Contents

Chapter 1

Interest Theory

1.1 Amount and Accumulation Functions

Amount function $A(t)$ refers to value of the investment at time t.

Accumulation function $a(t)$ refers to value of $1, which was invested at time 0, at time t.

Formula 1 relationship between A(t) and a(t) is

$$A(t) = A(0) \cdot a(t)$$

where

- A(0) is money you invested at time 0.
- a(t) tells how much **each dollar** grows to by time t.

- **Simple Interest**
  - Accumulation function: $a(t) = 1 + it$
  - Growth type: Linear
- **Compound Interest**
  - Accumulation function: $a(t) = (1 + i)^t$
  - Growth type: Exponential
- **Continuous (Force of Interest)**
  - Accumulation function: $a(t) = e^{\delta t}$
  - Growth type: Exponential (Smoothest)

1.2 Force of Interest

The **force of interest**, denoted by δ, measures how fast money grows in any specific time.

1.2.1 Constant Force of Interest

Start with a nominal compound interest $i^{(m)}$ converted m times per period. Let m go to infinity, money is compounded **continuously**:

$$\delta(t) = \lim_{m\to\infty} i^{(m)}$$

When using **compound interest**, δ_t is constant.

Formula 2 The initial amount of money grows exponentially with a constant rate δ.

$$A(t) = A(0) \cdot a(t) = A(0) \cdot e^{\delta t}$$

where accumulation function $a(t) = e^{\delta t}$.

Formula 3 To convert annual effective rate to **constant Force of Interest**, and conversely:

$$\delta = \lim_{m\to\infty} i^{(m)} = \ln(1+i)$$

$$i = e^{\delta} - 1$$

- i = effective annual interest rate
- $i^{(m)}$ = nominal rate compounded m times a year

Formula 4 When δ is constant, then $\delta(t) = \delta$ for all t.

$$\text{FV} = A(t) = A(0) \cdot e^{\delta t}$$

$$\text{PV} = A(t) \cdot e^{-\delta t}$$

where accumulation function $a(t) = e^{\delta t} = (1+i)^t$.

1.2.2 Varying Force of Interest

Force of Interest which **changes over time** is a variable and a function of time $\delta(t) = \delta_t$.

$$\delta_t = \frac{a'(t)}{a(t)} = \frac{A'(t)}{A(t)}$$

Given δ_t, we can recover accumulation function $a(t)$.

$$a(t) = e^{\int_0^t \delta_t dr}$$

Formula 5 (Present value and Future value)

$$\text{FV} = A(t) = A(0) \cdot e^{\int_0^t \delta_t dr}$$

$$\text{PV} = A(0) = A(t) \cdot e^{-\int_0^t \delta_t dr}$$

The amount of interest earned over n periods is

$$A(n) - A(0) = I_1 + I_1 + \cdots + I_n = \int_0^n A'(t)dt = \int_0^n A(t)\delta_t dt$$

1.3 Present value

1.3.1 Accumulation function with compound and simple interests

With **compound interest**, the accumulation function is

$$a(t) = (1+i)^t$$

Reminds that $a(t)$ tells how \$1 grows over t periods at (compound) interest rate i.

With **simple interest**, the accumulation function is

$$a(t) = 1 + it$$

Reminds that $a(t)$ tells how \$1 grows over t periods at (simple) interest rate i.

1.3.2 Discounting

1. **Discounting** What is \$1 in future worth today? $\rightarrow (1+i)^t$ after t periods
2. **Accumulation** What does \$1 today grow to in future? $\rightarrow \frac{1}{(1+i)^t}$ after t periods

1.3.3 Discount factor

Discount factor converts future money into present value. For t periods, the discount factor is

$$v^t = \frac{1}{(1+i)^t}$$

Discount factor during nth period is

$$(1+i_n)^{-1} = \frac{A(n-1)}{A(n)}$$

1.3.4 Present Value

(**Lump Sum**) Present Value of an Lump sum amount with compound interest:

$$\text{PV} = \frac{\text{FV}}{(1+i)^t} = \text{FV} \cdot v^t = \text{FV} \cdot \frac{1}{a(t)}$$

(**Cash Flow**) Present Value of a cash flow:

$$\text{PV}_{t=0} = \sum_{k=1}^{n} C_k (a(t_k))^{-1}$$

1.4 Future Value

With cash flows $C_1, C_2, \ldots, C_n$ received at times $t_1, t_2, \ldots, t_n$, then

$$\text{FV}_{t=n} = \sum_{k=1}^{n} C_k a(t_k)$$

- C_k: cash flow received at time t_k
- $a(t_k)$: accumulation factor from time t_k to the final time $t = n$
- $\text{FV}_{t=n}$: future value at time $t = n$

Formula 6 (**Accumulation factor and Deposits at Different time**) of \$k is deposited at time s, and you want to know its value at time $t > s$, use

$$\text{Future Value} = k \cdot \frac{a(t)}{a(s)}$$

where $\frac{a(t)}{a(s)}$ is an accumulation factor.

Chapter 2

Annuities with non-contigent payments

> An annuity is a series of payments made at equal time intervals.

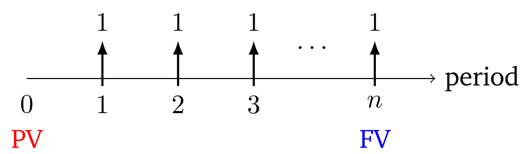
Types of Annuities (Based on Payment Structure)

1. **Amount of Payments**
   - **Level payments**: Equal payment each period
   - **Non-level payments**: Varying payment amounts
2. **Timing of Payments**
   - **Immediate annuity**: Payment at *end* of each period
   - **Due annuity**: Payment at *beginning* of each period
3. **Number of Payments**
   - **Term annuity**: Fixed number of payments
   - **Perpetuity**: Payments continue *forever*
4. **Deferral of Payments**
   - **Deferred annuity**: Payments start *after a delay*

2.1 Level annuity

2.1.1 Immediate Annuity

Consider a level annuity-immediate where each payment is **1 unit**, made at the **end of each period**. There are n total payments, and the effective rate of interest **per unit of time** is i. One unit of time equals one period.



The present value (sum of discounted payments) of that annuity at period t=0 is defined as:

$$PV = \frac{1}{1+i} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^n} = \sum_{k=1}^{n} \frac{1}{(1+i)^k} = \frac{1-v^n}{i}, \quad \text{where } v = \frac{1}{1+i}$$

The future value (value at time n) of the same annuity is:

$$FV = 1 + (1+i) + (1+i)^2 + \cdots + (1+i)^{n-1} = \sum_{k=0}^{n-1} (1+i)^k$$

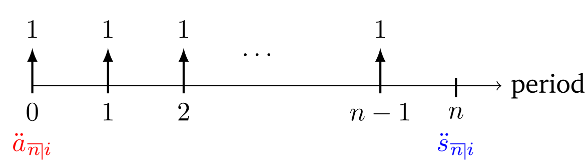
Formula 7 (Immediate Annuity)

$$a_{\overline{n}|i} = PV = \frac{1-v^n}{i}, \quad \text{where } v = \frac{1}{1+i}$$

$$s_{\overline{n}|i} = FV = \sum_{k=1}^{n} (1+i)^k = \frac{(1+i)^n - 1}{i}$$

2.1.2 Annuity Due

Consider a level annuity-due where each payment is **1 unit**, made at the **beginning of each period**. There are n total payments, and the effective rate of interest **per unit of time** is i. One unit of time equals one period.



The present value (sum of discounted payments) of that annuity at period t=0 is:

$$PV = 1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^{n-1}} = \sum_{k=0}^{n-1} \frac{1}{(1+i)^k} = \frac{1-v^n}{i} \cdot (1+i), \quad \text{where } v = \frac{1}{1+i}$$

The future value (value at time n) of the same annuity is:

$$FV = (1+i) + (1+i)^2 + \cdots + (1+i)^n = \sum_{k=1}^{n} (1+i)^k$$

Formula 8 (Annuity Due)

$$\ddot{a}_{\overline{n}|i} = PV = \frac{1-v^n}{i} \cdot (1+i) = \frac{1-v^n}{d}, \quad \text{where } d = \frac{1+i}{i}$$

$$\ddot{s}_{\overline{n}|i} = FV = \sum_{k=1}^{n} (1+i)^k$$

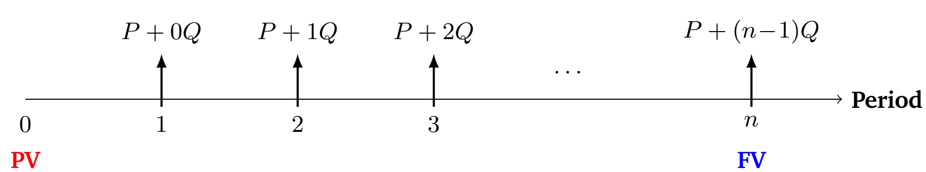
2.2 Varrying-Payments Annuity

Payments in an annuity that changes instead of staying level.

2.2.1 Payments in Arithmetic Progression

1. Arithmetic Increasing Annuity

Let the first payment = **P**, each following payment increases by **Q**, and there are total **n** payments.



Formula 9 (Arithmetic Increasing Annuity)
Present Value at time (period) t=0:

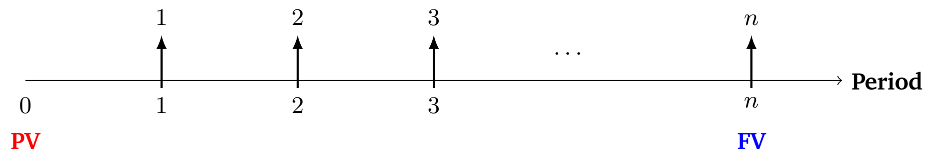
$$PV = P \cdot a_{\overline{n}|i} + Q \cdot \frac{a_{\overline{n}|i} - n \cdot v^n}{i}$$

Accumulated Value at time (period) t=n:

$$AV = P \cdot s_{\overline{n}|i} + Q \cdot \frac{s_{\overline{n}|i} - n}{i}$$

2. Increasing Annuity

When P=Q=1, payments become 1,2,...,n.



Formula 10 (Increasing Annuity)
PV at time (period) t = 0:

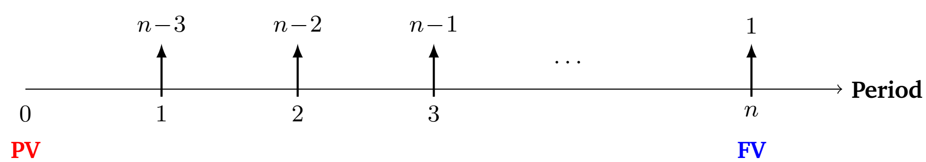
$$(Ia)_{\overline{n}|i} = a_{\overline{n}|i} + \frac{a_{\overline{n}|i} - nv^n}{i} = \frac{(1+i)a_{\overline{n}|i-nv^n}}{i} = \frac{\ddot{a}_{\overline{n}|}i - nv^n}{i}$$

FV at time (period) t = n:

$$(Is)_{\overline{n}|i} = (1+i)^n \cdot (Ia)_{\overline{n}|i} = \frac{\ddot{s}_{\overline{n}|i} - n}{i} = \frac{s_{\overline{n+1}|i} - (n+1)}{i}$$

3. Decreasing Annuity

When P=n and Q=1, payments become n, n-1, n-2, ..., 1.



Formula 11 (Decreasing Annuity)

$$\text{PV}_{t=0} = (Da)_{\overline{n}|} = na_{\overline{n}|i} - \frac{a_{\overline{n}|i} - nv^n}{i} = \frac{n - nv^n - a_{\overline{n}|i} + nv^n}{i} = \frac{n - a_{\overline{n}|i}}{i}$$

$$\text{FV}_{t=n} = (Ds)_{\overline{n}|} = (1+i)^n (Da)_{\overline{n}|} = \frac{n(1+i)^n - s_{\overline{n}|i}}{i} = (n+1)a_{\overline{n}|i} - (Ia)_{\overline{n}|i}$$

4. Varying Perpetuity-Immediate with Arithmetic growth

The payments follow an Arithmetic progression with constants P 0 and Q 0. Payments are P, P+Q, P+2Q,... First payment is at the end of first period.

Annuity	PV	FV
Arithmetic Increasing Annuity	$P \cdot a_{\overline{n}\rvert i} + Q \cdot \frac{a_{\overline{n}\rvert i} - n \cdot v^n}{i}$	$P \cdot s_{\overline{n}\rvert i} + Q \cdot \frac{s_{\overline{n}\rvert i} - n}{i}$

Table 2.1: Summary

Present Value of Perpetuity-Immediate with payments of arithmetic progression:

$$\text{PV} = \lim_{n \to \infty} \left(P a_{\overline{n}\rvert} + Q \cdot \frac{a_{\overline{n}\rvert} - n\nu^n}{i} \right)$$

Since n goes to infinity,

$$\lim_{n \to \infty} a_{\overline{n}\rvert} = a_{\overline{\infty}\rvert} = \frac{1}{i}$$

$$\lim_{n \to \infty} n\nu^n = 0 \quad \text{(via L'Hôpital's Rule)}$$

Hence,

$$\text{PV} = \frac{P}{i} + \frac{Q}{i^2}$$

Formula 12 (Increasing Perpetuity-Immediate - Arithmetic Growth)
When $P = Q = 1$:

$$(Ia)_{\overline{\infty}\rvert} = \frac{1}{i} + \frac{1}{i^2}$$

5. Varying Perpetuity-Due with Arithmetic growth

The PV of a perpetuity-immediate with arithmetic growth is:

$$\text{PV}_{\text{immediate}} = \frac{P}{i} + \frac{Q}{i^2}$$

For perpetuity-due, we shift everything 1 period earlier:

$$\text{PV}_{\text{due}} = (\frac{P}{i} + \frac{Q}{i^2})(1+i) = (\frac{1}{i} + \frac{1}{i^2})(1+i) = \frac{(1+i)^2}{i^2} = \frac{1}{d^2}$$

(Increasing Perpetuity-Due - Arithmetic Growth) When P = Q = 1:

$$(I\ddot{a})_{\overline{\infty}\rvert} = \frac{1}{d^2}$$

2.2.2 Payments in Geometric Progression

A **geometric annuity** has **n payments**, each payment grows by a constant percentage, or **growht rate k**. If the first payment is 1, and growth rate is k, then the payments are

$$\text{Payments} = 1, (1+k), (1+k)^2, \ldots, (1+k)^{n-1}$$

For each payment in n payments of the annuity, each is discounted back to time 0 by discount factor $\frac{1}{1+i}$.

$$\begin{aligned} PV &= v + v^2(1+r) + \cdots + v^n(1+k)^{n-1} \\ &= v(1 + v(1+k) + \cdots + v^{n-1}(1+k)^{n-1}) \end{aligned}$$

The RHS follows a geometric series with common ratio $r = \frac{1+k}{1+i}$, hence its sum follows:

$$\begin{aligned} \text{PV} &= v\left[\frac{1-\left(\frac{1+k}{1+i}\right)^n}{1-\frac{1+k}{1+i}}\right] = v\left[\frac{1-\left(\frac{1+k}{1+i}\right)^n}{\frac{i-k}{1+i}}\right] \\ &= v\left[\frac{1-\left(\frac{1+k}{1+i}\right)^n}{v(i-k)}\right] = \frac{1-\left(\frac{1+k}{1+i}\right)^n}{i-k} \end{aligned}$$

Formula 13 (PV of an annuity-immediate of geometric progression)

$$\text{PV} = \frac{1-\left(\frac{1+k}{1+i}\right)^n}{i-k}$$

Example 1. (Geometric annuity with unknown interest rate) A 10-year annuity-immediate has:

- First payment: \$11
- Subsequent payments: 10% increase each year
- Accumulated value: \$220.8

Find the annual effective interest rate i.

1. **AV as Geometric Series:**

$$AV = 11(1.1)^9 \left[1 + \frac{1+i}{1.1} + \left(\frac{1+i}{1.1}\right)^2 + \cdots + \left(\frac{1+i}{1.1}\right)^9\right]$$

2. **Sum the Series:**

$$220.8 = 11(1.1)^9 \cdot \frac{1 - \left(\frac{1+i}{1.1}\right)^{10}}{1 - \frac{1+i}{1.1}}$$

3. **Substitute Values:**

$$220.8 = 25.937 \cdot \frac{1-(1+j)^{10}}{-j}, \quad j = \frac{1.1}{1+i} - 1$$

4. **Solve for j:**

$$\frac{(1+j)^{10} - 1}{j} = 8.513 \implies j \approx 0.03773$$

5. **Find i:**

$$j = \frac{1.1}{1+i} - 1 \implies i = \boxed{0.06}\ (6\%)$$

2.3 Deferred Annuity

A deferred annuity is an annuity where the payments start at a future date, not immediately.

There are 2 parts in a deferred annuity:

1. Deferral period: Time you wait before payments start.
2. Payment period: Regular payments begin (like a normal annuity).

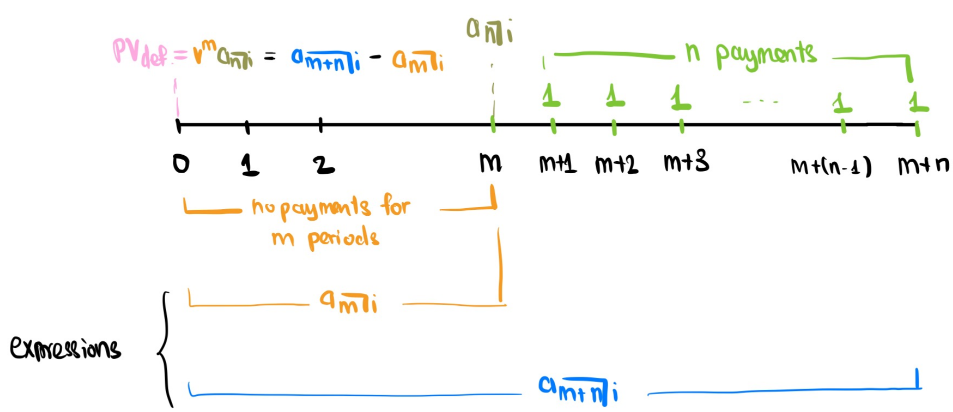


Figure 2.1: Deferred Annuity

Three annuity valuation cases:

- **Before 1st Payment (Deferred Annuity)**: $PV = v^m a_{\overline{n}|i} = a_{\overline{m+n}|i} - a_{\overline{m}|i}$
- **After Last Payment**: AV = move future value forward using interest.
- **Between Payments**: Split into past and future parts: some payments made, some still due.

Calculate the present value of the n-period annuity (a_n) as if payments were starting now. Then discount it back m periods using $v^m = (1+i)^{-m}$.

2.3.1 PV before first payment of an annuity-immediate

Theorem 1. The present value is: PV $= v^m a_n$

We proof the above theorem by derviving from standard annuity formulas:

$a_{\overline{m+n}|i}$ is the PV of an annuity-immediate including all payments over m+n periods (at this point, we assume that there are m+n payments from period t=1 to t=m+n but the actual number of payments is only n, as in 0-m period, there are zero payments).

$a_{\overline{m}|i}$ is the PV of an annuity-immediate over m periods with m payments (this is an assumption as in reality there is zero payment at this point since this is the deferred period).

Proof.

$$a_{\overline{m+n}|i} - a_{\overline{m}|i} = \frac{1-\nu^{m+n}}{i} - \frac{1-\nu^{m}}{i}$$

$$= \frac{\nu^m - \nu^{m+n}}{i} = \frac{\nu^m(1-\nu^n)}{i} = \nu^m a_n$$

$$\Rightarrow a_{\overline{m+n}|i} - a_{\overline{m}|i} = v^m a_n$$

This means:

- Calculate the present value of the n-period annuity (a_n) at time t=m (1st payment starts at time t=m+1.
- Then discount it back m periods using $v^m = (1+i)^{-m}$.

Note: those expressions are for convinient calculation.

□

2.4 Varying-Interest Annuity

An annuity has an interest that can vary in each period. Let i_k be the interest rate from time k-1 to k.

2.4.1 Rate i_k is applied for period k (date depends on calender year)

Starting from when the 1st payment made at time 0, up to time k^{th} or later, the future payments are discounted/accumulated using **all previous rate** $i_1, i_2, i_3, ..., i_k$.

Annuity-immediate

The PV of an annuity-immediate is:

$$a_{\overline{n}|i} = \frac{1}{1+i_1} + \frac{1}{(1+i_1)(1+i_2)} + \cdots + \frac{1}{(1+i_1)(1+i_2)\cdots(1+i_n)}$$

The AV of an anntuiy-immediate is:

$$s_{\overline{n+1}|i} = \ddot{s}_{\overline{n}|i} + 1$$

Annuity-due

The PV of an annuity-due is:

$$\ddot{a}_{\overline{n}|i} = 1 + a_{\overline{n-1}|i}$$

The AV of an anntuiy-due is:

$$\ddot{s}_{\overline{n}|i} = (1+i_1)(1+i_2)\cdots(1+i_n) + \cdots + (1+i_{n-1})(1+i_n) + (1+i_n)$$

2.4.2 Rate i_k is applied for period k and before/later (rate depends on deposit time)

Regardless of when the payments are made, any payments in the period k is discounted using only rate i_k. The rate i_k is the effective rate for period $i \le k$ (PV) and $i \ge k$. Intuition: The rate depends on when you invest - it locks in the rate.

Annuity-Immediate

The PV of an annuity-immediate is:

$$a_{\overline{n}|i} = \frac{1}{(1+i_1)^1} + \frac{1}{(1+i_2)^2} + \cdots + \frac{1}{(1+i_n)^n}$$

The FV of an annuity-immediate is:

$$s_{\overline{n+1}|i} = \ddot{s}_{\overline{n}|i} + 1$$

Annuity-due

The PV of an annuity-due is:

$$\ddot{a}_{\overline{n}|i} = 1 + a_{\overline{n-1}|i}$$

The FV of an annuity-due is:

$$\ddot{s}_{\overline{n}|i} = (1+i_1)^n + (1+i_2)^{n-1} + \cdots + (1+i_n)^1$$

2.5 Non-coinciding Frequencies Annuity

An annuity where **payments** made and **interest** compounded at a different frequency.

$$(1+j)^m = (1+i)^n$$

- j = rate per payment period (what we want)
- m = number of payment periods
- i = rate per compounding period
- n = number of compounding periods

Term	Applies To	Rate For	Found From
i	Compounding frequency	Compounding periods (e.g., quarterly)	Given by nominal rate
j	Payments frequency	Payment periods (e.g., monthly)	Must be **calculated** to match payment frequency

Formula 14 If **nominal rate** $i_{\textbf{nom}}$ is compounded m times per year, then:

$$i_{\text{eff}} + 1 = (1 + \frac{i_{\text{nom}}^{(m)}}{m})^{mn}$$

Formula 15 (Accumulated Value with Nominal interest rate)

$$\text{AV} = P \cdot (1 + \frac{i^{(m)}}{m})^{mn}$$

Formula 16 (Present Value with Nominal interest rate)

$$\text{PV} = \frac{P}{(1 + \frac{i^{(m)}}{m})^{mn}}$$

2.6 Perpetuity

Perpetuity is an annuity that pays forever. Perpetuity-immediate has payments start 1 period from now. Perpetuity-due has payments which start immediately (now).

Formula 17 (Present Value)
Perpetuity-Immediate:

$$\text{PV} = \text{a}_{\overline{\infty}|} = v + v^2 + \cdots = \frac{1}{i}$$

Perpetuity-Due:

$$\text{PV} = \ddot{a}_{\overline{\infty}|} = 1 + v + v^2 + \cdots = \frac{1}{d}$$

Relationship: $\ddot{a}_{\overline{\infty}|} = 1 + \text{a}_{\overline{\infty}|}$

2.7 Continuous Annuity

A continuous annuity has:

- A finite term
- An infinite frequency of payments

In other words, each interest conversion period has a total payment of $\$1$ at periodic interest rate i. Each period has m intervals of payment, where m goes to infinity. Payments are made continuously. Total amount paid during each period is:

$$\int_{k-1}^{k} dt = [t]_{k-1}^{k} = \$1$$

2.7.1 Level continuous annuity

Formula 18 (Present Value of an annuity payable continuously) For the annuity with n interest conversion periods, each period has a total amount of $\$1$. The present value is:

$$\bar{a}_{\overline{n}|} = \int_0^n v^t dt$$

2.7.2 Continuous annuity with Constant force of interest

Formula 19 (Present Value)

$$\bar{a}_{\overline{n}|} = \lim_{m \to \infty} a_{\overline{n}|} = \frac{1 - v^n}{\delta} = \frac{1 - e^{-n\delta}}{\delta} = \frac{i}{\delta} a_{\overline{n}|}$$

2.7.3 Continuous annuity with Variable force of interest

Formula 20 (Present Value)

$$\bar{a}_{\overline{n}|} = \int_0^n e^{-\int_0^t \delta_r dr} dt$$

Chapter 3

Loan

3.1 Debt instruments

> A debt instrument is a contract that requires the borrower to repay principal and usually interest at a future date.

Examples of **Debt Instruments:**

1. Bonds (corporate, government)
2. Loans

3.2 Outstanding Balance Calculation for Level Payments

Notation:

- I_t: Interest paid during the k-th period.
- P_t: Principal (capital) repaid during the k-th period.
- B_t: Outstanding balance immediately after the t-th payment.
- R_t: Total payment made during the t-th period (interest + principal).

Type	**Formula**	**Interpretation**
Prospective	$B_t = R \cdot a_{n-t}$	Present value of remaining (future) level payments
Retrospective	$B_t = R \cdot s_t$	Accumulated value of past payments

Loan Equation of Value: You can express the loan's total value L using the equation of value:

$$L = B_0 = R_1 v + R_2 v^2 + \cdots + R_n v^n$$

3.3 Loan Amortization (Level/Non-level payments

A loan can be interpreted as an annuity with payments made in regular intervals, each payment consists of two parts:

- **Interest on the loan**: A cost for borrowing the money.
- **Principal**: The amount of the loan that you borrowed.

Over time, interest portion decreases and principal portion increases. Total payments stay the same.

3.3.1 Interest portion in period t

At any given time, the interest due for the next payment period will depend on the outstanding balance before t-th payment.

$$I_t = i \cdot B_{t-1} = i \cdot R \cdot a_{\overline{n-t+1}|} = R \cdot (1 - v^{n-t+1})$$

3.3.2 Principal portion in period t

$$\begin{aligned} P_t &= R - I_t \\ &= R - R \cdot (1 - v^{n-t+1} \\ &= R \cdot v^{n-t+1} \end{aligned}$$

Only for level payments:

$$\text{FV of principal } P_t \text{ after } k \text{ periods at interest rate } i = P_t \cdot (1+i)^k = P_{t+k}$$

3.3.3 Balance

- **Interest:** Paid to lender (does **not** reduce balance)
- **Principal:** Reduces the loan balance

$$\begin{aligned} \text{Balance after t-th payment} &= \text{Balance before t-th payment} - \text{Principal repaid at time t} \\ &= B_{t-1} - P_t \end{aligned}$$

$$\begin{aligned} \text{Balance after t-th payment} &= \text{Balance at time t with interest} - \text{Payment made at time t} \\ &= B_{t-1}(1+i) - R_t \end{aligned}$$

3.4 Summary

Component	**Formula at Period t**
Total Payment R	$R = 1$
Interest I_t	$I_t = 1 - v^{n-t+1}$
Principal Repaid P_t	$P_t = v^{n-t+1}$

Chapter 4

Bond

4.1 Bond

A **bond** is a type of debt instrument made by investors to a borrower (government/corporation). The borrower promises to pay: interest (coupons) and principal (face value) at maturity.

4.1.1 Key terms

- **Term**: Time from issue to maturity.
- **Maturity Date**: Final payment date.
- **Yield**: Actual return for the investor, depending on the price.

Formula 21 Regarding to bond pricing formulas, key notations are as follow:

- **Par Value / Face Value** F: Original issue price of the bond which does not change over time.
- **Coupon Rate** r: Interest rate applied to the face value and is set by issuer (fixed).
- **Price** P: What the investor pays for the bond.
- **Redemption Value of Bond** C: $F = C$ until otherwise stated. C is the amount the bondholder gets at maturity. If the bons is called early, C = Call price.
- **Interest rate per payment period** i: Fluctuates based on market.
- **Number of coupon payments** n

4.2 Book Value

Book Value of a bond is the price (=present value) of the bond at any time between the purchase and maturity date. Define B_k is the book value (amortized value) of the bond immediately after the k^{th} payment.

Prospectively, the book value of a bond is given by:

$$B_k = \text{PV(Remaining Payments)} = Fr \cdot a_{\overline{n-k}|i} + C \cdot v^{n-k}$$

Retrospectively, the book value of a bond is given by:

$$B_k = \text{FV(P)} - \text{FV(Past Payments)} = P \cdot (1+i)^k - Fr \cdot s_{\overline{k}|i}$$

4.2.1 Principal and Interest

Principal: This is the original amount the bond issuer promises to pay back at maturity. It is the face value/par value and investor get this once, at the end of the bond's life.

$$\text{Principal} = \text{Redemption Value}$$

Interest: Interest each period is the coupon payment.

$$\text{Interest each period} = \text{Coupon rate x Face Value}$$

4.2.2 Principal and Interest of Bond in Book Value

In book value calculations, we break coupon into 2 parts:

- **(Expected) Interest:** Interest earned at yield rate = Yield rate x Book Value
- **Principal adjustment at the k^{th} coupon** = Coupon - Interest

Formula 22 Interest earned at yield rate:

$$I_k = i \cdot B_{k-1}$$

Principal adjustment after the k^{th} coupon payment:

$$P_k = Fr - I_k = B_{k-1} - B_k$$

- Premium: $P_k > 0$ Book value decreases
- Discount: $P_k < 0$ Book value increases

Formula 23 Total principal adjustment from time k to m:

$$\sum_{j=k+1}^{m} P_{k+1} = B_k - B_m$$

Total interest earned from k to m:

$$\sum_{j=k+1}^{m} I_j = \sum C - \sum P_j = (m-k) \cdot C - (B_k - B_m)$$

4.3 Bond yield

Bond Yield/Yield to Maturity (YTM) is the rate of return i you'll earn if you hold the bond to maturity, assuming all coupons are reinvested at the same rate. It is expressed as percentage $(\%)$. Components of YTM:

1. Current bond price P
2. Coupon payments Fr
3. Time to maturity n
4. Face Value/Par value F

Formula 24 (Nominal Yield)

$$\text{Nominal Yield} = \frac{\text{Annual Coupon}}{\text{Par Value}} = \frac{Fr}{F}$$

Nominal Yield is the **annualized** rate of return you earn based on the bond's Face value. Unit of r is %/year.

Formula 25 (Current Yield)

$$g = \text{Current Yield} = \frac{\text{Annual Coupon}}{\text{Bond Price}}$$

Current yield is the **annualized** rate you earn based on what you actually paid for the bond.

- $g > i$: Premium bond
- $g < i$: Discount bond

Formula 26 You buy a bond at the current market price and holding it to maturity. **Yield to Maturity** is the annualized rate of return you expect to have. In other words, it discounts the future cash flows to be equal to the market price of the bond.

4.4 Bond pricing formula

Formula 27 (Bond Price) For a bond with coupons, its price is

$$P(i) = \text{PV of Coupon payments} + \text{PV of Redemption value} = F \cdot r \cdot a_{\overline{n}|i} + Cv^n$$

4.4.1 Base amount formula

Formula 28 Base amount formula

$$\text{P} = \text{Base} + \text{Adjustment} = Fr + (C - Fr)v^n$$

4.4.2 Bonds with Geometric Coupon Payments

Formula 29 If a bond has a geometrically varying coupons (i.e., the first coupon is X and each subsequent coupon is (1+k) times the preceding one) with the yield rate i, PV = Price of the bond is:

$$\text{PV} = \text{PV}_{\text{Redemption value}} + \text{PV}_{\text{coupon annuity}} \tag{4.1}$$

$$= C \cdot v^t + X \cdot \frac{1 - (\frac{1+k}{1+i})^n}{i - k} \tag{4.2}$$

4.4.3 Bonds with Different Frequencies for Coupon and Yield rates

Case	Formula	Interpretation
$n < k$ (Few coupons, many yield periods)	$P = Fr \cdot a_{\overline{n}\vert}^{(k)} + Cv^n$	Coupon rate is compounded less frequently. Total payment periods is nk.
$n > k$ (Frequent coupons, fewer yield periods)	$P = \frac{Fr \cdot a_{\overline{n}\vert}}{s_k} + Cv^n$	Coupons occur more frequently. Total payment periods is n/k
$n = k$	$P = Fr \cdot a_{\overline{n}} + Cv^n$	Payments match discounting intervals.

4.5 Premium/Discount

Bonds dont always trade at **par value** (i.e. P is not always equal to F). This is because coupon rate r is different from (market) yield rate/YTM i. In addition, price of bond P is dependent on market yield i.

- If $r = i$ or $Fr = Ci$, bond is sold at par.
- If $r > i$ or $Fr > Ci$, bond is sold at a premium.
- If $r < i$ or $Fr < Ci$, bond is issed at a discount.

Note that $F = C$ by default.

If $r < i$, the bond is less attractive to investors. So the issuer must lower the price to make it appealing and the bond is sold at discount. Similarly, if $r > i$, the issuer will raise the price as investors are willing to pay higher price for the bond.

Condition	Price Compared to C	Called
$r < i$	$P < C(= F)$	**Discount bond**
$r = i$	$P = C(= F)$	**Par**
$r > i$	$P > C(= F)$	**Premium bond**

Formula 30 Price of a bond can be calculated using Premium/Discount:

$$P(i) = C + (F \cdot r - C \cdot i)a_{\overline{n}|i}$$

where $Fr - Ci$ is the premium or discount **per period**. They are cash flows over n periods and seen as an annuity of coupon payments.

- **Premium**: $P - C = (Fr - Ci)a_{\overline{n}|i}$
- **Discount**: $C - P = (Ci - Fr)a_{\overline{n}|i}$

Note that Fr is the coupon payment, and Ci is the market interest payment.

4.6 Bond duration

4.6.1 Par Bond Duration

When the bond is sold at par (i.e., when the yield $i = r$), its Macaulay Duration is:

$$\begin{aligned}\text{MacD} &= \frac{F(r(Ia)_{\overline{n}|} + nv^n)}{Fra_{\overline{n}|} + Fv^n} \\ &= \frac{r(Ia)_{\overline{n}|} + nv^n}{ra_{\overline{n}|} + v^n} \\ &= \frac{r(1+i)a_{\overline{n}|i} + (i-r)nv^n}{r + (i-r)v^n}\end{aligned}$$

Since $i = r$ (bond sold at par), the equation becomes

$$\text{MacD} = \frac{r(1+i)a_{\overline{n}|i}}{r} = (1+i)a_{\overline{n}|i} = \ddot{a}_{\overline{n}|i}$$

Formula 31 The Macaulay Duration of a par bond is

$$\text{MacD} = \ddot{a}_{\overline{n}|i}$$

4.7 Callable Bond

Bonds that can be redeemed by the issuer before maturity. T he issuer decides when to call, often to save on interest costs.

Callable bonds can be redeemed early by the issuer. The issuer decides when to call (redeem) the bond, but you, the investor, don't get to choose. So when you're trying to figure out how much to pay for a callable bond, you must assume the issuer will act in their own best interest, not yours.

Case	What's Happening	What You Assume as Investor	Why?
Premium Bond	Coupon rate $>$ yield rate	Bond is called **early**	Issuer wants to stop overpaying and call the bond at the first possible date
Discount Bond	Coupon rate $<$ yield rate	Bond is called **late**	Issuer benefits from paying low coupons longer and hold the bond until the final maturity date

When you're buying a callable bond, you already know:

- Call price = Redemption value

- Call schedule (when it can be called)
- Coupon rate
- Market yield you want (your target of return)

So you can **adjust the purchase price you're willing to pay**.

4.7.1 Endpoints shortcut

For callable bond, all possible call dates are evaluated when investor consier the purchase price. You only need to eavluate the bond price at the endpoints of the time interval in which the bond is possibly called.

Formula 32 (Lowest yield - worst case) at the endpoints: You get lowest yield and this is the worst vase when the callable bond is called early"

- Earliest call on left-endpoint - Premium bond
- Earliest call on right-endpoint - Discount bond

Chapter 5

Portfolio

5.1 Duration of Portfolio

5.1.1 Modified duration of Portfolio

Given a portfolio with **n bonds**, each bond k has price $P_k(i)$ and **Modified duration** v_k. Total portfolio value is

$$P(i) = P_1(i) + P_2(i) + \cdots + P_n(i)$$

Formula 33 The Modified duration of the portfolio is

$$\text{MacD}_{\text{Portfolio}} = \frac{P_1(i)}{P(i)}v_1 + \frac{P_2(i)}{P(i)}v_2 + \cdots + \frac{P_n(i)}{P(i)}v_n = \frac{P_1 v_1 + P_2 v_2 + \cdots + P_n v_n}{P_{\text{total}}}$$

$$\text{MacD}_{\text{Portfolio}} = \sum_{k=1}^{n} \left(\frac{P_k(i)}{P(i)} \cdot v_k \right)$$

It's the weighted average of each bond's modified duration, using current price as weight.

5.1.2 Macaulay duration of Portfolio

Formula 34 Macauly duration of a portfolio is

$$\text{MacD}_{\text{Portfolio}} = \frac{\sum_0^n (t \cdot \text{PV of cash flow at time t})}{\text{Total present value of the portfolio}}$$

Chapter 6

Portfolio Performance

6.1 Discounted Cash Flow Technique

DCF is a method to measure the **profitability** of investment projects. Unlike fixed annuities (same payment pattern), DCF allows any pattern of cash inflows (returns) and outflows (costs).

Feature	**Annuities**	**Investments**
Payments	Regular intervals, level payment	May vary in time and amount
Risk	Low/fixed	High/low
Examples	Pensions, loans	Stocks, real estate

There are **two** DCF measures:

1. Net Present Value (NPV) – Present value of all net cash flows.
2. Internal Rate of Return (IRR) – Interest rate that makes NPV = 0.

6.1.1 Cash flows

Notations with Cash flows:

- C_t = contributions/outflows (money invested)
- R_t = returns/inflows (money received)
- $c_t = R_t - C_t$ = net cash flow at time t

$c_t > 0$: Net deposit (inflow), $c_t < 0$: Net withdrawal (outflow)

Example 2. Project Description:
A company plans to develop and sell a new product. The cash flows are as follows:

- Initial investment of $80,000 at year 0.
- Additional investments of $10,000 in years 1, 2, and 3.
- A contribution of $20,000 in year 4 to launch the product.
- Maintenance costs of $2,000 per year from years 5 to 9.
- Returns: $12,000 in year 4, $30,000 in year 5, $40,000 in year 6, $35,000 in year 7, $25,000 in year 8, $15,000 in year 9, and $8,000 in year 10.

Cash Flow Table:

Year	Contributions	Returns	Net Cash Flow (c_t)
0	80,000	0	-80,000
1	10,000	0	-10,000
2	10,000	0	-10,000
3	10,000	0	-10,000
4	20,000	12,000	-8,000
5	2,000	30,000	28,000
6	2,000	40,000	38,000
7	2,000	35,000	33,000
8	2,000	25,000	23,000
9	2,000	15,000	13,000
10	0	8,000	8,000

Net Present Value (NPV):
Let i be the interest rate (cost of capital), and let $v = \frac{1}{1+i}$. Then the NPV of the project is:

$$\text{NPV}(i) = \sum_{t=0}^{10} c_t v^t = \sum_{t=0}^{10} \frac{c_t}{(1+i)^t}$$

Where c_t is the net cash flow in year t.
Interpretation:

- If NPV$(i) > 0$: the investment is profitable.
- If NPV$(i) = 0$: the investment breaks even.
- If NPV$(i) < 0$: the investment is not worth it.

6.2 Net Present Value

Net present value (NPV) is the difference between the present value of **cash inflows** and the present value of **cash outflows** over a period of time. Choose the investment with the greatest positive NPV.

Formula:

$$\text{NPV(i)} = \sum (R_t - C_t) v^t = \sum c_t \cdot v^t$$

- $v^t = \frac{1}{(1+i)^t}$ is the discount factor
- i = rate of interest per period = required return of the investment = cost of capital

6.3 Yield Rate - Internal Rate of Return (IRR)

Yield rate or IRR is the rate such that the PV of cash inflows is equal to the PV of cash outflows. Choose the investment with the greatest IRR.

Formula:

$$\text{IRR} = \text{value of } i \text{ such that } \sum (A_t - L_t) v^t = 0$$

Interpretation:
Yield Rate (IRR) = The interest rate where an investment neither gains nor loses money.

- NPV > 0 : Profit
- NPV < 0 : Loss
- NPV $= 0$: Break-even (Yield rate achieved)

Should you still invest when NPV $= 0$? When NPV $= 0$, there is no net gain, but no loss either. Reject the investment if there are better opportunities (i.e. another project with NPV > 0). Accept it when the Yield rate matched the **inflation rate**, so your money can keep its real value.

Connection Between NPV and IRR

- NPV is a function of the interest rate: $P(i)$
- IRR is the rate where $P(i) = 0$

6.4 Reinvestment

6.4.1 Lump Sum Investment + Interest reinvested

Intuition:
1. Invest 1 unit of money - **a lump sum/principal amount** - for n periods at rate i.
2. Interest is **reinvested** at rate j.

Reinvesting is like planting a tree (investment) and using its seeds (interest) to grow more trees instead of eating them. Over time, you get a forest!
Illustration:

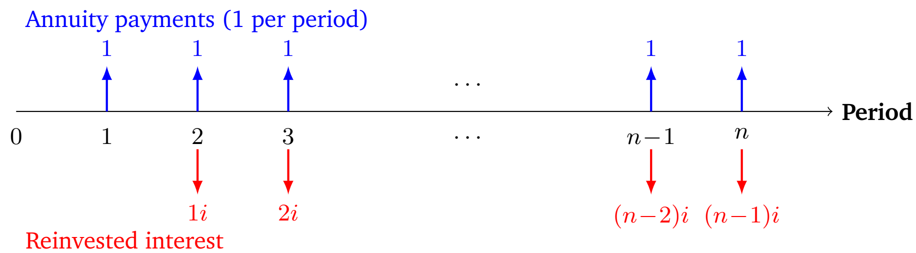
1. Start with 1 unit of money - that principal stays in the account for **n periods.**
2. Interest earned per period is 1$ * i = $i. The principal gets no compounding effect.

3. At the end of year 1,2,...,n-1, we reinvest \$i at each end of the year. This pattern follows an annuity-immediate with **n payments** of \$i, and rate per period is j.
4. The **AV** of that annuity is: i * $s_{\overline{n}|j}$.
5. Add the principle: Total AV = Principle + Reinvested Interest = 1 + i * $s_{\overline{n}|j}$.

Formula 35 (Lump Sum Reinvestment) Total Accumulated Value = 1 + i * $s_{\overline{n}|j}$
Special case: $i = j$, then AV = $(1+i)^n$

6.4.2 Annuity + Interest reinvested

Annuity-immediate



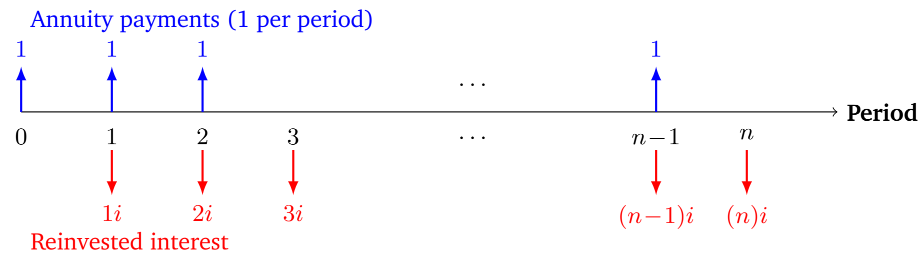
AV = sum of annual payments + reinvested interests as annuity
- Reinvested interests as an annuity: i, 2i, 3i, ..., (n-2)i, (n-1)i. This acts as an increasing annuity with i as a common difference.
- Sum of annual payments is n.

Formula 36 (Interest Reinvestment as Annuity-immediate)

$$AV = n + i(I_s)_{\overline{n-1}|j} = n + i[\frac{s_{\overline{n}|j} - n}{j}]$$

$$AV = s_{\overline{n}|i} \text{ when } i = j$$

Annuity-due



Formula 37 (Interest Reinvestment as Annuity-due)

$$AV = n + i(I_s)_{\overline{n}|j} = n + i[\frac{s_{\overline{n+1}|j} - (n+1)}{j}]$$

6.5 Dollar-weighted Rate of Interest

Concept	DWR	IRR (Rate for NPV = 0)
Purpose	Measures fund performance	Evaluates project profitability
Method	Based on future value	Based on present value
Time orientation	Grows cash to end	Discounts cash to start
Terminology	Used in actuarial & finance	Used in finance & investment

The **dollar-weighted interest rate** i is the average rate of how fast the money grew during a period, including the **timing and size of money in/out**("dollar-weighted"). It measures how well **you** did as an **investor**.

$$i \approx \frac{I}{A + \sum_{0 \le t \le 1} c_t(1-t)} = \frac{\text{Interest earned}}{\text{Exposure}}$$

The approximation is good when each contribution c_t is small compared to the amount A.

Formula 38 Potforlio at the end of investment periods:

$$B = A + C + I$$

Key terms:

- A: Amount at the beginning of the period.
- B: Amount at the end of the period.
- I: Total interest earned during the period.
- c_t: Net contribution (deposit - withdrawal) at time $t \in [0,1]$
- $C = \sum c_t$: Total net contributions.
- $(1+i)^{1-t} - 1$ is the effective rate for period from t to 1.

Concept:
1. Total amount at the end is: $B = A + C + I$
2. Interest earned: $I = iA + \sum_{0 \le t \le 1} c_t[(1+i)^{1-t} - 1]$

where iA is the interest on initial amount A and $\sum_{0\le t\le 1} c_t[(1+i)^{1-t}-1]$ is the interest on total contributions made at time $t\in[0,1]$.
3. Substitute into

$$\begin{aligned} B &= A + C + I \\ &= A + C + iA + \sum_{0\le t\le 1} c_t\left[(1+i)^{1-t}-1\right] \\ &= A(1+i) + \sum_{0\le t\le 1} c_t(1+i)^{1-t} \end{aligned}$$

4. Approximate the compound interest using simple interest:

$$(1+i)^{1-t} \approx 1 + (1-t)i \text{ then } (1+i)^{1-t} - 1 \approx (1-t)i$$

5. $I = iA + \sum_{0\le t\le 1} c_t[(1-t)i] = i[A + \sum_{0\le t\le 1} c_t(1-t)]$ then $i = \frac{I}{A+\sum_{0\le t\le 1} c_t(1-t)}$

Formula 39 (Exposure)

$$A + \sum_{0\le t\le 1} c_t(1-t)$$

Exposure is as a weighted sum of how much money was active in the fund and for how long.

The denominator is called **exposure associated with** i & represents **total time-weighted amount of money at risk**:

- A: Initial fund amount that earns interest for the full year - its weight = 1.
- $\sum_{0\le t\le 1} c_t(1-t)$: time-weighted contributions, giving how long each contribution had to earn interest (c_t it the contribution made at time t and $(1-t)$= weight is the fraction of the year left in which the contribution earns interests.)

Example 3. At the beginning of a year, an investment fund was established with an initial deposit of \$3,000. At the end of six months, a new deposit of \$1,500 was made. Withdrawals of \$500 and \$800 were made at the end of four months and eight months respectively. The amount in the fund at the end of the year is \$3,876. Set up the equation of value to calculate the dollar-weighted rate of interest.

Solution: To find the interest rate i that makes the future value of all cash flows = \$3876.
General formula:

$$\text{Future Value} = \sum c_t(1+i)^{1-t}$$

Set up the equation of value to calculate the dollar-weighted rate of interest i:

$$3000(1+i) + 1500(1+i)^{0.5} - 500(1+i)^{(1-\frac{4}{12})} - 800(1+i)^{(1-\frac{8}{12})} = 3876$$

6.6 Time-weighted Rate of Interest

Time-weighted rate of return isolates fund performance and ignores investor actions.

$$i = (1 + j_1)(1 + j_2)...(1 + j_m) - 1$$

where j_k is the rate of return for each sub-period in an interval (in this case, a year) which has m sub-periods.

Set up:

- The year is split into m intervals (sub-periods)
- At each time t_k,
  C_{t_k} = net contribution
  B_{t_k} = fund value just before that contribution

then for each subinterva; $k = 1, 2, ..., m$, the rate of return j_k for each sub-period is

$$B_{t_k} = (1 + j_k)(B_{t_{k-1}} + C_{t_{k-1}})$$

The overall yield rate i for the entire year is given by

$$i + 1 = (1 + j_1)(1 + j_2)...(1 + j_m)$$

We call i the **time-weighted rate of return.**

6.7 TWR and DWR

Example scenario:

- Year 1: Spend $1000 to buy 1000 shares at $1
- Year 2: Spend $2000 to buy 1000 shares at $2
- Year 3: Gain $2500 to sell 2000 shares at $1.25

Dollar-weighted Return Rate is -12.77% per year. It reflects your actual loss from your money decisions (you lost $500 as you invested $3000 but only received $2500).

Time-weighted Return Rate is 11.80% per year. The stock price went up from $1 to $2, then fell to $1.25, but it was still higher than it started. Therefore, the stock itself had a positive return over time.

Goals of the 2 return methods:

1. **Dollar-weighted Return Rate**: measures how well the investment performs, excluding investors actions.
2. **Time-weighted Return Rate**: measures how well the investors did, including timing and size of money flows and reflecting real experience.

Chapter 7

Measures of Interest Rate Sensitivity

7.1 Inflation

Inflation = general rise in prices of goods and services over time. It reduces purchasing power of money.

Inflation and Interest Rates: inflation and interest rates move together over time. Investors demand higher interest to compensate for future inflation.

Formula 40 Let π is the inflation rate.

$$1 + i_{\text{real}} = \frac{1 + i_{\text{nominal}}}{1 + \pi}$$

7.1.1 Payments grow with inflation

Present Value

Each future payment grows by constant **inflation rate**.

Formula 41 (PV iwth adjusted payments)

$$PV = R\left[\frac{1+r}{1+i} + \left(\frac{1+r}{1+i}\right)^2 + \cdots + \left(\frac{1+r}{1+i}\right)^n\right] = R(1+r)\cdot\frac{1-\left(\frac{1+r}{1+i}\right)^n}{i-r}$$

We discount the annuity with **nominal rate** as payments have been adjusted with inflation rate.

Formula 42 (PV with ray payments, while inflation exists)

$$PV = R\left[\frac{1}{1+i_0} + \frac{1}{(1+i_0)^2} + \cdots + \frac{1}{(1+i_0)^n}\right] = R\cdot a_{\overline{n}|i_0}$$

We discount the annuity with **real interest rate.**

7.1.2 Summary

Scenario	**Formula**	**When to Use**
Payments grow with inflation	$PV = R(1+r)\cdot\frac{1-\left(\frac{1+r}{1+i}\right)^n}{i-r}$	Use nominal rate i
Payments fixed in real terms	$PV = R\cdot a_{\overline{n}\|i_0}$	Use real rate i_0

Accumulated Value

Formula 43 (AV in nominal dollars (not adjusted for inflation))

$$\text{AV} = P(1+i_{\text{nominal}})^n$$

This is the raw future value of your investment.

Formula 44 (AV adjusted for inflation)

$$\text{AV} = P(\frac{1+i}{1+r})^n = P(1+i_{\text{real}})$$

This refects the true purchasing power of your money.

7.2 The Term structure of Interest Rates and Yield Curves

7.2.1 Term

Term: The length of time until an investment/loan matures/ends. It is the duration until you get your money back.

7.2.2 Spot rate

Spot rate is the yield to maturity/(single rate of annual return) of a zero-coupon bond/(no cash flows). Spot rate is always based on time zero. It is a rate for one-time future payment.

$$v_t = \frac{1}{(1+s_t)^t}$$

- v_t: discount factor
- s_t: spot rate

Generally — the longer the investment term, the higher the interest rate, because

- More time = more risk (like inflation, uncertainty, default).
- Investors want extra return for locking money up longer, hence they charge higher rates for longer loans.

Key differences between: **Zero-coupon bond with spot/forward rates** and **Annuity using varying spot rates**

Concept	Zero-Coupon Bond	Annuity with Varying Rates
Cash Flows	Single payment at end	Multiple payments/cash flows each year
Discounting Method	Compound using forward rates	Discount each payment with spot rates
Formula Used	$(1+s_n)^n = \prod_{i=0}^{n}(1+f_{[i,i+1]})$	$PV = \sum \frac{C_t}{(1+s_t)^t}$
Use Case	Zero-coupon bond pricing / yield	Valuing pension plans, loans, etc.

7.2.3 Yield

Extend the table to a continous graph, where y-axis is **yield** (interest rates/spot rates of risk-free bonds) and x-axis is **maturity**, we obtain a yield curve. Yield curve can be upward-sloping (rates expected to rise), flat (all terms have same rate), and inverted (short-term ¿ long-term, a signal of recession).

Length of investment (years)	Interest rate (Spot rate)
1 year	3%
2 year	4%
3 year	6%
4 year	7%

Yield Curve is a graph of **spot rates** versus maturity time.

Yield to Maturity: A **single average rate** that discounts all cash flows of a bond.
- The bond is held to maturity.
- The bond does not default.
- Reinvestment of the bond and all coupons is executed at the original YTM.

Formula 45 When spot rates i_t vary by year, NPV is

$$\text{NPV} = \sum_{t=0}^{n} \frac{c_t}{(1+i_t)^t}$$

7.2.4 Forward Rate

Forward rate: the **interest rate** agreed on today for borrowing or investing money in the future from time n to time n+m. It tells what the market expects interest rates to be.

$$v_{[n,m-n]} = \frac{1}{(1+f_{[n,n+m]})^{m-n}}$$

Formula 46 (Connect Spot rate with Forward rates)
- Spot Rate (s): Set by the current market — changes daily with supply/demand.
- Forward Rate (f): Calculated from spot rates.

$$(1+s_n)^n \dot{(}1+f_{[n,n+m]})^m = (1+s_{n+m})^{n+m}$$
$$(1+s_{[0,n]})^n = (1+f_{[0,1]}) \cdot (1+f_{[1,2]}) \ldots (1+f_{[n-1,n]})$$

7.3 Macaulay and Modified Durations

Why **Duration** matters?
- Duration measures the sensitivity of a bond's price to changes in interest rates.
- Duration reflects the timing of cash flowe (i.e. when you'll get your money back).

Types of Duration	Definition	Formula
Term to Maturity	Time until final payment (not very useful with coupons).	—
Equated Time	Weighted average of payment times (weights = cash flows).	$\bar{t} = \frac{\sum tR_t}{\sum R_t}$
Macaulay Duration	Weighted average of present values of payments.	$d = \frac{\sum t\nu^t R_t}{\sum \nu^t R_t}$
Modified Duration	Measures price sensitivity to interest changes.	ModDur $= \frac{d}{1+i}$

7.3.1 Average term-to-maturity

Average Term-to-Maturity: ”On average, when do I receive my money back?”

Setup: For zero-coupon bond, there is only one payment (at maturity). It means shorter maturity has faster cash back. But for coupon bons with multiple cash flows over time, term-to-maturity ignores earlier coupon payments. Let’s say you’re paid \$100 in year 1 (coupon), \$200 in year 2 (coupon), and \$700 (final coupon + principal) in year 3. The average term-to-maturity might be around 2.6 years - it tells on average, you get back total \$1000, not just \$700 in year 3.

Formula 47 If a bond pays cash flows $C_0, C_1, \ldots, C_n$ at times $t_0, t_1, \ldots, t_n$, then the **Equated Time** is given by:

$$\text{Equated Time} = \frac{\sum_{i=0}^{n} C_i \cdot t_i}{\sum_{i=0}^{n} C_i}$$

It tells how quickly your investment is returned, on average.

7.3.2 Macaulay Duration

It improves the upon equated time by using **present values** instead of just raw cash flows. Each cash flow is now discounted to present, thus the weighted average time of cash flows is more precise.

Formula 48 The **Macaulay Duration** $MacD$ is given by:

$$MacD(i) = \frac{\sum_{t=0}^{n} t \cdot \nu^t R_t}{\sum_{t=0}^{n} \nu^t R_t}$$

- R_t be the cash flow at time t
- $\nu^t = \frac{1}{(1+i)^t}$ is the discount factor at time t
- i is the effective rate of interest per period.

Note that duration depends on i. When $i = 0$, the MacD is equal to the equated time formula. When there is only 1 future payment, duration is equal to time of payment (MadD = t).

7.3.3 Macaulay Duration and Cauchy-Schwarz Inequality

Recall the Macaulay duration:

$$d = \frac{\sum_{t=0}^{n} t \cdot \nu^t R_t}{\sum_{t=0}^{n} \nu^t R_t}, \quad \text{where } \nu^t = \frac{1}{(1+i)^t}$$

By Cauchy-Schwarz inequality, we see that

$$\frac{d}{di} MacD(i) < 0$$

So, Macaulay duration decreases as the interest rate increases.

7.4 Modified Duration (Volatility)

(Modified Duration) Measures how sensitive a **bond's price** is to a small change in its **yield-to-maturity (YTM)**. As yield rises, price falls, and vice versa - inverse relationship.

Setup:

The price of a bond is the present value of its cash flows:

$$P(i) = \sum_{t=0}^{n} \frac{C_t}{(1+i)^t}$$

Take the derivative of P(i) w.t. i (yield rate). This is the **rate of change** of bond price when i changes:

$$\frac{dP(i)}{di}$$

We want to express the percentage change in price of bond by dividing the derivative by the price, with the minus sign as price drops when i increases. We obtained **volatility** which tells how sensitive the PV of bond's price to interest rate changes.

$$\text{Volatility} = -\frac{1}{P(i)} \cdot \frac{dP(i)}{di} = -\frac{P'(i)}{P(i)}$$

Real world meaning: if Volatility = 5, then if i increases by 1% bond price drops about 5%. It's a linear approximation of the price-yield curve i.e. the % change of $P(i)$.

The standard derivative identity is:

$$\frac{d}{di}[lnP(i)] = \frac{P'(i)}{P(i)}$$

Thus, volatility, denoted by $\bar{v}$, becomes:

$$\bar{v} = -\frac{d}{di}[lnP(i)] = -\frac{P'(i)}{P(i)}$$

Volatility is often called **modified duration**. Now we derive $P(i)$ by taking the derivative with respect to i :

$$P'(i) = \frac{d}{di}\left[\sum_{t=0}^{n} \frac{R_t}{(1+i)^t}\right] = -\sum_{t=0}^{n} t(1+i)^{-t-1} R_t = -\sum_{t=1}^{n} \frac{t \cdot R_t}{(1+i)^{t+1}}$$

Then plug into the volatility formula:

$$\bar{v} = -\frac{P'(i)}{P(i)} = \frac{\sum_{t=0}^{n} \frac{t \cdot R_t}{(1+i)^{t+1}}}{\sum_{t=0}^{n} \frac{R_t}{(1+i)^t}} = \frac{\sum_{t=0}^{n} t \cdot v^{t+1} \cdot R_t}{\sum_{t=0}^{n} t \cdot v^t \cdot R_t}$$

Express $\bar{v}$ in terms of MacD:

$$\bar{v} = \frac{\sum_{t=0}^{n} t \cdot v^{t+1} \cdot R_t}{\sum_{t=0}^{n} t \cdot v^t \cdot R_t} = v \cdot \frac{\sum_{t=0}^{n} t \cdot v^t \cdot R_t}{\sum_{t=0}^{n} t \cdot v^t \cdot R_t} = \text{MacD} \cdot v = \frac{\text{MacD}}{1+i}$$

Formula 49

$$\text{Modified Duration} = \text{Macaulay Duration} \cdot v = \frac{\sum_{t=0}^{n} t \cdot v^{t+1} \cdot R_t}{\sum_{t=0}^{n} t \cdot v^t \cdot R_t}$$

Remark: we assume that cash flows (payments) are fixed - they do not change with the interest rate changes.

7.5 MacaulayD vs. ModifiedD

Interpretation:

- $\frac{P'(i)}{P(i)}$ = change in PV per unit change in i. (in percentage)
- $\frac{P'(\delta)}{P(\delta)}$ = cheng in PV per unit change in δ. (in time)

Table 7.1: Comparison of Macaulay Duration and Modified Duration

Feature	**Macaulay Duration**	**Modified Duration**
Definition	Weighted average time until all payments in a series are made	Sensitivity of bond price to interest rate changes
Formula	$\text{MacD} = -\frac{P'(\delta)}{P(\delta)}$	$\text{ModD} = -\frac{P'(i)}{P(i)} = \text{MacD} \cdot (1+i)$
Units	Time (usually in years)	Percentage change per 1% interest rate change
Interpretation	"When" you get your money back (on average)	"How much" the price changes when interest changes
Rate sensitivity?	Indirectly	Directly
Dependence on Interest Rate	No (once cash flows are fixed)	Yes (through denominator $1+i$)

7.6 Passage of Time

As time passes, the cash flows are getting closer. So naturally, the duration decreases.

7.6.1 Macaulay Duration

(Passage of Time)

Formula 50

$$\text{MacD}_{new} = \text{MacD}_{old} - (t_1 - t_0)$$

MacD changes over time, as cash flows are getting closer. The difference between these two MacD_{old} and MacD_{new} is just the time has passed $(t_1) - t_0$.

7.6.2 Modified Duration

(Passage of Time)
Convert $\text{ModD}_{\text{new}} = \text{MacD}_{\text{new}} \cdot v = [\text{MacD}_{old} - (t_1 - t_0)] \cdot v = \text{ModD}_{\text{old}} - v(t_1 - t_0)$

Formula 51

$$\text{ModD}_{\text{new}} = \text{ModD}_{\text{old}} - v(t_1 - t_0)$$

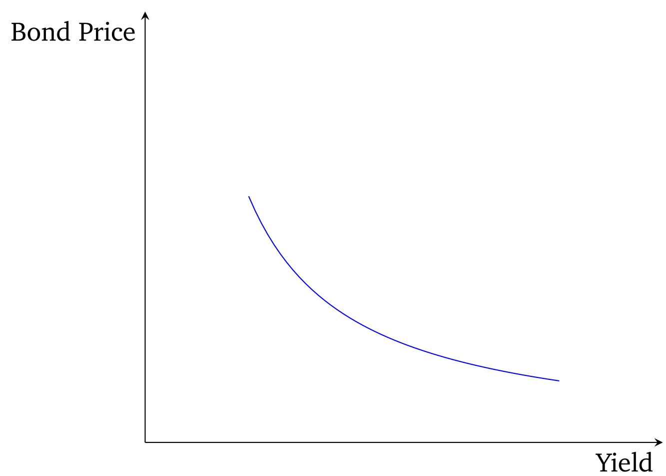
7.7 Convexity

- **Duration** gives a linear approximation of how bond price changes with interest rates (yield).
- **Convexity** gives a curvature of how bond price (non-linear) changes with interest rates (yield).

In other words, convexity measures the rate of change of the volatility with respect to interest changes. High convexity bonds: Lose less when yields go up, and gain more when yields go down.

Factors that increase convexity:

- Maturity
- Coupon rate
- YTM
- Cash flow spread



7.7.1 Macaulay Convexity

Macaulay convexity is the weighted average of the squares of the time t^2, using present values as weights.

$$\text{MacC} = \frac{P''(\delta)}{P(\delta)} = \sum_{t\geq 0}^{\infty}\left(\frac{C_t(1+i)^{-t}}{P(i)}\right)t^2 = \frac{\sum_{t=0}^{n} t^2 \cdot v^t \cdot \text{CF}_t}{\sum_{t=0}^{n} v^t \cdot \text{CF}_t}$$

7.7.2 Modified Convexity

Formula 52 (Modified Convexity)

$$\text{Convexity} = \frac{P''(i)}{P(i)} = \frac{\sum_{t=0}^{n} t \cdot (t+1) \cdot v^{t+2} \cdot R_t}{\sum_{t=0}^{n} v^t \cdot R_t}$$

where

- $P(i)$ is the PV of net cash flows at interest i
- $P''(i)$ tells how fast duration itself changes (i.e., rate of curvature)

7.8 Approximation of Bond Price

We want to approximate how the bond price changes when the interest rate changes slightly from i to Δi.

$$P(i + \Delta i) \approx P(i) + \Delta i \cdot P'(i)$$

To get percentage change, divide both sides by P(i):

$$\frac{P(i + \Delta i)}{P(i)} \approx 1 + \Delta i \cdot \frac{P'(i)}{P(i)} = 1 - \Delta i \cdot \text{ModD}$$

The approximation is:

$$P(i + \Delta i) \approx P(i) \cdot [1 - \text{ModD} \cdot \Delta i]$$

Formula 53 (1st-order Modified Approximation)

$$P(i_n) \approx P(i_0) \cdot [1 - (i_n - i_0)(\text{ModD})]$$

$$\Delta P = -\Delta i \cdot \text{ModD}$$

Now, we want the approximation in terms of MacD. Start with bond price as the sum of discounted cash flows:

$$P(i) = \sum_{t=0}^{n} \frac{C_t}{(1+i)^t}$$

We approximate the bond price as a single lump-sum (K) as total amount of cash flows at average time which is MacD:

$$P(i) \approx \frac{K}{(1+i)^{\text{MacD}}}$$

When i changes, $P(i)$ also changes. We take the ratio of new price to old price:

$$\frac{P(i_new)}{P(i_old)} \approx \frac{(1 + i_{old})^{\text{MacD}}}{(1 + i_{new})^{\text{MacD}}} = (\frac{1 + i_{old}}{1 + i_{new}})^{\text{MacD}}$$

Formula 54 (1st-order Macaulay Approximation)

$$P(i_{\text{new}}) \approx P(i_{\text{old}}) \cdot (\frac{1+i_{old}}{1+i_{new}})^{\text{MacD}}$$

7.9 Bond Duration

Bond duration is a measure of the bond's Sensitivity to interest changes.

Formula 55 For a bond of n annual coupons, face amount F, coupon rate r, and annual yield rate i:

- Annuity-immediate: F · r
- Redemption value at maturity date t = n: F = C

$$\text{MacD} = \frac{\sum_{t=1}^{n} t \cdot PV(\text{CF}_t)}{P}$$

where

- $PV(CF_t)$ is the PV of the cash flow at time t (coupon or principal)
- P is the total PV (price) of the bond

7.10 Note

1. Bond
   - Buying a bond = outflow.
   - Coupon payments = inflow.
   - Maturity value = inflow.
2. Annuity
   - Saving: regular payments = outflow.
3. Loan
   - Taking the loan = inflow.
   - Loan repayments = outflow.

Chapter 8

Immunization

8.1 Assets and liabilities

Cash flows:

- Asset inflows: $A_0, A_1, ..., A_n$
- liability outflows: $L_0, L_1, ..., L_n$
- At each time t, $R_t = A_t - L_t$ = Net cash flow at time t

8.2 Redington Immunization

Redington immunization is a strategy to **protect a portfolio** (assets vs liabilities) from small changes in interest rates.

Let $P(i)$ be the present value of all the net cash flows at interest rate i. Since we want the value of portfolio to **not drop** when i changes slightly, that means $P(i)$ should be at a minimum at the current target rate i_0.

Table 8.1: 3 Conditions for Immunization

Condition	**Meaning**	**Purpose**
$P(i) = 0$	PV of assets = PV of liabilities	Start balanced
$P'(i) = 0$	Modified durations match	No change for small Δi
$P''(i) > 0$	Positive convexity	Changes in rate increase value

Criterion	**Redington Immunization**
Present Value Match	$\text{PV}_A = \text{PV}_L$
Duration Match	$MacD_A = MacD_L$
Convexity (to ensure minimum at rate)	$C_A > C_L$ or $P''_A > P''_L$

8.3 Full Immunization

(Full immunization) Full immunization protects a portfolio from **any** interest rate changes, not just small changes.

Formula 56 Full Immunization Conditions at $i = i_0$:

1. $PV_A(i_0) = PV_L(i_0)$ or PV of assets equals to PV of liabilities
2. $PV'_A(i_0) = PV'_L(i_0)$ or $\text{ModD}_A(i_0) = \text{ModD}_L(i_0)$
3. $PV''_A(i_0)\ PV''_L(i_0)$ or $\text{ModC}_A(i_0) = \text{ModC}_L(i_0)$

The 3rd condition is the **timing condition**, which means that there has to be asset cash flow **before and after** each liability cash flow. It helps to reduce interest risk: no matter how interest rates move, your total asset value will always be enough to cover the liability.

8.4 Asset-Liability Exact Matching

Absolute Matching: is a **strategy** to ensure that every payment required (liability) is backed by a cash flow from the asset at the **same time and amount**.

- **Asstes**: Incoming payments A_t
- **Liabilities**: outgoing payments L_t

Formula 57 Absolute Matching: Ensure that $A_1 = L_1, A_2 = L_2, \ldots$ at same TIME and AMOUNT.

Challenges of **Dedication**:

1. Uncertain cash flows
2. Long-term liabilities
3. Lower returns: Dedication is very strict - you can't invest in potentially higher-return but flexible assets.

Chapter 9

Summary

9.1 Loan, Bond, and Annuity

Field	**Loan**	**Bond**	**Annuity**
N	Loan term	Time to maturity	Duration of annuity
I/Y	Loan interest rate	Bond yield (market rate)	Interest rate per period
PV	Loan amount (positive)	Bond price (negative if investing)	PV of payments (negative if paying)
PMT	Monthly payment (negative)	Coupon payment (positive cash inflow)	Payment (depends on in/outflow)
FV	0 (loan fully repaid)	Redemption value (money recedived at maturity date)	Usually 0 unless final lump sum