

Evolutionary Dynamics

Exercises 6

Prof. Dr. Niko Beerenwinkel
Dr. Katharina Jahn
Dr. Rob Noble
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Problem 1: One-dimensional Fokker-Planck equation

Consider the one-dimensional Fokker-Planck equation with constant coefficients,

$$\partial_t \psi(p, t) = -m \partial_p \psi(p, t) + \frac{v}{2} \partial_p^2 \psi(p, t), \quad (1)$$

with $p \in \mathbb{R}$, $v > 0$.

- (a) Show that for vanishing selection, $m = 0$, (1 point)

$$\psi(p, t) = \frac{1}{\sqrt{2\pi vt}} \exp\left(-\frac{p^2}{2vt}\right) \quad (2)$$

solves the Fokker-Planck equation. To which initial condition does this solution correspond?

- (b) Simulate a random walk, starting at 0 where at each time step the position is either increased by 1 with probability $\frac{1}{2}$ or decreased by 1 otherwise. Simulate many walks for $N = 10$, $N = 100$ and $N = 1000$ steps. Compare the results to (2). Find the relationship between the means and variances. (1.5 points)
- (c) Construct a solution for constant selection, $m \neq 0$, by substituting $z = p - mt$ for p in (1). What is the mean and variance? (0.5 points)

Problem 2: Diffusion approximation of the Moran process

Derive a diffusion approximation for the Moran process of two species. Assume the first species has a small selective advantage s .

- (a) The general definition for the drift coefficient is

$$M(p) = E[X(t) - X(t-1) \mid X(t-1) = i]/N,$$

where $p = i/N$ and $X(t)$ denotes the abundance of the first allele. Evaluate this expression for the Moran process with selection. Show that this yields the result for the Wright-Fisher process from the lecture, divided by N . (1 point)

- (b) By a similar argument calculate the diffusion coefficient $V(p)$. (1 point)
- (c) Use your results from (a) and (b) in the forward Kolmogorov equation to present a diffusion equation for the Moran model. (0.5 points)
- (d) Now assume that $s \ll 1$. Approximate your results from (a) and (b) and use the general expression for the fixation probability $\rho(p_0)$ to show that the fixation probability is given by: (1 point)

$$\rho(p_0 = 1/N) = \frac{1 - e^{-s}}{1 - e^{-Ns}}. \quad (3)$$

- (e) Take the limit to derive a result for the fixation probability of a neutral allele, $s = 0$. Evaluate (3) for $N = 10$ and $N = 1000$ for both positive, $s = 1\%$, and negative selection, $s = -1\%$, respectively. Compare your results with ρ_1 of the exact Moran process. (0.5 points)

Problem 3: Absorption time in the diffusion approximation

In the diffusion approximation of a process with absorbing states 0 and 1 the expected fixation time, conditioned on absorption in state 1, reads:

$$\tau_1(p_0) = 2(S(1) - S(0)) \left(\int_{p_0}^1 \frac{\rho(p)(1 - \rho(p))}{e^{-A(p)}V(p)} dp + \frac{1 - \rho(p_0)}{\rho(p_0)} \int_0^{p_0} \frac{\rho(p)^2}{e^{-A(p)}V(p)} dp \right),$$

where $\rho(p)$ denotes the fixation probability, $A(p) = \int_0^p 2M(p)/V(p)dp$, and $S(p) = \int_0^p \exp(-A(p))dp$.

- (a) Calculate the *conditional expected waiting time for fixation*, $\tau_1(p_0)$, of an allele of frequency p_0 in the neutral Wright-Fisher process. Approximate the result for small p_0 . **(2 points)**
- (b) Compute τ_0 , the conditional expected waiting time until *extinction* (absorption in state 0) in the neutral Wright-Fisher process. Also derive the unconditioned expected waiting time $\bar{\tau}$ until *either fixation or extinction*. **(1 point)**