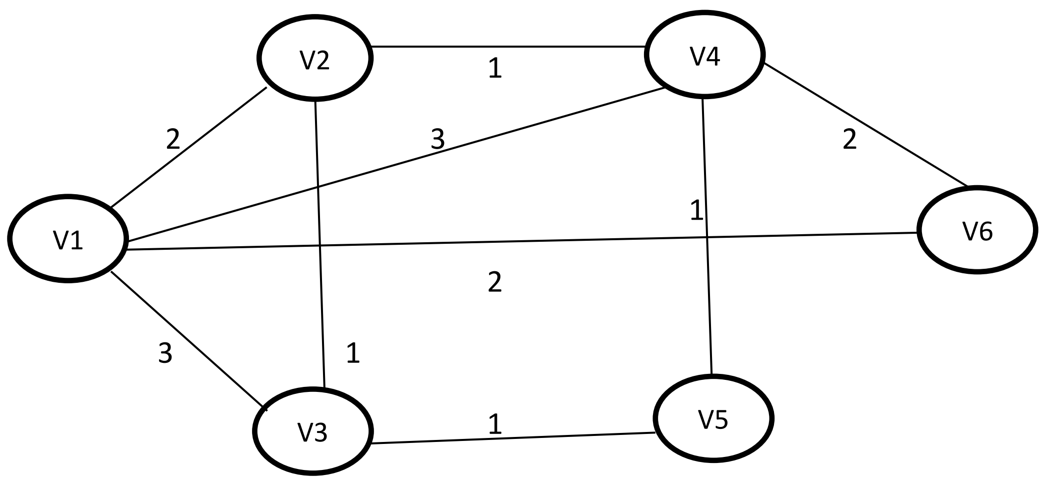
HW3 Solution

1. **Given the following adjacency matrix:**

$$\begin{bmatrix} 0 & 2 & 3 & 3 & 0 & 2 \\ 2 & 0 & 1 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 2 & 0 & 0 \end{bmatrix}$$

a. **Draw and label the graph**



	Shortest Path	Length
V1-V2	V1-V2	2
V1-V3	V1-V3	3
	V1-V2-V3	3
V1-V4	V1-V4	3
	V1-V2-V4	3
V1-V5	V1-V3-V5	4
	V1-V4-V5	4
	V1-V2-V4-V5	4
	V1-V2-V3-V5	4
V1-V6	V1-V6	2
V2-V3	V2-V3	1
V2-V4	V2-V4	1
V2-V5	V2-V3-V5	2
	V2-V4-V5	2
V2-V6	V2-V4-V6	3

V3-V4	V3-V2-V4	2
	V3-V5-V4	2
V3-V5	V3-V5	2
V3-V6	V3-V2-V4-V6	4
	V3-V5-V4-V6	4
V4-V5	V4-V5	1
V4-V6	V4-V6	2
V5-V6	V5-V4-V6	3

b. **What is the probability of a node in this graph having degree 3?**

$C_d(V1)$= 4
$C_d(V2)$= 3
$C_d(V3)$= 3
$C_d(V4)$= 4
$C_d(V5)$= 2
$C_d(V6)$= 2

So the **probability ($C_d = 3$) = $\frac{2}{6}$ = $\frac{1}{3}$**

c. **What is the diameter of the graph?**

Diameter = The longest shortest path of the graph

Diameter **= 4**

d. **Provide a table with the betweenness centrality, closeness centrality and strength for each node**

	Betweenness Centrality	**Closeness Centrality**	**Strength**
V1	0	$\frac{5}{14}$	10
V2	5	$\frac{5}{9}$	4
V3	2	$\frac{5}{11}$	5
V4	8	$\frac{5}{9}$	7
V5	2	$\frac{5}{11}$	2
V6	0	$\frac{5}{14}$	4

Betweenness Centrality:-

$$C_b(v_i) = \sum_{s \neq t \neq v_i} \frac{\sigma_{st}(v_i)}{\sigma_{st}}$$

σ_{st} The number of shortest paths from vertex s to t – *a.k.a.* information pathways

$\sigma_{st}(v_i)$ The number of **shortest paths** from s to t that pass through v_i

$C_b(V1)$= **0**

$C_b(V2)$= $(\frac{1}{2} + \frac{1}{2} + \frac{2}{4} + \frac{1}{2} + \frac{1}{2})$ * 2 = **5**

$C_b(V3)$= $(\frac{2}{4} + \frac{1}{2})$ * 2 = **2**

$C_b(V4)$= $(\frac{2}{4} + \frac{1}{2} + \frac{1}{1} + \frac{2}{2} + \frac{1}{1})$ * 2 = **8**

$C_b(V5)$= $(\frac{1}{2} + \frac{1}{2})$ * 2 = **2**

$C_b(V6)$= **0**

Closeness Centrality:-

Closeness centrality: $C_c(v_i) = \frac{1}{\bar{l}_{v_i}}$

$C_c(V1) = \frac{1}{\frac{2+3+3+4+2}{5}} = \frac{5}{14}$

$C_c(V2) = \frac{1}{\frac{2+1+1+2+3}{5}} = \frac{5}{9}$

$C_c(V3) = \frac{1}{\frac{3+1+2+1+4}{5}} = \frac{5}{11}$

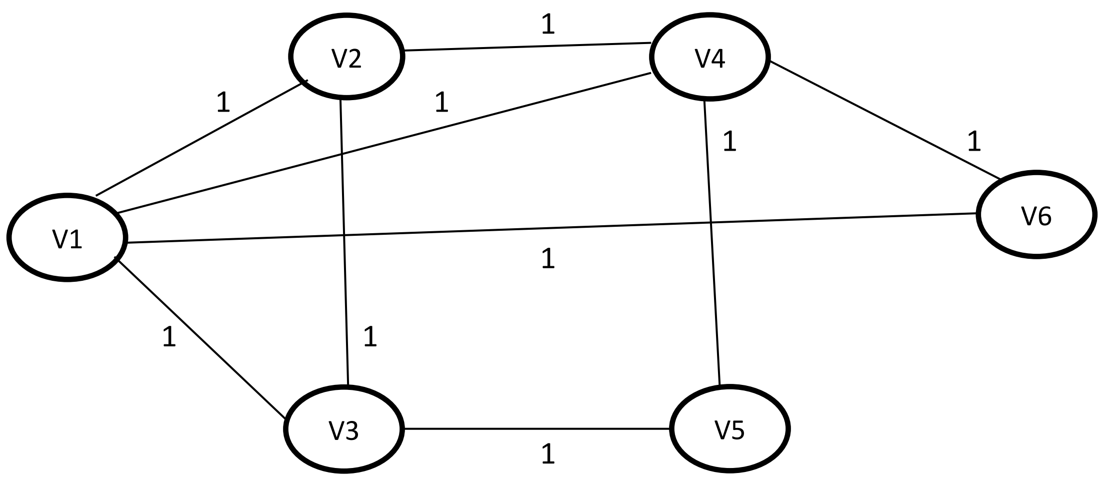
$C_c(V4) = \frac{1}{\frac{3+1+2+1+2}{5}} = \frac{5}{9}$

$C_c(V5) = \frac{1}{\frac{4+2+1+1+3}{5}} = \frac{5}{11}$

$C_c(V6) = \frac{1}{\frac{2+3+4+2+3}{5}} = \frac{5}{14}$

2. **Given the following adjacency matrix:**

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$



a. **Provide the local clustering coefficient for each node**

For a node i of degree $\boldsymbol{k_i}$, let $\boldsymbol{b_i}$ be the **number of edges among the neighbors of i**. The local clustering coefficient is

$$C_{local}(i) = \frac{b_i}{\binom{k}{2}} \qquad \text{with } 0 <= C_{local}(i) <= 1$$

Where,

$$\binom{k}{2} = \frac{k!}{2!(k-2)!} = \frac{k!}{2(k-2)!}$$

(let's find out $\boldsymbol{k_i}$, $\boldsymbol{b_i}$ for each node

- **V1** => neighbors {V2,V3,V4,V6} ,so $\boldsymbol{k_1}$**= 4**

  edges among neighbors = **3** (V2-V3 ,V2-V4, V4-V6)

  $\boldsymbol{b_1}$**=3**

- **V2** => neighbors {V1,V3,V4} ,so $\boldsymbol{k_2}$**= 3**

  edges among neighbors = **2** (V1-V3 ,V1-V4)

  $\boldsymbol{b_2}$**= 2**

- **V3** => neighbors {V1,V2,V5} ,so $\boldsymbol{k_3}$**= 3**

  edges among neighbors = **1** (V1-V2)

  $\boldsymbol{b_3}$**= 1**

- **V4** => neighbors {V1,V2,V5,V6} ,so $\boldsymbol{k_4}$**= 4**

  edges among neighbors = **2** (V1-V2 ,V1-V6)

  $\boldsymbol{b_4}$**= 2**

- **V5** => neighbors {V3,V4} ,so $\boldsymbol{k_5}$**= 2**

  edges among neighbors = **0**

  $\boldsymbol{b_5}$**= 0**

- **V6** => neighbors {V1,V4} ,so $\boldsymbol{k_6}$**= 2**

  edges among neighbors = **1** (V1-V4)

  $\boldsymbol{b_6}$**= 1**

)

Now let's find out $\boldsymbol{C_{local}}\,(\boldsymbol{Vi})$:-

$$\boldsymbol{C_{local}}\,(\boldsymbol{V1}) = \frac{b_1}{\binom{k_1}{2}} = \frac{3}{\binom{4}{2}} = \frac{3}{6} = \boldsymbol{\frac{1}{2}}$$

$$\boldsymbol{C_{local}}\,(\boldsymbol{V2}) = \frac{b_2}{\binom{k_2}{2}} = \frac{2}{\binom{3}{2}} = \boldsymbol{\frac{2}{3}}$$

$$\boldsymbol{C_{local}}\,(\boldsymbol{V3}) = \frac{b_3}{\binom{k_3}{2}} = \frac{1}{\binom{3}{2}} = \boldsymbol{\frac{1}{3}}$$

$$\boldsymbol{C_{local}}\,(\boldsymbol{V4}) = \frac{b_4}{\binom{k_4}{2}} = \frac{2}{\binom{4}{2}} = \frac{2}{6} = \boldsymbol{\frac{1}{3}}$$

$$\boldsymbol{C_{local}}\,(\boldsymbol{V5}) = \frac{b_5}{\binom{k_5}{2}} = \frac{0}{\binom{2}{2}} = \boldsymbol{0}$$

$$\boldsymbol{C_{local}}\,(\boldsymbol{V6}) = \frac{b_6}{\binom{k_6}{2}} = \frac{1}{\binom{2}{2}} = \boldsymbol{1}$$

b. **Provide the global clustering coefficient for the graph**

Average value of the local clustering coefficient $\boldsymbol{C_{local}}\,(\boldsymbol{i})$over all nodes in the network

$$C_{\text{global}} = \frac{1}{n} \sum_{i=1}^{n} C_{local}(i)$$

Now sum the all $\boldsymbol{C_{local}}\,(\boldsymbol{i})$:-

$\sum_{i=1}^{n} \boldsymbol{C_{local}}(\boldsymbol{i}) = \frac{1}{2} + \frac{2}{3} + \frac{1}{3} + \frac{1}{3} + 0 + 1 = \boldsymbol{\frac{17}{6}}$

$\boldsymbol{C_{global}} = \frac{1}{6} \cdot \frac{17}{6} = \frac{17}{36} \approx \mathbf{0.47}$

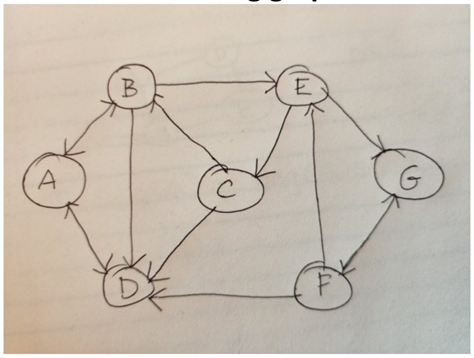
c. **Provide the degree sequence for each node**

The degree $\boldsymbol{k_{v_i}}$ of a node is the number of neighbors it has (row/column sum in the undirected adjacency matrix).

$\boldsymbol{k_{V1}} = \mathbf{4}$
$\boldsymbol{k_{V2}} = \mathbf{3}$
$\boldsymbol{k_{V3}} = \mathbf{3}$
$\boldsymbol{k_{V4}} = \mathbf{4}$
$\boldsymbol{k_{V5}} = \mathbf{2}$
$\boldsymbol{k_{V6}} = \mathbf{2}$

In- node = (4,3,3,4,2,2)
Non- increasing order = (4,4,3,3,2,2)

3. **Given the following graph:**



Using the simple PageRank algorithm from class, iterate until you have a stable rank. For each iteration, show the PageRank values and rank for each node.

Rank for initialization

$$PR_i^0 = \frac{1}{n}, \forall i$$ n = number of nodes/pages

A node's PageRank is found by summing up the portions of PageRank passed to it by the nodes that link to it in the previous step.

$$PR_i^{\text{t}} = \sum_{j=1}^{n} a_{ji} \frac{PR_j^{\text{t-1}}}{d_j^{out}}, \forall t \in \{1,k\}$$ t = iteration

where d_j^{out} is the number of hyperlinks on the node/page j.

<u>So the first iteration will be:-</u>

$$PR_i^1 = \frac{1}{n} \sum_{j=1}^{n} a_{ji} \frac{1}{d_j^{out}}$$

$$PR_a^1 = \frac{\mathbf{1}}{\boldsymbol{n}} \left(\frac{a_{aa}}{d_a out} + \frac{a_{ba}}{d_b out} + \frac{a_{ca}}{d_c out} + \frac{a_{da}}{d_d out} + \frac{a_{ea}}{d_e out} + \frac{a_{fa}}{d_f out} + \frac{a_{ga}}{d_g out} \right)$$
$$= \frac{\mathbf{1}}{\mathbf{7}} \left(0 + \frac{1}{3} + 0 + \frac{1}{1} + 0 + 0 + 0\right)$$
$$= \frac{\mathbf{4}}{\mathbf{21}}$$

$$PR_b^1 = \frac{\mathbf{1}}{\boldsymbol{n}} \left(\frac{a_{ab}}{d_a out} + \frac{a_{bb}}{d_b out} + \frac{a_{cb}}{d_c out} + \frac{a_{db}}{d_d out} + \frac{a_{eb}}{d_e out} + \frac{a_{fb}}{d_f out} + \frac{a_{gb}}{d_g out} \right)$$
$$= \frac{\mathbf{1}}{\mathbf{7}} \left(\frac{1}{2} + 0 + \frac{1}{1} + 0 + 0 + 0 + 0\right)$$
$$= \frac{\mathbf{3}}{\mathbf{21}}$$

$$PR_c^1 = \frac{\mathbf{1}}{\boldsymbol{n}}\left(\frac{a_{ac}}{d_a out} + \frac{a_{bc}}{d_b out} + \frac{a_{cc}}{d_c out} + \frac{a_{dc}}{d_d out} + \frac{a_{ec}}{d_e out} + \frac{a_{fc}}{d_f out} + \frac{a_{gc}}{d_g out}\right)$$
$$= \frac{\mathbf{1}}{\mathbf{7}}\left(0 + 0 + \frac{1}{2} + 0 + 0 + 0 + 0\right)$$
$$= \frac{\mathbf{1}}{\mathbf{14}}$$

$$PR_d^1 = \frac{\mathbf{1}}{\boldsymbol{n}}\left(\frac{a_{ad}}{d_a out} + \frac{a_{bd}}{d_b out} + \frac{a_{cd}}{d_c out} + \frac{a_{dd}}{d_d out} + \frac{a_{ed}}{d_e out} + \frac{a_{fd}}{d_f out} + \frac{a_{gd}}{d_g out}\right)$$
$$= \frac{\mathbf{1}}{\mathbf{7}}\left(\frac{1}{3} + \frac{1}{1} + \frac{1}{1} + 0 + 0 + \frac{1}{3} + 0\right)$$
$$= \frac{\mathbf{5}}{\mathbf{21}}$$

$$PR_e^1 = \frac{\mathbf{1}}{\boldsymbol{n}}\left(\frac{a_{ae}}{d_a out} + \frac{a_{be}}{d_b out} + \frac{a_{ce}}{d_c out} + \frac{a_{de}}{d_d out} + \frac{a_{ee}}{d_e out} + \frac{a_{fe}}{d_f out} + \frac{a_{ge}}{d_g out}\right)$$
$$= \frac{\mathbf{1}}{\mathbf{7}}\left(0 + \frac{1}{3} + 0 + 0 + 0 + \frac{1}{3} + 0\right)$$
$$= \frac{\mathbf{2}}{\mathbf{21}}$$

$$PR_f^1 = \frac{\mathbf{1}}{\boldsymbol{n}}\left(\frac{a_{af}}{d_a out} + \frac{a_{bf}}{d_b out} + \frac{a_{cf}}{d_c out} + \frac{a_{df}}{d_d out} + \frac{a_{ef}}{d_e out} + \frac{a_{ff}}{d_f out} + \frac{a_{gf}}{d_g out}\right)$$
$$= \frac{\mathbf{1}}{\mathbf{7}}\left(0 + 0 + 0 + 0 + 0 + \frac{1}{1} + 0\right)$$
$$= \frac{\mathbf{1}}{\mathbf{7}}$$

$$PR_g^1 = \frac{\mathbf{1}}{\boldsymbol{n}}\left(\frac{a_{ag}}{d_a out} + \frac{a_{bg}}{d_b out} + \frac{a_{cg}}{d_c out} + \frac{a_{dg}}{d_d out} + \frac{a_{eg}}{d_e out} + \frac{a_{fg}}{d_f out} + \frac{a_{gg}}{d_g out}\right)$$
$$= \frac{\mathbf{1}}{\mathbf{7}}\left(0 + 0 + 0 + 0 + \frac{1}{2} + \frac{1}{3} + 0\right)$$
$$= \frac{\mathbf{5}}{\mathbf{42}}$$

<u>So the second iteration will be:-</u>

$$\boldsymbol{PR_a^2} = \frac{PR_b^1}{d_a out} + \frac{PR_g^1}{d_a out}$$
$$= \frac{\frac{3}{21}}{3} + \frac{\frac{5}{21}}{1}$$
$$= \frac{\mathbf{6}}{\mathbf{21}}$$

$$\boldsymbol{PR_b^2} = \frac{PR_a^1}{d_a out} + \frac{PR_c^1}{d_c out}$$
$$= \frac{\frac{4}{21}}{2} + \frac{\frac{1}{14}}{2}$$
$$= \frac{\mathbf{11}}{\mathbf{84}}$$

$$\boldsymbol{PR_c^2} = \frac{PR_e^1}{d_e out}$$
$$= \frac{\frac{2}{21}}{2}$$
$$= \frac{\mathbf{1}}{\mathbf{21}}$$

$$PR_d^2 = \frac{PR_a^1}{d_a out} + \frac{PR_b^1}{d_b out} + \frac{PR_c^1}{d_a out} + \frac{PR_f^1}{d_f out}$$
$$= \frac{\frac{4}{21}}{2} + \frac{\frac{1}{14}}{3} + \frac{\frac{1}{14}}{2} + \frac{\frac{1}{7}}{3}$$
$$= \frac{19}{84}$$

$$PR_e^2 = \frac{PR_b^1}{d_b out} + \frac{PR_f^1}{d_f out}$$
$$= \frac{\frac{3}{21}}{3} + \frac{\frac{1}{7}}{3}$$
$$= \frac{2}{21}$$

$$PR_f^2 = \frac{PR_g^1}{d_g out}$$
$$= \frac{\frac{5}{42}}{1}$$
$$= \frac{5}{42}$$

$$PR_g^2 = \frac{PR_e^1}{d_e out} + \frac{PR_f^1}{d_f out}$$
$$= \frac{\frac{2}{21}}{2} + \frac{\frac{1}{7}}{3}$$
$$= \frac{2}{21}$$

<u>So the third iteration will be:-</u>

$$PR_a^3 = \frac{PR_b^2}{d_a out} + \frac{PR_g^2}{d_a out}$$
$$= \frac{\frac{11}{21}}{3} + \frac{\frac{19}{84}}{1}$$
$$= \frac{68}{252} = \frac{17}{63}$$

$$PR_b^3 = \frac{PR_a^2}{d_a out} + \frac{PR_c^2}{d_c out}$$
$$= \frac{\frac{6}{21}}{2} + \frac{\frac{1}{21}}{2}$$
$$= \frac{7}{42}$$

$$PR_c^3 = \frac{PR_e^2}{d_e out}$$
$$= \frac{\frac{2}{21}}{2}$$
$$= \frac{1}{21}$$

$$PR_d^3 = \frac{PR_a^2}{d_a out} + \frac{PR_b^2}{d_b out} + \frac{PR_c^2}{d_a out} + \frac{PR_f^2}{d_f out}$$
$$= \frac{\frac{6}{21}}{2} + \frac{\frac{11}{84}}{3} + \frac{\frac{1}{21}}{2} + \frac{\frac{5}{42}}{3}$$
$$= \frac{55}{252}$$

$$\boldsymbol{PR_e^3} = \frac{PR_b^2}{d_b out} + \frac{PR_f^2}{d_f out}$$
$$= \frac{\frac{11}{84}}{2} + \frac{\frac{5}{42}}{3}$$
$$= \boldsymbol{\frac{21}{252}}$$

$$\boldsymbol{PR_f^3} = \frac{PR_g^2}{d_g out} +$$
$$= \frac{\frac{2}{21}}{1}$$
$$= \boldsymbol{\frac{2}{21}}$$

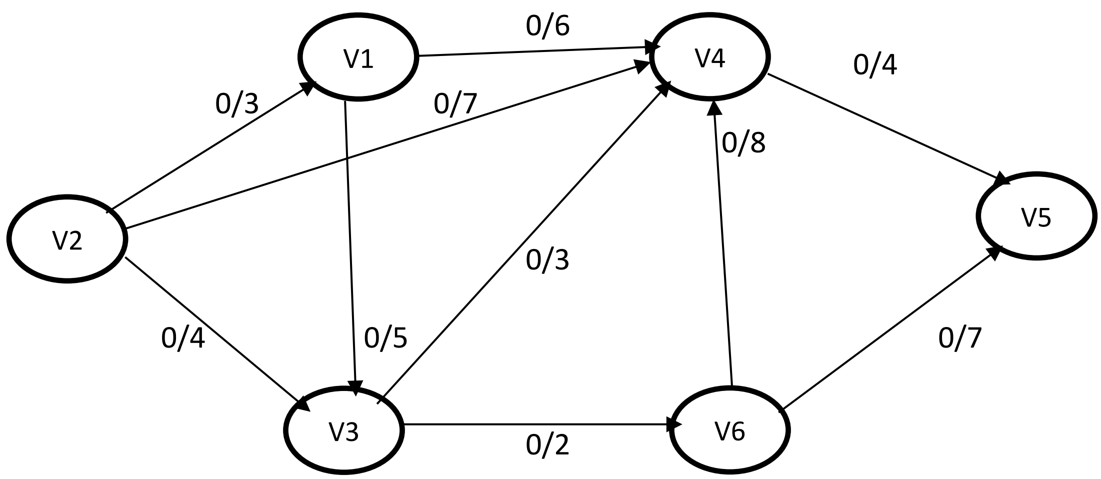
$$\boldsymbol{PR_g^3} = \frac{PR_e^2}{d_e out} + \frac{PR_f^2}{d_f out}$$
$$= \frac{\frac{2}{21}}{2} + \frac{\frac{5}{42}}{3}$$
$$= \boldsymbol{\frac{11}{126}}$$

	3rd iteration	**Rank**
A	$\frac{17}{63} \approx 0.269$	**1**
B	$\frac{7}{42} \approx 0.166$	**3**
C	$\frac{1}{21} \approx 0.047$	**7**
D	$\frac{55}{252} \approx 0.218$	**2**
E	$\frac{21}{252} \approx 0.083$	**6**
F	$\frac{2}{21} \approx 0.09$	**4**
G	$\frac{11}{126} \approx 0.087$	**5**

4. Given the following adjacency matrix representing a flow network from v2 (source) to v5 (sink)

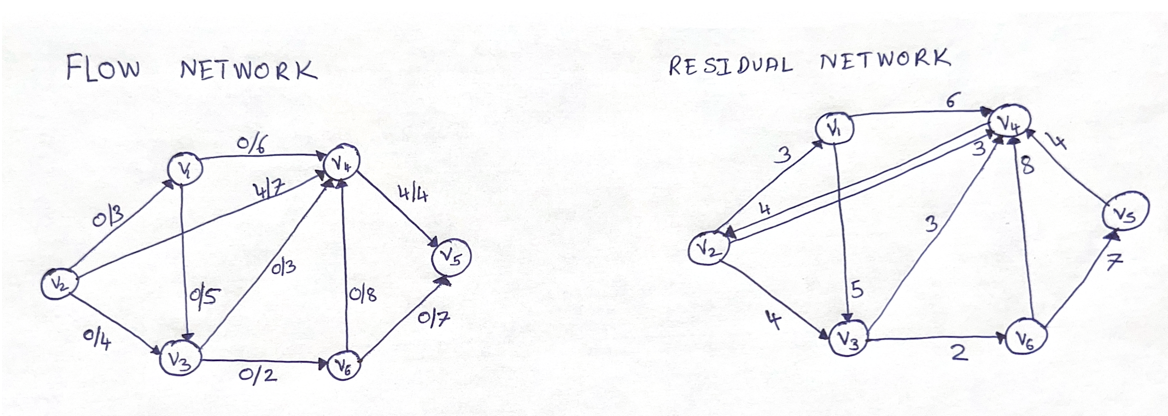
$$\begin{bmatrix} 0 & 0 & 5 & 6 & 0 & 0 \\ 3 & 0 & 4 & 7 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 7 & 0 \end{bmatrix}$$

a. **Draw the initial flow network**

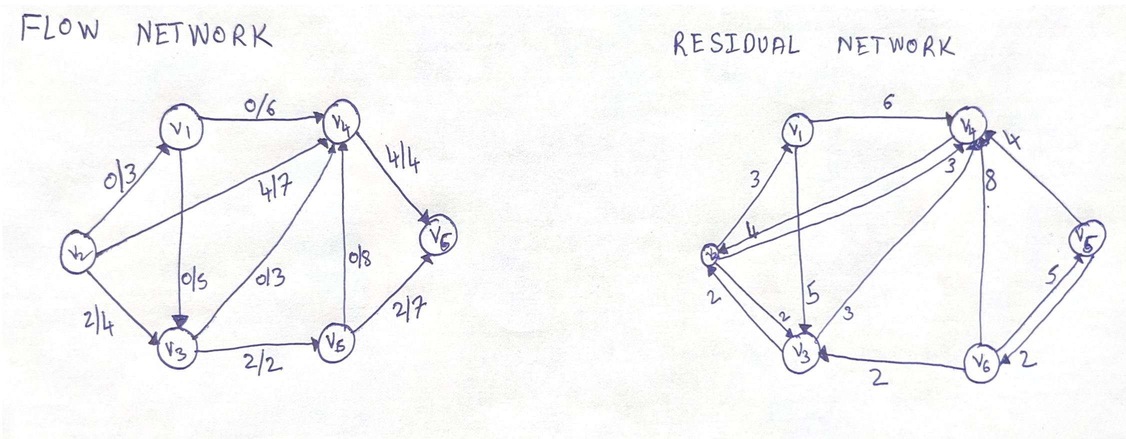


b. **Use the Ford-Fulkerson algorithm to determine the maximum flow through the network. For each iteration, show (1) the augmenting path, (2) the resulting flow network, (3) the residual network and (4) the flow value.**

Augmenting Path → V2 → V4 → V5
Flow :- 4



Augmenting Path:- V2 ➝V3 ➝ V6 ➝V5
Flow :- 2



Total Flow = 4+2 = 6