

# A Comparative Study of Fuzzy Sets and Rough Sets

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This paper reviews and compares theories of fuzzy sets and rough sets. Two approaches for the formulation of fuzzy sets are reviewed, one is based on many-valued logic and the other is based on modal logic. Two views of rough sets are presented, set-oriented view and operator-oriented view. Rough sets under set-oriented view are closely related to fuzzy sets, which leads to non-truth-functional fuzzy set operators. Both of them may be considered as deviations of classical set algebra. In contrast, rough sets under operator-oriented view are different from fuzzy sets, and may be regarded as an extension of classical set algebra.

*Key words:* approximation operators, fuzzy sets, interval fuzzy sets, modal logic, many-valued logic, possible-world semantics, product systems, rough sets.

## 1 INTRODUCTION

Theories of fuzzy sets and rough sets are generalizations of classical set theory for modeling vagueness and uncertainty [19,33]. A fundamental question concerning both theories is their connections and differences [20,35]. There have been many studies on this topic. While some authors argued that one theory is more general than the other [20,27], it is generally accepted that they are related but distinct and complementary theories [2,5,7,13,26]. The two theories model different types of uncertainty [7]. The rough set theory takes into consideration the indiscernibility between objects. The indiscernibility is typically characterized by an equivalence relation. Rough sets are the results of approximating crisp sets using equivalence classes. The fuzzy set theory deals with the ill-definition of the boundary of a class through a continuous generalization of set characteristic functions. The indiscernibility between objects is not used in fuzzy set theory [2]. A fuzzy set may be viewed as a class

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with unsharp boundaries, whereas a rough set is a crisp set which is coarsely described [35].

Klir [7] compared the roles played by non-classical logics, such as many-valued logics and modal logics, for interpreting fuzzy sets and rough sets. According to Haack [6], a non-classical logic is a deviation of classical two-valued logic, i.e., a deviant logic, if the two logics have the same logical vocabulary but different axioms or rules. Many-valued logics may be viewed as deviant logics. A non-classical logic is an extension, i.e., an extended logic, if it adds new vocabulary along with new axioms or rules for the new vocabulary [6]. Modal logics may be viewed as extended logics. Classical set-theoretic operators reflect the corresponding logic connectives in classical two-valued logic [7]. Similar correspondence may also be established between non-classical set-theoretic operators and non-classical logic connectives [7]. Non-classical set theories may therefore be similarly viewed as deviations and extensions of classical set theory. From such a point of view, this paper presents a comparative study of theories of fuzzy sets and rough sets.

As pointed out recently by Zadeh [34], fuzzy logic has many facets: the logical facet, the set-theoretic facet, the relational facet, and the epistemic facet. Each of these facets may be further divided. In the same way, there are many different formulations and interpretations of the theory of rough sets [29]. It is very important to realize that our comparisons of two theories are based on very specific interpretations of each theory. Furthermore, many issues involved in both theories are not taken into consideration. Although conclusions drawn from such comparisons should be read cautiously, the examination may provide more insights into both theories.

## 2 OVERVIEW OF FUZZY SETS AND ROUGH SETS

There are many formulations and interpretations of theories of fuzzy sets and rough sets [9,21]. This section reviews some of the commonly used systems that are closely related to classical set algebra.

### 2.1 FUZZY SETS

The notion of fuzzy sets provides a convenient tool for representing vague concepts by allowing partial memberships. Among many formulations of fuzzy sets, we choose two systems that are related to many-valued logic and modal logic [7]. In both systems, a fuzzy set can be interpreted by a family of crisp sets, and fuzzy set operators can be defined using standard set operators.

Let  $U$  be a finite and non-empty set called universe. A fuzzy set  $\mathcal{A}$  of  $U$  is defined by a membership function:

$$\mu_{\mathcal{A}} : U \longrightarrow [0, 1]. \quad (1)$$

The membership values may be interpreted in terms of truth values of certain propositions, and fuzzy set operators in terms of logic connectives in many-valued logic. This provides a formulation of fuzzy set theory based on many-valued logic [7]. In the study of many-valued logic, there are many definitions for logic connectives [24]. Similarly, there are many definitions for fuzzy set complement, intersection, and union. With the min-max system proposed by Zadeh [33], fuzzy set operators are defined component-wise as:

$$\begin{aligned} \mu_{\neg \mathcal{A}}(x) &= 1 - \mu_{\mathcal{A}}(x), \\ \mu_{\mathcal{A} \cap \mathcal{B}}(x) &= \min[\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)], \\ \mu_{\mathcal{A} \cup \mathcal{B}}(x) &= \max[\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)]. \end{aligned} \quad (2)$$

Let  $\mathcal{F}(U)$  denote the set of all fuzzy sets, i.e., the set of all functions from  $U$  to  $[0, 1]$ . The min-max fuzzy set algebra is a system  $(\mathcal{F}(U), 1 - \cdot, \min, \max)$ , where  $1 - \cdot$ ,  $\min$ , and  $\max$  are defined component-wise by equation (2). It can be algebraically characterized by a completely distributive lattice [17]. In general, fuzzy set intersection and union may be defined in terms of t-norms and t-conorms [9]. By choosing different pairs of t-norms and t-conorms, one can derive distinct fuzzy set systems. The mathematical structures of the corresponding fuzzy set algebras are not entirely clear. In many fuzzy set systems, membership functions of complement, intersection, and union of fuzzy sets are defined based solely on membership functions of the fuzzy sets involved. In fact, studies on fuzzy sets have been focused mainly on such truth-functional fuzzy set operators.

A crisp set can be regarded as a degenerated fuzzy set in which the membership function is restricted to the extreme points  $\{0, 1\}$  of  $[0, 1]$ . In this case, the membership function is also referred to as a characteristic function. A fuzzy set can be related to a family of crisp sets through the notions of  $\alpha$ -level sets. Given a number  $\alpha \in [0, 1]$ , an  $\alpha$ -cut, or  $\alpha$ -level set, of a fuzzy set is defined by:

$$\mathcal{A}_{\alpha} = \{x \in U \mid \mu_{\mathcal{A}}(x) \geq \alpha\}, \quad (3)$$

which is a subset of  $U$ . A strong  $\alpha$ -cut is defined by:

$$\mathcal{A}_{\alpha+} = \{x \in U \mid \mu_{\mathcal{A}}(x) > \alpha\}. \quad (4)$$

Using either  $\alpha$ -cuts or strong  $\alpha$ -cuts, a fuzzy set determines a family of nested subsets of  $U$ . Conversely, a fuzzy set  $\mathcal{A}$  can be reconstructed from its  $\alpha$ -level sets as follows:

$$\mu_{\mathcal{A}}(x) = \sup\{\alpha \mid x \in \mathcal{A}_{\alpha}\}. \quad (5)$$

This observation is commonly summarized by a representation theorem of fuzzy sets, which states that there is an one-to-one relationship between a fuzzy set and a family of crisp sets satisfying certain conditions [9,13,17,18]. Therefore, one can use either definition of fuzzy sets. Each of the two methods, i.e., functional approach using membership functions and set-based approach using families of  $\alpha$ -level sets, has its advantages in the study of fuzzy sets. One of the main advantages of the set based representation is that it explicitly establishes a connection between fuzzy sets and crisp sets. Such a linkage shows the inherent structure of a fuzzy set.

An implication of the min-max system is that fuzzy set operators can be defined by set operators on  $\alpha$ -level sets. They can be expressed by:

$$\begin{aligned} (\neg \mathcal{A})_{\alpha} &= \neg \mathcal{A}_{(1-\alpha)^+}, \\ (\mathcal{A} \cap \mathcal{B})_{\alpha} &= \mathcal{A}_{\alpha} \cap \mathcal{B}_{\alpha}, \\ (\mathcal{A} \cup \mathcal{B})_{\alpha} &= \mathcal{A}_{\alpha} \cup \mathcal{B}_{\alpha}. \end{aligned} \quad (6)$$

The  $\alpha$ -level sets of fuzzy sets for intersection and union are obtained from the same  $\alpha$ -level sets of the fuzzy sets involved. When an arbitrary pair of t-norm and t-conorm is used, it may be difficult to define such operators using set operators on  $\alpha$ -level sets of fuzzy sets.

Klir [7] proposed another formulation of fuzzy sets based on modal logic. A slightly different development is presented below by clearly identifying the underlying system being adopted. In this model, a vague concept is characterized by some, possibly different, crisp sets. Let  $W = \{w_1, \dots, w_n\}$  denote a set of  $n$  possible worlds or states. With respect to  $W$ , a vague concept is represented by  $n$  crisp sets:

$$\mathcal{A} = (A_{w_1}, \dots, A_{w_n}). \quad (7)$$

In each world  $w_i$ , the vague concept is described precisely by a crisp set  $A_{w_i}$ . The vagueness is captured by distinct representations of the same concept in different worlds. A similar approach was also used by Kruse, Schwecke, and Heinsohn [11], in which each possible world is referred to as a context in a framework consisting of layered contexts. The set all families of  $n$  crisp sets is given by the  $n$ -fold Cartesian product of  $2^U$ , namely,  $\prod_n 2^U = 2^U \times \dots \times 2^U$  ( $n$

repetitions). We can define set-theoretic operators on  $\prod_n 2^U$  component-wise as follows: for  $\mathcal{A} = (A_{w_1}, \dots, A_{w_n})$  and  $\mathcal{B} = (B_{w_1}, \dots, B_{w_n})$ ,

$$\begin{aligned}\neg \mathcal{A} &= (\neg A_{w_1}, \dots, \neg A_{w_n}), \\ \mathcal{A} \cap \mathcal{B} &= (A_{w_1} \cap B_{w_1}, \dots, A_{w_n} \cap B_{w_n}), \\ \mathcal{A} \cup \mathcal{B} &= (A_{w_1} \cup B_{w_1}, \dots, A_{w_n} \cup B_{w_n}).\end{aligned}\tag{8}$$

The system  $(\prod_n 2^U, \neg, \cap, \cup)$  may be interpreted as the  $n$ -fold product of classical set algebra  $(2^U, \neg, \cap, \cup)$ . It corresponds to the  $n$ -fold product of classical two-valued logic discussed by Rescher [24]. From this system, one can develop a constructive method for defining a fuzzy set using a family of crisp sets and a weighting function.

Suppose  $\Omega : W \longrightarrow [0, 1]$  is a weighting function satisfying the condition:

$$\sum_{i=1}^n \Omega(w_i) = \sum_{i=1}^n \omega_i = 1,\tag{9}$$

where the simplified notation  $\omega_i = \Omega(w_i)$  is used. For an element of  $\prod_n 2^U$  representing a vague concept, a fuzzy set can be defined by:

$$\mu_{\mathcal{A}}(x) = \sum_{i=1}^n \omega_i \mu_{A_{w_i}}(x),\tag{10}$$

where  $\mu_{A_{w_i}}$  is the characteristic function of  $A_{w_i}$ . If each crisp set  $A_{w_i}$  represents one view of the vague concept, a fuzzy set may be interpreted as a weighted combined view. Fuzzy sets corresponding to complement, intersection, and union can be constructed from  $\neg \mathcal{A}$ ,  $\mathcal{A} \cap \mathcal{B}$ , and  $\mathcal{A} \cup \mathcal{B}$  by combining equations (8) and (10). They may be regarded to be the results of fuzzy set complement, intersection, and union. An important feature of the constructive formulation of fuzzy sets is that operators  $\cap$  and  $\cup$  are no longer truth-functional. They obey the following properties:

- (f1)  $\mu_{\neg \mathcal{A}}(x) = 1 - \mu_{\mathcal{A}}(x),$
- (f2)  $\mu_{\mathcal{A} \cup \mathcal{B}}(x) = \mu_{\mathcal{A}}(x) + \mu_{\mathcal{B}}(x) - \mu_{\mathcal{A} \cap \mathcal{B}}(x),$
- (f3)  $\max(0, \mu_{\mathcal{A}}(x) + \mu_{\mathcal{B}}(x) - 1) \leq \mu_{\mathcal{A} \cap \mathcal{B}}(x) \leq \min(\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)),$
- (f4)  $\max(\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)) \leq \mu_{\mathcal{A} \cup \mathcal{B}}(x) \leq \min(1, \mu_{\mathcal{A}}(x) + \mu_{\mathcal{B}}(x)).$

Property (f2) links together fuzzy set intersection and union. Properties (f3) and (f4) provide the ranges of membership values of intersection and union, based on membership values of the two component fuzzy sets.

For a given fuzzy set  $\mathcal{A}$  in  $\mathcal{F}(U)$ , there exists at least a set of possible worlds, a weighting function, and a family of crisp sets, from which the fuzzy set can be obtained. Consider a set of possible worlds with  $|U|$  elements, where  $|\cdot|$  denotes the cardinality of a set. With each possible world  $w_i$ , one associates a unique singleton subset  $\{x_i\}$  of  $U$ . Let the weight function be defined by  $\omega_i = \mu_{\mathcal{A}}(x_i)$ , where  $\mu_{\mathcal{A}}(x_i)$  is the membership value of  $x_i$ . It can be easily seen that they produce the fuzzy set  $\mathcal{A}$ . For a fixed set of possible worlds and a weighting function, one can define at most  $2^{|U|}$  distinct fuzzy sets. Each of them corresponds to a distinct family of  $n$  crisp sets. For an arbitrary fuzzy set in  $\mathcal{F}(U)$ , if the set of possible worlds and the weighting function are fixed, it may not be possible to find a family of  $n$  crisp sets that produces the given fuzzy set. Furthermore, each family of  $n$  crisp sets corresponds to a fuzzy set in  $\mathcal{F}(U)$ , but the converse is not necessarily true. It may happen that a fuzzy set of  $\mathcal{F}(U)$  is produced by two different elements of  $\prod_n 2^U$ .

## 2.2 ROUGH SETS

The theory of rough sets is motivated by practical needs to interpret, characterize, represent, and process indiscernibility of individuals. For example, if a group of patients are described by using several symptoms, many patients would share the same symptoms, and hence are indistinguishable. This forces us to think a subset of the patients as one unit, instead of many individuals. **Rough set theory provides a systematic method for representing and processing vague concepts caused by indiscernibility in situations with incomplete information or a lack of knowledge.** At least two views can be used to interpret this theory, operator-oriented view and set-oriented view [29].

Formally, indiscernibility may be described by an equivalence relation  $\mathfrak{R} \subseteq U \times U$  on a finite and non-empty universe  $U$ , namely,  $\mathfrak{R}$  is reflexive, symmetric and transitive. The relation  $\mathfrak{R}$  partitions  $U$  into a family of disjoint subsets  $U/\mathfrak{R}$  called a quotient set of  $U$ . For two elements  $x, y \in U$ , if  $x\mathfrak{R}y$  we say that  $x$  and  $y$  are indistinguishable. Elements of  $U/\mathfrak{R}$  are called elementary or atomic sets. The empty set and the union of one or more elementary sets are called composed or definable sets.

The underlying system for constructing rough sets is the set algebra  $(2^U, \neg, \cap, \cup)$ . An element  $A$  of  $2^U$  represents a non-vague concept. When such a non-vague concept is viewed with respect to elements of the quotient set  $U/\mathfrak{R}$ , i.e., the equivalence classes of  $\mathfrak{R}$ , it becomes vague and uncertain. Consider two elements  $x, y \in U$  and a subset  $A \subseteq U$  with  $x\mathfrak{R}y$ ,  $x \in A$ , and  $y \notin A$ . The information given by the equivalence relation suggests that they are indistinguishable. On the other hand, only one of them belongs to  $A$ . From the point view of  $A$ ,  $x$  and  $y$  are distinguishable. This inconsistency causes

the non-vague concept to be vague and uncertain. For a subset  $A \subseteq U$ , one may describe it by a pair of lower and upper approximations:

$$\begin{aligned}\underline{apr}(A) &= \{x \in U \mid [x]_{\mathfrak{R}} \subseteq A\}, \\ &= \bigcup \{[x]_{\mathfrak{R}} \in U/\mathfrak{R} \mid [x]_{\mathfrak{R}} \subseteq A\}, \\ \overline{apr}(A) &= \{x \in U \mid [x]_{\mathfrak{R}} \cap A \neq \emptyset\}, \\ &= \bigcup \{[x]_{\mathfrak{R}} \in U/\mathfrak{R} \mid [x]_{\mathfrak{R}} \cap A \neq \emptyset\},\end{aligned}\tag{11}$$

where

$$[x]_{\mathfrak{R}} = \{y \mid x\mathfrak{R}y\},\tag{12}$$

is the equivalence class containing  $x$ . The lower approximation  $\underline{apr}(A)$  is the union of elementary sets which are subsets of  $A$ , and the upper approximation  $\overline{apr}(A)$  is the union of elementary sets which have a non-empty intersection with  $A$ . That is,  $\underline{apr}(A)$  is the greatest definable set contained by  $A$ , while  $\overline{apr}(A)$  is the least definable set containing  $A$ .

The lower and upper approximations can be understood as a pair of additional unary set-theoretic operators  $\underline{apr}, \overline{apr} : 2^U \longrightarrow 2^U$ , called approximation operators. They play a similar role as that of set complement [15,30]. By combining them with other set-theoretic operators, we have a system  $(2^U, \neg, \cap, \cup, \underline{apr}, \overline{apr})$ , where  $(2^U, \neg, \cap, \cup)$  is the classical set algebra. Operators  $\underline{apr}$  and  $\overline{apr}$  obey the following properties: for any subsets  $A, B \subseteq U$ ,

$$\begin{aligned}(\text{R0}) \quad & \underline{apr}(A) = \neg \overline{apr}(\neg A), \\ & \overline{apr}(A) = \neg \underline{apr}(\neg A), \\ (\text{R1}) \quad & \underline{apr}(A) \subseteq A \subseteq \overline{apr}(A), \\ (\text{R2}) \quad & \underline{apr}(\emptyset) = \overline{apr}(\emptyset) = \emptyset, \\ (\text{R3}) \quad & \underline{apr}(U) = \overline{apr}(U) = U, \\ (\text{R4}) \quad & \underline{apr}(A \cap B) = \underline{apr}(A) \cap \underline{apr}(B), \\ & \overline{apr}(A \cap B) \subseteq \overline{apr}(A) \cap \overline{apr}(B), \\ (\text{R5}) \quad & \underline{apr}(A \cup B) \supseteq \underline{apr}(A) \cup \underline{apr}(B), \\ & \overline{apr}(A \cup B) = \overline{apr}(A) \cup \overline{apr}(B), \\ (\text{R6}) \quad & \underline{apr}(A) = \underline{apr}(\underline{apr}(A)) = \overline{apr}(\underline{apr}(A)), \\ & \overline{apr}(A) = \underline{apr}(\overline{apr}(A)) = \overline{apr}(\overline{apr}(A)).\end{aligned}$$

Property (R0) shows that  $\underline{apr}$  and  $\overline{apr}$  are dual operators with respect to set complement  $\neg$ . Property (R1) says that the two operators produce a range in which lies the given set. Properties (R2) and (R3) are the boundary conditions that the operators must meet at the two extreme points of  $2^U$ , the minimum

element  $\emptyset$  and the maximum element  $U$ . Properties (R4) and (R5) may be viewed as weak distributivity and distributivity of operators  $\underline{apr}$  and  $\overline{apr}$  over set intersection and union. Property (R6) indicates that the result of iterative applications of approximation operators is the same as that of the last one.

The above formulation is referred to as the operator-oriented view of rough sets. Another interpretation, called set-oriented view, can be developed based on the notion of rough membership functions [22,26,29]. For a subset  $A \subseteq U$ , we define a rough set characterized by the membership function:

$$\mu_A(x) = \frac{|A \cap [x]_{\mathfrak{R}}|}{|[x]_{\mathfrak{R}}|}, \quad (13)$$

which represents a vague concept. One can easily see the similarity between rough membership functions and conditional probabilities. The rough membership value  $\mu_A(x)$  may be interpreted as the probability that an arbitrary element of  $[x]_{\mathfrak{R}}$  belongs to  $A$ .

There does not exist an one-to-one relationships between rough sets and subsets of  $U$ . Two distinct subsets of  $U$  may define the same rough membership function. Consequently, rough set operators cannot be defined directly using rough membership functions. Membership functions of rough sets corresponding to  $\neg A$ ,  $A \cap B$ , and  $A \cup B$  must be computed using set operators and equation (13). By laws of probability, intersection and union of rough sets are not truth-functional. Nevertheless, we have:

- (m0)  $y \in [x]_{\mathfrak{R}} \implies \mu_A(x) = \mu_A(y)$ ,
- (m1)  $\mu_{\neg A}(x) = 1 - \mu_A(x)$ ,
- (m2)  $\mu_{A \cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_{A \cap B}(x)$ ,
- (m3)  $\max(0, \mu_A(x) + \mu_B(x) - 1) \leq \mu_{A \cap B}(x) \leq \min(\mu_A(x), \mu_B(x))$ ,
- (m4)  $\max(\mu_A(x), \mu_B(x)) \leq \mu_{A \cup B}(x) \leq \min(1, \mu_A(x) + \mu_B(x))$ .

Property (m0) states that elements in the same equivalence class have the same degree of membership. Properties (m1)-(m4) are similar to properties (f1)-(f4) of fuzzy sets formulated based on modal logic.

The theory of rough sets can be easily generalized by using an arbitrary binary relation, instead of an equivalence relation. Let  $\mathfrak{R}$  be any binary relation on the universe  $U$ . The relation  $\mathfrak{R}$  may be more conveniently represented by  $\mathfrak{R}$ -related elements through a mapping  $\mathfrak{R}_s : U \longrightarrow 2^U$ :

$$\mathfrak{R}_s(x) = \{y \in U \mid x \mathfrak{R} y\}. \quad (14)$$

The set  $\mathfrak{R}_s(x)$  may be viewed as a neighborhood of  $x$  defined by the binary



relation  $\mathfrak{R}$  [14,30]. For a subset  $A$  of the universe, a pair of approximation operators can be defined by substituting  $[x]_{\mathfrak{R}}$  with  $\mathfrak{R}_s(x)$  in equation (11) as follows [29,30]:

$$\begin{aligned} \underline{apr}(A) &= \{x \mid \mathfrak{R}_s(x) \subseteq A\}, \\ \overline{apr}(A) &= \{x \mid \mathfrak{R}_s(x) \cap A \neq \emptyset\}. \end{aligned} \quad (15)$$

They can be equivalently defined by:

$$\begin{aligned} \underline{apr}(A) &= \{x \mid \forall y \in U[y \in \mathfrak{R}_s(x) \implies y \in A]\}, \\ \overline{apr}(A) &= \{x \mid \exists y \in U[y \in \mathfrak{R}_s(x), y \in A]\}, \end{aligned} \quad (16)$$

which relates approximation operators to necessity and possibility operators of modal logics. Approximation operators do not satisfy all properties (R0)-(R6). By imposing additional constraints on the binary relation, one can obtain approximation operators that satisfying each of these properties [30].

### 3 COMPARISONS OF FUZZY SETS AND ROUGH SETS

This section compares theories of fuzzy sets and rough sets based on the models presented earlier. The **set-oriented** view of rough sets is related to fuzzy sets and offers an deviation of classical set theory. In contrast, **operator-oriented** view of rough sets is complementary to fuzzy sets and offers an extension of classical set theory. The arguments may be extended to other models of these theories.

#### 3.1 FUZZY SETS, **SET-ORIENTED VIEW** OF ROUGH SETS: **DEVIATIONS** OF CLASSICAL SET THEORY

A fuzzy set is defined by a membership function from a universe  $U$  to the unit interval  $[0, 1]$ . This introduces generalized notions of sets and members of sets, compared with classical sets. In order to accommodate these notions, meanings of classical sets and set-theoretic operators have to be modified. Many proposals have been made for defining fuzzy set operators [4,25]. Typically, no new set-theoretic operators are introduced, although set complement may be extended into several forms [1]. From this observation, the theory of fuzzy sets may be viewed as being a *deviation* of classical set theory. Both theories share the same vocabulary, but have different interpretations for the vocabulary.

A fundamental difficulty with fuzzy set theory is the **semantical interpretations of the degrees of membership**, similar to difficulties in the semantical interpretations of truth values in many-valued logics [24]. This in turn leads to difficulties in the semantical interpretations of set-theoretic operators. Although one can define fuzzy set operators in the min-max system using set operators through  $\alpha$ -level sets, it is mainly a technical result rather than a semantical interpretation. In the context of many-valued logics, Rescher [24] reviewed a number of solutions to these difficulties. One solution is the use of product logic which interprets a many-valued logic based on the semantics of lesser-valued systems. Another solution is the probabilistic approach to many-valued logic. Within the framework of fuzzy sets, the model proposed by Klir [7] corresponds to the former solution, while the set-oriented view of rough set corresponds to the latter solution.

In developing modal logic based fuzzy sets, we start from the product system  $(\prod_n 2^U, \neg, \cap, \cup)$ . The system can be explained by using the semantical interpretation of classical set theory. With respect to a specific possible world, every vague concept is represented by a crisp subset of  $U$ . The interpretations of membership and set-theoretic operators are exactly the same as that of classical set theory. The vagueness is described by differences in representations of the same vague concept in various possible worlds. A fuzzy set is a weighted combination. This model captures one possible source of vagueness, and provides a semantical interpretation of fuzzy set theory based on the semantical interpretations of classical set theory.

The set-oriented view of rough sets starts from classical set algebra  $(2^U, \neg, \cap, \cup)$ , and associates a fuzzy set with each subset of the universe. Vagueness in concept formation and representation comes from our inability to describe a precisely defined concept in situations with incomplete information, where a group of individuals cannot be distinguished. This model captures another source of vagueness. Rough membership functions may be interpreted as a special type of fuzzy membership functions, which can be interpreted in terms of probabilities defined simply by cardinalities of sets [22,26]. In general, one may use a probability function on  $U$  to define rough membership functions [32]. With this view, rough set theory may be regarded as being a *deviation* of classical set theory, in the same way that fuzzy set theory is viewed. Using terminologies of fuzzy sets, lower and upper approximations are the core and support of fuzzy set  $\mu_A$ :

$$\begin{aligned} core(\mu_A) &= \{x \mid \mu_A(x) = 1\} = \underline{apr}(A), \\ support(\mu_A) &= \{x \mid \mu_A(x) > 0\} = \overline{apr}(A). \end{aligned} \tag{17}$$

In fact, lower approximation is the 1-level set, while upper approximation is the strong 0-level set.

The formulation and interpretation of fuzzy sets and set-theoretic operators are inseparable parts of theories of modal logic based fuzzy sets and rough sets. In the former model, fuzzy sets and set-theoretic operators are based on representations of vague concepts in various worlds. In the latter model, rough sets and set-theoretic operators must be computed based on the sets involved and their interaction with equivalence classes. Fuzzy set operators defined in both models satisfy similar properties as shown by (f1)-(f4) and (m1)-(m4), in spite of the fact that two entirely different interpretations are used. The representations of vague concepts in a set of possible worlds, or the equivalence relation  $\mathcal{R}$ , may be referred to as the context that provides semantical interpretations for membership functions of, and operators on, fuzzy sets. In many studies, the formulation and interpretation of fuzzy sets and set-theoretic operators are not incorporated into the theory. This may be one of the main causes for difficulties in semantical interpretations of fuzzy set theory. From the above discussions, one may say that a plausible solution to these difficulties is to design a theory that incorporates semantics information about the fuzzy concepts being modeled. One may formulate various sub-theories of the general theory of fuzzy sets, each of them is intended for specific situations with different semantics of membership values and set-theoretic operators. Modal logic based fuzzy sets and rough sets may be considered as two such sub-theories.

Modal logic based fuzzy set model and rough set model provide *constructive* approaches for the development of fuzzy set theory, in which both fuzzy sets and set-theoretic operators are constructed based on well-known concepts. This avoids the difficulties in the semantical interpretations of the theory. However, they can deal with only a subset of the set of all fuzzy sets  $\mathcal{F}(U)$ . One may say that they are more restrictive than the general fuzzy set theory. It should be taken as an advantage, rather than a limitation, of the theory. By explicitly stating the underlying assumptions and interpretations, one may provide guidelines regarding the correct uses of the theory. Fuzzy set systems, such as the min-max system, are more general. Unfortunately, they do not provide such guidelines. It might be more fruitful to examine sub-theories of fuzzy sets, each has its own clearly stated assumptions and intended domain of applications. In the development of fuzzy set theory, truth-functional operators have been studied and applied extensively, although non-truth-functional systems have been studied in many-valued logics [24]. Klir [7] pointed out that it is important to study non-truth-functional fuzzy set operators. With regard to this, both modal logic based fuzzy sets and rough sets use non-truth-functional operators.

There are additional features of rough set theory. In the theory of fuzzy sets, the membership value of an element does not depend on other elements. In contrast, with respect to an equivalence relation, the membership value of an element depends on other elements in the theory of rough sets [2]. Property (m0) reflects this observation. Such a property does not have a counterpart

in the modal logic based fuzzy sets. In the study of fuzzy sets, many types of fuzzy set membership functions have been proposed and applied [9]. They implicitly specify the membership value of one element with respect to other elements. For example, the membership value of a 20 years old person being young is related to the membership values of 19 and 21 years old persons. Semantics constraints on fuzzy membership functions should be explicitly and formally stated and incorporated into fuzzy set theory. There may be other interpretations of rough membership functions. In relation to a three-valued logic and the corresponding three-valued fuzzy sets, one can define a rough membership function:

$$\mu_A(x) = \begin{cases} 1, & [x]_{\mathfrak{R}} \subseteq A, \\ 1/2, & [x]_{\mathfrak{R}} \cap A \neq \emptyset \text{ and } [x]_{\mathfrak{R}} \not\subseteq A, \\ 0, & [x]_{\mathfrak{R}} \cap A = \emptyset. \end{cases} \quad (18)$$

Set-theoretic operators can be similarly defined and interpreted. The choice of 1/2 is arbitrary. One may in fact choose any value other than 0 and 1 as the third value. The resulting three-valued logic is not truth-functional.

### 3.2 OPERATOR-ORIENTED VIEW OF ROUGH SETS: EXTENSION OF CLASSICAL SET THEORY

Under operator-oriented view of rough sets, we start from a binary relation and construct a pair of approximation operators. The result is a system  $RS = (2^U, \neg, \cap, \cup, \underline{apr}, \overline{apr})$ , called a rough set algebra. Alternatively, we may define a rough set algebra by specifying two operators  $\underline{apr}, \overline{apr} : 2^U \longrightarrow 2^U$  and axioms that must be satisfied by the operators [15,29]. In a rough set algebra  $(2^U, \neg, \cap, \cup, \underline{apr}, \overline{apr})$ ,  $\neg$ ,  $\cap$ , and  $\cup$  are standard set operators. Operators  $\underline{apr}$  and  $\overline{apr}$  are two additional unary set-theoretic operators. They are non-truth-functional operators, and cannot be defined by standard set-theoretic operators. The theory of rough sets introduces new vocabulary to classical set theory and additional rules for the vocabulary. Unlike fuzzy sets and set-oriented view, operator-oriented view may be regarded as being an *extension* of classical set algebra with a pair of additional operators [15,30].

Approximation operators  $\underline{apr}$  and  $\overline{apr}$  may be related to operators in other mathematical structures. A Pawlak rough set algebras built from an equivalence relation is related to a special topological space in which an open set is closed and vice versa [19]. If a reflexive and transitive binary relation is used, lower and upper approximations defined by equation (15) are exactly interior and closure operators satisfying Kuratowski axioms for topological

spaces [10,15,29]. There are other ways for defining approximation operators if a non-equivalence relation is used [15,29]. However, exact mathematical structures of the corresponding rough set algebras are not entirely clear.

Rough set theory extends set theory in the same way that modal logic extends classical logic. Algebraically speaking, both rough set algebra and propositional modal logic can be related to Boolean algebra with added operators [3,12]. A Pawlak rough set can be algebraically characterized by a special class of topological Boolean algebra in which an open element is closed and vice versa [3,23]. The relationships between Boolean algebra with added operators, rough set algebras, and propositional modal logics imply that every theorem in any one of these theories has a counterpart in the other theory. They enable us to cover all these theories by developing one of them [7].

#### 4 COMBINATION OF FUZZY SETS AND ROUGH SETS

Two views of rough set theory provide distinct generalizations of classical set theory, namely, deviation and extension. In the study of non-classical logic, systems have been studied which are both deviation and extension of classical logic [7]. In set-theoretic framework, this may be achieved by the combination of fuzzy sets and rough sets, or the combination of two views of rough sets [28]. By using an equivalence relation on  $U$ , one can introduce lower and upper approximations in fuzzy set theory to obtain an extended notion called rough fuzzy sets [5,13,28]. Alternatively, a fuzzy similarity relation can be used to replace an equivalence relation, the result is a deviation of rough set theory called fuzzy rough sets [5,16]. A more general framework can be obtained which involves the approximation of fuzzy sets based on fuzzy similarity relations. The results may have several interpretations [28]. Based on our comparisons of two theories, further results on this topic can be obtained.

Klir [7] suggested that one may use necessity and possibility modal operators to define and interpret interval fuzzy sets. This is related to the notions of lower and upper approximations in operator-oriented view of rough sets, but the formulation is different. Let  $\mathfrak{R}$  be a binary relation on the set of possible worlds  $W = \{w_1, \dots, w_n\}$ . For any world  $w_i$ , the set of  $\mathfrak{R}$ -related worlds is given by:

$$\mathfrak{R}_s(w_i) = \{w_j \mid w_i \mathfrak{R} w_j\}. \quad (19)$$

A vague concept is represented by three elements of  $\prod_n 2^U$  as follows:

$$\begin{aligned}\mathcal{A} &= (A_{w_1}, \dots, A_{w_n}), \\ \Box \mathcal{A} &= (\underline{A}_{w_1}, \dots, \underline{A}_{w_n}), \\ \Diamond \mathcal{A} &= (\overline{A}_{w_1}, \dots, \overline{A}_{w_n}).\end{aligned}\tag{20}$$

The families  $\Box \mathcal{A}$  and  $\Diamond \mathcal{A}$  of  $n$  crisp subsets of  $U$  are defined from  $\mathcal{A}$  by:

$$\begin{aligned}\underline{A}_{w_i} &= \{x \in U \mid \forall w_j \in W [w_j \in \mathfrak{R}_s(w_i) \implies x \in A_{w_j}]\}, \\ \overline{A}_{w_i} &= \{x \in U \mid \exists w_j \in W [w_j \in \mathfrak{R}_s(w_i), x \in A_{w_j}]\}.\end{aligned}\tag{21}$$

Similarly,  $\Box$  and  $\Diamond$  may be understood as being a pair of unary operators on the product system  $\prod_n 2^U$ . They are derived from a binary relation on  $W$ , instead of a binary relation on  $\prod_n 2^U$  or a binary relation on  $U$ . From  $\Box \mathcal{A}$  and  $\Diamond \mathcal{A}$ , one can define two fuzzy membership functions in a way similar to equation (10). They define an interval fuzzy set. This model of interval fuzzy sets is an extension of fuzzy set theory.

In Klir's definition of interval fuzzy sets, operators  $\Box$  and  $\Diamond$  are not defined using operators on  $2^U$ , which is different from the definition of operators  $\neg$ ,  $\cap$ ,  $\cup$ . A solution to this problem can be obtained by considering the  $n$ -fold product system  $\prod_n RS = RS \times \dots \times RS$  ( $n$  repetitions) of a rough set algebra  $RS = (2^U, \neg, \cap, \cup, \underline{apr}, \overline{apr})$ . In this case, similar to the definition given by equation (8), a pair of lower and upper approximations are defined component-wise by:

$$\begin{aligned}\underline{apr}(\mathcal{A}) &= (\underline{apr}(A_{w_1}), \dots, \underline{apr}(A_{w_n})), \\ \overline{apr}(\mathcal{A}) &= (\overline{apr}(A_{w_1}), \dots, \overline{apr}(A_{w_n})).\end{aligned}\tag{22}$$

The product system  $(\prod_n RS, \neg, \cap, \cup, \underline{apr}, \overline{apr})$  can be explained using the semantical interpretations of rough set algebras. By combining equations (22) and (10), we have another definition of interval fuzzy sets. This notion of interval fuzzy sets is the result of combining modal logic based fuzzy sets and rough sets. In general, one may consider a product system in which different rough set algebras are used for different possible worlds.

In the set-oriented view of rough sets, lower and upper approximations are not used. One may in fact use them to define interval fuzzy sets as follows:

$$\begin{aligned}\mu_{\underline{apr}(A)}(x) &= \frac{|\underline{apr}(A) \cap \mathfrak{R}_s(x)|}{|\mathfrak{R}_s(x)|}, \\ \mu_{\overline{apr}(A)}(x) &= \frac{|\overline{apr}(A) \cap \mathfrak{R}_s(x)|}{|\mathfrak{R}_s(x)|}.\end{aligned}\tag{23}$$

If  $\mathfrak{R}$  is an equivalence relation, the results are not so interesting, as they are in fact the characteristic functions of  $\underline{apr}(A)$  and  $\overline{apr}(A)$ . If  $\mathfrak{R}$  is only a serial

relation ( $\mathfrak{R}$  is serial if for all  $x \in U$ ,  $\mathfrak{R}_s(x) \neq \emptyset$ ),  $\mu_{apr(A)}$  and  $\mu_{\overline{apr}(A)}$  can be interpreted using belief and plausibility functions [31]. If  $\mathfrak{R}$  is transitive and connected ( $\mathfrak{R}$  is connected is for all  $x, y \in U$ , either  $x\mathfrak{R}y$  or  $y\mathfrak{R}x$ ), they can be interpreted using necessity and possibility functions [8].

## 5 CONCLUSION

In this paper, we have examined relationships and differences between theories of fuzzy sets and rough sets with respect to two formulations of fuzzy sets and two views of rough sets. A fuzzy set is defined by a membership function from a universe  $U$  to the unit interval  $[0, 1]$ . Fuzzy set intersection and union are typically defined using t-norms and t-conorms, which are generalizations of set intersection and union [9]. Such a fuzzy set theory is based on many-valued logic and may be considered as a deviation of classical set theory. Klir [7] used modal logic as a basis for the development of fuzzy set theory. The possible-world semantics is used to construct fuzzy sets and operators on fuzzy sets, rather than introducing new set-theoretic operators. The resulting fuzzy set theory is also a deviation of classical set theory. The investigation of fuzzy sets based on modal logic suggests an important research direction, in which fuzzy sets with non-truth-functional set-theoretic operators are studied.

There are at least two views for interpreting the theory of rough sets [29]. Depending on the views adopted, one may regard rough set theory as either a deviation or an extension of classical set theory. In set-oriented view, a rough set is defined by using a rough membership function [22]. One may treat rough sets as a special class of fuzzy sets, in which membership functions are interpreted in terms of conditional probabilities [26]. This view of rough sets can be related to a special many-valued logic known as probabilistic logic [24]. In this case, no additional operators are introduced and classical set-theoretic operators are used to define rough set operators. Like fuzzy set theory, rough set theory under set-oriented view is a *deviation* of classical set theory. Similar to the modal logic based fuzzy sets, it may be considered to be a more concrete sub-theory of fuzzy sets. A salient feature of both models is that the formulation and interpretation of membership functions and set-theoretic operators are embodied in the theory. Furthermore, rough set operators are no longer truth-functional. In operator-oriented view, a set is approximated by a pair of sets called lower and upper approximations. They can be understood through two unary set-theoretic operators [15]. This view is related to modal logic [30]. Since rough set theory under this view only introduces two additional operators and does not change the meaning of other set-theoretic operators, one may consider it as an extension of classical set theory. It is different from, and complementary to, fuzzy set theory.

Many attempts have been made to combine theories of fuzzy sets and rough sets in order to have an algebra which is both an extension and a deviation of classical set algebra [5,16]. One may introduce additional set-theoretic operators in the theory of fuzzy sets, or use graded binary relations in the theory of rough sets [28]. Our comparisons of fuzzy sets and rough sets lead to further interesting results. The notion of interval fuzzy sets have been introduced for both modal logic based fuzzy sets and set-oriented view of rough sets.

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