EC508: Econometrics
Nonstationarity and Unit-Roots

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Nonstationarity

- Recall the definition of a stationary time-series:
  The distribution of $(Y_t, Y_{t+1}, \ldots, Y_{t+s}, X_t, \ldots, X_{t+s})$ **does not depend on** t **for any** $s \geq 0$
- This implies that $\mathbb{E}(Y_t), var(Y_t)$ does not vary with t
- However, many economic variables exhibit some form of non-stationarity:
  1. Deterministic trend: $Y_t = \delta_0 + \delta_1 t + \beta_1 X_t + u_t$
  2. Stochastic trend: $Y_t = \delta_0 + Y_{t-1} + u_t$
  3. Structural Break: $Y_t = \delta_0 + \delta_1 \mathbb{1}_{t \geq t_1} + \beta_1 X_t + u_t$
- In these cases, the LLN and CLT discussed before do not apply directly

Deterministic trend

- Suppose we have:

$$\begin{aligned} Y_t &= \delta_0 + \delta_1 t + \beta_1 X_t + u_t \\ X_t &= \gamma_0 + \gamma_1 t + v_t \end{aligned}$$

- u_t, v_t are stationary and weakly dependent
- Then regress Y_t on $(1, t)$ and X_t on $(1, t)$ using OLS; compute the residuals $\tilde{Y}_t, \tilde{X}_t$ and regress $\tilde{Y}_t$ on $\tilde{X}_t$.
- The resulting estimator $\hat{\beta}_1$ is consistent and has the same limiting distribution as if $\delta_0, \delta_1, \gamma_0, \gamma_1$ were known
- Why? Because $\hat{\delta}, \hat{\gamma}$ are <u>superconsistent</u> (converge at a T-rate rather than usual $\sqrt{T}$-rate)

Stochastic trend

- Suppose we have:

$$Y_t = Y_{t-1} + u_t$$

- u_t has mean 0 and variance σ_u^2; $Y_0 = 0$
- Then $\mathbb{E}(Y_t) = 0$, $var(Y_t) = t \times \sigma_u^2$ for all $t \geq 0$
- Y_t cannot be stationary, its variance increases over time
- This can cause some issues:
  1. AR coefficients are strongly biased towards zero. This leads to poor forecasts
  2. Some t-statistics don't have a standard normal distribution, even in large samples
  3. If Y and X both have random walk trends then they can look related even if they are not: **spurious regressions**.

A Spurious Regression

- Regress US unemployment on the log of Japanese Industrial Production
- Run two regressions; data 1965-1981 and 1986-2012:

$$(1965/1981) \quad US\_Rate_t = -2.37\,(1.19) + 2.22\,(0.32) \times \log(IP)$$
$$(1986/2012) \quad US\_Rate_t = 41.78\,(1.19) - 7.78\,(1.75) \times \log(IP)$$

- The coefficients are very different yet both appear to be significant using the usual t-test and normal approximation...
- The solution is to regress ΔY_t on ΔX_t since

$$Y_t = Y_{t-1} + u_t \Rightarrow \Delta Y_t = u_t$$

- which is stationary so that previous results are valid

Detecting Stochastic trends

- There is a test for a unit-root called the Dickey-Fuller test:

$$Y_t = \beta_0 + \beta_1 Y_{t-1} + u_t$$

- We have a unit root if $\beta_1 = 1$, the model is stationary if $\beta_1 < 1$ (but > -1); re-write:

$$\Delta Y_t = \beta_0 + \delta Y_{t-1} + u_t$$

- The unit-root test is:

$$H_0 : \delta = 0 \text{ vs. } H_1 : \delta < 0$$

- NB: one-sided test
- We use the t-statistic computed from the OLS estimator $\hat{\delta}$
- We can also allow for deterministic trends and serial correlation...

The augmented Dickey-Fuller test

- Without Deterministic trend, estimate using OLS

$$\Delta Y_t = \beta_0 + \delta Y_{t-1} + \gamma_1 \Delta Y_{t-1} + \cdots + \gamma_p \Delta Y_{t-p} + u_t$$

- With a Deterministic trend, estimate using OLS

$$\Delta Y_t = \beta_0 + \delta Y_{t-1} + \beta_1 t + \gamma_1 \Delta Y_{t-1} + \cdots + \gamma_p \Delta Y_{t-p} + u_t$$

- Compute the t-statistic for $\delta = 0$ and compare with the Augmented Dickey-Fuller critical values

ADF Critical Values

	Without trend		With trend	
Sample size	1%	5%	1%	5%
T = 25	3.75	3.00	4.38	3.60
T = 25	3.75	3.00	4.38	3.60
T = 50	3.58	2.93	4.15	3.50
T = 100	3.51	2.89	4.04	3.45
T = 250	3.46	2.88	3.99	3.43
T = 500	3.44	2.87	3.98	3.42
T = ∞	3.43	2.86	3.96	3.41

Structural Breaks

- The mean of Y_t can change over time: GDP growth was higher on average before the 2000s
- The relationship between variables can also change over time: change in monetary policy target, change in exchange rate policy, etc.
- These are called structural breaks, they can be policy induced or be the result of some other mechanism (e.g. lower population growth rate)
- One way to represent this is:

$$Y_t = \beta_0 + \beta_1 X_t + \delta_0 \mathbb{1}_{t \geq t_1} + \delta_1 X_t \mathbb{1}_{t \geq t_1} + u_t$$

- The effect of X on Y was β_1 before t_1 and is $\beta_1 + \delta_1$ after t_1

Structural Breaks, cont'd

- Structural breaks are important for forecasting, estimating dynamic causal effects
- There are two situations:
  1. The break date t_1 is known
  2. The break date t_1 is unknown
- Case 1. if the break date t_1 is known, then estimate by OLS

$$Y_t = \beta_0 + \beta_1 X_t + \delta_0 \mathbb{1}_{t \geq t_1} + \delta_1 X_t \mathbb{1}_{t \geq t_1} + u_t$$

  and compute the Wald-statistic for $H_0 : \delta_0 = \delta_1 = 0$
- This is called the Chow test

Structural Breaks with an Unknown Break Date

- Case 2. the break date t_1 is unknown, we need to estimate it. . .
- Take a range $\tau_0 = 0.15 \times T, \tau_1 = 0.85 \times T$ and for each $\tau \in (\tau_0, \tau_1)$ compute the Chow test $W(\tau)$ assuming the break date is τ
- Compute the following:

$$QLR = \max_{\tau \in (\tau_0, \tau_1)} W(\tau)$$

- The QLR statistic determines if we have a break date with knowing t_1 and $\hat{\tau}$ at which the maximum is attained is the estimated break date
- The distribution is not χ^2! The critical values are given in SW 14 for $F = W/q$, $q = \#$ coefficients with a break