ALGORITHMS FOR CONSTRUCTING VORONOI DIAGRAMS

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Naive algorithm

Constructing Voronoi diagrams

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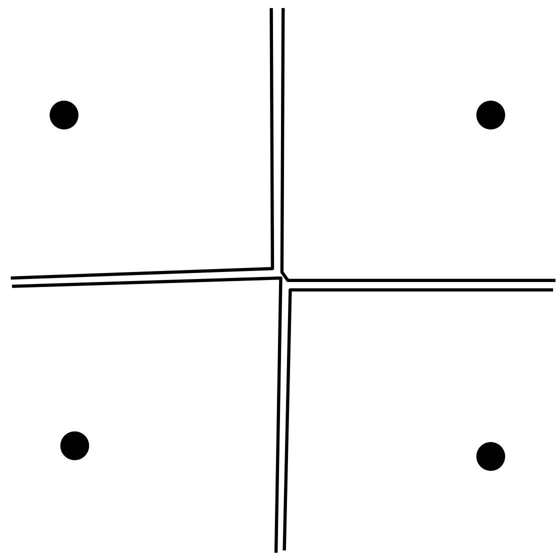

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The fact that each Voronoi region, $Vor(p_i)$, is built in optimal $\Theta(n \log n)$ time does not imply that the construction of the entire diagram, $Vor(P)$, requires $\Omega(n^2 \log n)$ time, as we will see.

incremental algorithm

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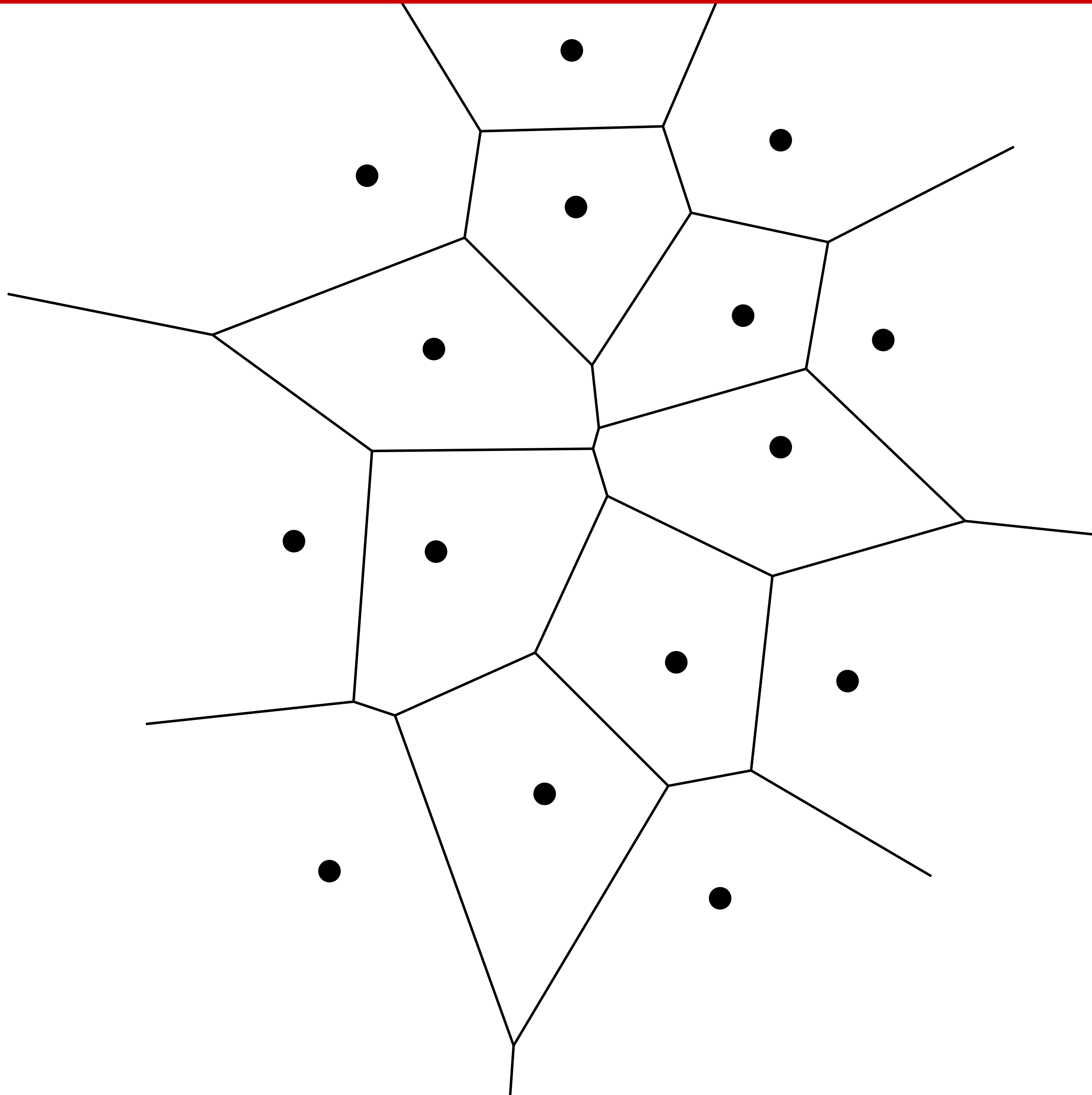

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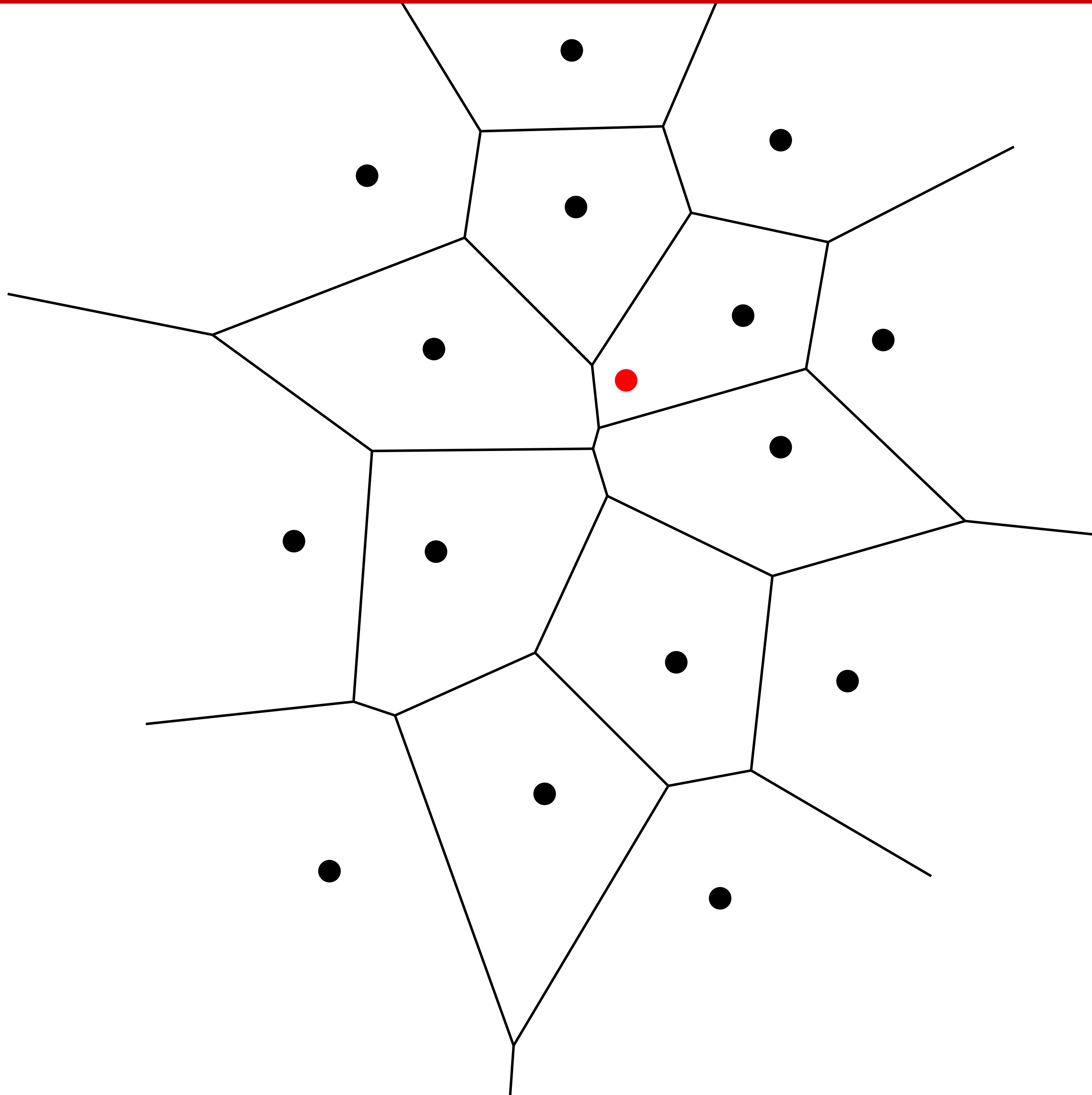

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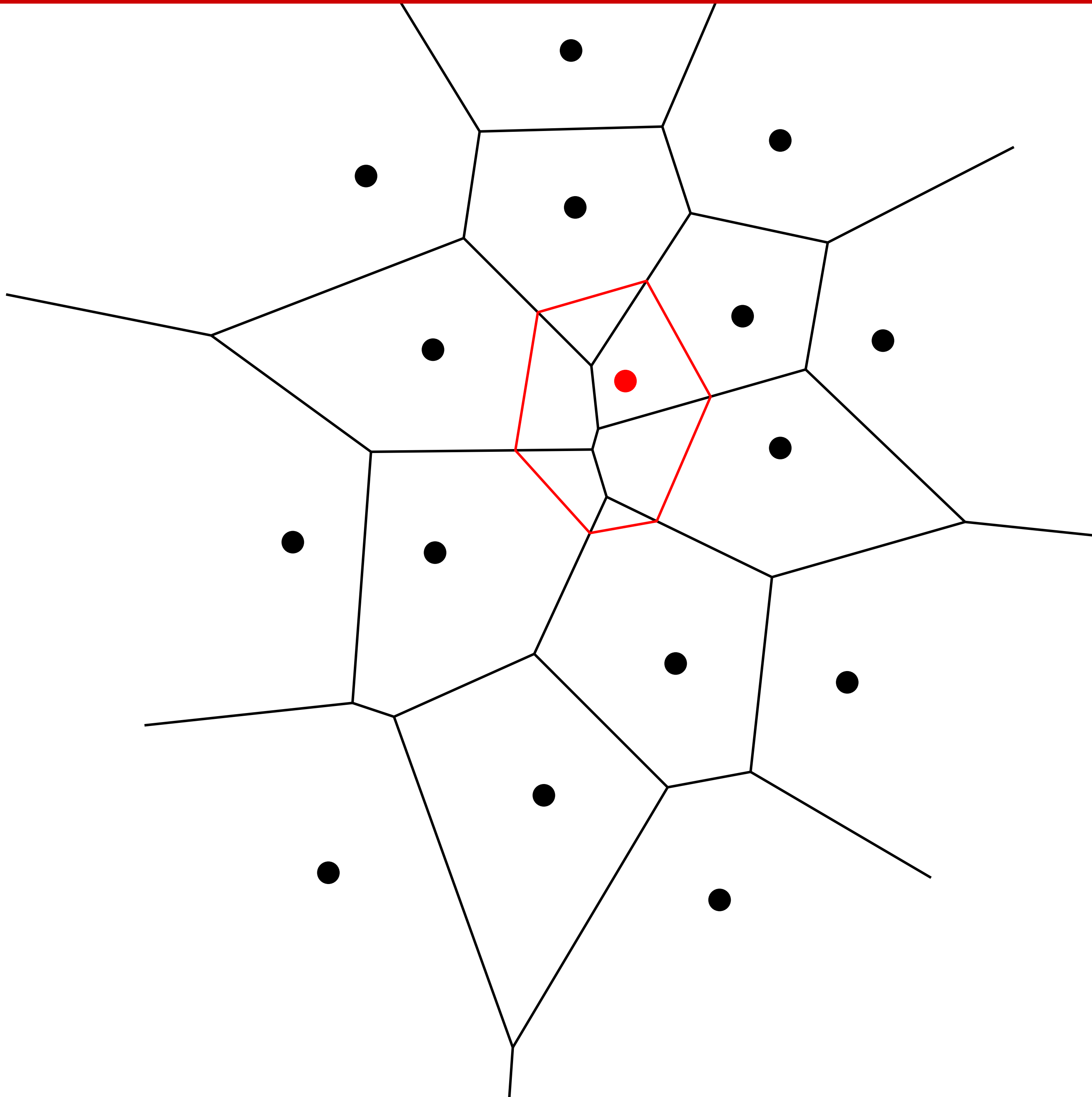

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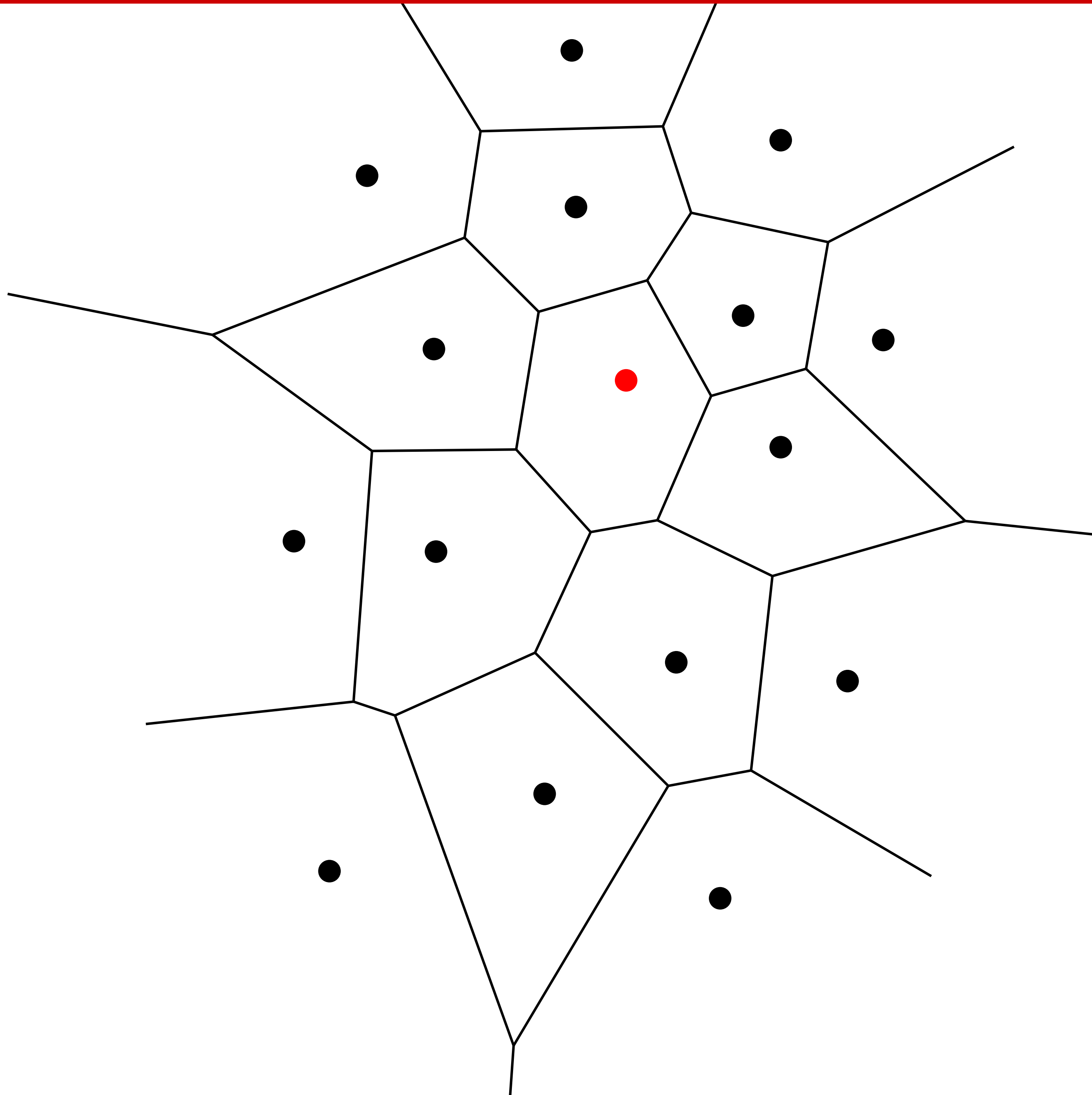

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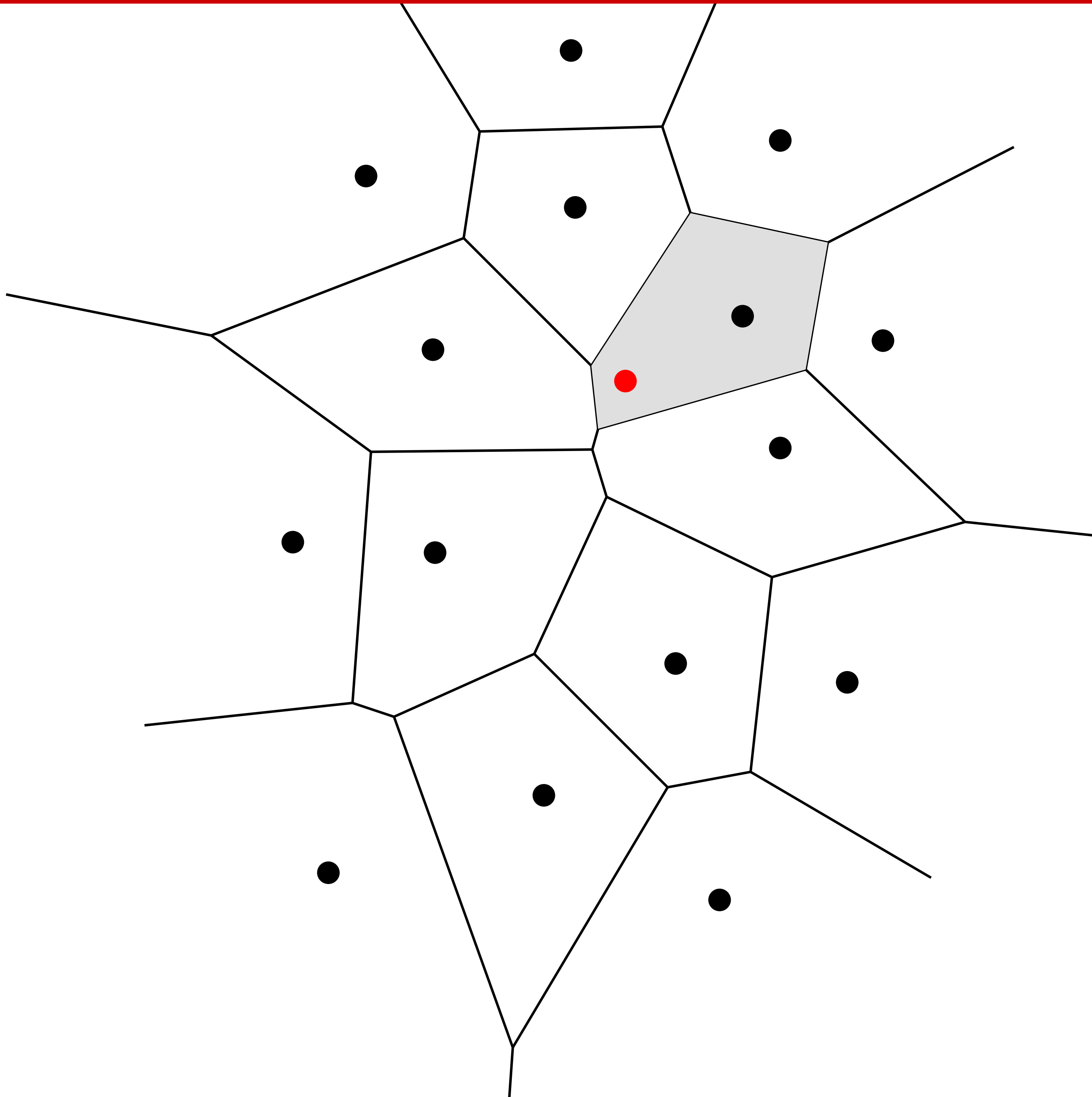

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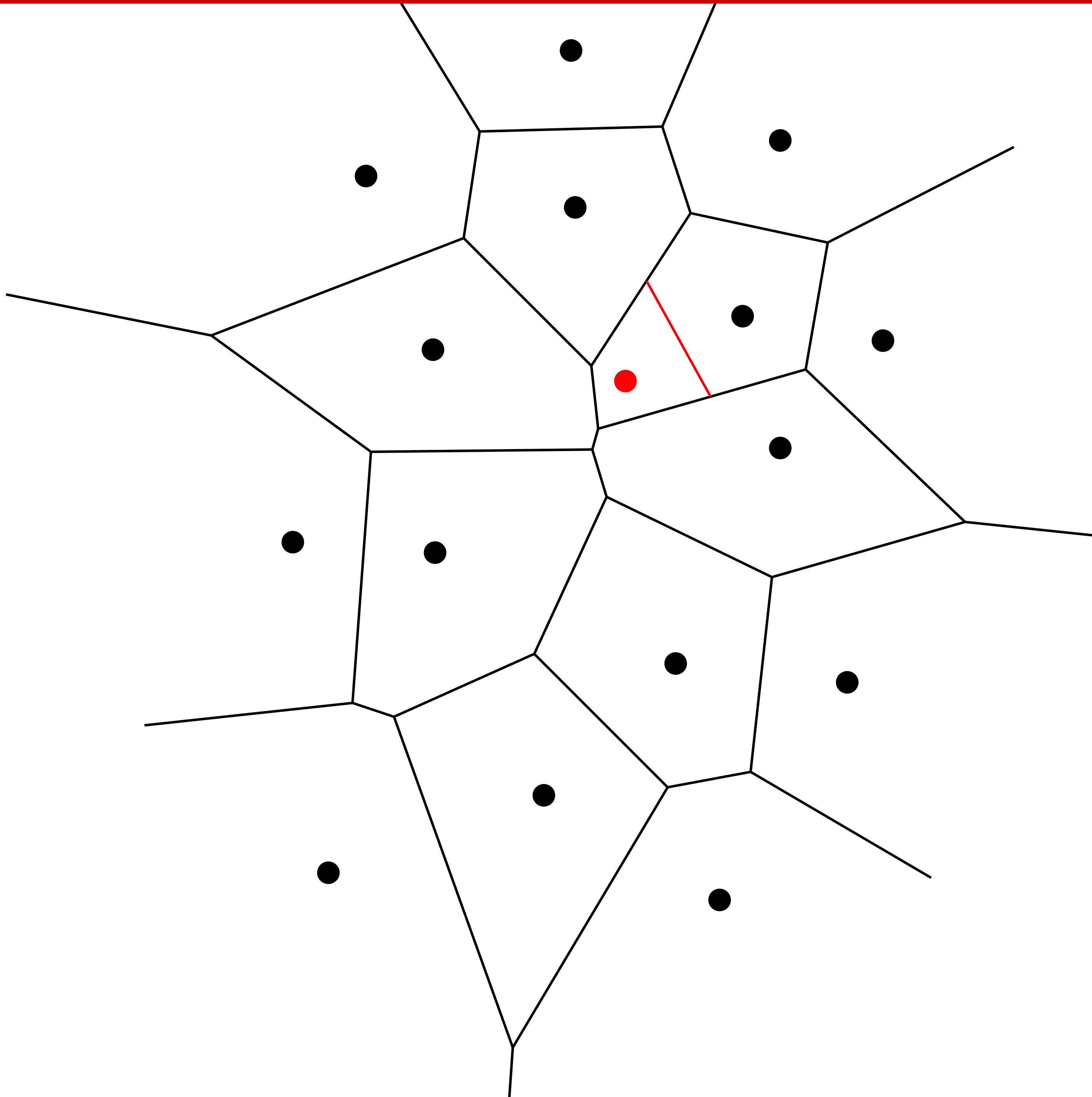

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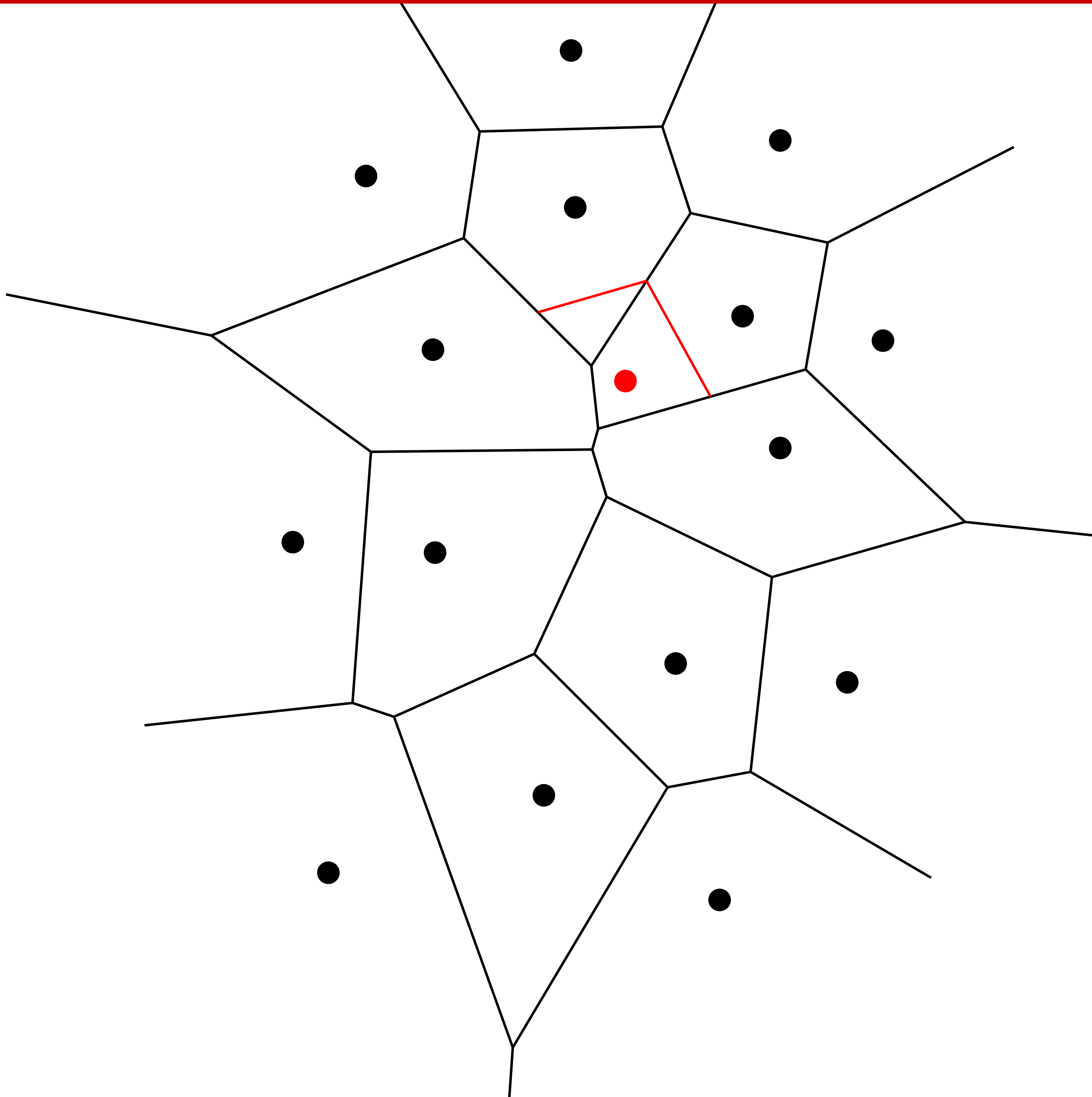

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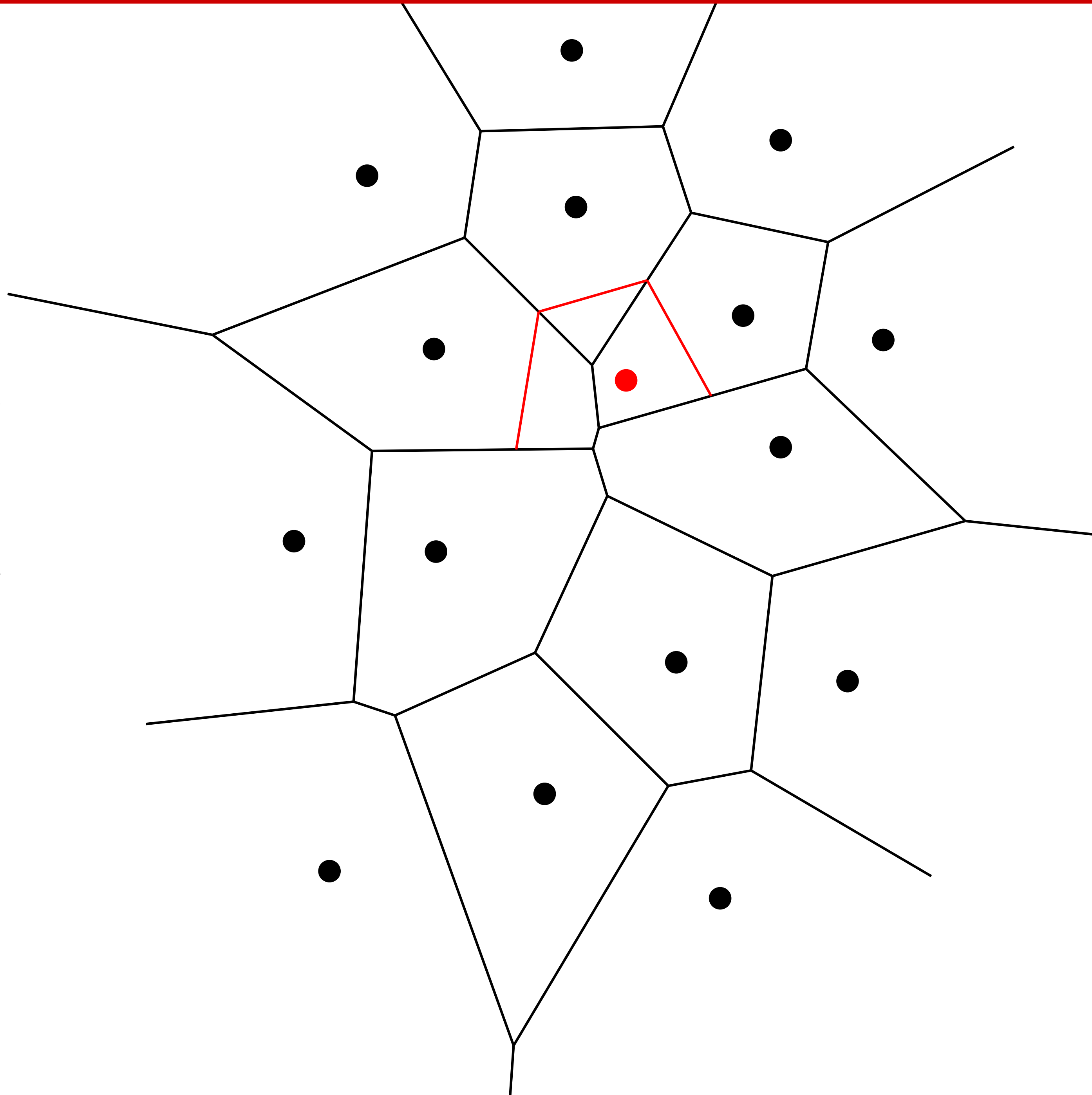

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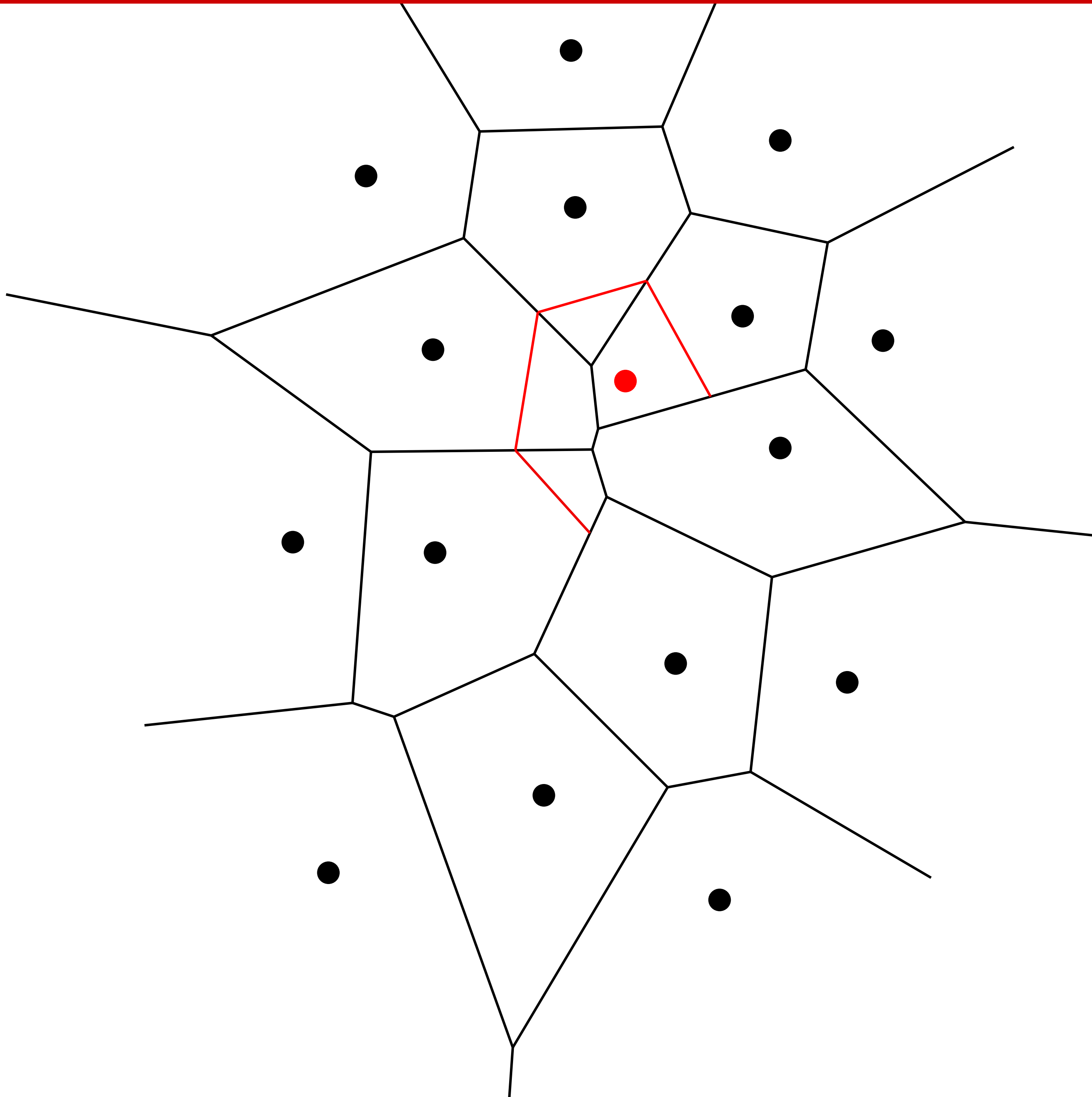

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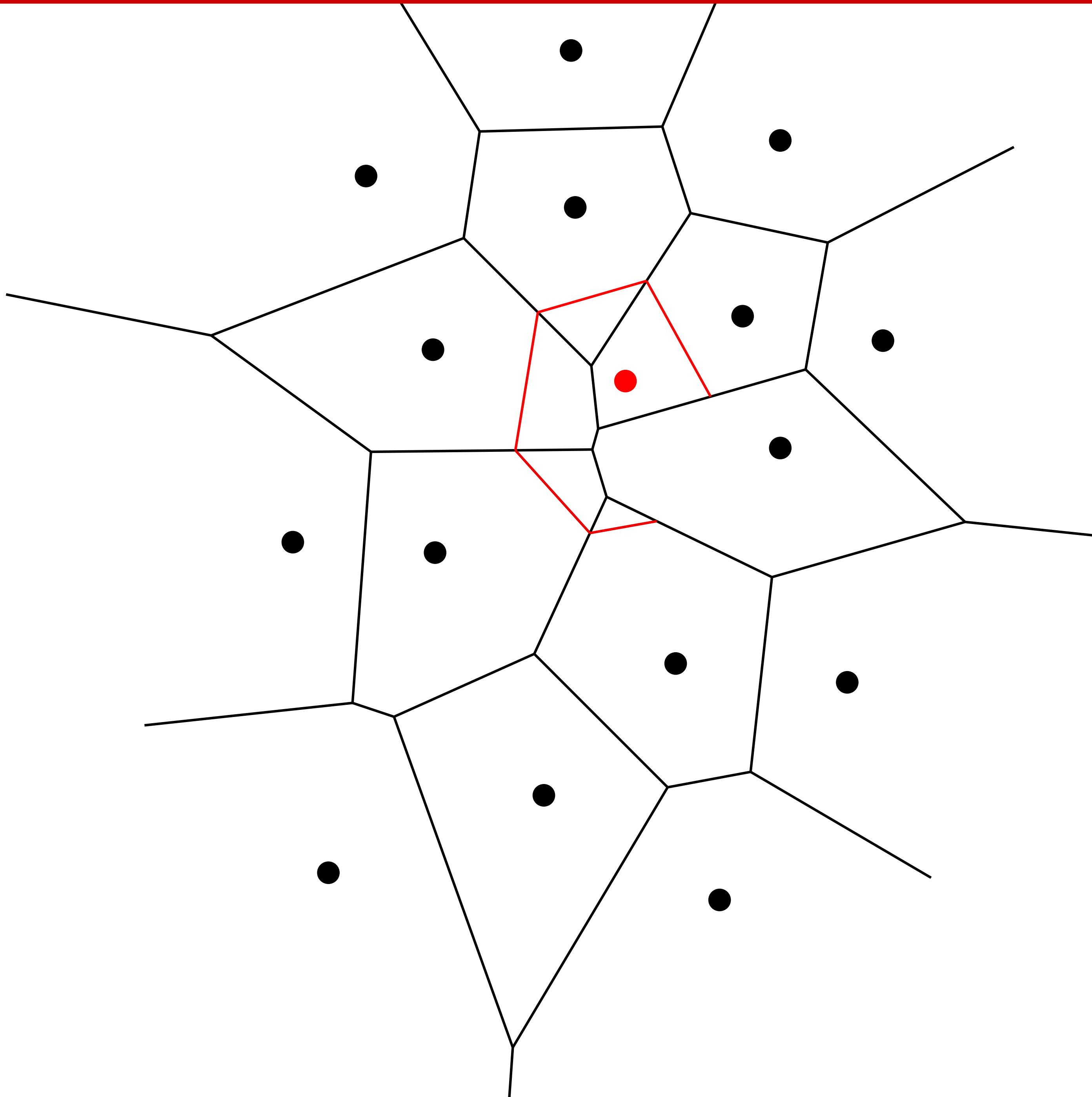

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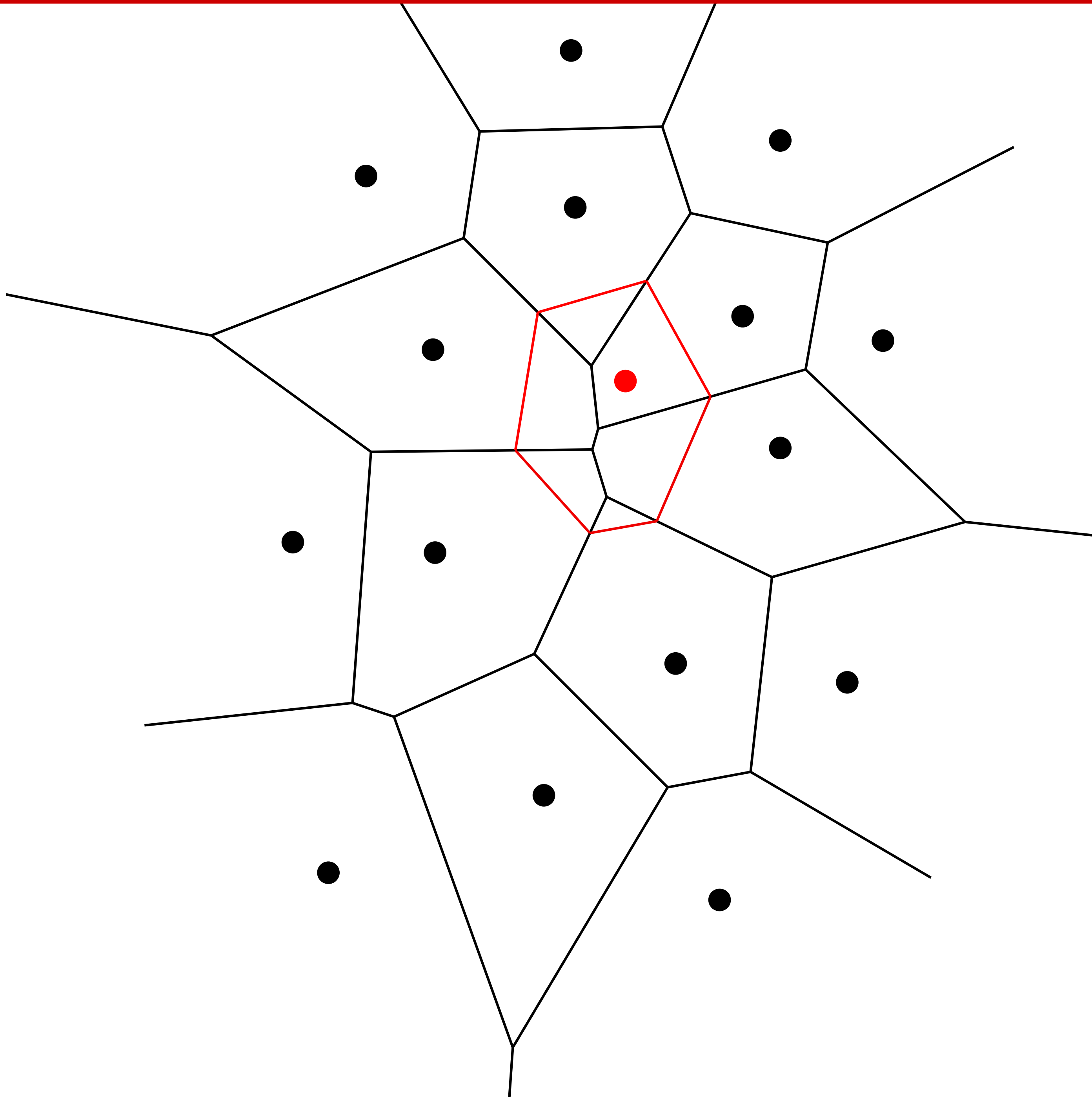

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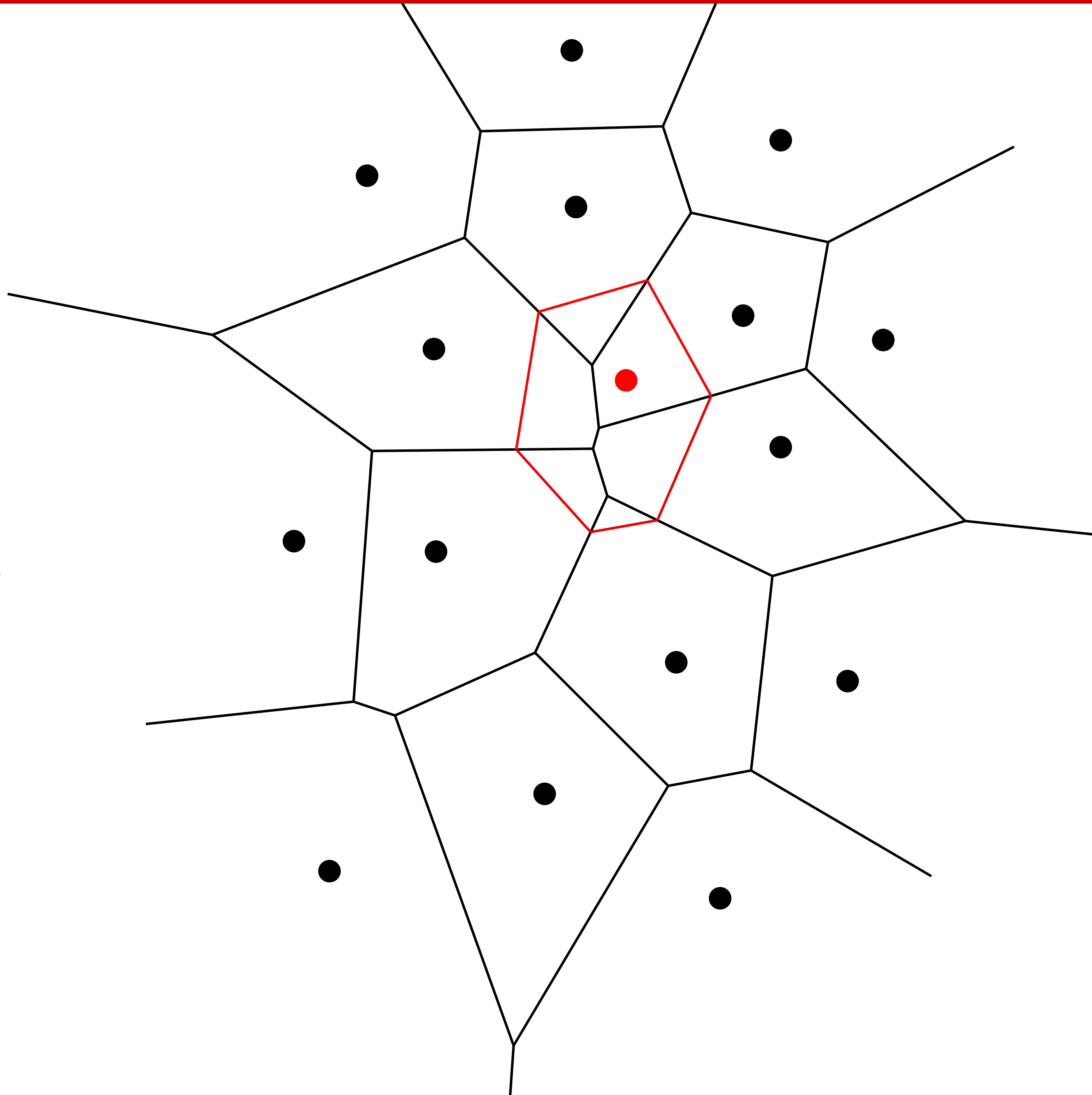

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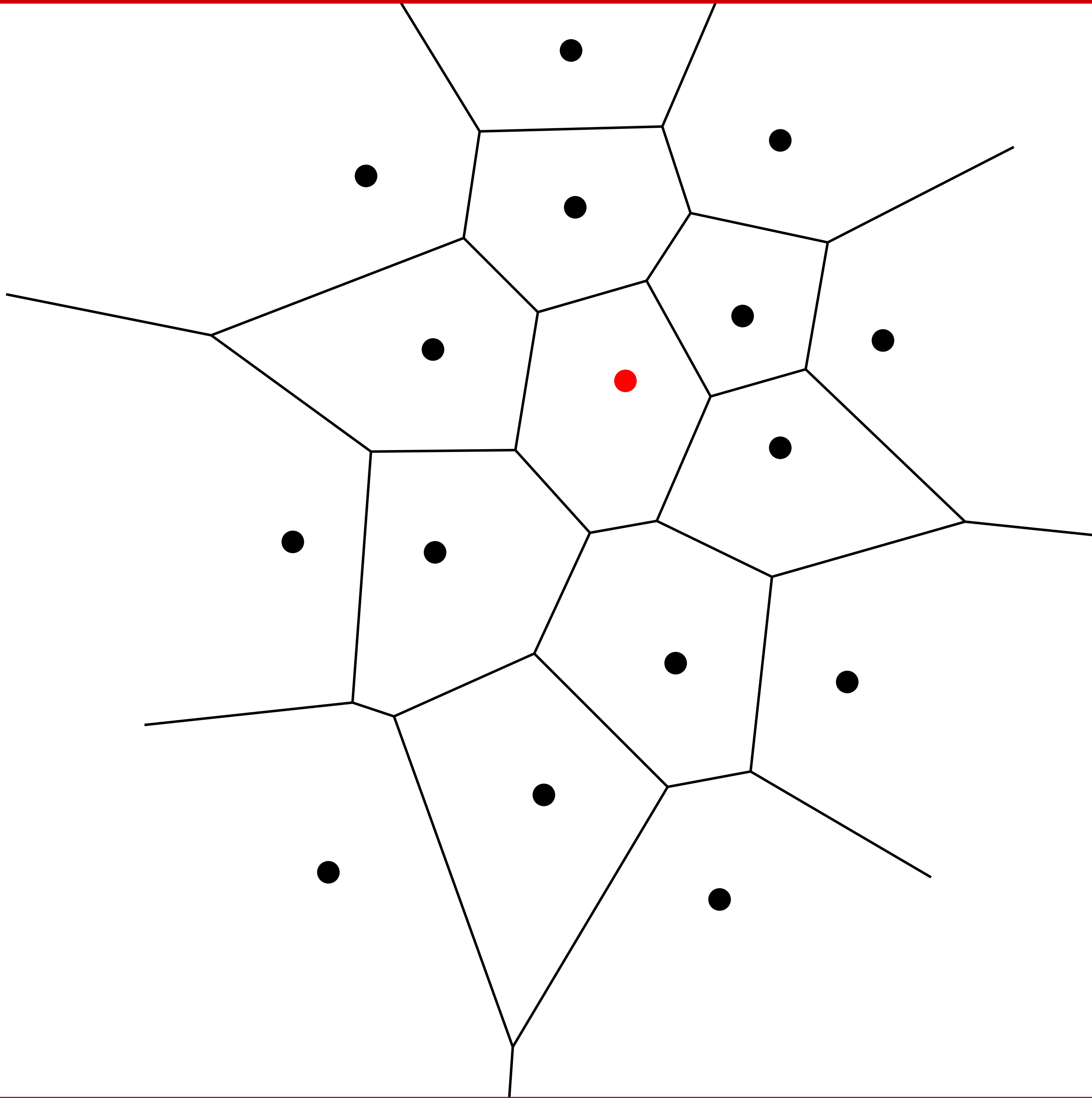

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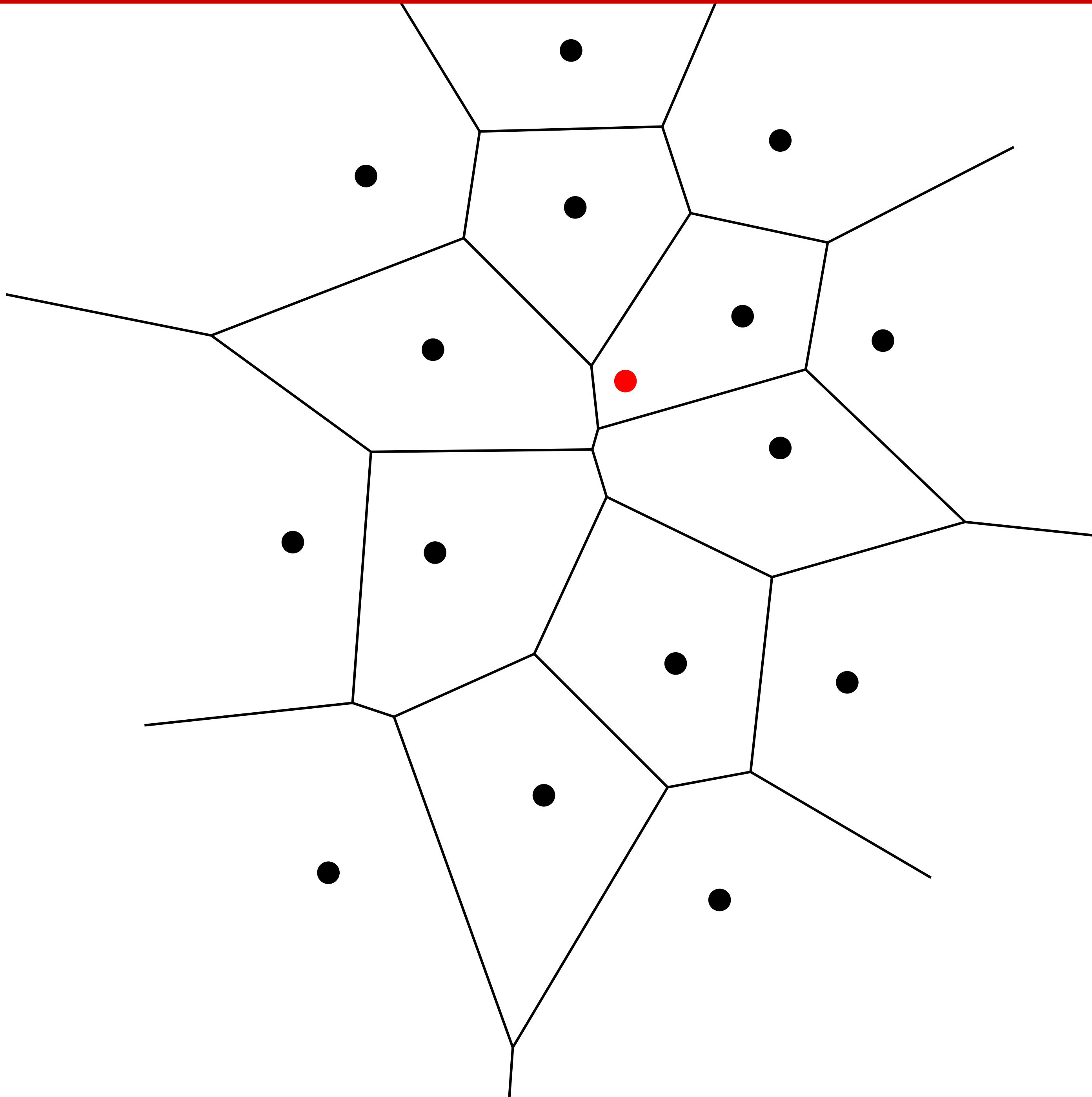
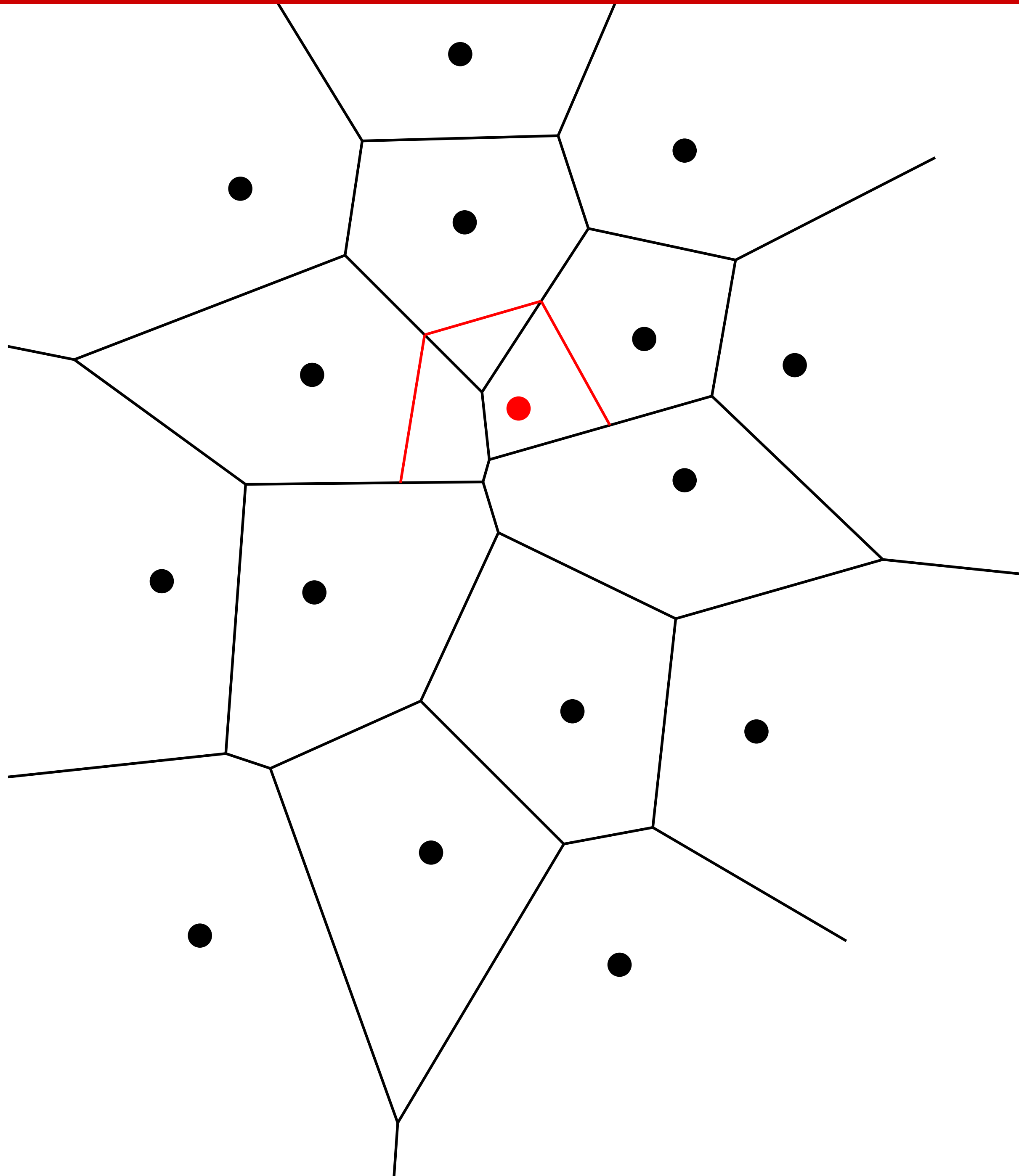


Constructing Voronoi diagrams

How to update the DCEL

Each time an edge e, generated by p_{i+1} and p_j, intersects a preexistent edge, e', a new vertex v is created and a new edge starts, $e+1$. Then, these are the tasks to perform:

- Create v with $e(v) = e$
- Assign $v_E(e) = v$, $e_N(e) = e'$, $f_L(e) = i+1$, $f_R(e) = j$
- Create $e+1$ and assign $v_B(e+1) = v$, $e_P(e+1) = e$
- Delete all edges of the region of p_j, that lie between $v_B(e)$ and $v_E(e)$ in clockwise order
- Update $v_*(e') = v$ and $e_*(e') = e{+}1$
- Update $e(p_j) = e$



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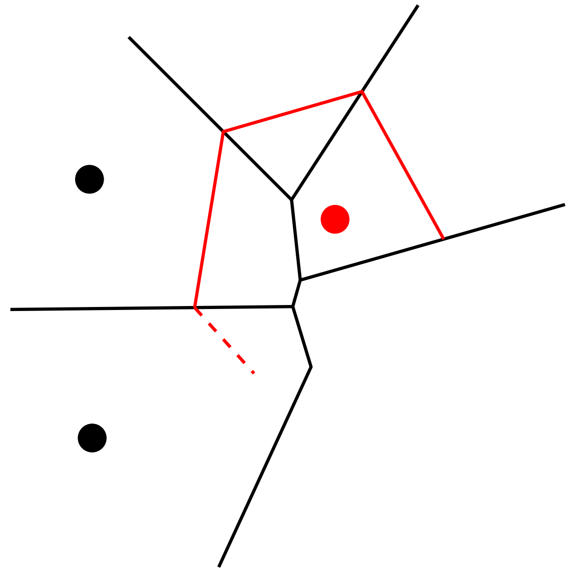

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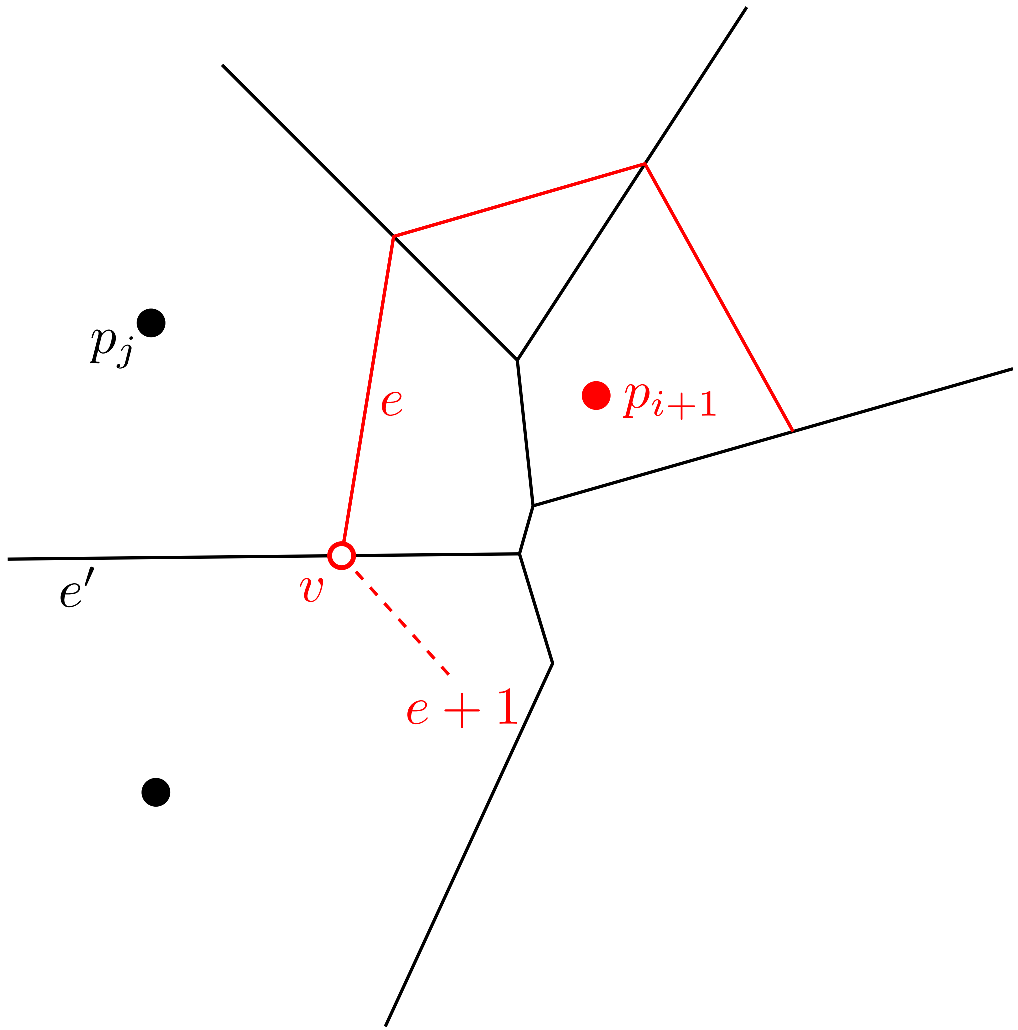

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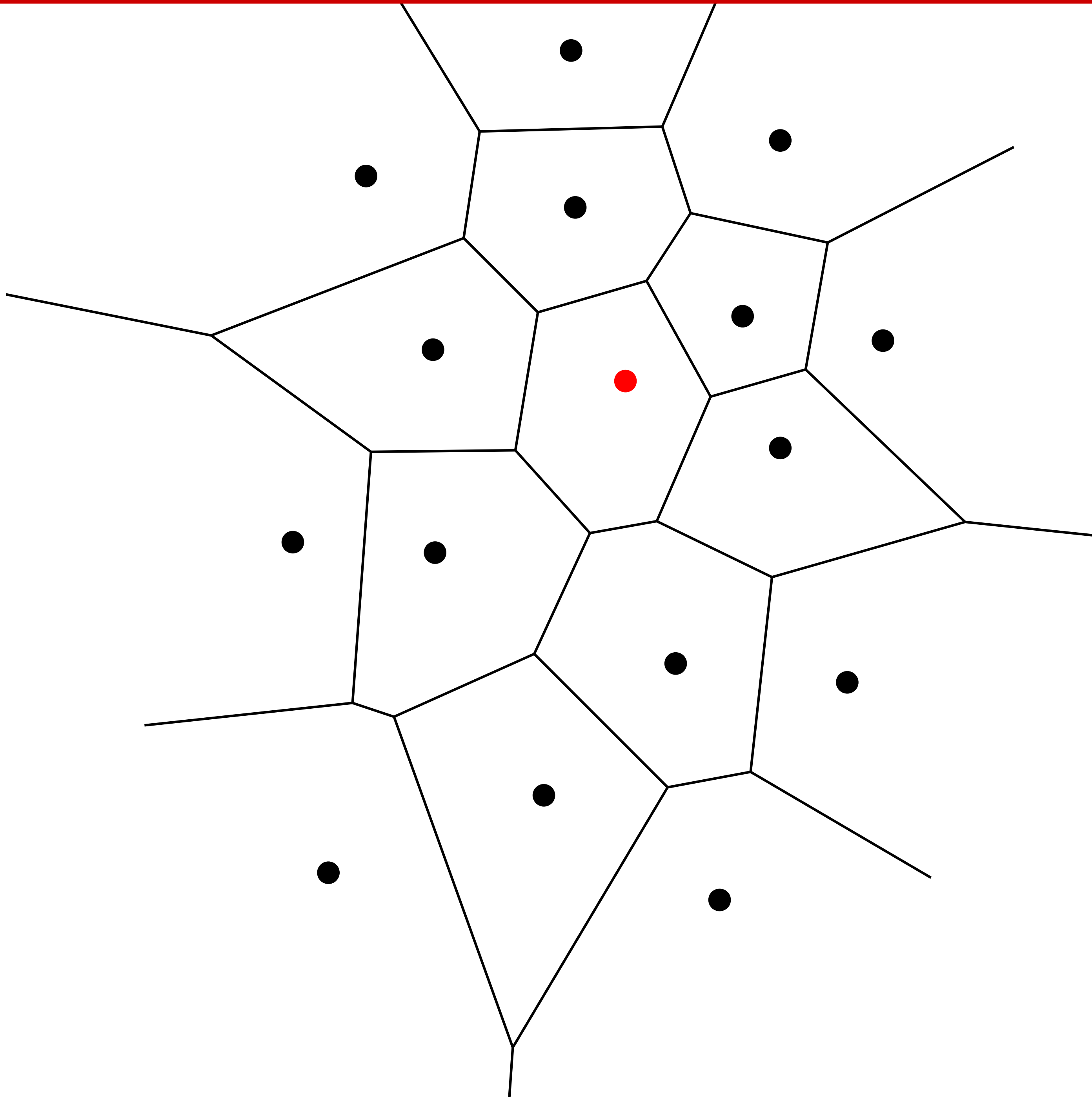

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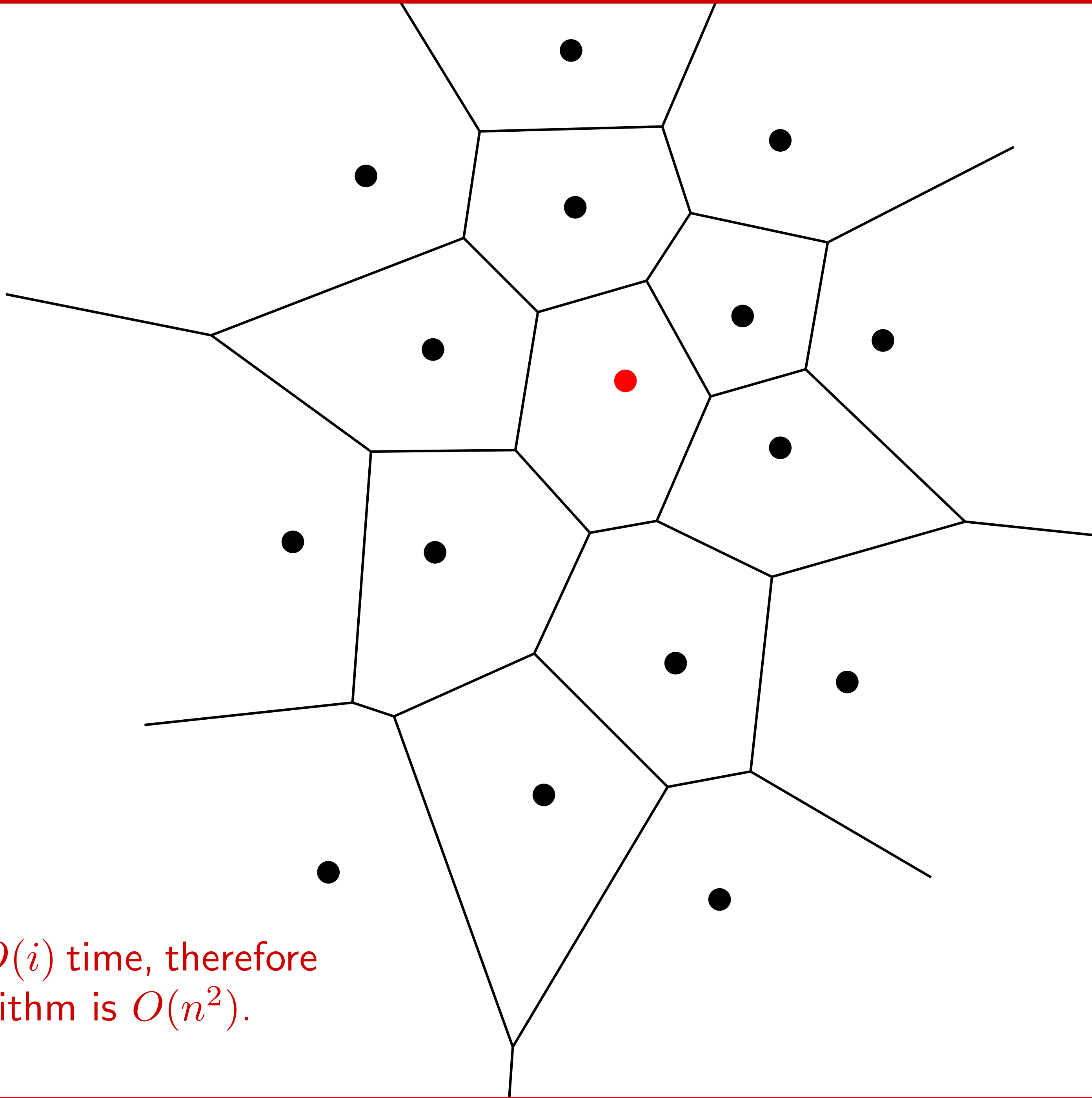

Running time: Each step runs in $O(i)$ time, therefore the total running time of the algorithm is $O(n^2)$.

divide and conquer algorithm

Constructing Voronoi diagrams

DIVIDE AND CONQUER IS POSSIBLE

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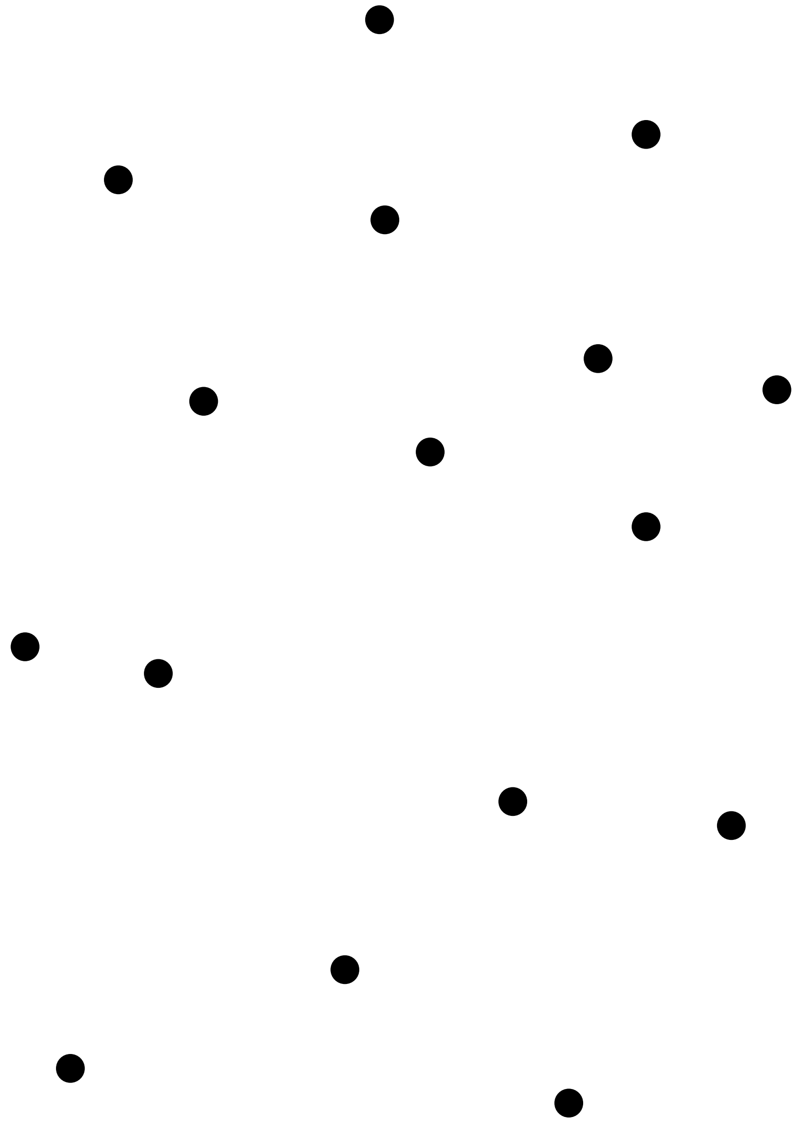

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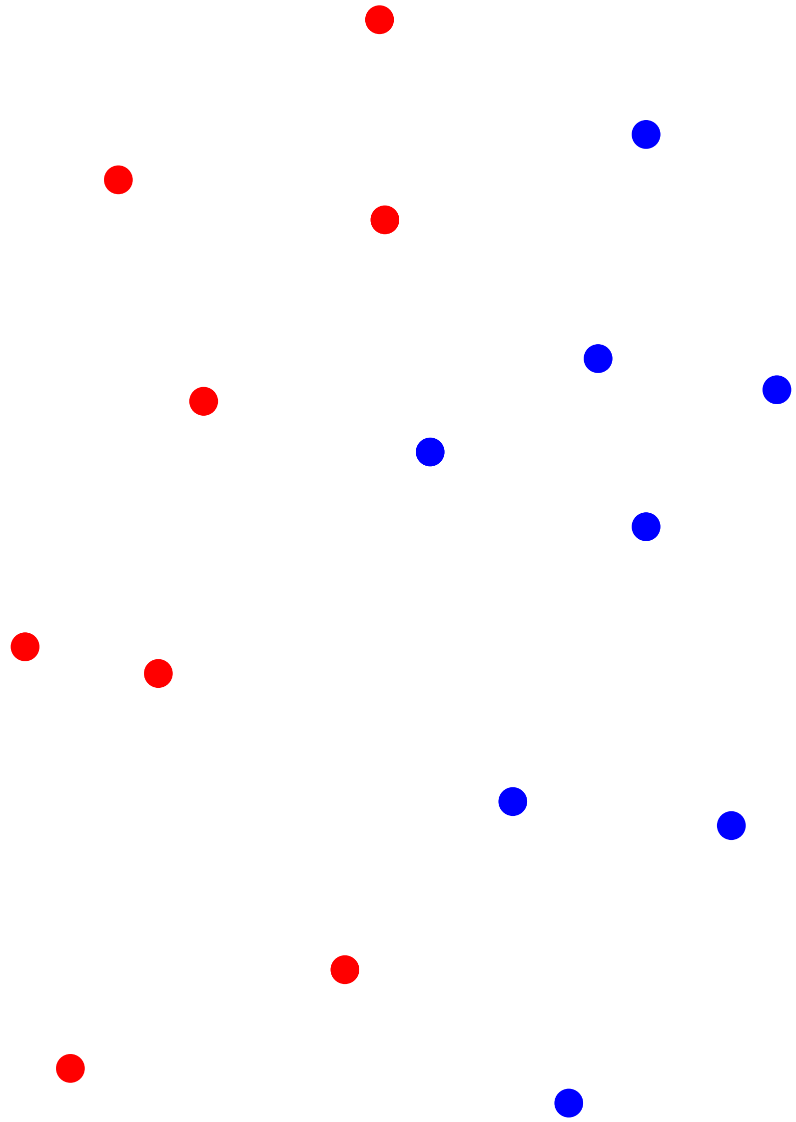

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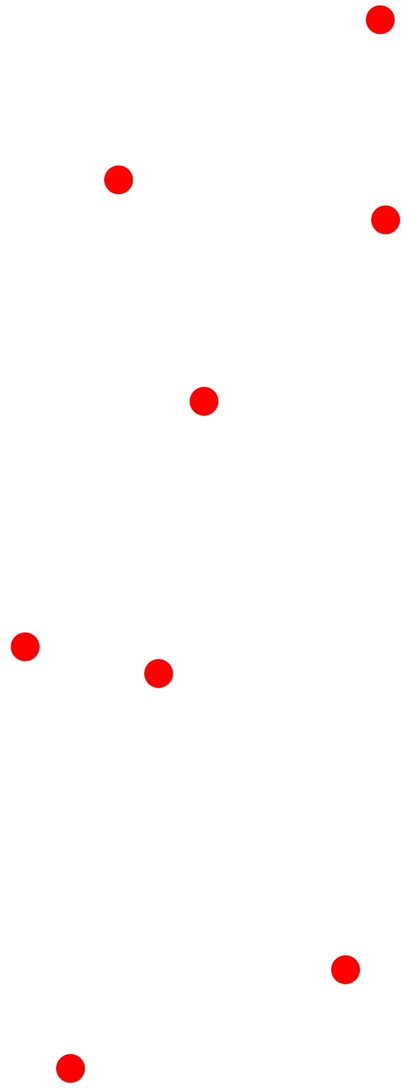

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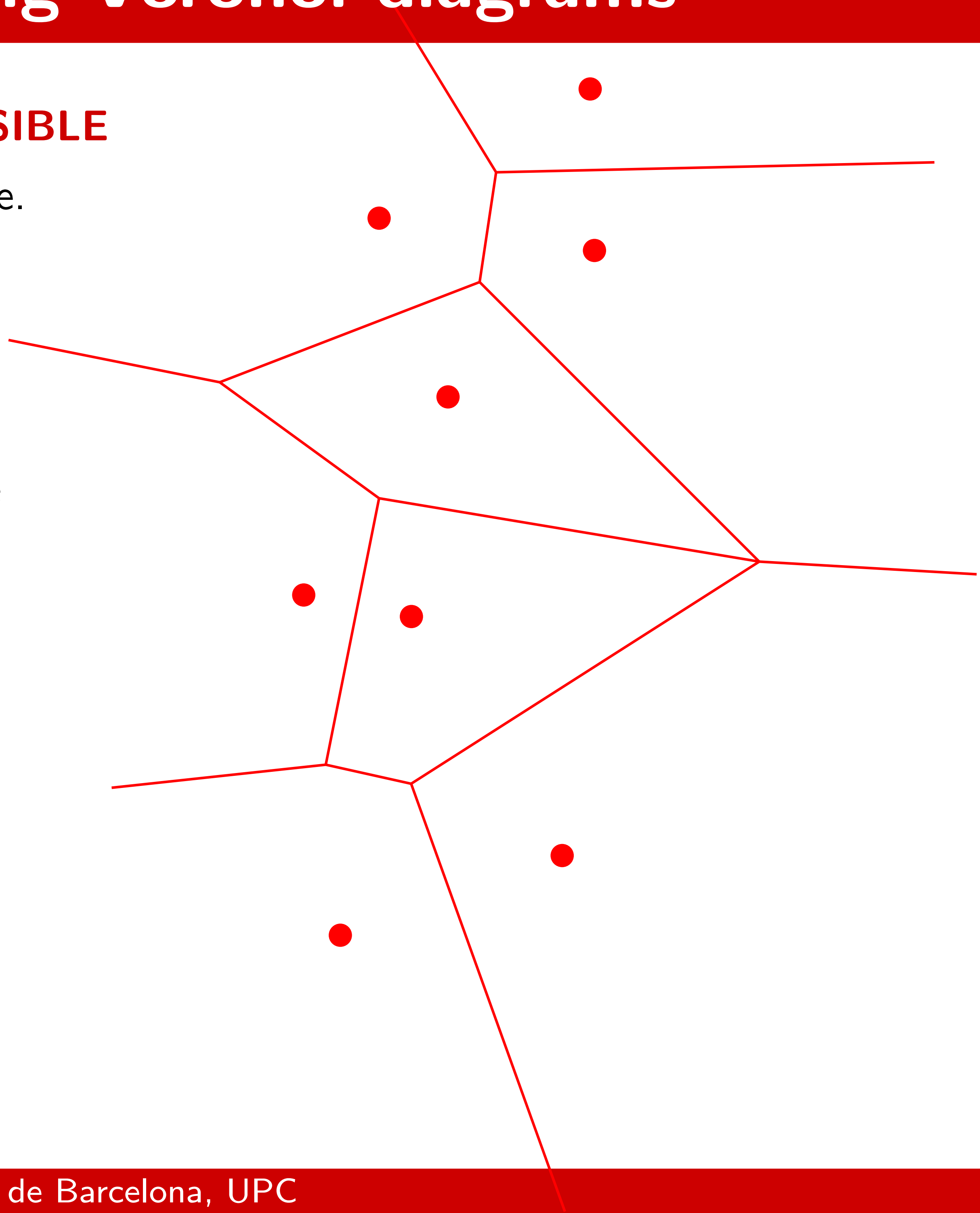

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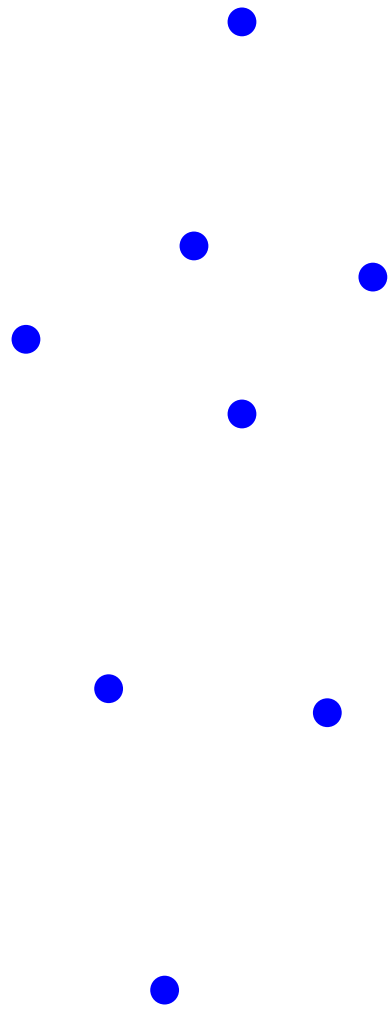

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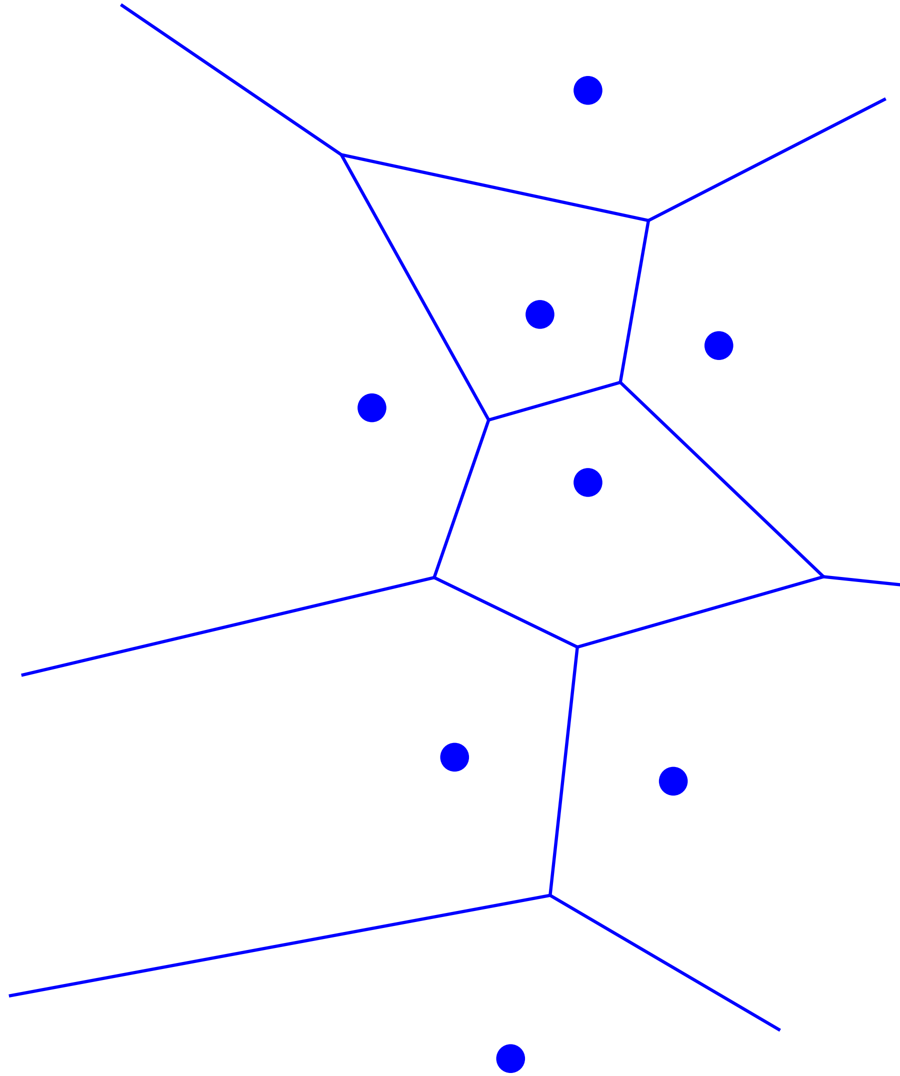

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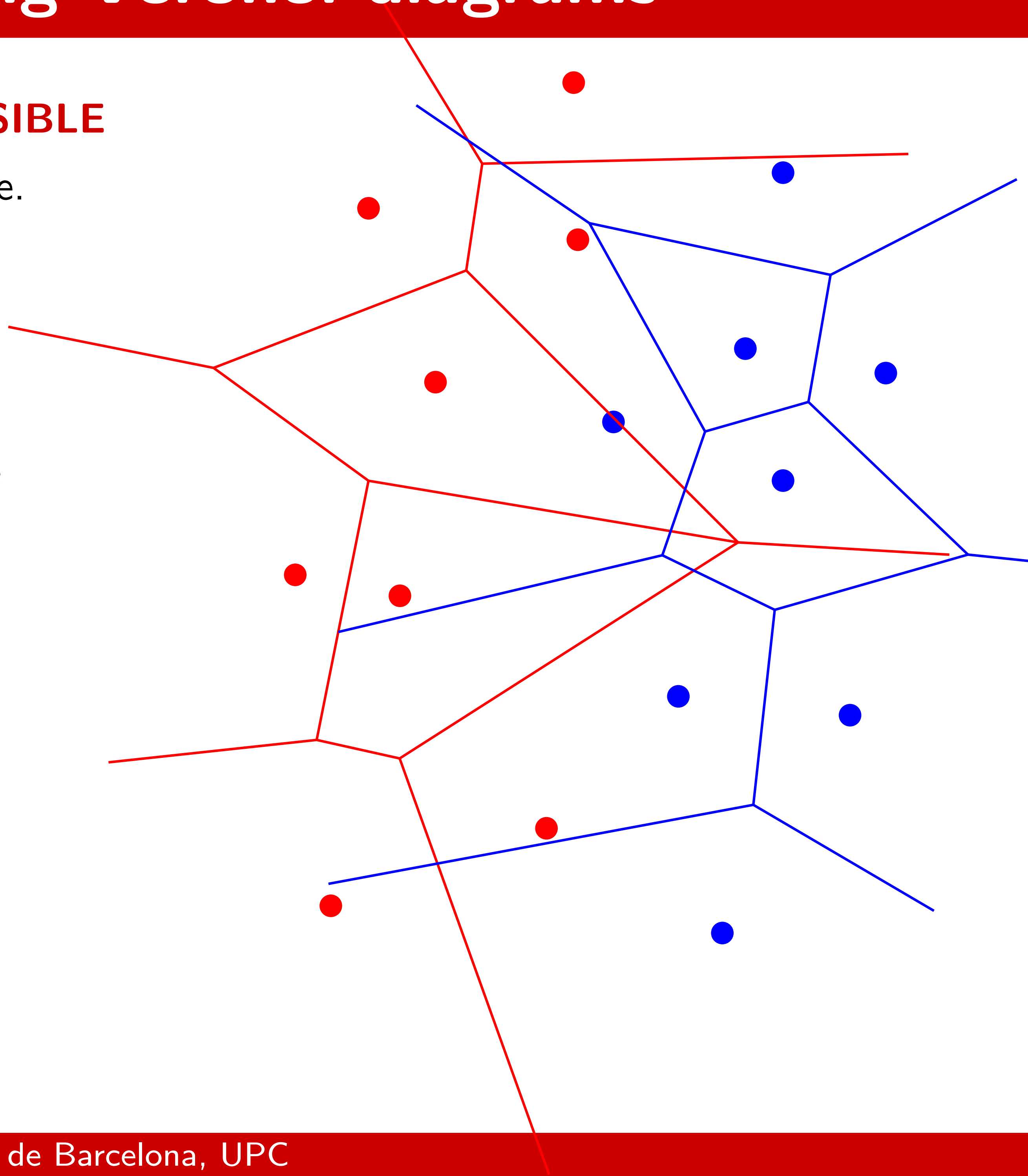

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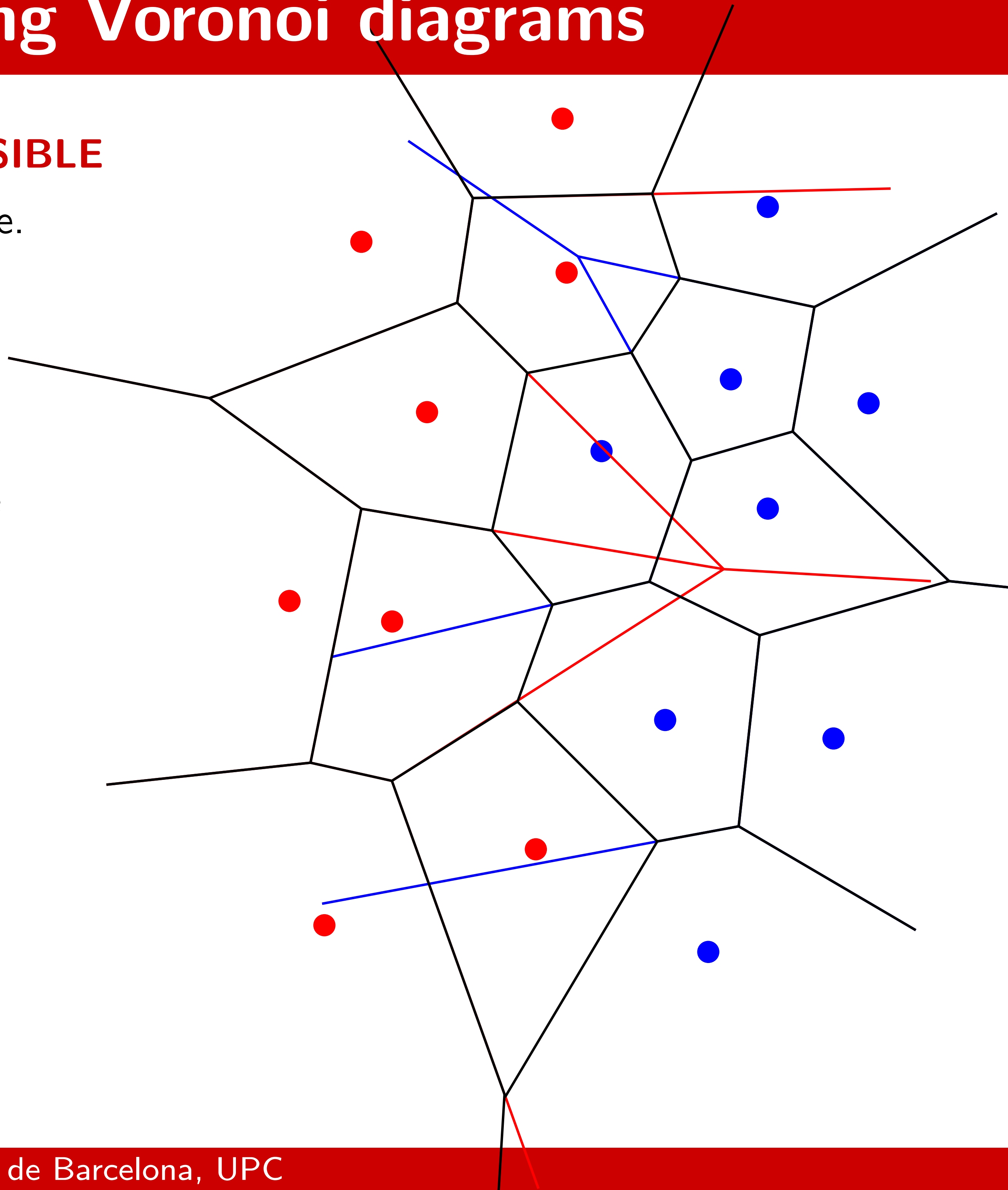

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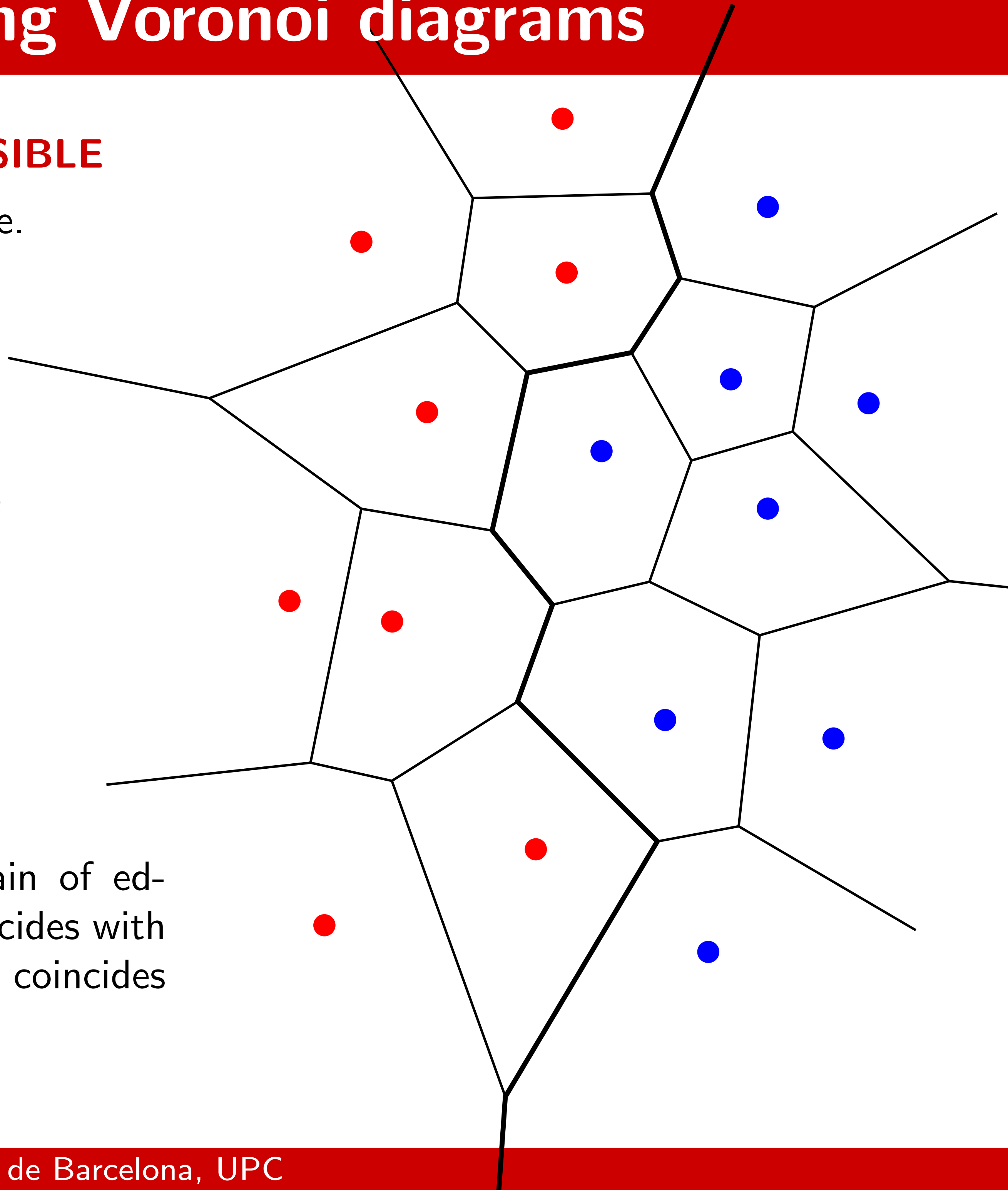

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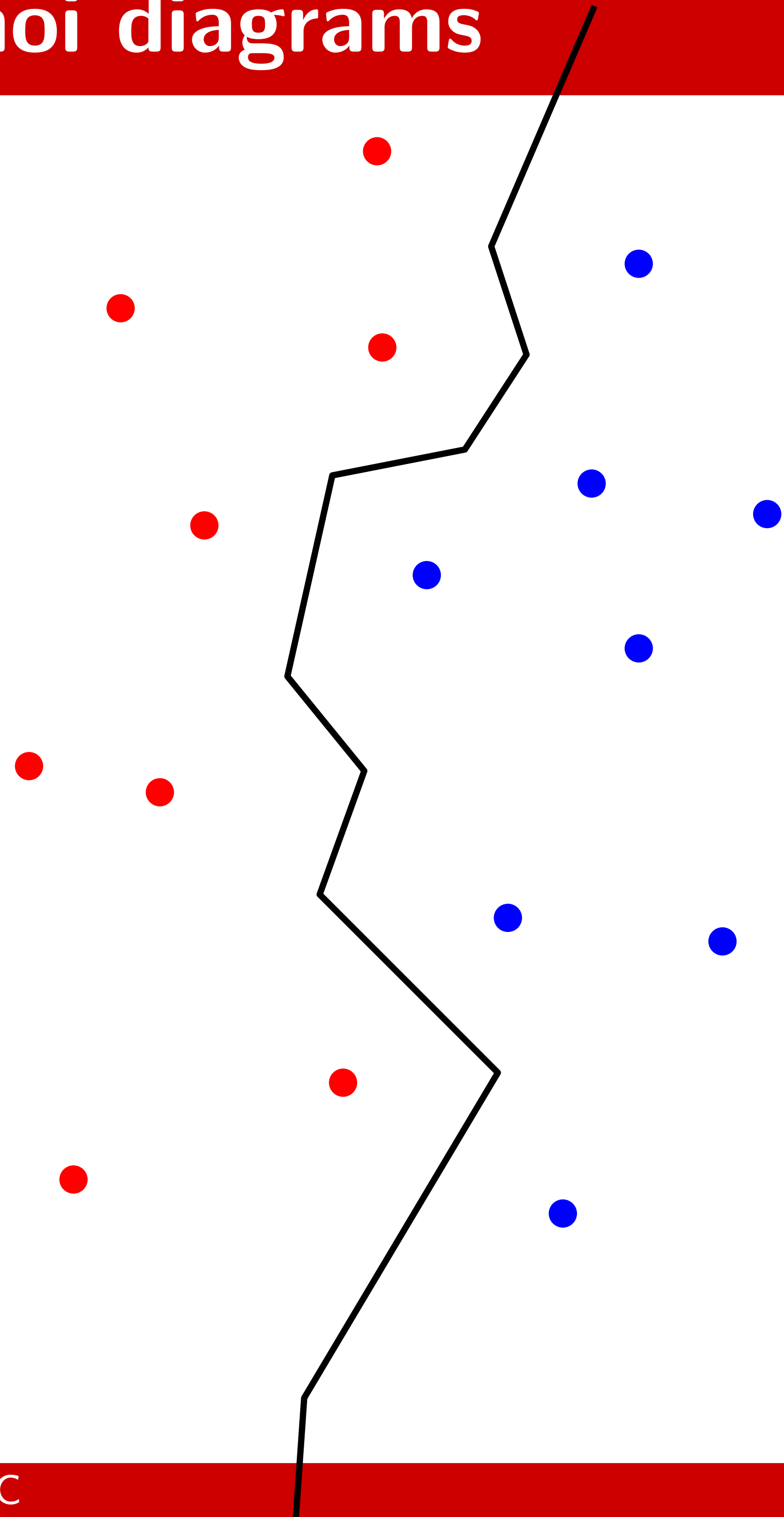

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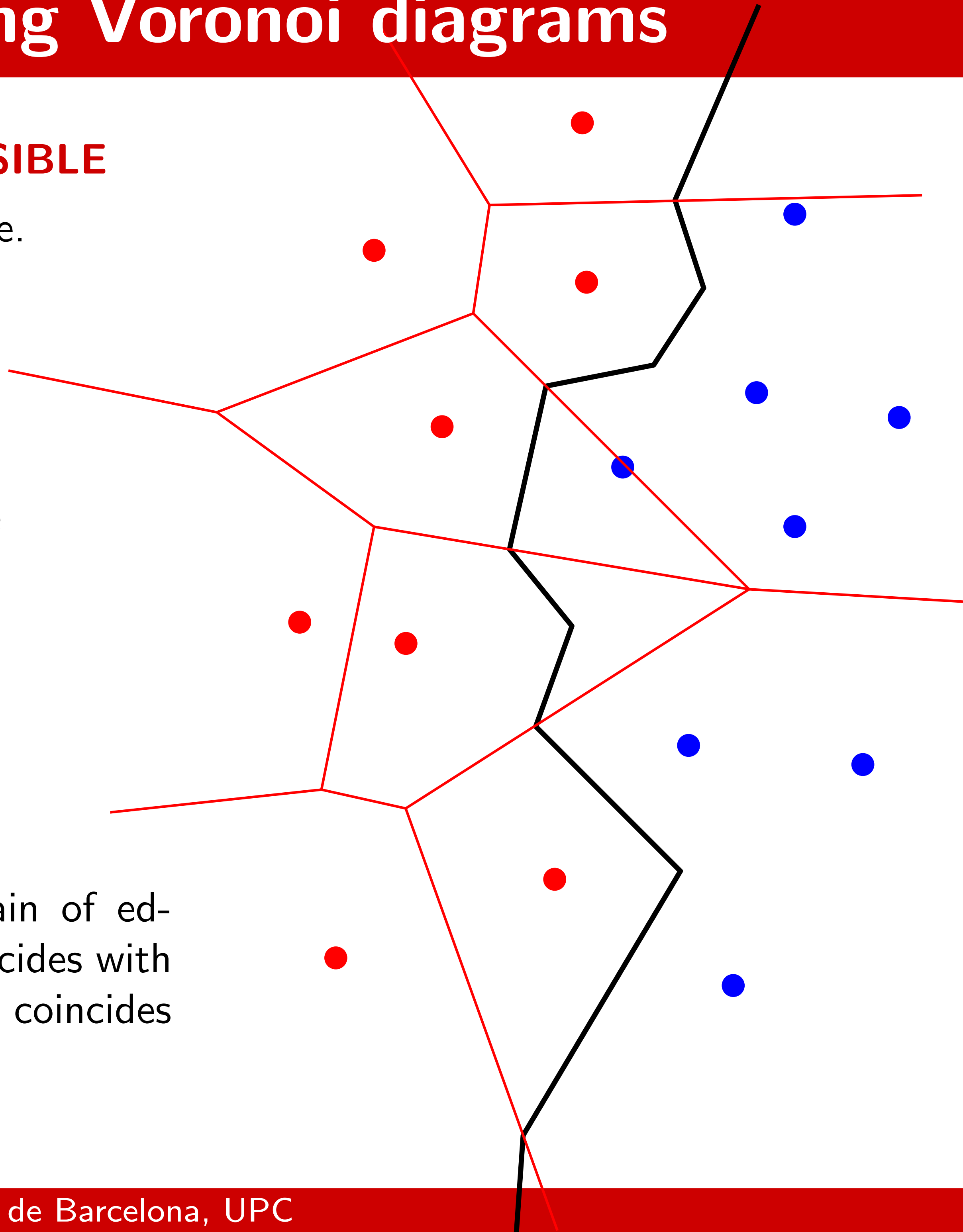

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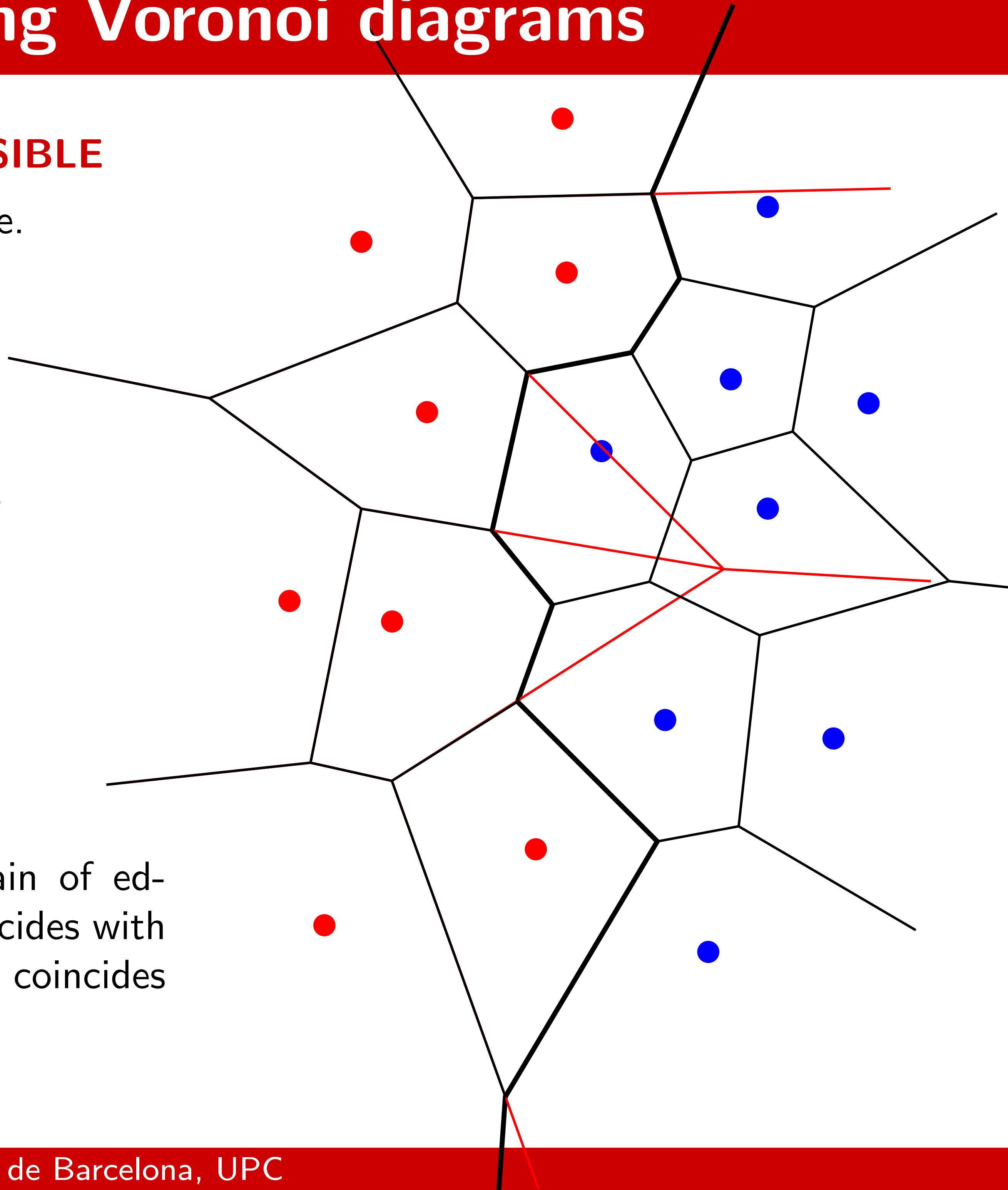

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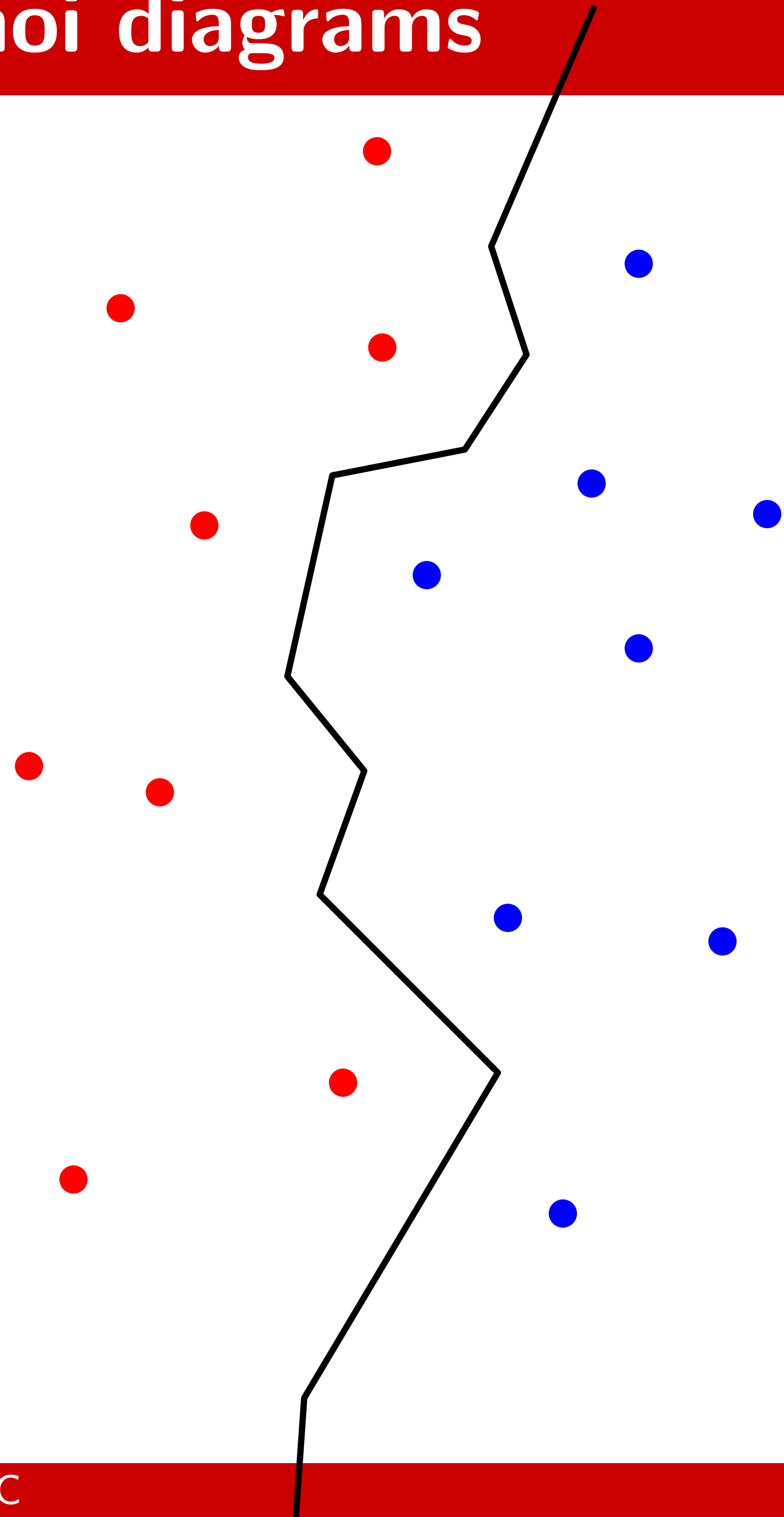

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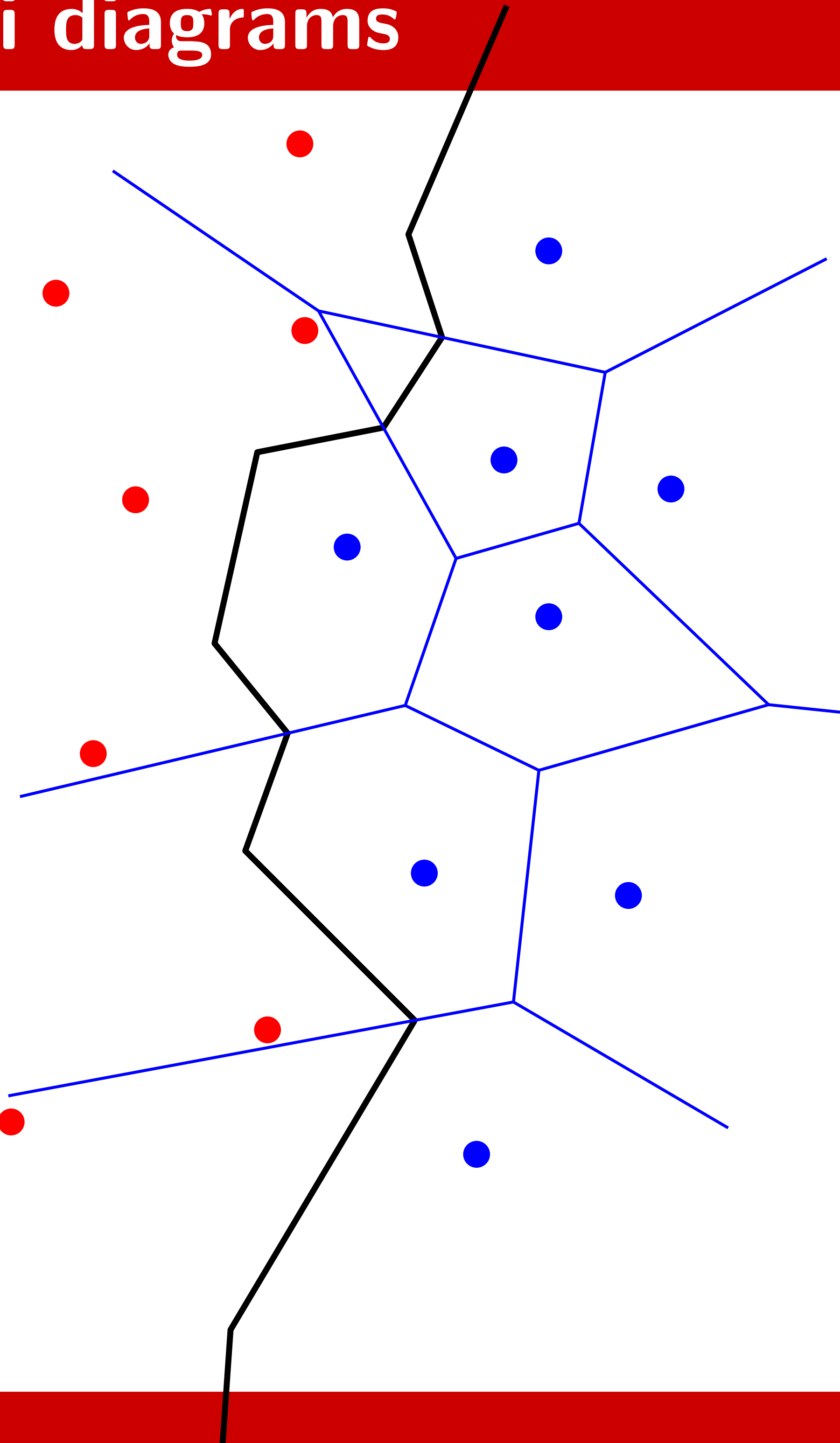

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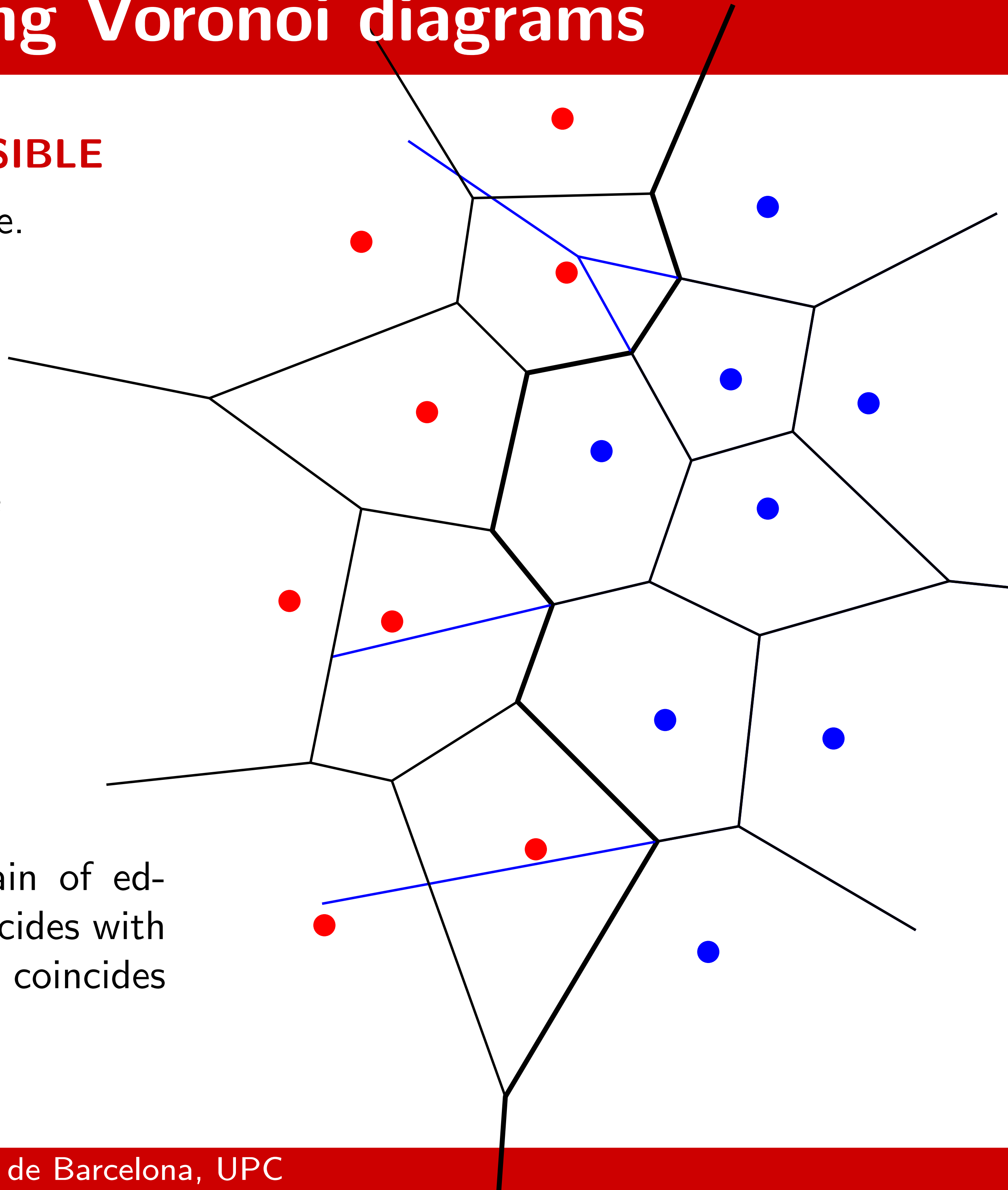
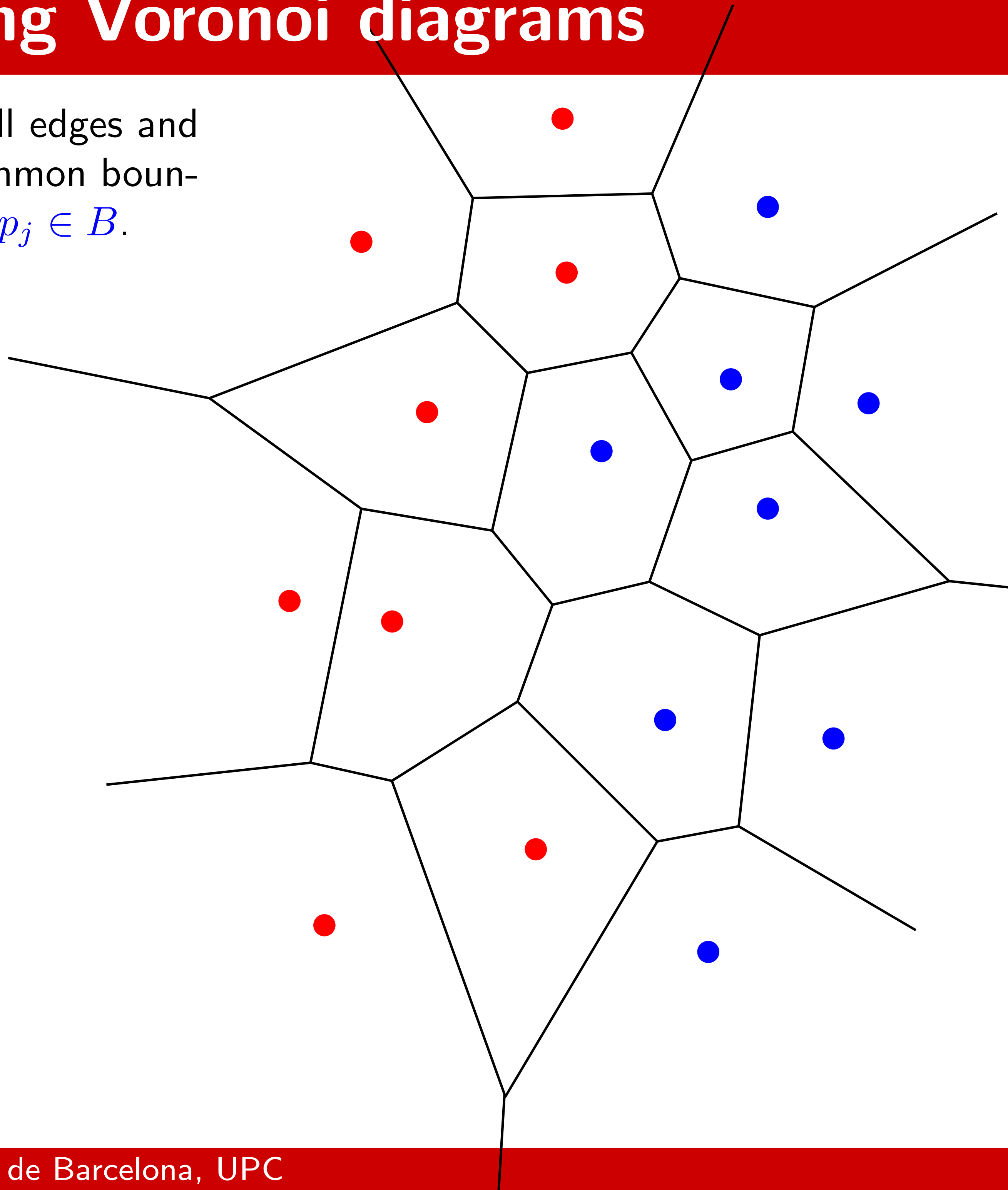


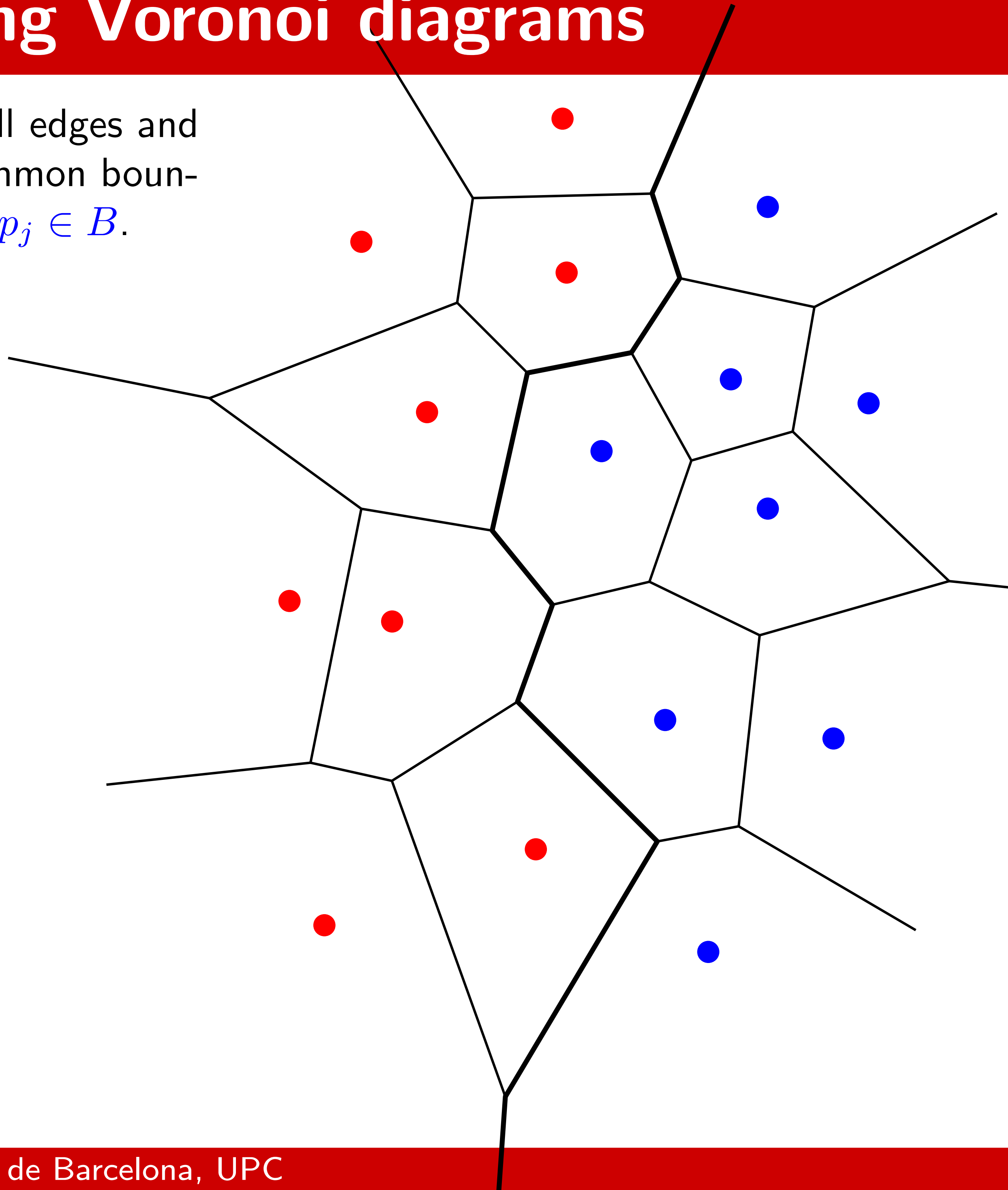
Constructing Voronoi diagrams

Definition. Let $b(R, B)$ be the set of all edges and vertices of $Vor(P)$ belonging to the common boundary of the regions of some $p_i \in R$ and $p_j \in B$.



Constructing Voronoi diagrams

Definition. Let $b(R, B)$ be the set of all edges and vertices of $Vor(P)$ belonging to the common boundary of the regions of some $p_i \in R$ and $p_j \in B$.



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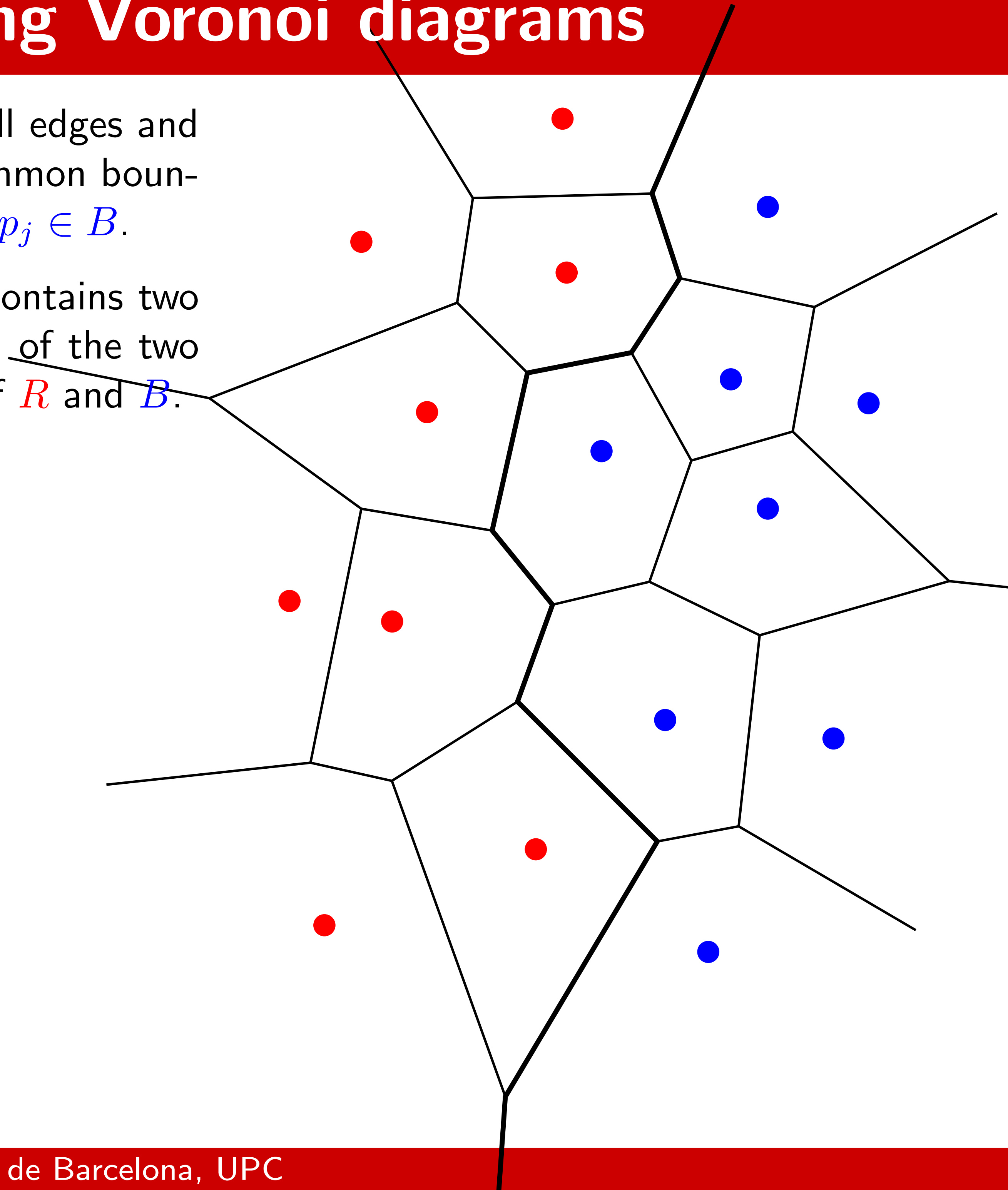
Observation 1. The bisector $b(R, B)$ contains two half-lines, belonging to the bisectors b_{ij} of the two "bridges" connecting the convex hulls of R and B.



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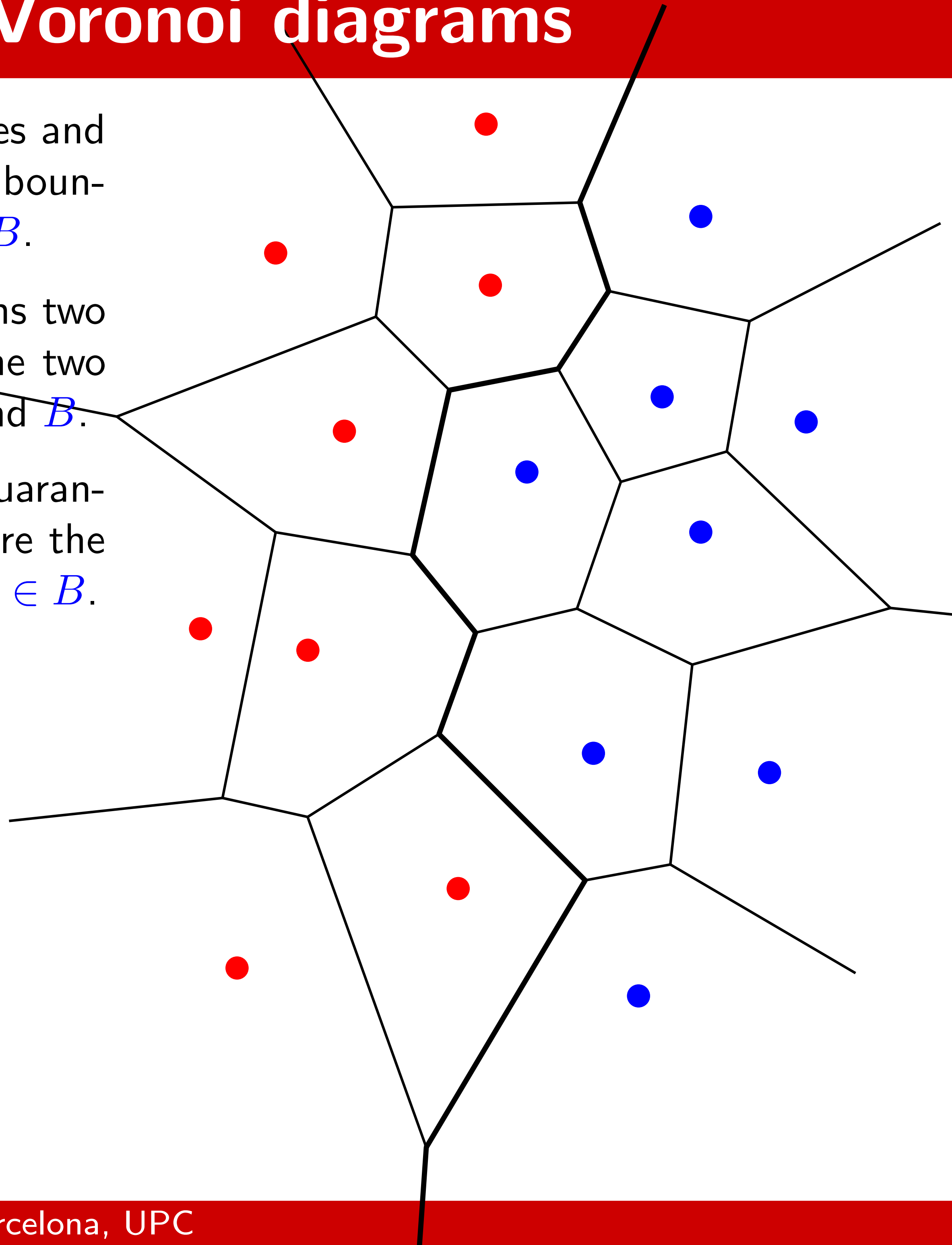
Proof. The vertical separation of R and B guarantees the existence of the "bridges", which are the edges of $ch(P)$ connecting a $p_i \in R$ to a $p_j \in B$.



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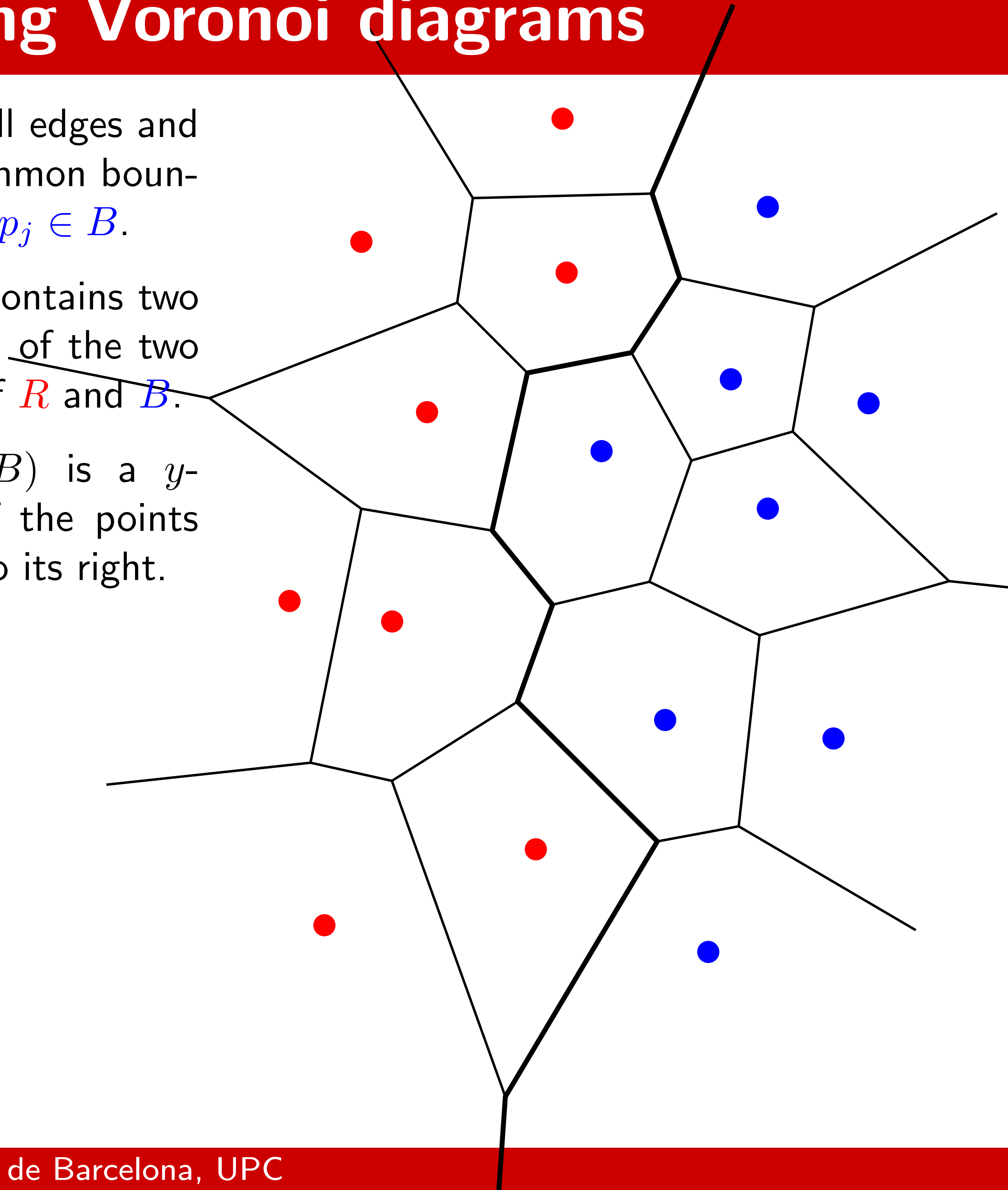

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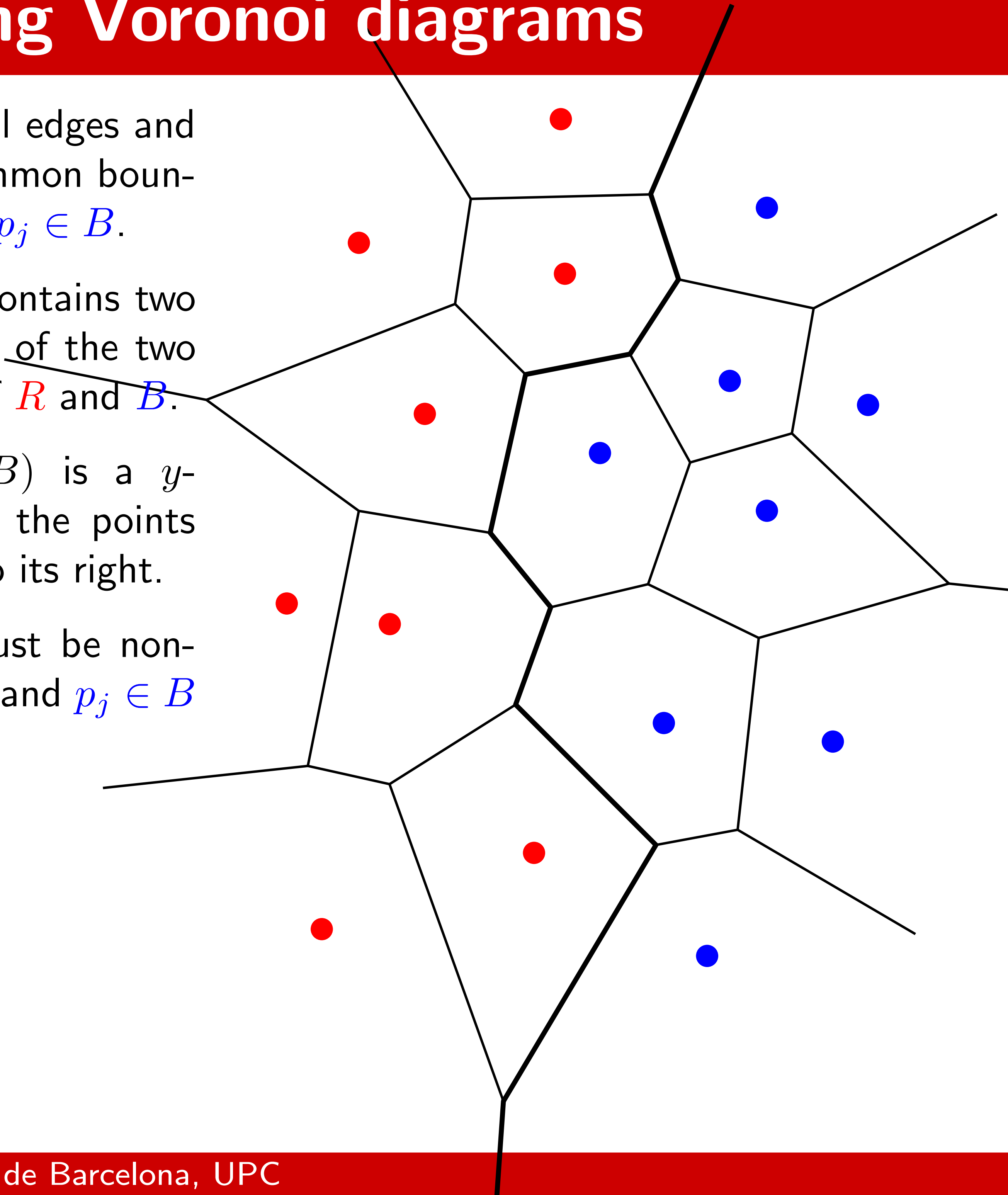
Proof. Every edge e_{ij} of $b(R, B)$ must be non-horizontal, and leave $p_i \in R$ to its left and $p_j \in B$ to its right.



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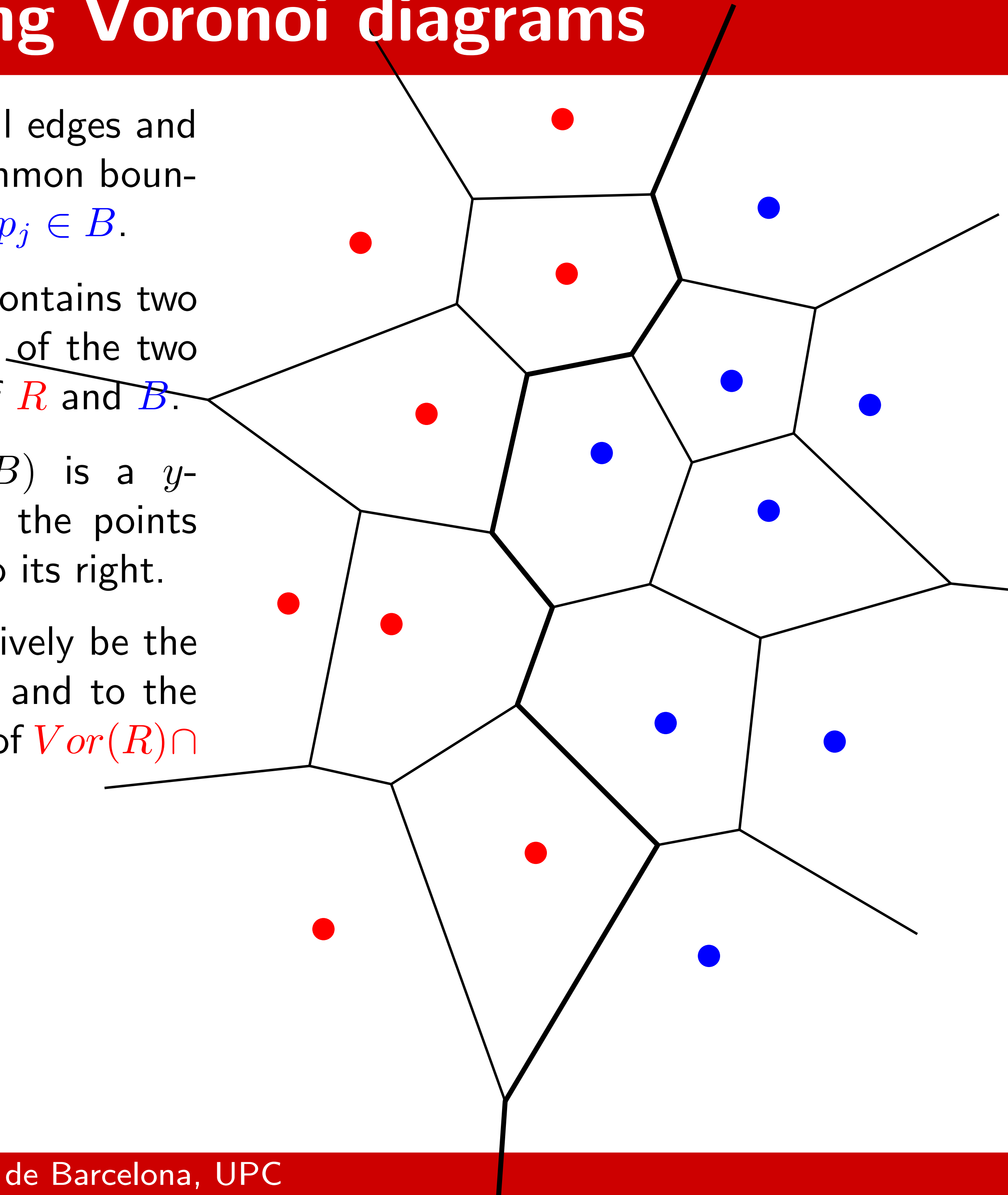

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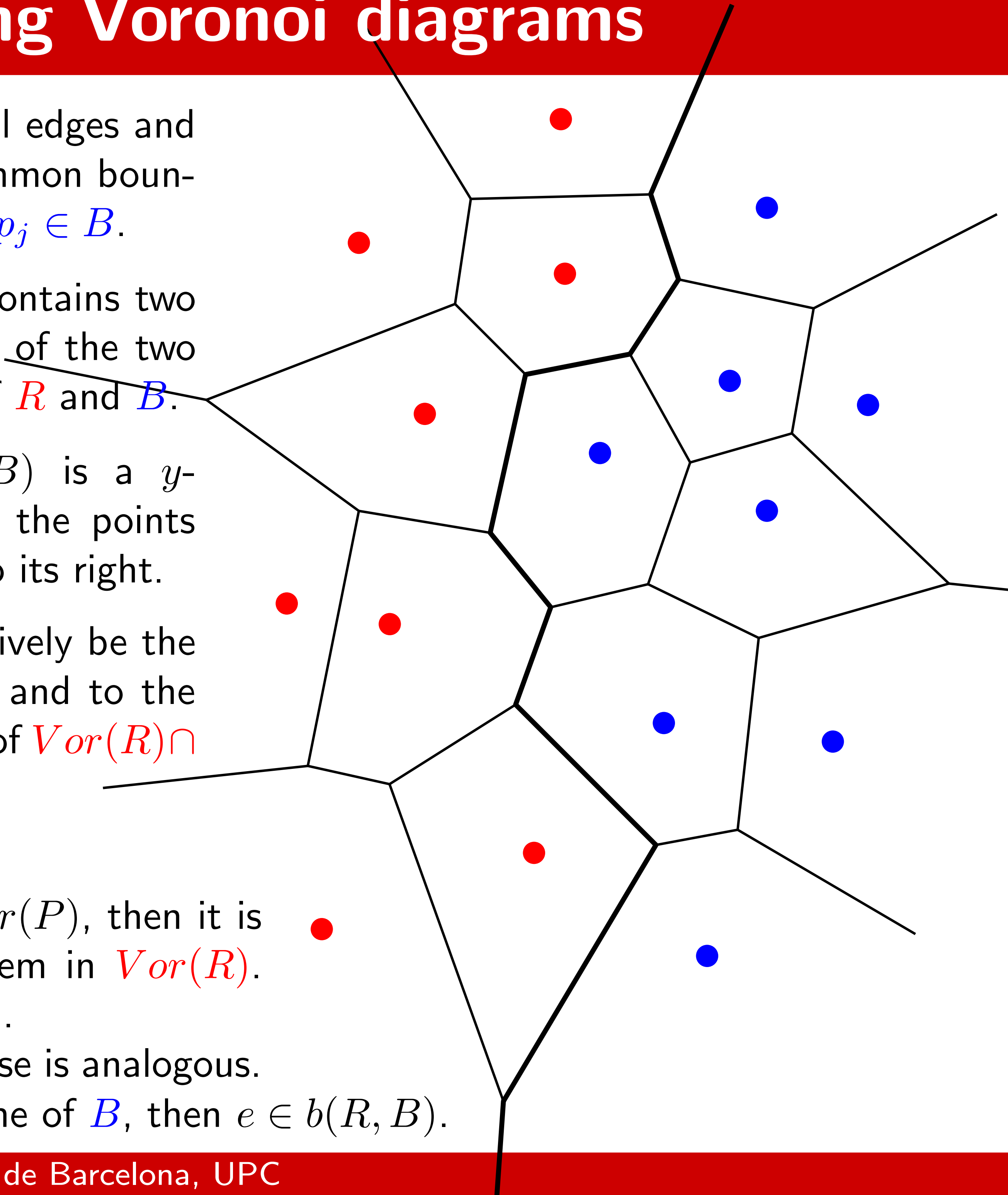
Observation 3. Let π_R and π_B respectively be the regions of the plane located to the left and to the right of $b(R, B)$. Then $Vor(P)$ consists of $Vor(R) \cap \pi_R$, $Vor(B) \cap \pi_B$ and $b(R, B)$.

Constructing Voronoi diagrams



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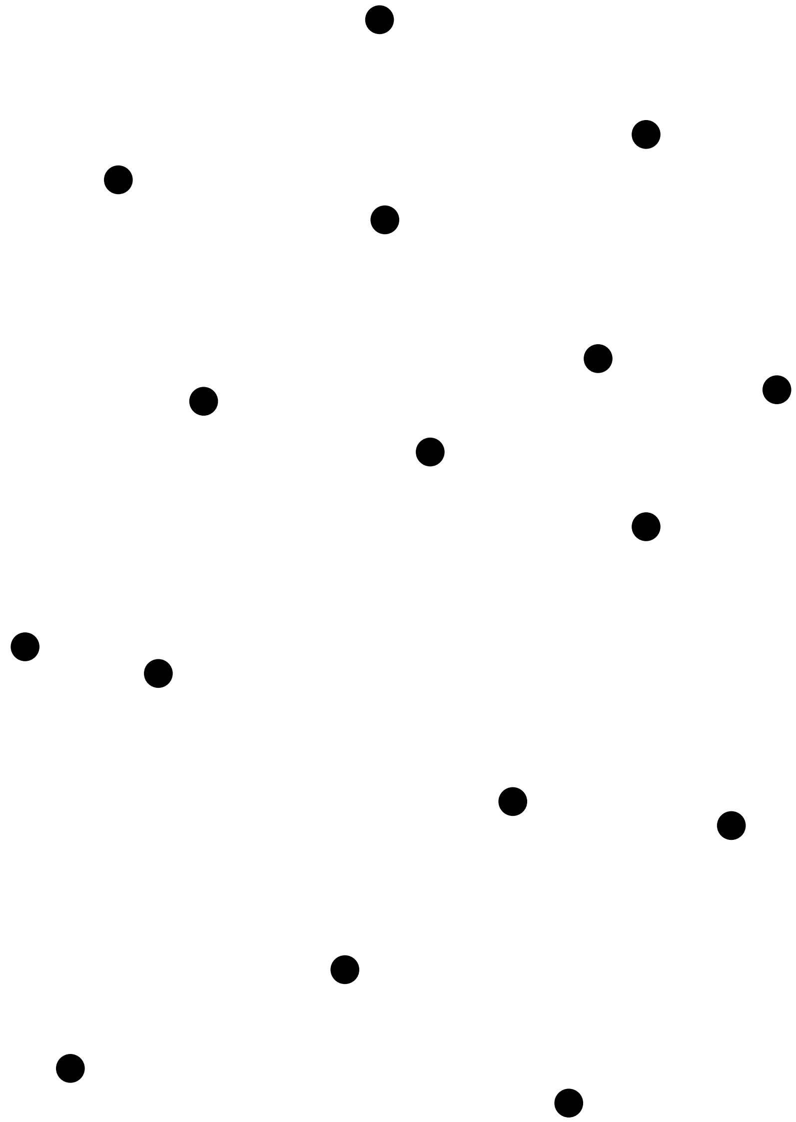
Proof. Let e be an edge of $Vor(P)$:

- If e separates two points of R in $Vor(P)$, then it is (a portion of) the edge separating them in $Vor(R)$. Due to Obs. 2, e cannot belong to π_B.
- If e separates two points of B, the case is analogous.
- If e separates one point of R from one of B, then $e \in b(R, B)$.

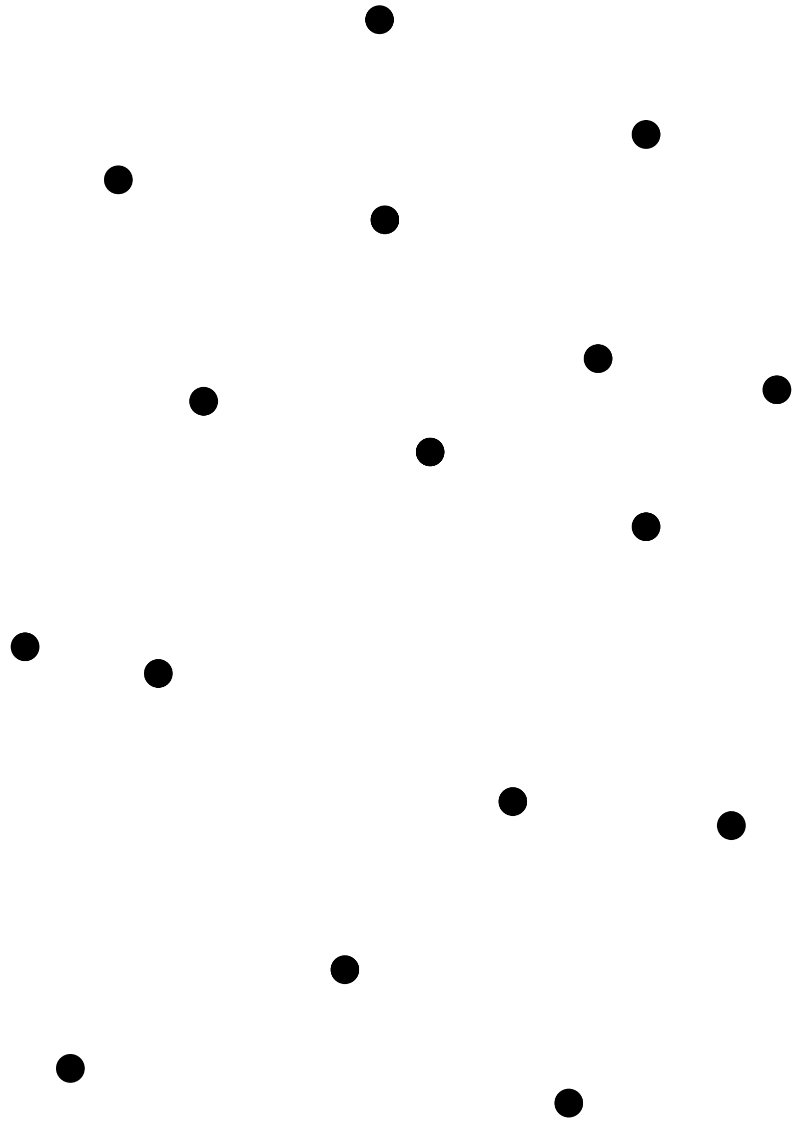
DIVIDE AND CONQUER ALGORITHM



Constructing Voronoi diagrams

DIVIDE AND CONQUER ALGORITHM

1. Preprocess: Sort the points of P by abscissa (only once).



DIVIDE AND CONQUER ALGORITHM

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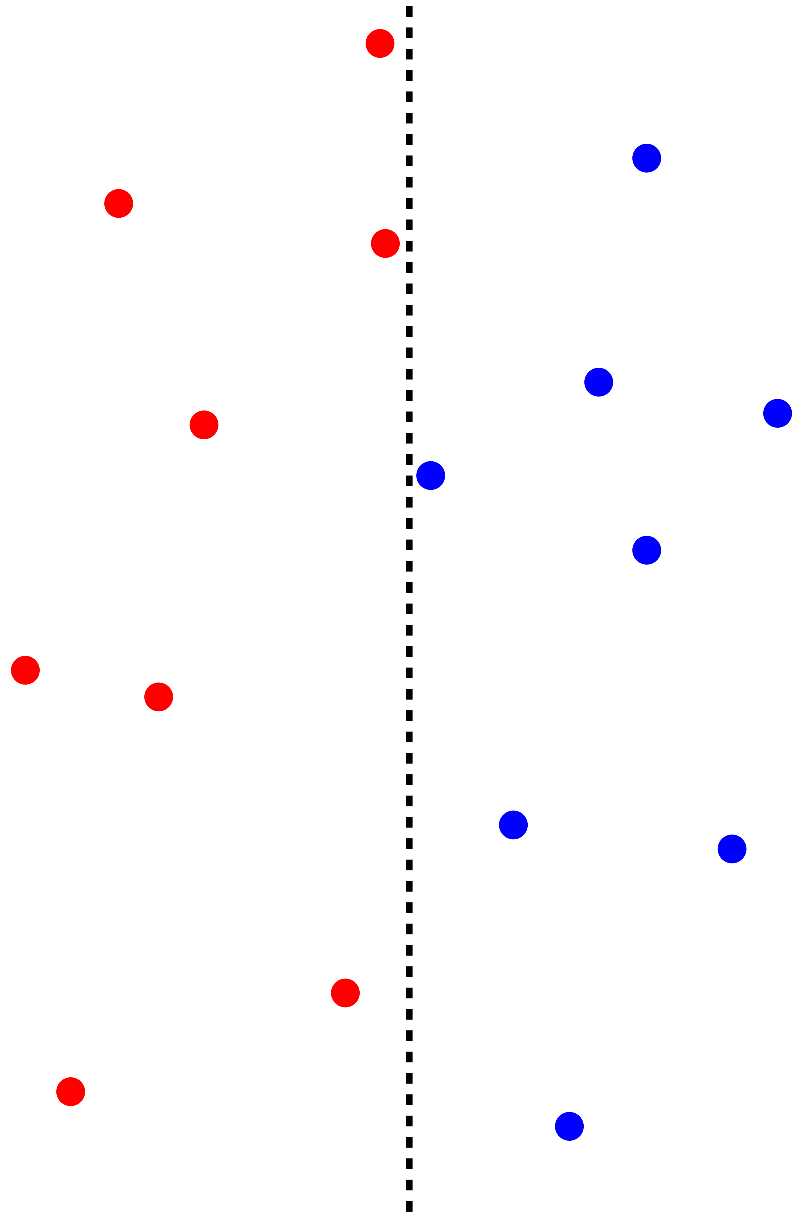
2. Division: Vertically partition P into two subsets R and B, of approximately the same size.



Constructing Voronoi diagrams

DIVIDE AND CONQUER ALGORITHM

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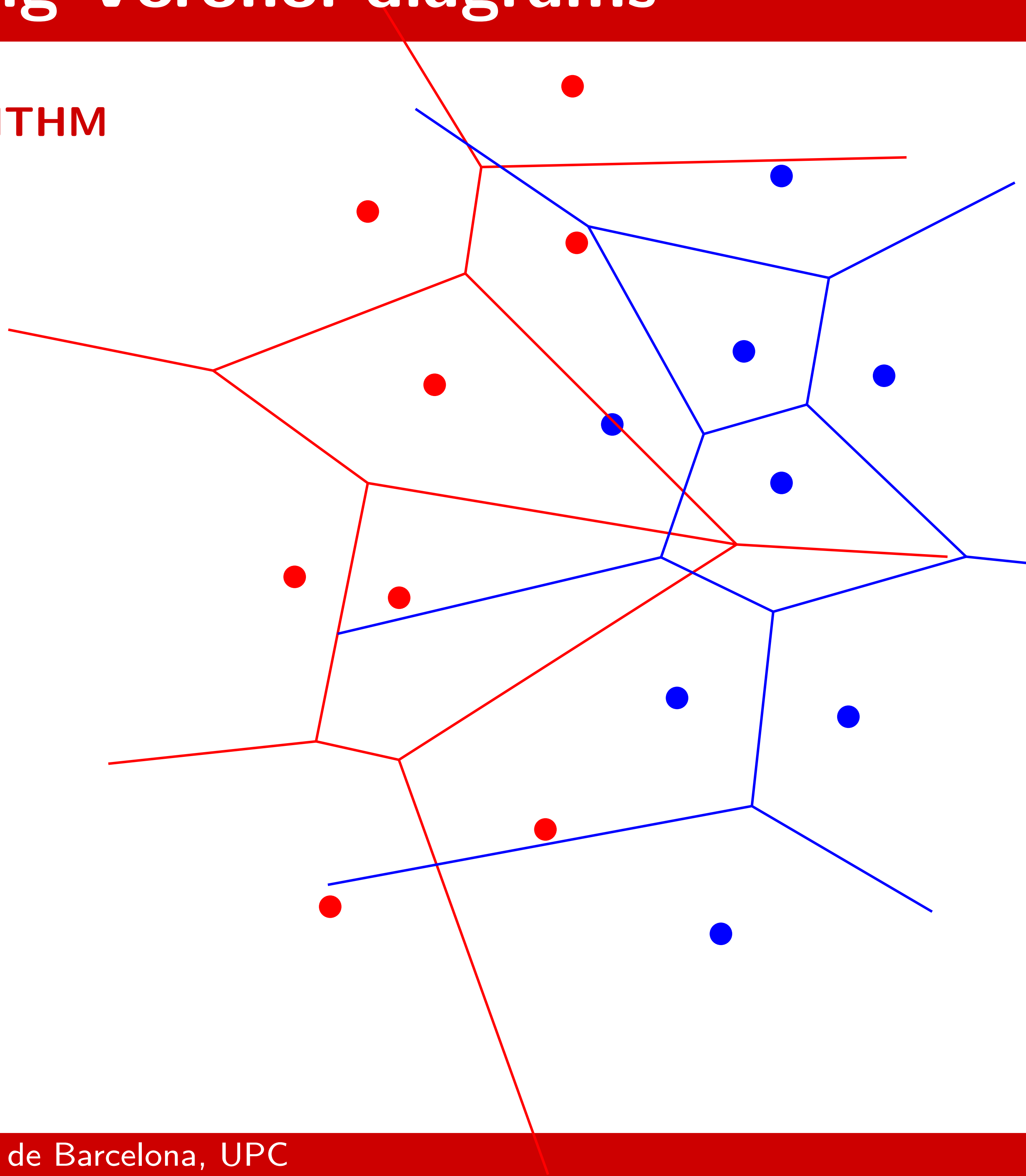
3. Recursion: Recursively compute $Vor(R)$ and $Vor(B)$.



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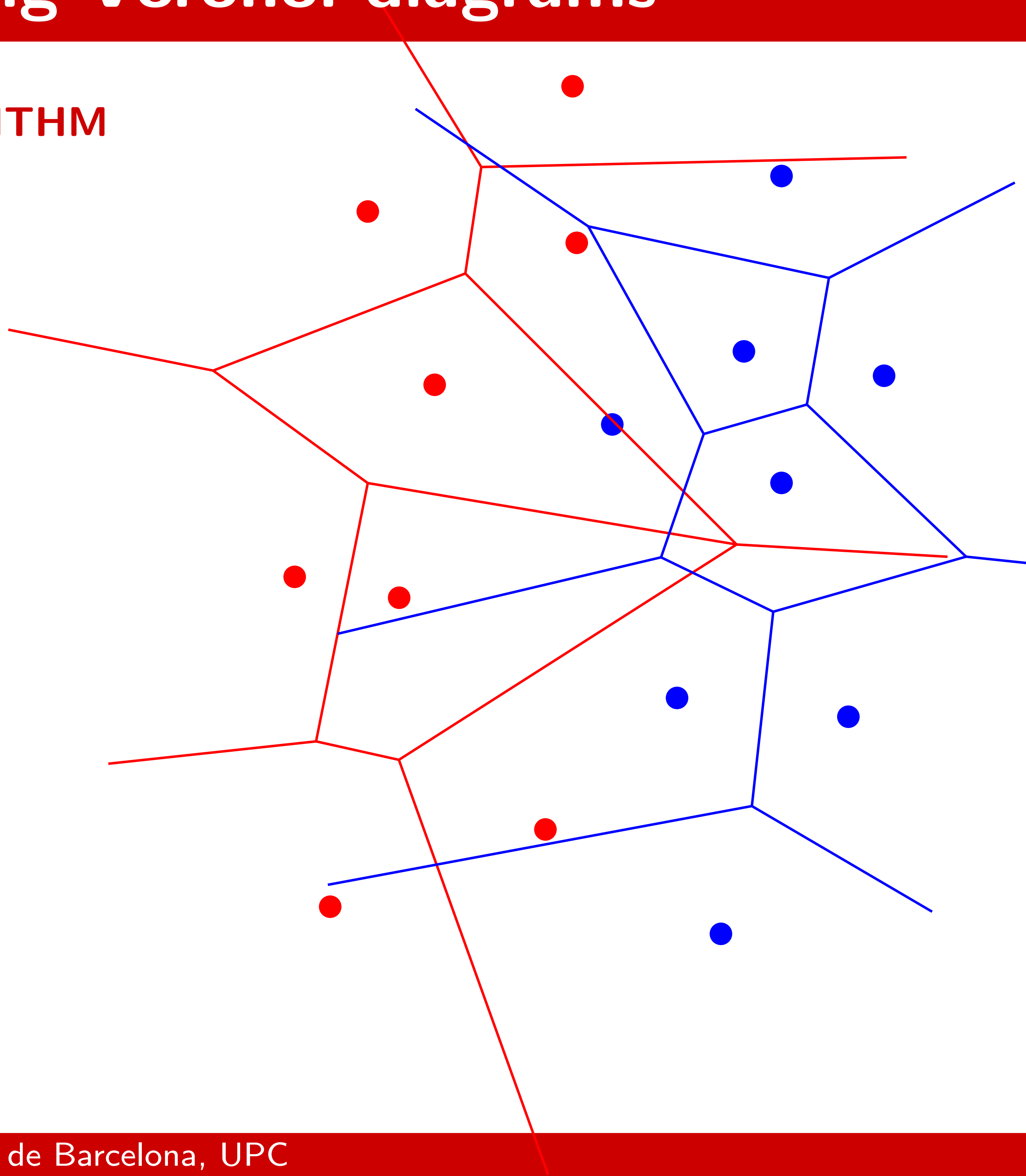

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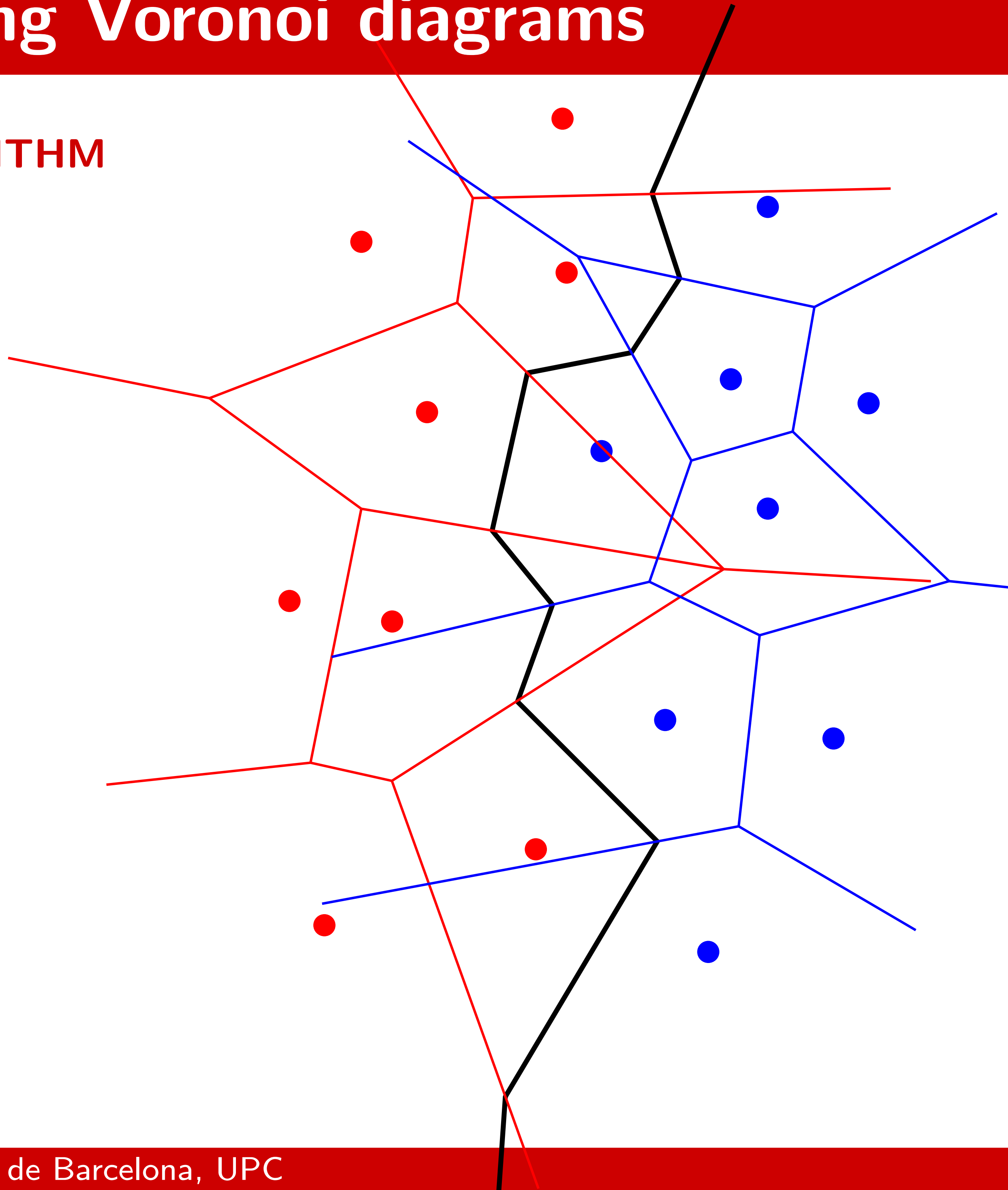
4. Merging:

Compute the separating chain.



Constructing Voronoi diagrams

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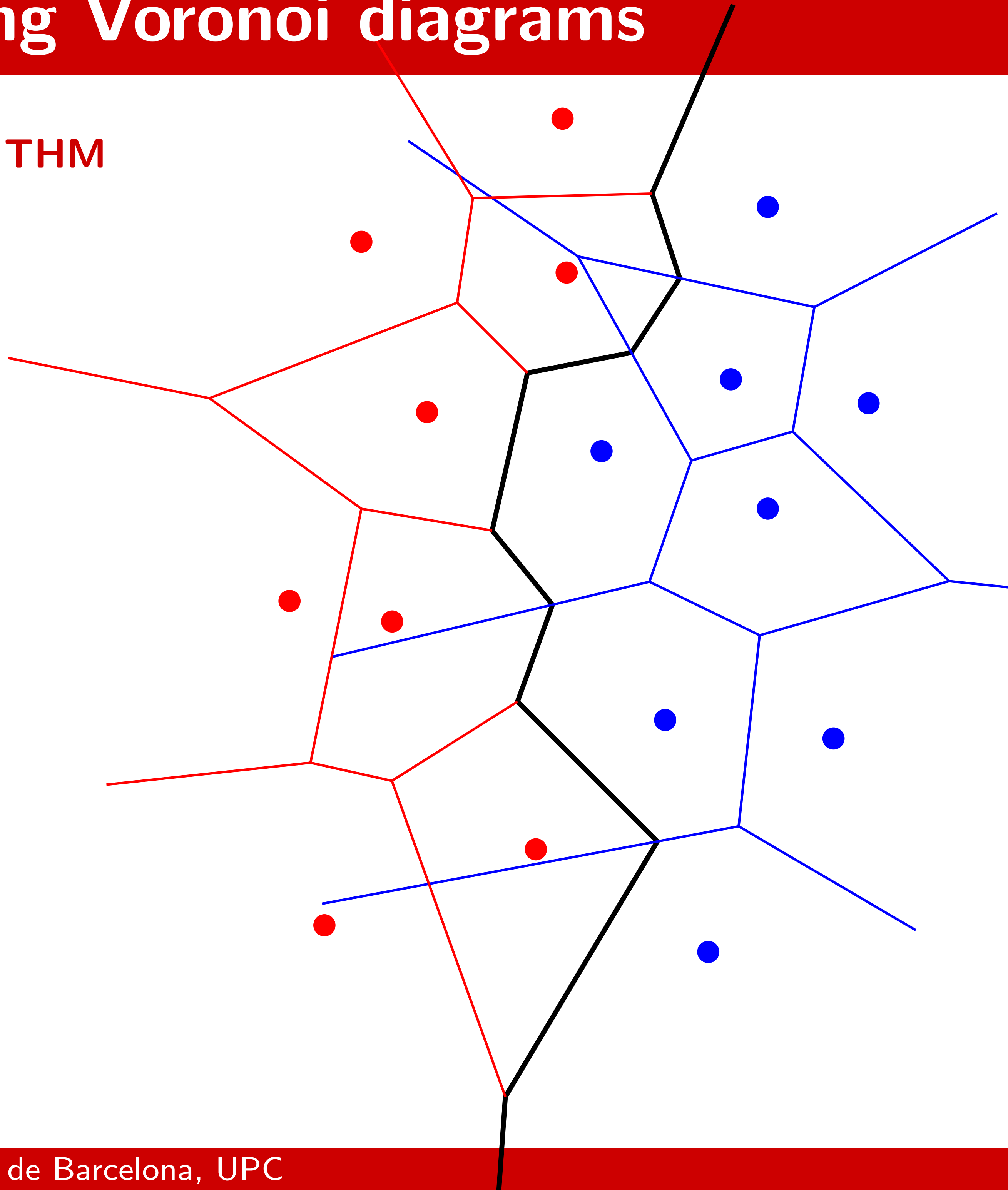
Compute the separating chain.

Prune the portion of $Vor(R)$ lying to the right of the chain and the portion of $Vor(B)$ lying to its left.



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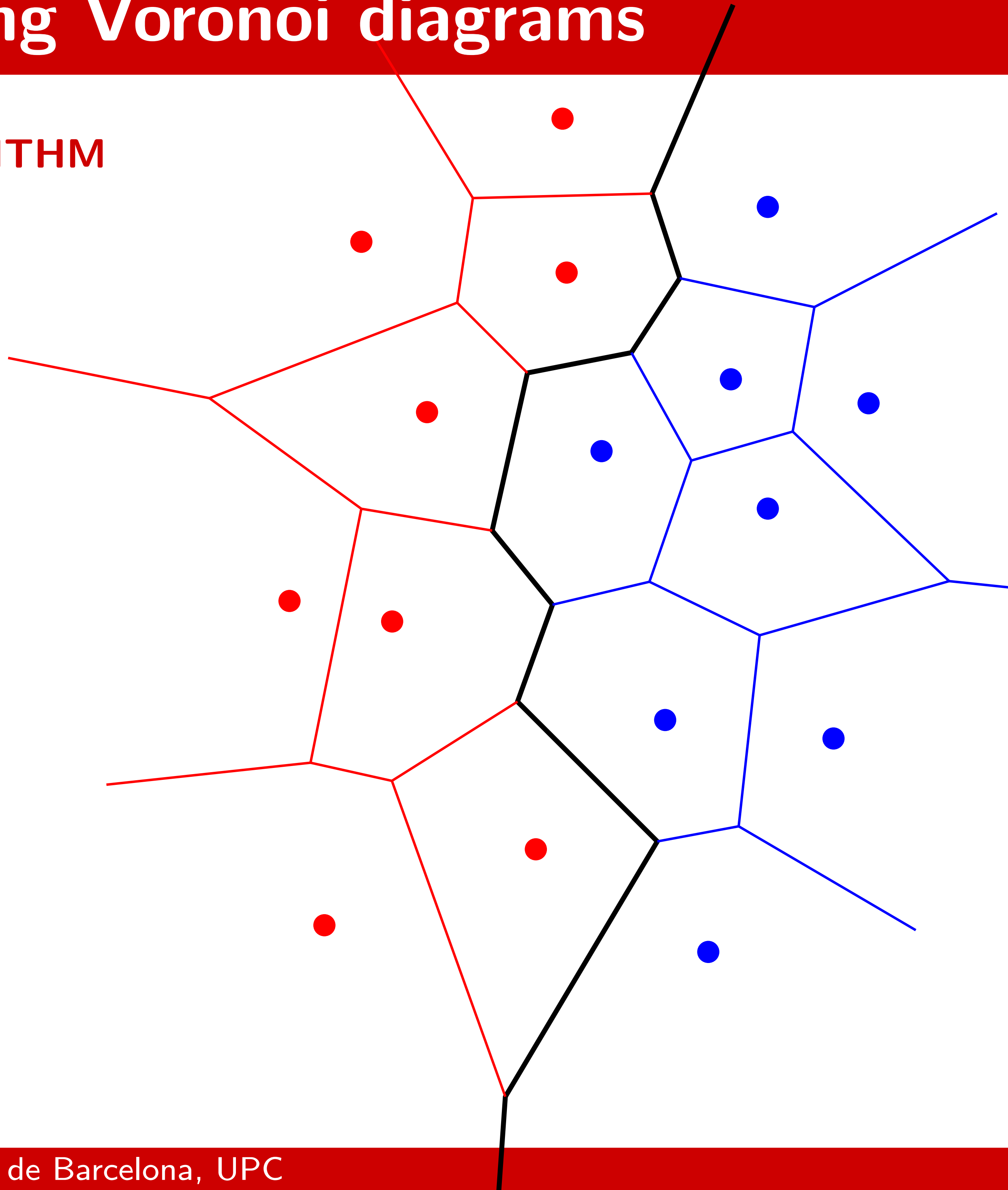

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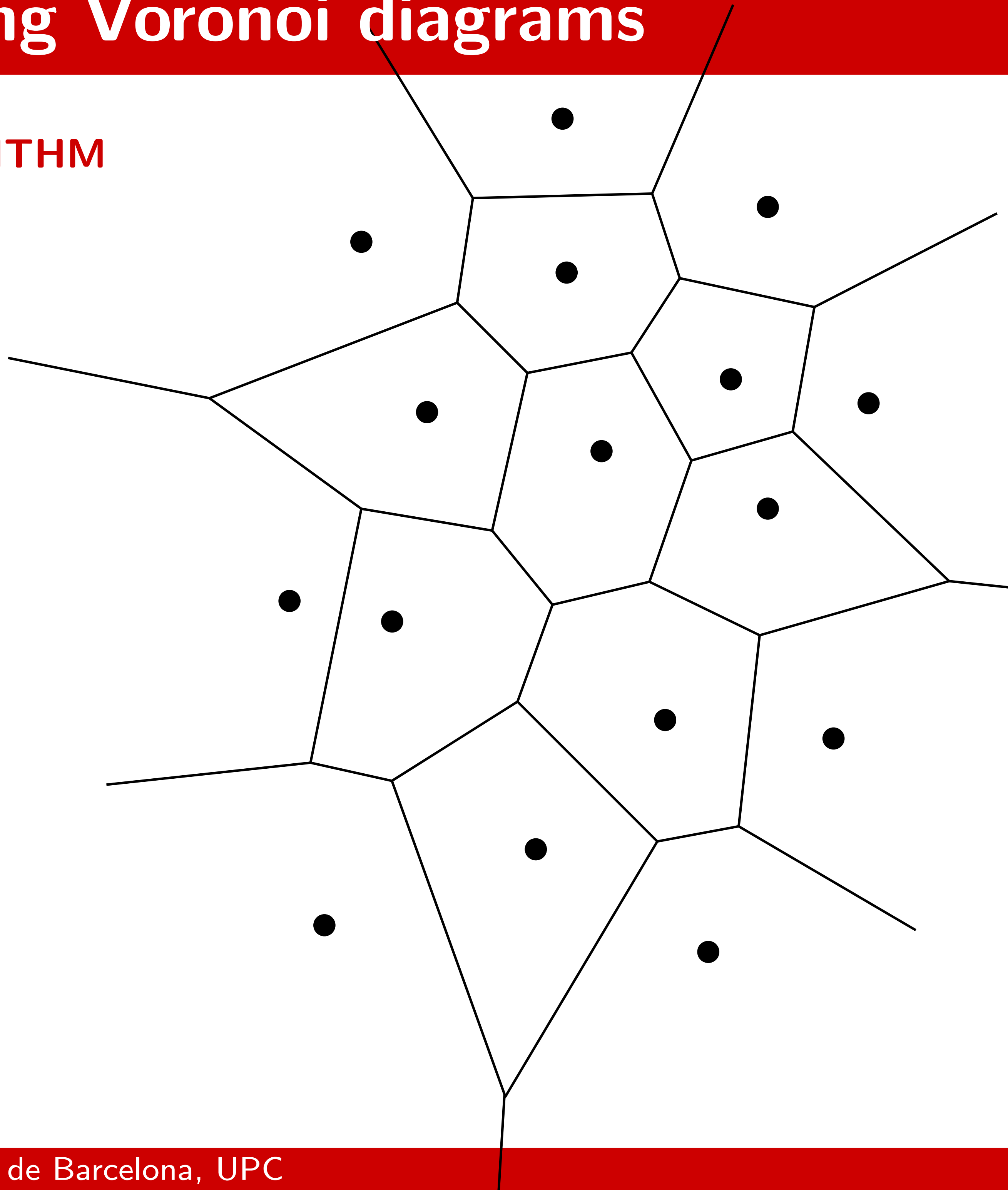
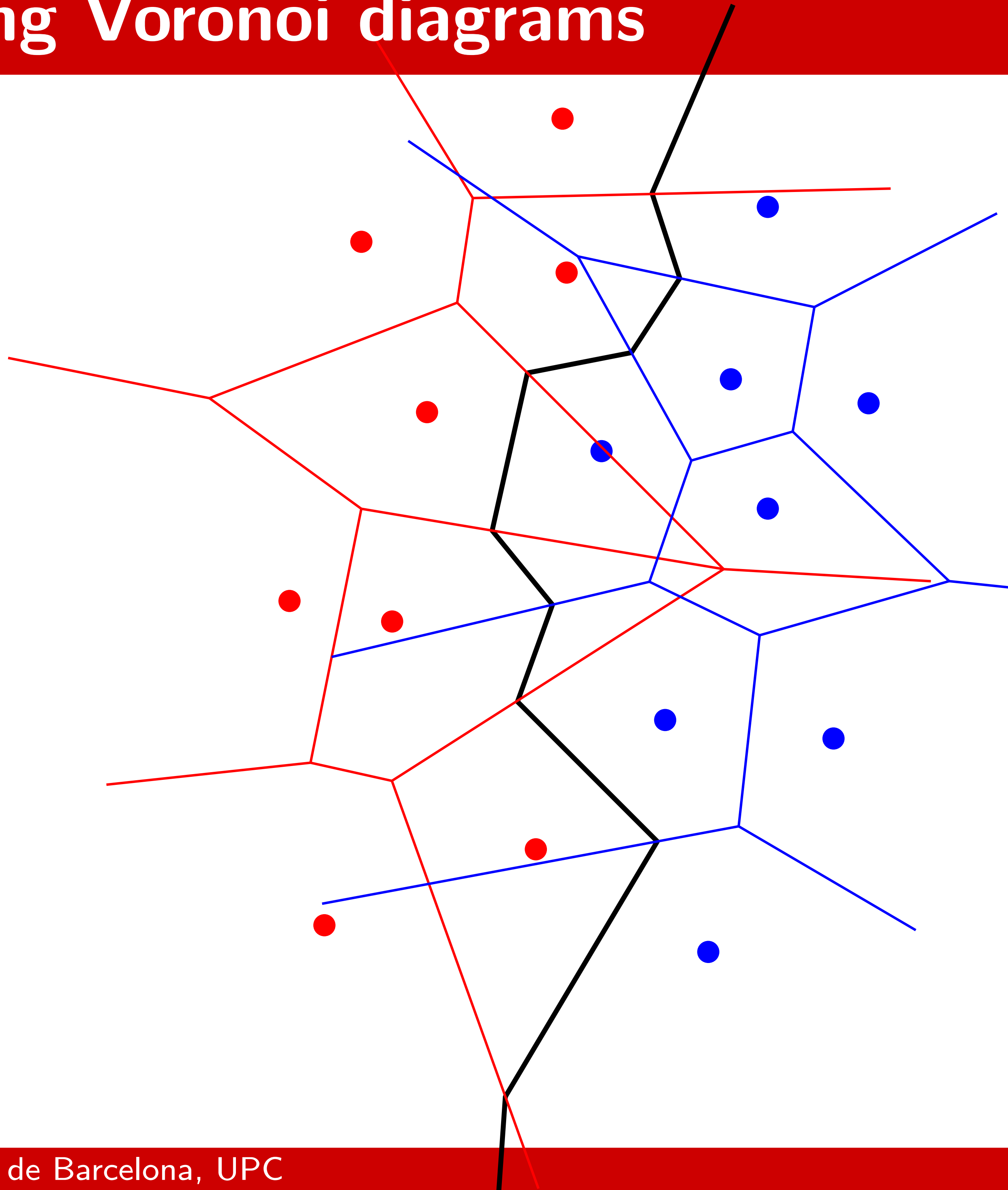


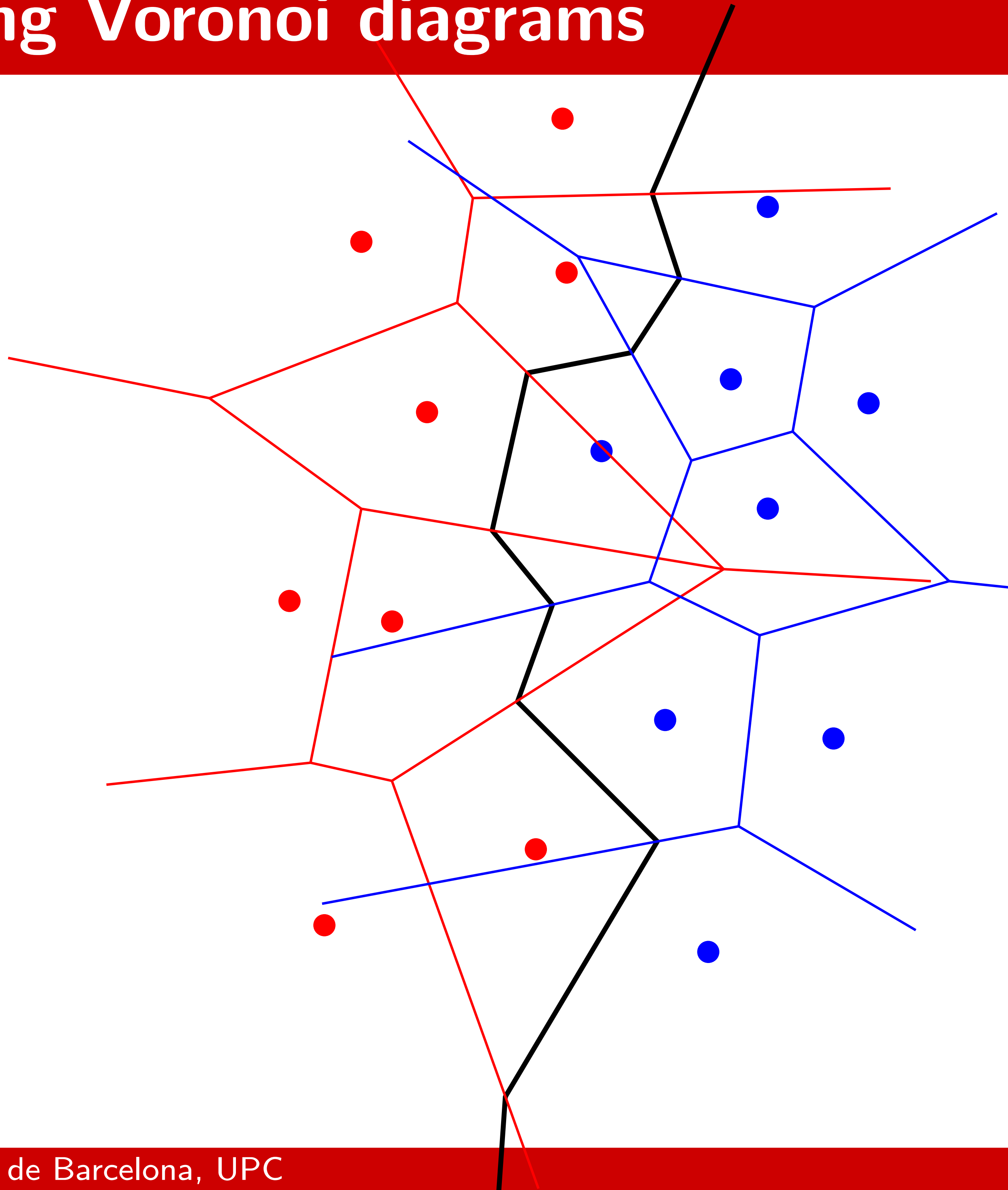
How to compute the chain?



How to compute the chain?

Initialization

Find the two halflines



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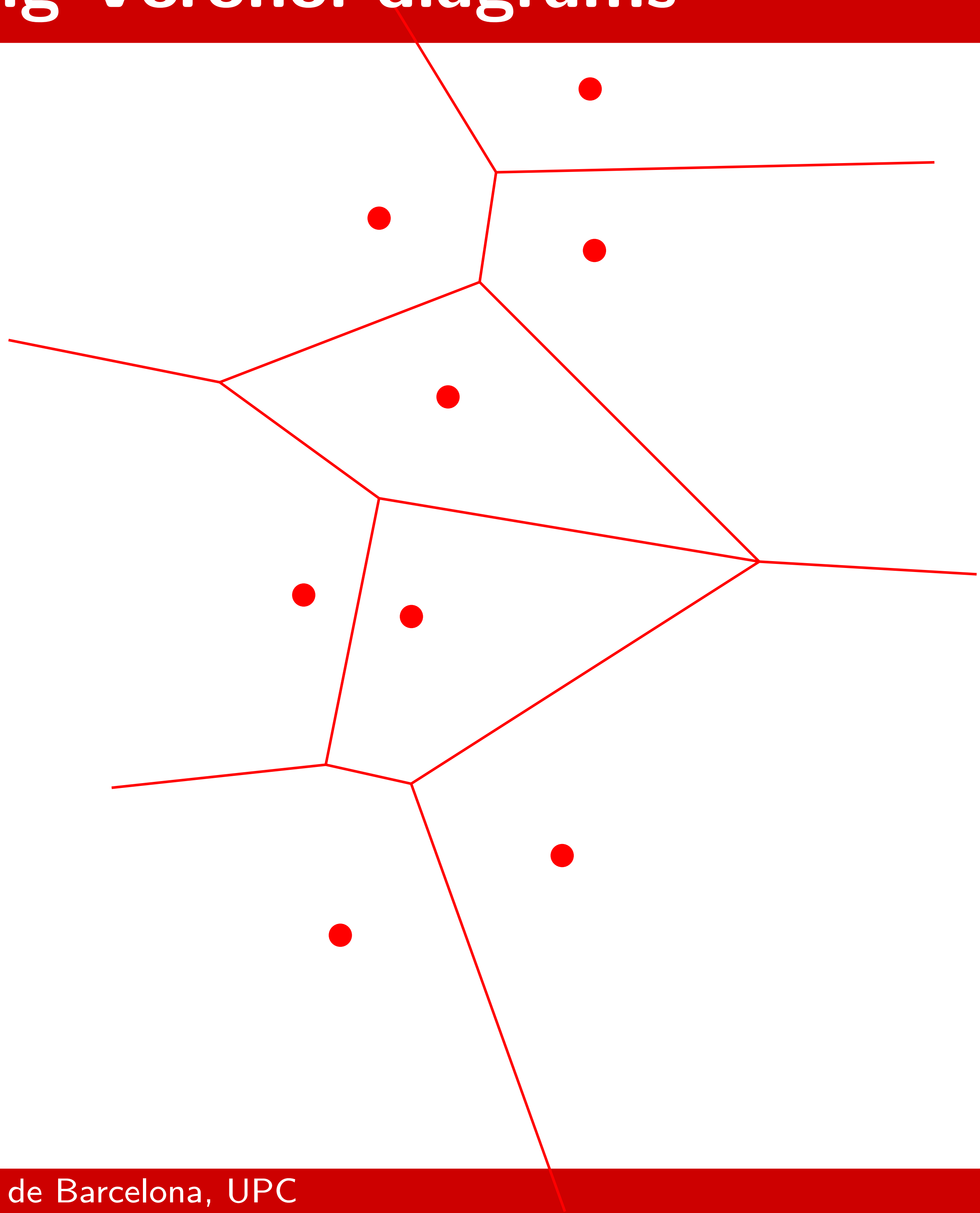

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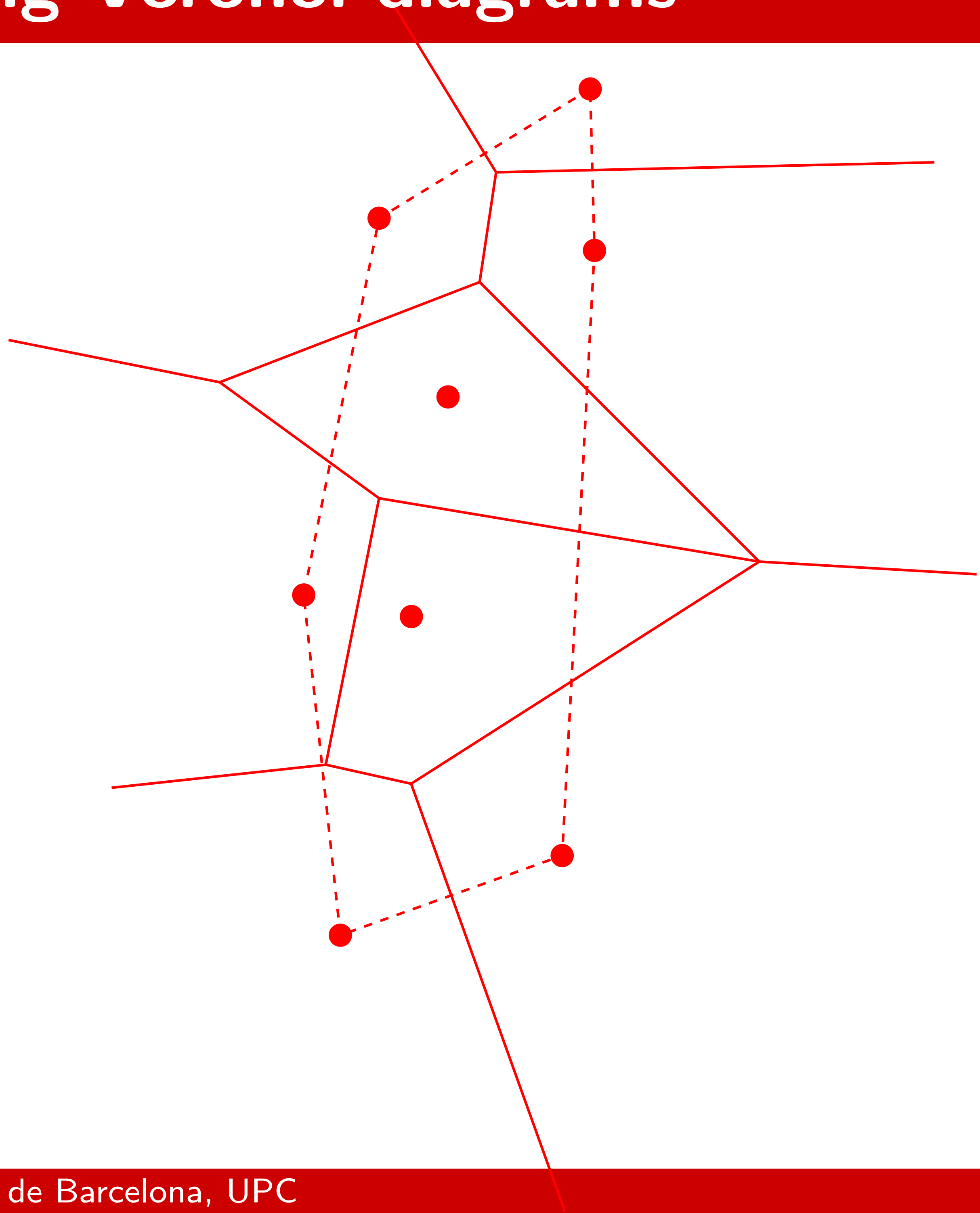

Constructing Voronoi diagrams

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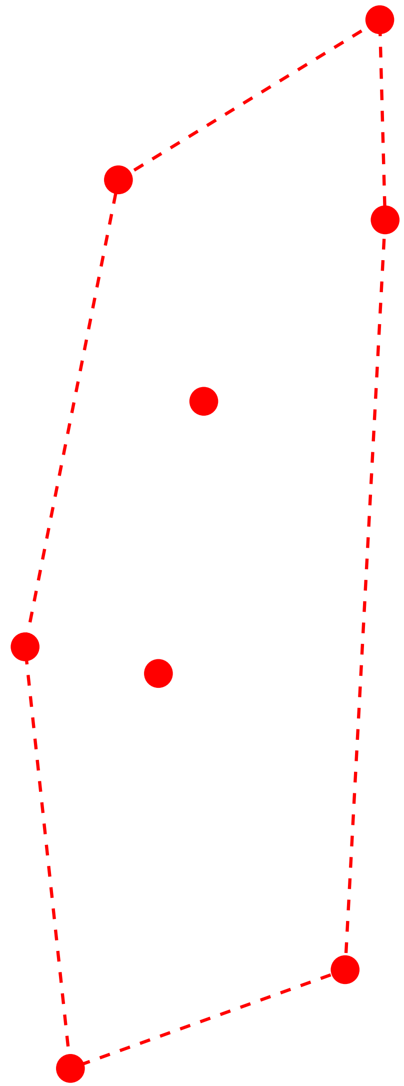

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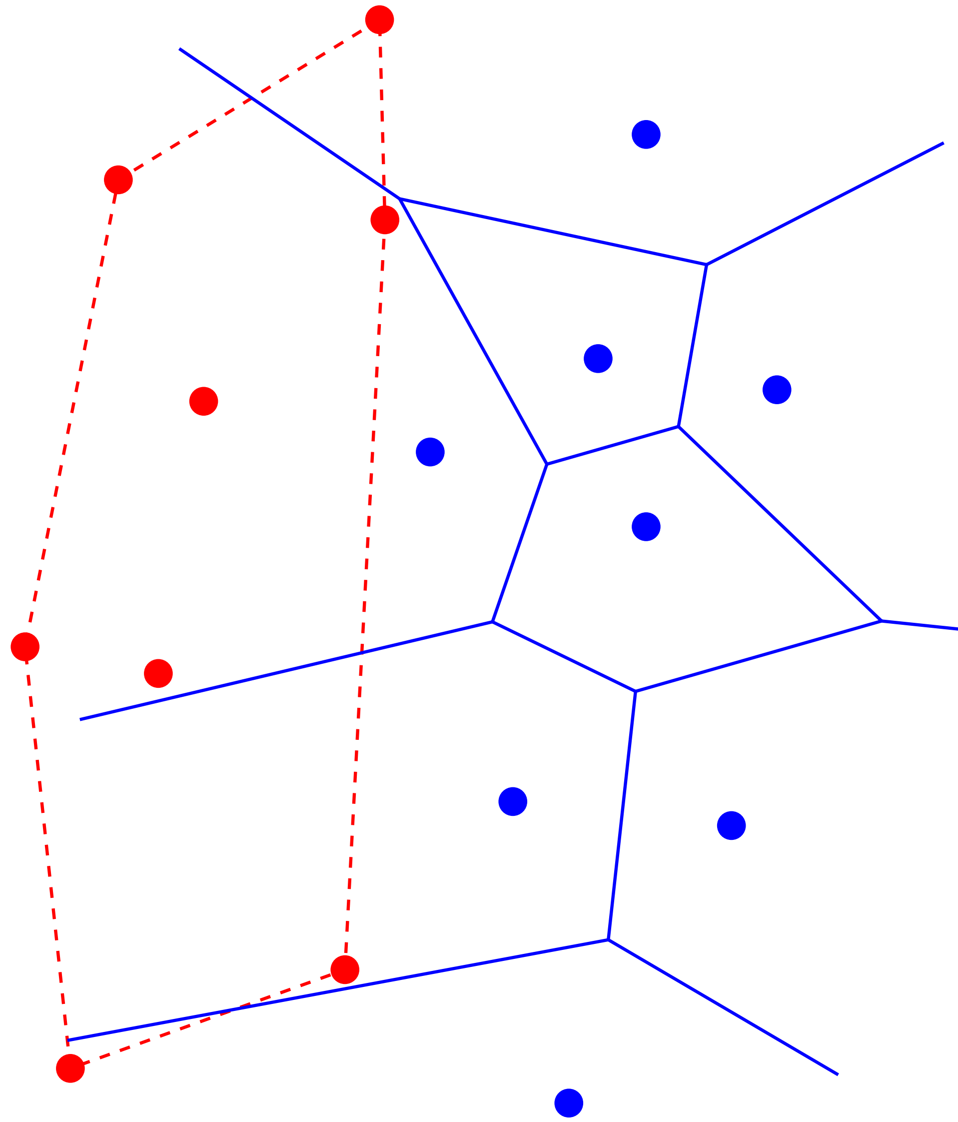

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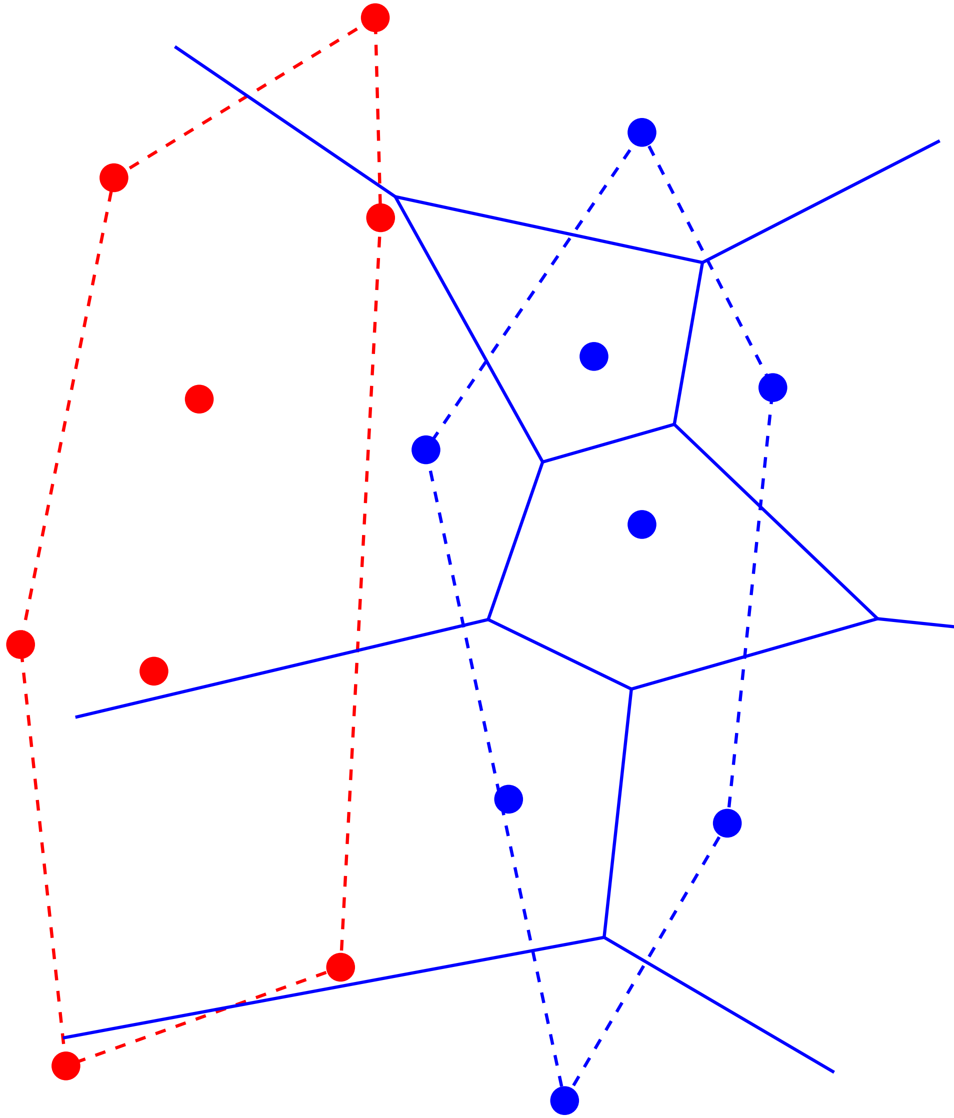

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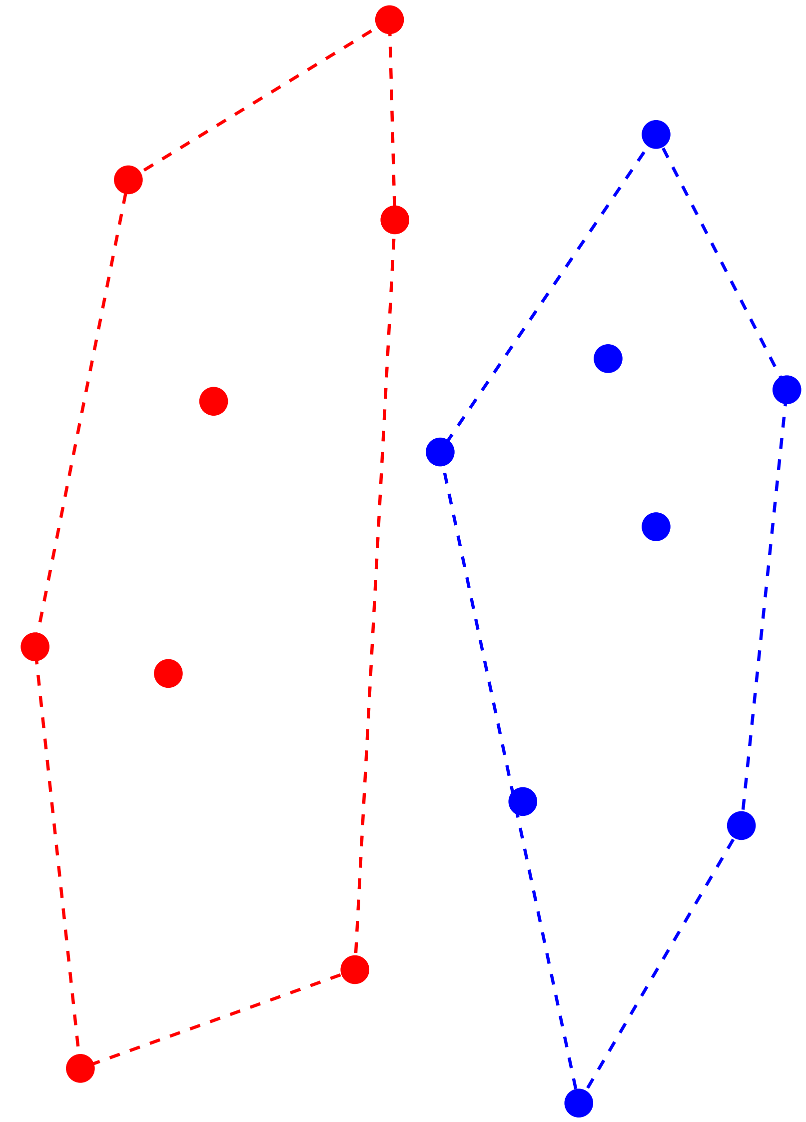

Constructing Voronoi diagrams

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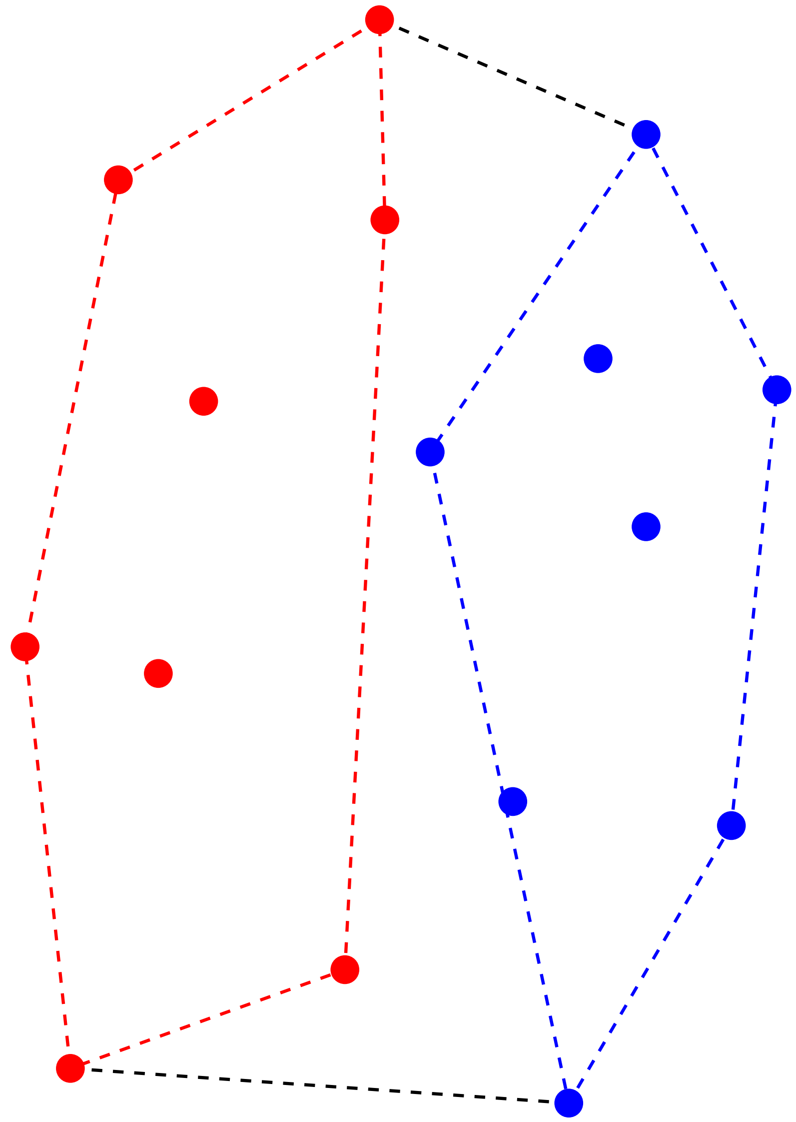

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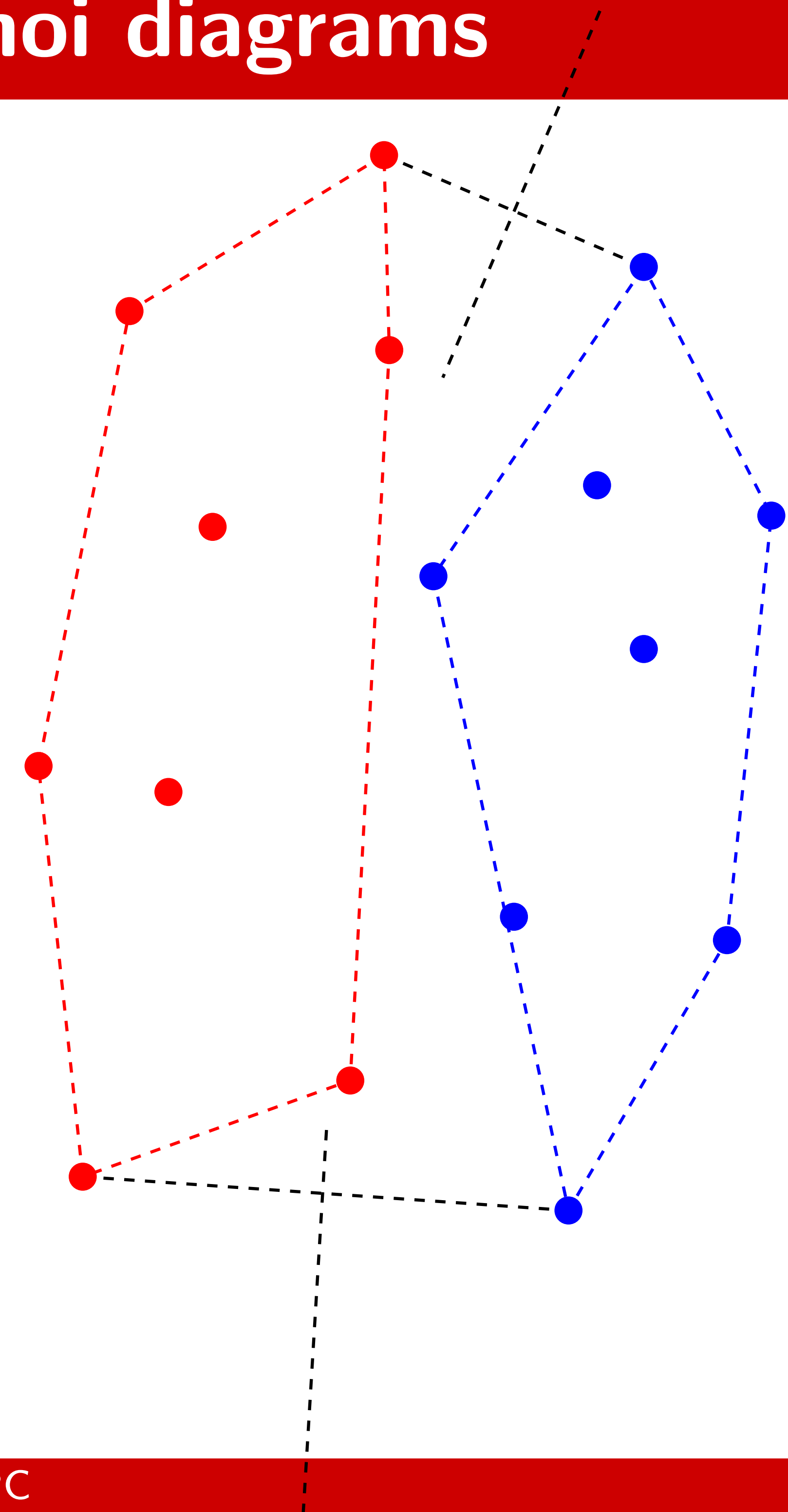

Constructing Voronoi diagrams

How to compute the chain?

Initialization

Find the two halflines

Advance

Starting with one of the halflines,
and until getting to the other one, do:

Each time an edge $e \in b(R, B)$ begins, such that $e \subset b_{ij}$, $p_i \in R$ and $p_j \in B$, do:

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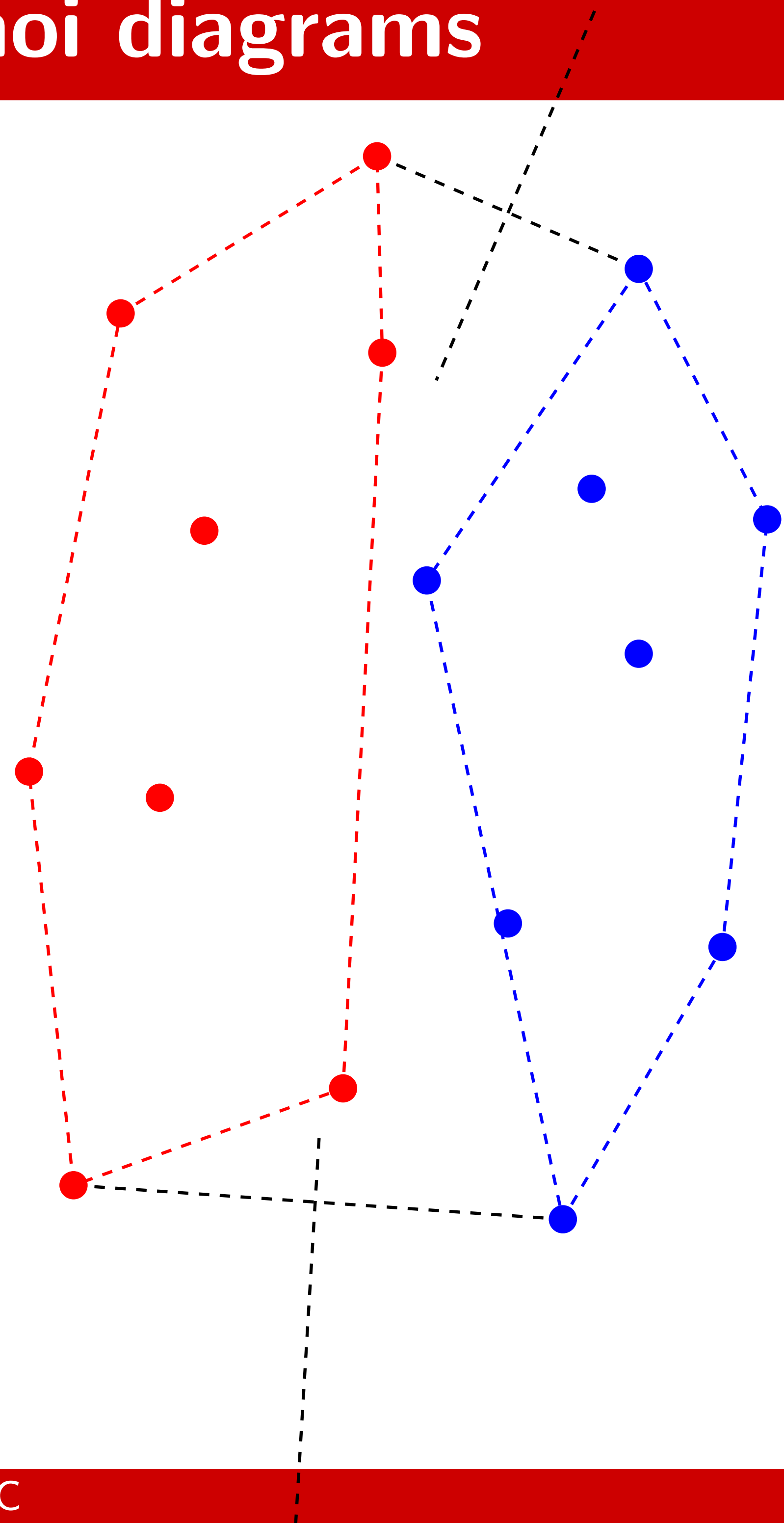

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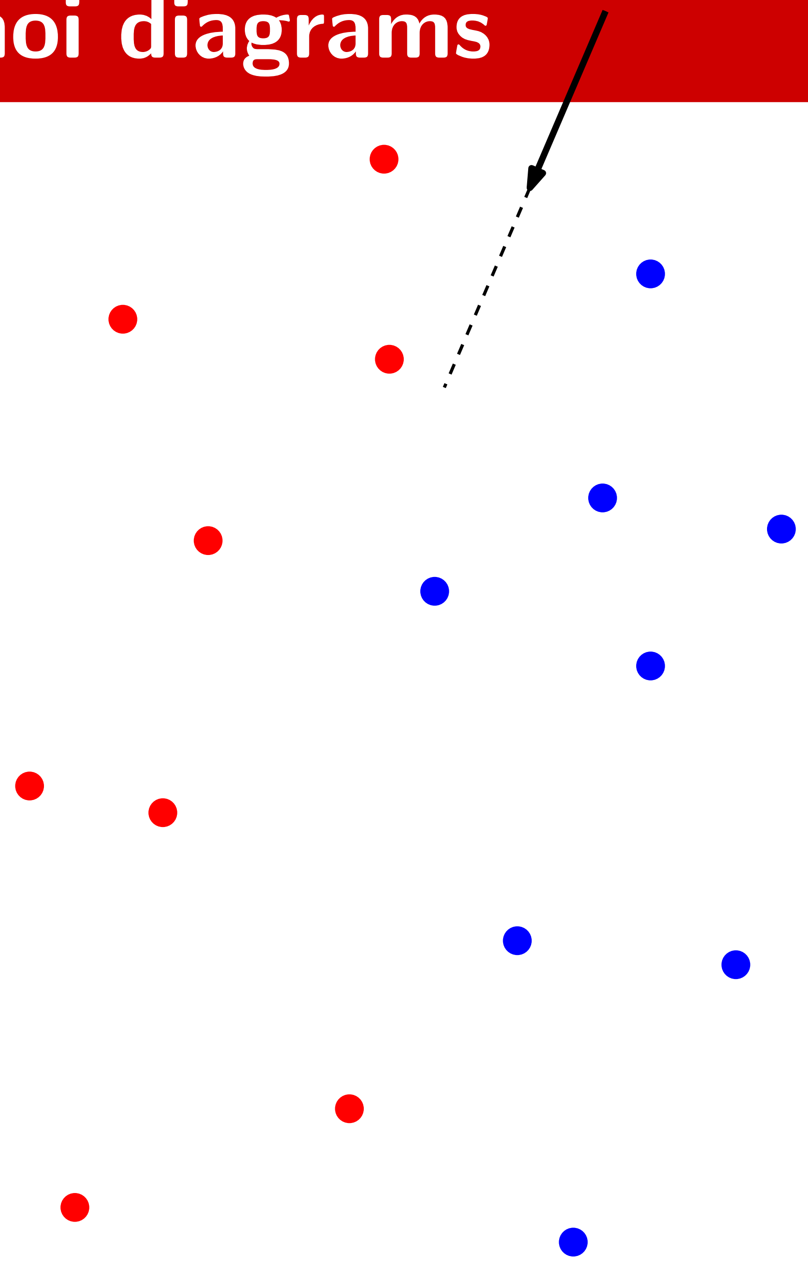

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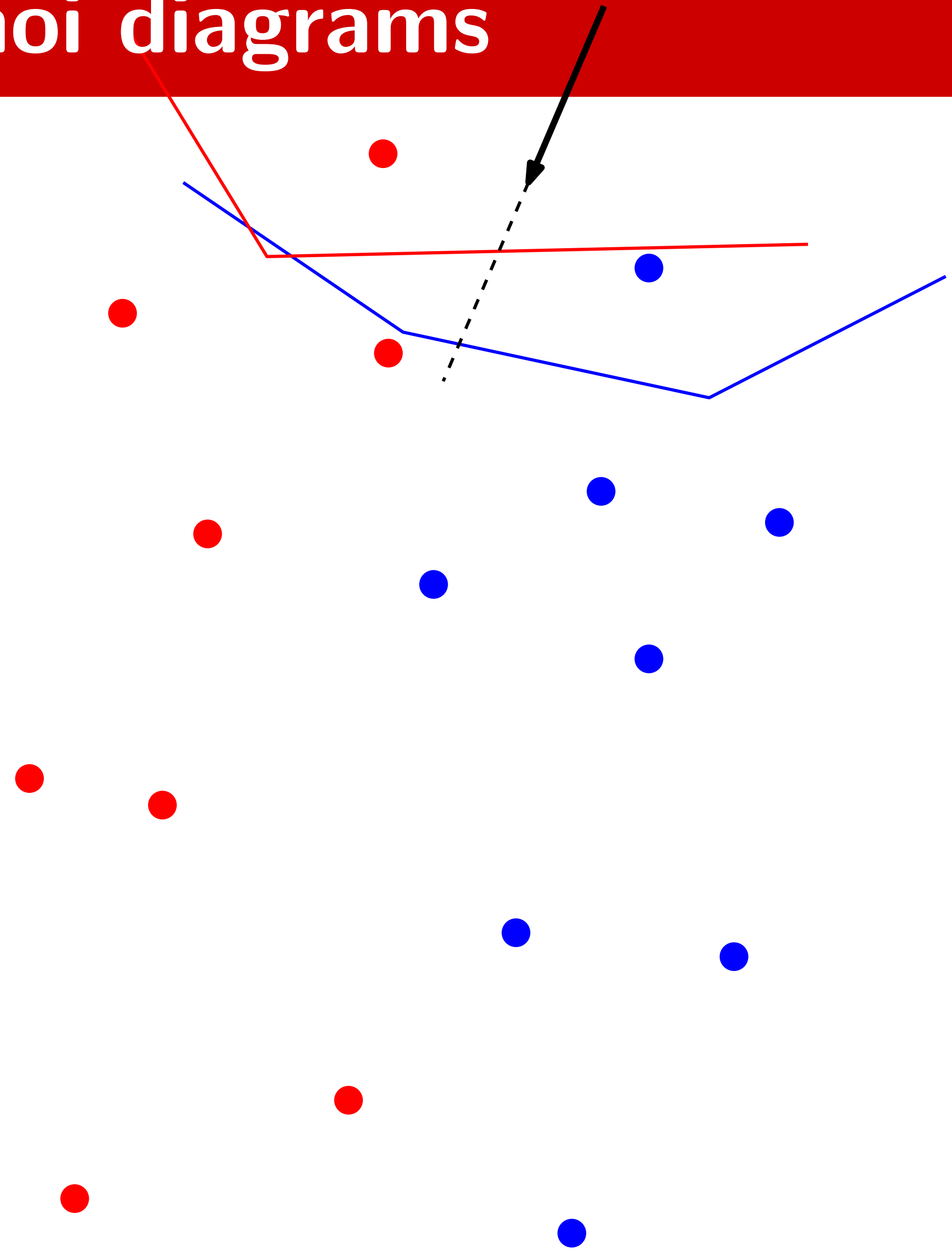

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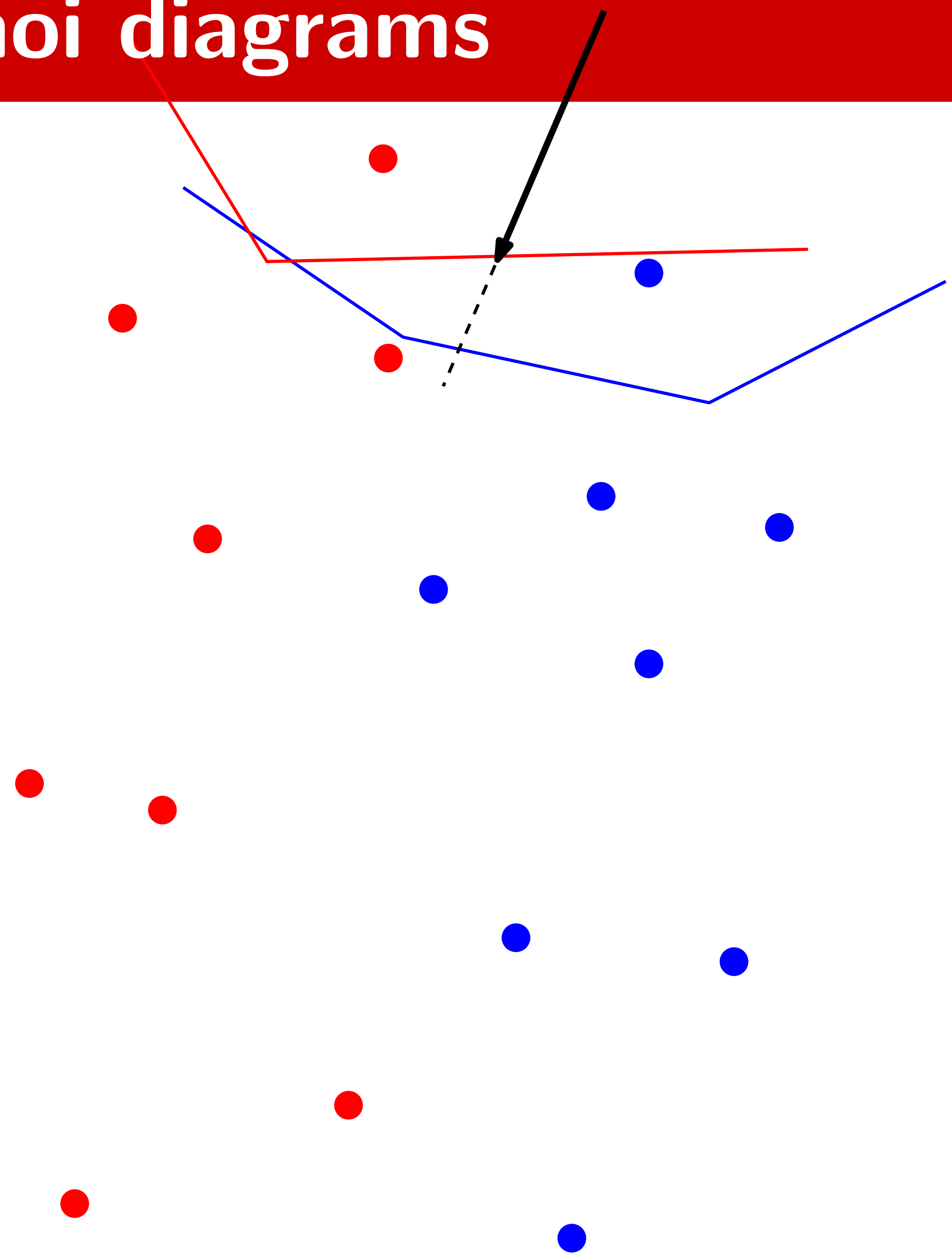

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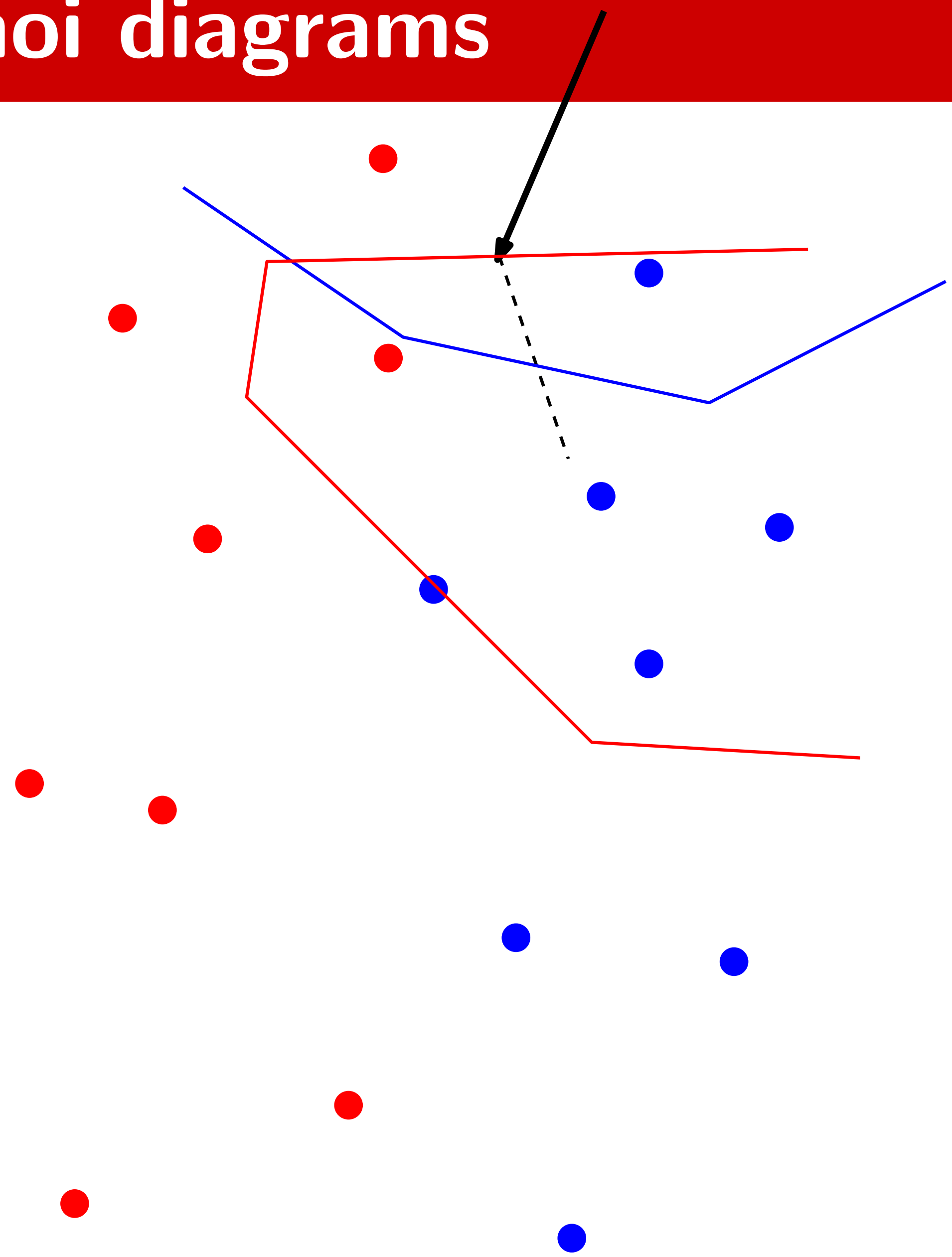

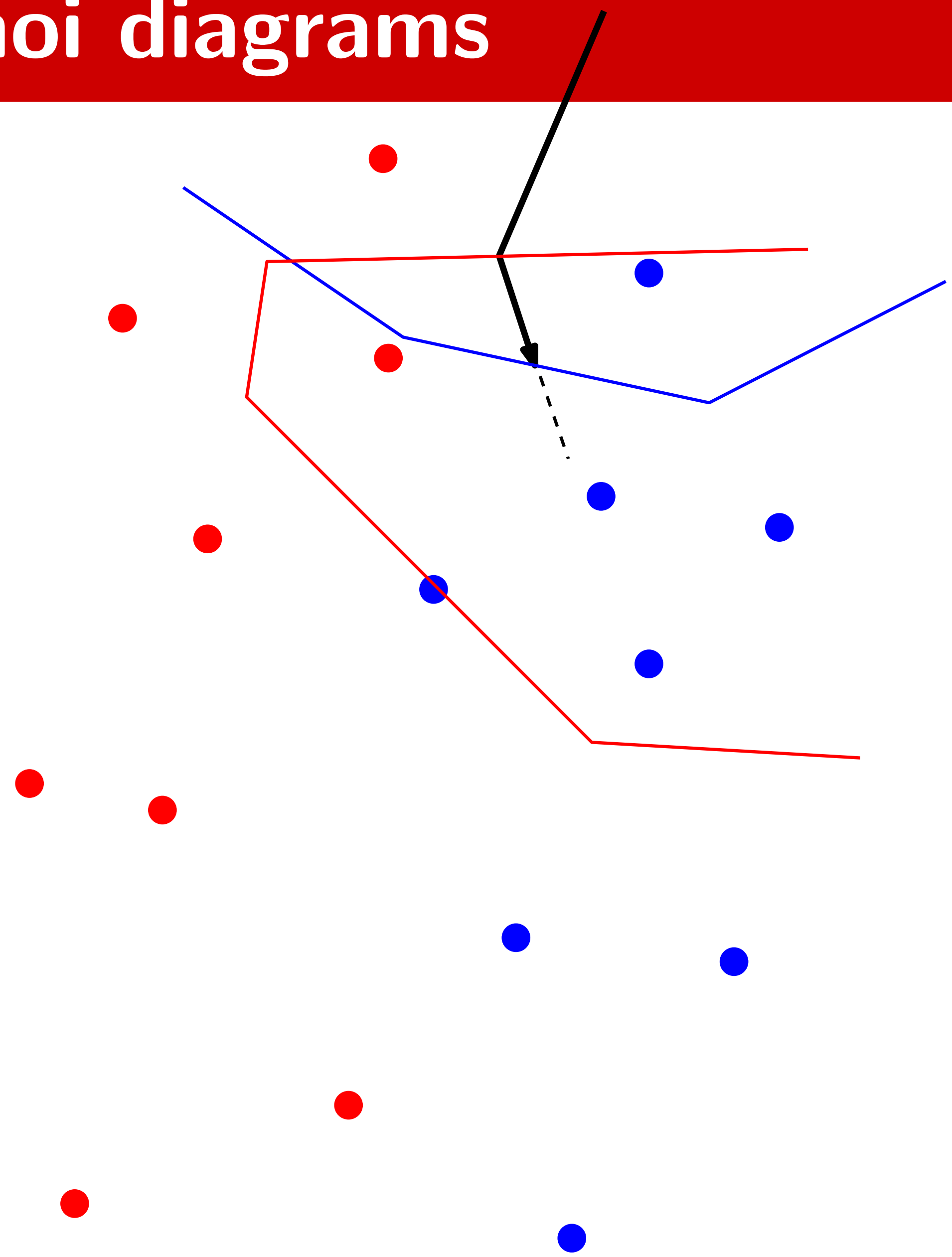

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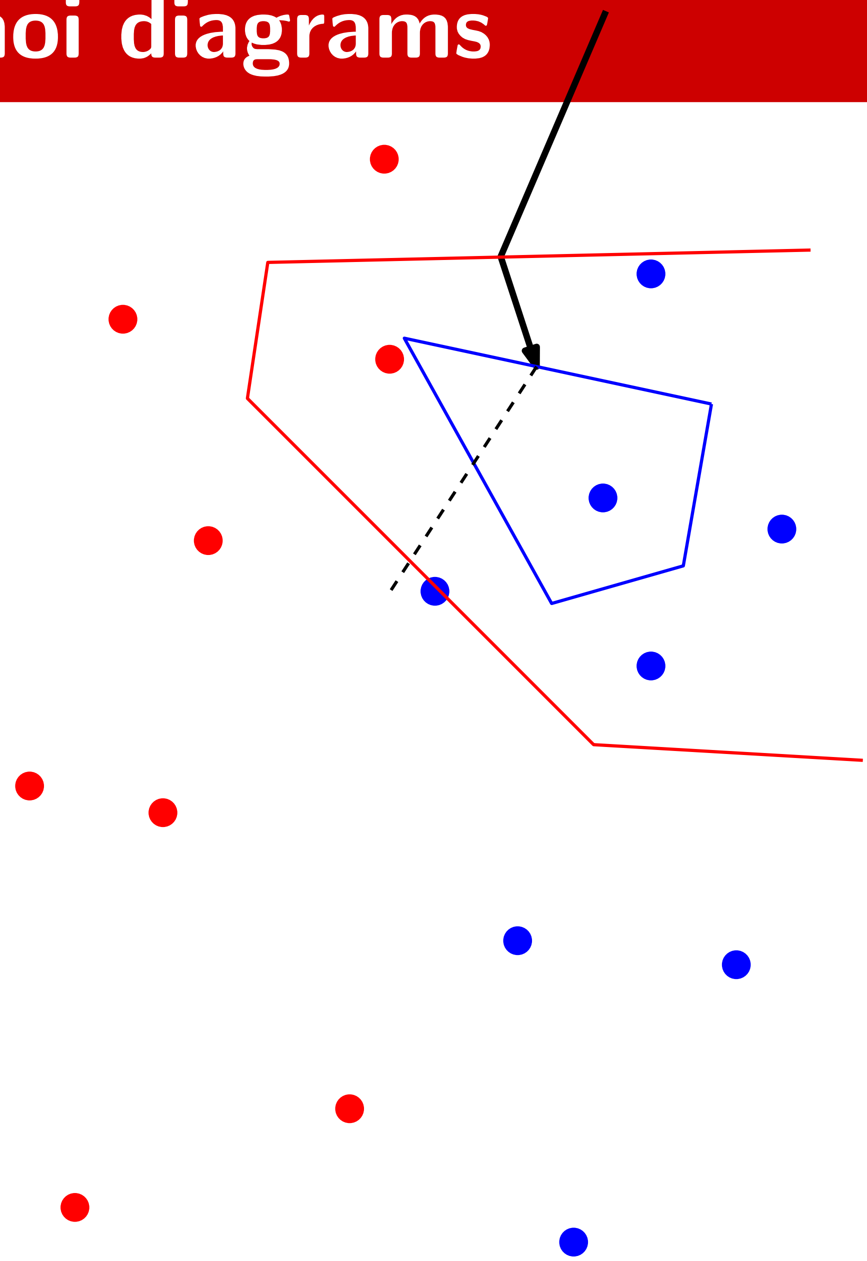

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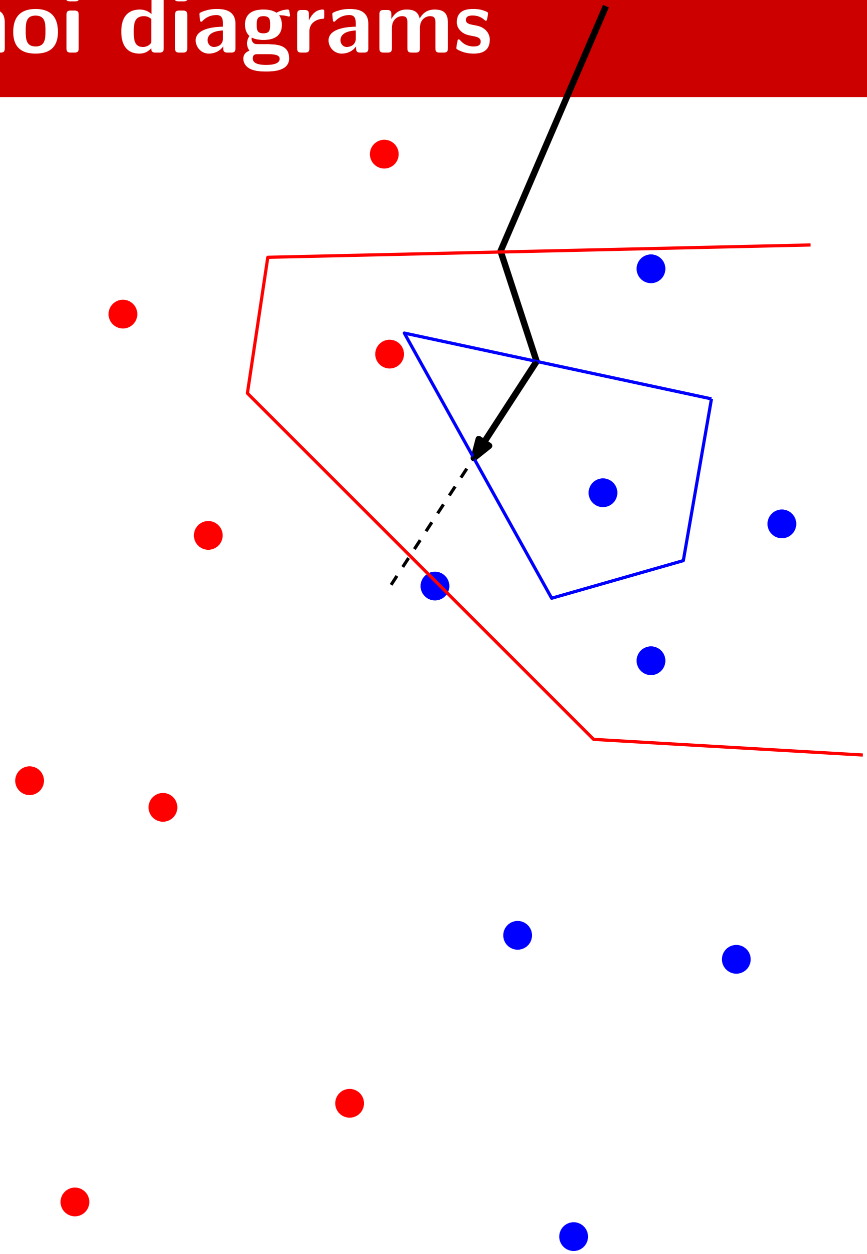

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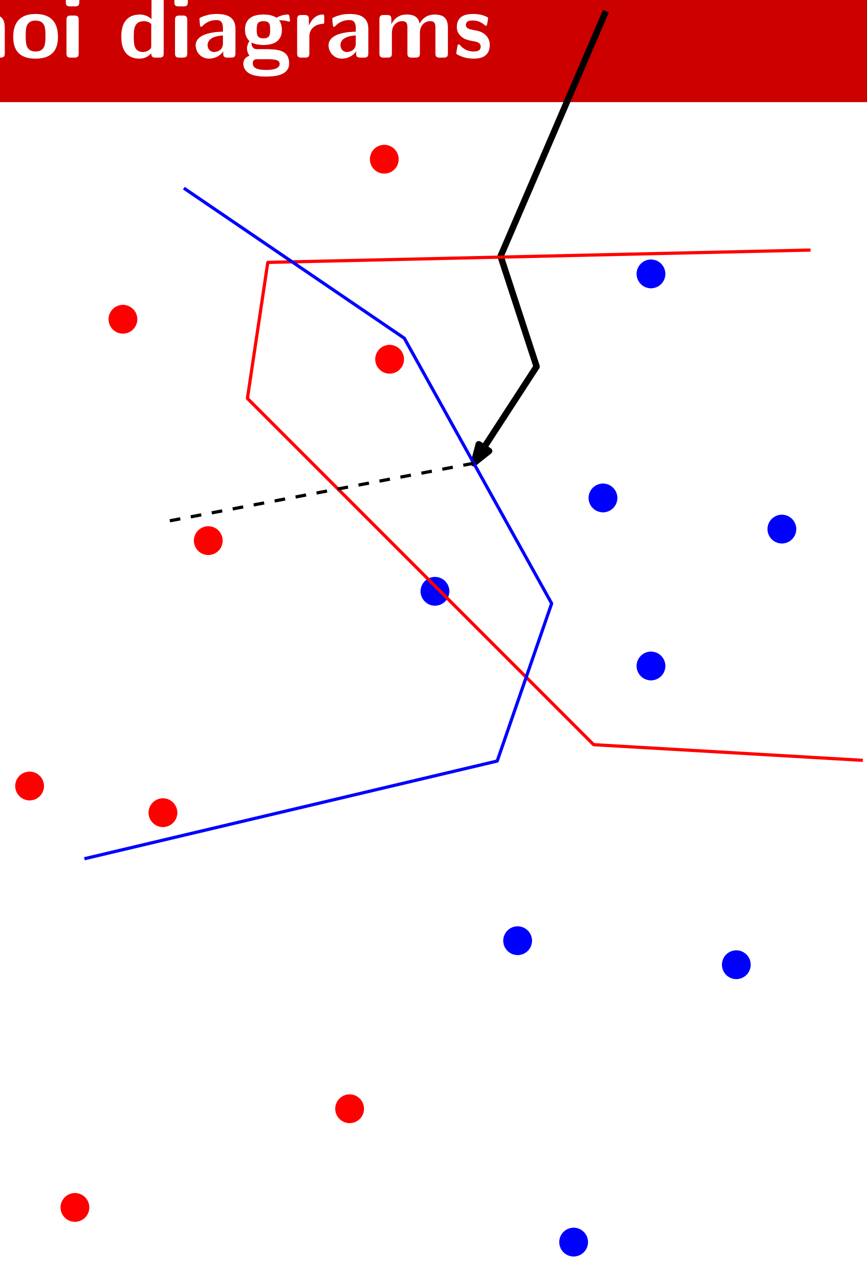

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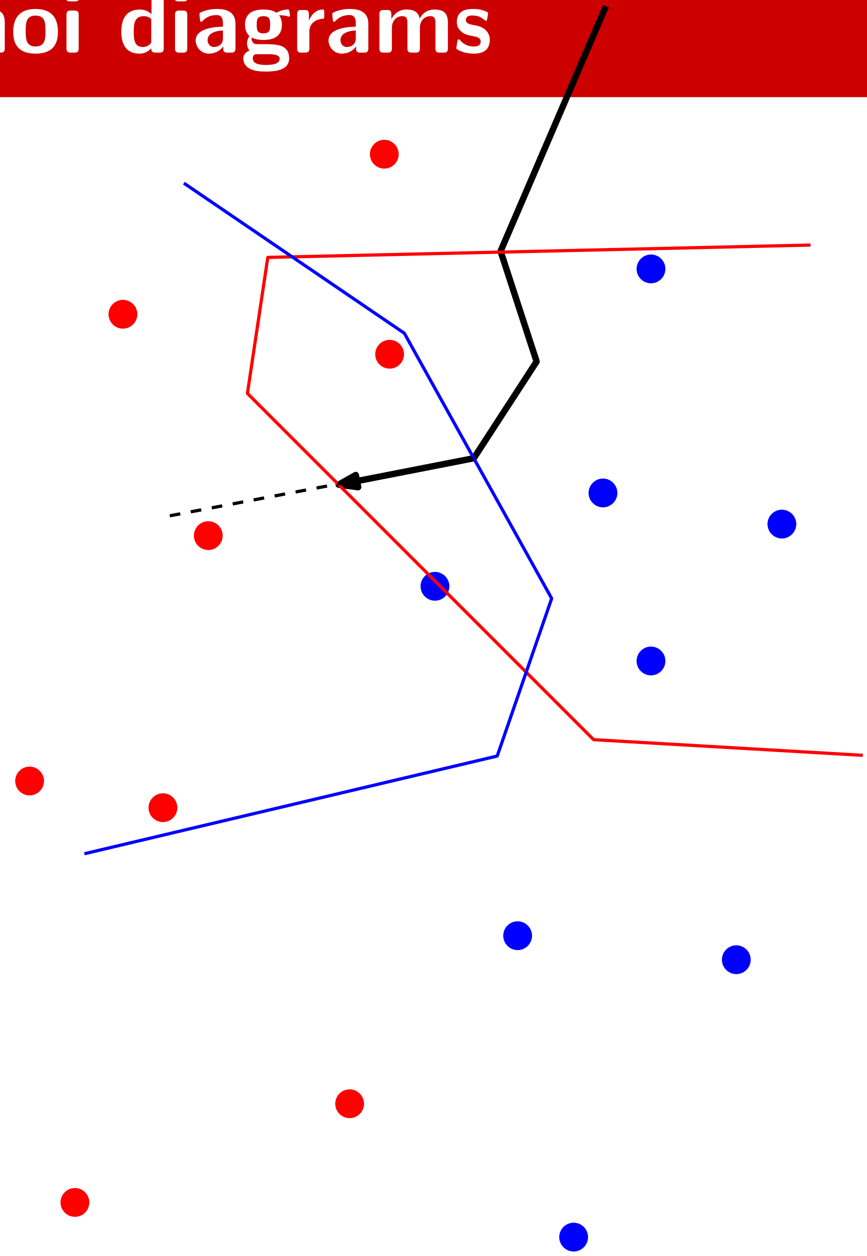

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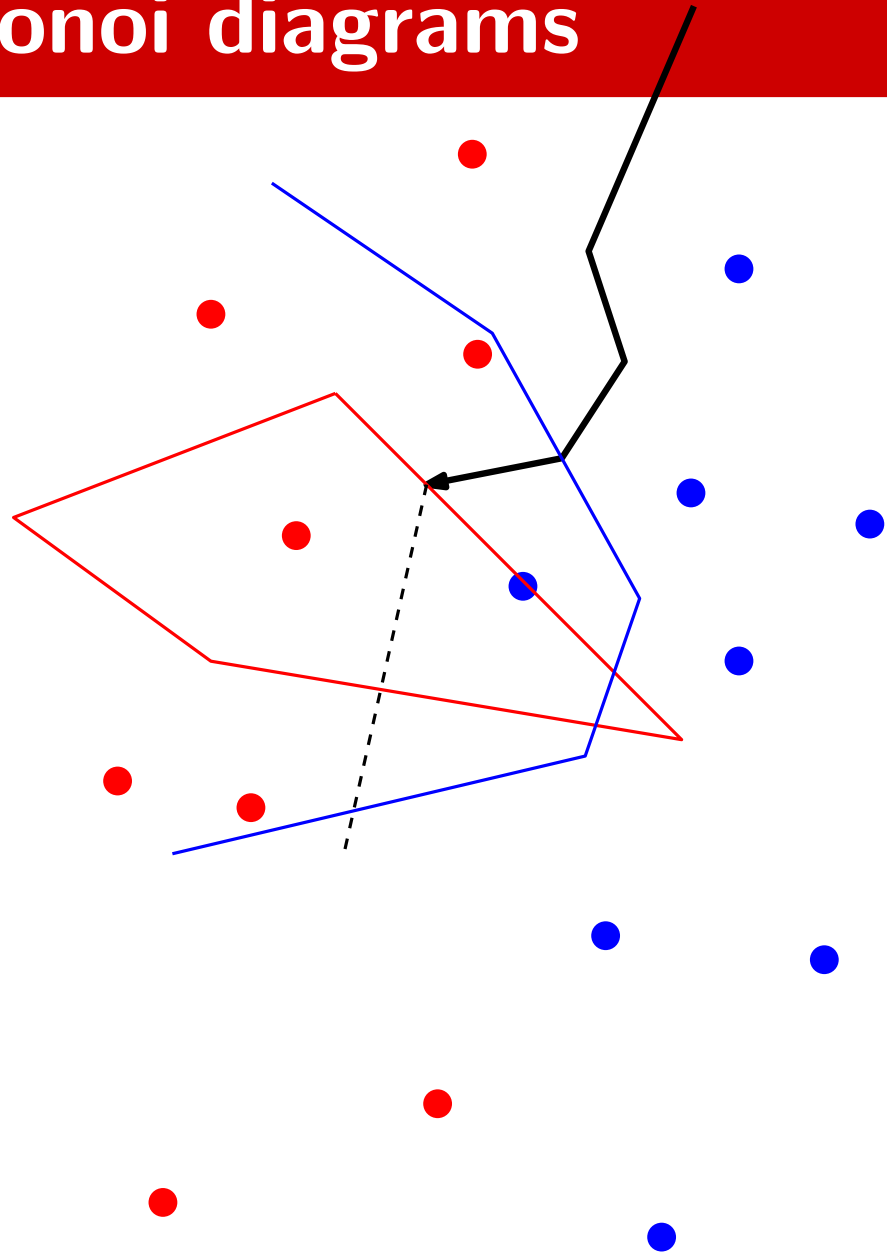

Constructing Voronoi diagrams

How to compute the chain?

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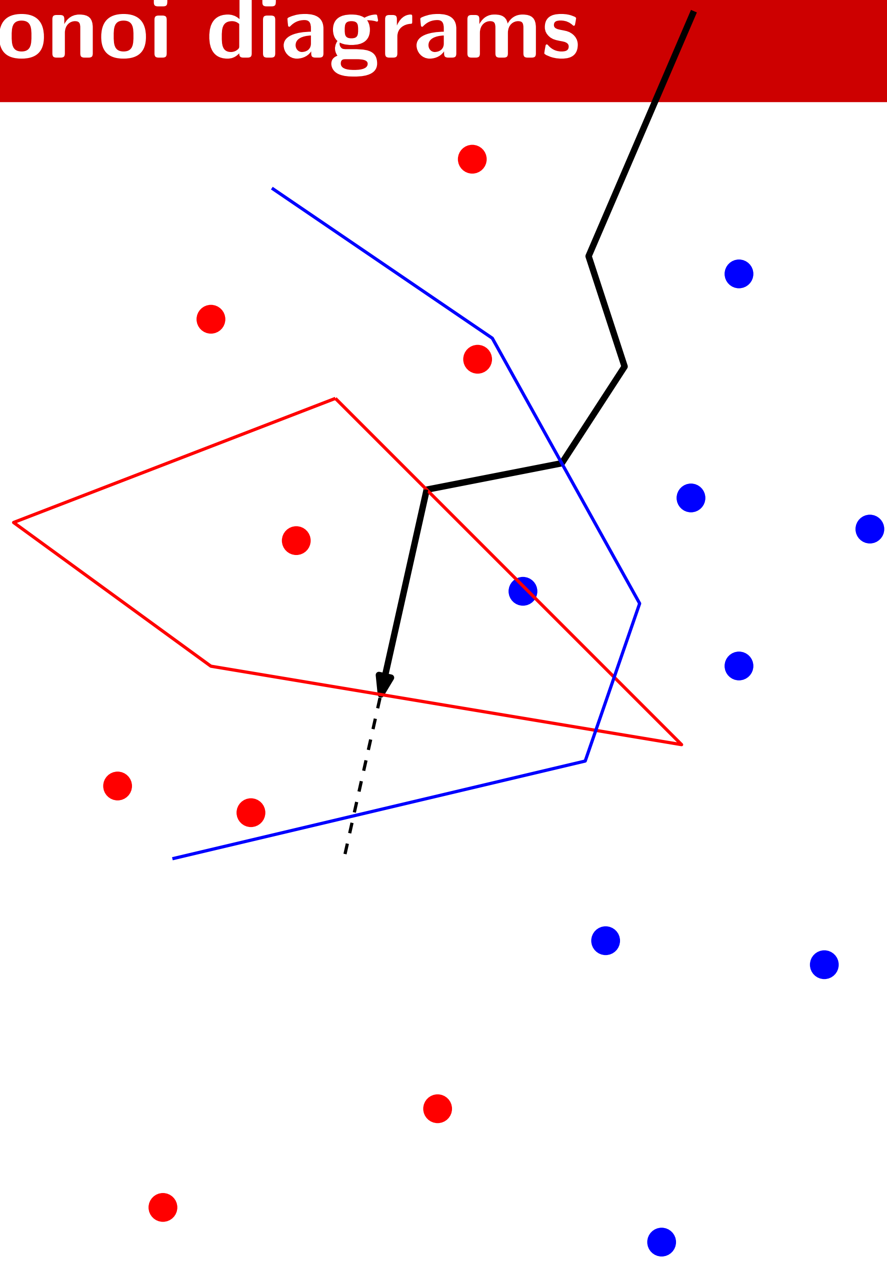

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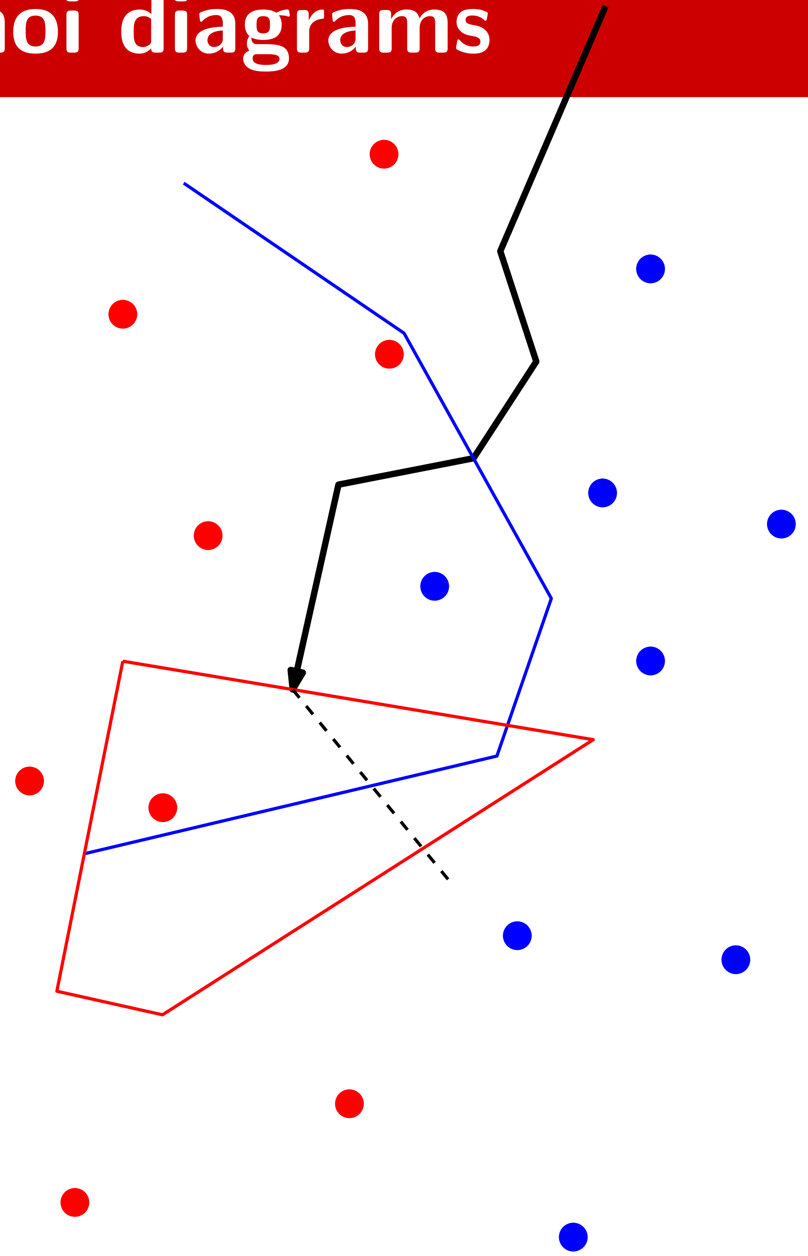

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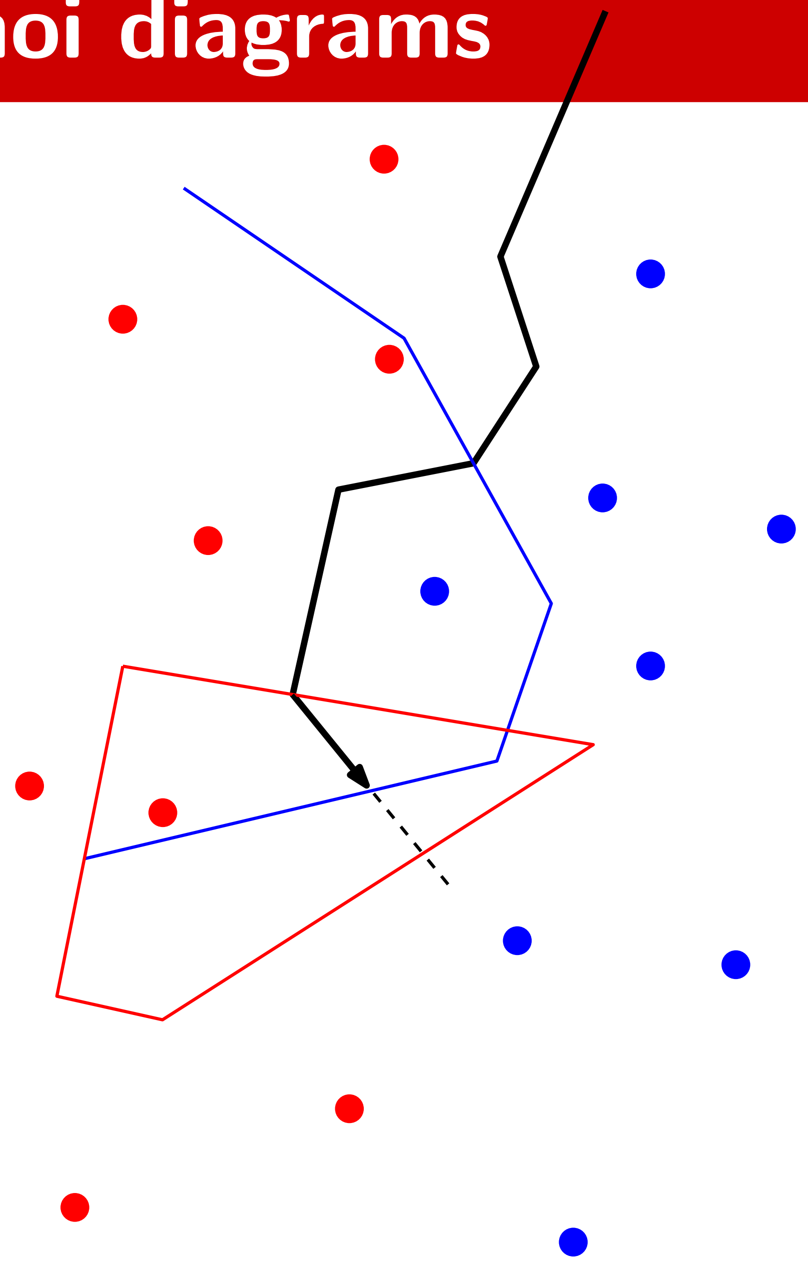

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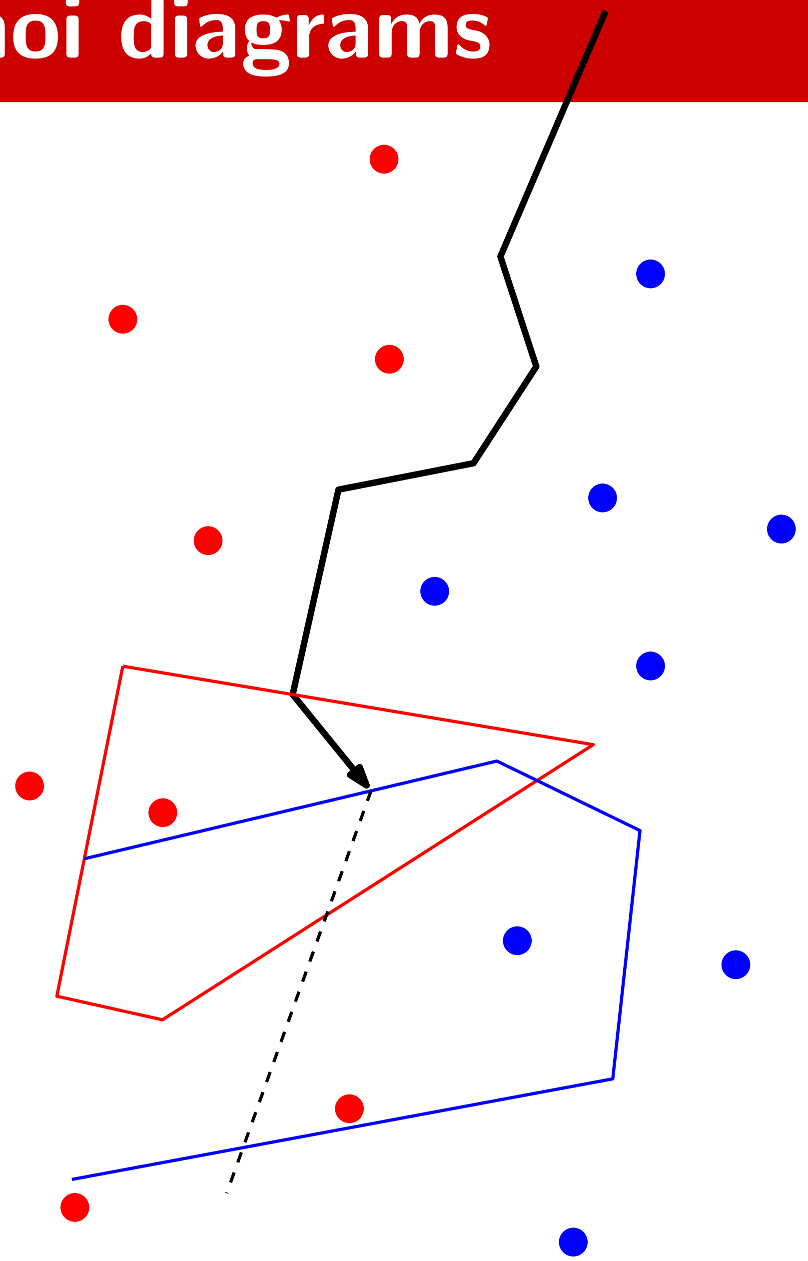

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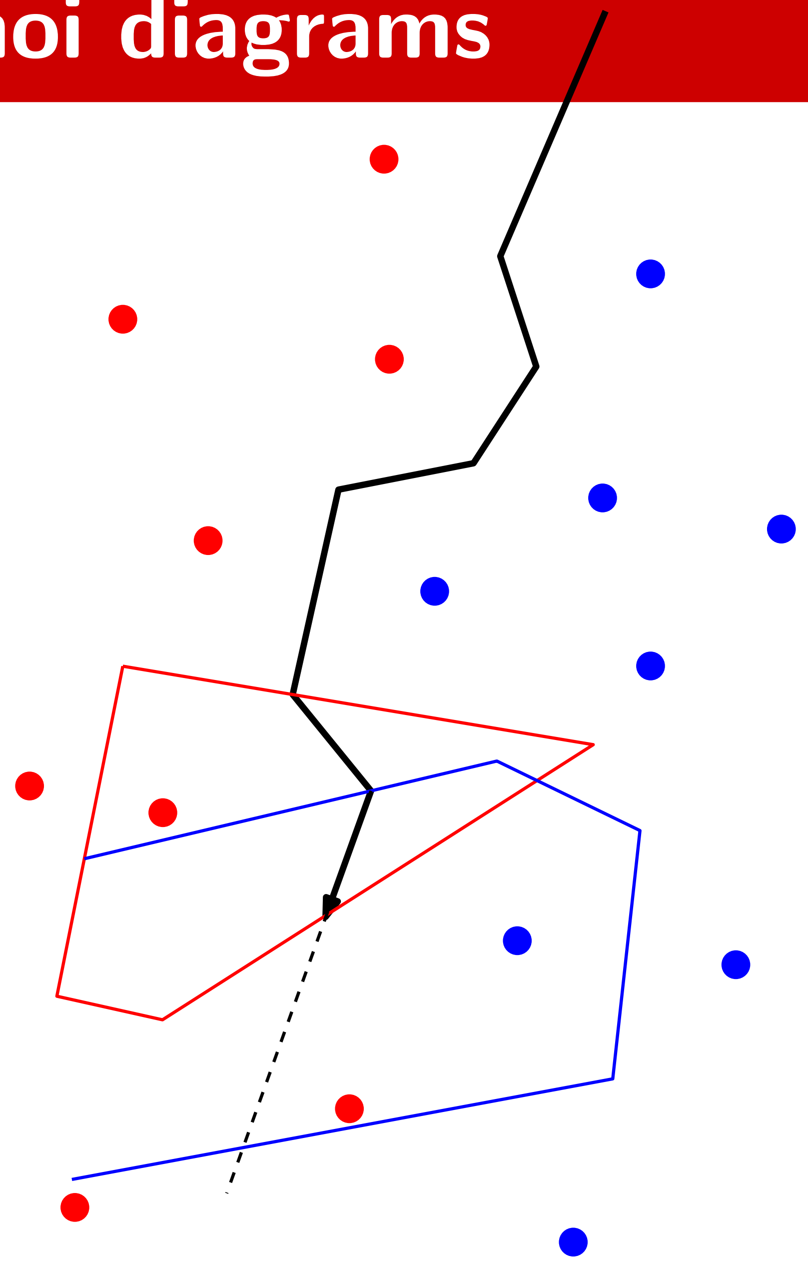

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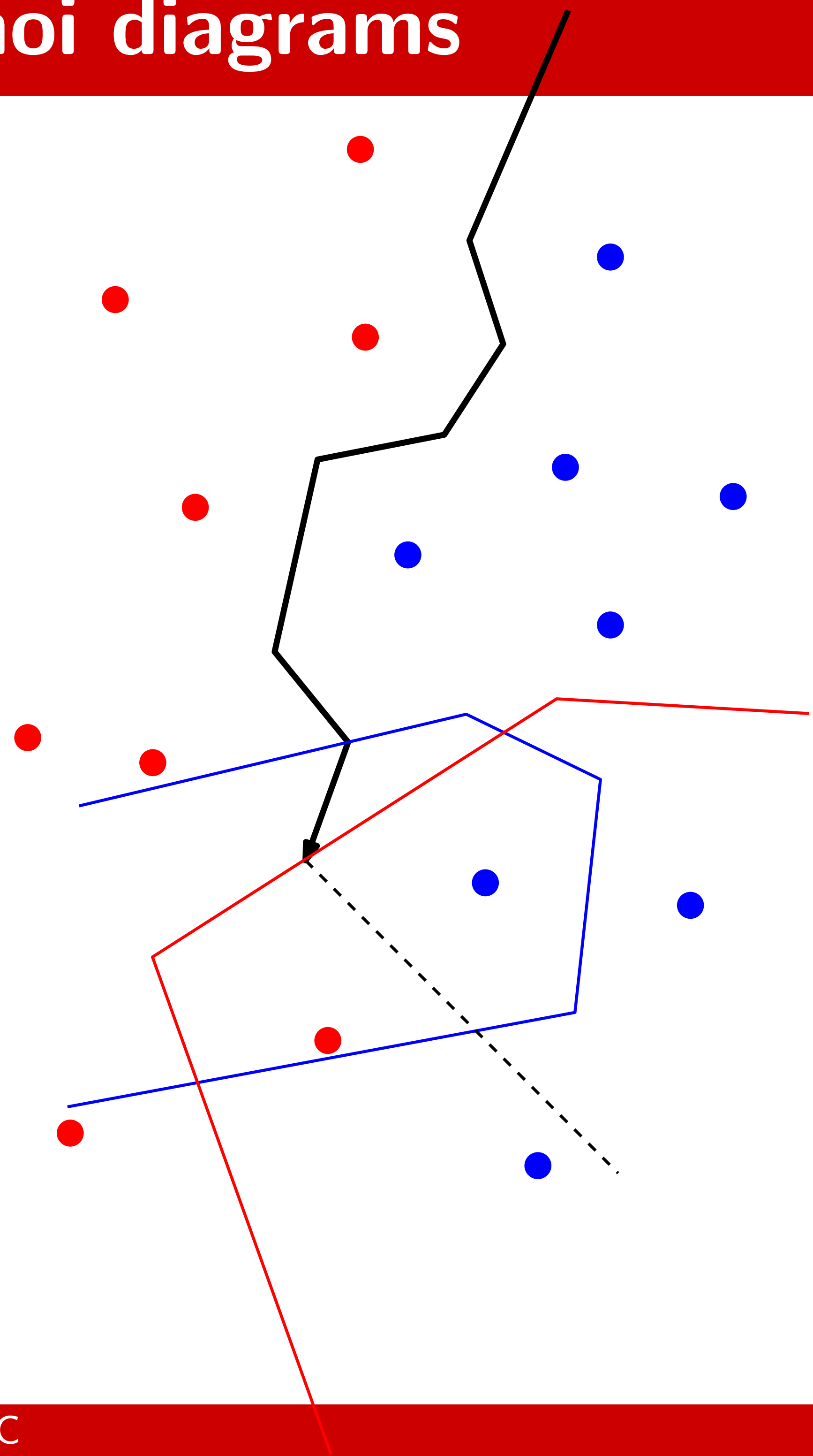

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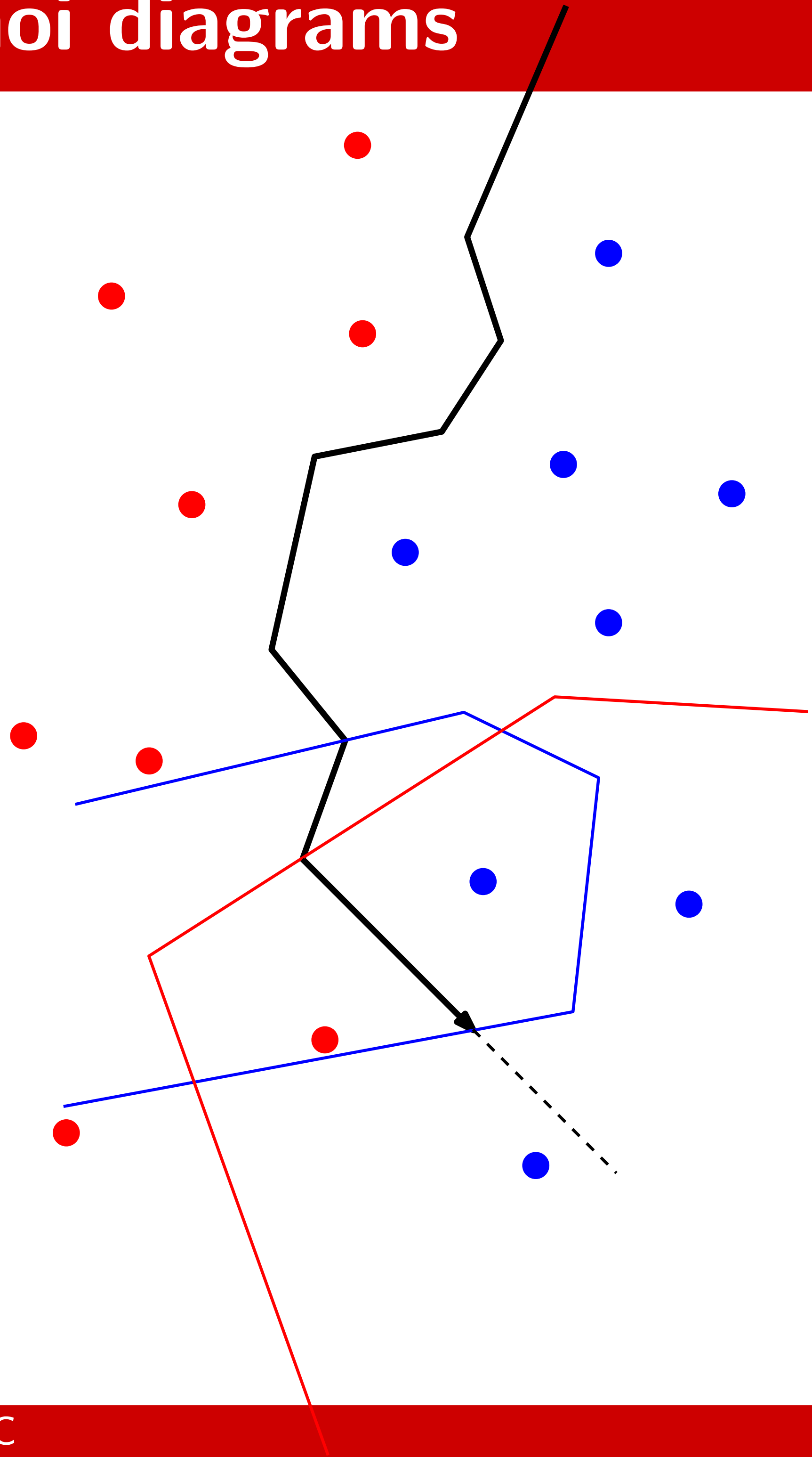

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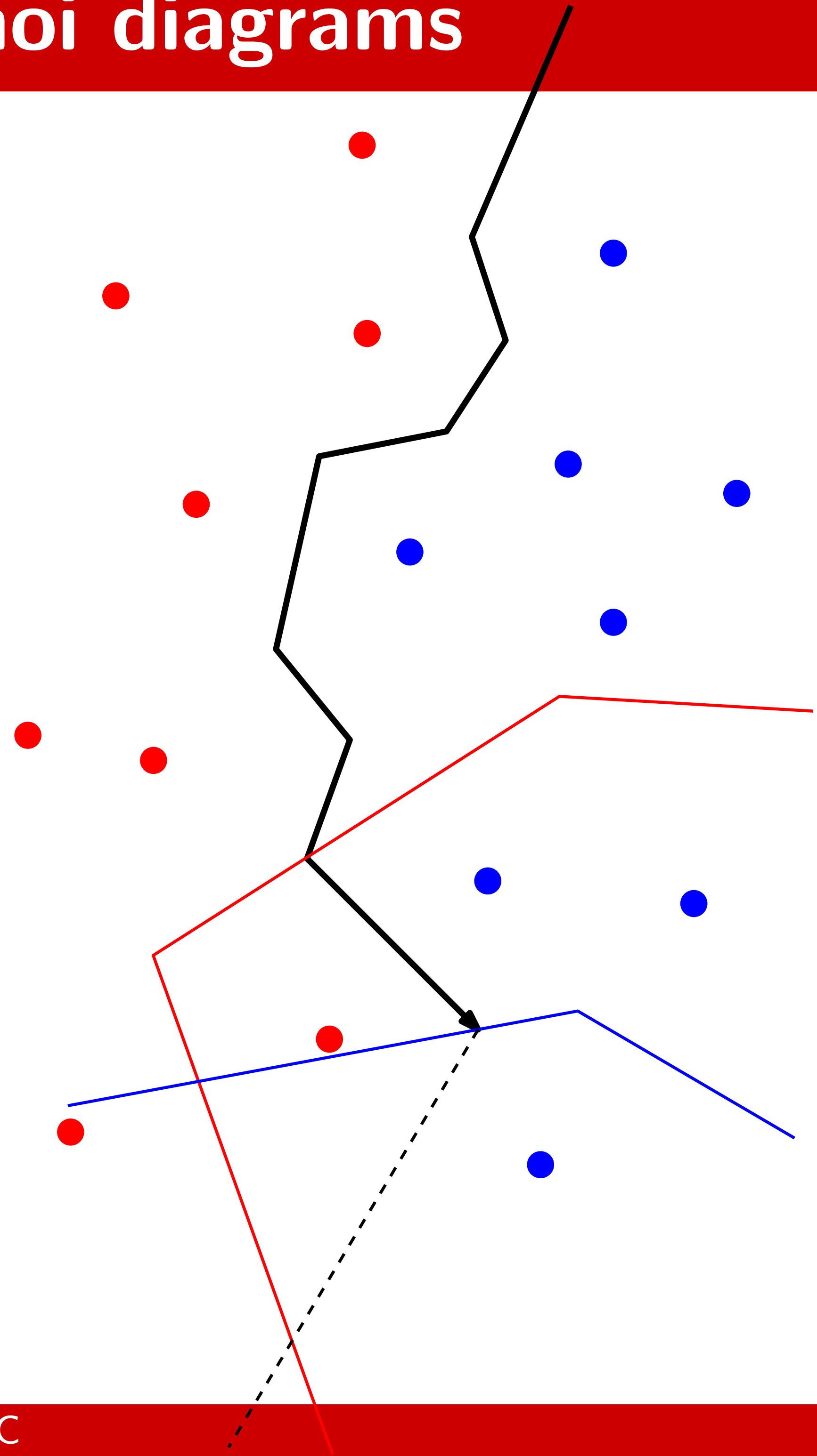

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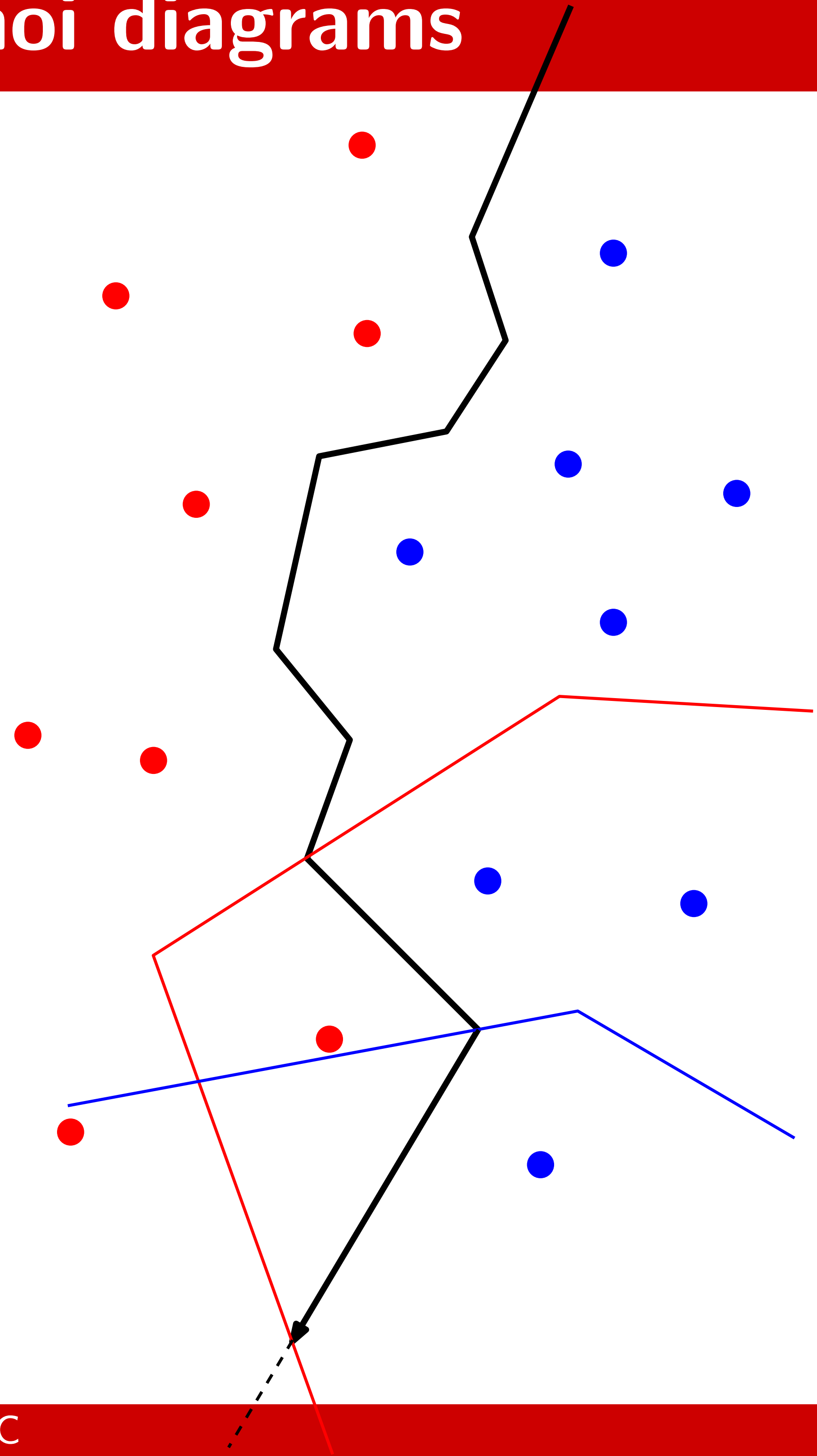

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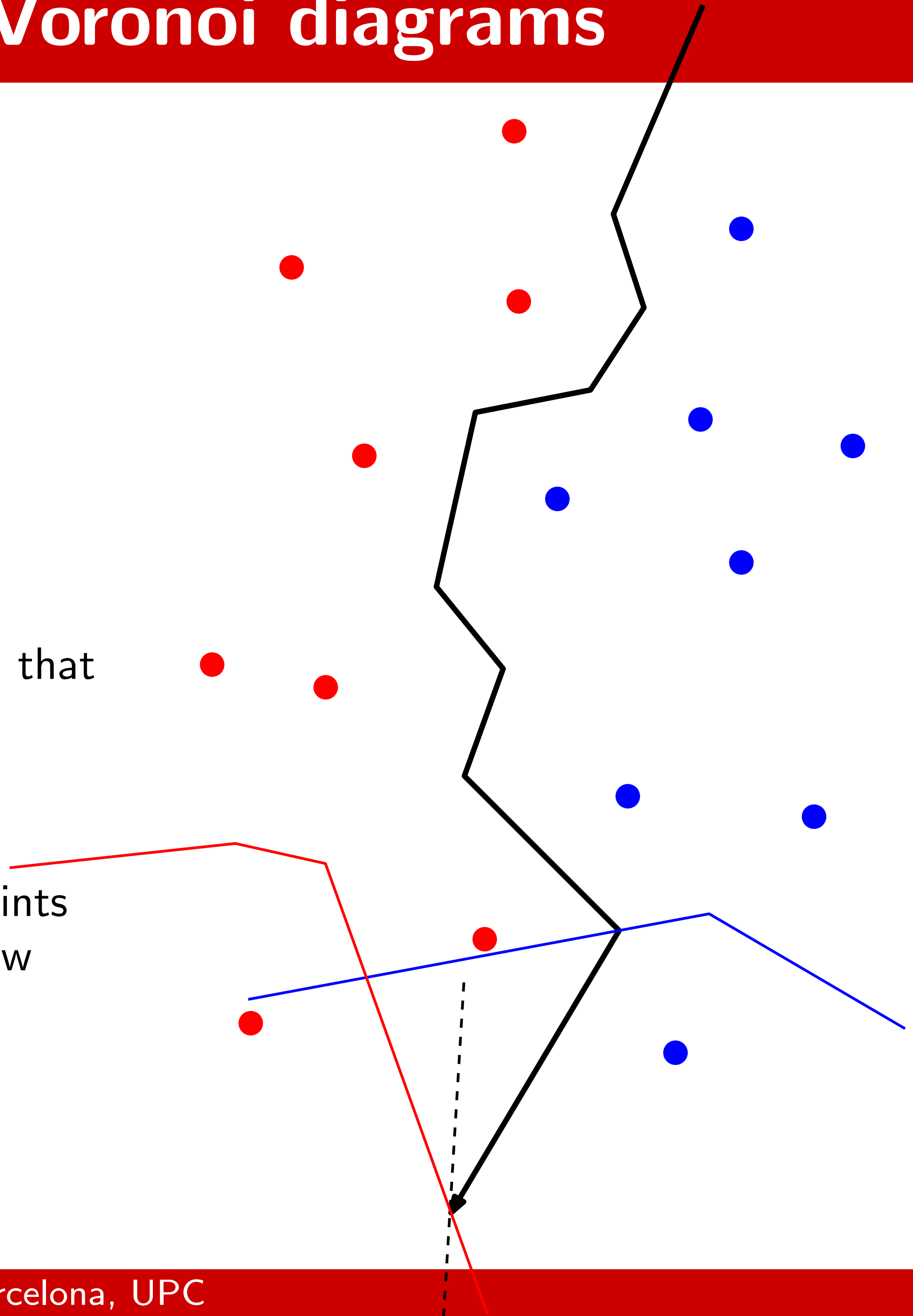

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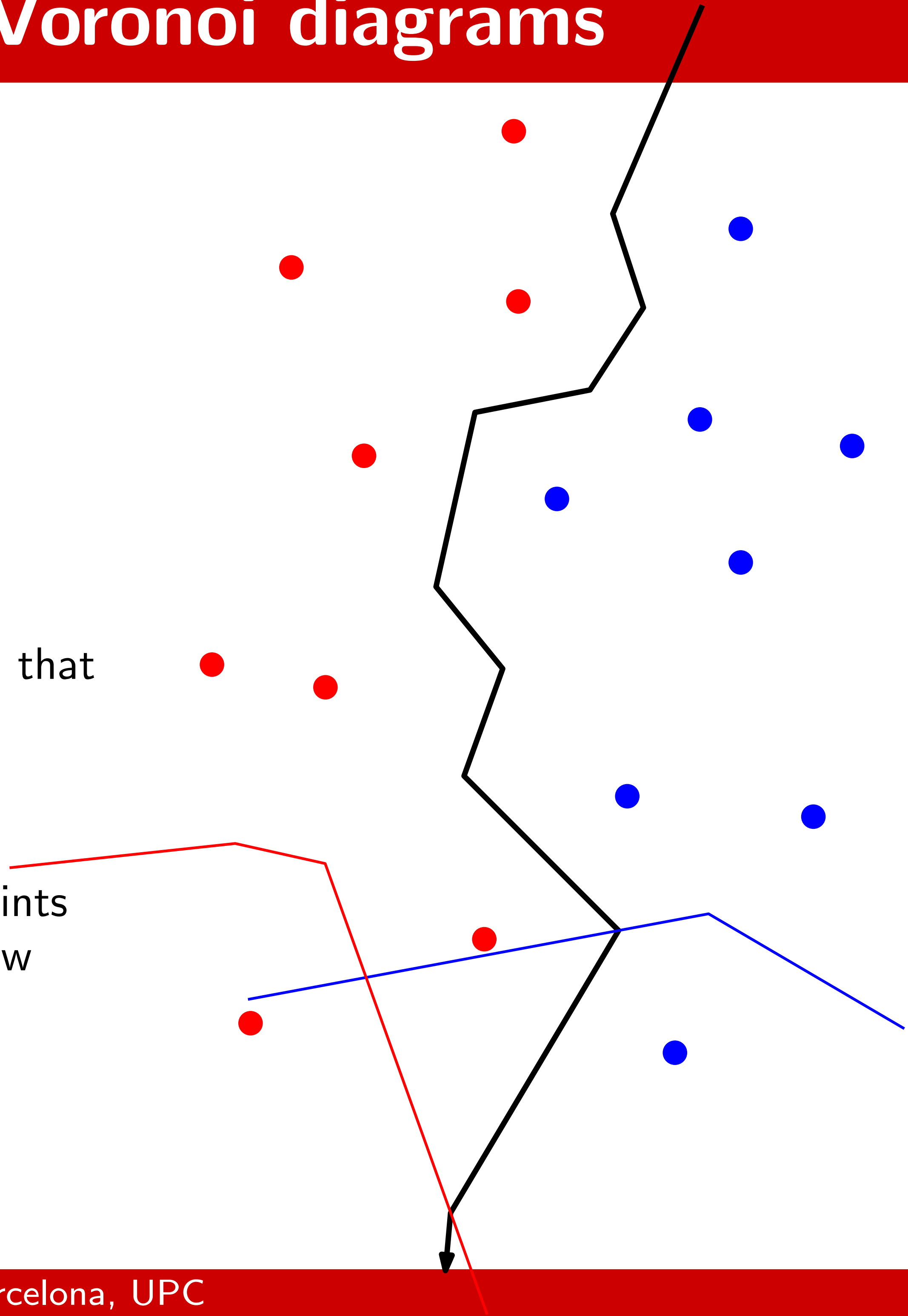

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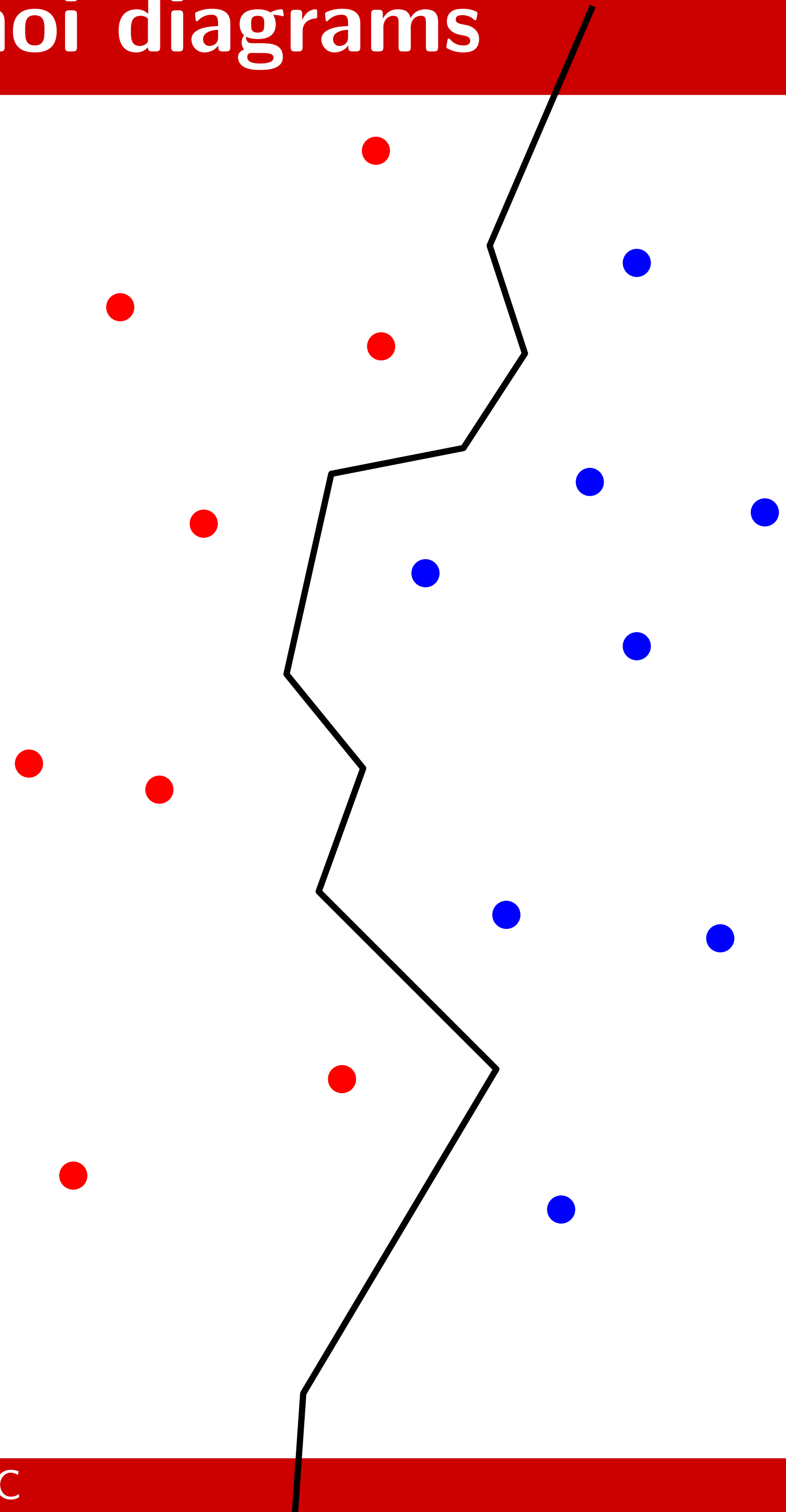
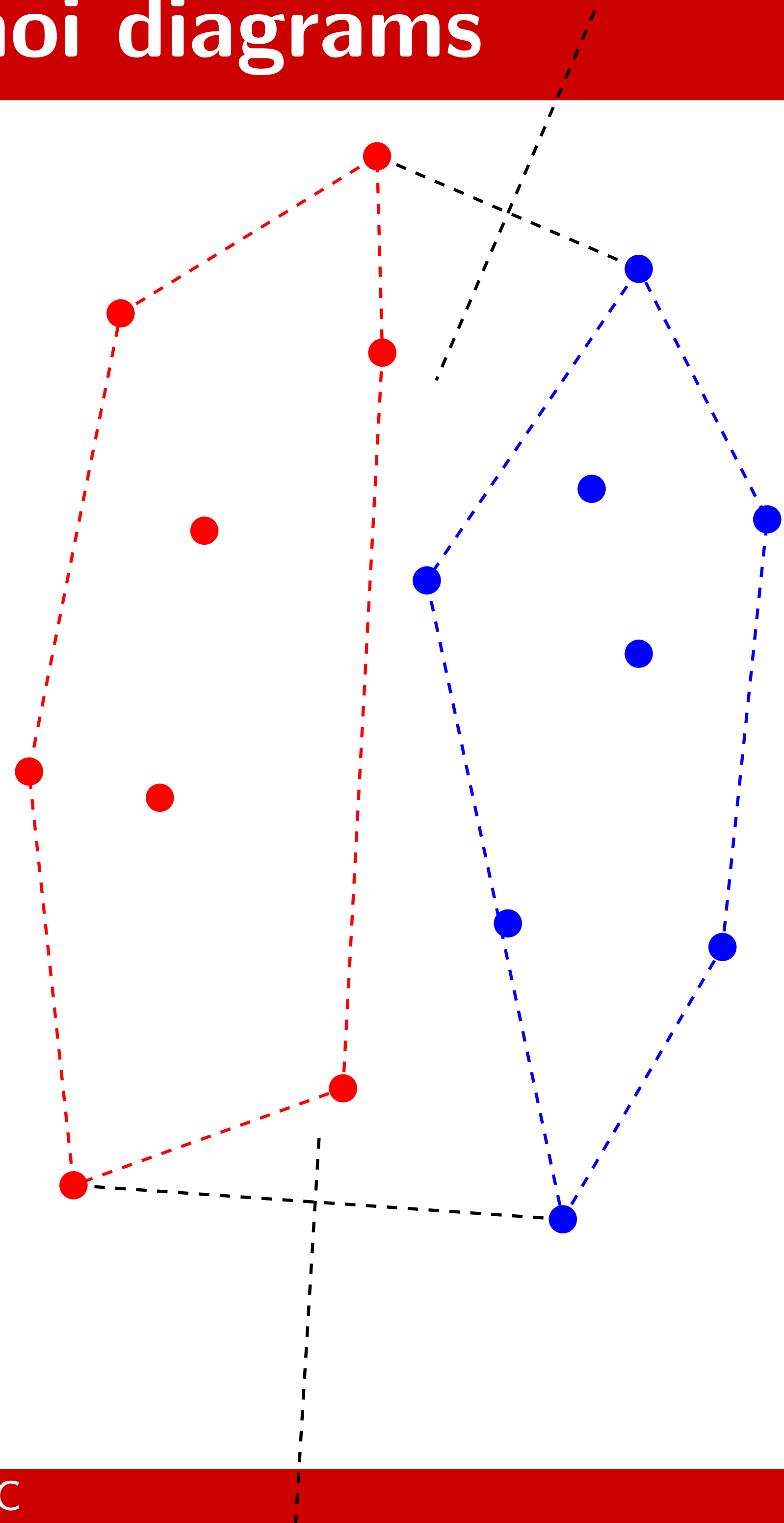


Constructing Voronoi diagrams

How to compute the chain?

Initialization running time: $O(n)$

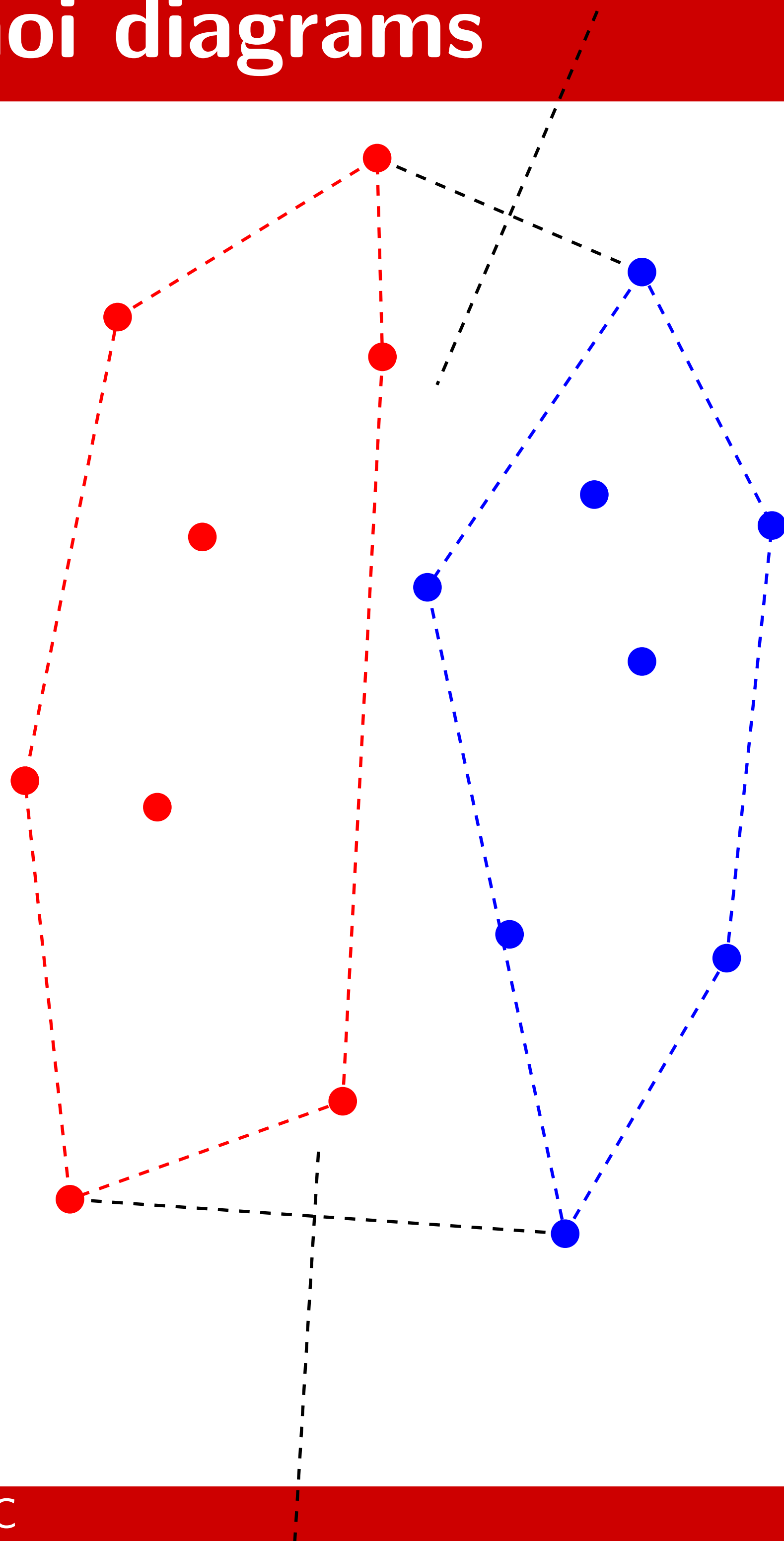


Constructing Voronoi diagrams

How to compute the chain?

Initialization running time: $O(n)$

From $Vor(R)$ and $Vor(B)$.



Constructing Voronoi diagrams

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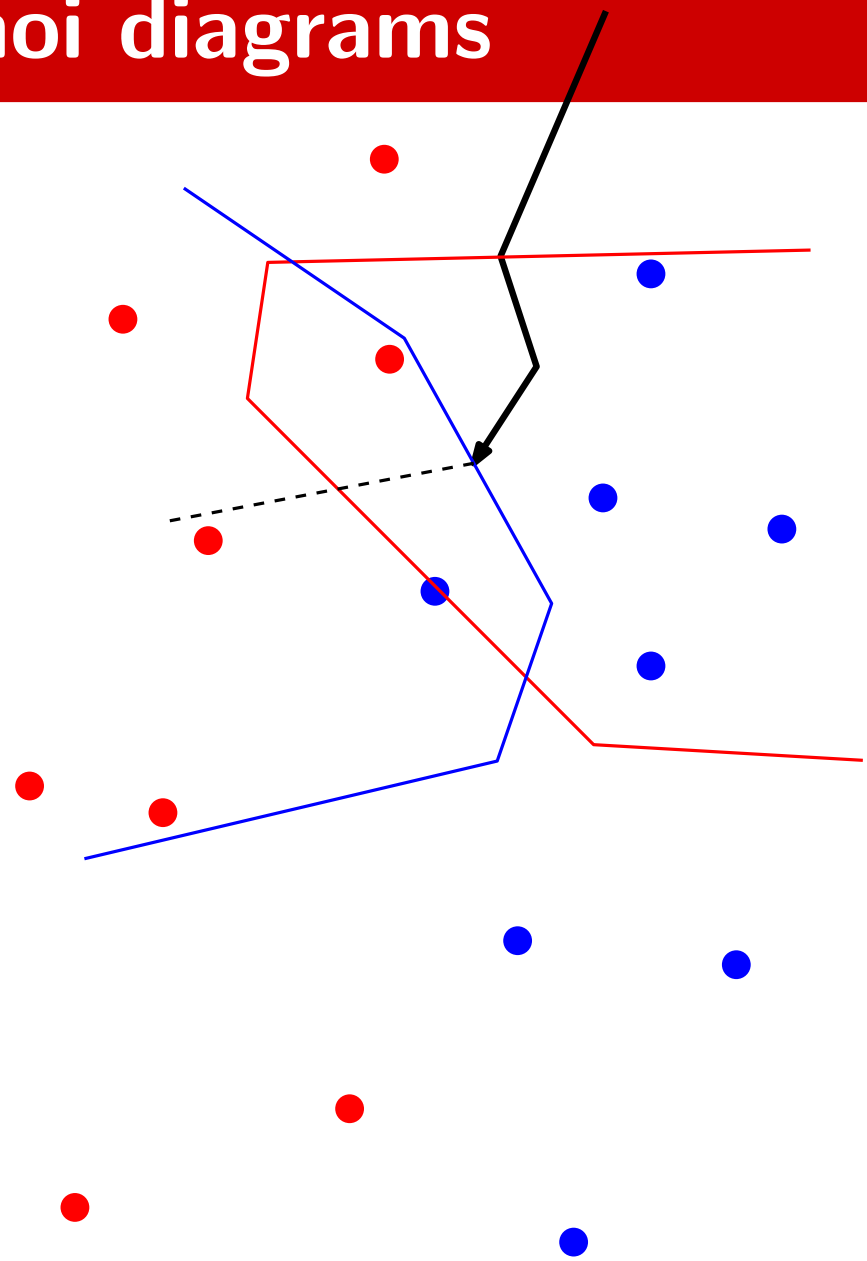

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Initialization running time: $O(n)$

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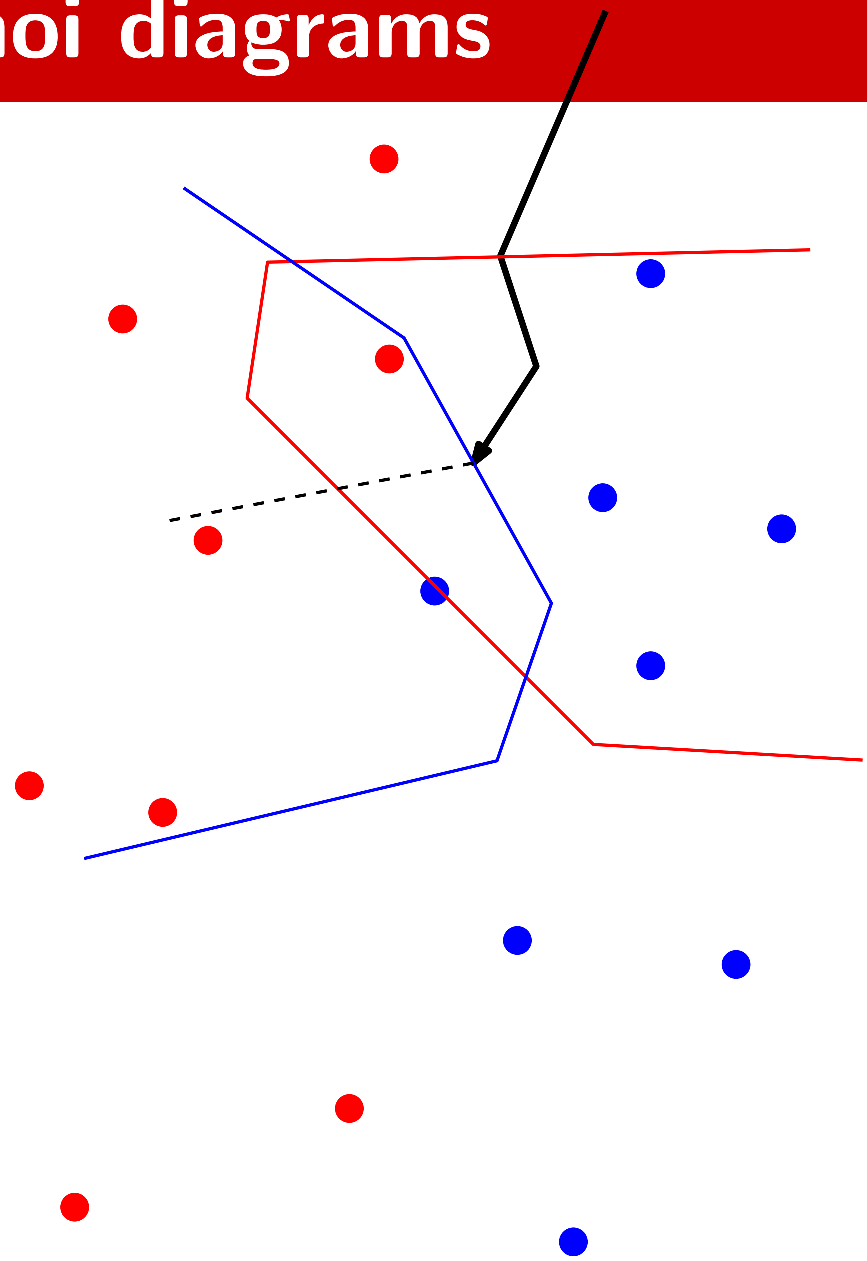
If e is an edge of $b(R, B)$ that entered $Vor_R(p_i)$ through some vertex $v \in Vor(P)$, then the exit point of $b(R, B)$ is found clockwise along the boundary of $Vor_R(p_i)$.



Constructing Voronoi diagrams

How to do the merging?

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It consists in updating the DCEL:

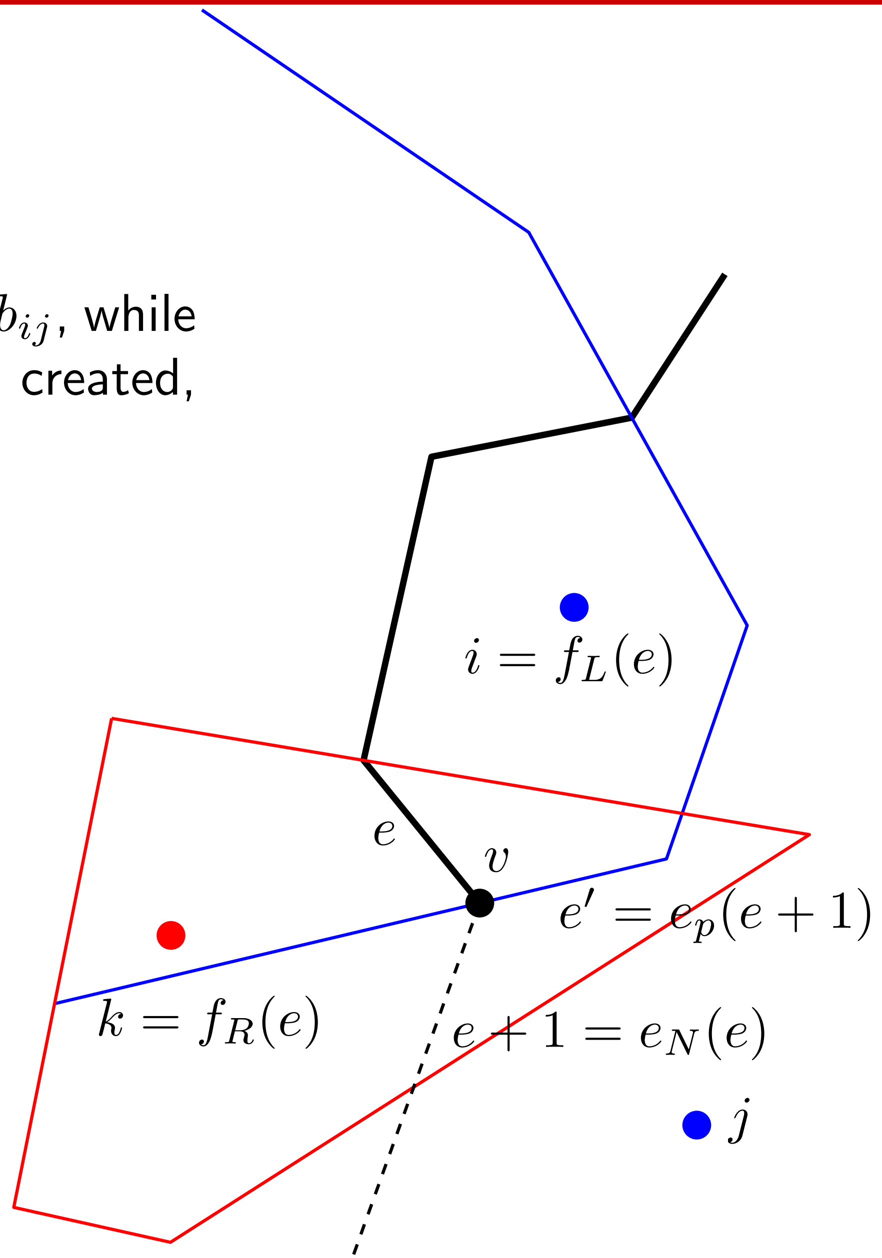
Constructing Voronoi diagrams

How to do the merging?

It consists in updating the DCEL:

Each time a face $Vor_B(p_i)$ is left through an edge $e' \in b_{ij}$, while staying in the same face $Vor_R(p_k)$, a new vertex v is created, an edge e ends and another edge $e+1$ begins:

- Create the new vertex v and assign $e(v) = e$
- Create $e+1$ and assign to it $v_B = v$ and $e_P = e'$
- Assign to e: $v_E = v$, $e_N = e+1$, $f_L = i$ and $f_R = k$
- Delete all edges of $Vor_B(p_i)$ found in counter-clockwise order between the entry and exit points
- Update for e': $v_* = v$, $e_* = e$
- Update $e(p_i) = e$



The procedure is analogous when exiting a face $Vor_R(p_i)$.

Constructing Voronoi diagrams

DIVIDE AND CONQUER ALGORITHM

1. Preprocess: Sort the points of P by abscissa (only once).

2. Division: Vertically partition P into two subsets R and B, of approximately the same size.

3. Recursion: Recursively compute $Vor(R)$ and $Vor(B)$.

4. Merging:

Compute the separating chain.

Prune the portion of $Vor(R)$ lying to the right of the chain and the portion of $Vor(B)$ lying to its left.

The total running time of the algorithm is $O(n \log n)$

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OTHER ALGORITHMS

There exist other algorithms with the same running time:
- Fortune's Algorithm (sweep)
- 3D projection algorithm

TWO BOOKS WITH MUCH MORE INFORMATION

A. Okabe, B. Boots, K. Sugihara, S. N. Chiu
Spatial Tessellations
2nd ed., J. Wiley & Sons, 2000.

F. Aurenhammer, R. Klein, D.-T. Lee
Voronoi Diagrams and Delaunay Triangulations
World Scientific, 2013.