Last name (print), First name (print): Student #:

UNIVERSITY OF TORONTO
Faculty of Arts and Science

DECEMBER 2011 EXAMINATIONS

STA257H1F

Duration - 3 hours

Examination Aids:
Non-Programmable Calculator. No aid sheet allowed.

Show your work in the space provided. You may use page backs for preparations.
In every part of a question, clearly circle your final answer.

Read questions carefully; results without work shown are not acceptable. Algebraic expressions should be simplified. Numerical results that are fractions need not be given in decimal form, but should be reduced, whenever possible.

Circle your tutorial section

Section	A	B	C	D	E	F
Room:	SS2106	SS1083	SS2110	UC328	UC144	MP188

Marking

Question	1	2	3	4	5	6	
Max Mark	12	12	12	12	12	16	
Mark earned							
Question	7	8	9				Total
Max Mark	12	10	10				108
Mark earned							

Question 1. **[12]**
A box contains *m* black and *n* white balls. Two balls are selected at random. Then one of these two balls is selected at random and returned to the box.

(a) Find probabilities of all possible contents of the box at the end of the experiment, in terms of numbers of black and white balls. **[4]**

S: There will be $n+m-1$ balls at the end of the experiment. B – black ball, W – white ball:

Balls selected	ball returned	#B at end	#W at end	probability
B B	B	m-1	n	$\frac{\binom{m}{2}}{\binom{m+n}{2}}$
BW	B	m	n-1	$\frac{\binom{m}{1}\binom{n}{1}}{\binom{m+n}{2}}\times\frac{1}{2}$
BW	W	m-1	n	$\frac{\binom{m}{1}\binom{n}{1}}{\binom{m+n}{2}}\times\frac{1}{2}$
W W	W	m	n-1	$\frac{\binom{n}{2}}{\binom{m+n}{2}}$

Final: P(m-1 B, n W) = $\frac{\binom{m}{2}}{\binom{m+n}{2}} + \frac{\binom{m}{1}\binom{n}{1}}{\binom{m+n}{2}}\times\frac{1}{2} = \frac{m(m-1)}{(m+n)(m+n-1)} + \frac{mn}{(m+n)(m+n-1)} = \frac{m}{m+n}$, **[2]**

P(m B, n-1 W) = 1- P(m-1 B, n W) = $\frac{n}{m+n}$. **[2]**

(b) Find the expected number of black balls in the box at the end. **[4]**

S: Y = # of B at the end, P(Y = m – 1) = $\frac{m}{m+n}$, P(Y = m) = $\frac{n}{m+n}$, **[2]**
E(Y) = $(m-1)\frac{m}{m+n} + m\,\frac{n}{m+n} = \frac{m(m+n-1)}{m+n} = m(1-\frac{1}{m+n})$. **[2]**

(c) If it is known that at least one white ball was selected in the first selection of two balls, what is the probability that a black ball was returned to the box? **[4]**

S: Event C: at least one W selected, Event D: B returned,
P(C) = 1 – P(both B) = $1-\frac{\binom{m}{2}}{\binom{m+n}{2}}$, P(CD) = P(BW, B returned) = $\frac{\binom{m}{1}\binom{n}{1}}{\binom{m+n}{2}}\times\frac{1}{2}$, **[2]**
P(D|C) = $\frac{P(CD)}{P(C)} = \frac{\frac{\binom{m}{1}\binom{n}{1}}{\binom{m+n}{2}}\times\frac{1}{2}}{1-\frac{\binom{m}{2}}{\binom{m+n}{2}}} = \frac{1}{2}\frac{\binom{m}{1}\binom{n}{1}}{\binom{m+n}{2}-\binom{m}{2}} = \frac{mn}{(m+n)(m+n-1)-m(m-1)} = \frac{m}{2m+n-1}$. **[2]**

Question 2. [12]

Two squares are chosen at random on a chessboard (8 x 8 grid of squares). What is the probability that

(a) They have a side in common? **[4]**

S: There are 8 x 8 = 64 squares, and then $\binom{64}{2} = 64 \times 63/2$ possible selections of two squares. **[1]**

1	2	3	4	5	6	7	8
2							
3							
4							
5							
6							
7							
8							

To count number of pairs of squares with a side in common, we can start from the top of each column down, which gives 7 different pairs of squares with the common side between them. E.g., in the first column, pairs are (1,2), (2,3),…,(7,8). This applies to each of 8 columns, so there are 8 x 7 pairs, counting from top to bottom. The same reasoning applies to each of 8 rows, by counting from left to right. E.g., the pairs in the first row are (1,2), (2,3),…,(7,8). The total number of the pairs of squares with a side in common is then 8 x 7 x 2. **[1]**

P(A) = P(side in common) = (8x7x2)/(64x63/2) = ... = 7/126. **[2]**

Warning: In this approach, squares in a pair are not ordered, so a common mistake could be to count the same pair twice, e.g., using pairs (1,2) and (2,1). If ordering is used, such as in selecting squares one at a time, without replacement, the total number of possible pairs would be 64 x 63, and then the pairs (1,2) and (2,1) would be different, but the result will be same.

(b) Have a corner in common? **[4]**

11	12	13	14	15	16	17	18
21	22	23	24	25	26	27	28
31							
41							
51							
61							
71							78
81	82					87	

S: Here we should be more careful. In every row, the squares 1,2,...,7 have one square on the right with a corner in common, e.g., (11,22), (12,23),...,(17,28), and squares 2,3,...,8, have one square on the left with a corner in common, e.g., (12,21), (13,22),...,(18,27), which makes 7 x 2 pairs of squares with a corner in common. This applies to rows 1, 2, ..., 7, so there are 7 x 2 x7 pairs with a corner in common. **[2]**

P(B) = P(corner in common) = (7x2x7)/(64x63/2) = ... = 7/144. **[2]**

(c) They are apart? **[4]**

S: Two squares either have a side in common, or a corner in common, or they are apart. Events A and B are disjoint. Then **[2]**

P(C) = P(squares are apart) = 1 – P(A) – P(B) = 1- 7/126 - 7/144 = ... = 43/48. **[2]**

Question 3. **[12]**
A player shoots a ball to try to score a point. Probability of a successful shoot is *p*. The player shoots up to 4 times and stops, and also stops after two consecutive points. Let *X* be the total number of unsuccessful shots, and *Y* the total number of successful shots in the game (experiment).

(a) Describe the sample space of the experiment and write all sample points. **[3]**

S: The trials stops either after two S, or after four trials. Sample space is:
Ω = {SS, SFSS, SFFS, SFSF, SFFF, FSS, FSFF, FSFS, FFSS, FFSF, FFFS, FFFF},
12 points in total. **[3]**

(b) What is the range of possible values for (*X*, *Y*)? **[3]**

S: All trials could be successful, or unsuccessful, i.e., $0 \leq X \leq 4$. The number of successes is maximum 3, i.e., $0 \leq Y \leq 3$. The total number of trials is minimum two, and maximum 4, i.e., $2 \leq X + Y \leq 4$. **[1]** All possible pairs of X and Y can be obtained from the sample space:

e	SS	SFSS	SFFS	SFSF	SFFF	FSS	FSFF	FSFS	FFSS	FFSF	FFFS	FFFF
X	0	1	2	2	3	1	3	2	2	3	3	4
Y	2	3	2	2	1	2	1	2	2	1	1	0
P(e)	p^2	p^3q	p^2q^2	p^2q^2	p^1q^3	p^2q^1	p^1q^3	p^2q^2	p^2q^2	p^1q^3	p^1q^3	q^4

$R_{X,Y}$ = {(0,2),(1,2),(1,3),(2,2),(3,1),(4,0)}. **[2]**

(c) Find the joint probability distribution function for (*X*,*Y*). (**continued**) **[3]**
S: See table above:

		Y				Marg.
		0	1	2	3	X
X	0			p^2		p^2
	1			p^2q	p^3q	$p^2q(1+p)$
	2			$4p^2q^2$		$4p^2q^2$
	3		$4pq^3$			$4pq^3$
	4	q^4				q^4
Marg.	Y	q^4	$4pq^3$	$p^2(1+q+4q^2)$	p^3q	1

[Students need not to include marginal distributions] **[3]**

(d) Are *X* and *Y* independent random variables? **[3]**

S: Obviously, not.
E.g., P(X=0, Y=0) = 0, P(X=0)P(Y=0) = $p^2q^4 \neq 0$, if $0 < p < 1$. **[3]**

Question 4. [12]
From the box with *n* marbles, all of different color, *j* marbles are drawn at random one at the time, with replacement (observed and returned).

(a) What is the probability that all *j* marbles selected are of different color? **[4]**

S: A_j – an event that all j marbles are of different color. $P(A_j) = 0$, if $j > n$, because at most n can be of different color. Assume $j \le n$. **[1]**
First marble selected could be anyone. Second marble should not be one that was previously selected, etc, the i-th marble selected cannot be one of (i-1) previously selected. In every selection there are n marbles, so the probability for i-th selection is (n-i+1)/n. **[1]**

$P(A_j) = \frac{n-1}{n}\frac{n-2}{n}\cdots\frac{n-j+1}{n}$. **[2]**

(b) What is the probability that the last drown marble will be one that was previously selected? (**continued**) **[4]**

S: Denote this event by B_j.
We will first find
$P(\bar{B}_j) = P(the\ marble\ selected\ in\ the\ j-th\ selection\ was\ not\ previously\ selected)$
$= (1-\frac{1}{n})^{j-1}$, **[2]**
$P(B_j) = 1-(1-\frac{1}{n})^{j-1}$. **[2]**

(c) What is the expected number of different colors which will appear in this selection? If *j* = *n*, and *n* is large, show that this expected value is approximately 0.632 *n*. (this part (c) may be tricky; if you cannot do it fast, leave it for later) **[4]**

S: Let $X_i = 0, if\ the\ i-th\ marble\ (color)\ was\ never\ selected\ in\ j\ trials,$
$X_i = 1, if\ the\ i-th\ marble\ (color)\ was\ selected\ at\ least\ once.$
$P(X_i = 0) = (1-\frac{1}{n})^j, P(X_i = 1) = 1-(1-\frac{1}{n})^j, i = 1,2,\ldots,n$,
$E(X_i) = 1-(1-\frac{1}{n})^j$. **[1]**
Y = # of different colors = $X_1 + X_2 + \cdots + X_n$, **[1]**
$E(Y) = E(X_1) + E(X_2) + \cdots + E(X_n) = n\left(1-\left(1-\frac{1}{n}\right)^j\right)$. **[1]**

If $j = n$, $Y = n\left(1-\left(1-\frac{1}{n}\right)^n\right) \sim n(1-e^{-1}) \approx n \times 0.6321$. **[1]**

[Another option would be to define
$X_i = 1, if\ the\ marble\ selected\ in\ the\ i-th\ selection\ was\ not\ previously\ selected,$
$P(X_i = 1) = P(\bar{B}_i), P(\bar{B}_i)$ calculated in (b), Y = # of different colors = $X_1 + X_2 + \cdots + X_j$.]

Question 5. **[12]**
Let X be a randomly selected real number form the interval [0,1]. Let Y be a randomly selected real number from the interval [X,1].

(a) Find the joint density function for X and Y. **[4]**

S: $X : U(0,1)$, $Y|X = x : U(x,1), f_1(x) = 1, 0 \le x \le 1, f_2(y|x) = \frac{1}{1-x}, x \le y \le 1,$ **[2]**
$f(x,y) = f_1(x)f_2(y|x) = \frac{1}{1-x}, 0 \le x \le y \le 1.$ **[2]**

(b) Find the marginal density for Y. (**continued**) **[4]**

S: $f_2(y) = \int_{-\infty}^{+\infty} f(x,y)dx = \int_0^y \frac{1}{1-x}dx = -\ln(1-x)\,|_0^y = -\ln(1-y) = \ln\frac{1}{(1-y)}, 0 \le y < 1.$

[4]

(c) Does $E(Y)$ exist? Explain without calculation. Then find $E(Y)$. (hint: using definition of $E(Y)$ directly could be tricky) **[4]**

S: Y is a bounded random variable, $0 \le Y \le 1$, so $0 \le E(Y) \le 1$ exists. **[1]**

You may try to find $E(Y)$ using formula $(Y) = \int_0^1 y\, f_2(y)dy$, but this leads you to a problem with integration at y = 1, where $f_2(y)$ is not defined.

If you use conditioning, it works simply:
$E(Y) = E[E(Y|X)] = E\left[\frac{1+X}{2}\right] = \frac{1}{2}E(1+X) = \frac{1}{2}(1+E(X)) = \frac{1}{2}\left(1+\frac{1}{2}\right) = \frac{3}{4}$. **[3]**

Question 6. **[16]**
The joint density function for continuous random variables X and Y is $f(x,y) = k(x\text{-}y)$, for $0 < y < x < 1$.

(a) Find constant k. **[4]**

S:
$1 = \iint f(x,y)dxdy = \iint_{0<y<x<1} k(x-y)dxdy = \int_0^1 [\int_0^x k(x-y)dy]dx = k\int_0^1 [\int_0^x (x-y)dy]dx = k\int_0^1 (x^2 - \frac{x^2}{2})dx = \frac{k}{2}\frac{x^3}{3}|_0^1 = \frac{k}{6}$
, k = 6. **[4]**

(b) Find the marginal densities for X and Y. (**continued**) **[4]**

S: $f_1(x) = \int_{-\infty}^{+\infty} f(x,y)dy = \int_0^x 6(x-y)dy = 6\frac{x^2}{2} = 3x^2, 0 \le x \le 1,$ **[2]**
$f_2(y) = \int_{-\infty}^{+\infty} f(x,y)dx = \int_y^1 6(x-y)dx = 6\left(\frac{x^2}{2} - yx\right)|_y^1 = 6\left(\frac{1}{2} - y + \frac{y^2}{2}\right), 0 \le y \le 1.$ **[2]**

(c) Find $E(Y|X)$. **[4]**

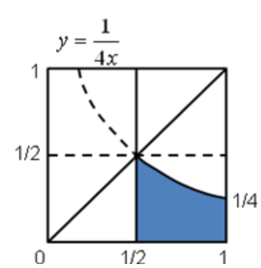
S: First find $f_2(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{6(x-y)}{3x^2} = 2\frac{(x-y)}{x^2}, 0 < y < x.$ **[2]**

$E(Y|X=x) = \int_0^x y f_2(y|x)dy = 2\int_0^x y\frac{(x-y)}{x^2}dy = \frac{2}{x^2}\left[x\frac{y^2}{2} - \frac{y^3}{3}\right]_0^x = \frac{x}{3}.$

$E(Y|X) = \frac{X}{3}.$ **[2]**

(d) Find $P(XY < 1/4|\ X > 1/2)$. **[4]**

S: $P\left(XY < \frac{1}{4}\middle|X > \frac{1}{2}\right) = \frac{P\left(XY<\frac{1}{4}, X>\frac{1}{2}\right)}{P\left(X>\frac{1}{2}\right)}$. $P\left(X > \frac{1}{2}\right) = \int_{1/2}^1 3x^2\,dx = x^3|_{1/2}^1 = 1 - (\frac{1}{2})^3 = 7/8.$ **[1]**
$P\left(XY < \frac{1}{4}, X > \frac{1}{2}\right) = P\left(\frac{1}{2} < X < 1, 0 < Y < \frac{1}{4X}\right) = \int_{1/2}^1 [\int_0^{1/(4x)} 6(x-y)dy]dx =$
$6\int_{1/2}^1 (xy - \frac{y^2}{2})\Big|_0^{1/(4x)} dx = 6\int_{1/2}^1 \frac{1}{4}\left(1 - \frac{1}{8x^2}\right)dx = \frac{3}{2}\left[x + \frac{1}{8x}\right]_{1/2}^1 = \frac{9}{16}$
. **[2]**



$P\left(XY < \frac{1}{4}\middle|X > \frac{1}{2}\right) = \frac{9/16}{7/8} = \frac{9}{14}.$ **[1]**

Question 7. **[12]**
Random variables X and Y are independent and with $E(1)$ (exponential) distributions ($f(x) = \exp(-x)$, $x \geq 0$).

(a) Find the joint PDF for random variables $Y_1 = X + Y$ and $Y_2 = X - Y$ (be careful with range of (Y_1, Y_2) (**continued**) **[4]**

S: We can find the joint CDF for Y_1 and Y_2 , and then the pdf, which requires more calculation. Using method of transformations with Jacobian, we can do it very simply.
$Y_1 = X + Y, Y_2 = X - Y, X = \frac{Y_1+Y_2}{2}, Y = \frac{Y_1-Y_2}{2}$. Obviously, $Y_1 \geq 0$, and from
$X \geq 0, Y \geq 0, Y_1 + Y_2 \geq 0, Y_1 - Y_2 \geq 0$, or

the range of Y_1, Y_2 is $Y_1 \geq 0, -Y_1 \leq Y_2 \leq Y_1$. **[1]**

$$J = \begin{vmatrix} \frac{\partial x}{\partial y_1} & \frac{\partial x}{\partial y_2} \\ \frac{\partial y}{\partial y_1} & \frac{\partial y}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}, |J| = \frac{1}{2}. \textbf{ [1]}$$

Random variables X and Y are independent, and then the joint pdf of Y_1 and Y_2 is
$$f_{Y_1,Y_2}(y_1, y_2) = f_X(x) f_Y(y)|J| = \frac{1}{2} f_X\left(\frac{y_1 + y_2}{2}\right) f_Y\left(\frac{y_1 - y_2}{2}\right)$$
$$= \frac{1}{2} e^{-\frac{y_1+y_2}{2}} e^{-\frac{y_1-y_2}{2}} = \frac{1}{2} e^{-y_1}, y_1 \geq 0, -y_1 \leq y_2 \leq y_1, \text{ or } y_1 \geq |y_2|. \textbf{ [2]}$$

(b) Find the marginal PDFs for Y_1 and Y_2 . **[4]**

S: We could find marginal pdf-s directly, using "conditioning", or from joint pdf above, which is now simpler.
Marginal pdf for Y_1 :

$$f_{Y_1}(y_1) = \int_{-y_1}^{+y_1} f_{Y_1,Y_2}(y_1, y_2) dy_2 = \int_{-y_1}^{+y_1} \frac{1}{2} e^{-y_1} dy_2 = \frac{1}{2} e^{-y_1} \int_{-y_1}^{+y_1} dy_2 = \frac{1}{2} e^{-y_1} \times 2y_1$$

$$= y_1 e^{-y_1}, y_1 \geq 0. \textbf{ [2]}$$

Marginal pdf for Y_2 :

$$f_{Y_2}(y_2) = \int_{|y_2|}^{+\infty} f_{Y_1,Y_2}(y_1, y_2) dy_1 = \int_{|y_2|}^{+\infty} \frac{1}{2} e^{-y_1} dy_1 = \frac{1}{2} \int_{|y_2|}^{+\infty} e^{-y_1} dy_1 = \frac{1}{2} e^{-|y_2|}, -\infty < y_2 < +\infty.$$

[2]

(c) Find $Cov(Y_1,Y_2) = E[(Y_1 - E(Y_1))(Y_2 - E(Y_2))]$ in terms of X and Y, and then apply it to this case. Are Y_1 and Y_2 independent? **[4]**

S:
$Cov(Y_1,Y_2) = Cov(X+Y, X-Y) = Cov(X,X) - Cov(X,Y) + Cov(Y,X) - Cov(Y,Y) = Var(X) - Var(Y).$
[2]
But, X and Y have the same distribution, E(1), so Var(X) = Var(Y), and then $Cov(Y_1, Y_2) = 0.$
[1]

Y_1 and Y_2 are not independent, because the joint pdf is not a product of marginal pdf-s. **[1]**

Question 8. **[10]**
A soft drink machine can be regulated so that it discharges an average of μ millilitres (mL) per cup (30 mL = 1 ounce). (hint: $\Phi(1.96) = 0.975$, $\Phi(2.326) = 0.99$)

(a) If the mLs of fill are normally distributed with standard deviation 10 mL, give the setting for μ so that 250 mL - cups will overflow when filled only 1% of the time. **[4]**

S: Let Y be that random amount of fill. Then $Y \sim N(\mu, 10^2)$. We need to find μ so that

$P(Y > 250) = 0.01$. **[1]**
If we use standardized $Z = \frac{Y-\mu}{10}$, then $P\left(Z > \frac{250-\mu}{10}\right) = 1 - \Phi(\frac{250-\mu}{10}) = 0.01$, or

$\Phi\left(\frac{250-\mu}{10}\right) = 1 - 0.01 = 0.99$, **[1]** or $\frac{250-\mu}{10} = 2.326$, and then
$\mu = 250 - 10 \times 2.326 = 226.74$. **[2]**

(b) The machine actually has standard deviation σ of fill that can be fixed at certain level by carefully adjusting the machine. What is the largest value of σ that will allow the actual amount dispensed to fall within 30 mL of the mean with probability at least 0.95. **[6]**

S: In this case we have to find the smallest σ so that $P(|Y-\mu| \leq 30) \geq 0.95$, **[1]** or when we use the standardized Y, $P\left(\frac{|Y-\mu|}{\sigma} \leq \frac{30}{\sigma}\right) = P\left(|Z| \leq \frac{30}{\sigma}\right) = 2\Phi\left(\frac{30}{\sigma}\right) - 1 = 0.95$. **[2]**Then

$\Phi\left(\frac{30}{\sigma}\right) = \frac{1.95}{2} = 0.975$, **[1]** or $\frac{30}{\sigma} = 1.96$, and then $\sigma = \frac{30}{1.96} = 15.306$. **[2]**

Question 9. **[10]**
Two events, *A* and *B*, are defined on the same sample space. Show

(a) $P(A\bar{B} \cup B\bar{A}) = P(A) + P(B) - 2P(AB)$ **[4]**

S: Obviously, $A\bar{B}$ and $B\bar{A}$ are disjoint. Then $P(A\bar{B} \cup B\bar{A}) = P(A\bar{B}) + P(B\bar{A})$, **[1]**

$P(A) = P(AB) + P(A\bar{B}), P(A\bar{B}) = P(A) - P(AB),$

$P(B) = P(AB) + P(\bar{A}B), P(\bar{A}B) = P(B) - P(AB)$. **[1]**

$P(A\bar{B} \cup B\bar{A}) = P(A\bar{B}) + P(B\bar{A}) = P(A) - P(AB) + P(B) - P(AB)$
$= P(A) + P(B) - 2P(AB)$ **[2]**

(b) If $P(A) \geq 1 - \epsilon, 0 < \epsilon < 1$, then $|P(B) - P(B|A)| \leq \epsilon$. **[6]**

S: We have to prove that $-\varepsilon \leq P(B) - P(B|A) \leq \epsilon$. From $P(A) \geq 1 - \epsilon, P(\bar{A}) \leq \epsilon$. **[1]**
$P(B) = P(AB) + P(\bar{A}B) \leq P(AB) + P(\bar{A}) \leq P(AB) + \epsilon, P(B) - P(AB) \leq \epsilon$. **[1]**

$P(B) - P(B|A) = P(B) - \frac{P(AB)}{P(A)} \leq P(B) - P(AB) \leq \epsilon$. **[2]**
If we apply this last inequality to $(\bar{B})$, we get $P(\bar{B}) - P(\bar{B}|A) \leq \epsilon$, or

$1 - P(B) - (1 - P(B|A)) \leq \epsilon$, or $P(B) - P(B|A) \geq -\epsilon$. **[2]**

Total marks = 108