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# Chapter 1

# Interest Theory

## 1.1 Amount and Accumulation Functions

**Amount function** $A(t)$ refers to value of the investment at time t.

**Accumulation function** $a(t)$ refers to value of $1, which was invested at time 0, at time t.

**Formula 1** relationship between A(t) and a(t) is

$$A(t) = A(0) \cdot a(t)$$

where

- A(0) is money you invested at time 0.
- a(t) tells how much **each dollar** grows to by time t.

## 1.2 Force of Interest

The **force of interest**, denoted by $\delta$, measures how fast monet grows in a particular instant in time. By definition, the **force of interest** at time t, denoted by $\delta(t)$, is:

$$\delta(t) = \frac{A'(t)}{A(t)}$$

or it is also called as **instantaneous rate of growth** of the investment.

### 1.2.1 Constant Force of Interest

**Formula 2** When $\delta$ is constant, then $\delta(t) = \delta$. Continuous compounding is

$$A(t) = A(0) \cdot e^{\delta t}$$

**Formula 3** Effective interest rate and force of interest:

$$i = e^{\delta} - 1$$

and

$$\delta = \ln(1 + i)$$

## 1.3 Present value

### 1.3.1 Accumulation function with compound and simple interests

With **compound interest**, the accumulation function is

$$a(t) = (1 + i)^t$$

Reminds that $a(t)$ tells how $1 grows over t periods at (compound) interest rate $i$.

With **simple interest**, the accumulation function is

$$a(t) = 1 + it$$

Reminds that $a(t)$ tells how $1 grows over t periods at (simple) interest rate $i$.

### 1.3.2 Discounting

1. **Discounting** What is $1 in future worth today? $\rightarrow (1 + i)^t$ after t periods
2. **Accumulation** What does $1 today grow to in future? $\rightarrow \frac{1}{(1+i)^t}$ after t periods

### 1.3.3 Discount factor

**Discount factor** converts future money into present value. For $t$ periods, the discount factor is

$$v^t = \frac{1}{(1+i)^t}$$

Discount factor during nth period is

$$(1+i_n)^{-1} = \frac{A(n-1)}{A(n)}$$

### 1.3.4 Present Value

Present Value with compound interest:

$$\text{PV} = \frac{\text{FV}}{(1+i)^t} = \text{FV} \cdot v^t$$

Present Value with compound interest and cash flows:

$$\text{PV}_{t=0} = \sum_{k=1}^{n} C_k v(t_k)$$

## 1.4 Future Value

With cash flows $C_1, C_2, \ldots, C_n$ received at times $t_1, t_2, \ldots, t_n$, then

$$\text{FV}_{t=n} = \sum_{k=1}^{n} C_k a(t_k)$$

- $C_k$: cash flow received at time $t_k$
- $a(t_k)$: accumulation factor from time $t_k$ to the final time $t = n$
- $\text{FV}_{t=n}$: future value at time $t = n$

# Chapter 2

# Annuities with non-contigent payments

An annuity is a series of payments made at equal time intervals.

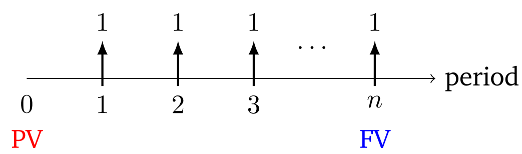
Types of Annuities (Based on Payment Structure)

1. **Amount of Payments**
   - **Level payments**: Equal payment each period
   - **Non-level payments**: Varying payment amounts
2. **Timing of Payments**
   - **Immediate annuity**: Payment at *end* of each period
   - **Due annuity**: Payment at *beginning* of each period
3. **Number of Payments**
   - **Term annuity**: Fixed number of payments
   - **Perpetuity**: Payments continue *forever*
4. **Deferral of Payments**
   - **Deferred annuity**: Payments start *after a delay*

## 2.1 Level annuity

### 2.1.1 Immediate Annuity

Consider a level annuity-immediate where each payment is **1 unit**, made at the **end of each period**. There are n total payments, and the effective rate of interest **per unit of time** is $i$. One unit of time equals one period.




The present value (sum of discounted payments) of that annuity at period t=0 is defined as:

$$PV = \frac{1}{1+i} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^n} = \sum_{k=1}^{n} \frac{1}{(1+i)^k} = \frac{1-v^n}{i}, \quad \text{where } v = \frac{1}{1+i}$$

The future value (value at time n) of the same annuity is:

$$FV = 1 + (1+i) + (1+i)^2 + \cdots + (1+i)^{n-1} = \sum_{k=0}^{n-1} (1+i)^k$$

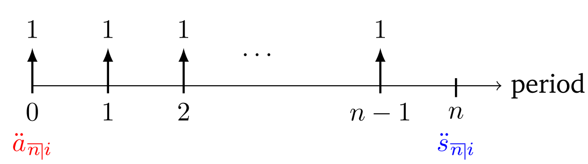
**Formula 4** (Immediate Annuity)

$$a_{\overline{n}|i} = PV = \frac{1-v^n}{i}, \quad \text{where } v = \frac{1}{1+i}$$

$$s_{\overline{n}|i} = FV = \sum_{k=1}^{n} (1+i)^k = \frac{(1+i)^n - 1}{i}$$

### 2.1.2 Annuity Due

Consider a level annuity-due where each payment is **1 unit**, made at the **beginning of each period**. There are n total payments, and the effective rate of interest **per unit of time** is $i$. One unit of time equals one period.




The present value (sum of discounted payments) of that annuity at period t=0 is:

$$PV = 1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^{n-1}} = \sum_{k=0}^{n-1} \frac{1}{(1+i)^k} = \frac{1-v^n}{i} \cdot (1+i), \quad \text{where } v = \frac{1}{1+i}$$

The future value (value at time n) of the same annuity is:

$$FV = (1+i) + (1+i)^2 + \cdots + (1+i)^n = \sum_{k=1}^{n}(1+i)^k$$

**Formula 5** (Annuity Due)

$$\ddot{a}_{\overline{n}|i} = PV = \frac{1-v^n}{i} \cdot (1+i) = \frac{1-v^n}{d}, \quad \text{where } d = \frac{1+i}{i}$$

$$\ddot{s}_{\overline{n}|i} = FV = \sum_{k=1}^{n}(1+i)^k$$

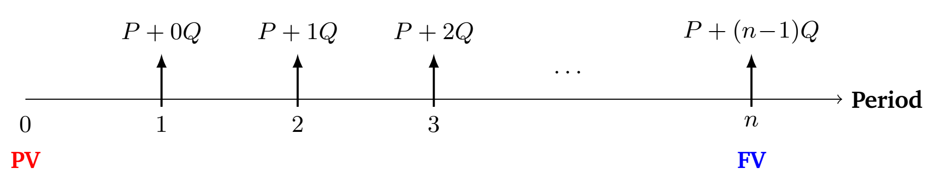
## 2.2 Varrying-Payments Annuity

Payments in an annuity that changes instead of staying level.

### 2.2.1 Payments in Arithmetic Progression

**1. Arithmetic Increasing Annuity**

Let the first payment = **P**, each following payment increases by **Q**, and there are total **n** payments.




**Formula 6** (Arithmetic Increasing Annuity)
Present Value at time (period) t=0:

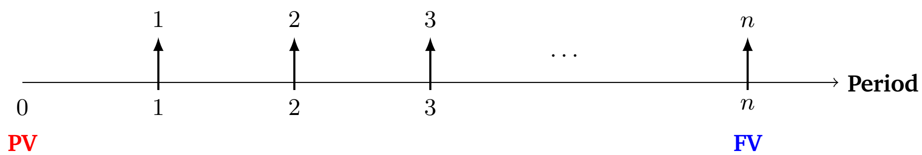
$$PV = P \cdot a_{\overline{n}|i} + Q \cdot \frac{a_{\overline{n}|i} - n \cdot v^n}{i}$$

Accumulated Value at time (period) t=n:

$$AV = P \cdot s_{\overline{n}|i} + Q \cdot \frac{s_{\overline{n}|i} - n}{i}$$

### 2. Increasing Annuity

When P=Q=1, payments become 1,2,...,n.




**Formula 7** (Increasing Annuity)
PV at time (period) t = 0:

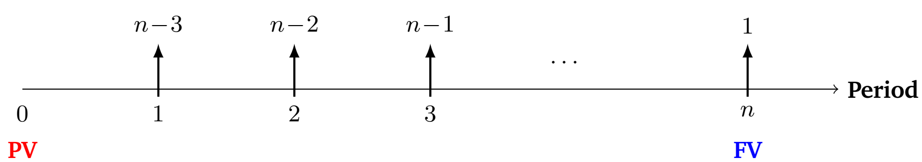
$$(Ia)_{\overline{n}|i} = a_{\overline{n}|i} + \frac{a_{\overline{n}|i} - nv^n}{i} = \frac{(1+i)a_{\overline{n}|i-nv^n}}{i} = \frac{\ddot{a}_{\overline{n}|}i - nv^n}{i}$$

FV at time (period) t = n:

$$(Is)_{\overline{n}|i} = (1+i)^n \cdot (Ia)_{\overline{n}|i} = \frac{\ddot{s}_{\overline{n}|i} - n}{i} = \frac{s_{\overline{n+1}|i} - (n+1)}{i}$$

### 3. Decreasing Annuity

When P=n and Q=1, payments become n, n-1, n-2, ..., 1.




**Formula 8** (Decreasing Annuity)

$$\text{PV}_{t=0} = (Da)_{\overline{n}|} = na_{\overline{n}|i} - \frac{a_{\overline{n}|i} - nv^n}{i} = \frac{n - nv^n - a_{\overline{n}|i} + nv^n}{i} = \frac{n - a_{\overline{n}|i}}{i}$$

$$\text{FV}_{t=n} = (Ds)_{\overline{n}|} = (1+i)^n (Da)_{\overline{n}|} = \frac{n(1+i)^n - s_{\overline{n}|i}}{i} = (n+1)a_{\overline{n}|i} - (Ia)_{\overline{n}|i}$$

### 4. Varying Perpetuity-Immediate with Arithmetic growth

The payments follow an Arithmetic progression with constants P 0 and Q 0. Payments are P, P+Q, P+2Q,... First payment is at the end of first period.

| Annuity | PV | FV |
|---|---|---|
| Arithmetic Increasing Annuity | $P \cdot a_{\overline{n}\vert i} + Q \cdot \frac{a_{\overline{n}\vert i} - n \cdot v^n}{i}$ | $P \cdot s_{\overline{n}\vert i} + Q \cdot \frac{s_{\overline{n}\vert i} - n}{i}$ |

Table 2.1: Summary

Present Value of Perpetuity-Immediate with payments of arithmetic progression:

$$\text{PV} = \lim_{n\to\infty} \left( Pa_{\overline{n}|} + Q \cdot \frac{a_{\overline{n}|} - n\nu^n}{i} \right)$$

Since n goes to infinity,

$$\lim_{n\to\infty} a_{\overline{n}|} = a_{\overline{\infty}|} = \frac{1}{i}$$

$$\lim_{n\to\infty} n\nu^n = 0 \quad \text{(via L'Hôpital's Rule)}$$

Hence,

$$\text{PV} = \frac{P}{i} + \frac{Q}{i^2}$$

**Formula 9** (Increasing Perpetuity-Immediate - Arithmetic Growth)
When $P = Q = 1$:

$$(Ia)_{\overline{\infty}|} = \frac{1}{i} + \frac{1}{i^2}$$

**5. Varying Perpetuity-Due with Arithmetic growth**

The PV of a perpetuity-immediate with arithmetic growth is:

$$\text{PV}_{\text{immediate}} = \frac{P}{i} + \frac{Q}{i^2}$$

For perpetuity-due, we shift everything 1 period earlier:

$$\text{PV}_{\text{due}} = (\frac{P}{i} + \frac{Q}{i^2})(1+i) = (\frac{1}{i} + \frac{1}{i^2})(1+i) = \frac{(1+i)^2}{i^2} = \frac{1}{d^2}$$

(Increasing Perpetuity-Due - Arithmetic Growth) When P = Q = 1:

$$(I\ddot{a})_{\overline{\infty}|} = \frac{1}{d^2}$$

### 2.2.2 Payments in Geometric Progression

A **geometric annuity** has **n payments**, each payment grows by a constant percentage, or **growht rate k**. If the first payment is 1, and growth rate is $k$, then the payments are

$$\text{Payments} = 1, (1+k), (1+k)^2, \dots, (1+k)^{n-1}$$

For each payment in n payments of the annuity, each is discounted back to time 0 by discount factor $\frac{1}{1+i}$.

$$\begin{aligned} PV &= v + v^2(1+r) + \cdots + v^n(1+k)^{n-1} \\ &= v(1 + v(1+k) + \cdots + v^{n-1}(1+k)^{n-1}) \end{aligned}$$

The RHS follows a geometric series with common ratio $r = \frac{1+k}{1+i}$, hence its sum follows:

$$\begin{aligned} \text{PV} = v\left[\frac{1-\left(\frac{1+k}{1+i}\right)^n}{1-\frac{1+k}{1+i}}\right] &= v\left[\frac{1-\left(\frac{1+k}{1+i}\right)^n}{\frac{i-k}{1+i}}\right] \\ = v\left[\frac{1-\left(\frac{1+k}{1+i}\right)^n}{v(i-k)}\right] &= \frac{1-\left(\frac{1+k}{1+i}\right)^n}{i-k} \end{aligned}$$

**Formula 10** (PV of an annuity-immediate of geometric progression)

$$\text{PV} = \frac{1-\left(\frac{1+k}{1+i}\right)^n}{i-k}$$

**Example 1.** (Geometric annuity with unknown interest rate) A 10-year annuity-immediate has:

- First payment: \$11
- Subsequent payments: 10% increase each year
- Accumulated value: \$220.8

Find the annual effective interest rate $i$.

1. **AV as Geometric Series:**
$$AV = 11(1.1)^9 \left[1 + \frac{1+i}{1.1} + \left(\frac{1+i}{1.1}\right)^2 + \cdots + \left(\frac{1+i}{1.1}\right)^9\right]$$
2. **Sum the Series:**
$$220.8 = 11(1.1)^9 \cdot \frac{1 - \left(\frac{1+i}{1.1}\right)^{10}}{1 - \frac{1+i}{1.1}}$$
3. **Substitute Values:**
$$220.8 = 25.937 \cdot \frac{1-(1+j)^{10}}{-j}, \quad j = \frac{1.1}{1+i} - 1$$
4. **Solve for $j$:**
$$\frac{(1+j)^{10}-1}{j} = 8.513 \implies j \approx 0.03773$$
5. **Find $i$:**
$$j = \frac{1.1}{1+i} - 1 \implies i = \boxed{0.06}\ (6\%)$$

## 2.3 Deferred Annuity

A deferred annuity is an annuity where the payments start at a future date, not immediately.

There are 2 parts in a deferred annuity:

1. Deferral period: Time you wait before payments start.
2. Payment period: Regular payments begin (like a normal annuity).

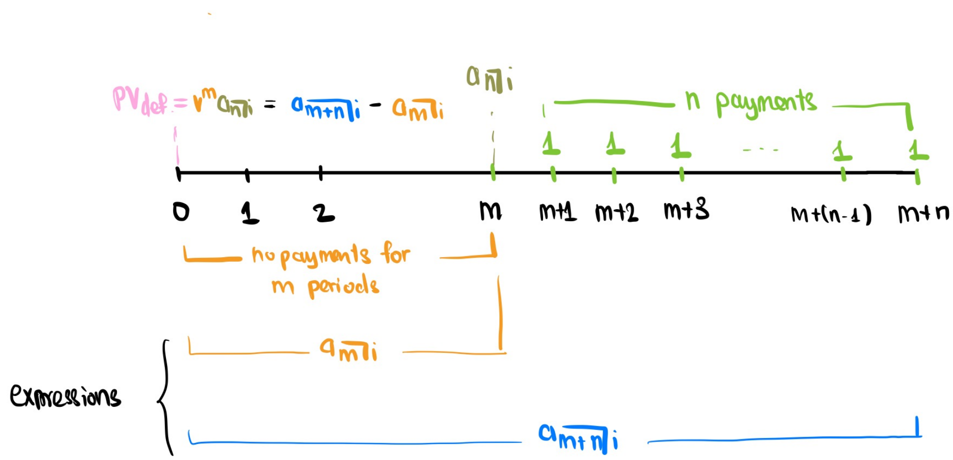



Figure 2.1: Deferred Annuity

Three annuity valuation cases:

- **Before 1st Payment (Deferred Annuity)**: $PV = v^m a_{\overline{n}|i} = a_{\overline{m+n}|i} - a_{\overline{m}|i}$
- **After Last Payment**: AV = move future value forward using interest.
- **Between Payments**: Split into past and future parts: some payments made, some still due.

Calculate the present value of the $n$-period annuity ($a_n$) as if payments were starting now. Then discount it back $m$ periods using $v^m = (1+i)^{-m}$.

### 2.3.1 PV before first payment of an annuity-immediate

**Theorem 1.** The present value is: PV $= v^m a_n$

We proof the above theorem by derviving from standard annuity formulas:

$a_{\overline{m+n}|i}$ is the PV of an annuity-immediate including all payments over m+n periods (at this point, we assume that there are m+n payments from period t=1 to t=m+n but the actual number of payments is only n, as in 0-m period, there are zero payments).

$a_{\overline{m}|i}$ is the PV of an annuity-immediate over m periods with m payments (this is an assumption as in reality there is zero payment at this point since this is the deferred period).

*Proof.*

$$a_{\overline{m+n}|i} - a_{\overline{m}|i} = \frac{1-\nu^{m+n}}{i} - \frac{1-\nu^{m}}{i}$$

$$= \frac{\nu^{m}-\nu^{m+n}}{i} = \frac{\nu^{m}(1-\nu^{n})}{i} = \nu^{m}a_{n}$$

$$\Rightarrow a_{\overline{m+n}|i} - a_{\overline{m}|i} = v^{m}a_{n}$$

This means:

- Calculate the present value of the $n$-period annuity ($a_n$) at time t=m (1st payment starts at time t=m+1.
- Then discount it back $m$ periods using $v^m = (1+i)^{-m}$.

Note: those expressions are for convinient calculation.

□

## 2.4 Varying-Interest Annuity

An annuity has an interest that can vary in each period. Let $i_k$ be the interest rate from time k-1 to k.

### 2.4.1 Rate $i_k$ is applied for period k (date depends on calender year)

Starting from when the 1st payment made at time 0, up to time $k^{th}$ or later, the future payments are discounted/accumulated using **all previous rate** $i_1, i_2, i_3, ..., i_k$.

**Annuity-immediate**

The PV of an annuity-immediate is:

$$a_{\overline{n}|i} = \frac{1}{1+i_1} + \frac{1}{(1+i_1)(1+i_2)} + \cdots + \frac{1}{(1+i_1)(1+i_2)\cdots(1+i_n)}$$

The AV of an anntuiy-immediate is:

$$s_{\overline{n+1}|i} = \ddot{s}_{\overline{n}|i} + 1$$

**Annuity-due**

The PV of an annuity-due is:

$$\ddot{a}_{\overline{n}|i} = 1 + a_{\overline{n-1}|i}$$

The AV of an anntuiy-due is:

$$\ddot{s}_{\overline{n}|i} = (1+i_1)(1+i_2)\cdots(1+i_n) + \cdots + (1+i_{n-1})(1+i_n) + (1+i_n)$$

### 2.4.2 Rate $i_k$ is applied for period k and before/later (rate depends on deposit time)

Regardless of when the payments are made, any payments in the period $k$ is discounted using only rate $i_k$. The rate $i_k$ is the effective rate for period $i \leq k$ (PV) and $i \geq k$. Intuition: The rate depends on when you invest - it locks in the rate.

**Annuity-Immediate**

The PV of an annuity-immediate is:

$$a_{\overline{n}|i} = \frac{1}{(1+i_1)^1} + \frac{1}{(1+i_2)^2} + \cdots + \frac{1}{(1+i_n)^n}$$

The FV of an annuity-immediate is:

$$s_{\overline{n+1}|i} = \ddot{s}_{\overline{n}|i} + 1$$

**Annuity-due**

The PV of an annuity-due is:

$$\ddot{a}_{\overline{n}|i} = 1 + a_{\overline{n-1}|i}$$

The FV of an annuity-due is:

$$\ddot{s}_{\overline{n}|i} = (1+i_1)^n + (1+i_2)^{n-1} + \cdots + (1+i_n)^1$$

## 2.5 Non-coinciding Frequencies Annuity

An annuity where **payments** made and **interest** compounded at a different frequency.

$$(1+j)^m = (1+i)^n$$

- $j$ = rate per payment period (what we want)
- $m$ = number of payment periods
- $i$ = rate per compounding period
- $n$ = number of compounding periods

| Term | Applies To | Rate For | Found From |
|---|---|---|---|
| $i$ | Compounding frequency | Compounding periods (e.g., quarterly) | Given by nominal rate |
| $j$ | Payments frequency | Payment periods (e.g., monthly) | Must be **calculated** to match payment frequency |

**Formula 11** If **nominal rate** $i_{\textbf{nom}}$ is compounded $n$ times per year, then:

$$i_{\text{eff}} + 1 = (1 + \frac{i_{\text{nom}}}{n})$$

## 2.6 Perpetuity

Perpetuity is an annuity that pays forever. Perpetuity-immediate has payments start 1 period from now. Perpetuity-due has payments which start immediately (now).

**Formula 12** (Present Value)
Perpetuity-Immediate:

$$\text{PV} = \text{a}_{\overline{\infty}|} = v + v^2 + \cdots = \frac{1}{i}$$

Perpetuity-Due:

$$\text{PV} = \ddot{a}_{\overline{\infty}|} = 1 + v + v^2 + \cdots = \frac{1}{d}$$

Relationship: $\ddot{a}_{\overline{\infty}|} = 1 + \text{a}_{\overline{\infty}|}$

# Chapter 3

# Loan

## 3.1 Debt instruments

A debt instrument is a contract that requires the borrower to repay principal and usually interest at a future date.

Examples of **Debt Instruments:**

1. Bonds (corporate, government)
2. Loans

## 3.2 Outstanding Balance Calculation for Level Payments

**Notation:**

- $I_t$: Interest paid during the $k$-th period.
- $P_t$: Principal (capital) repaid during the $k$-th period.
- $B_t$: Outstanding balance immediately after the $t$-th payment.
- $R_t$: Total payment made during the $t$-th period (interest + principal).

| Type | Formula | Interpretation |
|---|---|---|
| Prospective | $B_t = R \cdot a_{n-t}$ | Present value of remaining (future) level payments |
| Retrospective | $B_t = R \cdot s_t$ | Accumulated value of past payments |

**Loan Equation of Value:** You can express the loan's total value $L$ using the equation of value:

$$L = B_0 = R_1 v + R_2 v^2 + \cdots + R_n v^n$$

## 3.3 Loan Amortization (Level/Non-level payments

A loan can be interpreted as an annuity with payments made in regular intervals, each payment consists of two parts:

- **Interest on the loan**: A cost for borrowing the money.
- **Principal**: The amount of the loan that you borrowed.

Over time, interest portion decreases and principal portion increases. Total payments stay the same.

### 3.3.1 Interest portion in period t

At any given time, the interest due for the next payment period will depend on the outstanding balance before t-th payment.

$$I_t = i \cdot B_{t-1} = i \cdot R \cdot a_{\overline{n-t+1}|} = R \cdot (1 - v^{n-t+1})$$

### 3.3.2 Principal portion in period t

$$\begin{aligned} P_t &= R - I_t \\ &= R - R \cdot (1 - v^{n-t+1} \\ &= R \cdot v^{n-t+1} \end{aligned}$$

Only for level payments:

$$\text{FV of principal } P_t \text{ after } k \text{ periods at interest rate } i = P_t \cdot (1+i)^k = P_{t+k}$$

### 3.3.3 Balance

- **Interest:** Paid to lender (does **not** reduce balance)
- **Principal:** Reduces the loan balance

$$\begin{aligned} \text{Balance after t-th payment} &= \text{Balance before t-th payment} - \text{Principal repaid at time t} \\ &= B_{t-1} - P_t \end{aligned}$$

$$\begin{aligned} \text{Balance after t-th payment} &= \text{Balance at time t with interest} - \text{Payment made at time t} \\ &= B_{t-1}(1+i) - R_t \end{aligned}$$

## 3.4 Summary

| Component | Formula at Period $t$ |
|---|---|
| Total Payment $R$ | $R = 1$ |
| Interest $I_t$ | $I_t = 1 - v^{n-t+1}$ |
| Principal Repaid $P_t$ | $P_t = v^{n-t+1}$ |

# Chapter 4

# Bond

## 4.1 Bond

A **bond** is a type of debt instrument made by investors to a borrower (government/corporation). The borrower promises to pay: interest (coupons) and principal (face value) at maturity.

### 4.1.1 Key terms

- **Term**: Time from issue to maturity.
- **Maturity Date**: Final payment date.
- **Yield**: Actual return for the investor, depending on the price.

**Formula 13** Regarding to bond pricing formulas, key notations are as follow:

- **Par Value / Face Value** $F$: Original issue price of the bond which does not change over time.
- **Coupon Rate** $r$: Interest rate applied to the face value and is set by issuer (fixed).
- **Price** $P$: What the investor pays for the bond.
- **Redemption Value of Bond** $C$: $F = C$ until otherwise stated. $C$ is the amount the bondholder gets at maturity.
- **Interest rate per payment period** $i$: Fluctuates based on market.
- **Number of coupon payments** $n$

## 4.2 Bond yield

**Bond Yield/Yield to Maturity (YTM)** is the rate of return $i$ you'll earn if you hold the bond to maturity, assuming all coupons are reinvested at the same rate. It is expressed as percentage (%). Components of YTM:

1. Current bond price $P$
2. Coupon payments $Fr$
3. Time to maturity $n$
4. Face Value/Par value $F$

**Formula 14 (Nominal Yield)**

$$\text{Nominal Yield} = \frac{\text{Annual Coupon}}{\text{Par Value}}$$

**Formula 15 (Current Yield)**

$$\text{Current Yield} = \frac{\text{Annual Coupon}}{\text{Bond Price}}$$

**Formula 16 (Yield to Maturity (YTM))** is the effective return based on what the bond costs today.

## 4.3 Bond pricing formula

**Formula 17** (Bond Price) For a bond with coupons, its price is

$$P(i) = \text{PV of Coupon payments} + \text{PV of Redemption value} = F \cdot r \cdot a_{\overline{n}|i} + Cv^n$$

## 4.4 Premium/Discount

Bonds dont always trade at **par value** (i.e. $P$ is not always equal to $F$). This is because coupon rate $r$ is different from (market) yield rate/YTM $i$. In addition, price of bond $P$ is dependent on market yield $i$.

- If $r = i$ or $Fr = Ci$, bond is sold at par.
- If $r > i$ or $Fr > Ci$, bond is sold at a premium.
- If $r < i$ or $Fr < Ci$, bond is issed at a discount.

Note that $F = C$ by default.

If $r < i$, the bond is less attractive to investors. So the issuer must lower the price to make it appealing and the bond is sold at discount. Similarly, if $r > i$, the issuer will raise the price as investors are willing to pay higher price for the bond.

| **Condition** | **Price Compared to $C$** | **Called** |
|---|---|---|
| $r < i$ | $P < C(= F)$ | **Discount bond** |
| $r = i$ | $P = C(= F)$ | **Par** |
| $r > i$ | $P > C(= F)$ | **Premium bond** |

**Formula 18** Price of a bond can be calculated using Premium/Discount:

$$P(i) = C + (F \cdot r - C \cdot i)\mathsf{a}_{\overline{n}|i}$$

where $Fr - Ci$ is the premium or discount **per period**. They are cash flows over $n$ periods and seen as an annuity of coupon payments.

- **Premium**: $P - C = (Fr - Ci)a_{\overline{n}|i}$
- **Discount**: $C - P = (Ci - Fr)a_{\overline{n}|i}$

Note that $Fr$ is the coupon payment, and $Ci$ is the market interest payment.

## 4.5 Bond duration

### 4.5.1 Par Bond Duration

When the bond is sold at par (i.e., when the yield $i = r$), its Macaulay Duration is:

$$\begin{aligned}
\text{MacD} &= \frac{F(r(Ia)_{\overline{n}|} + nv^n)}{Fra_{\overline{n}|} + Fv^n} \\
&= \frac{r(Ia)_{\overline{n}|} + nv^n}{ra_{\overline{n}|} + v^n} \\
&= \frac{r(1+i)a_{\overline{n}|i} + (i-r)nv^n}{r + (i-r)v^n}
\end{aligned}$$

Since $i = r$ (bond sold at par), the equation becomes

$$\text{MacD} = \frac{r(1+i)a_{\overline{n}|i}}{r} = (1+i)a_{\overline{n}|i} = \ddot{a}_{\overline{n}|i}$$

**Formula 19** The Macaulay Duration of a par bond is

$$\text{MacD} = \ddot{a}_{\overline{n}|i}$$

# Chapter 5

# Portfolio

## 5.1 Duration of Portfolio

### 5.1.1 Modified duration of Portfolio

Given a portfolio with **n bonds**, each bond $k$ has price $P_k(i)$ and **Modified duration** $v_k$. Total portfolio value is

$$P(i) = P_1(i) + P_2(i) + \cdots + P_n(i)$$

**Formula 20** The Modified duration of the portfolio is

$$\text{MacD}_{\text{Portfolio}} = \frac{P_1(i)}{P(i)}v_1 + \frac{P_2(i)}{P(i)}v_2 + \cdots + \frac{P_n(i)}{P(i)}v_n = \frac{P_1v_1 + P_2v_2 + \cdots + P_nv_n}{P_{\text{total}}}$$

$$\text{MacD}_{\text{Portfolio}} = \sum_{k=1}^{n} \left( \frac{P_k(i)}{P(i)} \cdot v_k \right)$$

It's the weighted average of each bond's modified duration, using current price as weight.

### 5.1.2 Macaulay duration of Portfolio

**Formula 21** Macauly duration of a portfolio is

$$\text{MacD}_{\text{Portfolio}} = \frac{\sum_0^n (t \cdot \text{PV of cash flow at time t})}{\text{Total present value of the portfolio}}$$

# Chapter 6

# Portfolio Performance

## 6.1 Discounted Cash Flow Technique

DCF is a method to measure the **profitability** of investment projects. Unlike fixed annuities (same payment pattern), DCF allows any pattern of cash inflows (returns) and outflows (costs).

| **Feature** | **Annuities** | **Investments** |
|---|---|---|
| Payments | Regular intervals, level payment | May vary in time and amount |
| Risk | Low/fixed | High/low |
| Examples | Pensions, loans | Stocks, real estate |

There are **two** DCF measures:

1. Net Present Value (NPV) – Present value of all net cash flows.
2. Internal Rate of Return (IRR) – Interest rate that makes NPV = 0.

### 6.1.1 Cash flows

Notations with Cash flows:

- $C_t$ = contributions/outflows (money invested)
- $R_t$ = returns/inflows (money received)
- $c_t = R_t - C_t$ = net cash flow at time $t$

$c_t > 0$: Net deposit (inflow), $c_t < 0$: Net withdrawal (outflow)

**Example 2. Project Description:**
A company plans to develop and sell a new product. The cash flows are as follows:

- Initial investment of $80,000 at year 0.
- Additional investments of $10,000 in years 1, 2, and 3.
- A contribution of $20,000 in year 4 to launch the product.
- Maintenance costs of $2,000 per year from years 5 to 9.
- Returns: $12,000 in year 4, $30,000 in year 5, $40,000 in year 6, $35,000 in year 7, $25,000 in year 8, $15,000 in year 9, and $8,000 in year 10.

**Cash Flow Table:**

| Year | Contributions | Returns | Net Cash Flow ($c_t$) |
|---|---|---|---|
| 0 | 80,000 | 0 | -80,000 |
| 1 | 10,000 | 0 | -10,000 |
| 2 | 10,000 | 0 | -10,000 |
| 3 | 10,000 | 0 | -10,000 |
| 4 | 20,000 | 12,000 | -8,000 |
| 5 | 2,000 | 30,000 | 28,000 |
| 6 | 2,000 | 40,000 | 38,000 |
| 7 | 2,000 | 35,000 | 33,000 |
| 8 | 2,000 | 25,000 | 23,000 |
| 9 | 2,000 | 15,000 | 13,000 |
| 10 | 0 | 8,000 | 8,000 |

**Net Present Value (NPV):**
Let $i$ be the interest rate (cost of capital), and let $v = \frac{1}{1+i}$. Then the NPV of the project is:

$$\text{NPV}(i) = \sum_{t=0}^{10} c_t v^t = \sum_{t=0}^{10} \frac{c_t}{(1+i)^t}$$

Where $c_t$ is the net cash flow in year $t$.
**Interpretation:**

- If NPV$(i) > 0$: the investment is profitable.
- If NPV$(i) = 0$: the investment breaks even.
- If NPV$(i) < 0$: the investment is not worth it.

## 6.2 Net Present Value

**Net present value (NPV)** is the difference between the present value of **cash inflows** and the present value of **cash outflows** over a period of time. Choose the investment with the greatest positive NPV.

**Formula:**

$$\text{NPV(i)} = \sum (R_t - C_t) v^t = \sum c_t \cdot v^t$$

- $v^t = \frac{1}{(1+i)^t}$ is the discount factor
- $i$ = rate of interest per period = required return of the investment = cost of capital

## 6.3 Yield Rate - Internal Rate of Return (IRR)

**Yield rate** or IRR is the rate such that the PV of cash inflows is equal to the PV of cash outflows. Choose the investment with the greatest IRR.

**Formula:**

$$\text{IRR} = \text{value of } i \text{ such that } \sum (A_t - L_t) v^t = 0$$

**Interpretation:**
Yield Rate (IRR) = The interest rate where an investment neither gains nor loses money.

- NPV $> 0$ : Profit
- NPV $< 0$ : Loss
- NPV $= 0$ : Break-even (Yield rate achieved)

Should you still invest when NPV $= 0$ ? When NPV $= 0$, there is no net gain, but no loss either. Reject the investment if there are better opportunities (i.e. another project with NPV $> 0$). Accept it when the Yield rate matched the **inflation rate**, so your money can keep its real value.

### Connection Between NPV and IRR

- NPV is a function of the interest rate: $P(i)$
- IRR is the rate where $P(i) = 0$

## 6.4 Reinvestment

### 6.4.1 Lump Sum Investment + Interest reinvested

Intuition:
1. Invest 1 unit of money - **a lump sum/principal amount** - for n periods at rate $i$.
2. Interest is **reinvested** at rate $j$.

Reinvesting is like planting a tree (investment) and using its seeds (interest) to grow more trees instead of eating them. Over time, you get a forest!
Illustration:

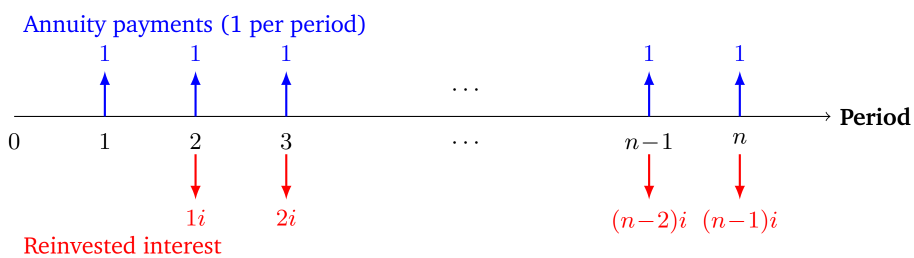
1. Start with 1 unit of money - that principal stays in the account for **n periods.**
2. Interest earned per period is 1$ * $i$ = $i. The principal gets no compounding effect.

3. At the end of year 1,2,...,n-1, we reinvest \$i at each end of the year. This pattern follows an annuity-immediate with **n payments** of \$i, and rate per period is $j$.
4. The **AV** of that annuity is: i * $s_{\overline{n}|j}$.
5. Add the principle: Total AV = Principle + Reinvested Interest = 1 + i * $s_{\overline{n}|j}$.

> **Formula 22** (Lump Sum Reinvestment) Total Accumulated Value = 1 + i * $s_{\overline{n}|j}$
> Special case: $i = j$, then AV = $(1+i)^n$

### 6.4.2 Annuity + Interest reinvested

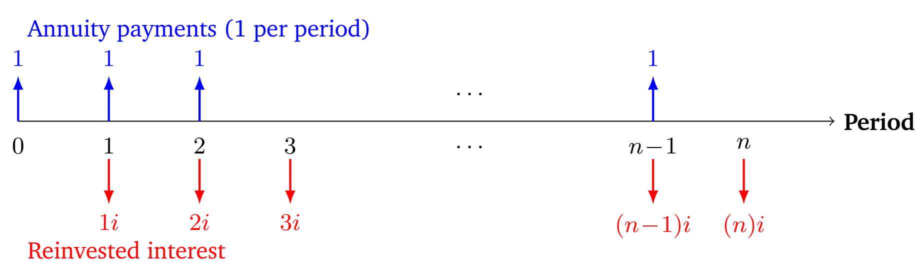
**Annuity-immediate**




AV = sum of annual payments + reinvested interests as annuity
- Reinvested interests as an annuity: i, 2i, 3i, ..., (n-2)i, (n-1)i. This acts as an increasing annuity with i as a common difference.
- Sum of annual payments is n.

> **Formula 23** (Interest Reinvestment as Annuity-immediate)
>
> $$AV = n + i(I_s)_{\overline{n-1}|j} = n + i[\frac{s_{\overline{n}|j} - n}{j}]$$
>
> $$AV = s_{\overline{n}|i} \text{ when } i = j$$

**Annuity-due**



**Formula 24** (Interest Reinvestment as Annuity-due)

$$AV = n + i(I_s)_{\overline{n}|j} = n + i[\frac{s_{\overline{n+1}|j} - (n+1)}{j}]$$

## 6.5 Dollar-weighted Rate of Interest

| **Concept** | **DWR** | **IRR (Rate for NPV = 0)** |
|---|---|---|
| Purpose | Measures fund performance | Evaluates project profitability |
| Method | Based on future value | Based on present value |
| Time orientation | Grows cash to end | Discounts cash to start |
| Terminology | Used in actuarial & finance | Used in finance & investment |

The **dollar-weighted interest rate** $i$ is the average rate of how fast the money grew during a period, including the **timing and size of money in/out**("dollar-weighted"). It measures how well **you** did as an **investor**.

$$i \approx \frac{I}{A + \sum_{0 \le t \le 1} c_t(1-t)} = \frac{\text{Interest earned}}{\text{Exposure}}$$

The approximation is good when each contribution $c_t$ is small compared to the amount $A$.

**Formula 25** Potforlio at the end of investment periods:

$$B = A + C + I$$

Key terms:

- $A$: Amount at the beginning of the period.
- $B$: Amount at the end of the period.
- $I$: Total interest earned during the period.
- $c_t$: Net contribution (deposit - withdrawal) at time $t \in [0, 1]$
- $C = \sum c_t$: Total net contributions.
- $(1+i)^{1-t} - 1$ is the effective rate for period from $t$ to 1.

Concept:

1. Total amount at the end is: $B = A + C + I$
2. Interest earned: $I = iA + \sum_{0 \le t \le 1} c_t[(1+i)^{1-t} - 1]$

where $iA$ is the interest on initial amount $A$ and $\sum_{0 \le t \le 1} c_t[(1+i)^{1-t} - 1]$ is the interest on total contributions made at time $t \in [0,1]$.
3. Substitute into

$$\begin{aligned} B &= A + C + I \\ &= A + C + iA + \sum_{0 \le t \le 1} c_t \left[(1+i)^{1-t} - 1\right] \\ &= A(1+i) + \sum_{0 \le t \le 1} c_t (1+i)^{1-t} \end{aligned}$$

4. Approximate the compound interest using simple interest:

$$(1+i)^{1-t} \approx 1 + (1-t)i \text{ then } (1+i)^{1-t} - 1 \approx (1-t)i$$

5. $I = iA + \sum_{0 \le t \le 1} c_t[(1-t)i] = i[A + \sum_{0 \le t \le 1} c_t(1-t)]$ then $i = \frac{I}{A + \sum_{0 \le t \le 1} c_t(1-t)}$

**Formula 26** (Exposure)

$$A + \sum_{0 \le t \le 1} c_t(1-t)$$

Exposure is as a weighted sum of how much money was active in the fund and for how long.

The denominator is called **exposure associated with** $i$ & represents **total time-weighted amount of money at risk**:

- $A$: Initial fund amount that earns interest for the full year - its weight = 1.
- $\sum_{0 \le t \le 1} c_t(1-t)$: time-weighted contributions, giving how long each contribution had to earn interest ($c_t$ it the contribution made at time $t$ and $(1-t)$= weight is the fraction of the year left in which the contribution earns interests.)

**Example 3.** At the beginning of a year, an investment fund was established with an initial deposit of $3,000. At the end of six months, a new deposit of $1,500 was made. Withdrawals of $500 and $800 were made at the end of four months and eight months respectively. The amount in the fund at the end of the year is $3,876. Set up the equation of value to calculate the dollar-weighted rate of interest.

*Solution:* To find the interest rate $i$ that makes the future value of all cash flows = $3876.
General formula:

$$\text{Future Value} = \sum c_t(1+i)^{1-t}$$

Set up the equation of value to calculate the dollar-weighted rate of interest $i$:

$$3000(1+i) + 1500(1+i)^{0.5} - 500(1+i)^{(1-\frac{4}{12})} - 800(1+i)^{(1-\frac{8}{12})} = 3876$$

## 6.6 Time-weighted Rate of Interest

**Time-weighted rate of return** isolates fund performance and ignores investor actions.

$$i = (1 + j_1)(1 + j_2)...(1 + j_m) - 1$$

where $j_k$ is the rate of return for each sub-period in an interval (in this case, a year) which has m sub-periods.

**Set up:**

- The year is split into m intervals (sub-periods)
- At each time $t_k$,
  $C_{t_k}$ = net contribution
  $B_{t_k}$ = fund value just before that contribution

then for each subinterva; $k = 1, 2, ..., m$, the rate of return $j_k$ for each sub-period is

$$B_{t_k} = (1 + j_k)(B_{t_{k-1}} + C_{t_{k-1}})$$

The overall yield rate $i$ for the entire year is given by

$$i + 1 = (1 + j_1)(1 + j_2)...(1 + j_m)$$

We call $i$ the **time-weighted rate of return.**

## 6.7 TWR and DWR

Example scenario:

- Year 1: Spend \$1000 to buy 1000 shares at \$1
- Year 2: Spend \$2000 to buy 1000 shares at \$2
- Year 3: Gain \$2500 to sell 2000 shares at \$1.25

**Dollar-weighted Return Rate** is $-12.77\%$ per year. It reflects your actual loss from your money decisions (you lost \$500 as you invested \$3000 but only received \$2500).

**Time-weighted Return Rate** is $11.80\%$ per year. The stock price went up from \$1 to \$2, then fell to \$1.25, but it was still higher than it started. Therefore, the stock itself had a positive return over time.

Goals of the 2 return methods:

1. **Dollar-weighted Return Rate**: measures how well the investment performs, excluding investors actions.
2. **Time-weighted Return Rate**: measures how well the investors did, including timing and size of money flows and reflecting real experience.

# Chapter 7

# Measures of Interest Rate Sensitivity

## 7.1 Inflation

Inflation = general rise in prices of goods and services over time. It reduces purchasing power of money.

Inflation and Interest Rates: inflation and interest rates move together over time. Investors demand higher interest to compensate for future inflation.

**Formula 27** Let $\pi$ is the inflation rate.

$$1 + i_{\text{real}} = \frac{1 + i_{\text{nominal}}}{1 + \pi}$$

### 7.1.1 Payments grow with inflation

**Present Value**

Each future payment grows by constant **inflation rate**.

**Formula 28** (PV iwth adjusted payments)

$$PV = R\left[\frac{1+r}{1+i} + \left(\frac{1+r}{1+i}\right)^2 + \cdots + \left(\frac{1+r}{1+i}\right)^n\right] = R(1+r)\cdot\frac{1-\left(\frac{1+r}{1+i}\right)^n}{i-r}$$

We discount the annuity with **nominal rate** as payments have been adjusted with inflation rate.

**Formula 29** (PV with ray payments, while inflation exists)

$$PV = R\left[\frac{1}{1+i_0} + \frac{1}{(1+i_0)^2} + \cdots + \frac{1}{(1+i_0)^n}\right] = R\cdot a_{\overline{n}|i_0}$$

We discount the annuity with **real interest rate.**

### 7.1.2 Summary

| Scenario | Formula | When to Use |
|---|---|---|
| Payments grow with inflation | $PV = R(1+r)\cdot\frac{1-\left(\frac{1+r}{1+i}\right)^n}{i-r}$ | Use nominal rate $i$ |
| Payments fixed in real terms | $PV = R\cdot a_{\overline{n}\|i_0}$ | Use real rate $i_0$ |

**Accumulated Value**

**Formula 30** (AV in nominal dollars (not adjusted for inflation))

$$\text{AV} = P(1+i_{\text{nominal}})^n$$

This is the raw future value of your investment.

**Formula 31** (AV adjusted for inflation)

$$\text{AV} = P(\frac{1+i}{1+r})^n = P(1+i_{\text{real}})$$

This refects the true purchasing power of your money.

## 7.2 The Term structure of Interest Rates and Yield Curves

### 7.2.1 Term

**Term**: The length of time until an investment/loan matures/ends. It is the duration until you get your money back.

### 7.2.2 Spot rate

**Spot rate** is the yield to maturity/(single rate of annual return) of a zero-coupon bond/(no cash flows). Spot rate is always based on time zero. It is a rate for one-time future payment.

$$v_t = \frac{1}{(1+s_t)^t}$$

- $v_t$: discount factor
- $s_t$: spot rate

Generally — the longer the investment term, the higher the interest rate, because

- More time = more risk (like inflation, uncertainty, default).
- Investors want extra return for locking money up longer, hence they charge higher rates for longer loans.

Key differences between: **Zero-coupon bond with spot/forward rates** and **Annuity using varying spot rates**

| Concept | Zero-Coupon Bond | Annuity with Varying Rates |
|---|---|---|
| Cash Flows | Single payment at end | Multiple payments/cash flows each year |
| Discounting Method | Compound using forward rates | Discount each payment with spot rates |
| Formula Used | $(1+s_n)^n = \prod_{i=0}^{n}(1+f_{[i,i+1]})$ | $PV = \sum \frac{C_t}{(1+s_t)^t}$ |
| Use Case | Zero-coupon bond pricing / yield | Valuing pension plans, loans, etc. |

### 7.2.3 Yield

Extend the table to a continous graph, where y-axis is **yield** (interest rates/spot rates of risk-free bonds) and x-axis is **maturity**, we obtain a yield curve. Yield curve can be upward-sloping (rates expected to rise), flat (all terms have same rate), and inverted (short-term ¿ long-term, a signal of recession).

| Length of investment (years) | Interest rate (Spot rate) |
|---|---|
| 1 year | 3% |
| 2 year | 4% |
| 3 year | 6% |
| 4 year | 7% |

**Yield Curve** is a graph of **spot rates** versus maturity time.

**Yield to Maturity:** A **single average rate** that discounts all cash flows of a bond.
- The bond is held to maturity.
- The bond does not default.
- Reinvestment of the bond and all coupons is executed at the original YTM.

**Formula 32** When spot rates $i_t$ vary by year, NPV is

$$\text{NPV} = \sum_{t=0}^{n} \frac{c_t}{(1+i_t)^t}$$

### 7.2.4 Forward Rate

**Forward rate:** the **interest rate** agreed on today for borrowing or investing money in the future from time n to time n+m. It tells what the market expects interest rates to be.

$$v_{[n,m-n]} = \frac{1}{(1+f_{[n,n+m]})^{m-n}}$$

**Formula 33** (Connect Spot rate with Forward rates)
- Spot Rate (s): Set by the current market — changes daily with supply/demand.
- Forward Rate (f): Calculated from spot rates.

$$(1+s_n)^n \dot{(}1+f_{[n,n+m]})^m = (1+s_{n+m})^{n+m}$$
$$(1+s_{[0,n]})^n = (1+f_{[0,1]}) \cdot (1+f_{[1,2]}) \ldots (1+f_{[n-1,n]})$$

## 7.3 Macaulay and Modified Durations

Why **Duration** matters?
- Duration measures the sensitivity of a bond's price to changes in interest rates.
- Duration reflects the timing of cash flowe (i.e. when you'll get your money back).

| Types of Duration | Definition | Formula |
|---|---|---|
| Term to Maturity | Time until final payment (not very useful with coupons). | — |
| Equated Time | Weighted average of payment times (weights = cash flows). | $\bar{t} = \frac{\sum tR_t}{\sum R_t}$ |
| Macaulay Duration | Weighted average of present values of payments. | $d = \frac{\sum t\nu^t R_t}{\sum \nu^t R_t}$ |
| Modified Duration | Measures price sensitivity to interest changes. | $\text{ModDur} = \frac{d}{1+i}$ |

### 7.3.1 Average term-to-maturity

Average Term-to-Maturity: ”On average, when do I receive my money back?”

Setup: For zero-coupon bond, there is only one payment (at maturity). It means shorter maturity has faster cash back. But for coupon bons with multiple cash flows over time, term-to-maturity ignores earlier coupon payments. Let’s say you’re paid \$100 in year 1 (coupon), \$200 in year 2 (coupon), and \$700 (final coupon + principal) in year 3. The average term-to-maturity might be around 2.6 years - it tells on average, you get back total \$1000, not just \$700 in year 3.

**Formula 34** If a bond pays cash flows $C_0, C_1, \ldots, C_n$ at times $t_0, t_1, \ldots, t_n$, then the **Equated Time** is given by:

$$\text{Equated Time} = \frac{\sum_{i=0}^{n} C_i \cdot t_i}{\sum_{i=0}^{n} C_i}$$

It tells how quickly your investment is returned, on average.

### 7.3.2 Macaulay Duration

It improves the upon equated time by using **present values** instead of just raw cash flows. Each cash flow is now discounted to present, thus the weighted average time of cash flows is more precise.

**Formula 35** The **Macaulay Duration** $MacD$ is given by:

$$MacD(i) = \frac{\sum_{t=0}^{n} t \cdot \nu^t R_t}{\sum_{t=0}^{n} \nu^t R_t}$$

- $R_t$ be the cash flow at time $t$
- $\nu^t = \frac{1}{(1+i)^t}$ is the discount factor at time $t$
- $i$ is the effective rate of interest per period.

Note that duration depends on $i$. When $i = 0$, the MacD is equal to the equated time formula. When there is only 1 future payment, duration is equal to time of payment (MadD = t).

### 7.3.3 Macaulay Duration and Cauchy-Schwarz Inequality

Recall the Macaulay duration:

$$d = \frac{\sum_{t=0}^{n} t \cdot \nu^t R_t}{\sum_{t=0}^{n} \nu^t R_t}, \quad \text{where } \nu^t = \frac{1}{(1+i)^t}$$

By Cauchy-Schwarz inequality, we see that

$$\frac{d}{di} MacD(i) < 0$$

So, Macaulay duration decreases as the interest rate increases.

## 7.4 Modified Duration (Volatility)

(Modified Duration) Measures how sensitive a **bond's price** is to a small change in its **yield-to-maturity (YTM)**. As yield rises, price falls, and vice versa - inverse relationship.

Setup:

The price of a bond is the present value of its cash flows:

$$P(i) = \sum_{t=0}^{n} \frac{C_t}{(1+i)^t}$$

Take the derivative of P(i) w.t. $i$ (yield rate). This is the **rate of change** of bond price when $i$ changes:

$$\frac{dP(i)}{di}$$

We want to express the percentage change in price of bond by dividing the derivative by the price, with the minus sign as price drops when $i$ increases. We obtained **volatility** which tells how sensitive the PV of bond's price to interest rate changes.

$$\text{Volatility} = -\frac{1}{P(i)} \cdot \frac{dP(i)}{di} = -\frac{P'(i)}{P(i)}$$

Real world meaning: if Volatility = 5, then if $i$ increases by 1% bond price drops about 5%. It's a linear approximation of the price-yield curve i.e. the % change of $P(i)$.

The standard derivative identity is:

$$\frac{d}{di}[lnP(i)] = \frac{P'(i)}{P(i)}$$

Thus, volatility, denoted by $\bar{v}$, becomes:

$$\bar{v} = -\frac{d}{di}[lnP(i)] = -\frac{P'(i)}{P(i)}$$

Volatility is often called **modified duration**. Now we derive $P(i)$ by taking the derivative with respect to $i$ :

$$P'(i) = \frac{d}{di}\left[\sum_{t=0}^{n} \frac{R_t}{(1+i)^t}\right] = -\sum_{t=0}^{n} t(1+i)^{-t-1}R_t = -\sum_{t=1}^{n} \frac{t \cdot R_t}{(1+i)^{t+1}}$$

Then plug into the volatility formula:

$$\bar{v} = -\frac{P'(i)}{P(i)} = \frac{\sum_{t=0}^{n} \frac{t \cdot R_t}{(1+i)^{t+1}}}{\sum_{t=0}^{n} \frac{R_t}{(1+i)^t}} = \frac{\sum_{t=0}^{n} t \cdot v^{t+1} \cdot R_t}{\sum_{t=0}^{n} t \cdot v^t \cdot R_t}$$

Express $\bar{v}$ in terms of MacD:

$$\bar{v} = \frac{\sum_{t=0}^{n} t \cdot v^{t+1} \cdot R_t}{\sum_{t=0}^{n} t \cdot v^t \cdot R_t} = v \cdot \frac{\sum_{t=0}^{n} t \cdot v^t \cdot R_t}{\sum_{t=0}^{n} t \cdot v^t \cdot R_t} = \text{MacD} \cdot v = \frac{\text{MacD}}{1+i}$$

**Formula 36**

$$\text{Modified Duration} = \text{Macaulay Duration} \cdot v = \frac{\sum_{t=0}^{n} t \cdot v^{t+1} \cdot R_t}{\sum_{t=0}^{n} t \cdot v^t \cdot R_t}$$

Remark: we assume that cash flows (payments) are fixed - they do not change with the interest rate changes.

## 7.5 MacaulayD vs. ModifiedD

Interpretation:

- $\frac{P'(i)}{P(i)}$ = change in PV per unit change in $i$. (in percentage)
- $\frac{P'(\delta)}{P(\delta)}$ = cheng in PV per unit change in $\delta$. (in time)

Table 7.1: Comparison of Macaulay Duration and Modified Duration

| **Feature** | **Macaulay Duration** | **Modified Duration** |
|---|---|---|
| Definition | Weighted average time until all payments in a series are made | Sensitivity of bond price to interest rate changes |
| Formula | $\text{MacD} = -\frac{P'(\delta)}{P(\delta)}$ | $\text{ModD} = -\frac{P'(i)}{P(i)} = \text{MacD} \cdot (1+i)$ |
| Units | Time (usually in years) | Percentage change per 1% interest rate change |
| Interpretation | "When" you get your money back (on average) | "How much" the price changes when interest changes |
| Rate sensitivity? | Indirectly | Directly |
| Dependence on Interest Rate | No (once cash flows are fixed) | Yes (through denominator $1+i$) |

## 7.6 Passage of Time

As time passes, the cash flows are getting closer. So naturally, the duration decreases.

### 7.6.1 Macaulay Duration

(Passage of Time)

**Formula 37**

$$\text{MacD}_{new} = \text{MacD}_{old} - (t_1 - t_0)$$

MacD changes over time, as cash flows are getting closer. The difference between these two $\text{MacD}_{old}$ and $\text{MacD}_{new}$ is just the time has passed $(t_1) - t_0$.

### 7.6.2 Modified Duration

(Passage of Time)
Convert $\text{ModD}_{\text{new}} = \text{MacD}_{\text{new}} \cdot v = [\text{MacD}_{old} - (t_1 - t_0)] \cdot v = \text{ModD}_{\text{old}} - v(t_1 - t_0)$

**Formula 38**

$$\text{ModD}_{\text{new}} = \text{ModD}_{\text{old}} - v(t_1 - t_0)$$

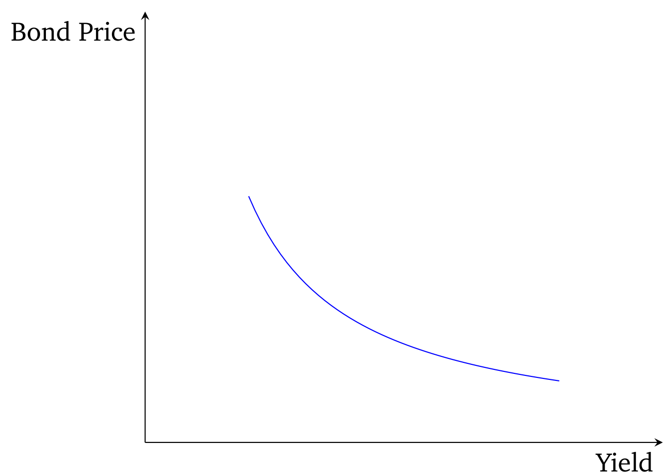
## 7.7 Convexity

- **Duration** gives a linear approximation of how bond price changes with interest rates (yield).
- **Convexity** gives a curvature of how bond price (non-linear) changes with interest rates (yield).

In other words, convexity measures the rate of change of the volatility with respect to interest changes. High convexity bonds: Lose less when yields go up, and gain more when yields go down.

Factors that increase convexity:

- Maturity
- Coupon rate
- YTM
- Cash flow spread




### 7.7.1 Macaulay Convexity

**Macaulay convexity** is the weighted average of the squares of the time $t^2$, using present values as weights.

$$\text{MacC} = \frac{P''(\delta)}{P(\delta)} = \sum_{t\geq 0}^{\infty}\left(\frac{C_t(1+i)^{-t}}{P(i)}\right)t^2 = \frac{\sum_{t=0}^{n} t^2 \cdot v^t \cdot \text{CF}_t}{\sum_{t=0}^{n} v^t \cdot \text{CF}_t}$$

### 7.7.2 Modified Convexity

**Formula 39** (Modified Convexity)

$$\text{Convexity} = \frac{P''(i)}{P(i)} = \frac{\sum_{t=0}^{n} t \cdot (t+1) \cdot v^{t+2} \cdot R_t}{\sum_{t=0}^{n} v^t \cdot R_t}$$

where

- $P(i)$ is the PV of net cash flows at interest $i$
- $P''(i)$ tells how fast duration itself changes (i.e., rate of curvature)

## 7.8 Approximation of Bond Price

We want to approximate how the bond price changes when the interest rate changes slightly from $i$ to $\Delta i$.

$$P(i+\Delta i) \approx P(i) + \Delta i \cdot P'(i)$$

To get percentage change, divide both sides by P(i):

$$\frac{P(i+\Delta i)}{P(i)} \approx 1 + \Delta i \cdot \frac{P'(i)}{P(i)} = 1 - \Delta i \cdot \text{ModD}$$

The approximation is:

$$P(i+\Delta i) \approx P(i) \cdot [1 - \text{ModD} \cdot \Delta i]$$

**Formula 40** (1st-order Modified Approximation)

$$P(i_n) \approx P(i_0) \cdot [1 - (i_n - i_0)(\text{ModD})]$$

$$\Delta P = -\Delta i \cdot \text{ModD}$$

Now, we want the approximation in terms of MacD. Start with bond price as the sum of discounted cash flows:

$$P(i) = \sum_{t=0}^{n} \frac{C_t}{(1+i)^t}$$

We approximate the bond price as a single lump-sum (K) as total amount of cash flows at average time which is MacD:

$$P(i) \approx \frac{K}{(1+i)^{\text{MacD}}}$$

When $i$ changes, $P(i)$ also changes. We take the ratio of new price to old price:

$$\frac{P(i_new)}{P(i_old)} \approx \frac{(1+i_{old})^{\text{MacD}}}{(1+i_{new})^{\text{MacD}}} = (\frac{1+i_{old}}{1+i_{new}})^{\text{MacD}}$$

**Formula 41** (1st-order Macaulay Approximation)

$$P(i_{\text{new}}) \approx P(i_{\text{old}}) \cdot (\frac{1+i_{old}}{1+i_{new}})^{\text{MacD}}$$

## 7.9 Bond Duration

**Bond duration** is a measure of the bond's Sensitivity to interest changes.

**Formula 42** For a bond of n annual coupons, face amount F, coupon rate r, and annual yield rate $i$:

- Annuity-immediate: F · r
- Redemption value at maturity date t = n: F = C

$$\text{MacD} = \frac{\sum_{t=1}^{n} t \cdot PV(\text{CF}_t)}{P}$$

where

- $PV(CF_t)$ is the PV of the cash flow at time $t$ (coupon or principal)
- $P$ is the total PV (price) of the bond

## 7.10 Note

1. Bond
   - Buying a bond = outflow.
   - Coupon payments = inflow.
   - Maturity value = inflow.
2. Annuity
   - Saving: regular payments = outflow.
3. Loan
   - Taking the loan = inflow.
   - Loan repayments = outflow.

# Chapter 8

# Immunization

## 8.1 Assets and liabilities

Cash flows:

- Asset inflows: $A_0, A_1, ..., A_n$
- liability outflows: $L_0, L_1, ..., L_n$
- At each time $t$, $R_t = A_t - L_t$ = Net cash flow at time $t$

## 8.2 Redington Immunization

Redington immunization is a strategy to **protect a portfolio** (assets vs liabilities) from small changes in interest rates.

Let $P(i)$ be the present value of all the net cash flows at interest rate $i$. Since we want the value of portfolio to **not drop** when $i$ changes slightly, that means $P(i)$ should be at a minimum at the current target rate $i_0$.

Table 8.1: 3 Conditions for Immunization

| **Condition** | **Meaning** | **Purpose** |
|---|---|---|
| $P(i) = 0$ | PV of assets = PV of liabilities | Start balanced |
| $P'(i) = 0$ | Modified durations match | No change for small $\Delta i$ |
| $P''(i) > 0$ | Positive convexity | Changes in rate increase value |

## 8.3 Full Immunization

(Full immunization) Full immunization protects a portfolio from **any** interest rate changes, not just small changes.

**Formula 43** Full Immunization Conditions at $i = i_0$:

1. $PV_A(i_0) = PV_L(i_0)$ or PV of assets equals to PV of liabilities
2. $PV'_A(i_0) = PV'_L(i_0)$ or $\text{ModD}_A(i_0) = \text{ModD}_L(i_0)$
3. $PV''_A(i_0)\ PV''_L(i_0)$ or $\text{ModC}_A(i_0) = \text{ModC}_L(i_0)$

The 3rd condition is the **timing condition**, which means that there has to be asset cash flow **before and after** each liability cash flow. It helps to reduce interest risk: no matter how interest rates move, your total asset value will always be enough to cover the liability.

## 8.4 Asset-Liability Exact Matching

**Absolute Matching**: is a **strategy** to ensure that every payment required (liability) is backed by a cash flow from the asset at the **same time and amount**.

- **Asstes**: Incoming payments $A_t$
- **Liabilities**: outgoing payments $L_t$

**Formula 44 Absolute Matching:** Ensure that $A_1 = L_1, A_2 = L_2, \ldots$ at same TIME and AMOUNT.

Challenges of **Dedication**:

1. Uncertain cash flows
2. Long-term liabilities
3. Lower returns: Dedication is very strict - you can't invest in potentially higher-return but flexible assets.

# Chapter 9

# Summary

## 9.1 Loan, Bond, and Annuity

| Field | Loan | Bond | Annuity |
|---|---|---|---|
| **N** | Loan term | Time to maturity | Duration of annuity |
| **I/Y** | Loan interest rate | Bond yield (market rate) | Interest rate per period |
| **PV** | Loan amount (positive) | Bond price (negative if investing) | PV of payments (negative if paying) |
| **PMT** | Monthly payment (negative) | Coupon payment (positive cash inflow) | Payment (depends on in/outflow) |
| **FV** | 0 (loan fully repaid) | Redemption value (money recedived at maturity date) | Usually 0 unless final lump sum |