# Exercises 3

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## Problem 1

### a)

By the properties of the neutral Moran process, we know that the population size is constant and both A and B has the same of probablity of reproduction and death.

Assumed that at time $t-1$, the number of of A in the population is $(X(t-1) = i, t > 1)$.

The transition probability for individuals of allele A in the population: $P_{i,i-1} = P_{i,i+1} = p(1-p), P_{i,i} = p^2 + (1-p)^2, p = \frac{i}{N}$

$$E[X(t) \mid X(t-1) = i] = i \cdot P_{i,i-1} + (i-1) \cdot P_{i,i+1} + (i+1) \cdot P_{i,i} = i = X(t-1)$$

By law of total expectation, $E[X(t)] = E_{X(t-1)}[E_{X(t)}[X(t) \mid X(t-1)]] = i$ and hence the stationary mean.

### b)

$$\begin{aligned}
V_1 &= Var[X(1) \mid X(0) = i] \\
&= E[X(1)^2 \mid X(0) = i] - (E[X(1) \mid X(0) = i])^2 \\
&= i^2 \cdot P_{i,i-1} + (i-1)^2 \cdot P_{i,i+1} + (i+1)^2 \cdot P_{i,i} - i^2 \\
&= -\frac{i}{N}(1 - \frac{i}{N}) \cdot i^2 + ((i+1)^2 + (i-1)^2) \cdot \frac{i}{N}(1 - \frac{i}{N}) \\
&= 2\frac{i}{N}(1 - \frac{i}{N})
\end{aligned}$$

$Var[X(1) \mid X(0) = i] = 2\frac{i}{N}(1 - \frac{i}{N}),\ \forall t > 0.$

By the law of total variance:

$$\begin{aligned}
Var[X(t)] &= E_{X(t-1)}[Var_{X(t)}[X(t) \mid X(t-1)]] + Var_{X(t-1)}[E_{X(t)}[X(t) \mid X(t-1)]] \\
&= E_{X(t-1)}[2\frac{X(t-1)}{N}(1 - \frac{X(t-1)}{N})] + Var[X(t-1)] \\
&= 2\frac{E_{X(t-1)}}{N}(1 - \frac{E_{X(t-1)}}{N}) - \frac{2}{N^2}Var[X(t-1)] + Var[X(t-1)] \\
&= 2\frac{E_{X(t-1)}}{N}(1 - \frac{E_{X(t-1)}}{N}) + (1 - \frac{2}{N^2})Var[X(t-1)] \\
&= 2V_1 + (1 - \frac{2}{N})Var[X(t-1)]
\end{aligned}$$

if $X(0) = i$, we can rewrite the equation as

$$\begin{aligned}
Var[X(t)] - \frac{V_1}{\frac{2}{N^2}} &= (1 - \frac{2}{N^2})(Var[X(t-1)] - \frac{V_1}{\frac{2}{N^2}}) \\
&= (1 - \frac{2}{N^2})^{t-1}(V_1 - \frac{V_1}{\frac{2}{N^2}})
\end{aligned}$$

We have $Var[X(t)] = V_1 \frac{1-(1-\frac{2}{N^2})^t}{\frac{2}{N^2}}$.

### c)

Given the expression of variance in b), we can show

$$\begin{aligned}\lim_{N\to\infty} Var[X(t)] &= \lim_{N\to\infty} 2\frac{i}{N}(1-\frac{i}{N})\frac{1-(1-\frac{2}{N^2})^t}{\frac{2}{N^2}} \\ &= (1-\frac{2}{N^2})^{t-1}(V_1 - \frac{V_1}{\frac{2}{N^2}})\end{aligned}$$

### d)

## Problem 2

### a)

We know from the problem above that the $X(t)$ is a Markov chain. We can aslo know that $X(t)$ will reach a stationary distribution thanks to ergodicity.

Let $y_i = x_i - x_{i-1}$, we have $x_j = \sum_{i=1}^{j} = y_i$.

we also have: $x_i = P_{i,i-1}x_{i-1} + P_{i,i+1}x_{i+1} + (1 - P_{i,i+1} - P_{i,i-1})x_i$ by the stationary distribution.

This can be further simplified into: $\beta_i(x_i - x_{i-1}) = \alpha_i(x_{i+1} - x_i) \Rightarrow \beta_i y_i = \alpha_i y_{i+1}$.

Let $\gamma_i = \frac{\beta_i}{\alpha_i}$, we have $y_j = x_1 \cdot \frac{\beta_1}{\alpha_1} \cdots \frac{\beta_{j-1}}{\alpha_{j-1}} = x_1 \prod_{k=1}^{j-1} \gamma_k$

We have:

$$x_j = \sum_{i=1}^{j} y_i = x_1 + x_1 \cdot \sum_{i=1}^{j-1} \prod_{k=1}^{i} \gamma_k$$

Also by the fact that $x_N = 1$, $x_1 + x_1 \cdot \sum_{i=1}^{N-1} \prod_{k=1}^{i} \gamma_k = 1$, we can obtain that:

$$x_1 = \frac{1}{1 + \sum_{i=1}^{N-1} \prod_{k=1}^{i} \gamma_k}$$

Combining all that above, yielding:

$$x_j = \frac{1 + \sum_{i=1}^{j-1} \prod_{k=1}^{i} \gamma_k}{1 + \sum_{i=1}^{N-1} \prod_{k=1}^{i} \gamma_k}$$

### b)

From the subsection above, we can know that the fitness of A and B depends on the abundance of each type. Assuming that the A individual has a reproduce rate $r$ times as the B individuals.

In this case $\gamma_i = \frac{1}{r}$ remain constant.

Therefore, the eqauation above can be written as $x_i = \frac{1-r^{-i}}{1-r^{-N}}$

In the case of $i = 1$, $x_1 = \frac{1-r^{-i}}{1-r^{-N}}$

The limit can be calculated

$$\lim_{r\to 1} \frac{1-r^{-1}}{1-r^{-N}} = \lim_{r\to 1} \frac{-\log(r)}{Nr^{-N-1}} = 0$$