Contents

Chapter 1

Annuities with non-contigent payments

An annuity is a series of payments made at equal time intervals.

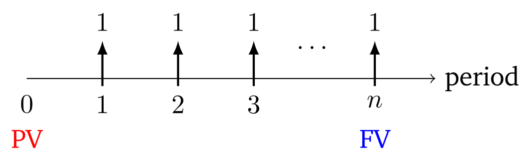
Types of Annuities (Based on Payment Structure)

1. **Amount of Payments**
   - **Level payments**: Equal payment each period
   - **Non-level payments**: Varying payment amounts
2. **Timing of Payments**
   - **Immediate annuity**: Payment at *end* of each period
   - **Due annuity**: Payment at *beginning* of each period
3. **Number of Payments**
   - **Term annuity**: Fixed number of payments
   - **Perpetuity**: Payments continue *forever*
4. **Deferral of Payments**
   - **Deferred annuity**: Payments start *after a delay*

1.1 Level annuity

1.1.1 Immediate Annuity

Consider a level annuity-immediate where each payment is **1 unit**, made at the **end of each period**. There are n total payments, and the effective rate of interest **per unit of time** is i. One unit of time equals one period.



The present value (sum of discounted payments) of that annuity at period t=0 is defined as:

$$PV = \frac{1}{1+i} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^n} = \sum_{k=1}^{n} \frac{1}{(1+i)^k} = \frac{1-v^n}{i}, \quad \text{where } v = \frac{1}{1+i}$$

The future value (value at time n) of the same annuity is:

$$FV = 1 + (1+i) + (1+i)^2 + \cdots + (1+i)^{n-1} = \sum_{k=0}^{n-1} (1+i)^k$$

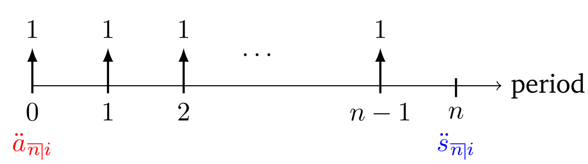
Formula 1 (Immediate Annuity)

$$a_{\overline{n}|i} = PV = \frac{1-v^n}{i}, \quad \text{where } v = \frac{1}{1+i}$$

$$s_{\overline{n}|i} = FV = \sum_{k=1}^{n} (1+i)^k = \frac{(1+i)^n - 1}{i}$$

1.1.2 Annuity Due

Consider a level annuity-due where each payment is **1 unit**, made at the **beginning of each period**. There are n total payments, and the effective rate of interest **per unit of time** is i. One unit of time equals one period.



The present value (sum of discounted payments) of that annuity at period t=0 is:

$$PV = 1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^{n-1}} = \sum_{k=0}^{n-1} \frac{1}{(1+i)^k} = \frac{1-v^n}{i} \cdot (1+i), \quad \text{where } v = \frac{1}{1+i}$$

The future value (value at time n) of the same annuity is:

$$FV = (1+i) + (1+i)^2 + \cdots + (1+i)^n = \sum_{k=1}^{n} (1+i)^k$$

Formula 2 (Annuity Due)

$$\ddot{a}_{\overline{n}|i} = PV = \frac{1-v^n}{i} \cdot (1+i) = \frac{1-v^n}{d}, \quad \text{where } d = \frac{1+i}{i}$$

$$\ddot{s}_{\overline{n}|i} = FV = \sum_{k=1}^{n} (1+i)^k$$

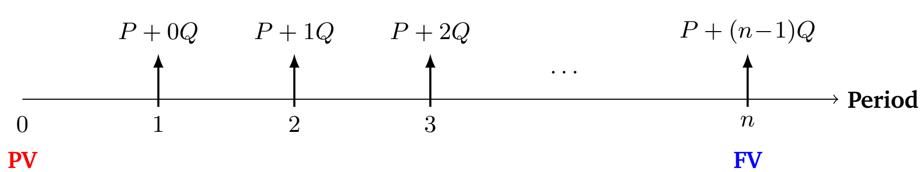
1.2 Varrying-Payments Annuity

Payments in an annuity that changes instead of staying level.

1.2.1 Payments in Arithmetic Progression

1. Arithmetic Increasing Annuity

Let the first payment = **P**, each following payment increases by **Q**, and there are total **n** payments.



Formula 3 (Arithmetic Increasing Annuity)
Present Value at time (period) t=0:

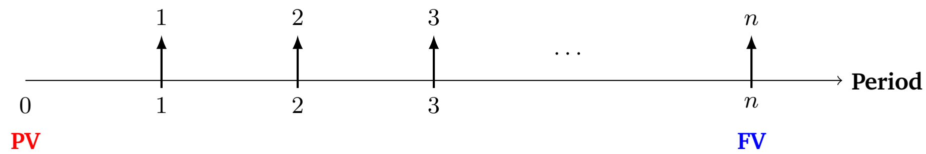
$$PV = P \cdot a_{\overline{n}|i} + Q \cdot \frac{a_{\overline{n}|i} - n \cdot v^n}{i}$$

Accumulated Value at time (period) t=n:

$$AV = P \cdot s_{\overline{n}|i} + Q \cdot \frac{s_{\overline{n}|i} - n}{i}$$

2. Increasing Annuity

When P=Q=1, payments become 1,2,...,n.



Formula 4 (Increasing Annuity)
PV at time (period) t = 0:

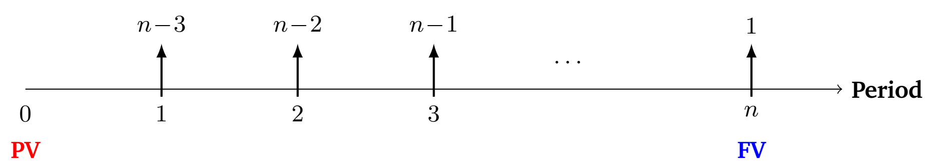
$$(Ia)_{\overline{n}|i} = a_{\overline{n}|i} + \frac{a_{\overline{n}|i} - nv^n}{i} = \frac{(1+i)a_{\overline{n}|i-nv^n}}{i} = \frac{\ddot{a}_{\overline{n}|}i - nv^n}{i}$$

FV at time (period) t = n:

$$(Is)_{\overline{n}|i} = (1+i)^n \cdot (Ia)_{\overline{n}|i} = \frac{\ddot{s}_{\overline{n}|i} - n}{i} = \frac{s_{\overline{n+1}|i} - (n+1)}{i}$$

3. Decreasing Annuity

When P=n and Q=1, payments become n, n-1, n-2, ..., 1.



Formula 5 (Decreasing Annuity)

$$\text{PV}_{t=0} = (Da)_{\overline{n}|} = na_{\overline{n}|i} - \frac{a_{\overline{n}|i} - nv^n}{i} = \frac{n - nv^n - a_{\overline{n}|i} + nv^n}{i} = \frac{n - a_{\overline{n}|i}}{i}$$

$$\text{FV}_{t=n} = (Ds)_{\overline{n}|} = (1+i)^n(Da)_{\overline{n}|} = \frac{n(1+i)^n - s_{\overline{n}|i}}{i} = (n+1)a_{\overline{n}|i} - (Ia)_{\overline{n}|i}$$

4. Varying Perpetuity-Immediate with Arithmetic growth

The payments follow an Arithmetic progression with constants P¿0 and Q¿0. Payments are P, P+Q, P+2Q,... First payment is at the end of first period.

Annuity	PV	FV
Arithmetic Increasing Annuity	$P \cdot a_{\overline{n}\vert i} + Q \cdot \frac{a_{\overline{n}\vert i} - n \cdot v^n}{i}$	$P \cdot s_{\overline{n}\vert i} + Q \cdot \frac{s_{\overline{n}\vert i} - n}{i}$

Table 1.1: Summary

Present Value of Perpetuity-Immediate with payments of arithmetic progression:

$$\text{PV} = \lim_{n \to \infty} \left(P a_{\overline{n}|} + Q \cdot \frac{a_{\overline{n}|} - n\nu^n}{i} \right)$$

Since n goes to infinity,

$$\lim_{n \to \infty} a_{\overline{n}|} = a_{\overline{\infty}|} = \frac{1}{i}$$

$$\lim_{n \to \infty} n\nu^n = 0 \quad \text{(via L'Hôpital's Rule)}$$

Hence,

$$\text{PV} = \frac{P}{i} + \frac{Q}{i^2}$$

Formula 6 (Increasing Perpetuity-Immediate - Arithmetic Growth)
When $P = Q = 1$:

$$(Ia)_{\overline{\infty}|} = \frac{1}{i} + \frac{1}{i^2}$$

1.3 Deferred Annuity

A deferred annuity is an annuity where the payments start at a future date, not immediately.

There are 2 parts in a deferred annuity:

1. Deferral period: Time you wait before payments start.
2. Payment period: Regular payments begin (like a normal annuity).

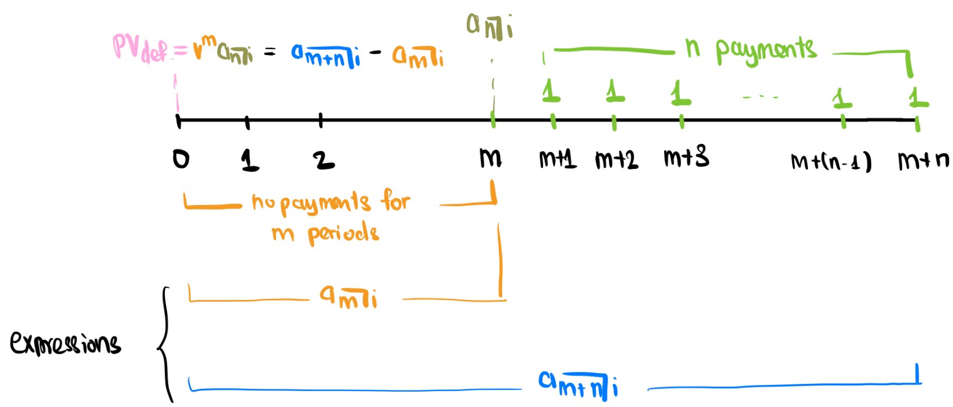


Figure 1.1: Deferred Annuity

Three annuity valuation cases:

- **Before 1st Payment (Deferred Annuity)**: $PV = v^m a_{\overline{n}|i} = a_{\overline{m+n}|i} - a_{\overline{m}|i}$
- **After Last Payment**: AV = move future value forward using interest.
- **Between Payments**: Split into past and future parts: some payments made, some still due.

Calculate the present value of the n-period annuity (a_n) as if payments were starting now. Then discount it back m periods using $v^m = (1+i)^{-m}$.

1.3.1 PV before first payment of an annuity-immediate

Theorem 1. The present value is: PV $= v^m a_n$

We proof the above theorem by derviving from standard annuity formulas:

$a_{\overline{m+n}|i}$ is the PV of an annuity-immediate including all payments over m+n periods (at this point, we assume that there are m+n payments from period t=1 to t=m+n but the actual number of payments is only n, as in 0-m period, there are zero payments).

$a_{\overline{m}|i}$ is the PV of an annuity-immediate over m periods with m payments (this is an assumption as in reality there is zero payment at this point since this is the deferred period).

Proof.

$$a_{\overline{m+n}|i} - a_{\overline{m}|i} = \frac{1-\nu^{m+n}}{i} - \frac{1-\nu^{m}}{i}$$

$$= \frac{\nu^{m} - \nu^{m+n}}{i} = \frac{\nu^{m}(1-\nu^{n})}{i} = \nu^{m} a_{n}$$

$$\Rightarrow a_{\overline{m+n}|i} - a_{\overline{m}|i} = v^{m} a_{n}$$

This means:

- Calculate the present value of the n-period annuity (a_n) at time t=m (1st payment starts at time t=m+1.
- Then discount it back m periods using $v^m = (1+i)^{-m}$.

Note: those expressions are for convinient calculation.

□

1.4 Varying-Interest Annuity

An annuity has an interest that can vary in each period. Let i_k be the interest rate from time k-1 to k.

1.4.1 Rate i_k is applied for period k (date of payments matter

Starting from when the 1st payment made at time 0, up to time k^{th} or later, the future payments are discounted/accumulated using **all previous rate** $i_1, i_2, i_3, ..., i_k$.

Annuity-immediate

The PV of an annuity-immediate is:

$$a_{\overline{n}|i} = \frac{1}{1+i_1} + \frac{1}{(1+i_1)(1+i_2)} + \cdots + \frac{1}{(1+i_1)(1+i_2)\cdots(1+i_n)}$$

The AV of an anntuiy-immediate is:

$$s_{\overline{n+1}|i} = \ddot{s}_{\overline{n}|i} + 1$$

Annuity-due

The PV of an annuity-due is:

$$\ddot{a}_{\overline{n}|i} = 1 + a_{\overline{n-1}|i}$$

The AV of an anntuiy-due is:

$$\ddot{s}_{\overline{n}|i} = (1+i_1)(1+i_2)\cdots(1+i_n) + \cdots + (1+i_{n-1})(1+i_n) + (1+i_n)$$

1.4.2 Rate i_k is applied for period k and before/later (date of payment does not matter)

Regardless of when the payments are made, any payments in the period k is discounted using only rate i_k. The rate i_k is the effective rate for period $i \leq k$ (PV) and $i \geq k$.

Annuity-Immediate

The PV of an annuity-immediate is:

$$a_{\overline{n}|i} = \frac{1}{(1+i_1)^1} + \frac{1}{(1+i_2)^2} + \cdots + \frac{1}{(1+i_n)^n}$$

The FV of an annuity-immediate is:

$$s_{\overline{n+1}|i} = \ddot{s}_{\overline{n}|i} + 1$$

Annuity-due

The PV of an annuity-due is:

$$\ddot{a}_{\overline{n}|i} = 1 + a_{\overline{n-1}|i}$$

The FV of an annuity-due is:

$$\ddot{s}_{\overline{n}|i} = (1+i_1)^n + (1+i_2)^{n-1} + \cdots + (1+i_n)^1$$

1.5 Annuity with non-coinciding frequencies

An annuity where **payments** made and **interest** compounded at a different frequency.

1.6 Non-level Annuity

1.6.1 Geometric annuity

A **geometric annuity** is an annuity in which the payments form a geometric progression.

Consider a non-level annuity with **n** payments, growth rate of each payment **k**, and the first payment to be 1 unit. The present value (sum of discounted payments) of that annuity at period t=0 is defined as:

$$
\begin{aligned}
PV &= v + v^2(1+r) + \cdots + v^n(1+k)^{n-1} \\
&= v\left[\frac{1-\left(\frac{1+k}{1+i}\right)^n}{1-\frac{1+k}{1+i}}\right] = v\left[\frac{1-\left(\frac{1+k}{1+i}\right)^n}{\frac{i-k}{1+i}}\right] \\
&= v\left[\frac{1-\left(\frac{1+k}{1+i}\right)^n}{v(i-k)}\right] \\
&= \frac{1-\left(\frac{1+k}{1+i}\right)^n}{i-k}
\end{aligned}
$$

Since the right-hand side of the equation above is the sum of a finite geometric series.

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

Example 1. (Geometric annuity with unknown interest rate) A 10-year annuity-immediate has:

- First payment: \$11
- Subsequent payments: 10% increase each year
- Accumulated value: \$220.8

Find the annual effective interest rate i.

1. **AV as Geometric Series:**

$$AV = 11(1.1)^9 \left[1 + \frac{1+i}{1.1} + \left(\frac{1+i}{1.1}\right)^2 + \cdots + \left(\frac{1+i}{1.1}\right)^9\right]$$

2. **Sum the Series:**

$$220.8 = 11(1.1)^9 \cdot \frac{1 - \left(\frac{1+i}{1.1}\right)^{10}}{1 - \frac{1+i}{1.1}}$$

3. **Substitute Values:**

$$220.8 = 25.937 \cdot \frac{1 - (1+j)^{10}}{-j}, \quad j = \frac{1.1}{1+i} - 1$$

4. **Solve for j:**

$$\frac{(1+j)^{10} - 1}{j} = 8.513 \implies j \approx 0.03773$$

5. **Find i:**

$$j = \frac{1.1}{1+i} - 1 \implies i = \boxed{0.06}\ (6\%)$$

Chapter 2

Loan

2.1 Debt instruments

A debt instrument is a contract that requires the borrower to repay principal and usually interest at a future date.

Examples of **Debt Instruments:**

1. Bonds (corporate, government)
2. Loans

2.2 Outstanding Balance Calculation for Level Payments

Notation:

- I_t: Interest paid during the k-th period.
- P_t: Principal (capital) repaid during the k-th period.
- B_t: Outstanding balance immediately after the t-th payment.
- R_t: Total payment made during the t-th period (interest + principal).

Type	Formula	Interpretation
Prospective	$B_t = R \cdot a_{n-t}$	Present value of remaining (future) level payments
Retrospective	$B_t = R \cdot s_t$	Accumulated value of past payments

Loan Equation of Value: You can express the loan's total value L using the equation of value:

$$L = B_0 = R_1 v + R_2 v^2 + \cdots + R_n v^n$$

2.3 Loan Amortization (Level/Non-level payments

A loan can be interpreted as an annuity with payments made in regular intervals, each payment consists of two parts:

- **Interest on the loan**: A cost for borrowing the money.
- **Principal**: The amount of the loan that you borrowed.

Over time, interest portion decreases and principal portion increases. Total payments stay the same.

2.3.1 Interest portion in period t

At any given time, the interest due for the next payment period will depend on the outstanding balance before t-th payment.

$$I_t = i \cdot B_{t-1} = i \cdot R \cdot a_{\overline{n-t+1}|} = R \cdot (1 - v^{n-t+1})$$

2.3.2 Principal portion in period t

$$\begin{aligned} P_t &= R - I_t \\ &= R - R \cdot (1 - v^{n-t+1} \\ &= R \cdot v^{n-t+1} \end{aligned}$$

Only for level payments:

$$\text{FV of principal } P_t \text{ after } k \text{ periods at interest rate } i = P_t \cdot (1+i)^k = P_{t+k}$$

2.3.3 Balance

- **Interest:** Paid to lender (does **not** reduce balance)
- **Principal:** Reduces the loan balance

$$\begin{aligned} \text{Balance after t-th payment} &= \text{Balance before t-th payment} - \text{Principal repaid at time t} \\ &= B_{t-1} - P_t \end{aligned}$$

$$\begin{aligned} \text{Balance after t-th payment} &= \text{Balance at time t with interest} - \text{Payment made at time t} \\ &= B_{t-1}(1+i) - R_t \end{aligned}$$

2.4 Summary

Component	**Formula at Period t**
Total Payment R	$R = 1$
Interest I_t	$I_t = 1 - v^{n-t+1}$
Principal Repaid P_t	$P_t = v^{n-t+1}$

Chapter 3

Rate of return of an Investment

3.1 Discounted Cash Flow Technique

DCF is a method to measure the **profitability** of investment projects. Unlike fixed annuities (same payment pattern), DCF allows any pattern of cash inflows (returns) and outflows (costs).

Feature	**Annuities**	**Investments**
Payments	Regular intervals, level payment	May vary in time and amount
Risk	Low/fixed	High/low
Examples	Pensions, loans	Stocks, real estate

There are **two** DCF measures:

1. Net Present Value (NPV) – Present value of all net cash flows.
2. Internal Rate of Return (IRR) – Interest rate that makes NPV = 0.

3.1.1 Cash flows

Notations with Cash flows:

- C_t = contributions/outflows (money invested)
- R_t = returns/inflows (money received)
- $c_t = R_t - C_t$ = net cash flow at time t

$c_t > 0$: Net deposit (inflow), $c_t < 0$: Net withdrawal (outflow)

Example 2. Project Description:
A company plans to develop and sell a new product. The cash flows are as follows:

- Initial investment of $80,000 at year 0.
- Additional investments of $10,000 in years 1, 2, and 3.
- A contribution of $20,000 in year 4 to launch the product.
- Maintenance costs of $2,000 per year from years 5 to 9.
- Returns: $12,000 in year 4, $30,000 in year 5, $40,000 in year 6, $35,000 in year 7, $25,000 in year 8, $15,000 in year 9, and $8,000 in year 10.

Cash Flow Table:

Year	Contributions	Returns	Net Cash Flow (c_t)
0	80,000	0	-80,000
1	10,000	0	-10,000
2	10,000	0	-10,000
3	10,000	0	-10,000
4	20,000	12,000	-8,000
5	2,000	30,000	28,000
6	2,000	40,000	38,000
7	2,000	35,000	33,000
8	2,000	25,000	23,000
9	2,000	15,000	13,000
10	0	8,000	8,000

Net Present Value (NPV):
Let i be the interest rate (cost of capital), and let $v = \frac{1}{1+i}$. Then the NPV of the project is:

$$\text{NPV}(i) = \sum_{t=0}^{10} c_t v^t = \sum_{t=0}^{10} \frac{c_t}{(1+i)^t}$$

Where c_t is the net cash flow in year t.
Interpretation:

- If NPV$(i) > 0$: the investment is profitable.
- If NPV$(i) = 0$: the investment breaks even.
- If NPV$(i) < 0$: the investment is not worth it.

3.2 Net Present Value

Net present value (NPV) is the difference between the present value of **cash inflows** and the present value of **cash outflows** over a period of time. Choose the investment with the greatest positive NPV.

Formula:

$$\text{NPV(i)} = \sum (R_t - C_t) v^t = \sum c_t \cdot v^t$$

- $v^t = \frac{1}{(1+i)^t}$ is the discount factor
- i = rate of interest per period = required return of the investment = cost of capital

3.3 Yield Rate - Internal Rate of Return (IRR)

Yield rate or IRR is the rate such that the PV of cash inflows is equal to the PV of cash outflows. Choose the investment with the greatest IRR.

Formula:

$$\text{IRR} = \text{value of } i \text{ such that } \sum (A_t - L_t) v^t = 0$$

Interpretation:
Yield Rate (IRR) = The interest rate where an investment neither gains nor loses money.

- NPV > 0 : Profit
- NPV < 0 : Loss
- NPV $= 0$: Break-even (Yield rate achieved)

Should you still invest when NPV $= 0$? When NPV $= 0$, there is no net gain, but no loss either. Reject the investment if there are better opportunities (i.e. another project with NPV > 0). Accept it when the Yield rate matched the **inflation rate**, so your money can keep its real value.

Connection Between NPV and IRR

- NPV is a function of the interest rate: $P(i)$
- IRR is the rate where $P(i) = 0$

3.4 Reinvestment

3.4.1 Lump Sum Investment + Interest reinvested

Intuition:
1. Invest 1 unit of money - **a lump sum/principal amount** - for n periods at rate i.
2. Interest is **reinvested** at rate j.

Reinvesting is like planting a tree (investment) and using its seeds (interest) to grow more trees instead of eating them. Over time, you get a forest!
Illustration:

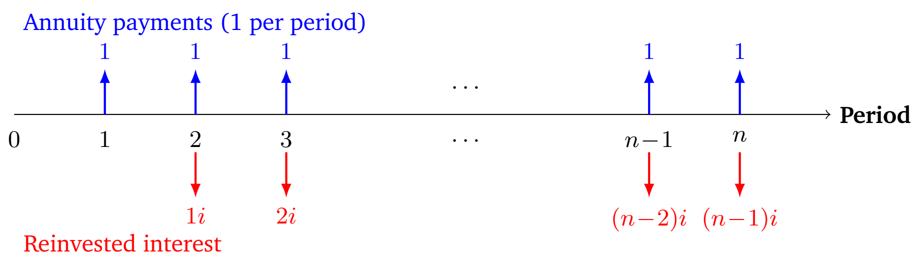
1. Start with 1 unit of money - that principal stays in the account for **n periods.**
2. Interest earned per period is 1$ * i = $i. The principal gets no compounding effect.

3. At the end of year 1,2,...,n-1, we reinvest $i at each end of the year. This pattern follows an annuity-immediate with **n payments** of $i, and rate per period is j.
4. The **AV** of that annuity is: i * $s_{\overline{n}|j}$.
5. Add the principle: Total AV = Principle + Reinvested Interest = 1 + i * $s_{\overline{n}|j}$.

> **Formula 7** (Lump Sum Reinvestment) Total Accumulated Value = 1 + i * $s_{\overline{n}|j}$ Special case: $i = j$, then AV = $(1+i)^n$

3.4.2 Annuity + Interest reinvested

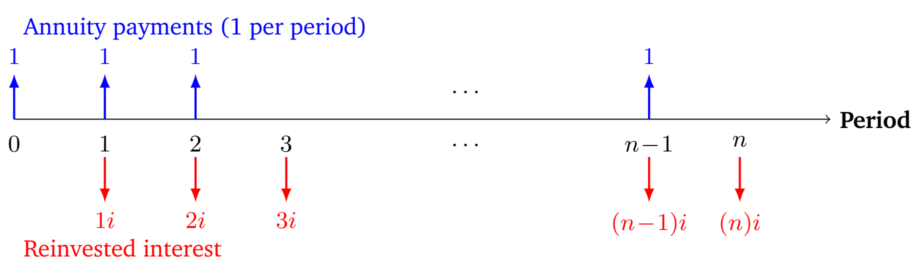
Annuity-immediate



AV = sum of annual payments + reinvested interests as annuity
- Reinvested interests as an annuity: i, 2i, 3i, ..., (n-2)i, (n-1)i. This acts as an increasing annuity with i as a common difference.
- Sum of annual payments is n.

> **Formula 8** (Interest Reinvestment as Annuity-immediate)
>
> $$AV = n + i(I_s)_{\overline{n-1}|j} = n + i[\frac{s_{\overline{n}|j} - n}{j}]$$
>
> $$AV = s_{\overline{n}|i} \text{ when } i = j$$

Annuity-due



Formula 9 (Interest Reinvestment as Annuity-due)

$$AV = n + i(I_s)_{\overline{n}|j} = n + i[\frac{s_{\overline{n+1}|j} - (n+1)}{j}]$$

3.5 Dollar-weighted Rate of Interest

Concept	**DWR**	**IRR (Rate for NPV = 0)**
Purpose	Measures fund performance	Evaluates project profitability
Method	Based on future value	Based on present value
Time orientation	Grows cash to end	Discounts cash to start
Terminology	Used in actuarial & finance	Used in finance & investment

(Dollar-weighted rate of interest) The **interest rate** i is the average rate of how fast the money grew during a period, based on all the deposits, withdrawals, and interest earned ("dollar-weighted").

Key terms:

- A: Amount at the beginning of the period.
- B: Amount at the end of the period.
- I: Total interest earned during the period.
- c_t: Net contribution (deposit - withdrawal) at time $t \in [0, 1]$
- $C = \sum c_t$: Total net contributions.
- $(1+i)^{1-t} - 1$ is the effective rate for period from t to 1.

Concept:

1. Total amount at the end is: $B = A + C + I$
2. Interest earned: $I = iA + \sum_{0 \le t \le 1} c_t[(1+i)^{1-t} - 1]$

 where iA is the interest on initial amount A and $\sum_{0 \le t \le 1} c_t[(1+i)^{1-t} - 1]$ is the interest on total contributions made at time $t \in [0, 1]$.

3. Substitute into

$$\begin{aligned} B &= A + C + I \\ &= A + C + iA + \sum_{0 \le t \le 1} c_t \left[(1+i)^{1-t} - 1\right] \\ &= A(1+i) + \sum_{0 \le t \le 1} c_t (1+i)^{1-t} \end{aligned}$$

4. Approximate the compound interest using simple interest:

$$(1+i)^{1-t} \approx 1 + (1-t)i \text{ then } (1+i)^{1-t} - 1 \approx (1-t)i$$

5. $I = iA + \sum_{0 \leq t \leq 1} c_t[(1-t)i] = i[A + \sum_{0 \leq t \leq 1} c_t(1-t)]$ then $i = \frac{I}{A + \sum_{0 \leq t \leq 1} c_t(1-t)}$

Formula 10

$$i = \frac{I}{A + \sum_{0 \leq t \leq 1} c_t(1-t)}$$

The approximation is good when each contribution c_t is small compared to the amount A.