Exercises1

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Problem 1

a)

The model reaches equilibrium points as the $x_{t+1} = f(x_t) = x_t$, let x^* be the equilibrium points of the system.

Solving $f(x) = rx(1-x) = x$, we have: $x^* = 0$ or $x^* = 1 - \frac{1}{r}$ when $r \neq 0$.

b)

The stability of the points can be checked by the gradient of $|f'(x^*)| = |r - 2rx^*|$.

$r = 0.5$, $x_1^* = 0, |f'(x_1^*)| = 0.5 < 1$ and hence **attractive and stable**.

$x_2^* = -1$, discarded since x can only take value larger or equal to zero.

$r = 1.5$, $x_1^* = 0, |f'(x_1^*)| = 1.5 > 1$ and hence **repelling and unstable**.

$x_2^* = \frac{1}{3}, |f'(x_2^*)| = 0.5 < 1$ and hence **attractive and stable**.

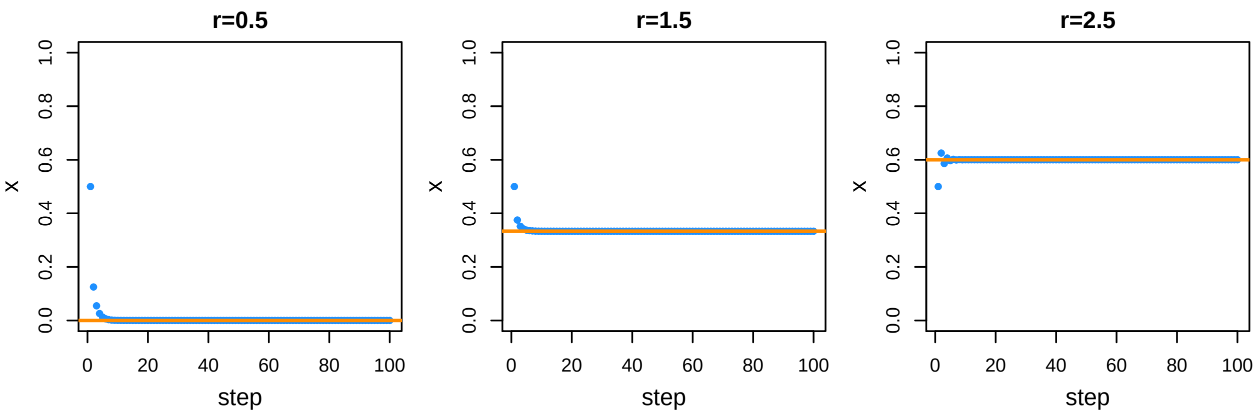
$r = 2.5$, $x_1^* = 0, |f'(x_1^*)| = 2.5 > 1$ and hence **repelling and unstable**.

$x_2^* = \frac{3}{5}, |f'(x_2^*)| = 0.5 < 1$ and hence **attractive and stable**.

c)

```r
getRep <- function(init, steps, r) {
  x <- rep(0, steps)
  x[1] <- init
  for (i in 2:steps) {
    x[i] <- r * x[i - 1] * (1 - x[i - 1])
  }
  return(x)
}

steps = 100
x1 = getRep(0.5, steps, 0.5)
x2 = getRep(0.5, steps, 1.5)
x3 = getRep(0.5, steps, 2.5)

par(mar = c(4, 4, 2, 0.5)) # margin size
par(mgp = c(2.5, 1, 0)) # axis localtion
par(cex.lab = 1.25) # size of y axis label
par(mfrow=c(1,3)) # 1x3 fig


plot(x1, xlab = "step", ylab = "x", main = "r=0.5", col = "dodgerblue",
     pch = 20, ylim = c(0, 1))
abline(h = 0, col = "#ff8c00", lwd = 2)
```

```r
plot(x2, xlab = "step", ylab = "x", main = "r=1.5", col = "dodgerblue",
     pch = 20, ylim = c(0, 1))
abline(h = 1/3, col = "#ff8c00", lwd = 2)

plot(x3, xlab = "step", ylab = "x", main = "r=2.5", col = "dodgerblue",
     pch = 20, ylim = c(0, 1))
abline(h = 3/5, col = "#ff8c00", lwd = 2)
```

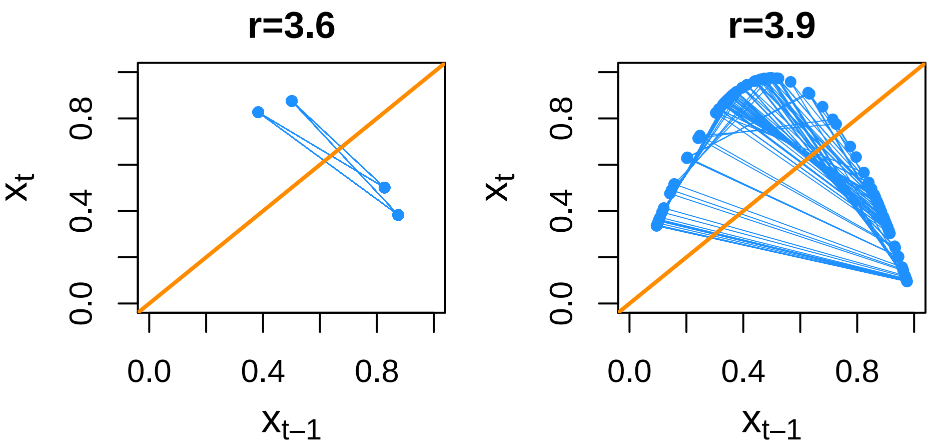


d)

e)

As seen in the figures below,

```r
x = getRep(0.5, 100, 3.5)
xstm1 <- x[-length(x)]
xst <- x[-1]
y = getRep(0.5, 100, 3.9)
ystm1 <- y[-length(y)]
yst <- y[-1]

par(mar = c(4, 4, 2, 0.5)) # margin size
par(mgp = c(2.5, 1, 0)) # axis localtion
par(cex.lab = 1.25) # size of y axis label
par(mfrow=c(1,2)) # 1x2

plot(xstm1, xst, xlab = expression(x[t - 1]), ylab = expression(x[t]),
     main = "r=3.6", col = "dodgerblue", pch = 20, xlim = c(0, 1),
     ylim = c(0,1))
lines(xstm1, xst, col = "dodgerblue", lwd = 0.5)
abline(b = 1, a = 0, col = "#ff8c00", lwd = 2)

plot(ystm1, yst, xlab = expression(x[t - 1]), ylab = expression(x[t]),
     main = "r=3.9", col = "dodgerblue", pch = 20, xlim = c(0, 1),
     ylim = c(0,1))
lines(ystm1, yst, col = "dodgerblue", lwd = 0.5)
abline(b = 1, a = 0, col = "#ff8c00", lwd = 2)
```



Problem 2

a)

$$\begin{aligned}
\int \frac{dx(t)}{x(t)(1-\frac{x(t)}{K})} &= \int \lambda dt \quad \text{separation of variables} \\
\int \frac{dx(t)}{x(t)} + \int \frac{dx(t)}{K-x(t)} &= \int \lambda dt \\
\ln|x(t)| - \ln|K-x(t)| &= \lambda t + C \\
\ln|\frac{K-x(t)}{x(t)}| &= -(\lambda t + C) \\
|\frac{K-x(t)}{x(t)}| &= e^{-(\lambda t + C)} \\
\frac{K-x(t)}{x(t)} &= e^{-\lambda t} C_0
\end{aligned}$$

From that we can get:

$$x(t) = \frac{K}{1+C_0 e^{-\lambda t}}, \quad C_0 = \frac{K-x_0}{x_0}$$

And hence the $x(t) = \frac{K x_0 e^{\lambda t}}{K + x_0(e^{\lambda t}-1)}$.

b)

The condition for the equilibrium is that $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$

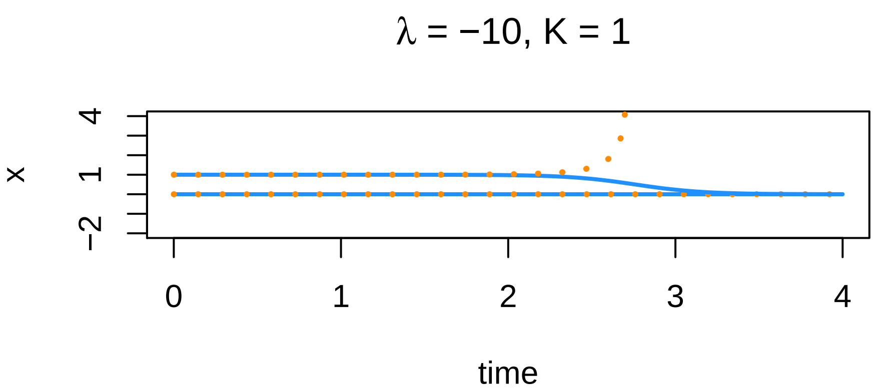
Solving $f'(x) = 0$, we have: $x^* = 0$ or $x^* = K$.

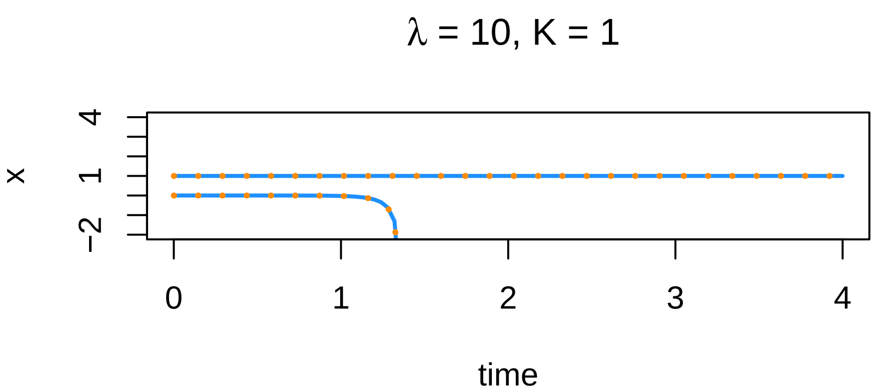
When $x^* = 0, f'(x^*) = \lambda$ and hence the point is **stable** if $\lambda < 0$ and **unstable** otherwise.

When $x^* = K, f'(x^*) = -\lambda$ and hence the point is **unstable** if $\lambda < 0$ and **stable** otherwise.

c)

```r
library(deSolve)
k = 1
lambda = -5
parms <- c()
my.atol <- c(1e-06)
times <- c(0:100)/25
sdiffeqns <- function(t, s, parms) {
  sd1 <- lambda * s[1] * (1 - s[1]/k)
  list(c(sd1))
}

# just below 0
out1m <- lsoda(c(0 - 1e-06), times, sdiffeqns, rtol = 1e-10, atol = my.atol)
# just above 0
out1p <- lsoda(c(0 + 1e-06), times, sdiffeqns, rtol = 1e-10, atol = my.atol)
# just below k
out2m <- lsoda(c(k - 1e-06), times, sdiffeqns, rtol = 1e-10, atol = my.atol)
# just above k
out2p <- lsoda(c(k + 1e-06), times, sdiffeqns, rtol = 1e-10, atol = my.atol)

plot(out1m, xlab = "time", ylab = "x", col = "dodgerblue", lty = 1, lwd = 2,
     ylim = c(-2, 4), xlim = c(0, 4),
     main = expression(" "*lambda*" = -10, "*K*" = 1"))
lines(out1m, col = "#ff8c00", lty = 3, lwd = 3)
lines(out2m, col = "dodgerblue", lty = 1, lwd = 2)
lines(out2p, col = "#ff8c00", lty = 3, lwd = 3)
```





Note

Codes for the solution above is inspired by the tutorials materials and available on github repo