Exercises 7

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Problem 1

a)

The derivative of x, y are zero at equilibrium/fixed point:

$$0 = x(a - by) \tag{1}$$
$$0 = y(-c + dx) \tag{2}$$

It's easy to see that: $(0, 0)$ and $(\frac{c}{d}, \frac{a}{b})$ are the fixed points.

b)

The Jacobian of the RHS:

$$J = \begin{bmatrix} a - by & -bx \\ dy & -c + dx \end{bmatrix} \tag{3}$$

For the non-trivial fixed points:

$$J = \begin{bmatrix} a - b\frac{a}{b} & -b\frac{c}{d} \\ d\frac{a}{b} & -c + d\frac{c}{d} \end{bmatrix} = \begin{bmatrix} 0 & \frac{-bc}{d} \\ \frac{ad}{b} & 0 \end{bmatrix} \tag{4}$$

Eigenvalues can be calculated easily: $\lambda_1 = i\sqrt{ac}, \lambda_2 = -i\sqrt{ac}$. As shown by the eigenvalues which both have a zero real part. This indicates that the equiblirum is not attriactive and is not repulsive.

Due to the fact that the eigenvalues have a non-zero imaginary part, the system will now oscillate awith a period of $\sqrt{ac}$

c)

When solving this question, we consulted this lecture note as well as some other books.

Consider the replicator equation in n variables:

$$\dot{x_i} = x_i \left(r_i + \sum_j a_{ij} x_j \right), \quad i = 1, \ldots, n+1 \tag{5}$$

This can be re-written into:

$$\dot{x_i} = x_i \left((\boldsymbol{A}\boldsymbol{x})_i - \boldsymbol{x} \cdot \boldsymbol{A}\boldsymbol{x} \right), \quad i = 1, \ldots, n+1 \tag{6}$$

We have x_i denoting the density of i-th species and r_i denoting the i-th species growth rate. Let a_{ij} denote the interaction between i-th and j-th and the matrix $\boldsymbol{A}$ with entry a_{ij} be the interaction matrix. With the last row of $\boldsymbol{A}$ being zeros.

Define $y_{n+1} = 1$ and define a mapping from $\boldsymbol{y}$ to $\boldsymbol{x}$ given by:

$$y_i = \frac{y_i}{\sum_{j=1}^{n} y_j}, \quad i = 1, \ldots, n+1 \tag{7}$$

The inverse mapping from $\boldsymbol{x}$ to $\boldsymbol{y}$ given by:

$$x_i = \frac{x_i}{x_n}, \quad i = 1, \ldots, n+1 \tag{8}$$

Since $(\boldsymbol{A}\boldsymbol{x})_n = 0$ and the ODE systems on x_i above. we have:

$$\begin{align}
\dot{y_i} &= \left(\frac{x_i}{x_n}\right) \cdot ((\boldsymbol{A}\boldsymbol{x})_i - (\boldsymbol{A}\boldsymbol{x})_n) \tag{9}\\
&= y_i \left(\sum a_{ij} x_j\right) \tag{10}\\
&= y_i \left(\sum_{j=1}^{n+1} a_{ij} y_i\right) x_n \quad \text{Since } x_j = y_j \cdot x_n \tag{11}\\
&= y_i (a_{in} + \sum_{j=1}^{n} a_{ij} y_j) \quad x_n \text{removed} \tag{12}
\end{align}$$

Problem 2

a)

A matrix is called a stochastics matrix if

1. it is a square matrix
2. $0 \leq A_{ij} \leq 1, \quad \forall\, i, j$
3. $\sum_j A_{ij} = 1, \quad \forall\, i, j$

(1) and (2) is trivial since transition can be made from any state to another and hence the square matrix. As $p, q, 1-p, 1-p \in [0, 1]$, (2) is also fulfilled.

By simple calculation, it is not hard to see that the row of matrix M equals to 1.

b)

To find the stationary distribution of the transition, let x_t be the distribution after t transition.

If x stated in the question were the stationary distribution,

$$\lim_{t \to \infty} x_t \cdot M = x \tag{13}$$

To verify (just the first component of x_t for simplicity, denoted by x_t^1) Provided that $x_t = (s_1 s_2, s_1(1-s_2), (1-s_1)s_2, (1-s_1)(1-s_2))$

$$
\begin{aligned}
x_{t+1}^1 &= x_t^1 \cdot M && (14)\\
&= s_1 s_2 \cdot p_1 p_2 + s_1(1-s_2) \cdot q_1 p_2 + (1-s_1)s_2 \cdot p_1 q_2 + (1-s_1)(1-s_2) \cdot q_1 q_2 && (15)\\
&= s_1 s_2 \Big(p_1 p_2 - q_1 p_2 + p_1 q_2 + q_1 q_2 \Big) s_1 q_1 p_2 + s_2 p_1 q_2 - (s_1+s_2) q_1 q_2 + q_1 q_2 && (16)\\
&= s_1 s_2 r_1 r_2 + s_1 q_1 r_2 + s_2 q_2 r_1 + q_1 q_2 && (17)\\
&= \Bigg[\Big((q_2 r_1 + q_1)(q_1 r_2 + q_2) r_1 r_2 \Big) + \Big((q_2 r_1 + q_1) r_2 q_1 (1 - r_1 r_2) \Big) && (18)\\
&\quad + \Big((q_1 r_2 + q_2) r_1 q_2 (1 - r_1 r_2) \Big) + \Big((1 - r_1 r_2)^2 q_1 q_2 \Big) \Bigg] \cdot \frac{1}{(1-r_1 r_2)^2} && (19)\\
&= \Bigg[\Big(q_2^2 r_1^2 r_2^2 + q_2^2 r_1^2 r_2 + q_1 q_2 r_2^2 r_1 + q_1 q_2 r_1 r_2 \Big) + \Big(q_1 q_2 r_1 r_2 + q_1^2 r_2 - q_1 q_2 r_1^2 r_2^2 \Big) && (20)\\
&\quad + \Big(q_1 q_2 r_1 r_2 + q_2^2 r_1 - q_1 q_2 r_1^2 r_2^2 - q_2^2 r_1^2 r_2 \Big) + \Big(q_1 q_2 - 2 q_1 q_2 r_1 r_2 + q_1 q_2 r_1^2 r_2^2 \Big) \Bigg] \cdot \frac{1}{(1-r_1 r_2)^2} && (21)\\
&= \frac{q_1 q_2 r_1 r_2 + q_1^2 r_2 + q_1 q_2 + q_2^2 r_1}{(1-r_1 r_2)^2} && (22)\\
&= s_1 s_2 && (23)
\end{aligned}
$$

c)

It is easy to see this strategy is tit-for-tat. From the results from (b), we can show the expected payoffs for both players with the help of s_1, s_2

In the setting of strategy $S_1(1, 0)$,

$$
\begin{aligned}
p_1 &= 1 && (24)\\
q_1 &= 0 && (25)
\end{aligned}
$$

$$
\begin{aligned}
s_1 &= \frac{q_2}{1 + q_2 - p_2} && (26)\\
s_2 &= \frac{q_2}{1 + q_2 - p_2} && (27)
\end{aligned}
$$

It is easy to see that $s_1 = s_2$ and hence the expected payoff at the stationary distribution of the correspounding Markov chain is the same for both players.

d)

To calculate the expected payoff for S_1 against S_2, calculate s_1 and s_2,

$$s_1 = \frac{\frac{1}{4}(1-0)+0}{1-(1-0)(1-\frac{1}{4})} = 1 \tag{28}$$

$$s_2 = \frac{0(1-\frac{1}{4})+\frac{1}{4}}{1-(1-0)(1-\frac{1}{4})} = 1 \tag{29}$$

The expected paoff at the stationary distribution:

$$E(S_1, S_2) = Rs_1s_2 + Ss_1(1-s_2) + T(1-s_1)s_2 + P(1-s_1)(1-s_2) = 3$$

The expected payoff for this game with S_1, S_2 is 3.

Code is available on github repo: (https://github.com/wyq977/evolutionary-dynamics-2019)