Weyl polynomial interpolates between the polynomials P_n^α for $\alpha \leqslant 1$ and $\alpha > 1$ [11].

One question, which does not appear to have yet been studied, is whether phase transitions also occur in some sense in the distribution of the complex roots at $\alpha = 1$, and whether, for $1 < \alpha < 2$, the distribution varies "continuously" along α, perhaps in terms of its density, just as $\mathbb{E}[N_n^\alpha]$ varies continuously in α. It is a simple consequence of Theorem 1 that when $0 < \alpha < 1$, the limiting distribution of the complex roots is uniform on the unit circle (note that the radius does not depend on α). When $\alpha = 1$, the distribution is uniform on the circle of radius $\sqrt{e}$ (since this is just a rescaling of the Kac polynomial). Just as in the asymptotics for the expected number of real roots, these polynomials also resemble the Kac polynomial (whose limiting distribution is the unit circle) in the limiting distribution of their complex roots.

As mentioned, the limiting distribution of the complex roots of the Weyl polynomial $W_n(z)$ (which we noted interpolates between the $\alpha > 1$ and $\alpha \leqslant 1$ cases) is uniform in the disc of radius $\sqrt{n}$. Since the limiting distribution depends on n, it cannot be the case that this result is directly derived from Theorem 1, since $\mathcal{L}$, which determines the limiting distribution

μ, does not depend on n. However, rescaling z by $\sqrt{n}$ eliminates the dependence on n and allows us to apply Theorem 1. That is, define:

$$\tilde{W}_n(z) = P_n(\sqrt{n}z) = \sum_{k=0}^{n} \sqrt{\frac{n^k}{k!}} z^k \xi_k.$$

Using the notation of Theorem 1, applied to $\tilde{W}_n(z)$, gives $\beta = 1$ and $f(t) = e^{-\frac{1}{2}t(\log t - 1)}$ when $t \in [0, 1]$ (note that $t \log t \to 0$ as $t \to 0$, so we define $f(0) = 1$). This yields $\mathcal{L}(s) = \frac{1}{2}e^{2s}$ when $s \leqslant 0$ and $\mathcal{L}(s) = \frac{1}{2} + s$ when $s \geqslant 0$, which in turn gives that the limiting distribution of $\tilde{W}_n$ is uniform in the disc of radius 1. Because we rescaled by a constant factor of $\sqrt{n}$, the limiting distribution of W_n will be uniform in the disc of radius $\sqrt{n}$ (Theorem 2.3 in [8]).

It is straightforward to see that when $1 < \alpha < 2$, the polynomials P_n^α do not satisfy the approximation condition, and so Theorem 1 cannot be applied directly. Suppose to the contrary that the approximation condition is satisfied. By definition of supremum, the approximation condition implies that

$$\lim_{n\to\infty} \left| |f_{cn,n}|^{1/n} - f(c) \right| \quad \text{for all } c > 0$$

Since P_n^α has n terms, $f(c) = 0$ when $c > 1$. When $c \leqslant 1$,

$$f(c) = \lim_{n\to\infty} |f_{cn,n}|^{1/n} = \lim_{n\to\infty} e^{-c^\alpha n^{\alpha-1}/2} = 0$$

because $1 < \alpha < 2$. Therefore $f \equiv 0$. We can therefore rewrite the approximation condition as

$$\lim_{n\to\infty} \sup_{k\in[0,cn]} |f_{k,n}|^{1/n} = 0.$$

But

$$0 = \lim_{n\to\infty} \sup_{k\in[0,cn]} |f_{k,n}|^{1/n} \geqslant \lim_{n\to\infty} |f_{0,n}|^{1/n} = 1,$$

which is a contradiction.

We also show that we cannot apply Theorem 3, despite the conditions being slightly weaker. Recall that the first condition of this theorem seeks a continuous function V in z such that

$$\lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{k=0}^{n} |f_{k,n}||z|^k \right) = V(z).$$

We will show that $V \equiv 0$. First, observe that $V(z) \geqslant 0$ for all $z \in \mathbb{C}\backslash\{0\}$, since

$$V(z) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{k=0}^{n} |f_{k,n}||z|^k \right) \geqslant \lim_{n\to\infty} \log(|f_{0,n}||z|^0) = \lim_{n\to\infty} \frac{1}{n} \log 1 = 0.$$

Now we may consider two cases.

Case 1 ($0 < |z| \leqslant 1$)**.** *In this case,*

$$V(z) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{k=0}^{n} e^{-k^\alpha/2}|z|^k \right) \leqslant \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{k=0}^{n} e^{-k^\alpha/2} \right) \leqslant \lim_{n\to\infty} \frac{1}{n} \log(n \cdot e^{-0^\alpha/2}) = 0$$

and since $V(z) \geqslant 0$, we have that $V(z) = 0$ for $0 < |z| \leqslant 1$.

Case 2 ($|z| > 1$)**.** *Note we view z as fixed because the limit is taken pointwise. In this case, we split the sum at $k = n^{1/\alpha}$, which we may do since $\frac{1}{2} < 1/\alpha < 1$. First, consider the second part of the sum:*

$$\sum_{k=n^{1/\alpha}}^{n} e^{-k^\alpha/2}|z|^k = \sum_{k=n^{1/\alpha}}^{n} \left(e^{-k^{\alpha-1}/2}|z| \right)^k$$

and note that $e^{-k^{\alpha-1}/2} \leqslant e^{-n^{1-1/\alpha}/2}$, which can be made smaller than $\frac{1}{2|z|}$ for large enough n. Thus for n large enough,

$$\sum_{k=n^{1/\alpha}}^{n} \left(e^{-k^{\alpha-1}/2}|z| \right)^k \leqslant \sum_{k=n^{1/\alpha}}^{n} \left(\frac{1}{2} \right)^k < 2.$$

Considering now both parts of the sum,

$$\sum_{k=0}^{n} e^{-k^\alpha/2}|z|^k \leqslant \sum_{k=0}^{n^{1/\alpha}} e^{-k^\alpha/2}|z|^k + \sum_{k=n^{1/\alpha}}^{n} e^{-k^\alpha/2}|z|^k \leqslant n^{1/\alpha} e^{-0^\alpha/2}|z|^{n^{1/\alpha}} + 2$$

where we have used the fact that $|z| > 1$ to bound the first sum and the bound of 2 from above for the second sum. Thus

$$V(z) \leqslant \lim_{n\to\infty} \frac{1}{n} \log\left(2n^{1/\alpha}|z|^{n^{1/\alpha}}\right) = \lim_{n\to\infty}\left(\frac{1}{n}\log 2 + \frac{1}{\alpha}\frac{1}{n}\log n + n^{1/\alpha-1}\log|z|\right) = 0$$

where we have used the fact that $-1/2 < \frac{1}{\alpha} - 1 < 0$.

Since $V(z) = 0$, $V(z) - \log|z|$ is not bounded as $z \to \infty$, violating the first condition of Theorem 3 (which is included to guarantee that $\frac{1}{2\pi}\Delta V(z)$ is a probability measure). Therefore, Theorem 3 will not apply to P_n^α.

Later, we will also show that it does not appear that the limiting distribution can be derived by rescaling z by a constant factor (depending on n) to eliminate the dependence of the limiting distribution on n, as was done in the Weyl polynomial, and then applying Theorem 1 to the rescaled polynomial.

Below, we will provide a conjecture for the limiting distribution of the polynomials P_n^α when $1 < \alpha < 2$ and show a heuristic argument used to derive it. This argument will reuse the tools used in the proof of Theorem 1, and rely on a generalization of the rescaling technique shown in the Weyl polynomial to one-to-one transformations (of which rescaling is a special case). Our conjectured distribution will be supported on an annulus whose inner radius is 1 and whose outer radius is $e^{\frac{\alpha}{2}n^{\alpha-1}}$. Before proceeding to the derivation of this conjectured distribution, we observe a polynomial which is related in form to P_n^α, and whose distribution is in fact supported on an annulus. We will use Schehr and Majumdar's terminology and refer to it as the randomized theta polynomial, which in its finite and infinite forms is given by

$$H_n(z) = \sum_{k=0}^{n} e^{-k^2/n} z^k \xi_k$$

$$H_n^\infty(z) = \sum_{k=0}^{\infty} e^{-k^2/n} z^k \xi_k$$

(Section 2.4 in [8]). Let ρ and ρ^∞ denote the densities of the limiting distributions. Then a direct application of Theorem 1 yields

$$\rho(z) = \frac{1}{4\pi|z|^2}\mathbf{1}_{1<|z|<e^2}$$

$$\rho^\infty(z) = \frac{1}{4\pi|z|^2}\mathbf{1}_{|z|\geqslant 1}$$

so that in the finite case the support of the limiting distribution is the annulus with inner and outer radii 1 and e^2, respectively (and since $R = \infty$ in the application of Theorem 1 to the finite case, we have that the density integrates to 1). Given the similarity between the forms of P_n^α and $H_n(z)$, we might expect similarities in their limiting distributions (such as being supported on an annulus). We also note that in Section 4, we approximated $f_{k,n}$ in the polynomial $P_n^{(1,2)}$, which had limiting distribution uniform on the annulus between radii 1 and 2, by $e^{-\frac{3}{2}\frac{k^2}{n}}$, which is nearly the same form of the coefficients as the randomized theta polynomial.

5.1 Heuristic Argument

Upper Bound: As in the strategy of the proof of Theorem 1, loosely speaking we would like to find $k^* = \arg\max_k |f_{k,n}z^k|$ and find a function $\mathcal{L}$, possibly depending on n, satisfying

$$|f_{k^*,n}z^{k^*}| = e^{n(\mathcal{L}_n(\log|z|)+\varepsilon)}$$

Given the failure to satisfy the approximation condition, we anticipate that the function $\mathcal{L}$ will depend on n. To maximize $f_{k^*,n}$, we consider the continuous extension of $|f_{k,n}z^k|$, as a function of k, given by

$$f(u) = e^{-u^\alpha/2}|z|^u.$$

To maximize f (and in turn $|f_{k,n}|$), it will suffice to maximize

$$g(u) = \log f(u) = -u^\alpha/2 + u\log|z|.$$

We find that f is maximized at

$$u^* = \left(\frac{2}{\alpha}\right)^{\frac{1}{\alpha-1}} (\log|z|)^{\frac{1}{\alpha-1}}$$

and $u^* \in [0, n]$ when

$$1 \leqslant |z| \leqslant e^{\frac{\alpha}{2}n^{\alpha-1}}$$

which we will refer to as the "main region." We suspect that the entire support of the limiting distribution will be contained in this region.

For convenience, set $\beta = \left(\frac{2}{\alpha}\right)^{\frac{1}{\alpha-1}}$. The upper bound on the log of the coefficients that we achieve is

$$g(u^*) = (\log|z|)^{\frac{\alpha}{\alpha-1}} \left(\frac{\alpha-1}{\alpha}\beta\right).$$

Setting $g(u^*) = n\mathcal{L}_n(\log|z|)$, we obtain, for $1 \leqslant |z| \leqslant e^{\frac{\alpha}{2}n^{\alpha-1}}$

$$\mathcal{L}_n(\log|z|) = \frac{1}{n}\left(\frac{\alpha}{\alpha-1}\beta\right)(\log|z|)^{\frac{\alpha}{\alpha-1}}$$

and for $1 < t < \frac{\alpha}{2}n^{\alpha-1}$,

$$\mathcal{L}_n(t) = \frac{1}{n}\left(\frac{\alpha}{\alpha-1}\beta\right)t^{\frac{\alpha}{\alpha-1}}$$

$$\mathcal{L}_n'(t) = \frac{1}{n}\beta t^{\frac{1}{\alpha-1}}.$$

$$\mathcal{L}_n'(\log r) = \frac{1}{n}\beta(\log r)^{\frac{1}{\alpha-1}}, \quad r \in (1, e^{\frac{\alpha}{2}n^{\alpha-1}})$$

This suggests that the support of μ_n is $1 < |z| < e^{\frac{\alpha}{2}n^{\alpha-1}}$. We perform two sanity checks. First, we check that $\mathcal{L}_n'(\log r)$ is increasing in r, for r in the main region. Indeed, this is the case, since

$$\frac{\partial}{\partial r}\mathcal{L}_n'(\log r) = \frac{1}{n}\frac{1}{\alpha-1}\beta(\log r)^{\frac{2-\alpha}{\alpha-1}}\frac{1}{r}$$

which is positive for $r > 1$. As a second sanity check, we observe that

$$\mathcal{L}_n'(\log e^{\frac{\alpha}{2}n^{\alpha-1}}) = 1.$$

This leads to the conjecture that

$$\mu_n(\mathbb{D}_r) \approx \mathcal{L}_n'(\log r), \quad r \in (1, e^{\frac{\alpha}{2} n^{\alpha-1}}).$$

We will describe preciesly what we mean by $\approx$ by performing an appropriate transformation to P_n^α that eliminates the dependence of $\mathcal{L}_n$ on n.

Lower Bound: To provide further support for the conjecture, we argue that the lower bound is satisfied, that is, there are $\Theta(h(n))$ values of k for which

$$|f_{k,n} z^k| \geqslant e^{n(\mathcal{L}_n(\log|z|)-\varepsilon)},$$

where $h(n)$ is strictly increasing in n and $h(n) \to \infty$. By construction of $\mathcal{L}$, it suffices to demonstrate $\Theta(h(n))$ values of $k > u^*$ for which

$$e^{-k^\alpha/2}|z|^k \geqslant e^{-(u^*)^\alpha/2}|z|^{u^*} e^{-n\varepsilon}$$

$$\iff n\varepsilon > \frac{1}{2}(k^\alpha - (u^*)^\alpha) - (k-u^*)\log|z|.$$

Write $\Delta = k - u^*$. Using a second-order Taylor expansion centered at u^*, we have that

$$k^\alpha - (u^*)^\alpha \approx \alpha(u^*)^{\alpha-1}\Delta + \frac{\alpha(\alpha-1)(u^*)^{\alpha-2}}{2}\Delta^2$$

so that we rewrite the above inequality as

$$n\varepsilon \geqslant \frac{1}{2}\left(\alpha(u^*)^{\alpha-1}\Delta + \frac{\alpha(\alpha-1)(u^*)^{\alpha-2}}{2}\Delta^2\right) - \Delta\log|z|$$

and using the definition of u^* we can further simplify this inequality as

$$n\varepsilon \geqslant \frac{\alpha(\alpha-1)}{4}(u^*)^{\alpha-2}\Delta^2$$

and thus we may take $h(n) = \sqrt{n}$. Note that u^* depends on $|z|$, which $\Theta(h(n))$ may depend on.

Transformation: Now our challenge is to eliminate the dependence of $\mathcal{L}_n$ on n. In other words, while we do conjecture that the limiting distribution of the zeros does depend on n and the limiting measure is specified by $\mu_n(\mathbb{D}_r) \approx \mathcal{L}_n'(\log r)$ for $r \in (1, e^{\frac{\alpha}{2}n^{\alpha-1}})$ as $n \to \infty$, the method of proof in Theorem 1 (the strategy we used in the above heuristic argument) requires $\mathcal{L}$ to be independent of n. Just as in the case of the Weyl polynomial when we had to transform $W_n(z)$ into $\tilde{W}_n(z)$ in order to find $\mathcal{L}$ independent of n, and *then* apply Theorem 1 to this scaled polynomial, we will likewise have to perform a transformation to $P_n^\alpha(z)$ to eliminate the dependence on n. This transformation will not be a simple rescaling, but it (necessarily) will be a one-to-one transformation. Briefly, we argue why a simple rescaling will not work. Further, this transformation will not result in a polynomial, as a rescaling does, so we will not be able to appeal to Theorem 1 for the transformation of P_n^α.

Recall that we have $\mathcal{L}_n(t) = \frac{1}{n}\left(\frac{\alpha}{\alpha-1}\beta\right) t^{\frac{\alpha}{\alpha-1}}$ for $0 < t < \frac{\alpha}{2}n^{\alpha-1}$. Write $R^{P_n^\alpha} = e^{\frac{\alpha}{2}n^{\alpha-1}}$ which is the conjectured outer radius for the support of the limiting distribution of P_n^α. We would like to see whether there exists A (depending only on n) such that if $Q_n(z) := P_n^\alpha(Az)$, then there exists a radius R^{Q^α} and a function $\mathcal{L}^{Q^\alpha}$ that do not depend on n such that for every $\varepsilon > 0$,

$$|Q_n(z)| = M(\varepsilon)e^{n\mathcal{L}(\log|z|)\pm n\varepsilon}, \quad |z| < R^{Q^\alpha}.$$

Without loss of generality, take $A = e^{\frac{\alpha}{2}n^{\alpha-1}} = R^{P_n^\alpha}$ (A may vary by a constant multiplicative factor). Then $R^{Q^\alpha} = 1$. Moreover,

$$|Q_n(z)| = |P_n(Az)| = e^{n(\mathcal{L}_n(\log|z|+\log A))\pm n\varepsilon}$$

so that

$$\mathcal{L}^{Q^\alpha}(t) = \mathcal{L}_n(t + \log A) = C_\alpha \frac{1}{n}\left(t + \frac{\alpha}{2}n^{\alpha-1}\right)^{\frac{\alpha}{\alpha-1}} \pm \varepsilon, \quad -\infty < t < 0$$

where C_α is a constant only depending on α. When t is near 0, for instance,

$$\mathcal{L}^{Q^\alpha}(t) = C'_\alpha n^{\alpha-1} \pm \varepsilon'$$

where again C'_α is a constant depending only on α. Then $\mathcal{L}^{Q^\alpha}(t)$ necessarily depends on n, since $0 < \alpha - 1 < 1$ and ε is arbitrarily small.

Now we describe the one-to-one transformation that we will use instead of a rescale.

$$\frac{1}{n}\mu_n(\mathbb{D}_r) = \mathcal{L}'_n(\log r), \quad r \in (1, e^{\frac{\alpha}{2}n^{\alpha-1}})$$

$$\iff \frac{1}{n}\mu_n(\mathbb{D}_{e^t}) = \mathcal{L}'_n(t), \quad t \in (0, \frac{\alpha}{2}n^{\alpha-1})$$

Now define a new measure ν_n by

$$\nu_n(\mathbb{D}_t) := \frac{1}{n}\mu_n(\mathbb{D}_{e^t}), \quad t \in (0, \frac{\alpha}{2}n^{\alpha-1})$$

More specifically, we describe this transformation via a function $\varphi(x) = \log x$, so that for any measurable set A,

$$\nu_n(A) = \frac{1}{n}\mu_n(\varphi^{-1}(A)).$$

Write $t_n = \frac{\alpha}{2}n^{\alpha-1}$, and thus the support of ν_n is $(0, t_n)$. Finally, we may eliminate the dependence on n by rescaling by t_n:

$$\tilde{\nu}(\mathbb{D}_t) = \nu_n(\mathbb{D}_{t \cdot t_n}), \quad t \in (0, 1)$$

so that $\tilde{\nu}$ does *not* depend on n, since

$$\tilde{\nu}(\mathbb{D}_t) = t^{\frac{1}{\alpha-1}}, \quad t \in (0, 1).$$

Unlike in the transformation of the Weyl polynomial, which was a simple rescaling that results in a new random polynomial $\tilde{W}$, the transformation of P_n^α to $P_n^\alpha(t_n \cdot \log|z|)$ results in

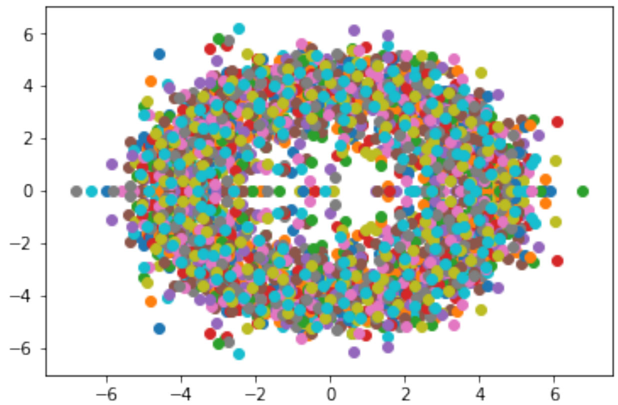

Figure 1: The roots of 20 different polynomials P_n^α for $n = 160$ and $\alpha = 1.2$. The plot agrees with the conjecture that the roots should concentrate in the annulus with radii 1 and $e^{\frac{\alpha}{2}n^{\alpha-1}}$.

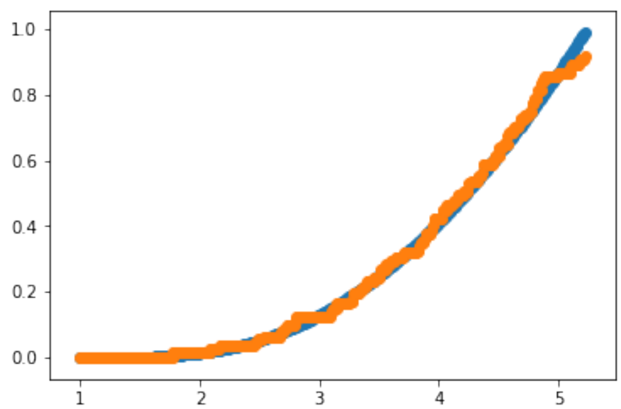

Figure 2: The function $\mathcal{L}_n'(\log r)$ for $1 < r < e^{\frac{\alpha}{2}n^{\alpha-1}}$ (blue) and $\frac{1}{n}\mu_n(\mathbb{D}_r)$ (orange) on the same interval for a polynomial P_n^α, $n = 160$, and $\alpha = 1.2$. As predicted, these curves align.

a random function that is no longer a random polynomial, and so we cannot apply Theorem 1 to it directly.

To formalize the conjecture that $\tilde{\nu}(\mathbb{D}_t) = t^{\frac{1}{\alpha-1}}, \quad t \in (0,1)$, and therefore the conjecture that, in the limit, $\mathcal{L}_n$ describes the limiting distribution of P_n^α, $1 < \alpha < 2$, we will have to exhibit the upper and lower bounds in conditions 3 and 4 for the transformed polynomial $P_n^\alpha(t_n \cdot \log|z|)$. See Figures 1 (above) and 2 (below) for experimental support of the conjecture (see code for generating these figures in the Appendix).

From the above, it does appear that the phase transitions in real root asymptotics are mimicked by phase transitions in the limiting distributions of the complex roots, where at $\alpha = 1$, the distribution transitions from being supported (and uniform) on a circle to being supported on an annulus; further, the conjectured outer radius of the annulus grows continuously as a function of α (while the inner radius is always 1), and the conjectured cumulative density is a continuous function in α. Interestingly, the Weyl polynomial, which we thought of as interpolating between P_n^α for $\alpha \leqslant 1$ and $\alpha > 1$ in terms of variance and real root asymptotics, has a limiting distribution that is uniform on the disc of radius $\sqrt{n}$ (which is not in any obvious sense an interpolation between the annulus and the circle).

6 References

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7 Appendix

The following code (used in a Python 3 Jupyter notebook) was written to generate the plots in Figures 1 and 2.

```
import numpy as np
import math
import random
import matplotlib.pyplot as plt
import scipy.special

#Kac
def formula_1(n,k,alpha):
    return 1, 'Kac'

#annulus
def formula_2(n,k,alpha):
    return math.pow(math.pow(1+3*k/n, -(3*k+n)/6)*
                    math.pow(math.e, (3*k+n)/6),2), 'annulus'
```

```
#annulus weak estimate
def formula_3(n,k,alpha):
    return math.pow(math.pow(math.e, -3*math.pow(k,2)/(2*n)),2),
                                                    'annulus weak estimate'

#annulus strong estimate
def formula_4(n,k,alpha):
    return math.pow(math.pow(1+3*k/n, -k/2),2), 'annulus strong estimate'

#Weyl
def formula_5(n,k,alpha):
    return math.pow(math.e, -k*math.log(k+1)), 'Weyl'

#Majumdar
def formula_6(n,k,alpha):
    return math.pow(math.e, -math.pow(k,alpha)), 'Majumdar'

#interpolation attempt log log
def formula_7(n,k,alpha):
    return math.exp(-k*math.log(math.log(k+2))), 'log log'

#interpolation attempt sqrt log
def formula_8(n,k,alpha):
    return math.exp(-k*math.sqrt(math.log(k+1))), 'sqrt log'

#binomial
def formula_9(n,k,alpha):
    return scipy.special.binom(n,k), 'binomial'
```

```
#majumdar scaled by alpha
def formula_10(n,k,alpha):
    return math.pow(math.pow(math.e, math.pow(k,alpha)/n),2), 'majscaled'

#iterate degree from lo to hi
#trials = number of trials per degree
#return a vector of vectors of vectors with the complex roots
#of all the trials for all the degrees
def simulate(lo, hi, formula, trial, alpha):
    roots = []
    real_roots = []
    real_roots_averages = []
    for m in range(hi-lo+1):
        degree_roots = []
        degree_real_roots = []
        n = lo + m
        total = 0
        for trials in range(trial):
            coeff = []
            for k in range(n+1):
                j = n-k
                var, name = formula(n,j,alpha)
                curr_coeff = np.random.normal(0, math.sqrt(var))
                coeff.append(curr_coeff)
            trial_roots = np.roots(coeff)
            trial_real_roots = trial_roots[np.isreal(trial_roots)]
            degree_roots.append(trial_roots)
            degree_real_roots.append(trial_real_roots)
```

```
                total = total + len(trial_real_roots)
            roots.append(degree_roots)
            real_roots.append(degree_real_roots)
            real_roots_averages.append(total/trial)
        return roots, real_roots, real_roots_averages, name

#takes in a vector of real_roots_averages vectors
#takes in a vector of names
#plots asymptotics along with labels
def real_roots_asymptotics(vector, name_vector):
    fig = plt.figure(figsize=(10,10))
    ax1 = fig.add_subplot(111)
    for i in range(len(vector)):
        ax1.plot(vector[i], label=name_vector[i])
    plt.legend(loc='upper left')
    plt.show()

#roots is a vector of roots to plot
def distribution_of_roots(roots, cutoff_upper, cutoff_lower, yes):
    roots_cutoff = []
    for i in range(len(roots)):
        if (abs(roots[i])<=cutoff_upper and abs(roots[i])>=cutoff_lower):
            roots_cutoff.append(roots[i])
    real_parts = []
    imag_parts = []
    for l in range(len(roots_cutoff)):
        real_parts.append(roots_cutoff[l].real)
        imag_parts.append(roots_cutoff[l].imag)
    plt.scatter(real_parts, imag_parts)
```

```
if yes == 'yes':
    return len(roots_cutoff)
```