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CS430

Spring 2023

Introduction to Algorithms

Lec 3

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Agenda

- Runtime Analysis–proof
  - examples
  - classworks

Asymptotic Notation

We have to investigate how the running time of an algorithm increases in the limit with the size of input going infinite.

For a given function g(n), we denote by Θ(n) the set of functions:

$\Theta(g(n))=$ { f(n), there exits positive constants c_1, c_2 and n_0 such that $0 <= c_1 g(n) <= f(n) <= c_2 g(n)$ for all $n > n_0$ }

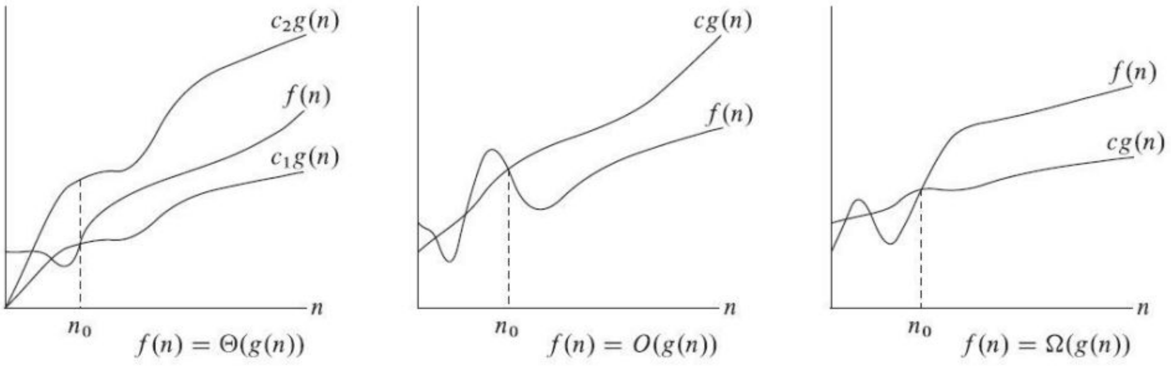
If the above definition stands, $f(n) \in \Theta(g(n))$.

We use $f(n)=\Theta(g(n))$ instead of $f(n) \in \Theta(g(n))$ for simplification.

- Asymptotic bound:

  For all $n>n_0$, the function f(n) is equal to g(n) to within a constant factor, we say that g(n) is an asymptotically tight bound for f(n)

Asymptotic Notation



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- Asymptotic upper bound:

When we only have an asymptotic **upper bound**, we use O notation for a given function g(n), we denote by O(g(n)) the set of functions:

O(g(n))={ f(n) there exists positive constant c, n_0 such that 0<=f(n)<=cg(n) for all n>=n_0}

T(n)=O(g(n))--the asymptotic upper bound of the algorithm is g(n)

- Asymptotic lower bound:

  When we only have an asymptotic **lower bound**, we use Ω notation for a given function g(n), we denote by $\Omega(g(n))$ the set of functions:

  $\Omega(g(n))=\{ f(n)$ there exists positive constant c, n_0 such that $0<=cg(n) <=f(n)$ for all $n>=n_0\}$

  $T(n)=\Omega(g(n))$--the asymptotic lower bound of the algorithm is g(n)

Examples

1. Compare the complexity of insertion and merge

$T_1(n)=an^2+bn+c$--n^2 ⇨

$T_2(n)=cnlgn+cn$--$nlgn$ ⇨

Theta Notation

- **Drop lower order terms**
- **Ignore leading constants**
- **Concentrates on the growth**

$$\lim_{n\to\infty}\left(\frac{nlgn}{n^2}\right) = ?$$

- $$\lim_{n\to\infty}\left(\frac{nlgn}{n^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{lgn}{n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1/n(ln2)}{1}\right)$$

$$\lim_{n\to\infty}\left(\frac{1}{ln2*n}\right)$$

=0

#1: Limit– O, or big theta or big omega

if the limit g(n)/f(n)=0, g(n) is big omega;
n→∞

if the limit g(n)/f(n)=∞, g(n) is big O;
n→∞

if the limit g(n)/f(n)=c, g(n) is big theta ;
n→∞

Rules:

$$\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = \lim_{n\to\infty}\left(\frac{f'(n)}{g'(n)}\right)$$

Where f'(n) is the derivative of f(n)

some basic derivatives:
https://www.dummies.com/article/academics-the-arts/math/calculus/the-most-important-derivatives-and-antiderivatives-to-know-188540/

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Examples (cnt)

2. $f(n)=n^3$, $g(n)=2^n$, choose the correct answer:

A. $f(n)=\Theta(g(n))$

B. $f(n)=O(g(n))$

C. $f(n)=\Omega(g(n))$

-
- $\lim_{n\to\infty}\left(\frac{2^n}{n^3}\right)$

$=\quad \lim_{n\to\infty}\left(\frac{ln2*2^n}{3n^2}\right)$

$= \lim_{n\to\infty}\left(\frac{c(ln2)2^n}{2n}\right)$

$\lim_{n\to\infty}\left(\frac{c(ln2)2^n}{1}\right)$

$=\infty$

B

$$3.\ \mathrm{Prove}\quad 2n^2 = O\left(n^3\right).$$

#2: definitions

4. Prove $T(n) = 3n^3 - 4n^2 + 3\lg n - n = O(n^3)$

$$3n^3 - 4n^2 + 3\lg n - n \le cn^3$$

$$3n^3 - \left(4n^2 - 3\lg n + n\right)$$

5. prove $\frac{1}{2}n^2 - 3n = \Theta(n^2)$

Proof:

$$c_1 n^2 \le \frac{1}{2}n^2 - 3n \le c_2 n^2 = c_1 \le \frac{1}{2} - \frac{3}{n} \le c_2$$

$$n = 6 \quad c_1 \le 0 \le c_2$$

$$n \to \infty$$

$$n = 30 \Rightarrow c_1 \le \frac{2}{5} \le c_2, holds$$

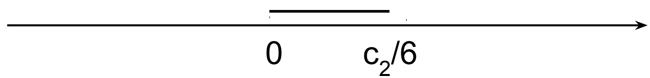
6. Show the Proof: $6n^3 \neq \Theta(n^2)$

#3: Contradictional Proof

$$Assume \quad that \quad c_1 n^2 \le 6n^3 \le c_2 n^2 = c_1 \le 6n \le c_2$$

$$n \le \frac{c_2}{6}$$

$$n \to \infty$$



$$it \quad doesn't \quad hold.$$

7. Show that: $5n^2 - 2n + 3 = \Theta(n^2)$

#4: Assumption Proof

$$Assume \quad that \quad 2n^2 \le 5n^2 - 2n + 3 \le 10n^2$$

(1) $2n^2 \le 5n^2 - 2n + 3 \Rightarrow 3n^2 - 2n + 3 \ge 0$

discriminant:

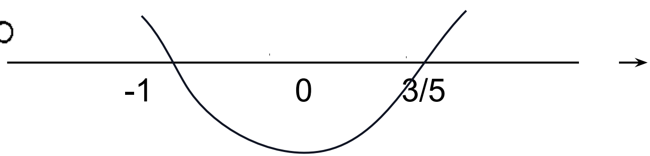
$$(-2)^2 - 4 \times 3 \times 3 < 0 \Rightarrow no\,\text{int}\,ercepts\ on \quad x - axis$$

$$\Rightarrow for \quad all \quad n, \quad it \quad holds.$$

(2) $5n^2 - 2n + 3 \le 10n^2 \Rightarrow 5n^2 + 2n - 3 \ge 0$

$\Rightarrow (5n-3)(n+1) \ge 0 \Rightarrow$ n>=3/5 or n<=-1

$n \to \infty$



it holds.

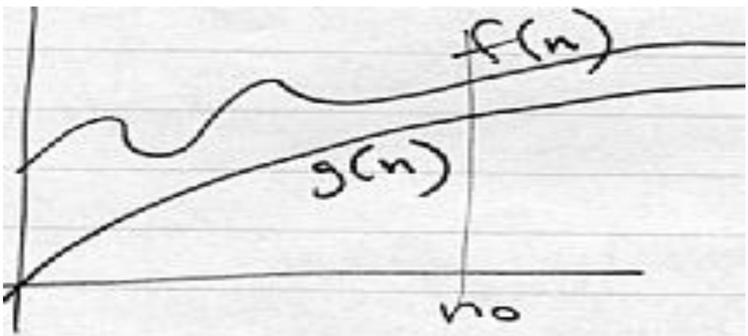
8. Show that

$$Omega\ \Omega - lower\ bound$$

$$f(n) = \Omega\left(g(n)\right)$$

$$proof: \quad 0 \le cg(n) \le f(n)$$

$c?$ $\qquad n_o < n \qquad c > 0$



$$n^{\frac{1}{2}} = \Omega(\lg n)$$

$$c \lg_2 n \le \sqrt{n} \qquad\qquad c = 1$$

$$c \lg_2 16 \le \sqrt{16} \qquad\qquad n > 16$$

$$(1) 4 \le 4 \qquad\qquad n_o$$

$$\lg_2 64 \le \sqrt{64}$$

$$(2) 6 \le 8$$

Meditate it with another approach!

Desmo Classroom

Classwork

1. go to Desmos.com and show you proof at the following link:

https://student.desmos.com/activitybuilder/student-greeting/63c5f41b6da6042e0f89b3b4

2. rewrite INSERTION and analyze the complexity.

https://student.desmos.com/activitybuilder/student-greeting/63befd4c35baa757b2656af5