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Evolutionary Dynamics

Tutorial 2

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1 Multiple variable models: Linear systems

A system is described by n independent quantities $x_1, \ldots, x_n$. For a linear differential equation, the dynamical behaviour can be written in matrix form as

$$\frac{dx}{dt} = \mathbf{M}x, \tag{1}$$

where x is the column vector $x_1, \ldots, x_n$.

1.1 Finding equilibrium points

Analogously to the single variable case, the equilibrium is given by

$$\frac{dx}{dt} = 0 \quad \Leftrightarrow \quad \mathbf{M}\, x = 0, \tag{2}$$

From a simple linear algebra argument, the null vector $x = 0$ is always a solution of equation 2. If $\det \mathbf{M} \neq 0$, it's also the **only** equilibrium point. If $\det \mathbf{M} = 0$, on the other hand, an infinite number of equilibrium points exist.

1.1.1 An example

Consider the system of linear and homogeneous equations

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\mathbf{M}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{3}$$

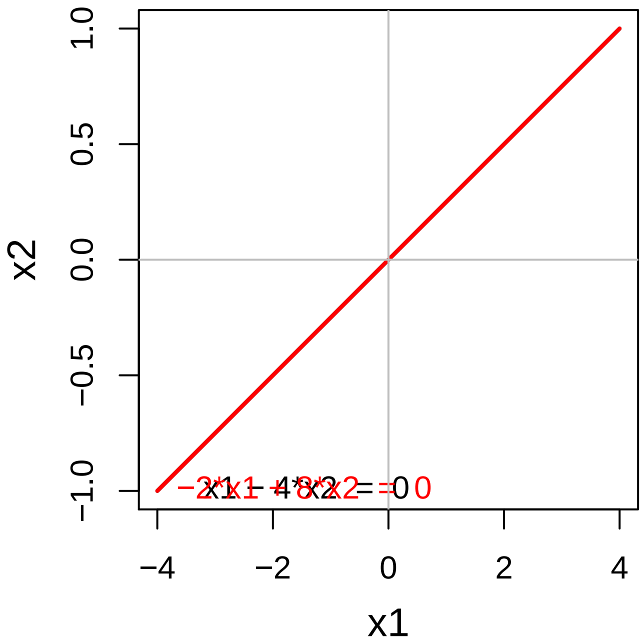
In two dimensions, the homogeneous system (3) represents two lines passing through the origin, with slopes $-a/b$ and $-c/d$, respectively. The solution of the system (3) corresponds the intersection of the two lines. There are two possible scenarios: (i) $a/b = c/d$, or (ii) $a/b \neq c/d$.

If the slopes are equal, the two lines coincide and there are infinite number of solutions. Furthermore, the determinant of matrix $\mathbf{M}$ is zero,

$$\det \mathbf{M} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = bd \underbrace{\left(\frac{a}{b} - \frac{c}{d}\right)}_{0} = 0.$$

For example, setting $a = 1$, $b = -4$, $c = -2$ and $d = 8$

```
library(matlib)
A <- matrix(c(1, -2, -4, 8), 2, 2)
b <- c(0, 0)
showEqn(A, b)

 1*x1 - 4*x2  =  0
-2*x1 + 8*x2  =  0

par(mar = c(4, 4, 0.5, 0.5))
par(mgp = c(2.5, 1, 0))
par(cex.lab = 1.25)
plotEqn(A, b)

   x1 - 4*x2  =  0
-2*x1 + 8*x2  =  0
```



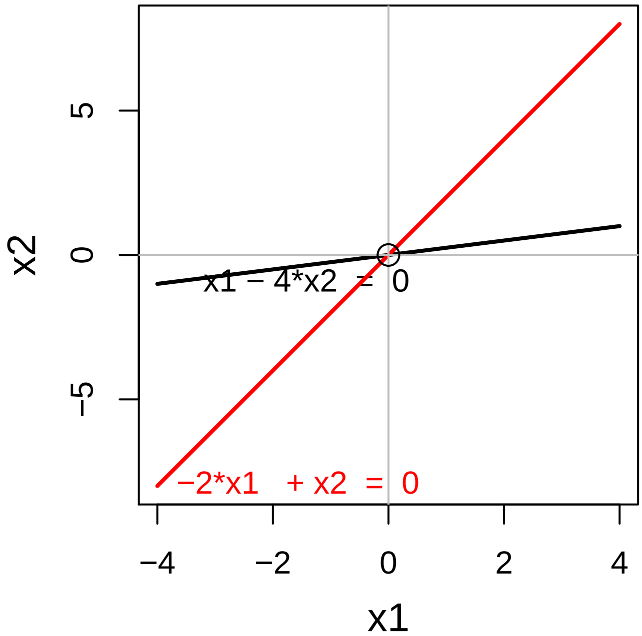
The solutions of the system are of the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c \begin{bmatrix} 4 \\ 1 \end{bmatrix} \text{ with } c \in \mathbb{R}.$$

On the other hand, if the slopes are different, the system of linear equations has a unique solution, the trivial solution. For example, setting $a = 1$, $b = -4$, $c = -2$ and $d = 1$,

```
A <- matrix(c(1, -2, -4, 1), 2, 2)
b <- c(0, 0)
showEqn(A, b)

 1*x1 - 4*x2  =  0
-2*x1 + 1*x2  =  0

plotEqn(A, b)
```

```
   x1 - 4*x2  =  0
-2*x1   + x2  =  0

solve(A, b)

[1] 0 0
```



Note that $\det \mathbf{M} = -7 \neq 0$.

If the model is described by

$$\frac{dx}{dt} = \mathbf{M}x + c, \tag{4}$$

with c a constant vector (the model is said to be *affine*), then the equilibrium point is given by $x^* = -\mathbf{M}^{-1}c$. Again, this argument is only valid if $\det \mathbf{M} \neq 0$, because the inverse of a matrix is defined only if the determinant is not zero.

1.1.2 An example

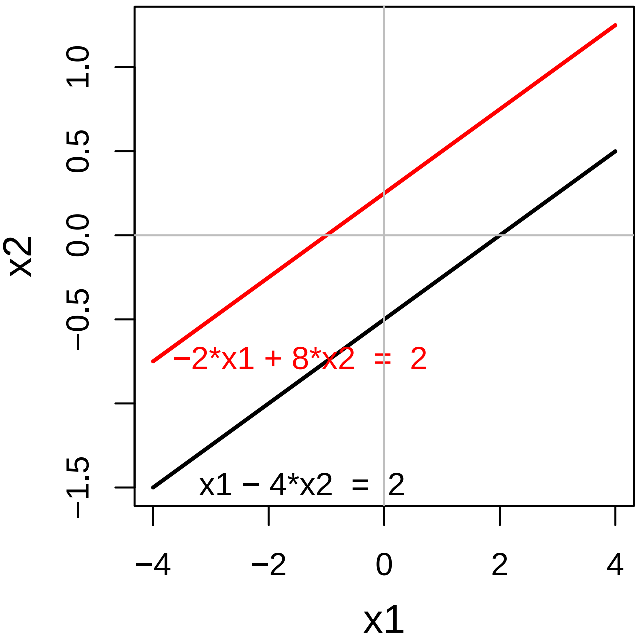
Consider the system of linear and non-homogeneous equations

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\mathbf{M}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} \tag{5}$$

with $e \neq 0 \vee f \neq 0$. In two dimensions, the non-homogeneous system (5) represents two lines with slopes $-a/b$ and $-c/d$, respectively. If the slopes are equal (and $\det \mathbf{M} = 0$), the lines are either identical or parallel. In the former case, there are infinite number of solutions, and in the latter case, no solution exists. For example, setting $a = 1$, $b = -4$, $c = -2$, $d = 8$, and $e = f = 2$,

```
library(matlib)
A <- matrix(c(1, -2, -4, 8), 2, 2)
b <- c(2, 2)
showEqn(A, b)
```

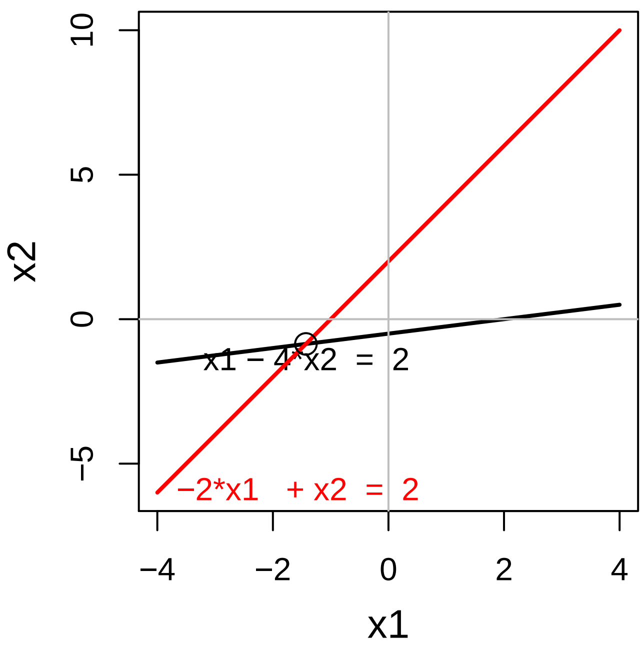
```
 1*x1 - 4*x2  =  2
-2*x1 + 8*x2  =  2

plotEqn(A, b)

   x1 - 4*x2  =  2
-2*x1 + 8*x2  =  2
```



Note that det $\mathbf{M} = 0$.

Now, if the slopes are different, the lines intersect at a unique point, i.e. , the system of linear equations has a unique solution. For example, setting $a = 1$, $b = -4$, $c = -2$, $d = 1$, and $e = f = 2$,

```
library(matlib)
A <- matrix(c(1, -2, -4, 1), 2, 2)
b <- c(2, 2)
showEqn(A, b)

 1*x1 - 4*x2  =  2
-2*x1 + 1*x2  =  2

plotEqn(A, b)

   x1 - 4*x2  =  2
-2*x1   + x2  =  2

solve(A, b)

[1] -1.4285714 -0.8571429
```



1.2 Determining the stability of equilibria

In the following paragraph we will admit that the determinant of matrix $\mathbf{M}$ is different from zero, then we will analyze the stability of the equilibrium point $x^* = 0$. Equation 1 can be, at least formally, integrated and admits the following solution

$$x(t) = e^{\mathbf{M}t}x(0). \tag{6}$$

Consider the transformation of matrix $\mathbf{M}$ to a diagonal matrix of eigenvalues $\mathbf{D}$, i.e., the eigendecomposition

$$\mathbf{M} = \mathbf{A}^{-1}\mathbf{D}\mathbf{A}. \tag{7}$$

Then, the evolution of $x(t)$ can be written as

$$x(t) = \mathbf{A}^{-1}e^{\mathbf{D}t}\mathbf{A}x(0), \tag{8}$$

or

$$v(t) = e^{\mathbf{D}t}v(0), \tag{9}$$

where we have introduced the transformed vector $v(t) = \mathbf{A}x(t)$. Let's also note that $\mathbf{A}$ transforms the null vector into itself. In other words, if we want to analyse the stability of the equilibrium point $x^* = 0$ we have to study the evolution of the vector $v^* = 0$. Since matrix $\mathbf{D}$ is diagonal, each entry of the vector v will be affected by the corresponding eigenvalue only. In symbols

$$v(t) = \begin{pmatrix} v_1(t) \\ \vdots \\ v_n(t) \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t}v_1(0) \\ \vdots \\ e^{\lambda_n t}v_n(0) \end{pmatrix}. \tag{10}$$

From this it should be clear that in order for the null vector to be a stable equilibrium point *the real parts of all eigenvalues must be negative*. The real part of the eigenvalues represent the decay rate at which each eigenvector is suppressed. If we project the evolution of the system on the different eigenvectors, we see that the dynamics are clearly dominated by the maximum eigenvector. Let's suppose that the eigenvalues $\lambda_1, \ldots, \lambda_n$ are ordered in decreasing order according to their real part, so that λ_1 has the maximum real part. The system is at time $t = 0$ in a state $x(t) = \sum_{i=1}^{n} \alpha_i a_i(t)$, with a_i the eigenvector corresponding to λ_i. The ratio of the contribution of any other eigenvector j with $j \neq 1$ evolves as

$$\frac{\alpha_j(t)}{\alpha_1(t)} = e^{(\lambda_j - \lambda_1)t} \to 0.$$

1.3 An example: the quasispecies equation

The equation can be written

$$\frac{dx}{dt} = \mathbf{M}x - \phi x = \mathbf{T}x. \tag{11}$$

If we consider the eigenvalues of $\mathbf{T}$ we find that they are exactly those of $\mathbf{M}$ shifted by ϕ. The stable fixed point will be the eigenvector corresponding to the maximum eigenvalue of the eigenvalue equation $\mathbf{M}x = \phi x$. Then, the stable fixed point of the system is the one that maximizes the average fitness ϕ.

Alternatively, using the following change of variables,

$$X_i(t) = x_i(t)e^{\psi(t)} \quad \text{with } \psi = \int_0^t \phi(s)ds,$$

Then, the quasispecies equation 11 can be written as

$$\frac{dX}{dt} = \mathbf{M}X, \tag{12}$$

Generically, the solution of equation 12 is of the form

$$X(t) = e^{\mathbf{M}t}X(0),$$

Using the eigendecomposition of matrix M and the transformation as above (see section 1.2)

$$v(t) = e^{\mathbf{D}t}v(0),$$

Now, assuming $\lambda_1 > \lambda_i \forall i$,

$$\frac{v(t)}{||v(t)||_1} = \frac{e^{\lambda_1}\left(\sum_i v_i(0)e^{(\lambda_i-\lambda_1)t}w_i\right)}{e^{\lambda_1}\left|\left|\sum_i v_i(0)e^{(\lambda_i-\lambda_1)t}w_i\right|\right|_1} \xrightarrow{t\to\inf} \frac{v_1(0)w_1}{||v_1(0)w_1||_1} = \frac{1}{||w_1||_1}w_1.$$

Thus, the largest eigenvalue and associated eigenvector dominates the dynamics of the quasispecies. Note that the eigenvectors of matrix $\mathbf{D}$ form the canonical basis of the space $\mathbb{R}^2$.

Perron–Frobenius Theorem Consider a irreducible non-negative $n \times n$ matrix $\mathbf{M}$. Matrix $\mathbf{M}$ has a positive eigenvalue λ_{max}, such that all other eigenvalues satisfy $|\lambda| < \lambda_{max}$. Furthermore, λ_{max} is simple and the components of the associated eigenvector w are all (strictly) positive, $w_i > 0\,\forall i$.

For example, consider the following quasispecies equation for two variants

$$\begin{bmatrix}\dot{x_0}\\ \dot{x_1}\end{bmatrix} = \begin{bmatrix} f_0q_{00} & f_1q10\\ f_0q_{01} & f_1q11\end{bmatrix}\begin{bmatrix}x_0\\ x_1\end{bmatrix} - \phi\begin{bmatrix}x_0\\ x_1\end{bmatrix},$$

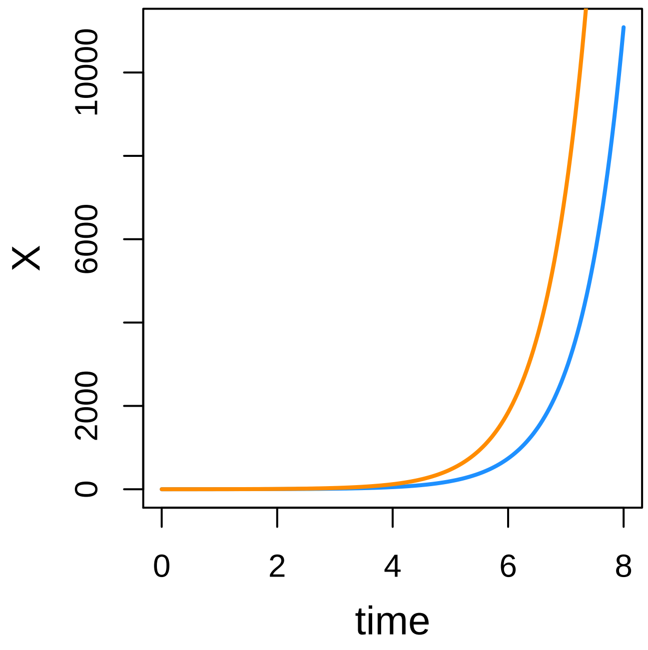
Asume, $f_0 = 1$ and $f_1 = 1.5$, and

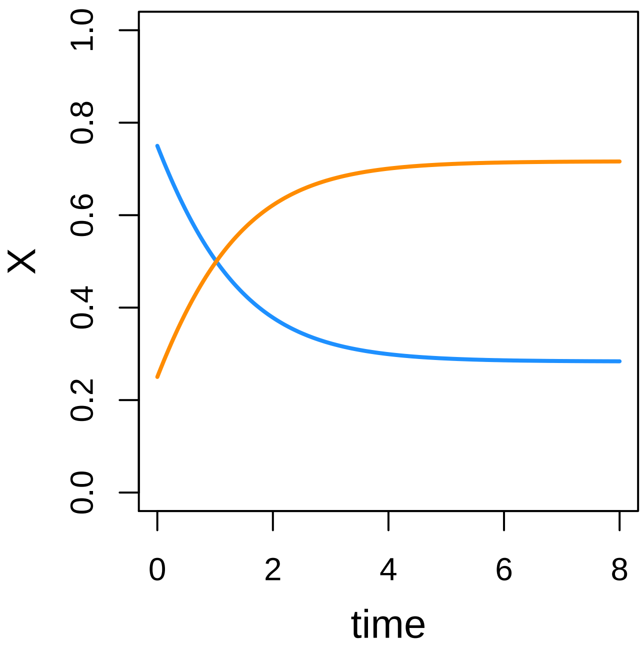
$$Q = \begin{bmatrix}0.6 & 0.2\\ 0.4 & 0.8\end{bmatrix}$$

The system of linear differential equations (see equation 12) can be solved formally

```
parms <- c(f0 = 1, f1 = 1.5, q_00 = 0.6, q_01 = 0.4, q_10 = 0.2,
    q_11 = 0.8)
times <- c(0:200)/25
initconds <- c(a = 0.75, b = 0.25)
M = matrix(c(parms["f0"] * parms["q_00"], parms["f0"] * parms["q_01"],
    parms["f1"] * parms["q_10"], parms["f1"] * parms["q_11"]),
    2, 2)
eig = eigen(M)
eig$values
```

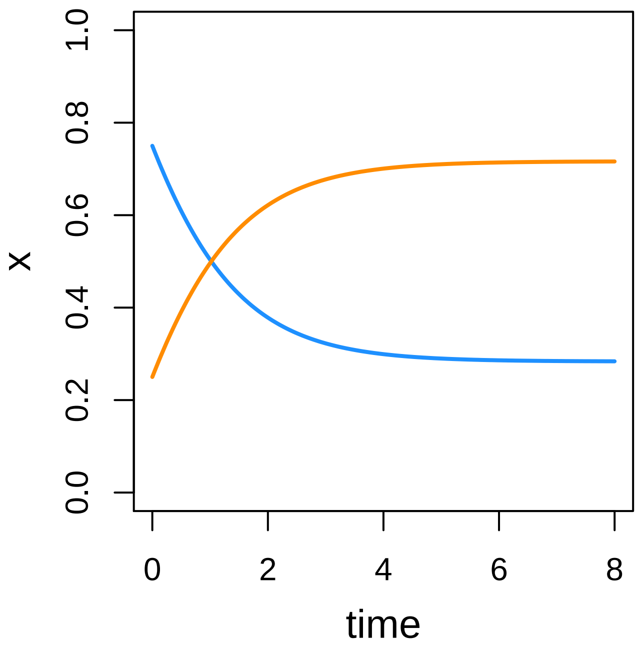
```
[1] 1.3582576 0.4417424

x_star = eig$vectors[, 1]/sum(eig$vectors[, 1])
x_star

[1] 0.2834849 0.7165151

# phi corresponds to largest eigenvalue
sum(x_star * c(parms["f0"], parms["f1"]))

[1] 1.358258

# Solve IVP
ivp = solve(eig$vectors, initconds)
Xa = ivp[1] * eig$vectors[1, 1] * exp(eig$values[1] * times) +
    ivp[2] * eig$vectors[1, 2] * exp(eig$values[2] * times)
Xb = ivp[1] * eig$vectors[2, 1] * exp(eig$values[1] * times) +
    ivp[2] * eig$vectors[2, 2] * exp(eig$values[2] * times)
plot(times, Xa, xlab = "time", ylab = expression(X), main = "",
    col = "dodgerblue", type = "l", lwd = 2)
lines(times, Xb, col = "#ff8c00", lwd = 2)
plot(times, Xa/(Xa + Xb), ylim = c(0, 1), xlab = "time", ylab = expression(X),
    main = "", col = "dodgerblue", type = "l", lwd = 2)
lines(times, Xb/(Xa + Xb), col = "#ff8c00", lwd = 2)
```





We can check this by numerically integrating the system of differential equation

```
library(deSolve)
my.atol <- c(1e-06)

sdiffeqns <- function(t, x, p) {
    dxa <- x[1] * p["f0"] * p["q_00"] + x[2] * p["f1"] * p["q_10"] -
        (x[1] * p["f0"] + x[2] * p["f1"]) * x[1]
    dxb <- x[1] * p["f0"] * p["q_01"] + x[2] * p["f1"] * p["q_11"] -
        (x[1] * p["f0"] + x[2] * p["f1"]) * x[2]
    list(c(dxa, dxb))
}

out <- lsoda(initconds, times, sdiffeqns, parms, rtol = 1e-10,
    atol = my.atol)
plot(out[, 1], out[, 2], xlab = "time", ylab = "x", main = "",
    col = "dodgerblue", lwd = 2, ylim = c(0, 1), xlim = c(0, 8),
    type = "l")
lines(out[, 1], out[, 3], col = "#ff8c00", lwd = 2)
```



2 Multivariate models: Non-linear systems

Suppose the general case

$$\frac{dx}{dt} = f(x) \tag{13}$$

with f being differentiable, $f \in C_1(\mathbb{R}^n)$. As in the one-dimensional case, the fixed points x^* are defined as the solution of

$$0 = f(x). \tag{14}$$

In order to determine the *local dynamics* of the system around x^*, consider the linearisation:

$$f(x) = f(x^* + \varepsilon) = f(x^*) + \mathbf{J}\underbrace{(x - x^*)}_{\varepsilon} + \cdots = 0 + \mathbf{J}\varepsilon + \ldots \tag{15}$$

to find

$$\frac{d\varepsilon}{dt} \approx \mathbf{J}\varepsilon. \tag{16}$$

Here, $\mathbf{J}_{ij} = \partial_j f_i$ is the Jacobian matrix of f. The nature of the fixed point x^* is given by the Jacobian matrix replacing $\mathbf{M}$ in Eq. (1). The fixed point is only stable if all eigenvalues of $\mathbf{J}$ have a negative real part. The dynamical behaviour then depends on the imaginary parts. In the case that the eigenvalues cannot be computed efficiently, it is sufficient to check the *Routh-Hurwitz* conditions. Note that no conclusions can be drawn if $\mathrm{Re}\,\lambda_i = 0$.