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Problem 1

The 1000 estimated beta creates a empirical distribution for beta. So calculate the standard deviation, sd(beta) from the 1000 estimators and compare it with the mean of the standard errors.

Problem 2

set $y = \Gamma^T z$, $\|z\|_2 = 1$

$$\|y\|_2 = \sqrt{y^T y} = \sqrt{(\Gamma^T z)^T (\Gamma^T z)} = \sqrt{z^T (\Gamma \Gamma^T) z}$$

$$= \sqrt{z^T z} \quad (\text{As } \Gamma \Gamma^T = I)$$

$$= \|z\|_2 = 1$$

For a diagonal matrix $D = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}$ $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ $y_1 + y_2^2 + \dots y_n = 1$

we have $y^T D y = \sigma_1 y_1^2 + \sigma_2 y_2^2 + \cdots \sigma_n y_n^2 \le \sigma_1$

As we know $A = A^T = \Gamma \Lambda \Gamma^T$

$$\|A\|_2 = \sqrt{(Az)^T (Az)} = \sqrt{z^T (A^T A) z}$$

$$= \sqrt{z^T (\Gamma \Lambda \Gamma^T \Gamma \Lambda \Gamma^T) z}$$

$$= \sqrt{z^T \Gamma \Lambda^2 \Gamma^T z}$$

As Λ is a diagonal matrix with eigenvectors of A

we have $\|A\|_2 = \sqrt{q \Lambda^2 q^T}$ $q = z^T \Gamma$

$$= \sqrt{\sigma_1^2 q_1^2 + \sigma_2^2 q_2^2 + \cdots \sigma_n^2 q_n^2}$$

$$\le \sqrt{\sigma_1^2} = |\sigma_1|$$

Problem 3

For a retangular matrix X $(n \times p)$

$X = UDV^T$ $\quad U: n \times n \quad D: n \times p$ diagonal matrix

$\qquad V: p \times p$

$$X^T X = (UDV^T)^T (UDV^T)$$
$$= VD^T U^T U D V^T$$
$$= VD^T D V^T \quad (U^T U = I)$$

As D is a diagonal matrix,

$D^T D$ is a diagonal matrix, whose first k diagonal entries are σ_i^2.

$X^T X = V(D^T D)V^T$ is the EVD of $X^T X$, V is the eigenvectors of the matrix $X^T X$ and the eigenvalue of $X^T X$ is the square of the singular value of X. which is σ_i^2

let $S = X^T X = VD^T D V^T$ V is the eigenvectors of S.

we can gurantee their orthogonality. so that

$v_j^T v_i = 0$ for $j \neq i$

$$(Xv_j)^T (Xv_i) = v_j^T X^T X v_i = v_j^T \sigma_i^2 v_i = \begin{cases} \sigma_i^2 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

Thus, $X^T X$ is positive semidefinite.

b) $\Sigma = \Gamma \Lambda \Gamma^T$

$cI = c\Gamma\Gamma^T = \Gamma C \Gamma^T \qquad C = \begin{bmatrix} c & & 0 \\ & c & \\ 0 & & \ddots & c \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_i & \\ 0 & & \ddots & \sigma_n \end{bmatrix}$

$$Z = \Sigma + cI = \Gamma \Lambda \Gamma^T + \Gamma C \Gamma^T = \Gamma(\Lambda + C)\Gamma^T$$

so the eigenvalue of Z is $\sigma_i + c$

the eigenvectors of Z is Γ

It is $O(n)$ operation as shown.

Problem 4

a) $X = QR$ (using QR decomposition)

$X^T = R^T Q^T$

$C = X^T X = R^T Q^T Q R \quad (Q^T Q = I) \qquad C \; p \times p$

$\quad = R^T R$

As R is a upper triangular matrix,

$C = R^T R$ is a upper triangular matrix as well.

① Calculate $d = X^T Y \qquad d$

② Calcalate $e = C^{-1} d$ (using backsolve(C, d)

$\hat{\beta} = e + C^{-1} A^T (A C^{-1} A^T)^{-1} (-Ae + b)$

$\quad = e + (C^{-1} A^T)(A^T)^{-1} C A^{-1} (-Ae + b)$

③ Calculate $f = -Ae + b$

④ calculate $g = A^{-1} f$

⑤ Calacualte $D = (C^{-1} A^T)^T$

calculate : E = crossprod(D) calculate Eg

Calculate from right to left to get $\hat{\beta}$

b)
```
X = matrix(rnorm(np), nrow=n)
X.qr = qr(X)
Q = qr.Q(X.qr)
R = qr.R(X.qr)
C <- crossprod(R)
d <- crossprod(X, Y)
e <- backsolve(C, d)
f <- b - A %*% e
```

```
e <- backsolve (C, a)
f <- b - A %*% e
g <- inverse(A) %*% f
h <- C %*% g
D <- backsolve (C, A, transpose = TRUE)
E <- crossprod( D )
j <- E %*% h
Beta_hat <- e + j
```

Problem 5

a) As $Z(Z^TZ)^{-1}Z$ is a 60 million by 60 million matrix the computer run out of memory, thus, we can not do the calculation in two stages as given.

b) ① Calculate $A = (Z^TZ)^{-1} \Rightarrow A$ is a 630×630 matrix

$630 \times 6e^7 \quad 6e^7 \times 630$

② Calculate $\hat{X}^T = (Z\Lambda Z^T X)^T = X^T Z \Lambda^T Z^T \rightarrow \hat{X}^T$ is $600 \times 6e^7$

③ calculate $B = \hat{X}^T\hat{X} = X^T Z \Lambda^T (Z^TZ) A Z^T X$

$= X^T Z \Lambda^T (A^{-1}A) Z^T X$

$= X^T Z \Lambda^T Z^T X$

$B\ \hat{X}^T$ is 600×600 matrix

④ calculate $C = \hat{X}^T \times y$ is 600×1 matrix

⑤ calculate $\beta = BC$ is 600×1 matrix