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# Chapter 1

## Annuities with non-contingent payments

An annuity is a series of payments made at equal time intervals.

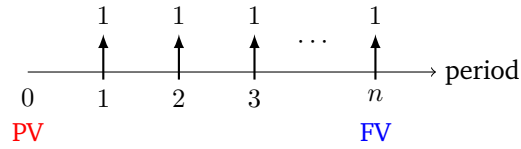
Types of Annuities (Based on Payment Structure)

1. **Amount of Payments**
  - **Level payments:** Equal payment each period
  - **Non-level payments:** Varying payment amounts
2. **Timing of Payments**
  - **Immediate annuity:** Payment at *end* of each period
  - **Due annuity:** Payment at *beginning* of each period
3. **Number of Payments**
  - **Term annuity:** Fixed number of payments
  - **Perpetuity:** Payments continue *forever*
4. **Deferral of Payments**
  - **Deferred annuity:** Payments start *after a delay*

### 1.1 Level annuity

#### 1.1.1 Immediate Annuity

Consider a level annuity-immediate where each payment is **1 unit**, made at the **end of each period**. There are  $n$  total payments, and the effective rate of interest **per unit of time** is  $i$ . One unit of time equals one period.



The present value (sum of discounted payments) of that annuity at period  $t=0$  is defined as:

$$PV = \frac{1}{1+i} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^n} = \sum_{k=1}^n \frac{1}{(1+i)^k} = \frac{1-v^n}{i}, \quad \text{where } v = \frac{1}{1+i}$$

The future value (value at time  $n$ ) of the same annuity is:

$$FV = 1 + (1+i) + (1+i)^2 + \cdots + (1+i)^{n-1} = \sum_{k=0}^{n-1} (1+i)^k$$

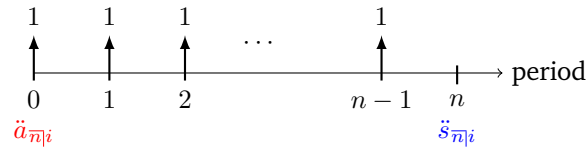
#### Formula 1 (Immediate Annuity)

$$a_{\overline{n}|i} = PV = \frac{1-v^n}{i}, \quad \text{where } v = \frac{1}{1+i}$$

$$s_{\overline{n}|i} = FV = \sum_{k=1}^n (1+i)^k = \frac{(1+i)^n - 1}{i}$$

### 1.1.2 Annuity Due

Consider a level annuity-due where each payment is **1 unit**, made at the **beginning of each period**. There are  $n$  total payments, and the effective rate of interest **per unit of time** is  $i$ . One unit of time equals one period.



The present value (sum of discounted payments) of that annuity at period  $t=0$  is:

$$PV = 1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^{n-1}} = \sum_{k=0}^{n-1} \frac{1}{(1+i)^k} = \frac{1-v^n}{i} \cdot (1+i), \quad \text{where } v = \frac{1}{1+i}$$

The future value (value at time  $n$ ) of the same annuity is:

$$FV = (1+i) + (1+i)^2 + \cdots + (1+i)^n = \sum_{k=1}^n (1+i)^k$$

#### Formula 2 (Annuity Due)

$$\ddot{a}_{\overline{n}|i} = PV = \frac{1-v^n}{i} \cdot (1+i) = \frac{1-v^n}{d}, \quad \text{where } d = \frac{1+i}{i}$$

$$\ddot{s}_{\overline{n}|i} = FV = \sum_{k=1}^n (1+i)^k$$

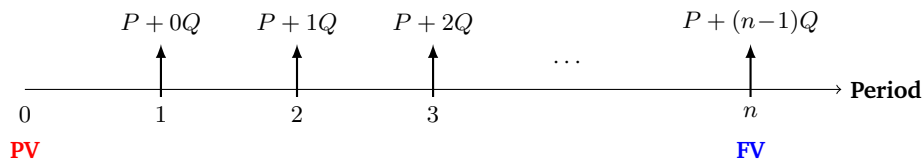
## 1.2 Varying-Payments Annuity

Payments in an annuity that changes instead of staying level.

### 1.2.1 Payments in Arithmetic Progression

#### 1. Arithmetic Increasing Annuity

Let the first payment =  $P$ , each following payment increases by  $Q$ , and there are total  $n$  payments.



#### Formula 3 (Arithmetic Increasing Annuity)

Present Value at time (period)  $t=0$ :

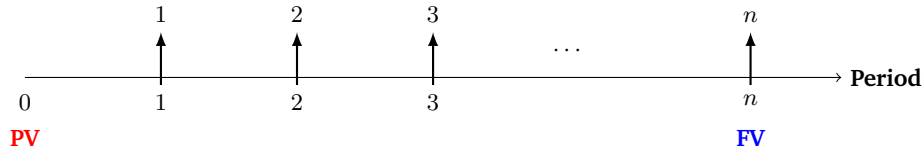
$$PV = P \cdot a_{\overline{n}|i} + Q \cdot \frac{a_{\overline{n}|i} - n \cdot v^n}{i}$$

Accumulated Value at time (period)  $t=n$ :

$$AV = P \cdot s_{\overline{n}|i} + Q \cdot \frac{s_{\overline{n}|i} - n}{i}$$

## 2. Increasing Annuity

When  $P=Q=1$ , payments become  $1, 2, \dots, n$ .



### Formula 4 (Increasing Annuity)

PV at time (period)  $t = 0$ :

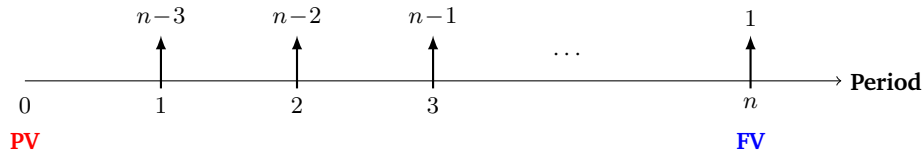
$$(Ia)_{\overline{n}|i} = a_{\overline{n}|i} + \frac{a_{\overline{n}|i} - nv^n}{i} = \frac{(1+i)a_{\overline{n}|i} - nv^n}{i} = \frac{\ddot{a}_{\overline{n}|i} - nv^n}{i}$$

FV at time (period)  $t = n$ :

$$(Is)_{\overline{n}|i} = (1+i)^n \cdot (Ia)_{\overline{n}|i} = \frac{\ddot{s}_{\overline{n}|i} - n}{i} = \frac{s_{\overline{n+1}|i} - (n+1)}{i}$$

## 3. Decreasing Annuity

When  $P=n$  and  $Q=1$ , payments become  $n, n-1, n-2, \dots, 1$ .



### Formula 5 (Decreasing Annuity)

$$PV_{t=0} = (Da)_{\overline{n}|i} = na_{\overline{n}|i} - \frac{a_{\overline{n}|i} - nv^n}{i} = \frac{n - nv^n - a_{\overline{n}|i} + nv^n}{i} = \frac{n - a_{\overline{n}|i}}{i}$$

$$FV_{t=n} = (Ds)_{\overline{n}|i} = (1+i)^n (Da)_{\overline{n}|i} = \frac{n(1+i)^n - s_{\overline{n}|i}}{i} = (n+1)a_{\overline{n}|i} - (Ia)_{\overline{n}|i}$$

## 4. Varying Perpetuity-Immediate with Arithmetic growth

The payments follow an Arithmetic progression with constants  $P \neq 0$  and  $Q \neq 0$ . Payments are  $P, P+Q, P+2Q, \dots$ . First payment is at the end of first period.

Annuity	PV	FV
Arithmetic Increasing Annuity	$P \cdot a_{\overline{n} i} + Q \cdot \frac{a_{\overline{n} i} - n \cdot v^n}{i}$	$P \cdot s_{\overline{n} i} + Q \cdot \frac{s_{\overline{n} i} - n}{i}$

Table 1.1: Summary

Present Value of Perpetuity-Immediate with payments of arithmetic progression:

$$PV = \lim_{n \rightarrow \infty} \left( P a_{\overline{n}|i} + Q \cdot \frac{a_{\overline{n}|i} - n v^n}{i} \right)$$

Since  $n$  goes to infinity,

$$\lim_{n \rightarrow \infty} a_{\overline{n}|i} = a_{\infty|i} = \frac{1}{i}$$

$$\lim_{n \rightarrow \infty} n v^n = 0 \quad (\text{via L'Hôpital's Rule})$$

Hence,

$$PV = \frac{P}{i} + \frac{Q}{i^2}$$

**Formula 6** (Increasing Perpetuity-Immediate - Arithmetic Growth)

When  $P = Q = 1$ :

$$(Ia)_{\infty|i} = \frac{1}{i} + \frac{1}{i^2}$$

### 1.3 Deferred Annuity

A deferred annuity is an annuity where the payments start at a future date, not immediately.

There are 2 parts in a deferred annuity:

1. Deferral period: Time you wait before payments start.
2. Payment period: Regular payments begin (like a normal annuity).

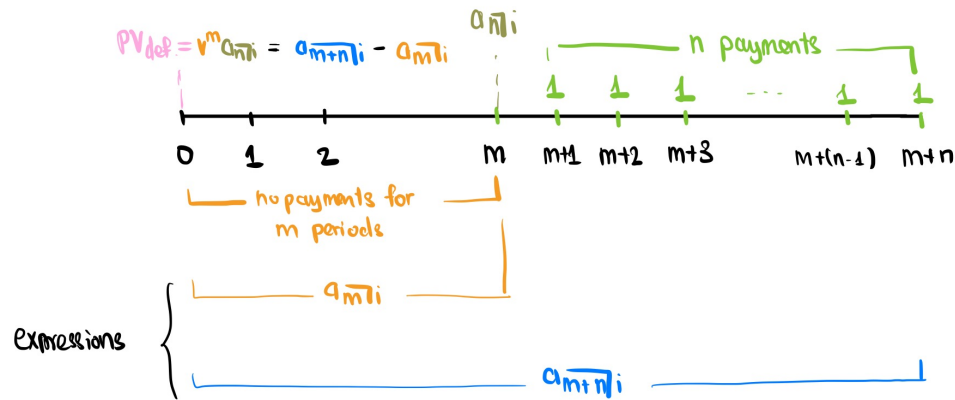


Figure 1.1: Deferred Annuity

Three annuity valuation cases:

- **Before 1st Payment (Deferred Annuity):**  $PV = v^m a_n|i = a_{m+n|i} - a_m|i$
- **After Last Payment:** AV = move future value forward using interest.
- **Between Payments:** Split into past and future parts: some payments made, some still due.

Calculate the present value of the  $n$ -period annuity ( $a_n$ ) as if payments were starting now. Then discount it back  $m$  periods using  $v^m = (1+i)^{-m}$ .

### 1.3.1 PV before first payment of an annuity-immediate

**Theorem 1.** The present value is:  $PV = v^m a_n$

We proof the above theorem by deriving from standard annuity formulas:

$a_{m+n|i}$  is the PV of an annuity-immediate including all payments over  $m+n$  periods (at this point, we assume that there are  $m+n$  payments from period  $t=1$  to  $t=m+n$  but the actual number of payments is only  $n$ , as in  $0-m$  period, there are zero payments).

$a_m|i$  is the PV of an annuity-immediate over  $m$  periods with  $m$  payments (this is an assumption as in reality there is zero payment at this point since this is the deferred period).



*Proof.*

$$\begin{aligned}
 a_{\overline{m+n}|i} - a_{\overline{m}|i} &= \frac{1 - v^{m+n}}{i} - \frac{1 - v^m}{i} \\
 &= \frac{v^m - v^{m+n}}{i} = \frac{v^m(1 - v^n)}{i} = v^m a_n \\
 \Rightarrow a_{\overline{m+n}|i} - a_{\overline{m}|i} &= v^m a_n
 \end{aligned}$$

This means:

- Calculate the present value of the  $n$ -period annuity ( $a_n$ ) at time  $t=m$  (1st payment starts at time  $t=m+1$ ).
- Then discount it back  $m$  periods using  $v^m = (1+i)^{-m}$ .

Note: those expressions are for convenient calculation.

□

## 1.4 Varying-Interest Annuity

An annuity has an interest that can vary in each period. Let  $i_k$  be the interest rate from time  $k-1$  to  $k$ .

### 1.4.1 Rate $i_k$ is applied for period $k$ (date of payments matter)

Starting from when the 1st payment made at time 0, up to time  $k^{th}$  or later, the future payments are discounted/accumulated using **all previous rate**  $i_1, i_2, i_3, \dots, i_k$ .

#### Annuity-immediate

The PV of an annuity-immediate is:

$$a_{\overline{n}|i} = \frac{1}{1+i_1} + \frac{1}{(1+i_1)(1+i_2)} + \dots + \frac{1}{(1+i_1)(1+i_2)\dots(1+i_n)}$$

The AV of an annuity-immediate is:

$$s_{\overline{n+1}|i} = \ddot{s}_{\overline{n}|i} + 1$$

**Annuity-due**

The PV of an annuity-due is:

$$\ddot{a}_{\overline{n}|i} = 1 + a_{\overline{n-1}|i}$$

The AV of an annuity-due is:

$$\ddot{s}_{\overline{n}|i} = (1 + i_1)(1 + i_2) \cdots (1 + i_n) + \cdots + (1 + i_{n-1})(1 + i_n) + (1 + i_n)$$

### 1.4.2 Rate $i_k$ is applied for period $k$ and before/after (date of payment does not matter)

Regardless of when the payments are made, any payments in the period  $k$  is discounted using only rate  $i_k$ . The rate  $i_k$  is the effective rate for period  $i \leq k$  (PV) and  $i \geq k$ .

**Annuity-Immediate**

The PV of an annuity-immediate is:

$$a_{\overline{n}|i} = \frac{1}{(1 + i_1)^1} + \frac{1}{(1 + i_2)^2} + \cdots + \frac{1}{(1 + i_n)^n}$$

The FV of an annuity-immediate is:

$$s_{\overline{n+1}|i} = \ddot{s}_{\overline{n}|i} + 1$$

**Annuity-due**

The PV of an annuity-due is:

$$\ddot{a}_{\overline{n}|i} = 1 + a_{\overline{n-1}|i}$$

The FV of an annuity-due is:

$$\ddot{s}_{\overline{n}|i} = (1 + i_1)^n + (1 + i_2)^{n-1} + \cdots + (1 + i_n)^1$$

## 1.5 Annuity with non-coinciding frequencies

An annuity where **payments** made and **interest** compounded at a different frequency.

## 1.6 Non-level Annuity

### 1.6.1 Geometric annuity

A **geometric annuity** is an annuity in which the payments form a geometric progression.

Consider a non-level annuity with  $n$  payments, growth rate of each payment  $k$ , and the first payment to be 1 unit. The present value (sum of discounted payments) of that annuity at period  $t=0$  is defined as:

$$\begin{aligned}
 PV &= v + v^2(1+k) + \cdots + v^n(1+k)^{n-1} \\
 &= v \left[ \frac{1 - \left(\frac{1+k}{1+i}\right)^n}{1 - \frac{1+k}{1+i}} \right] = v \left[ \frac{1 - \left(\frac{1+k}{1+i}\right)^n}{\frac{i-k}{1+i}} \right] \\
 &= v \left[ \frac{1 - \left(\frac{1+k}{1+i}\right)^n}{v(i-k)} \right] \\
 &= \frac{1 - \left(\frac{1+k}{1+i}\right)^n}{i-k}
 \end{aligned}$$

Since the right-hand side of the equation above is the sum of a finite geometric series.

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

**Example 1.** (Geometric annuity with unknown interest rate) A 10-year annuity-immediate has:

- First payment: \$11
- Subsequent payments: 10% increase each year
- Accumulated value: \$220.8

Find the annual effective interest rate  $i$ .

1. **AV as Geometric Series:**

$$AV = 11(1.1)^9 \left[ 1 + \frac{1+i}{1.1} + \left( \frac{1+i}{1.1} \right)^2 + \cdots + \left( \frac{1+i}{1.1} \right)^9 \right]$$

2. **Sum the Series:**

$$220.8 = 11(1.1)^9 \cdot \frac{1 - \left( \frac{1+i}{1.1} \right)^{10}}{1 - \frac{1+i}{1.1}}$$

3. **Substitute Values:**

$$220.8 = 25.937 \cdot \frac{1 - (1+j)^{10}}{-j}, \quad j = \frac{1.1}{1+i} - 1$$

4. **Solve for  $j$ :**

$$\frac{(1+j)^{10} - 1}{j} = 8.513 \implies j \approx 0.03773$$

5. **Find  $i$ :**

$$j = \frac{1.1}{1+i} - 1 \implies i = \boxed{0.06} \text{ (6\%)}$$

# Chapter 2

## Loan

### 2.1 Debt instruments

A debt instrument is a contract that requires the borrower to repay principal and usually interest at a future date.

Examples of **Debt Instruments**:

1. Bonds (corporate, government)
2. Loans

### 2.2 Outstanding Balance Calculation for Level Payments

**Notation:**

- $I_t$ : Interest paid during the  $k$ -th period.
- $P_t$ : Principal (capital) repaid during the  $k$ -th period.
- $B_t$ : Outstanding balance immediately after the  $t$ -th payment.
- $R_t$ : Total payment made during the  $t$ -th period (interest + principal).

Type	Formula	Interpretation
Prospective	$B_t = R \cdot a_{n-t}$	Present value of remaining (future) level payments
Retrospective	$B_t = R \cdot s_t$	Accumulated value of past payments

**Loan Equation of Value:** You can express the loan's total value  $L$  using the equation of value:

$$L = B_0 = R_1v + R_2v^2 + \cdots + R_nv^n$$

## 2.3 Loan Amortization (Level/Non-level payments)

A loan can be interpreted as an annuity with payments made in regular intervals, each payment consists of two parts:

- **Interest on the loan:** A cost for borrowing the money.
- **Principal:** The amount of the loan that you borrowed.

Over time, interest portion decreases and principal portion increases. Total payments stay the same.

### 2.3.1 Interest portion in period t

At any given time, the interest due for the next payment period will depend on the outstanding balance before t-th payment.

$$I_t = i \cdot B_{t-1} = i \cdot R \cdot a_{\overline{n-t+1}|i} = R \cdot (1 - v^{n-t+1})$$

### 2.3.2 Principal portion in period t

$$\begin{aligned} P_t &= R - I_t \\ &= R - R \cdot (1 - v^{n-t+1}) \\ &= R \cdot v^{n-t+1} \end{aligned}$$

Only for level payments:

$$\text{FV of principal } P_t \text{ after } k \text{ periods at interest rate } i = P_t \cdot (1 + i)^k = P_{t+k}$$

### 2.3.3 Balance

- **Interest:** Paid to lender (does **not** reduce balance)
- **Principal:** Reduces the loan balance

$$\begin{aligned} \text{Balance after } t\text{-th payment} &= \text{Balance before } t\text{-th payment} - \text{Principal repaid at time } t \\ &= B_{t-1} - P_t \end{aligned}$$

$$\begin{aligned} \text{Balance after } t\text{-th payment} &= \text{Balance at time } t \text{ with interest} - \text{Payment made at time } t \\ &= B_{t-1}(1 + i) - R_t \end{aligned}$$

## 2.4 Summary

Component	Formula at Period $t$
Total Payment $R$	$R = 1$
Interest $I_t$	$I_t = 1 - v^{n-t+1}$
Principal Repaid $P_t$	$P_t = v^{n-t+1}$





## Chapter 3

# Rate of return of an Investment

### 3.1 Discounted Cash Flow Technique

DCF is a method to measure the **profitability** of investment projects. Unlike fixed annuities (same payment pattern), DCF allows any pattern of cash inflows (returns) and outflows (costs).

Feature	Annuities	Investments
Payments	Regular intervals, level payment	May vary in time and amount
Risk	Low/fixed	High/low
Examples	Pensions, loans	Stocks, real estate

There are **two** DCF measures:

1. Net Present Value (NPV) – Present value of all net cash flows.
2. Internal Rate of Return (IRR) – Interest rate that makes  $NPV = 0$ .

#### 3.1.1 Cash flows

Notations with Cash flows:

- $C_t$  = contributions/outflows (money invested)
- $R_t$  = returns/inflows (money received)
- $c_t = R_t - C_t$  = net cash flow at time  $t$

$c_t > 0$ : Net deposit (inflow),  $c_t < 0$ : Net withdrawal (outflow)

**Example 2. Project Description:**

A company plans to develop and sell a new product. The cash flows are as follows:

- Initial investment of \$80,000 at year 0.
- Additional investments of \$10,000 in years 1, 2, and 3.
- A contribution of \$20,000 in year 4 to launch the product.
- Maintenance costs of \$2,000 per year from years 5 to 9.
- Returns: \$12,000 in year 4, \$30,000 in year 5, \$40,000 in year 6, \$35,000 in year 7, \$25,000 in year 8, \$15,000 in year 9, and \$8,000 in year 10.

**Cash Flow Table:**

Year	Contributions	Returns	Net Cash Flow ( $c_t$ )
0	80,000	0	-80,000
1	10,000	0	-10,000
2	10,000	0	-10,000
3	10,000	0	-10,000
4	20,000	12,000	-8,000
5	2,000	30,000	28,000
6	2,000	40,000	38,000
7	2,000	35,000	33,000
8	2,000	25,000	23,000
9	2,000	15,000	13,000
10	0	8,000	8,000

**Net Present Value (NPV):**

Let  $i$  be the interest rate (cost of capital), and let  $v = \frac{1}{1+i}$ . Then the NPV of the project is:

$$\text{NPV}(i) = \sum_{t=0}^{10} c_t v^t = \sum_{t=0}^{10} \frac{c_t}{(1+i)^t}$$

Where  $c_t$  is the net cash flow in year  $t$ .

**Interpretation:**

- If  $\text{NPV}(i) > 0$ : the investment is profitable.
- If  $\text{NPV}(i) = 0$ : the investment breaks even.
- If  $\text{NPV}(i) < 0$ : the investment is not worth it.

## 3.2 Net Present Value

**Net present value (NPV)** is the difference between the present value of **cash inflows** and the present value of **cash outflows** over a period of time. Choose the investment with the greatest positive NPV.

**Formula:**

$$\text{NPV}(i) = \sum (R_t - C_t)v^t = \sum c_t \cdot v^t$$

- $v^t = \frac{1}{(1+i)^t}$  is the discount factor
- $i$  = rate of interest per period = required return of the investment = cost of capital

### 3.3 Yield Rate - Internal Rate of Return (IRR)

**Yield rate** or IRR is the rate such that the PV of cash inflows is equal to the PV of cash outflows. Choose the investment with the greatest IRR.

**Formula:**

$$\text{IRR} = \text{value of } i \text{ such that } \sum (A_t - L_t)v^t = 0$$

**Interpretation:**

Yield Rate (IRR) = The interest rate where an investment neither gains nor loses money.

- $\text{NPV} > 0$  : Profit
- $\text{NPV} < 0$  : Loss
- $\text{NPV} = 0$  : Break-even (Yield rate achieved)

Should you still invest when  $\text{NPV} = 0$  ? When  $\text{NPV} = 0$ , there is no net gain, but no loss either. Reject the investment if there are better opportunities (i.e. another project with  $\text{NPV} > 0$ ). Accept it when the Yield rate matched the **inflation rate**, so your money can keep its real value.

#### Connection Between NPV and IRR

- NPV is a function of the interest rate:  $P(i)$
- IRR is the rate where  $P(i) = 0$

### 3.4 Reinvestment

#### 3.4.1 Lump Sum Investment + Interest reinvested

Intuition:

1. Invest 1 unit of money - a **lump sum/principal amount** - for  $n$  periods at rate  $i$ .
2. Interest is **reinvested** at rate  $j$ .

Reinvesting is like planting a tree (investment) and using its seeds (interest) to grow more trees instead of eating them. Over time, you get a forest!

Illustration:

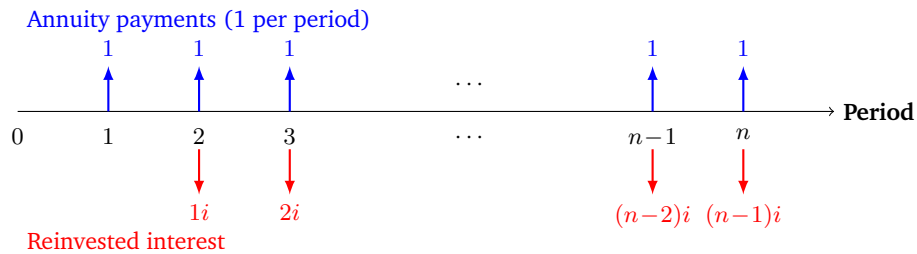
1. Start with 1 unit of money - that principal stays in the account for  **$n$  periods**.
2. Interest earned per period is  $1\$ * i = \$i$ . The principal gets no compounding effect.

3. At the end of year 1,2,...,n-1, we reinvest \$i at each end of the year. This pattern follows an annuity-immediate with **n payments** of \$i, and rate per period is  $j$ .
4. The AV of that annuity is:  $i * s_{\overline{n}|j}$ .
5. Add the principle: Total AV = Principle + Reinvested Interest =  $1 + i * s_{\overline{n}|j}$ .

**Formula 7** (Lump Sum Reinvestment) Total Accumulated Value =  $1 + i * s_{\overline{n}|j}$  Special case:  $i = j$ , then  $AV = (1 + i)^n$

### 3.4.2 Annuity + Interest reinvested

#### Annuity-immediate



AV = sum of annual payments + reinvested interests as annuity

- Reinvested interests as an annuity:  $i, 2i, 3i, \dots, (n-2)i, (n-1)i$ . This acts as an increasing annuity with  $i$  as a common difference.

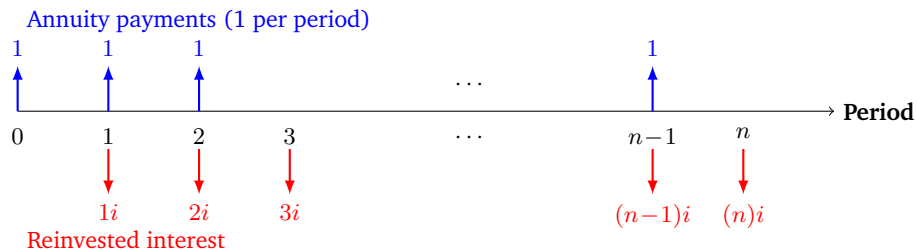
- Sum of annual payments is  $n$ .

**Formula 8** (Interest Reinvestment as Annuity-immediate)

$$AV = n + i(I_s)_{\overline{n-1}|j} = n + i \left[ \frac{s_{\overline{n}|j} - n}{j} \right]$$

$$AV = s_{\overline{n}|i} \text{ when } i = j$$

#### Annuity-due



**Formula 9** (Interest Reinvestment as Annuity-due)

$$AV = n + i(I_s)_{\overline{n}|j} = n + i \left[ \frac{s_{\overline{n+1}|j} - (n+1)}{j} \right]$$

### 3.5 Dollar-weighted Rate of Interest

Concept	DWR	IRR (Rate for NPV = 0)
Purpose	Measures fund performance	Evaluates project profitability
Method	Based on future value	Based on present value
Time orientation	Grows cash to end	Discounts cash to start
Terminology	Used in actuarial & finance	Used in finance & investment

(Dollar-weighted rate of interest) The **interest rate**  $i$  is the average rate of how fast the money grew during a period, based on all the deposits, withdrawals, and interest earned ("dollar-weighted").

Key terms:

- $A$ : Amount at the beginning of the period.
- $B$ : Amount at the end of the period.
- $I$ : Total interest earned during the period.
- $c_t$ : Net contribution (deposit - withdrawal) at time  $t \in [0, 1]$
- $C = \sum c_t$ : Total net contributions.
- $(1 + i)^{1-t} - 1$  is the effective rate for period from  $t$  to 1.

Concept:

1. Total amount at the end is:  $B = A + C + I$

2. Interest earned:  $I = iA + \sum_{0 \leq t \leq 1} c_t [(1 + i)^{1-t} - 1]$

where  $iA$  is the interest on initial amount  $A$  and  $\sum_{0 \leq t \leq 1} c_t [(1 + i)^{1-t} - 1]$  is the interest on total contributions made at time  $t \in [0, 1]$ .

3. Substitute into

$$\begin{aligned}
 B &= A + C + I \\
 &= A + C + iA + \sum_{0 \leq t \leq 1} c_t [(1 + i)^{1-t} - 1] \\
 &= A(1 + i) + \sum_{0 \leq t \leq 1} c_t (1 + i)^{1-t}
 \end{aligned}$$

4. Approximate the compound interest using simple interest:

$$(1+i)^{1-t} \approx 1 + (1-t)i \text{ then } (1+i)^{1-t} - 1 \approx (1-t)i$$

$$5. I = iA + \sum_{0 \leq t \leq 1} c_t[(1-t)i] = i[A + \sum_{0 \leq t \leq 1} c_t(1-t)] \text{ then } i = \frac{I}{A + \sum_{0 \leq t \leq 1} c_t(1-t)}$$

**Formula 10**

$$i \approx \frac{I}{A + \sum_{0 \leq t \leq 1} c_t(1-t)} = \frac{\text{Interest earned}}{\text{Exposure}}$$

The approximation is good when each contribution  $c_t$  is small compared to the amount  $A$ .

The denominator is called **exposure associated with  $i$**  & represents **total time-weighted amount of money at risk**:

- $A$ : Initial fund amount that earns interest for the full year - its weight = 1.
- $\sum_{0 \leq t \leq 1} c_t(1-t)$ : time-weighted contributions, giving how long each contribution had to earn interest ( $c_t$  is the contribution made at time  $t$  and  $(1-t)$  = weight is the fraction of the year left in which the contribution earns interests.)

**Formula 11 (Exposure)**

$$A + \sum_{0 \leq t \leq 1} c_t(1-t)$$

Exposure is as a weighted sum of how much money was active in the fund and for how long.

**Example 3.** At the beginning of a year, an investment fund was established with an initial deposit of \$3,000. At the end of six months, a new deposit of \$1,500 was made. Withdrawals of \$500 and \$800 were made at the end of four months and eight months respectively. The amount in the fund at the end of the year is \$3,876. Set up the equation of value to calculate the dollar-weighted rate of interest.

**Solution:** To find the interest rate  $i$  that makes the future value of all cash flows = \$3876.

General formula:

$$\text{Future Value} = \sum c_t(1+i)^{1-t}$$

Set up the equation of value to calculate the dollar-weighted rate of interest  $i$ :

$$3000(1+i) + 1500(1+i)^{0.5} - 500(1+i)^{(1-\frac{4}{12})} - 800(1+i)^{(1-\frac{8}{12})} = 3876$$

### 3.6 Time-weighted Rate of Interest

**Time-weighted rate of return** isolates fund performance and ignores investor actions.

$$i = (1 + j_1)(1 + j_2)\dots(1 + j_m) - 1$$

where  $j_k$  is the rate of return for each sub-period in an interval (in this case, a year) which has  $m$  sub-periods.

**Set up:**

- The year is split into  $m$  intervals (sub-periods)
- At each time  $t_k$ ,

$C_{t_k}$  = net contribution

$B_{t_k}$  = fund value just before that contribution

then for each subinterval;  $k = 1, 2, \dots, m$ , the rate of return  $j_k$  for each sub-period is

$$B_{t_k} = (1 + j_k)(B_{t_{k-1}} + C_{t_{k-1}})$$

The overall yield rate  $i$  for the entire year is given by

$$i + 1 = (1 + j_1)(1 + j_2)\dots(1 + j_m)$$

We call  $i$  the **time-weighted rate of return**.





## Chapter 4

# Measures of Interest Rate Sensitivity

### 4.1 Inflation

Inflation = general rise in prices of goods and services over time. It reduces purchasing power of money.

Inflation and Interest Rates: inflation and interest rates move together over time. Investors demand higher interest to compensate for future inflation.

**Formula 12** Let  $\pi$  is the inflation rate.

$$1 + i_{\text{real}} = \frac{1 + i_{\text{nominal}}}{1 + \pi}$$

#### 4.1.1 Payments grow with inflation

##### Present Value

Each future payment grows by constant **inflation rate**.

**Formula 13** (PV iwth adjusted payments)

$$PV = R \left[ \frac{1+r}{1+i} + \left( \frac{1+r}{1+i} \right)^2 + \cdots + \left( \frac{1+r}{1+i} \right)^n \right] = R(1+r) \cdot \frac{1 - \left( \frac{1+r}{1+i} \right)^n}{i-r}$$

We discount the annuity with **nominal rate** as payments have been adjusted with inflation rate.

**Formula 14** (PV with ray payments, while inflation exists)

$$PV = R \left[ \frac{1}{1+i_0} + \frac{1}{(1+i_0)^2} + \cdots + \frac{1}{(1+i_0)^n} \right] = R \cdot a_{\overline{n}|i_0}$$

We discount the annuity with **real interest rate**.

## Summary

Scenario	Formula	When to Use
Payments grow with inflation	$PV = R(1+r) \cdot \frac{1 - \left( \frac{1+r}{1+i} \right)^n}{i-r}$	Use nominal rate $i$
Payments fixed in real terms	$PV = R \cdot a_{\overline{n} i_0}$	Use real rate $i_0$

## Accumulated Value

**Formula 15** (AV in nominal dollars (not adjusted for inflation))

$$AV = P(1 + i_{\text{nominal}})^n$$

This is the raw future value of your investment.

**Formula 16** (AV adjusted for inflation)

$$AV = P \left( \frac{1+i}{1+r} \right)^n = P(1 + i_{\text{real}})$$

This reflects the true purchasing power of your money.

## 4.2 The Term structure of Interest Rates and Yield Curves

Term structure of Interest rate shows how interest rates depend on the time (term) to maturity. Generally, longer-term loans/investment results in higher interest rates. This relationship is visualized in a **yield curve**.

Generally — the longer the investment term, the higher the interest rate, because,...

- More time = more risk (like inflation, uncertainty, default).
- Investors want extra return for locking money up longer.
- So lenders charge higher rates for longer loans.

Extend the table to a continuous graph, where y-axis is **yield** (interest rates/spot rates of risk-free bonds) and x-axis is **maturity**, we obtain a yield curve. Yield curve can be upward-sloping (rates expected to rise), flat (all terms have same rate), and inverted (short-term > long-term, a signal of recession).

Length of investment (years)	Interest rate (Spot rate)
1 year	3%
2 year	4%
3 year	6%
4 year	7%

**Yield Curve** is a graph of **spot rates**.

- **Tern**: The length of time until an investment/loan matures/ends
- Term = duration until you get your money back.
- Spot rate = **current** interest rate you would earn if you invested in a **risk-free bond** today and held it until maturity.

**Formula 17** When spot rates  $i_t$  vary by year, NPV is

$$\text{NPV} = \sum_{t=0}^n \frac{c_t}{(1 + i_t)^t}$$

**Forward rate**: Interest rate agreed on today for investment starting in the future. It reflects: Current spot rates (interest rates today for different terms), market expectations of the future, and no-arbitrage principle (no free money).