UNIVERSITY OF TORONTO

Faculty of Arts and Science

DECEMBER 2017 EXAMINATIONS

STA257H1S

Duration - 3 hours

Examination Aids: A calculator

**First Name:** ______________________ **Surname:** ______________________

**Student Number:** ________________________________

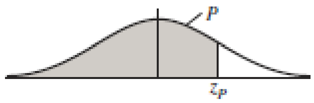
**Instructions:**

- Answer all questions, using only the booklets provided. If you use this exam paper to write on, note that it will not be looked at
- Print your name on each exam booklet
- Data from the normal distribution might be required for calculations. See the appendix
- Return this question set and your answer booklets

| Question | Value |
|---|---|
| 1 | 5 |
| 2 | 4 |
| 3 | 6 |
| 4 | 18 |
| 5 | 9 |
| 6 | 6 |
| 7 | 13 |
| 8 | 11 |
| 9 | 11 |
| 10 | 7 |
| Total | 90 |

This exam should have 7 pages including this page

**TABLE 2 Cumulative Normal Distribution—Values of $P$ Corresponding to $z_p$ for the Normal Curve**




$z$ is the standard normal variable. The value of $P$ for $-z_p$ equals 1 minus the value of $P$ for $+z_p$; for example, the $P$ for $-1.62$ equals $1 - .9474 = .0526$.

| $z_p$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| .0 | .5000 | .5040 | .5080 | .5120 | .5160 | .5199 | .5239 | .5279 | .5319 | .5359 |
| .1 | .5398 | .5438 | .5478 | .5517 | .5557 | .5596 | .5636 | .5675 | .5714 | .5753 |
| .2 | .5793 | .5832 | .5871 | .5910 | .5948 | .5987 | .6026 | .6064 | .6103 | .6141 |
| .3 | .6179 | .6217 | .6255 | .6293 | .6331 | .6368 | .6406 | .6443 | .6480 | .6517 |
| .4 | .6554 | .6591 | .6628 | .6664 | .6700 | .6736 | .6772 | .6808 | .6844 | .6879 |
| .5 | .6915 | .6950 | .6985 | .7019 | .7054 | .7088 | .7123 | .7157 | .7190 | .7224 |
| .6 | .7257 | .7291 | .7324 | .7357 | .7389 | .7422 | .7454 | .7486 | .7517 | .7549 |
| .7 | .7580 | .7611 | .7642 | .7673 | .7704 | .7734 | .7764 | .7794 | .7823 | .7852 |
| .8 | .7881 | .7910 | .7939 | .7967 | .7995 | .8023 | .8051 | .8078 | .8106 | .8133 |
| .9 | .8159 | .8186 | .8212 | .8238 | .8264 | .8289 | .8315 | .8340 | .8365 | .8389 |
| 1.0 | .8413 | .8438 | .8461 | .8485 | .8508 | .8531 | .8554 | .8577 | .8599 | .8621 |
| 1.1 | .8643 | .8665 | .8686 | .8708 | .8729 | .8749 | .8770 | .8790 | .8810 | .8830 |
| 1.2 | .8849 | .8869 | .8888 | .8907 | .8925 | .8944 | .8962 | .8980 | .8997 | .9015 |
| 1.3 | .9032 | .9049 | .9066 | .9082 | .9099 | .9115 | .9131 | .9147 | .9162 | .9177 |
| 1.4 | .9192 | .9207 | .9222 | .9236 | .9251 | .9265 | .9279 | .9292 | .9306 | .9319 |
| 1.5 | .9332 | .9345 | .9357 | .9370 | .9382 | .9394 | .9406 | .9418 | .9429 | .9441 |
| 1.6 | .9452 | .9463 | .9474 | .9484 | .9495 | .9505 | .9515 | .9525 | .9535 | .9545 |
| 1.7 | .9554 | .9564 | .9573 | .9582 | .9591 | .9599 | .9608 | .9616 | .9625 | .9633 |
| 1.8 | .9641 | .9649 | .9656 | .9664 | .9671 | .9678 | .9686 | .9693 | .9699 | .9706 |
| 1.9 | .9713 | .9719 | .9726 | .9732 | .9738 | .9744 | .9750 | .9756 | .9761 | .9767 |
| 2.0 | .9772 | .9778 | .9783 | .9788 | .9793 | .9798 | .9803 | .9808 | .9812 | .9817 |
| 2.1 | .9821 | .9826 | .9830 | .9834 | .9838 | .9842 | .9846 | .9850 | .9854 | .9857 |
| 2.2 | .9861 | .9864 | .9868 | .9871 | .9875 | .9878 | .9881 | .9884 | .9887 | .9890 |
| 2.3 | .9893 | .9896 | .9898 | .9901 | .9904 | .9906 | .9909 | .9911 | .9913 | .9916 |
| 2.4 | .9918 | .9920 | .9922 | .9925 | .9927 | .9929 | .9931 | .9932 | .9934 | .9936 |
| 2.5 | .9938 | .9940 | .9941 | .9943 | .9945 | .9946 | .9948 | .9949 | .9951 | .9952 |
| 2.6 | .9953 | .9955 | .9956 | .9957 | .9959 | .9960 | .9961 | .9962 | .9963 | .9964 |
| 2.7 | .9965 | .9966 | .9967 | .9968 | .9969 | .9970 | .9971 | .9972 | .9973 | .9974 |
| 2.8 | .9974 | .9975 | .9976 | .9977 | .9977 | .9978 | .9979 | .9979 | .9980 | .9981 |
| 2.9 | .9981 | .9982 | .9982 | .9983 | .9984 | .9984 | .9985 | .9985 | .9986 | .9986 |
| 3.0 | .9987 | .9987 | .9987 | .9988 | .9988 | .9989 | .9989 | .9989 | .9990 | .9990 |
| 3.1 | .9990 | .9991 | .9991 | .9991 | .9992 | .9992 | .9992 | .9992 | .9993 | .9993 |
| 3.2 | .9993 | .9993 | .9994 | .9994 | .9994 | .9994 | .9994 | .9995 | .9995 | .9995 |
| 3.3 | .9995 | .9995 | .9995 | .9996 | .9996 | .9996 | .9996 | .9996 | .9996 | .9997 |
| 3.4 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9998 |

**This aid material will be provided to you along with your exam, on the day.**

Number of ways to select $k$ elements out of $n$ elements: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ $\qquad$ $e^x = \sum_{j=0}^{\infty} x^j \,/\, j!$

Multiplication rule: $P(A_1 \cap A_2 \cap \cdots A_n) = P(A_1)\,P(A_2|A_1)\,P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap A_2 \cap \ldots A_{n-1})$

Total probability: $P(A) = P(A \cap B_1) + P(A \cap B_2) + \ldots + P(A \cap B_n) = P(A|B_1)P(B_1) + \ldots + P(A|B_n)P(B_n)$

Sum of $n$ terms in a geometric series: $S_n = a\frac{1-r^{n+1}}{1-r}$ $\qquad$ Bayes' Rule: $P(A|B) = \dfrac{P(B|A)P(A)}{P(B)}$

| Distribution | Probability mass/density function (pmf/pdf) | Notes |
|---|---|---|
| Bernoulli | $p(k\|p) = p^k(1-p)^{1-k}$ | $k \in \{0,1\},$ mean $= p$ |
| Binomial | $p(k\|n,p) = \binom{n}{k} p^k (1-p)^{n-k}$ | $k \in \{0,1,2,\ldots\}$ |
| Geometric | $p(k\|p) = (1-p)^{k-1}\,p$ | $k \in \{1,2,3,\ldots\},$ mean $= p^{-1}$ |
| Neg. binomial | $p(k\|r,p) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$ | $k \in \{r,\, r+1,\, r+2, \ldots\}$ |
| Hypergeometric | $p(k\|m,r,n) = \binom{r}{k}\binom{n-r}{m-k}/\binom{n}{m}$ | $k \in \{0,1,\ldots m\}$ |
| Poisson | $p(k\|\lambda) = \frac{\lambda^k}{k!}e^{-\lambda}$ | $k \in \{0,1,2,\ldots\}$ |
| Uniform | $p(x\|a,b) = \frac{1}{b-a}$ | $a \leq x \leq b$ |
| Exponential | $p(x\|\lambda) = \lambda e^{-\lambda x}$ | $x \geq 0$ |
| Gamma | $p(x\|\alpha,\lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$ | $x \geq 0,\ \Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du$ |
| Normal | $p(x\|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ | $x \in \mathbb{R}$ |
| Beta | $p(x\|a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)} x^{a-1}(1-x)^{b-1}$ | $0 \leq x \leq 1,$ mean $= a/(a+b)$ |
| Bivariate normal | $p(x,y \,\|\, \mu_1,\mu_2,\sigma_1,\sigma_2,\rho) = (*)$ | $x \in \mathbb{R},\ y \in \mathbb{R}$ |
| Cauchy | $p(x) = \frac{1}{\pi(x^2+1)}$ | $x \in \mathbb{R}$ |
| Rayleigh | $p(x) = xe^{-x^2/2}$ | $x \geq 0,$ mean $= \sqrt{\pi/2}$ |
| $\chi^2$ or $\chi^2_n$ | $p(x\|n) = \frac{1}{2^{n/2}\,\Gamma(n/2)}\, x^{\frac{n}{2}-1} e^{-x/2}$ | $x \geq 0$ |
| $t$ or $t_n$ | $p(x\|n) = \frac{\Gamma[(n+1)/2]}{\sqrt{n\pi}\,\Gamma(n/2)} \left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}}$ | $x \in \mathbb{R}$ |
| $F$ or $F_{m,n}$ | $p(x\|m,n) = \frac{\Gamma[(m+n)/2]}{\Gamma(m/2)\,\Gamma(n/2)} \left(\frac{m}{n}\right)^{m/2} x^{m/2-1} \left(1 + \frac{m}{n}x\right)^{-(m+2)/2}$ | $x \geq 0,$ mean $= n/(n-2)$ |

$$* \ \ p(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right]\right\}$$

*Aid material continued. Not all will be needed and the equations aren't necessarily a complete summary.*

**Joint distributions**

For independent $X$ and $Y$, with $Z = X + Y$, $\quad p_Z(z) = \sum_{x=-\infty}^{\infty} p(x, z-x) = \sum_{x=-\infty}^{\infty} p_X(x)\, p_Y(z-x) \quad$ or

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}\,(x, z-x)\, dx = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

For independent $X$ and $Y$, with $Z = Y/X$, $\quad f_Z(z) = \int_{-\infty}^{\infty} f\,(x, xz)\,|x|\, dx = \int_{-\infty}^{\infty} |x|\, f_X(x)\, f_Y(xz)\, dx$

$$f_{UV}(u,v) = f_{XY}(h_1(u,v),\, h_2(u,v))\ |J^{-1}(h_1(u,v),\, h_2(u,v))| \quad \text{where} \quad J\,(x,y) = \det \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{bmatrix}$$

**Expected values**

$\mathrm{E}(X) \;=\; \sum_{i=1}^{k} x_i\, p(x_i) \qquad \mathrm{var}(X) \;=\; \mathrm{E}\left\{[X - \mathrm{E}(X)]^2\right\} \qquad \mathrm{cov}(X,Y) \;=\; \mathrm{E}\left\{[X - \mathrm{E}(X)]\,[Y - \mathrm{E}(Y)]\right\}$

Markov's inequality: $P(X \geq t) \;\leq\; \mathrm{E}(X)/t \qquad$ Chebyshev's inequality: $P(|X - \mu| > t) \;\leq\; \sigma^2/t^2$

CoLiCo: $\mathrm{cov}\left(a + \sum_{i=1}^{n} b_i X_i,\, c + \sum_{j=1}^{m} d_j Y_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} b_i\, d_j\, \mathrm{cov}(X_i, Y_j)$

Correlation: $\rho = \mathrm{cov}(X,Y)/\sqrt{\mathrm{var}(X)\,\mathrm{var}(Y)}$

Variance and conditional expectation: $\mathrm{var}(Y) = \mathrm{var}\left[\mathrm{E}(Y|X)\right] + \mathrm{E}\left[\mathrm{var}(Y|X)\right]$

$\mathrm{MSE} = \mathrm{E}\left\{[Y - h(X)]^2\right\} \qquad \hat{Y} = \alpha + \beta X \quad \text{where} \quad \alpha = \mu_Y - \beta\mu_X \quad \text{and} \quad \beta = \rho\, \sigma_Y/\sigma_X$

$\mathrm{E}(X^{\,r}) = M^{(r)}(0)$, where $M(t) = \mathrm{E}\left(e^{\,tX}\right)$. Also, $M_X(s) = \mathrm{E}\left(e^{sX}\right) = \mathrm{E}\left(e^{sX + t(0)}\right) = M_{XY}(s, 0)$

**Limit theorems**

Types of convergence:

- $Z_n \xrightarrow{d} Z$ if $\lim_{n\to\infty} F_n(z) = F(z)$ at every point at which $F(z)$ is continuous
- $Z_n \xrightarrow{p} Z$ if $\lim_{n\to\infty} P\left(|Z_n - Z| > \varepsilon\right) = 0$
- $Z_n \xrightarrow{\text{a.s.}} Z$ if $P\left(\lim_{n\to\infty} Z_n = Z\right) = 1$

Weak Law of Large Numbers: For $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, $\mathrm{E}(X_i) = \mu$, and $\mathrm{var}(X_i) = \sigma^2$, with $X_i$ independent: $\bar{X}_n \xrightarrow{p} \mu$.

CLT: For $S_n = \sum_{i=1}^{n} X_i$, $\mathrm{E}(X_i) = 0$, and $\mathrm{var}(X_i) = \sigma^2$, with $X_i$ i.i.d.: $\lim_{n\to\infty} P\left(\frac{S_n}{\sigma\sqrt{n}} \leq x\right) = \Phi(x)$

Sample variance $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

**Additional help**

Integration by parts: $\int u\, dv = uv - \int v\, du$