Joint Probability Distributions

Definition 1 The joint probability distribution function (pdf) of a couple of random variables X and Y is defined by:

	discrete	continuous
pdf	$P(X=m,Y=n)$	$f(x,y)$
	$\sum_n \sum_m P(X=m,Y=n)=1$	$\int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)dxdy=1$
$P((X,Y)\in A)$	$\sum_{(m,n)\in A} P(X=m,Y=n)$	$\int\int_A f(x,y)dxdy$

$\diamond$

Example 1 A coin is tossed 4 times. Let X be the number of heads in the first toss, and Y be the number of heads in the 4 tosses. Find the joint pdf of (X,Y). $\triangle$

Example 2 Let (X,Y) be a couple random of variable with pdf

$$f(x,y)=cxy\ ,\ 0<x<y<1$$

Find the value of the constant c, and find $P(XY<1/2)$. $\triangle$

Definition 2 The marginal pdf of X and Y are defined by

<u>*discrete case:*</u> $P(X=m)=\sum_n P(X=m,Y=n)$, and

$$P(Y=n)=\sum_m P(X=m,Y=n)$$

<u>*continuous case:*</u> $g(x)=\int_{\mathbb{R}} f(x,y)\ dy$, and $h(y)=\int_{\mathbb{R}} f(x,y)\ dx$ $\diamond$

Proposition 1 Two random variables X and Y are independent if and only if:

<u>*discrete case:*</u> $P(X=m,Y=n)=P(X=m)\times P(Y=n)$ for all m and n

<u>*continuous case:*</u> $f(x,y)=g(x)\times h(y)$ $\square$

Example 3 Find the marginal pdf in examples 1 and 2. $\triangle$

Example 4 *(Trinomial distribution)* An urn contains 3 red balls, 3 white balls, and 4 blue balls. 15 balls are selected at random and with replacement form the urn. Let X be the number of red balls, and Y be the number of white balls among the 15 selected. Find the joint pdf of (X,Y), then find the marginal pdf of X and Y. $\triangle$

1 Transformation of Couples of Random Variables

Let (X, Y) be a couple of random variables with joint pdf $f(x, y)$, and let $U = g(X, Y)$, and $V = h(X, Y)$.

<u>*question:*</u> what is the joint pdf of the couple (U, V)?

Example 5 Let X and Y be a couple of random variables with joint pdf

$$f(x, y) = 2 \quad , \quad 0 < x < y < 1$$

Let $U = X/Y$ and $V = Y$, find the joint pdf of the couple (U, V). Are U and V independent. $\triangle$

Proposition 2 $k(u, v) = |J(u, v)| \times f(l(u, v), m(u, v))$ $\square$

Example 6 Let X and Y be independent $\chi^2(2)$ distributions. Find the joint pdf of $U = X + Y$ and $V = X - Y$, then find the marginal pdf of V.

solving the system of equations yields $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$

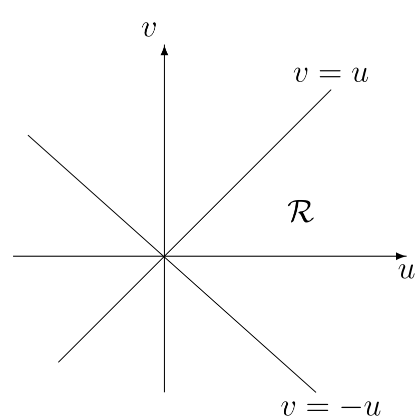
the Jacobian is $J(u, v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$

and hence

$$g(u, v) = \frac{1}{8}\, e^{-u/2} \qquad 0 \le u + v \le \infty \ , \ 0 \le u - v \le \infty$$

$$h(u) = \int_{-u}^{u} \frac{1}{8}\, e^{-u/2} dv = \frac{1}{4}\, u\, e^{-u/2} \qquad 0 < u < +\infty$$

$$k(v) = \begin{cases} \displaystyle\int_{-v}^{+\infty} \frac{1}{8}\, e^{-u/2}\, du = \frac{1}{4}\, e^{v/2} & -\infty < v < 0 \\ \displaystyle\int_{v}^{+\infty} \frac{1}{8}\, e^{-u/2}\, du = \frac{1}{4}\, e^{-v/2} & o < v < +\infty \end{cases}$$



$\triangle$

<u>*question:*</u> Let (X, Y) be a couple of random variables with joint pdf $f(x, y)$. How to find the pdf of $U = g(X, Y)$?

Exercise 1 Let X and Y be a couple of random variables with pdf

$$f(x, y) = \frac{1}{x^2 y^2} \qquad x \ge 1 \ , \ y \ge 1$$

Find the joint pdf of $U = XY$ and $V = X/Y$. Find the marginal pdf of U and V.

solving the system of equations yields $x = \sqrt{uv}$ and $y = \sqrt{\frac{u}{v}}$

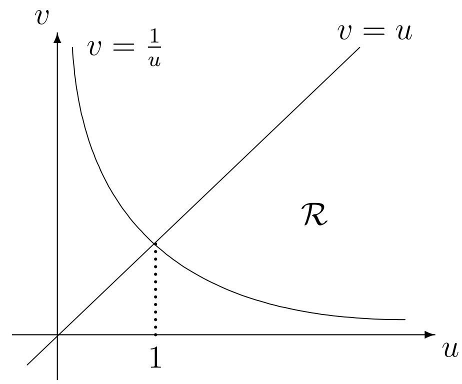
the Jacobian is $J(u,v) = \begin{vmatrix} \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \\ \frac{1}{2\sqrt{uv}} & -\frac{\sqrt{u}}{2v\sqrt{v}} \end{vmatrix} = -\frac{1}{2v}$

and hence

$$g(u,v) = \frac{1}{2u^2v} \qquad uv \geq 1\ ,\ \frac{u}{v} \geq 1$$

$$h(u) = \int_{\frac{1}{u}}^{u} \frac{1}{2u^2v} dv = \frac{1}{2u^2}\Big[\ln v\Big]_{\frac{1}{u}}^{u} = \frac{\ln u}{u^2} \qquad u \geq 1$$

$$k(v) = \begin{cases} \displaystyle\int_{1/v}^{+\infty} \frac{1}{2u^2v}\, du = \frac{1}{2} & 0 \leq v \leq 1 \\ \displaystyle\int_{v}^{+\infty} \frac{1}{2u^2v}\, du = \frac{1}{2v^2} & v \geq 1 \end{cases}$$



$\bigtriangledown$

Fisher distribution: Let X_1, X_2 be two independent chi-square random variables with r_1 and r_2 degrees of freedom, respectively. Let $Y_1 = (X_1/r_1)/(X_2/r_2)$ and $Y_2 = X_2$. Find the joint pdf of Y_1 and Y_2, then find the marginal pdf of Y_1.

$$f(x_1,x_2) = \frac{1}{\Gamma(r_1/2)\Gamma(r_2/2)2^{(r_1+r_2)/2}}\ x_1^{\frac{r_1}{2}-1} x_2^{\frac{r_2}{2}-1} e^{-\frac{x_1+x_2}{2}} \qquad 0 < x_1 < \infty\ ,\ 0 < x_2 < \infty$$

solving the system of equations yields $x_1 = \frac{r_1}{r_2}\, y_1 y_2$, and $x_2 = y_2$

the Jacobian is $J(y_1,y_2) = \begin{vmatrix} \frac{r_1}{r_2}\, y_2 & \frac{r_1}{r_2}\, y_1 \\ 0 & 1 \end{vmatrix} = \frac{r_1}{r_2}\, y_2$

and hence

$$g(y_1,y_2) = \frac{\left(\frac{r_1}{r_2}\right)^{r_1/2}}{\Gamma(r_1/2)\Gamma(r_2/2)2^{(r_1+r_2)/2}}\ y_1^{\frac{r_1}{2}-1} y_2^{\frac{r_1}{2}+\frac{r_2}{2}-1} e^{-\frac{\left(\frac{r_1}{r_2}\, y_1+1\right) y_2}{2}} \qquad 0 < y_1 < \infty\ ,\ 0 < y_2 < \infty$$

is the pdf of the couple (Y_1, Y_2)

$$h(y_1) = \frac{\left(\frac{r_1}{r_2}\right)^{r_1/2}}{\Gamma(r_1/2)\Gamma(r_2/2)2^{(r_1+r_2)/2}}\ y_1^{\frac{r_1}{2}-1} \int_0^\infty y_2^{\frac{r_1}{2}+\frac{r_2}{2}-1} e^{-\frac{\left(\frac{r_1}{r_2}\, y_1+1\right) y_2}{2}} dy_2$$

but $\displaystyle\int_0^\infty x^{\alpha-1} e^{-\frac{x}{\theta}} dx = \Gamma(\alpha)\theta^\alpha$ (pdf of a $G(\alpha,\theta)$ distribution)

by comparison, $\alpha = \frac{r_1+r_2}{2}$ and $\theta = \frac{2}{\frac{r_1}{r_2} y_1 + 1}$

hence $h(y_1) = \dfrac{{r_1}^{r_1/2}\Gamma((r_1+r_2)/2)}{\Gamma(r_1/2)\Gamma(r_2/2)} . \dfrac{y_1^{\frac{r_1}{2}-1}}{(\frac{r_1}{r_2}y_1+1)^{(r_1+r_2)/2}} \qquad 0 < y_1 < \infty$

(the distribution of Y_1 is called Fisher).

The Beta distribution: Let X and Y be independent random variables with distributions $G(\alpha, \theta)$ and $G(\beta, \theta)$ respectively. Let $U = \frac{X}{X+Y}$ and $V = X + Y$. Find the joint pdf of the couple (U, V), then find the marginal pdf of U and V.

$$f(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\theta^{\alpha+\beta}} x^{\alpha-1} y^{\beta-1} e^{-(x+y)/\theta} \qquad 0 < x < \infty \ , \ 0 < y < \infty$$

solving the system of equations yields $x = uv$ and $y = v(1-u)$

the Jacobian is $J(u, v) = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = v$

and hence

$$g(u, v) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\theta^{\alpha+\beta}} u^{\alpha-1}(1-u)^{\beta-1} v^{\alpha+\beta-1} e^{-v/\theta} \qquad 0 < u < 1 \ , \ 0 < v < \infty$$

$$\begin{aligned} h(u) &= \int_0^\infty \frac{1}{\Gamma(\alpha)\Gamma(\beta)\theta^{\alpha+\beta}} u^{\alpha-1}(1-u)^{\beta-1} v^{\alpha+\beta-1} e^{-v/\theta} \, dv \\ &= \frac{u^{\alpha-1}(1-u)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \frac{1}{\theta^{\alpha+\beta}} v^{\alpha+\beta-1} e^{-v/\theta} \, dv \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1}(1-u)^{\beta-1} \qquad 0 < u < 1 \end{aligned}$$

U is called Beta distribution with parameters α and β, denoted Beta(α, β)

$$\begin{aligned} k(v) &= \int_0^1 \frac{1}{\Gamma(\alpha)\Gamma(\beta)\theta^{\alpha+\beta}} u^{\alpha-1}(1-u)^{\beta-1} v^{\alpha+\beta-1} e^{-v/\theta} \, du \\ &= \frac{v^{\alpha+\beta-1}e^{-v/\theta}}{\Gamma(\alpha)\Gamma(\beta)\theta^{\alpha+\beta}} \int_0^1 u^{\alpha-1}(1-u)^{\beta-1} \, du \\ &= \frac{v^{\alpha+\beta-1}e^{-v/\theta}}{\Gamma(\alpha)\Gamma(\beta)\theta^{\alpha+\beta}} \times \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \frac{1}{\Gamma(\alpha+\beta)\theta^{\alpha+\beta}} v^{\alpha+\beta-1} e^{-v/\theta} \qquad 0 < v < \infty \end{aligned}$$

V is then a Gamma distribution with parameters $\alpha + \beta$ and θ.

Proposition 3 If $X \rightsquigarrow G(\alpha, \theta)$, and $Y \rightsquigarrow G(\beta, \theta)$, and if X and Y are independent, then

a. $\dfrac{X}{X+Y}$ have a Beta distribution with parameters α and β.

b. $\dfrac{Y}{X+Y}$ have a Beta distribution with parameters β and α.

c. $X + Y \rightsquigarrow G(\alpha + \beta, \theta)$.

□

2 Several Independent Random Variables

Definition 3 If $X_1, X_2, \ldots, X_n$ are n independent random variables, then

$$P(X_1 \in B_1 \cap X_2 \in B_2 \cap \ldots \cap X_n \in B_n) = P(X_1 \in B_1) \times \ldots \times P(X_n \in B_n) \qquad \diamond$$

Notation: If $X_1, X_2, \ldots, X_n$ are n independent random variables with the same pdf, then $X_1, X_2, \ldots, X_n$ is called a random sample of size n.

Property 1 If $X_1, X_2, \ldots, X_n$ are n independent random variables, and if $g_1, g_2, \ldots, g_n$ are n functions, then

$$E(g_1(X_1) \times \ldots \times g_n(X_n)) = E(g_1(X_1)) \times \ldots \times E(g_n(X_n)) \qquad \square$$

Proof:

$$\begin{aligned} E(g_1(X_1)\times\ldots\times g_n(X_n)) &= \int_{\mathbb{R}}\int_{\mathbb{R}}\ldots\int_{\mathbb{R}} g_1(x_1)\times\ldots\times g_n(x_n)\times f_1(x_1)\times\ldots\times f_n(x_n)dx_1\ldots dx_n \\ &= \left(\int_{\mathbb{R}} g_1(x_1)f_1(x_1)dx_1\right)\ldots\left(\int_{\mathbb{R}} g_n(x_n)f_n(x_n)dx_1\right) \\ &= E(g_1(X_1)) \times \ldots \times E(g_n(X_n)) \qquad \blacksquare \end{aligned}$$

Example 7 Let X_1, X_2, X_3 be a random sample of size 3 with exponential distribution with parameter $\lambda = 1$. Find

a) $P(0 < X_1 < 1, 0 < X_2 < 1/2, X_3 > 1)$,

b) $E(2X_1^2X_3)$ $\triangle$

Proposition 4 (*distribution of sum of independent random variables*)
Let $X_1, X_2, \ldots, X_n$ be n independent random variables, and let $Y = a_1X_1 + \ldots + a_nX_n$ (a_i are scalars), then

$$M_Y(t) = M_{X_1}(a_1t) \times \ldots \times M_{X_n}(a_nt) \qquad \square$$

Proof: $M_Y(t) = E(e^{tY}) = E(e^{t(a_1X_1+\ldots+a_nX_n)}) = E(e^{ta_1X_1} \times \ldots \times e^{ta_nX_n})$, hence by property 1,

$M_Y(t) = E(e^{ta_1X_1}) \times \ldots \times E(e^{ta_nX_n}) = M_{X_1}(a_1t) \times \ldots \times M_{X_n}(a_nt)$ $\blacksquare$

Proposition 5 Let $X_1, X_2, \ldots, X_n$ be a random sample of size n, and let $Y = X_1 + \ldots + X_n$, then

$$M_Y(t) = (M_{X_1}(t))^n \qquad \square$$

Proof: simple consequence of proposition 4, with all $a_i = 1$ and M_{X_i} are equal. $\blacksquare$

Property 2 If $X_i \rightsquigarrow G(\alpha_i\,;\theta)$ for $i = 1, 2, \ldots, n$, and if the X_i's are independent, then

$$Y = X_1 + \ldots + X_n \rightsquigarrow G(\alpha_1 + \alpha_2 + \ldots + \alpha_n\,;\theta) \qquad \square$$

Proof: $M_Y(t) = M_{X_1}(a_1 t) \times \ldots \times M_{X_n}(a_n t) = \left(\frac{1}{1-\theta t}\right)^{\alpha_1} \times \ldots \times \left(\frac{1}{1-\theta t}\right)^{\alpha_n}$

$$= \left(\frac{1}{1-\theta t}\right)^{\alpha_1+\alpha_2+\ldots+\alpha_n} \qquad \blacksquare$$

Exercise 2 Let $X_1, X_2, \ldots, X_n$ be a random sample of size n with pdf $f(x)$. Find the pdf of $Y = \max(X_1, X_2, \ldots, X_n)$. $\bigtriangledown$

Solution: We first start by finding G, the cdf of Y,

$$G(y) = P(Y \leq y) = P(\max(X_1, X_2, \ldots, X_n) \leq y) = P(X_1 \leq y \cap \ldots \cap X_n \leq y)$$
$$= P(X_1 \leq y) \times \ldots \times P(X_n \leq y) = (F(y))^n, \quad \text{where } F \text{ is the cdf of } X_i$$

Now to find $g(y)$, the pdf Y:

- continuous case:

$$g(y) = G'(y) = ((F(y))^n)' = n \times (F(y))^{n-1} \times f(y)\,, \quad \text{where } f \text{ is the pdf of } X_i$$

- discrete case:

$$P(Y = y) = P(Y \leq y) - P(Y \leq y-1) = (F(y))^n - (F(y-1))^n\,, \qquad \blacksquare$$

Exercise 3 Let $X_1, X_2, \ldots, X_n$ be a random sample of size n with pdf $f(x)$. Find the pdf of $Z = \min(X_1, X_2, \ldots, X_n)$. $\bigtriangledown$