Chromatic Number of String Graphs

Candidate Number: [redacted]

M.Sc. Mathematical Sciences

Trinity Term 2020

May 4, 2020

Contents

1 Notation

Throughout, we will use the following notations for a graph G.

- $V(G)$ denotes the vertex set of G and $E(G)$ denotes the edge set of G.
- $\chi(G)$ denotes the chromatic number of G, which is the fewest number of colours that can be used to colour $V(G)$ such that no two adjacent vertices are given the same colour.
- $\alpha(G)$ denotes the independence number of G, which is the largest number of pairwise non-adjacent vertices.
- $\omega(G)$ denotes the clique number of G, which is the largest size of a clique (a set of pairwise adjacent vertices). Where G is clear, we may simply write ω.

2 Introduction

Note: All figures produced throughout are my own.

One point of interest in the study of graph colouring is the relationship between chromatic number and clique number. The chromatic number, of course, is at least the clique number, and perfect graphs describe the tightest relationship: a graph G is said to be *perfect* if, for every induced subgraph H of G, $\chi(H) = \omega(H)$. The Weak Perfect Graph Theorem, proven by Lovász in 1972 [19], states that a graph is perfect if and only if its complement is perfect. A much harder recent result, the Strong Perfect Graph Theorem, proven in 2006 by Chudnovsky *et.al.* [5], provides a complete characterisation of perfect graphs as those graphs which do not contain an odd cycle on five or more vertices or the complement of one as an induced subgraph. Bipartite graphs and comparability graphs, for instance, are easily seen to be perfect.

Where a family of graphs does not satisfy perfection, it still may be the case that the chromatic number is controlled by the clique number. A weaker notion than perfection is χ-boundedness: a family $\mathcal{F}$ of graphs is said to be *χ-bounded* if there exists a function f such that for all $G \in \mathcal{F}$, $\chi(G) \leqslant f(\omega(G))$, and we say that f is a *χ-binding function* for $\mathcal{F}$. (For perfect graphs, we take $f(x) = x$.) There are several examples of families of graphs that are *not* χ-bounded, which take the family to be triangle-free ($\omega = 2$) and the chromatic number arbitrarily large. Many constructions are recursive and have the chromatic number increase by 1 at each step, such as that of Mycielski [25]. On the other hand, the shift graphs due to Erdős and Hajnal [10] are an example of a triangle-free recursively-defined family in which the chromatic number at iteration k is $\lceil \log_2 k \rceil$, and thus increases at a slower rate. There are also probabilistic existence proofs of families that are not χ-bounded, such as that of Erdős [9]

Demonstrating χ-boundedness of a family of graphs is a non-trivial problem. Where a family does satisfy χ-boundedness, there is further interest in finding the best possible order χ-binding function (polynomial, exponential, etc.). Esperet [11] conjectured that all hereditary families of graphs (families that are closed under isomorphism and under taking induced subgraphs) are *polynomially* χ-bounded, and there has yet to be a counterexample [32]. A main topic of current interest is forbidden subgraph characterisations for χ-bounded families. Most notably, Gyárfás in 1975 [13] and Sumner in 1981 [34] conjectured that, given a tree T, the family of graphs that do not contain T as an induced subgraph is χ-bounded, and this remains unresolved.

In this dissertation, we will be interested in χ-boundedness of geometrically-defined graphs.

Definition 1. *An intersection graph G of a collection $\mathcal{F}$ of sets is a graph such that $V(G) = \mathcal{F}$ and an edge between $F, F' \in \mathcal{F}$ if and only if $F \cap F' \neq \varnothing$.*

Definition 2. *A string graph is the intersection graph of a collection of curves in the plane.*

Definition 3. *A disjointness graph is the complement of an intersection graph.*

Throughout, we will consider χ-boundedness of intersection and disjointness graphs where we take $\mathcal{F}$ in the above to be a collection of geometric objects. There is a sizable literature on this matter. Interval graphs, which are intersection graphs of open intervals on $\mathbb{R}$, are easily seen to be perfect. Asplünd and Grunbaum [1] showed that intersection graphs of axis-aligned rectangles in $\mathbb{R}^2$ are quadratically χ-bounded while Burling [2] proved that intersection graphs of axis-aligned boxes in $\mathbb{R}^3$ are *not* χ-bounded. Here, in which all three families consist of axis-aligned boxes, the relationship between χ and ω weakens under increasing dimensionality.

In this dissertation, we explore χ-boundedness of families of string graphs. In Section 3, we present the 2014 result of Pawlik *et.al.* [29] that intersection graphs of line segments in the plane are not χ-bounded, and frame their construction as a realisation of Burling's graphs for axis-aligned boxes (see above). This is the first proof that the family of all string graphs is not χ-bounded. Moreover, we discuss how this result revived interest in Burling's construction, and how this realisation of Burling's construction can be used to derive results about forbidden subgraph characterisations for χ-boundedness.

In Section 4, we present the complementary result that disjointness graphs of line segments in the plane are χ-bounded, noting some errors in previous literature on this matter and filling in a gap. In particular, we follow the 2017 paper of Pach *et.al.* [26] to show this result generalises to disjointness graphs of projective lines, of lines, and of line segments in $\mathbb{R}^d$, $d \geqslant 3$, and trace how the first is used to deduce the second which in turn is used to deduce the third. We also include how the proofs of all three give polynomial-time algorithms for finding a colouring that achieves the χ-bounds. We analyse these results as a case study for why there is no analogue of the Weak Perfect Graph Theorem for χ-boundedness: it is *not* true that a family of graphs is χ-bounded if and only if the family of complements is χ-bounded.

In Section 5, we present the 2019 result of Davies and McCarty [6] that circle graphs (intersection graphs of chords on a circle) are *quadratically* χ-bounded. We follow their proof method but deviate from their structure and sequence, forego some of their definitions, and simplify certain propositions in order to present the proof in a more motivated way. We also compare the technique used in this result with the technique used by Gyárfás to show that circle graphs are *exponentially* χ-bounded [14].

3 Intersection graphs of line segments

The study of intersection graphs of curves in the plane began with Ehrlich *et.al.* [8], who showed that determining whether the intersection graph of a collection of line segments in the plane is k-colourable, for $k \geqslant 3$, is NP-complete. Since then, χ-boundedness has been demonstrated for subfamilies of intersection graphs of curves in the plane. Suk [33] showed that intersection graphs of simple, x-monotone curves (every vertical line intersects each curve in exactly one point) that each intersect a fixed vertical line are χ-bounded. Rok and Walczak in 2019 [30] generalised this to show that the intersection graph of any set of curves such that each crosses a fixed curve in at least one and at most t points is χ-bounded, without requiring that the curves be simple or x-monotone.

Here we present the 2014 result of Pawlik *et.al.* [29] proving that intersection graphs of line segments in the plane are not χ-bounded. ln Section 3.1 we review Burling's construction of triangle-free graphs with arbitritaly large chromatic number, used in [2] to prove that axis-aligned boxes in $\mathbb{R}^3$ are not χ-bounded, and show the construction of Pawlik *et.al.* realising Burling's construction. We discuss implications of this result, including a conjecture of Scott from [31], in Section 3.2.

3.1 String graphs are not χ-bounded

Burling's graphs [2, 12] are defined as follows:

1. Set G_1 to be the graph consisting of a single vertex v and $\mathcal{S}(G_1) = \{\{v\}\}$.
2. For $k = 2, 3, \dots$:
   (a) For each $S \in \mathcal{S}(G_{k-1})$:
      i. Let H_S be a copy of G_{k-1}.
      ii. For each $X \in \mathcal{S}(H_S)$:
         A. Let $v_{S,X}$ be a new vertex and make $v_{S,X}$ adjacent to all vertices in X and no others.
         B. Let $\mathcal{S}(S, X) = \{S \cup X, S \cup v_{S,X}\}$.
      iii. Let H'_S be the graph induced by $V(H_S)$ and the vertices $v_{S,X}$ for $X \in \mathcal{S}(H_S)$.
      iv. Let $\mathcal{S}(S) = \bigcup_{X \in \mathcal{S}(H_S)} \mathcal{S}(S, X)$.
   (b) Let G_k be the union of G_{k-1} and all H'_S $(S \in \mathcal{S}(G_{k-1}))$.
   (c) Let $\mathcal{S}(G_k) = \bigcup_{S \in \mathcal{S}(G_{k-1})} \mathcal{S}(S)$.

Theorem 1 ([2, 12]). *For $k \in \mathbb{N}$, let G_k and $\mathcal{S}(G_k)$ be as in Burling's construction. The following invariants hold:*

1. *$\mathcal{S}(G_k)$ is a collection of independent sets in G_k.*
2. *G_k is triangle-free.*
3. *$\chi(G_k) \geqslant k$. In particular, for any proper colouring ϕ of G_k, there exists $T \in \mathcal{S}(G_k)$ such that ϕ uses at least k colours on T.*

Proof of Theorem 1. Statements 1 and 2 are easily proven by induction. Using the notation from the construction, for each $S \in \mathcal{S}(G_{k-1})$ and each $X \in \mathcal{S}(H_S)$, S and X are independent sets in G_{k-1}, there are no edges between S and X in G_k, and no edges between S and $v_{S,X}$ in G_k. Thus, $S \cup X$ and $S \cup \{v_{S,X}\}$ are independent sets in G_k. For statement 2, note that G_{k-1} is triangle-free and the neighbourhood of $v_{S,X}$ in G_k is X, which is an independent set, so G_k is triangle-free.

To prove statement 3, assume inductively that it holds for G_{k-1}. Let ϕ be a proper colouring of G_k. Now inductively there exists $S \in \mathcal{S}(G_{k-1})$ such that ϕ uses at least $k-1$ colours on S, and there exists $X \in \mathcal{S}(H_S)$ such that ϕ uses at least $k-1$ colours on X (since H_S is a copy of G_{k-1}). Then ϕ uses at least k colours on $S \cup X$ or $S \cup \{v_{S,X}\}$. For, if only $k-1$ colours are used on $S \cup X$ (so that we cannot take $T = S \cup X$ in statement 3), then $v_{S,X}$ must use a new colour, and we may take $T = S \cup \{v_{S,X}\}$. $\square$

Remark 1. *The use of independent sets in Burling's construction is unique among constructions of triangle-free graphs with arbitrarily large chromatic number. The idea is that maintaining a collection of independent sets at each iteration allows us to avoid unwanted intersections, preserving the triangle-free condition, while the vertices $v_{S,X}$ serve as a "bridge" between independent sets, allowing us to inductively use information about colour usage on independent sets from the previous iteration.*

Theorem 2 (Pawlik *et.al.* [29])**.** *There exists a sequence of intersection graphs of line segments in the plane realising the sequence of graphs in Burling's construction.*

The line segments realising each graph in the sequence will be contained in the interior of the (closed) rectangle $R = [a, b] \times [c, d]$ where $a < b$ and $c < d$. Given a set of line segments L contained in the interior of R, we say that a rectangle P is a *probe* for the pair (R, L) and that V_P is the *vacant rectangle* of the probe P (*root* in [29]) if the following hold:

1. $P = [a', b] \times [c', d']$ where $a < a' < c$ and $c < c' < d' < d$.

2. If $l \in L$ and $l \cap P \neq \varnothing$, then l intersects the top and bottom boundaries of P, does *not* intersect the left boundary of P, and has both endpoints lying outside of the boundaries of P. We say that l *pierces* P, and let $L(P)$ denote the set of lines piercing P.

3. The line segments in $L(P)$ are pairwise disjoint. That is, in the intersection graph of L, they form an independent set.

4. Let V_P be the widest rectangle contained in P whose left, top, and bottom sides lie on the respective boundaries of P and whose interior is disjoint from all line segments piercing P.

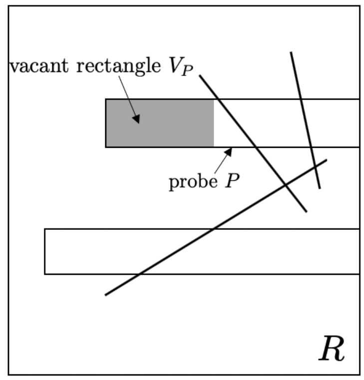


Figure 1: Probes and vacant rectangles. The line segments piercing each probe are pairwise disjoint.

See Figure 1. Note that necessarily $b' > a'$ since we have specified that no $l \in L$ intersects the left boundary of P. Also note that none of the line segments in L may be horizontal by definition of probe.

The sequence of realisations of Burling's graphs in Theorem 2 will generate triples $(G_k, L_k, \mathcal{P}(G_k))$ where G_k is the kth Burling graph, L_k is the set of lines realising G_k (informally, $V(G_k) = L_k$), and $\mathcal{P}(G_k)$ is a collection of probes for the pair (R, L_k). The collection of probes $\mathcal{P}(G_k)$ will correspond to the collection of independent sets $\mathcal{S}(G_k)$ generated at each step of Burling's construction; that is,

$$\mathcal{L}(G_k) := \{L(P) : P \in \mathcal{P}(G_k)\} = \mathcal{S}(G_k)$$

Hence, this is why we required in the definition of probe above that the lines piercing a probe be pairwise disjoint. The copies H_S of G_{k-1} referred to in Burling's construction will be placed inside the vacant rectangles of each probe corresponding to S, thereby preventing unwanted intersections between $V(H_S)$ and the existing lines from G_{k-1}. Below, we present the sequence satisfying Theorem 2, mirroring the notation of Burling's construction to establish the correspondence.

Proof of Theorem 2.

1. Let v be any non-horizontal line contained in the interior of R. Set $L_1 = \{v\}$. Let P be a probe of (R, L_1) such that v pierces P. Set G_1 to be the intersection graph of L_1 and set $\mathcal{P}(G_1) = \{P\}$.

2. For each $k = 2, 3, \dots$:

   (a) For each $P \in \mathcal{P}(G_{k-1})$:

      i. By rescaling, place a copy of (R, L_{k-1}) inside V_P, and temporarily draw the probes $\mathcal{P}(G_{k-1})$ inside V_P. Denote by H_P the copy of G_{k-1} corresponding to this copy of (R, L_{k-1}) and denote by $\mathcal{P}(H_P)$ the set of probes.

      ii. For each $Q \in \mathcal{P}(H_P)$:

         A. Draw the diagonal $v_{P,Q}$ of the rectangle Q, whose endpoints are the bottom left vertex and top right vertex of Q.

         B. Draw a probe inside Q that is pierced exactly by the line segments piercing Q (and disjoint from the diagonal $v_{P,Q}$). Extend the probe's right boundary to the right boundary of R, and call the resulting rectangle Q_l the *lower probe* of Q.

         C. Draw a probe inside Q that is pierced by the diagonal $v_{P,Q}$ (and disjoint from the line segments piercing Q). Extend the probe's right boundary to the right boundary of R, and call the resulting rectangle Q_u the *upper probe* of Q.

         D. Let $\mathcal{P}(P, Q) = \{Q_l, Q_u\}$.

      iii. Let H'_P be the graph induced by $V(H_P)$ and the vertices (line segments) $v_{P,Q}$ for $Q \in \mathcal{P}(H_P)$.

      iv. Let $\mathcal{P}(P) = \bigcup_{Q \in \mathcal{P}(H_P)} \mathcal{P}(P, Q)$.

   (b) Let G_k be the union of G_{k-1} and all H'_P $(P \in \mathcal{P}(G_{k-1}))$.

   (c) Let $\mathcal{P}(G_k) = \bigcup_{P \in \mathcal{P}(G_{k-1})} \mathcal{P}(P)$.

   (d) Let $L_k = V(G_k)$.

See Figure 2. Observe that the line segments in L_k consist of all of the line segments in L_{k-1}, the line segments in the $|\mathcal{P}(G_{k-1})|$ copies of L_{k-1}, and the $|\mathcal{P}(G_{k-1})|^2$ new diagonals drawn. On the other hand, all of the probes from $\mathcal{P}(G_{k-1})$, along with the copies $\mathcal{P}(H_P)$ of such probes, are discarded, and the probes in $\mathcal{P}(G_k)$ consist only of the newly added upper and lower probes. It is clear that the sequence of graphs generated by this construction realises

Burling's sequence once we make the following identifications.

$L(P)$	S
$L(Q)$	X
$v_{P,Q}$ intersects exactly $L(Q)$	$N(v_{S,X}) = X$
$v_{P,Q} \cup L(P)$ is a pairwise disjoint set piercing $Q_u \in \mathcal{P}(G_k)$	$v_{S,X} \cup S$ is an independent set in $S(G_k)$
$L(Q) \cup L(P)$ is a pairwise disjoint set piercing $Q_l \in \mathcal{P}(G_k)$	$X \cup S$ is an independent set in $S(G_k)$

□

Remark 2. *In the construction of Theorem 2, each iteration used a copy of the previous iteration along with rescaled and translated copies of the previous iteration. This is consistent with the corollary to the result of Suk [33] that states intersection graphs of unit segments in the plane are χ-bounded. Further, Pawlik et.al. [28] generalised their result to arc-connected, compact sets in the plane that are not axis-aligned rectangles: given such a set E, a triangle-free family of intersection graphs can be constructed from copies of E obtained by independent horizontal and vertical scalings and translations of E. They show that for certain sets E, such as circles, such a family may even be constructed using uniform scaling and translation (homothetic copies). In contrast, taking E to be a convex set, Kim et.al. [16] showed that the family of intersection graphs obtained from homothetic copies of E are linearly χ-bounded by* $6\omega - 6$ *and, additionally, are* $(6\omega - 7)$*-degenerate.*

3.2 Burling's Graphs and Conjectures for χ-boundedness

Scott [31] proved that for every tree T, the family of graphs that does not contain a subdivision of T as an induced subgraph is χ-bounded, and conjectured that this statement holds when T is replaced by any graph. It turns out that the realisation of Burling's graphs as intersection graphs of line segments in the plane provides counterexamples to this conjecture.

It is known that every graph obtained by subdividing every edge of a non-planar graph at least once *cannot* be realised as a string graph, and in particular, as the intersection graphs of segments in the plane [21]. Thus, Scott's conjecture is false for any subdivision of a non-planar graph. Esperet *et.al.* [4] have shown that Burling's construction can be exploited to generate further counterexamples. For instance, they prove Scott's conjecture is false for the graph obtained by subdividing every edge of K_4 at least once. Burling's graphs thus hold new promise for research on χ-boundedness.

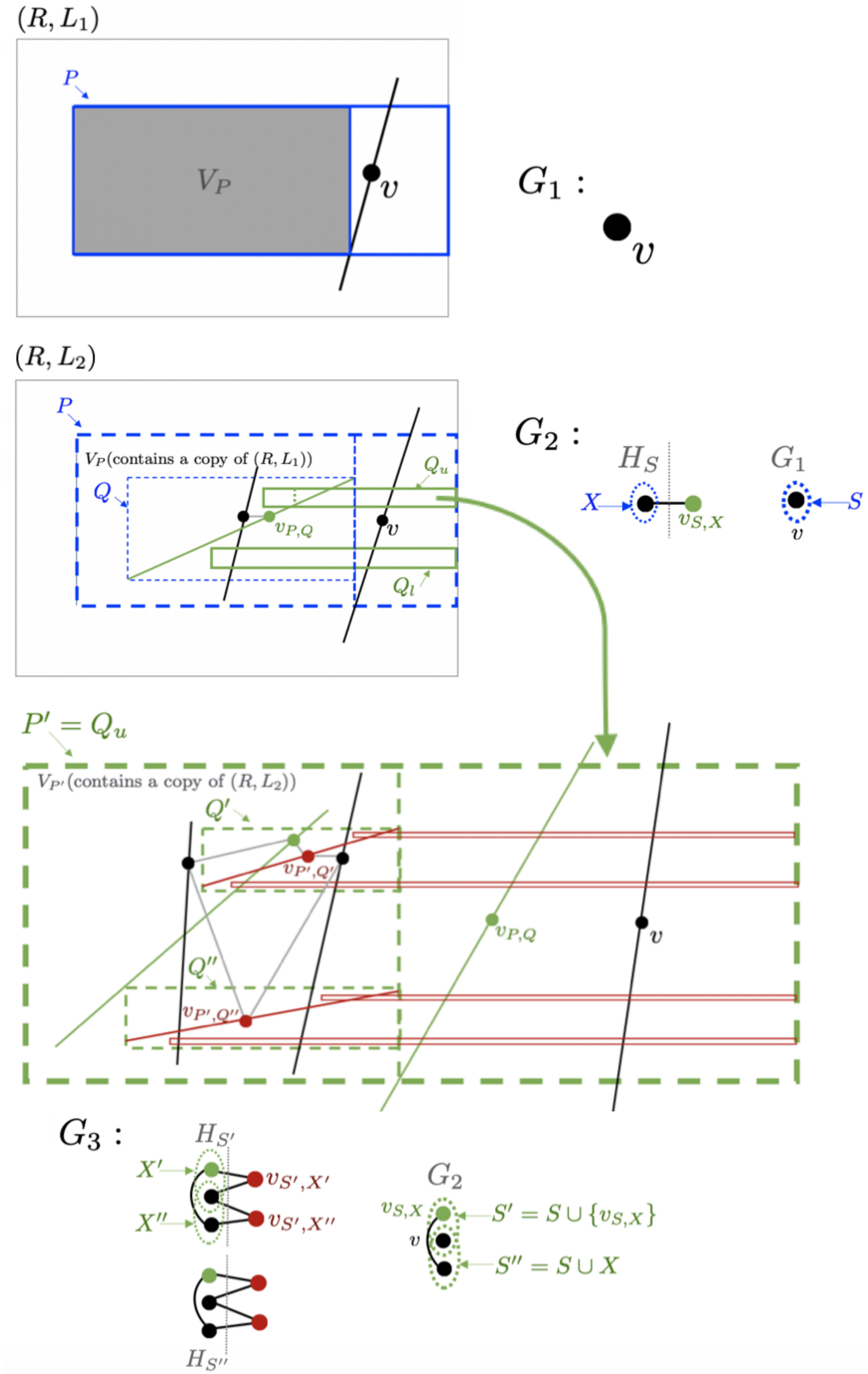


Figure 2: Three iterations of the construction of Pawlik *et.al.* [29], with the corresponding graphs of Burling, using the notation from the text. In the third iteration, we only show the recursion on one of the two green probes from the second iteration.

4 Disjointness graphs of lines and line segments

We have just seen that intersection graphs of line segments in the plane are not χ-bounded. On the other hand, their complements are in fact χ-bounded, and χ-boundedness, along with the specific χ-binding functions, are preserved under increasing dimensionality. Disjointness graphs of lines in the projective space $\mathbb{P}^d$, of lines in $\mathbb{R}^d$, and of line segments in $\mathbb{R}^d$, $d \geqslant 3$, are quadratically, cubically, and quartically χ-bounded, respectively, and each of the latter two bounds can be deduced from the previous bound [26]. We will use these disjointness graphs as a case study for why there is no analogue of the Weak Perfect Graph Theorem for χ-boundedness.

4.1 Disjointness graphs of line segments in $\mathbb{R}^d$

Theorem 3. *Let G be the disjointness graph of a set of line segments in the plane and let ω be its clique number. Then $\chi(G) \leqslant \omega^4$.*

In this section, we fill in a gap in the literature. While Theorem 3 has been referenced in the literature [26], its proof does appear to be explicitly written anywhere. First, recall the following well-known duality theorems for partially ordered sets.

Theorem 4 (Mirsky's Theorem [23])**.** *Let $(P, \prec)$ be a partially ordered set. The minimum number of antichains needed to partition P is equal to the maximum length of a chain.*

Theorem 5 (Dilworth's Theorem [7])**.** *Let $(P, \prec)$ be a partially ordered set. The minimum number of chains needed to partition P is equal to the maximum size of an antichain.*

Definition 4. *A comparability graph G is a graph such that there exists a partially ordered set $(P, \prec)$ which satisfies*

- $V(G) = P$
- $(u, v) \in E(G)$ *if and only if* $u \prec v$ *or* $v \prec u$.

It is straightforward to see that Mirsky's Theorem can be used to prove that comparability graphs are perfect, while Dilworth's Theorem can be used to prove that the complements of comparability graphs are perfect. In [18], a weaker version of Dilworth's Theorem, stating that any partially ordered set on n elements contains either a chain of length p or an antichain of size $\lceil n/p \rceil$, is applied four times on appropriately defined partial orders for convex sets to obtain the following result.

Theorem 6 (Larman *et.al.* [18])**.** *Let G be the disjointness graph of a set of line segments in the plane with independence number α and clique number ω. Then* $\max\{\alpha, \omega\} \geqslant n^{1/5}$.

In [26], it is misstated that Theorem 6 implies Theorem 3. This is not true (though it is easy to see that the converse holds); for if the implication were in this direction, by taking complements it would imply that intersection graphs of line segments in the plane are χ-bounded. It is the case, however, that a method similar to that used to prove Theorem 6 can be used to prove Theorem 3 by applying Mirsky's Theorem four times. Here, I fill in the gap and prove Theorem 3. The proof is my own.

We will use the four partial orders defined on convex sets in [18].

Let $\mathcal{C}$ be a collection of convex sets in $\mathbb{R}^2$, and let $A, B \in \mathcal{C}$. We define four partial orders $\prec_1, \prec_2, \prec_3, \prec_4$ on $\mathcal{C}$; in all four partial orders, A and B must be disjoint in order to be comparable. We let $\pi(A)$ denote the projection of A onto the x-axis, and similarly for $\pi(B)$.

1. $A \prec_1 B$ if $A \cap B = \varnothing$, $\pi(A) \subseteq \pi(B)$, and A lies below B with respect to the y-axis.

2. $A \prec_2 B$ if $A \cap B = \varnothing$, $\pi(A) \subseteq \pi(B)$, and B lies below A with respect to the y-axis.

3. $A \prec_3 B$ if $A \cap B = \varnothing$, $\pi(A)$ and $\pi(B)$ are not contained in each other, the left endpoint of $\pi(A)$ is left of the left endpoint of $\pi(B)$, and if $\pi(A)$ and $\pi(B)$ overlap, A lies above B with respect to the y-axis on the interval of overlap.

4. $A \prec_4 B$ if $A \cap B = \varnothing$, $\pi(A)$ and $\pi(B)$ are not contained in each other, the left endpoint of $\pi(A)$ is left of the left endpoint of $\pi(B)$, and if $\pi(A)$ and $\pi(B)$ overlap, A lies below B with respect to the y-axis on the interval of overlap.

It is straightforward to see that $\prec_1, \prec_2, \prec_3, \prec_4$ are partial orders. We also observe the following two propositions, which are straightforward to verify.

Proposition 1 ([18])**.** *If $\mathcal{C}$ is a collection of convex sets, $A, B \in \mathcal{C}$, and $A \cap B = \varnothing$, then there exists $i \in \{1, 2, 3, 4\}$ such that $A \prec_i B$.*

Observe that a chain with respect to $\prec_i$, for any i, is a set of pairwise disjoint sets. However, an antichain with respect to $\prec_i$ is not necessarily a set of pairwise intersecting sets. However, since the four partial orders are exhaustive by Proposition 1, the following proposition holds:

Proposition 2 ([18])**.** *If $\mathcal{C}$ is a collection of convex sets and $\mathcal{A} \subseteq \mathcal{C}$ is an antichain with respect to each of $\prec_1, \prec_2, \prec_3, \prec_4$, then the sets in $\mathcal{A}$ are pairwise intersecting.*

Proof of Theorem 3. Let $\mathcal{C}$ be the set of line segments corresponding to $V(G)$. It suffices to show that $\mathcal{C}$ can be partitioned into at most ω^4 sets, such that the line segments in each set are pairwise intersecting (and thus that the vertices corresponding to each set form an independent set in G). We apply Mirsky's Theorem four times. Let ω_i be the maximum length of a chain in $\mathcal{C}$ with respect to $\prec_i$, $i = 1, 2, 3, 4$. By Theorem 4, $\mathcal{C}$ can be partitioned into ω_1 antichains with respect to $\prec_1$. Each antichain with respect to $\prec_1$ can in turn be partitioned into ω_2 antichains with respect to $\prec_2$. Each antichain with respect to $\prec_2$ can be partitioned into ω_3 antichains with respect to $\prec_3$. Finally, each antichain with respect to $\prec_3$ can be partitioned into ω_4 antichains with respect to $\prec_4$. Thus, $\mathcal{C}$ can be partitioned into $\omega_1\omega_2\omega_3\omega_4$ sets such that each set is an antichain with respect to $\prec_1, \prec_2, \prec_3, \prec_4$. These $\omega_1\omega_2\omega_3\omega_4$ sets are each pairwise intersecting by Proposition 2, and thus are each independent sets in $V(G)$. Moreover, since a chain in each $\prec_i$ corresponds to a clique in G, $\omega_i \leqslant \omega$ for $i = 1, 2, 3, 4$. Thus, $\chi(G) \leqslant \omega_1\omega_2\omega_3\omega_4 \leqslant \omega^4$. $\square$

Remark 3. *Upon cursory inspection of the proof of Theorem 3, one might think that we could apply the dual of Mirsky's Theorem, Dilworth's Theorem, to show that intersection graphs of line segments are χ-bounded (which we know to be false), in analogy with proving perfection of comparability graphs and their complements. Here is why such an approach would ultimately break down. A chain with respect to a partial order corresponds to an independent set in the intersection graph, while an antichain with respect to **all four** partial orders corresponds to a clique. If we were to proceed as above, but now applying Dilworth's theorem four times, we would obtain a decomposition of $\mathcal{C}$ into $\omega_1\omega_2\omega_3\omega_4$ chains, where ω_i is an antichain with respect to $\prec_i$. But since an antichain with respect to $\prec_i$ is not necessarily a set of pairwise intersecting line segments, it is not necessarily the case that $\omega_i \leqslant \omega$. The proof for disjointness graphs, on the other hand, works because a chain with respect to $\prec_i$ is always a set of pairwise disjoint line segments.*

If we relax to disjointness graphs of *curves* in the plane, we lose χ-boundedness. Pach *et.al.* [26] showed that the shift graphs of Erdös and Hajnal [10] can be realised as the disjointness graphs of polygonal curves containing four line segments, and Mütze [24] improved this to polygonal curves containing three line segments. However, we can preserve χ-boundedness under additional restrictions. In 2020, Pach and Tomon [27] proved that the disjointness graphs of x-monotone curves, each of which intersects a fixed vertical line, have χ-binding function $\frac{\omega+1}{2}\binom{\omega+1}{3}$, recovering the quartic bound for line segments, and that this bound is indeed tight for this specific family.

Pach *et.al.* [26] show that Theorem 3 generalises to line segments in $\mathbb{R}^d$, $d \geqslant 2$.

Theorem 7 (Pach *et.al.* [26])**.** *Let G be the disjointness graph of a set of line segments in $\mathbb{R}^d$, $d \geqslant 2$, with clique number ω. Then $\chi(G) \leqslant \omega^4 + \omega^3$. Moreover, given the line segments realising G, there exists a polynomial-time algorithm that identifies a clique K in G and a proper colouring of G using $|V(K)|^4 + |V(K)|^3$ colours.*

The proof of Theorem 7 relies on the proof of Theorem 3 by considering two dimensional planes containing the line segments in $\mathbb{R}^d$, hence the ω^4 in the χ-bound. However, because the line segments lying in distinct planes may be disjoint, and therefore adjacent in G, a correction is needed. The correction is obtained by extending the line segments to lines and viewing those lines as embedded in the projective space $\mathbb{P}^d$. We now turn to results about χ-boundedness of disjointness graphs of lines in $\mathbb{R}^d$ and $\mathbb{P}^d$ for $d \geqslant 3$.

4.2 Disjointness graph of lines in $\mathbb{P}^d$ and $\mathbb{R}^d$

Lines that intersect contain segments that are disjoint. Therefore, if we use a χ-bound for disjointness graphs of lines in $\mathbb{R}^d$ to prove a χ-bound for disjointness graphs of line segments in $\mathbb{R}^d$, we should expect the former bound to be smaller. We follow Pach *et.al.* [26] in proving the following theorems.

Theorem 8 (Pach *et.al.* [26])**.** *Let G be the disjointness graph of a set of lines in $\mathbb{P}^d$, $d \geqslant 3$, with clique number ω. Then $\chi(G) \leqslant \omega^2$. Further, given the set of lines realising G, there exists a polynomial-time algorithm that identifies a clique K in G and a proper colouring of G using $|V(K)|^2$ colours.*

Theorem 9 (Pach *et.al.* [26])**.** *Let G be the disjointness graph of a set of lines in $\mathbb{R}^d$, $d \geqslant 3$, with clique number ω. Then $\chi(G) \leqslant \omega^3$. Further, given the set* *of lines realising G, there exists a polynomial-time algorithm that identifies a clique K in G and a proper colouring of G using $|V(K)|^3$ colours.*

We will use Theorem 8 to prove Theorem 9. By lines in $\mathbb{P}^d$ we mean lines in $\mathbb{R}^d$ as viewed in projective space. The only fact about projective space that we will use is that two parallel lines in $\mathbb{R}^d$ intersect when embedded in $\mathbb{P}^d$. Thus referring to projective space here is really just a way of formalizing the following alteration to the construction of the disjointness graph: two lines are adjacent if and only if they are disjoint and non-parallel. The key relationship between the disjointness graphs of lines in $\mathbb{R}^d$ and of the same lines as viewed in $\mathbb{P}^d$ is given in the next proposition, the proof of which we omit for brevity.

Proposition 3. *Let G be the disjointness graph of a set of lines in $\mathbb{R}^d$ and let G' be the disjointness graph of the same set of lines embedded in $\mathbb{P}^d$. Then an independent set in G' induces a (vertex and edge) disjoint union of complete graphs in G, where the vertex set of each complete graph is a set of pairwise parallel lines.*

Note that some of the complete graphs may be isolated vertices. The relationship between independent sets in G' and cliques in G will be useful because colour classes are independent sets, so we will be able to use a bound for $\chi(G')$ to obtain a bound for $\chi(G)$.

To prove Theorem 8, we will need the following proposition. We omit the proof for brevity.

Proposition 4 ([26])**.** *Let G be the disjointness graph of a set of lines in $\mathbb{P}^d$ and v be an isolated vertex of G. Then $G\backslash v$ is the complement of the line graph of a bipartite graph, and thus G is perfect.*

Proof of Theorem 8. Let C_0 be the vertex set of a maximal clique of G. Since C_0 is maximal, for every vertex $v \in V(G)\backslash C_0$, there exists $c \in C_0$ such that c and v are non-adjacent. Thus $V(G)$ can be partitioned into $|C|$ sets $V_1, \dots, V_{|C|}$ such that $G|V_i$ contains exactly one vertex c from C and c is isolated in $G|V_i$. Each $G|V_i$ is perfect by Proposition 4, so

$$\chi(G) \leqslant |C_0| \cdot \max\{\omega(G|V_i)|1 \leqslant i \leqslant c\} \leqslant \omega \cdot \omega,$$

as desired. The algorithmic claim follows from the fact that the maximal clique C_0 can be found efficiently, and the sets V_i can also be found efficiently. Since each graph $G|V_i$ is perfect, there exists an efficient algorithm to colour its vertices with $\omega(G|V_i)$ colours and to find a maximum clique C_i [20]. Thus in the statement of the theorem we may take $K = \arg\max_{i=0,1,\dots,|C|} |C_i|$. $\square$

Proof of Theorem 9. Take G as in the statement of the theorem and G' as in Proposition 3. By Theorem 8, $V(G)$ can be partitioned into sets $V_1, \dots, V_{\chi(G')}$ such that V_i is an independent set in G'. Let k be the maximum number of pairwise parallel lines in $V(G)$. Since each V_i induces the disjoint union of complete graphs in G, where each complete graph is a set of pairwise parallel lines, each complete graph has size at most k. Thus each V_i can be partitioned into k independent sets in G (some may be empty), by letting each set contain a vertex from each complete graph in V_i. Thus

$$\chi(G) \leqslant k \cdot \chi(G') \leqslant \omega \cdot (\omega(G'))^2 \leqslant \omega \cdot \omega^2 \leqslant \omega^3,$$

as desired. The algorithmic claim follows from the fact that the sets V_i can be found efficiently by colouring G', per Theorem 8, and the clique decomposition of each V_i can likewise be found efficiently. Letting K_0 be as in Theorem 8 and K_i be the largest clique in the decomposition of V_i, we may take $K = \arg\max_{i=0,1,\dots,\chi(G')} |K_i|$ to satisfy the algorithmic claim in the statement of the theorem. $\square$

Remark 4. *The proof of Theorem 8 rests on Proposition 4, since this allows for the maximal clique construction in the proof. In turn, the construction of an efficient algorithm in Theorems 8 and 9 rests on the maximal clique construction. This lends insight into why the case of disjointness graphs is quite different, for colouring purposes, from the case of intersection graphs: specifically, the presence of an isolated vertex in a disjointness graph of lines in $\mathbb{P}^d$ reveals global information about the structure of the graph. Proposition 4 thus helps us make sense of the algorithmic results in Theorems 8 and 9, which might seem surprising because computing ω and χ for disjointness graphs of lines in $\mathbb{R}^d$ and of lines in $\mathbb{P}^d$ are NP-hard problems [26].*

5 χ-boundedness of Circle Graphs

In this section, we will present the 2019 result of Davies and McCarty [6] that circle graphs (intersection graphs of chords on a circle) are quadratically χ-bounded. First, we mention an equivalent definition of circle graphs as overlap graphs, which shall be easier to work with, and some results about permutation graphs, which will be a key tool in the proof.

5.1 Preliminaries: Permutation Graphs

The below definitions and propositions can be found in [6].

Definition 5. *An interval system $\mathcal{I}$ is a finite collection of open subintervals of $(0,1)$ such that 0 and 1 do not appear as ends of any subintervals in $\mathcal{I}$ and no two distinct subintervals in $\mathcal{I}$ share an end.*

Definition 6. *The interval graph of an interval system $\mathcal{I}$, denoted $G = G(\mathcal{I})$, is the graph with vertex set and edge set defined by*

$$V(G) = \{I \in \mathcal{I}\}$$

$$E(G) = \{(I, I') \in \mathcal{I} : I \cap I' \neq \emptyset\}.$$

Definition 7. *The overlap graph of an interval system $\mathcal{I}$, denoted $G = G(\mathcal{I})$, is the graph with vertex set and edge set defined by*

$$V(G) = \{I \in \mathcal{I}\}$$

$$E(G) = \{(I, I') \in \mathcal{I} : I \text{ and } I' \text{ overlap}\}$$

where I and I' are said to overlap if they have non-empty intersection and neither interval contains the other.

Proposition 5. *Circle graphs and overlap graphs are equivalent.*

Definition 8. *A permutation graph is a graph that is isomorphic to the overlap graph of some interval system in which there exists a point contained in every interval of the interval system.*

Proposition 6. *Permutation graphs are a special case of comparability graphs, and are therefore perfect.*

Proposition 7. *The disjoint union of permutation graphs is a permutation graph.*

Proposition 8 (Claim 5.1 in [6])**.** *Let (a, b) be an open interval in $(0, 1)$, with $0 < p_1 < p_2$, and let $\mathcal{I}$ be a collection of intervals such that for every $I \in \mathcal{I}$, I overlaps with (p_1, p_2). Then the overlap graph $G(\mathcal{I})$ of $\mathcal{I}$ is isomorphic to a permutation graph.*

There is an error in the proof of Proposition 8 in [6] (Claim 5.1); the isomoprhism they define does not quite work. We alter it here to obtain a correct proof.

Proof of Proposition 8. Let $\mathcal{J}$ be the set of all intervals in $(0, 1)$ that contain p_2. We will demonstrate a one-to-one map $f : \mathcal{I} \to \mathcal{J}$ such that intervals $I_1, I_2 \in \mathcal{I}$ overlap if and only if their images $f(I_1)$ and $f(I_2)$ overlap. By definition, the overlap graph $G(f(\mathcal{I}))$ of $f(\mathcal{I})$ is a permutation graph, and by construction of f, the overlap graph $G(\mathcal{I})$ of $\mathcal{I}$ is isomorphic to $G(f(\mathcal{I}))$.

Let a be the smallest end of an interval in $\mathcal{I}$ and b the largest end of an interval in $\mathcal{I}$. Note $a > 0$ and $b < 1$ by definition of an interval system. Thus we may choose $\varepsilon > 0$ satisfying $1 - \varepsilon a^{-1} > b$. Define $\mathcal{I}_1 = \{I \in \mathcal{I} : p_1 \in I\}$ and $\mathcal{I}_2 = \{I \in \mathcal{I} : p_2 \in I\}$. We may assume $\mathcal{I}_1$ and $\mathcal{I}_2$ are non-empty, for otherwise the proposition is immediate. Since every interval in $\mathcal{I}$ overlaps with (p_1, p_2), we have that $(\mathcal{I}_1, \mathcal{I}_2)$ is a partition of $\mathcal{I}$. For $(c, d) \in \mathcal{I}_1$, define $f((c, d)) = (d, 1 - \varepsilon c^{-1})$. Clearly, $f(\mathcal{I}_1) \subset \mathcal{J}$, since $d < p_2$ and our choice of ε ensures that $1 - \varepsilon c^{-1} > b > p_2$. For $I \in \mathcal{I}_2$, define $f(I) = I$. For $(c, d), (c', d') \in \mathcal{I}_1$, $c < c'$, $f(c, d)$ and $f(c', d')$ overlap if and only if $d < d'$ and $1 - \varepsilon c^{-1} < 1 - \varepsilon(c')^{-1}$, equivalently, if and only if $d < d'$ and $c < c'$; that is, if and only if (c, d) and (c', d') overlap. For $(c, d) \in \mathcal{I}_1$ and $(c', d') \in \mathcal{I}_2$, $f(c, d)$ and $f(c', d')$ overlap if and only if $c' < d$ (by choice of ε); that is, if and only if (c, d) and (c', d') overlap. $\square$

5.2 Circle Graphs are Quadratically χ-bounded

Gyárfás [14, 15] gave the first proof that circle graphs are χ-bounded, proving the exponential χ-binding function $2^\omega(2^\omega - 2)\omega^2$. The proof uses a technique known as leveling. In leveling, a graph is partitioned into levels, each of which consists of all vertices at the same graph distance from a fixed root vertex. If each level can be coloured with c colours, the graph can be coloured with $2c$ colours by using different sets of colours on consecutive levels, and the same set of colours on levels with the same parity. We may iterate this procedure by applying leveling to the graphs induced by each level. Gyárfás's proof iterates ω times to get the above bound.

Prior to this recent result of Davies and McCarty [6] proving circle graphs are quadratically χ-bounded, modest improvements were made to Gyárfás's bound but maintained the exponential order, with Kostochka and Kratochvíl [17] in 1997 obtaining the χ-binding function $f(\omega) = 50 \cdot 2^\omega$ and Černý [3] in 2007 obtaining the χ-binding function $f(\omega) = 21 \cdot 2^\omega - 24\omega - 24$. As in Gyárfás's proof, both proofs use the leveling technique. It has also been known that $\chi(G) = \Omega(\omega \log \omega)$ [17], so while the quadratic upper bound represents a significant improvement relative to the lower bound, the task of potentially closing the gap remains. Notably, the proof of Davies and McCarty does *not* use the leveling technique, which turns out to colour quite wastefully.

Theorem 10 (Davies and McCarty [6])**.** *Let $\mathcal{I}$ be an interval system, $G = G(\mathcal{I})$ be its overlap graph, and ω be the clique number of G. Then $\chi(G) \leqslant 7\omega^2$.*

To prove this, we reorganise [6], presenting the material in a different sequence and simplifying certain propositions and definitions. For example, we will not explicitly define P-degree and $(P, \prec)$-degree, as defined in their proof, since we find that these definitions obscure the matter at hand. They are essentially replaced by Claim 1. The statement of the main theorem, Theorem 11, is not given explicitly in [6] but can be found in [22].

The strategy will be to partition $V(G)$ into at most 7ω sets of intervals, such that the subgraph of G induced by each set is a permutation graph. Since each permutation graph is perfect, each can be coloured with at most ω colours, giving the bound of $7\omega^2$. (Note that we say "at most" ω colours because the clique number of a subgraph of G is not necessarily equal to the clique number of G itself.)

Now we know by Proposition 8 that the overlap graph of a collection of intervals in which there exists a point contained in every interval is a permutation graph. Further, by Proposition 7, the disjoint union of permutation graphs is a permutation graph. These two ways of generating permutation graphs are the only ones we will use to generate the 7ω permutation graphs we seek.

Let us call any point $p \in (0, 1)$ a *pillar* of $\mathcal{I}$ if it is not an endpoint of any interval in $\mathcal{I}$. We would like to place pillars in $(0, 1)$, assign colours to pillars, and use the pillars to "colour" the intervals in $\mathcal{I}$, such that the subgraph of G induced by all intervals given the same colour is a permutation graph. Per the above, we would like to use at most 7ω colours. We will only allow a pillar to colour an interval if the interval contains the pillar (and each interval in $\mathcal{I}$ will be coloured by *exactly one* pillar). Note that in the above, "colour" is *not* used in the context of a graph colouring; we use this terminology because it is visually intuitive. For simplicity, assume the colours belong to $\mathbb{N}$.

We place pillars, assign them colours, and colour intervals by an iterative process: first, place a pillar p_1, assign a colour $c(p_1)$ to p_1, and greedily colour with $c(p_1)$ all intervals containing p_1. For each subsequent iteration i, place a new pillar p_i, assign a colour $c(p_i)$ to p_i, and greedily colour with $c(p_i)$ all intervals containing p_i *that have not already been coloured by one of* $p_1, \ldots, p_{i-1}$.

The challenge will be in selecting pillars and assigning colours to pillars such that

1. For each colour, the subgraph of G induced by the set of intervals given that colour is a permutation graph.

2. At most 7ω colours are assigned to pillars, so that we obtain at most 7ω permutation graphs, as desired.

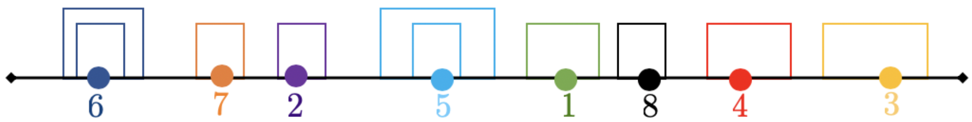
Suppose that for each $i = 1, \ldots, N$, where N is the number of iterations, we let $c(p_i) = i$, that is, we use a new colour for each newly placed pillar. Then each subgraph of G induced by each set of intervals receiving the same colour will be a permutation graph by construction, since there exists a point contained in all intervals in the set. However, we cannot guarantee that at most 7ω colours are used, since we may need arbitrarily many pillars (relative to ω) to reach all of $\mathcal{I}$ (see Figure 3a).

In order to reuse colours while maintaining 1, we will enforce the following condition on the assignment of colours to pillars.

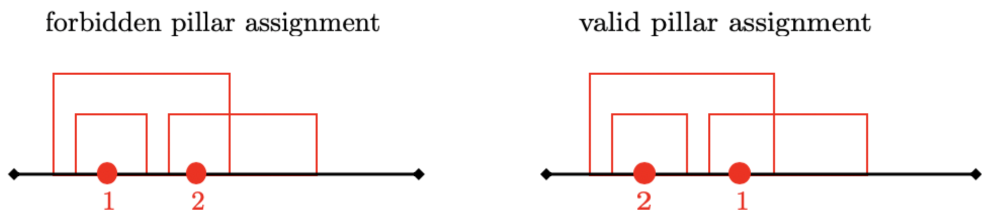
Condition 1. *At every iteration $j > 1$, we allow setting $c(p_j) = c$ if, for all $i < j$ such that $c(p_i) = c$, no interval that will be coloured by p_j overlaps with an interval already coloured by p_i. It is also allowed that we set $c(p_j)$ to an unused colour, even if the condition above can be satisfied with a previously used colour.*

Note that once p_j has been placed, the intervals to be coloured by p_j are well-defined, so that this condition makes sense. If we enforce this condition at every iteration $j = 2, \ldots, N$, it will be the case that for every colour used, the subgraph of G induced by the set of intervals given that colour will be a permutation graph. This is because the subgraph induced by these intervals is the disjoint union of permutation graphs by construction of Condition 1.

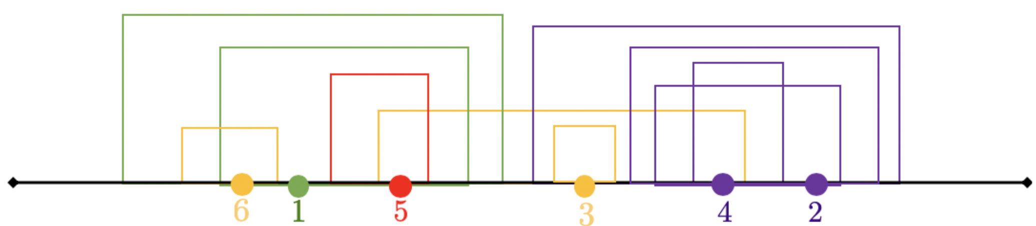
We call the pair consisting of the *ordered* sequence of pillars and the corresponding colours assigned to pillars (satisfying Condition 1) a *pillar assignment* for $\mathcal{I}$, and denote it $P(\mathcal{I}) = ((p_1, \ldots, p_N), (c(p_1), \ldots, c(p_N)))$. **We do not require every interval in $\mathcal{I}$ to be coloured in a pillar assignment. We call a pillar assignment that does colour every interval in $\mathcal{I}$ a *complete pillar assignment*. To prove Theorem 10, it suffices to find a *complete* pillar assignment for $\mathcal{I}$ using at most 7ω distinct colours.**



(a) Pillars numbered in the order in which they are placed. Each new pillar is given a distinct colour. Such a strategy cannot allow us to bound the number of colours used on pillars in terms of ω, because we may require arbitrarily many pillars in order to colour all intervals



(b) In both diagrams, the pillars are numbered in the order in which they are placed. The left is not a valid pillar assignment because the interval coloured by pillar 2 overlaps with an interval coloured by pillar 1, and pillars 1 and 2 are given the same colour (red). The right is a valid pillar assignment because the interval coloured by pillar 2 does not overlap with either interval coloured by pillar 1, so pillar 1 may be given the same colour (red) as pillar 2.



(c) A valid pillar assignment, in which pillars are numbered in the order in which they are placed. All intervals containing pillar i that have not already been coloured by one of pillars $1, \ldots, i-1$ are coloured by pillar i. Further, one can check that Condition 1 is satisfied. This pillar assignment is complete because every interval is coloured.

Figure 3: Pillar assignments.

See Figures 3b and 3c.

We say that a pillar assignment $P^*(\mathcal{I})$ *extends a pillar assignment* $P(\mathcal{I}) = ((p_1, \ldots, p_N), (c(p_1), \ldots, c(p_N)))$ if

$$P^*(\mathcal{I}) = ((p_1, \ldots, p_N, p_{N+1}, \ldots, p_{N+k}), (c(p_1), \ldots, c(p_N), c(p_{N+1}), \ldots, c(p_{N+k})))$$

for some $k \geqslant 1$. In other words, we continue to iterate past N, obtaining $P^*(\mathcal{I})$ from $P(\mathcal{I})$ by extending the sequence of existing pillars (while maintaining the rules for how pillars may be placed and their colours assigned). Speaking of extending a pillar assignment makes sense because a pillar assignment may not be complete, as stated above.

We first prove a result that implies circle graphs are *quadratically* χ-bounded but is weaker than Theorem 10, in that it does not achieve the coefficient of 7. We do this first because the calculations involved will be less cumbersome and the proof concept will thus be easier to see. Once we have proved this result, we will show the calculation that gives the coefficient of 7.

Theorem 11. *[22] Let $\mathcal{I}$ be an interval system, $G = G(\mathcal{I})$ be its overlap graph, and ω be the clique number of G. Assume $\omega \geqslant 12$. If $P(\mathcal{I})$ is a pillar assignment for $\mathcal{I}$ using exactly 2ω pillars, each given a distinct colour, then there exists a* ***complete*** *pillar assignment $P^*(\mathcal{I})$ extending $P(\mathcal{I})$ using at most $\lfloor 2.5\omega \rfloor$ colours.*

Since a pillar assignment may use any number of pillars for any $\mathcal{I}$ (we do not require that a pillar colour any interval), the theorem may be applied to any $\mathcal{I}$ such that $\omega \geqslant 12$. Since we already know that circle graphs are (exponentially) χ-bounded, it follows that circle graphs are quadratically χ-bounded, since Theorem 11 gives a χ-bounding function of $f(\omega) = 2.5\omega^2$ for $\omega \geqslant 12$, and we may obtain a χ-binding function for all ω by combining this bound with, say, Gyárfás's exponential χ-binding function for $\omega = 1, \ldots, 11$.

Given a pillar assignment, we will call the (open) intervals into which the pillars partition $(0, 1)$ *segments*. Further, we will say that a segment S *sees* a colour c if there is an interval with one end in S and one end outside of S that is given the colour c under the pillar assignment. Note that any interval with one end in S and one end outside of S necessarily has been coloured, because it contains at least one pillar. Likewise, an uncoloured interval is contained in one segment. See Figure 4a.

Proof of Theorem 11. If all intervals in $\mathcal{I}$ are coloured by the 2ω pillars, which use 2ω colours, then the theorem is satisfied and we are done. We consider the case that there exist intervals that are uncoloured under the pillar assignment $P(\mathcal{I})$. We may further assume that all uncoloured intervals are contained in one segment; for, given a segment, we can temporarily delete all uncoloured intervals not contained in that segment, and use the conclusion of the theorem to colour the uncoloured intervals in that segment. We may do this separately for each segment, since an interval contained in one segment by definition does not overlap with an interval contained in another segment, so no colour conflicts could arise. Further, since $P^*(\mathcal{I})$ in the theorem is an extension of $P(\mathcal{I})$, all intervals coloured by the initial 2ω pillars in $P(\mathcal{I})$ would remain that way.

Let S be the segment containing the uncoloured intervals. Add pillars inside S so that among the new segments in S formed by the new pillars, each except the rightmost sees exactly $\lceil 1.5\omega \rceil$ distinct colours, and the rightmost sees at most $\lceil 1.5\omega \rceil - 1$ colours. Let n be the number of new pillars. Thus there are $n+1$ new segments and the n leftmost each see $\lceil 1.5\omega \rceil$ distinct colours. In Figure 4b, $\omega = 2$, $\lceil 1.5\omega \rceil = 3$, and $n = 2$; the 2 new pillars are indicated by diamonds, and the 2 leftmost segments each see $3 = \lceil 1.5 \cdot 2 \rceil$ distinct colours. Note that if S sees strictly fewer than $\lceil 1.5\omega \rceil$ distinct colours, we do not add any pillars inside S. Let Q be the set of pillars added inside S. We will later assign an ordering to Q and use it to extend $P(\mathcal{I})$.

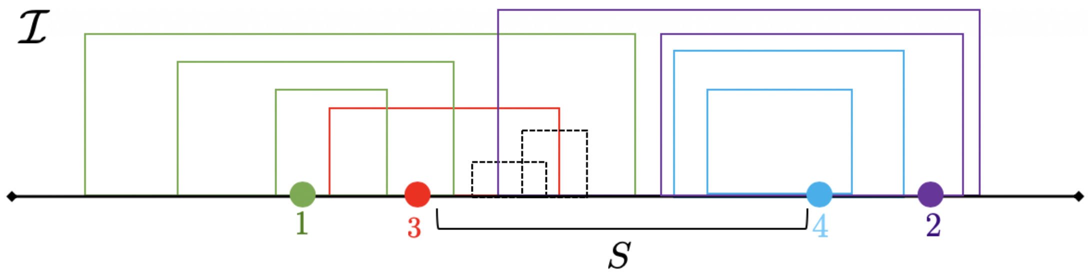
Delete the uncoloured intervals in S and all intervals which do not have an end in S, so that every interval remaining has exactly one end in S. Further, for each of the n leftmost segments of S, keep one interval of each colour and delete the rest. Also delete any intervals with an end in the rightmost segment of S. Let $\mathcal{I}'$ be the resulting interval system and H be its overlap graph. See Figure 4c.

We will now bound the number of newly placed pillars n in terms of the clique number ω. Note this is the only place in the proof where we shall use the clique number. By Proposition 8, H is perfect. Letting $\alpha(H)$ be the independence number of H, we have that

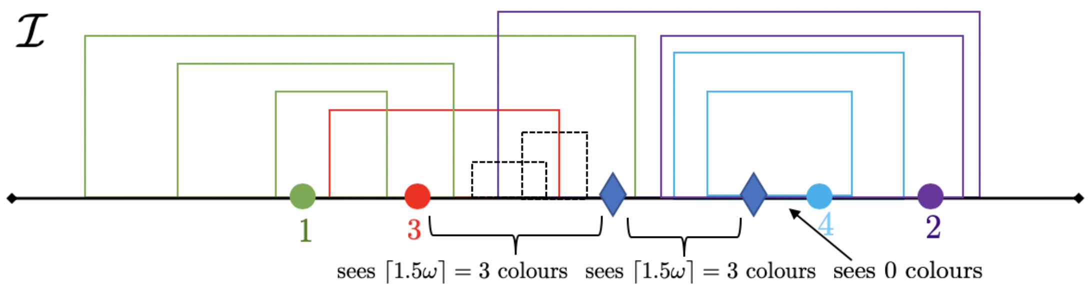
$$\lceil 1.5\omega \rceil n = V(H) \leqslant \alpha(H) \cdot \chi(H) \leqslant \alpha(H) \cdot \omega \tag{1}$$

where the first inequality follows from the fact that H is perfect and colour classes are independent sets.

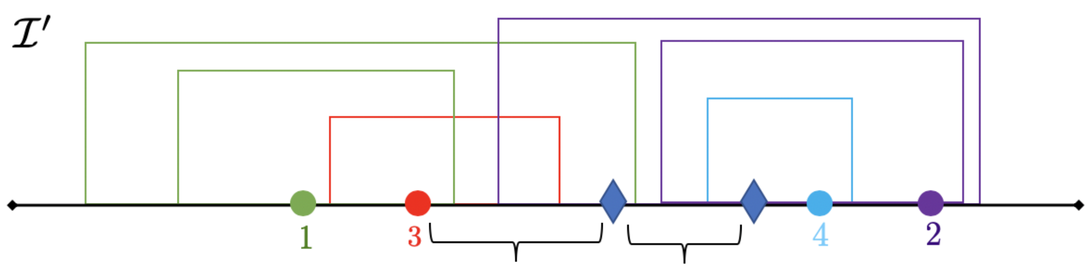
Claim 1. *Given H as above, $\alpha(H) \leqslant 2\omega + n$.*



(a) An interval system $\mathcal{I}$ and a pillar assignment $P(\mathcal{I})$ for $\mathcal{I}$ (pillars are numbered in the order in which they are placed). The intervals that remain uncoloured by $P(\mathcal{I})$ are shown in the dashed black lines, and are contained in the segment S.



(b) A set of pillars Q is added inside S, as indicated by the blue diamonds. They will later be assigned an ordering to extend $P(\mathcal{I})$. The two leftmost segments in S created by the new pillars each see 3 colours (green, purple, and red for the leftmost segment and green, purple, and blue for the second from leftmost segment). The rightmost segment in S sees 0 colours. Note $\omega = 2$.



(c) The interval system $\mathcal{I}'$ obtained from $\mathcal{I}$ through deletions. Specifically, all intervals that do not have an end in S are deleted. (In this case, one such interval is deleted). All (necessarily uncoloured) intervals contained in S (i.e., the dashed intervals in Figure 4b) are deleted. Finally, for each segment of S ("new segments"), deletions are made so that the segment sees only one interval of each colour that it saw in $\mathcal{I}$. For example, here, one of the two blue intervals that the second-from-leftmost segment in S saw was deleted.

Figure 4: Pillar additions and deletions in the proof of Theorem 11.

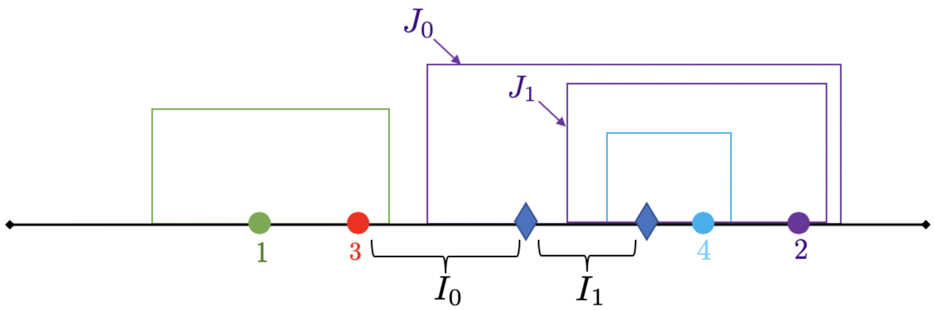


Figure 5: An independent set $\mathcal{J}$ in H (the overlap graph of $\mathcal{I}'$). The purple and light blue intervals form $\mathcal{J}_R$ (and the green interval is excluded). In the proof of Claim 1, we apply the inductive step by deleting the pillar (blue diamond) between I_0 and I_1 and deleting the maximal interval J_0.

Proof of Claim 1. We will refer to the segments inside S as *new segments* and the segments disjoint from S as *old segments*. Observe that every interval in $V(H)$ by construction has one end in a new segment and one end in an old segment. We will induct on the number of pillars, and we assume for smaller cases (i.e. under deleting pillars) that, as in H, no intervals with an end in the same new segment have the same colour. To prove the claim, we need to show that the independence number is at most the number of pillars, $2\omega+n$.

Let $\mathcal{J}$ be an independent set in H and assume for contradiction that $|\mathcal{J}| = 2\omega + n + 1$. Without loss of generality, consider $\mathcal{J}_R \subseteq \mathcal{J}$, the set of intervals in $\mathcal{J}$ whose left ends are in a new segment and whose right ends are in an old segment. Then for every pair of intervals in $\mathcal{J}_R$, one contains the other since $\mathcal{J}_R$ is an independent set and no two intervals in $\mathcal{J}_R$ are disjoint. So $\mathcal{J}_R$ is a set of nested intervals. Let J_0 be the maximal interval and let J_1 be the second most maximal interval in $\mathcal{J}_R$. See Figure 5.

Observe that no two distinct intervals can have both their left ends in the same segment and their right ends in the same segment, for this would imply they contain exactly the same pillars from the initial pillar assignment and thus were given the same colour; but no intervals with an end in the same segment of S have the same colour (by the deletions we used to construct H).

Thus, first assume that the left ends of J_0 and J_1 lie in different segments, call them I_0 and I_1, respectively. If there is only one pillar between I_0 and I_1, then we may delete this pillar and also delete J_0. Since J_0 was chosen maximally, the merged segment resulting from the pillar deletion sees the same intervals that I_1 saw and no other intervals, so we may apply the inductive hypothesis. So now there are $2\omega + n - 1$ pillars and $2\omega + n$ independent sets, which is a contradiction. Note that we must delete J_0 because it is possible that J_0 and J_1 have the same colour, violating the construction of H. See Figure 5.

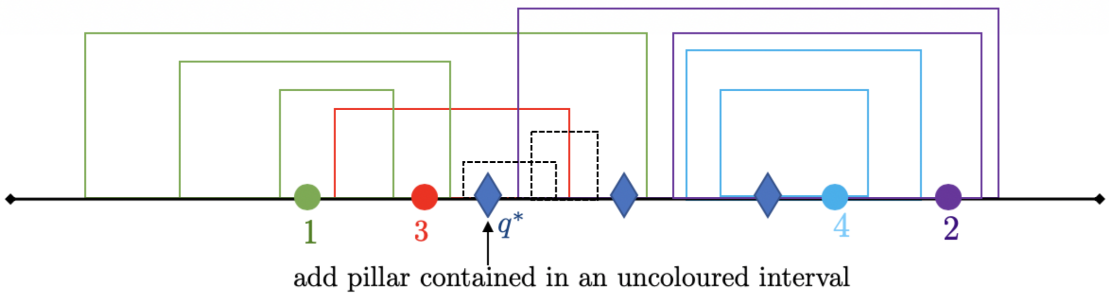
If there is more than one pillar between I_0 and I_1, then we may delete the leftmost pillar, so that there are now $2\omega + n - 1$ pillars and, as before, an independent set of size $2\omega + n + 1$. Since the merged segment resulting from the pillar deletion sees the same intervals as its constitutent segments by choice of J_0 and J_1, we may apply the inductive hypothesis and thus obtain a contradiction. The same contradiction is derived if we assume the right ends of J_0 and J_1 lie in different segments. Note that in this inductive proof we are able to consider $\mathcal{J}_R$ and $\mathcal{J}_L = \mathcal{J} \backslash \mathcal{J}_R$ separately because the intervals in $\mathcal{J} \cap \mathcal{J}_R$ are disjoint from the intervals in $\mathcal{J} \cap \mathcal{J}_L$ by construction of $\mathcal{J}_L, \mathcal{J}_R$ and the fact that $\mathcal{J}$ is an independent set. $\square$

Now we are ready to finish the proof. By Equation 1 and Claim 1,

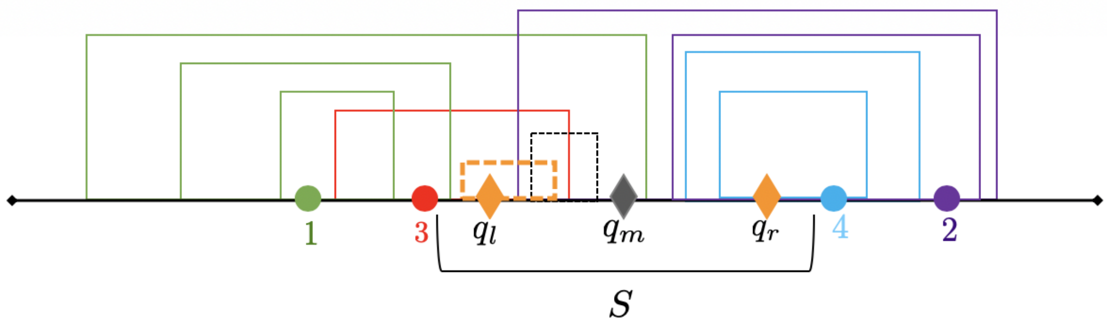
$$\begin{aligned} 1.5\omega n \leqslant \lceil 1.5w \rceil n \leqslant \alpha(H) \cdot \omega \leqslant (2\omega + n)\omega \\ \implies n + 1 \leqslant 4\omega + 1 < 2^{2\omega - \lceil 1.5\omega \rceil} \leqslant 2^{0.5\omega} \quad (\omega \geqslant 12). \end{aligned} \tag{2}$$

Now add an additional pillar q^* in S to ensure that some uncoloured interval contains a pillar in S (if there already exists an uncoloured interval that contains a pillar, we may still place another one). See Figure 6a. Let $Q^* = Q \cup q^*$ be the $n + 1$ new pillars added to S. We claim we may extend the pillar assignment $P(\mathcal{I})$ that used the initial 2ω pillars to a pillar assignment $P^*(\mathcal{I})$ using the additional $n + 1$ pillars in Q^* such that at most $2\omega - \lceil 1.5\omega \rceil$ new colours are used on the set of new pillars Q^*.

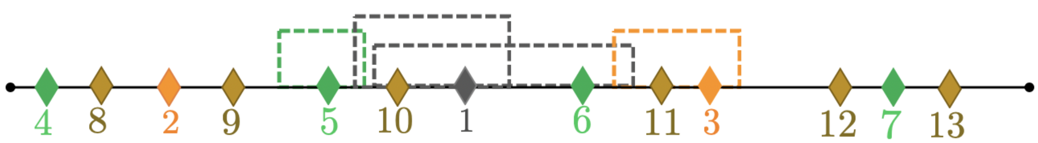
First, append the middle pillar q_m in S to the sequence of pillars from $P(\mathcal{I})$ and assign it a new colour. Then, (in either order) add the middle elements q_l and q_r of the pillars to the left and right of q_m to the sequence, and give them the same (new) colour. See Figure 6b. This gives a valid pillar assignment because any initially uncoloured interval that crosses q_m will already have been coloured by q_m, so that if an interval containing q_l and an interval containing q_r overlap, then they will both have been coloured by q_m (and thus given the same colour, as is required by Condition 1). We iterate this process until all pillars have been coloured. See Figure 6c. Using this recursion to colour the $n + 1$ pillars in Q^*, we will have used at most $\log(n + 1) + 1$ new colours, that is, by Equation 2, we will use at most $2\omega - \lceil 1.5\omega \rceil$ new colours. Further,



(a) A pillar q^* is added to ensure that at least one uncoloured interval becomes coloured in the extension of $P(\mathcal{I})$. The set $Q^* = Q \cup q^*$ consists of all three blue diamond pillars.



(b) Colouring the set of pillars Q^* in the segment S using the recursive method described in the proof of Theorem 11. First the middle pillar q_m is given a new colour (gray), and then the middle pillars to the left and right of q_m (here, q_l and q_r) are given the same new colour (orange). Observe that one of the previously uncoloured intervals in S is coloured orange by the pillar q_l. To see how this recusion works on a larger set Q^*, see Figure 6c.



(c) The recursive process for colouring the pillars in S, on a larger example. The pillars are labelled in the order in which they are assigned (and coloured). We first colour the middle pillar (gray), then colour the pillars that are in the middle of the pillars to the left and right of this pillar orange, and so on, using green and then brown. The intervals containing pillars are coloured in accordance with the order in which the pillars are assigned.

Figure 6: Ordering and colouring the new pillars, and the intervals containing them, in the proof of Theorem 11.

at least one uncoloured interval will be coloured by this process, by choice of q^*.

Finally, suppose there are still uncoloured intervals in S. Recall that each segment in S initially saw at most $\lceil 1.5\omega \rceil$ colours. After colouring the pillars in Q^* with at most $2\omega - \lceil 1.5\omega \rceil$ new colours, each segment in S now sees at most 2ω colours, so that we may now apply the same process above separately to any segment of S containing uncoloured intervals (possibly after deleting pillars and intervals so that there are at most 2ω pillars, as in our initial setting, and they represent all colours that the segment sees). Note that we may reuse the same $2\omega - \lceil 1.5\omega \rceil$ new colours on segments separately because the uncoloured intervals are contained in segments, which are disjoint. This process will necessarily terminate, because we have guaranteed using the above process that at least one uncoloured interval becomes coloured. This completes the proof, once we note that the 2ω existing colours and the $2\omega - \lceil 1.5\omega \rceil$ new colours gives at most $\lfloor 2.5\omega \rfloor$ colours in total. $\square$

Remark 5. *The only part of the proof of Theorem 11 where the clique number is used is in Equation 1, and even then we do not reference the structure of cliques explicitly (in contrast to in the proof of Gyárfás [14]), but need only use that permutation graphs are perfect.*

Finally, the more refined bound of $7\omega^2$ in Theorem 10 follows from the following theorem, which can be proved in the exact same way as Theorem 11 by changing the numbers used in the calculations in Equations 1 and 2.

Theorem 12. *Every pillar assignment using exactly $\omega + \lceil 2\log_2 \omega \rceil + 8$ pillars can be extended to a complete pillar assignment using at most $\omega(\omega + 2\lceil 2\log_2 \omega \rceil + 8) \leqslant 7\omega^2$ colours.*

Proof. In the proof of Theorem 11, replace $\lceil 1.5\omega \rceil$ with $\omega + 8$ (that is, each new segment in S now sees $\omega + 8$ colours). Equation 2 then amounts to

$$n \leqslant \frac{\omega(\omega + \lceil 2\log_2 \omega \rceil + 8)}{\omega + 8} < \omega^2$$

so that $\lceil \log_2 \omega^2 \rceil = \lceil 2\log_2 \omega \rceil$ colours can be used to colour the new segments in S. $\square$

Using Overleaf's word count function (https://www.overleaf.com/blog/word-count-2015-09-15), I obtained: 7450 words.

6 References

[1] E. Asplund and B. Grünbaum. On a coloring problem. *Mathematica Scandinavica*, 8:181–188, 1960.

[2] J. Burling. On coloring problems of families of prototypes (PhD thesis). *University of Colorado, Boulder*, 1965.

[3] J. Černỳ. Coloring circle graphs. *Electronic Notes in Discrete Mathematics*, 29:457–461, 2007.

[4] J. Chalopin, L. Esperet, Z. Li, and P. Ossona de Mendez. Restricted frame graphs and a conjecture of Scott. *The Electronic Journal of Combinatorics*, 23(1):P1.30, 2016.

[5] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas. The strong perfect graph theorem. *Annals of Mathematics*, 164:51–229, 2006.

[6] J. Davies and R. McCarty. Circle graphs are quadratically χ-bounded. *arXiv:1905.11578*, 2019.

[7] R. P. Dilworth. A decomposition theorem for partially ordered sets. In *Classic Papers in Combinatorics*, pages 139–144. Springer, 2009.

[8] G. Ehrlich, S. Even, and R. E. Tarjan. Intersection graphs of curves in the plane. *J. Comb. Theory, Ser. B*, 21:8–20, 1976.

[9] P. Erdős. Graph theory and probability. *Canadian Journal of Mathematics*, 11:34–38, 1959.

[10] P. Erdős and A. Hajnal. On chromatic number of graphs and set-systems. *Acta Mathematica Hungarica*, 17:61–99, 1966.

[11] L. Esperet. *Graph colorings, flows and perfect matchings.* Université Grenoble Alpes, 2017.

[12] S. Felsner, G. Joret, P. Micek, W. T. Trotter, and V. Wiechert. Burling graphs, chromatic number, and orthogonal tree-decompositions. *Electronic Notes in Discrete Mathematics*, 61:415–420, 2017.

[13] A. Gyárfás. On ramsey covering-numbers. *Infinite and Finite Sets*, 2:801–816, 1975.

[14] A. Gyárfás. On the chromatic number of multiple interval graphs and overlap graphs. *Discret. Math.*, 55:161–166, 1985.

[15] A. Gyárfás. Corrigendum. *Discret. Math.*, 62(3):333, 1986.

[16] S.-J. Kim, A. Kostochka, and K. Nakprasit. On the chromatic number of intersection graphs of convex sets in the plane. *The Electronic Journal of Combinatorics*, page R52, 2004.

[17] A. Kostochka and J. Kratochvíl. Covering and coloring polygon-circle graphs. *Discrete Mathematics*, 163(1-3):299–305, 1997.

[18] D. Larman, J. Matoušek, J. Pach, and J. Törőcsik. A ramsey-type result for convex sets. *Bulletin of the London Mathematical Society*, 26(2):132–136, 1994.

[19] L. Lovász. A characterization of perfect graphs. *J. Comb. Theory, Ser. B*, 13(2):95–98, 1972.

[20] L. Lovász, M. Grötschel, and A. Schrijver. *Geometric algorithms and combinatorial optimization*. Springer, 1988.

[21] J. Matoušek. String graphs and separators. *arXiv:1311.5048*, 2013.

[22] R. McCarty. *Circle graphs are quadratically chi-bounded*. IBS Discrete Mathematics Group, Apr 2019. https://www.youtube.com/watch?v=Ed5NlyJv8AM.

[23] L. Mirsky. A dual of dilworth's decomposition theorem. *The American Mathematical Monthly*, 78(8):876–877, 1971.

[24] T. Mütze, B. Walczak, and V. Wiechert. Realization of shift graphs as disjointness graphs of 1-intersecting curves in the plane. *arXiv:1802.09969*, 2018.

[25] J. Mycielski. Sur le coloriage des graphes. *Colloq. Math*, 3:161–162, 1955.

[26] J. Pach, G. Tardos, and G. Tóth. Disjointness graphs of segments. *arXiv:1704.01892*, 2017.

[27] J. Pach and I. Tomon. On the chromatic number of disjointness graphs of curves. *J. Comb. Theory, Ser. B*, 2020. https://doi.org/10.1016/j.jctb.2020.02.003.

[28] A. Pawlik, J. Kozik, T. Krawczyk, M. Lasoń, P. Micek, W. T. Trotter, and B. Walczak. Triangle-free geometric intersection graphs with large chromatic number. *Discrete & Computational Geometry*, 50(3):714–726, 2013.

[29] A. Pawlik, J. Kozik, T. Krawczyk, M. Lason, P. Micek, W. T. Trotter, and B. Walczak. Triangle-free intersection graphs of line segments with large chromatic number. *J. Comb. Theory, Ser. B*, 105:6–10, 2014.

[30] A. Rok and B. Walczak. Coloring curves that cross a fixed curve. *Discrete & Computational Geometry*, 61(4):830–851, 2019.

[31] A. Scott. Induced trees in graphs of large chromatic number. *Journal of Graph Theory*, 24(4):297–311, 1997.

[32] A. Scott and P. Seymour. A survey of χ-boundedness. *arXiv:1812.07500*, 2018.

[33] A. Suk. Coloring intersection graphs of x-monotone curves in the plane. *Combinatorica*, 34(4):487–505, 2014.

[34] D. Sumner. Subtrees of a graph and chromatic number. In *The Theory and Applications of Graphs*, pages 557–576, New York, 1981. John Wiley and Sons.