EC508: Econometrics
Hypothesis Testing using the Wald Test

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Joint Hypotheses

$$H_0: R\beta = c \text{ vs. } H_1: H_0: R\beta \neq c$$

- R is a non-zero $q \times (k+1)$ matrix, $q \geq 1$
- c is a $q \times 1$ vector
- Examples:
  - $H_0 : \beta_0 = \beta_1 = \cdots = \beta_k = 0$ then $R = I_{k+1}, c = 0_{k+1}$
  - $H_0 : \beta_1 + \beta_2 = 1$ then

$$R = \begin{pmatrix} 0 & 1 & 1 & 0 & \dots & 0 \end{pmatrix}, \quad c = 0_{k+1}$$

  - What is $H_0 : \beta_1 + \beta_2 = 1, \beta_1 - \beta_2 = -1$
- **Idea:** If H_0 is true then $R\hat{\beta} - c$ must be close to zero

Chi-squared distribution

$$H_0 : \ R\beta = c \text{ vs. } H_1 : \ H_0 : \ R\beta \neq c$$

- Let $Z \sim \mathcal{N}(\mu_Z, \Sigma_Z)$, normally distributed
- Then
$$(Z - \mu_Z)' \Sigma_Z^{-1} (Z - \mu_Z) \sim \chi^2_{\dim(Z)}$$
where $\chi^2_{\dim(Z)}$ is a Chi-squared distribution with $\dim(Z)$ degrees of freedom
- Suppose R is a $q \times \dim(Z)$ matrix with rank $q \leq \dim(Z)$ then:
$$(RZ - R\mu_Z)'[R\Sigma_Z R']^{-1}(RZ - R\mu_Z) \sim \chi^2_q$$
- Continuous Mapping Theorem if $Z_n - \mu_Z \xrightarrow{d} \mathcal{N}(0, \Sigma_Z)$ then
$$(RZ_n - R\mu_Z)'(R\Sigma_Z R')^{-1}(RZ_n - R\mu_Z) \xrightarrow{d} \chi^2_q$$

The Wald statistic in large samples

$$H_0:\ R\beta = c \text{ vs. } H_1:\ H_0:\ R\beta \neq c$$

The Wald statistic is defined as

$$w = n \times (R\hat{\beta} - c)'(R\Sigma R')^{-1}(R\hat{\beta} - c)$$

it measures the distance between H_0 and 0

- Under H_0, we have

  $$\sqrt{n}\left(R\hat{\beta} - c\right) \xrightarrow{d} \mathcal{N}(0, R\Sigma R')$$

  because $c = R\beta$ if H_0 holds

- under H_0, the Wald statistic satisfies

  $$w = n \times (R\hat{\beta} - c)'(R\Sigma R')^{-1}(R\hat{\beta} - c) \xrightarrow{d} \chi^2_q$$

  assuming R has rank q

Hypothesis testing with the Wald statistic

1. Compute $w = n \times (R\hat{\beta} - c)'(R\Sigma R')^{-1}(R\hat{\beta} - c)$
2. If $w > c_{1-\alpha}$, we can reject H_0 at the $1 - \alpha$ confidence level
   $c_{1-\alpha}$ is the $1 - \alpha$ (e.g. 95%) quantile of a χ^2_q distribution; in R, it is computed using *qchisq(0.95,q)*

What is q for:

- $H_0 : \beta_1 = \beta_2 = 0$?
- $H_0 : \beta_1 + \beta_2 + 2 = 0$?

Hint: write it down in matrix form

Rejection and critical value $c_{1-\alpha}$: [Drawing]

Confidence sets

- The $1 - \alpha$ confidence set is the set of β that cannot be rejected at the $1 - \alpha$ confidence level:

$$\mathrm{CS}_{1-\alpha} = \{\beta,\, n \times (\hat{\beta} - \beta)' \Sigma^{-1} (\hat{\beta} - \beta) \leq c_{1-\alpha}\}$$

- The quadratic equation $n \times (\hat{\beta} - \beta)' \Sigma^{-1} (\hat{\beta} - \beta)$ defines an ellipsoid in $\mathbb{R}^{k+1}$
- For a simultaneous confidence set for 2 coefficients this yields an ellipse: [drawing]

Computing the asymptotic variance matrix Σ

- Recall that $\Sigma = \Sigma_X^{-1}\Sigma_v\Sigma_X^{-1}$
- First compute: $\hat{\Sigma}_X = \frac{1}{n}\sum_{i=1}^n X_i X_i'$
- Then compute: $\hat{\Sigma}_v = \frac{1}{n}\sum_{i=1}^n \hat{v}_i\hat{v}_i'$, $\hat{v}_i = X_i\hat{u}_i$
- Finally: $\hat{\Sigma} = \hat{\Sigma}_X^{-1}\hat{\Sigma}_v\hat{\Sigma}_X^{-1}$
- You can implement the Wald test using:

$$w = n \times (R\hat{\beta} - c)'(R\hat{\Sigma}R')^{-1}(R\hat{\beta} - c)$$

- Under homoskedasticity, $\mathbb{E}(u_i^2|X_i) = \sigma_u^2$ so that $\Sigma_v = \sigma_u^2\Sigma_X$. This implies that $\Sigma = \sigma_u^2\Sigma_X^{-1}$ which can be computed similarly. In general we don't want to assume homoskedasticity so we compute the general formula above called the heteroskedasticity robust asympotic variance estimator.