UNIVERSITY OF TORONTO

Faculty of Arts and Science

DECEMBER 2002 EXAMINATIONS

STA 257H1F

Duration - 3 hours

No Aids Allowed

NAME: ______________________________

STUDENT NUMBER: ______________________________

- There are 15 pages including this page. The last two pages are formulae that may be useful.
- Answers that are algebraic expressions should be simplified. Series and integrals should be evaluated. Numerical answers need not be expressed in decimal form.
- Total marks: 110

1	2	3	4	5	6	7	8

9	10	11	12	13	14	15

1. (10 marks) A, B, and C are events in the probability space $(S, \mathcal{F}, P)$.

   (a) Prove that $P(A \cup B \cup C) \leq P(A) + P(B) + P(C)$.

   (b) If A is a subset of B, is it possible for A and B to be independent events? If impossible, explain why. If it is possible, give an example.

2. (5 marks) Suppose X is a continuous random variable with density function

$$f(x) = \begin{cases} 2(1-x) & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the density of $Y = 1/X$.

3. (5 marks) Suppose that a random variable X has a strictly increasing cumulative distribution function $F(x)$. Show that the random variable $Y = F(X)$ has a uniform distribution on $(0, 1)$.

Continued

4. (12 marks) Let X be a continuous random variable with density function $f(x) = \frac{3}{2}x^2$, $-1 \le x \le 1$ and 0 otherwise. Sketch the following functions, clearly indicating at least three important points on each horizontal axis.

(a) the density function for X

(b) the cumulative distribution function for X

(c) the approximate density function of $(X_1 + X_2 + \cdots + X_{100})/100$ where the X_i are independent random variables with density f

5. (10 marks) Suppose that X and Y are jointly distributed discrete random variables with probability function

$$p(x, y) = kq^2 p^{x+y}, \quad x, y = 0, 1, 2, \ldots, \quad 0 < p < 1, \quad q = 1 - p$$

(a) Determine the value of the constant k.

(b) Are X and Y independent? Why or why not?

(c) Find $P(X + Y = t)$.

Continued

6. (5 marks) A fair coin is flipped 20 times. Given that the total number of heads is 12, what is the probability function for the number of heads in the first 10 flips?

7. (5 marks) Prove that, for a Poisson random variable X, if the parameter λ is not fixed and is itself an exponential random variable with parameter 1, then

$$P(X = x) = \left(\frac{1}{2}\right)^{x+1}$$

8. (10 marks) Let X and Y be two independent random variables.

(a) Show that $\text{Cov}(X, XY) = E(Y)V(X)$.

(b) Prove that

$$\rho(X+Y, X-Y) = \frac{V(X) - V(Y)}{V(X) + V(Y)}$$

9. (7 marks) Let X be a nonnegative random variable with $E(X) = 5$ and $E(X^2) = 42$. Find an upper bound for $P(X \geq 11)$ using

(a) Markov's inequality

(b) Chebyshev's inequality

10. (10 marks) Let X and Y be independent Gamma random variables with parameters (α_1, λ) and (α_2, λ), respectively. Let $U = X + Y$ and $V = X/(X + Y)$.

   (a) Find the joint density function of U and V.

   (b) Identify the marginal distributions of U and V.

11. (7 marks) $\pi(t)$ is the probability generating function of a non-negative integer-valued random variable X.

(a) What is $\pi(1)$?

(b) What is $\pi(0)$?

(c) What is $\frac{1}{2}(\pi(1) + \pi(-1))$?

(d) If $0 < p < 1$ and $q = 1 - p$ then $p/(1 - qt)$ is a probability generating function. But $\pi(t) = p/(1+qt)$ is not a probability generating function; for one thing, $\pi(1)$ does not have the right value. However, $\pi(t) = \alpha/(1 + qt)$ does have the right value at $t = 1$ if α is chosen correctly. Why is it still not a probability generating function?

12. (6 marks) For a random variable X, its moment generating function is $m_X(t) = (1/81)(e^t + 2)^4$.

    (a) Find $P(X \leq 2)$.

    (b) Find EX.

13. (5 marks) Let X_1, X_2, X_3, X_4 be independent and identically distributed exponential random variables with parameter λ. Let $X_{(4)} = \max\{X_1, X_2, X_3, X_4\}$. Find $P(X_{(4)} \geq 3\lambda)$.

14. (5 marks) Suppose the random variable X has a $N(3, 9)$ distribution and the random variable Y has a $N(1, 4)$ distribution and X and Y are independent.

(a) Give an expression for $P(X + 2Y \leq 6)$ in terms of Φ, the cumulative distribution function for the standard normal distribution.

(b) Find a random variable Z that is a function of both X and Y such that Z has a Chi-square distribution with parameter 2.

15. (8 marks) $X_1, X_2, \ldots, X_n$ is a random sample of a Bernoulli random variable with parameter p.

(a) Find the value of the constant a that makes

$$a(X_1 + X_1^2 + X_2 + X_2^2 + \ldots + X_n + X_n^2)$$

an unbiased estimator for p.

(b) Is the estimator in part (a) consistent? Explain.

The Gamma Function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

The Beta Function

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Some Important Discrete Probability Distributions

Distribution	Probability Function	Mean	Variance
Binomial(n, p)	$p(x) = \binom{n}{x} p^x (1-p)^{n-x}$ for $x = 0, 1, 2, \ldots, n$	np	$np(1-p)$
Bernoulli(p)	same as Binomial$(1, p)$		
Poisson(λ)	$p(x) = \frac{\lambda^x e^{-\lambda}}{x!}$ for $x = 0, 1, 2, \ldots$	λ	λ
Geometric(p)	$p(x) = p(1-p)^x$ for $x = 0, 1, 2, \ldots$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$

Continued

Some Important Continuous Probability Distributions

Distribution	Density Function	Mean	Variance
Uniform(a, b)	$f(x) = \frac{1}{b-a}$ for $a < x < b$	$\frac{b+a}{2}$	$\frac{(b-a)^2}{12}$
Normal(μ, σ^2)	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$ for $x \in \mathbb{R}$	μ	σ^2
Standard Normal	same as Normal$(0, 1)$		
Exponential(λ)	$f(x) = \lambda e^{-\lambda x}$ for $x > 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma(α, λ)	$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$ for $x > 0$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$
Beta(α, β)	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$ for $0 < x < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
Chi-square(n)	same as Gamma$\left(\frac{n}{2}, \frac{1}{2}\right)$		

Total pages 15
Total marks 110