ADVANCED ECONOMIC THEORY - GAME THEORY

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Contents

1 Strategic Games with Ordinal preferences

1.1 Prisoner's Dilemma

Game model 1.1.1 (Prisoner's Dilemma)**.**

Players: 2 suspects
Actions: $A = \{\text{Quite, Fink}\}$
Payoffs: Payoff matrix

		Suspect 2	
		Quite	Fink
Suspect 1	Quite	$2, 2$	$0, 3$
	Fink	$3, 0$	$1, 1$

Nash equilibrium: (stable) (Fink, Fink)
Preferences:

- For player 1: $u_1(\text{Fink}, \text{Quiet}) > u_1(\text{Quiet}, \text{Quiet}) > u_1(\text{Fink}, \text{Fink}) > u_1(\text{Quiet}, \text{Fink})$

Observation 1.1.2. Cooperation (Quiet, Quiet) gives a better outcome than mutual betrayal (Fink, Fink), but each suspect has an incentive to defect. It is the essence of the dilemma.

1.1.1 Working on a Joint Project

Example 1.1.3. You and your friend are working on a joint project. If your friend works hard, then you prefer to goof off (you can work hard but the increment in outcome is now worth the extra effort). You prefer the outcome that both working hard to the outcome you both goofing off (in case nothing gets accomplished), and the worst outcome is that you work hard and your friend goofs off.

Players: 2 collaborators
Actions: $A = \{\text{Work hard, Goof off}\}$
Payoffs: Payoff matrix

		Fiend 2	
		Work hard	Goof off
Friend 1	Work hard	2, 2	0, 3
	Goof off	3, 0	1, 1

Preferences:

- Player 1: $u_1(G, K) > u_1(K, K) > u_1(G, G) > u_1(K, G)$
- Player 2: $u_2(K, G) > u_2(K, K) > u_2(G, G) > u_2(G, K)$

Note. We are **not** claiming that this situation has the structure of Prisoner's Dilemma.

Exercise (16.1)**.** The situation is the same as Working on a joint project except that both friends prefer **mutual cooperation strongly.** Their preferences change, thus the payoff matric for this strategic game is now:

		Fiend 2	
		Work hard	Goof off
Friend 1	Work hard	3, 3	0, 3
	Goof off	3, 0	1, 1

1.2 Dominated actions

Exercise (47.1)**.**

- Player 1: T is weakly dominated by M (when $a_2 =$ C) and strictly dominated by B.
- Player 2: No action is weakly or dominated strategy.

	L	C	R
T	0, 0	1, 0	1, 1
M	1, 1	1, 1	3, 0
B	1, 1	2, 1	2, 2

1.2.1 Classic illustration: Voting

Intuition. The voting example illustrates how **weakly** (NOT dominated) actions can appear in NE.

- When switching to a weakly dominated strategy, the outcome does not change (thus same payoff).
- A weakly dominated action can still be a best response to the others' actions.
- Core idea: the **only** weakly dominated action is voting for your least favorite.

Exercise (49.1)**.** Setup:

(1) Three candidates: A, B, C.
(2) Every citizen has a strict ranking of the three candidates.
(3) Preferences: The candidate with the most votes wins. Ties for first place are possible. If tie between x and y, each citizen ranks this outcome between a win for x and a win for y.
(4) Question: Show that a citizen's only weakly dominated action is a vote for her least preferred candidate (C).

If a citizen i has preferences $A \succ B \succ C$. Claim: Voting for C is the only weakly dominated action.

- If he votes for C (the least favorite), then the outcome is either worse (when A or B wins) or the same (C wins). So C is weakly dominated by A and B.

- Compare A and B: one counter-example that voting for B is strictly better than voting for A. If there is a tie between B and C, if he votes for A, the outcome is tie(B,C), and if she votes for B, the outcome is B winning. Since tie(B,C) is strictly dominated by B, voting for A is strictly dominated by voting for B in this scenario. By definition, a choice is only weakly dominated by the other one where it is either worse or equal to that choice. In this scenario, voting for B is strictly better than voting for A, hence B is not weakly dominated by A.

Exercise (49.2)**.** Setup:

(1) Approval voting system: Each voter can vote for **any number of candidates** (including none, one, or multiple candidates).
(2) Consider a case of 2 candidates A and B. A voter can vote for:
    - Neither A nor B ($\emptyset$)
    - Only A (A)
    - Only B (B)
    - Both A and B ({A, B})

    Show that any action that includes voting for the voter's least preferred candidate is weakly dominated. Show that any action that does *not* include a vote for the voter's most preferred candidate is weakly dominated.

Solution:

- Define a_i as action that player i votes for B. Define a_i' as action that player i does not vote for B.
- a_i is weakly dominated by a_i' iff:

$$\forall a_{-i}, \quad (a_i', a_{-i}) \succeq_i (a_i, a_{-i}) \quad \text{and} \quad \exists a_{-i} \quad \text{such that} \quad (a_i', a_{-i}) \succ_i (a_i, a_{-i})$$

- Two scenarios
  (1) $(a_i', a_{-i}) =_i (a_i, a_{-i})$: Set of winning candidates are the same for both (a_i', a_{-i}) and (a_i, a_{-i}).
  (2) Candidate k wins in (a_i, a_{-i}) and does not win in (a_i', a_{-i}). Then

$$(a_i', a_{-i}) \succ (a_i, a_{-i})$$

  because the outcome when k does not win strictly dominates the one when k wins.

There is no scenario that k wins in (a_i', a_{-i}) and does not win in (a_i, a_{-i}) because the approval is monotonically increasing; it is impossible that one extra vote from player i makes k not win but when player i gets back the vote, k wins (as the more number of votes, the highe chance that k wins).

2 Bargaining game

Definition 2.0.1. Two or more players must agree on how to divide something valuable.

2.1 Rationalizable actions in a Barganing game

Example 2.1.1. Two players split \$4 using the following procedure:

- Each announces an integral number of dollars.
- If the sum of the amounts named is at most \$4, then each player receives the amount she names.
- If the sum exceeds \$4 and both players name the same amount, then each receives \$2.
- If the sum exceeds \$4 and the players name different amounts:
  - The player who names the smaller amount receives that amount plus a small amount proportional to the difference between the amounts.
  - The other player receives the remainder of the \$4.

That is, there is a small penalty for making a demand that is "excessive" relative to that of the other player.

(1) Actions that are never best responses:
    - Worst: Demand = \$0 -¿ total amount always ≤ 4 -¿ no bonus sum

- Not optimal: Demand = \$1 $\longrightarrow$Opponent demands \$0 or \$1 $\longrightarrow$You get \$1
- Not optimal: Demand = \$1 $\longrightarrow$Opponent demands more than \$1, then the small gain is not worth

(2) Actions that might be optimal
- Demand = \$2
- Demand = \$3
- Demand = \$4

In summary, the payoff of each player i is given by

$$u_i(a_i, a_j) = \begin{cases} a_i & \text{if } a_1 + a_2 \leq 4 \\ 2 & \text{if } a_1 + a_2 > 4 \text{ and } a_i = a_j \\ 4 - a_j - (a_i - a_j)\epsilon & \text{if } a_1 + a_2 > 4 \text{ and } a_i > a_j \\ a_i + (a_j - a_i)\epsilon & \text{if } a_1 + a_2 > 4 \text{ and } a_i < a_j \end{cases}$$

where $\varepsilon > 0$ is a small amount (less than 30 cents).

Level of Assumption	**Rational Demands**
Just you are rational	\$2, \$3, or \$4
You + Opponent rational	\$2 or \$3
Common knowledge of rationality	✓ Only \$2

2.2 Finding rationalizable actions

Exercise (358.1)**.** Find the set of rationalizable actions of each player in the game in Figure 358.2.

	L	C	R
T	2, 1	1, 4	0, 3
B	1, 8	0, 2	1, 3

Solution 2.2.1. Action R is suspicious since no matter player 1 choose, it always give player 2 payoff of 3. Test R first.

(1) Test R if R is strictly dominated by constructing a mixed strategy for player 2.
(2) Eliminate R. Eliminate B. Eliminate L.

Exercise (391.2)**.** Setup

- Strategy sets: Each player chooses an integer from 0 to 10.
- Payoffs are determined based on the sum of the two players' demands:
  - If the sum is less than or equal to 10, both players receive exactly what they demanded.
  - If the sum exceeds 10 and the demands are different, then:
    (1) The player who demanded less receives their full amount.
    (2) The player who demanded more receives the remainder (i.e., 10− the lower demand).
  - If the sum exceeds 10 and the demands are equal, then both players receive 5.

3 Quiz

3.1 Hotelling model of politicians

You and your opponent are politicians competing in an election. Each of you must choose a number between 0 and 10 — this represents your political platform (policy position).

- 0 = most right-wing
- 10 = most left-wing

Your payoff = number of votes you receive. There are total 21 voters. Distribution of voters by favorite policy is,

Favorite policy	0	1	2	3	4	5	6	7	8	9	10
Number of voters	1	2	3	4	3	3	1	1	1	1	1

- If you choose 3 and your opponent chooses 6: You get voters at positions 0, 1, 2, 3, 4. Your total votes: $1+2+3+4+3=13$. Opponent's votes = 8 (from the remaining voters).
- If both choose 3: You tie exactly — each gets $\frac{21}{2}=10.5$ votes

4 Problem Set 2

4.1 Question 5

Exercise. There are more than 10 players. Each player chooses a number between 0 and 100. After everyone picks, the average of all chosen numbers is calculated. The target number is 2/3 of this average.

- Whoever's chosen number is closest to that target wins a payoff of 1. If several players are equally close, they all win and each gets 1.
- All others get 0.

Find all strategies that remain after iterated elimination of strictly and weakly dominated strategies.

Solution 4.1.1. There are N players. Each player chooses an integer

$$s_i \in \{0,1,2,\ldots,m\}.$$

- Let target $T=\frac{2}{3}\bar{s}=\frac{2}{3}\left(\frac{s_1+s_2+\cdots+s_N}{N}\right)$.
- Let maximum possible guess

$$m^*(m) = \text{the natural number closest to } \frac{2}{3}m.$$

(1) Pick $s_i > m$: Show that $u_i(m^*(m), s_{-i}) \geq u_i(s, s_{-i}) \quad$ for all s_{-i}
  - For any $\bar{s}$ (combination of others'plays), $\bar{s} \leq m \implies T=\frac{2}{3}\bar{s} \leq \frac{2}{3}m$
  - Distance $|s_i - T| \geq |m^*(m) - T|$
  - Hence $u_i(m^*(m), s_{-i}) \geq u_i(s_i, s_{-i}) \quad \forall s_{-i}$ or all $s_i > m$ is weakly dominated by $m^*(m)$.

(2) Pick $s_i \leq m$: Check if s_i is weakly dominated.
  - For any $s_i < m^*(m)$ we can construct a situation that s_i is closer to target T than any s_{-i}.
  - Show there exists at least one action $\forall s_i' \neq s$, $\exists s_{-i} : \left|s-\frac{2}{3}\bar{s}\right| < \left|s_i'-\frac{2}{3}\bar{s}\right|$. When others choose larger numbers, making $T=\frac{2}{3}\bar{s}$ is high but still $T \leq m^*(m)$, we can pick s_i as high as $m^*(m)$, so that $|s_i - T| < |s_{-i} - T|$.
  - Hence for any $s_i < m^*(m)$ is not weakly dominated.

(3) Strict domination: To be strictly dominated, a strategy must be worse for every possible average ($\forall s_{-i}$. For any numbers s_{-i}, you can pick a s_i which is close to target T. Thus, there is not strict domination.

Solution 4.1.2.
- Action set: each player chooses $s \in \{0,1,\ldots,100\}$.
- Define $m^*(m)$ as the integer closest to $\frac{2}{3}m$ (if two, take the larger one).
- For any $m>1$, every $s > m^*(m)$ is **weakly dominated** by $m^*(m)$, since

$$u_i(m^*(m), s_{-i}) \geq u_i(s, s_{-i}) \quad \forall s_{-i},$$

  and the inequality is strict when all others play s.
- There are **no strictly dominated** strategies, since for any smaller $s' < s \leq m^*(m)$, we can find s_{-i} such that $u_i(s, s_{-i}) > u_i(s', s_{-i})$.
- Apply iterated elimination of weakly dominated strategies:

$$m_{t+1} = m^*(m_t), \quad m_0 = 100.$$

- Since $m^*(m) < m$ for all $m > 1$, the sequence $\{m_t\}$ strictly decreases until $m_t \leq 1$.
- When $m_t = 1$, $m^*(1) = 1$ (fixed point), so elimination stops.
- **Surviving strategies:** $\{0, 1\}$.

5 Problem Set 3

5.1 37.1 - Finding Nash equilibria using best response functions

Exercise (37.1)**.** Part a. Find Nash equilibria by identifying the best response functions for players in different games.

(1) Prisoner's Dilemma (Figure 15.1)
(2) Battle of Sexes (BoS) (Figure 19.1)

	Bach	Stravinsky
Bach	$\underline{2}, \underline{1}$	$0, 0$
Stravinsky	$0, 0$	$\underline{1}, \underline{2}$

19.1 - Battle of Sexes

(3) Matching Pennies (Figure 19.2)

	Head	Tail
Head	$\underline{1}, -1$	$-1, \underline{1}$
Tail	$-1, \underline{1}$	$\underline{1}, -1$

19.2 - Matching Pennies

(4) Two-Player Stag Hunt (Figure 21.1)

	Stag	Hare
Stag	$\underline{2}, \underline{2}$	$0, 1$
Hare	$1, 0$	$\underline{1}, \underline{1}$

21.1 - Stag Hunt

Part b. Fine Nash Equilibria for the Game in Figure 38.1:

	L	C	R
T	$\underline{2}, \underline{2}$	$\underline{1}, \underline{3}$	$0, 1$
M	$\underline{3}, \underline{1}$	$0, 0$	$\underline{0}, \underline{0}$
B	$1, \underline{0}$	$0, \underline{0}$	$0, \underline{0}$

38.1

5.2 38.1 - Draw a Diagram for NE

5.3 38.2 - Dividing money

Exercise (38.1)**.** (Constructing best response functions) Draw a diagram for best responses for 37.1 part b.

Solution 5.3.1. Fix player 2's choice and find player 1's sets of best responses.

(1) Player 2 chooses \$0, then $B_1(0) = \{10\}$.
(2) Player 2 chooses \$1, then $B_1(1) = \{9, 10\}$.
(3) Player 2 chooses \$2, then $B_1(2) = \{8, 9, 10\}$.
(4) Player 2 chooses \$3, then $B_1(3) = \{7, 8, 9, 10\}$.
(5) Player 2 chooses \$4, then $B_1(4) = \{6, 7, 8, 9, 10\}$.
(6) Player 2 chooses \$5, then $B_1(5) = \{5, 6, 7, 8, 9, 10\}$.
(7) Player 2 chooses \$6, then $B_1(6) = \{5, 6\}$.
(8) Player 2 chooses \$7, then $B_1(7) = \{6\}$.
(9) Player 2 chooses \$8, then $B_1(8) = \{7\}$.
(10) Player 2 chooses \$9, then $B_1(9) = \{8\}$.
(11) Player 2 chooses \$10, then $B_1(10) = \{9\}$.

There are 4 nash equilibria (5, 5), (5, 6), (6, 5), and (6, 6).

5.4 27.1 - Variant of Prisoner's Dilemma with altruistic preferences

Exercise (27.1)**.** A modified version of the Prisoner's Dilemma, where each player's preferences include not only their own payoff but also a weighted value of the other player's payoff, introducing altruism into the model. The degree of altruism is controlled by the parameter α.

(a) Formulate a strategic game that models this situation in the case $\alpha = 1$. Is this game the Prisoner's Dilemma?
(b) Find the range of values of α for which the resulting game is the Prisoner's Dilemma. For values of α for which the game is not the Prisoner's Dilemma, find the Nash equilibria.

Player 1 \ Player 2	Quiet	Fink
Quiet	$\underline{2}, \underline{2}$	$0, \underline{3}$
Fink	$\underline{3}, 0$	$\underline{1}, \underline{1}$

Solution 5.4.1.

$$u_i(a) = m_i(a) + \alpha\, m_j(a)$$

where $\alpha \geq 0$ measures how **altruistic** a player is.

- $\alpha = 0$: purely selfish
- $\alpha = 1$: completely altruistic — I value your payoff just as much as mine

When $\alpha = 1$,

	Quiet	Fink
Quiet	$\underline{4}, \underline{4}$	$\underline{3}, 3$
Fink	$3, \underline{3}$	$2, 2$

Part a.

(1) Preference order of Prisoners' Dilemma:

$$(F_i, Q_{-i}) \succ_i (Q_i, Q_{-i}) \succ_i (F_i, F_{-i}) \succ_i (Q_i, F_{-i})$$

(2) Preference order of modified version with altruism:

$$(Q_i, Q_{-i}) \succ_i (F_i, Q_{-i}) \sim_i (Q_i, F_{-i}) \succ_i (F_i, F_{-i})$$

Hence this is not a prisoner's dilemma.

Part b.

To remain a true Prisoner's Dilemma (PD), player i's ranking must satisfy:

$$u_i(F_i, Q_{-i}) > u_i(Q_i, Q_{-i}) > u_i(F_i, F_{-i}) > u_i(Q_i, F_{-i})$$

Action pair	Payoff to i
(F, Q)	$3 + 0\alpha = 3$
(Q, Q)	$2 + 2\alpha$
(F, F)	$1 + \alpha$
(Q, F)	$0 + 3\alpha = 3\alpha$

Take the inequalities, then the PD structure holds only when:

$$0 \leq \alpha < \frac{1}{2}$$

Range of α	Type of Game	Nash Equilibrium
$0 \leq \alpha < \frac{1}{2}$	Prisoner's Dilemma	Both Fink (mutual defection)
$\alpha = \frac{1}{2}$	Neutral / indifferent	Every pair of actions is a NE
$\alpha > \frac{1}{2}$	Cooperative (Altruistic)	Both Quiet (mutual cooperation)

- When α is small, players mostly care about themselves, then **defection dominates**.
- As α increases, players begin to value the other's payoff, then less incentive to betray.
- Once $\alpha > 0.5$, caring about the other outweighs the temptation to defect, then **cooperation becomes dominant**.

5.5 Problem Set 4

5.6 63.1 - Interaction among resource users

Exercise (63.1)**.** There are n firms, each using a shared resource (like water or forest). Each firm's action is how much it uses, denoted $x_i \geq 0$.

The total usage is:

$$X = x_1 + x_2 + \cdots + x_n.$$

Each firm's payoff is:

$$u_i(x_1, \ldots, x_n) = \begin{cases} x_i(1 - X), & \text{if } X \leq 1, \\ 0, & \text{if } X > 1. \end{cases}$$

- x_1: how much resources firm i uses
- (1-X): the quality or productivity left from the resource
- The more total use X, the lower the productivity.

Solution 5.6.1. Formalize the question in standard strategic game:

(1) Players $N = \{frm[o]--, 2, \ldots, n\}$
(2) Action sets:
   - For each firm $i \in N$, the set of possible actions is:
   $$A_i = [0, \infty) \quad \text{(nonnegative real numbers representing resource usage)}.$$
   - An action profile is
   $$x = (x_1, x_2, \ldots, x_n) \in A_1 \times A_2 \times \cdots \times A_n = [0, \infty)^n.$$
(3) Payoff function:
$$u_i(x_1, \ldots, x_n) = \begin{cases} x_i(1 - X), & \text{if } X \leq 1, \\ 0, & \text{if } X > 1. \end{cases}$$

Solve for NE:

(1) At NE Symmetric equilibrium: If each firm uses
$$(x_1, \ldots, x_n) = \left(\frac{1}{n+1}, \ldots, \frac{1}{n+1}\right)$$

then,

$$u_i^* = x_i(1 - X^*) = \frac{1}{n+1}\left(1 - \frac{n}{n+1}\right) = \frac{1}{(n+1)^2}$$

(2) At NE Overuse equilibrium: If the total output $X > 1$, then each firm's output is 0.

$$(x_1, \dots, x_n) = (0, \dots, 0) \quad \forall i, \sum_{j \neq 1} x_j \geq 1$$

Intuition:

- No firm has incentive to deviate, because of x_i increases that the total output $X > 1$, then best response for each firm is to produce $x_i = 0$.
- A candidate profile where everyone is better off is when each firm uses less the resources, e.g. $x_i = \frac{1}{2n}$. Then at equilibrium, $u_i = \frac{1}{4n}$ means that each firm's output u_i^* is higher than it is at NE.

6 Extensive game

6.1 Extensive game with perfect information

Definition 6.1.1 (Extensive game with perfect information)**.** An *extensive game with perfect information* consists of:

- a set of players;
- a set of sequences (terminal histories) such that no sequence is a proper subhistory of any other sequence;
- a *player function* that assigns a player to every sequence that is a proper subhistory of some terminal history;
- for each player, preferences over the set of terminal histories.

6.1.1 Subgame of an Extensive Game with perfect information

Definition 6.1.2. Let Γ be an extensive game with perfect information, with player function P. For any nonterminal history h of Γ, the *subgame* $\Gamma(h)$ following the history h is the extensive game defined as follows.

- **Players:** the players in Γ.
- **Terminal histories:** the set of all sequences h' of actions such that (h, h') is a terminal history of Γ.
- **Player function:** the player at history h' in the subgame is

$$P(h, h') \quad \text{for each proper subhistory } h' \text{ of a terminal history.}$$

- **Preferences:** each player prefers h' to h'' in the subgame if and only if she prefers (h, h') to (h, h'') in Γ.

6.1.2 Strategic form of an Extensive Game

Definition 6.1.3 (Strategic form of an Extensive game with perfect information)**.** Given an extensive game with perfect information, its *strategic form* is the strategic game defined as follows:

- **Players:** the same players as in the extensive game.
- **Actions:** each player's set of actions is her set of strategies in the extensive game.
- **Payoffs:** for any strategy profile s, each player's payoff is the payoff of the terminal history generated by s in the extensive game.

In this strategic form,

$$\boxed{\text{Nash equilibria of the extensive game} = \text{Nash equilibria of its strategic form.}}$$

6.2 Strategy

Definition 6.2.1 (Strategy)**.** A strategy of player i in an extensive game with perfect information is a function

$$s_i : \{\, h \in H : P(h) = i \,\} \longrightarrow A(h),$$

that assigns to each history h with $P(h) = i$ an action $s_i(h) \in A(h)$.

Note. Strategy vs Action:

(1) Strategic-form (matrix): A single action (because moves are simultaneous)
(2) Extensive-form (tree): A complete plan for every decision point

Note.
- Normal form = actions.
- Strategic form = strategies.

6.3 Outcome

Definition 6.3.1 (Outcome of a Strategy profile)**.** The terminal history generated when we start at the root and let each player take the action prescribed by her strategy whenever it is her turn. It is the actual path of play.

Let $s = (s_1, \ldots, s_n)$ be a strategy profile in an extensive game with perfect information. Define the outcome $O(s)$ recursively as follows.

(1) Start at the empty history: $h_0 = \emptyset$.
(2) At step k, let player $i = P(h_{k-1})$ move.
(3) That player's strategy specifies the action

$$a_k = s_i(h_{k-1}).$$

(4) Update the history:

$$h_k = (h_{k-1}, a_k).$$

Stop when h_K is a terminal history. The *outcome* of the strategy profile s is then

$$O(s) = h_K.$$

6.4 Subgame Perfect Equilibrium

Definition 6.4.1 (Subgame Perfect Equilibrium)**.** A strategy profile s^* in an extensive game with perfect information is a subgame perfect equilibrium if starting from every history h in the game, no player can gain from deviating.

Formally, for every player i, every history h with $P(h) = i$, and every alternative strategy r_i of player i,

$$u_i\big(O_h(s^*)\big) \;\geq\; u_i\big(O_h(r_i, s^*_{-i})\big),$$

Note. There can be multiple SP Equilibria s^*, but each of them must give the same payoff $O_h(s^*) = aconstant$.

6.4.1 NE of Subgame

Proposition 6.4.2 (SPE and NE)**.** Let s^* be a strategy profile in an extensive game with perfect information.

(i) Since $O_\emptyset(s) = O(s)$, the definition of subgame perfection implies

$$s^* \text{ is SPE} \implies s^* \text{ is NE}.$$

(ii) If s^* is an SPE, then for every history h and every player i,

$$u_i\big(O_h(s^*)\big) \;\geq\; u_i\big(O_h(r_i, s^*_{-i})\big) \qquad \text{for all } r_i,$$

so the continuation strategies induced by s^* form a Nash equilibrium in every subgame.

(iii) Conversely, if a strategy profile induces a Nash equilibrium in every subgame, then

$$\forall i, \forall h: \quad u_i\big(O_h(s^*)\big) \;\geq\; u_i\big(O_h(r_i, s^*_{-i})\big) \quad \forall r_i,$$

hence the strategy profile is a subgame perfect equilibrium.

Proposition 6.4.3 (SPE and Backward Induction)**.** For any finite-horizon extensive game with perfect information, the set of subgame perfect equilibria is exactly the set of strategy profiles obtained by backward induction. Formally,

$$\{\text{SPE}\} \;=\; \{\text{strategy profiles generated by backward induction}\}.$$

Proposition 6.4.4 (Existence of SPE)**.** Every finite extensive game with perfect information has at least one subgame perfect equilibrium. Formally,

$$\Gamma \text{ finite} \implies \exists\, s^* \in \text{SPE}(\Gamma).$$

6.5 Other comments

Note (Backward induction)**.** A tiny change at the end of the game can dramatically change the beginning, because backward induction propagates that change all the way back.

6.6 Ultimatum game

Definition 6.6.1 (Ultimatum game)**.** Game setup:

(1) Players: Two players divide a pie of size c.
(2) Strategies:
- Player 1: choose an offer x.
- Player 2: choose Y or N for *every* possible x.
- Payoffs:
  - If Yes, $(c - x,\ x)$
  - If No, $(0,\ 0)$

6.6.1 1-stage

Backward Induction

- If $x > 0$: Player 2's best response is Y.
- If $x = 0$: both Y and N are optimal; one optimal strategy is "always accept."

Given "always accept," Player 1 maximizes $c - x$ by choosing $x = 0$.

SPE

$$s_1^* = 0, \qquad s_2^*(x) = Y \quad \text{for all } x.$$

Outcome

Player 1 keeps the entire pie; Player 2 accepts.

Note. P2 has no future bargaining power, so P1 extracts the entire pie.

6.6.2 2-stage (with Discounting)

- players take turns making offers
- each round of delay makes both players worse off (because of discounting)

Let each player i discount the future by δ_i, with

$$0 < \delta_i < 1.$$

- More patient = larger δ_i.
- More impatient = smaller δ_i.

Full SPE includes

(1) Period 1: P2 offers $(\delta_1,\ 1 - \delta_1)$; P1 accepts
(2) Period 2 subgame:
- P1 would offer $(1, 0)$
- P2 would accept

SPE Outcome:

- P2 offers $(\delta_1,\ 1 - \delta_1)$
- P1 accepts
- Game ends

(all these strategies must form NEs in every subgame)

6.6.3 3-stage

Period	Prop.	If reject →	Cont. value	Offer must give	Keeps
3 (last)	P1	Game ends; P1 gets all	P2: 0	$(1, 0)$	1
2	P2	P1 rejects → go to 3; P1 gets 1 discounted by δ_1	P1: δ_1	$(\delta_1,\ 1 - \delta_1)$	$1 - \delta_1$
1	P1	P2 rejects → go to 2; P2 gets $(1 - \delta_1)$ discounted by δ_2	P2: $\delta_2(1 - \delta_1)$	$\big(1 - \delta_2(1 - \delta_1),\ \delta_2(1 - \delta_1)\big)$	$1 - \delta_2 + \delta_1\delta_2$

7 Repeated Game

7.1 Game is repeated

- The *same players* interact over and over.
- Each player *observes all past actions.*
- They can *reward or punish* based on history (reputation effects).
- Future payoffs matter, but are *discounted* by the factor $\delta \in (0, 1)$.

7.2 Payoff

For a path of actions

$$a_1,\, a_2,\, a_3,\, \ldots,$$

(1) Different actions: Discounted payoff to player i is

$$u_i(a_1) \;+\; \delta\, u_i(a_2) \;+\; \delta^2 u_i(a_3) \;+\; \cdots .$$

(2) Same actions: Discounted payoff to player i is:

$$u_i(a) + \delta\, u_i(a) + \delta^2 u_i(a) + \cdots = \frac{u_i(a)}{1-\delta}.$$

7.3 Strategies

7.3.1 Cooperate every period

history = {(C,C), (C,C), (C,C),...}

7.3.2 Defect every period

history = {(D,D), (D,D), (D,D),...}

7.3.3 Unrelenting punishment (Grim Trigger)

- Start with C
- One player deviates to D
- The opponent punishes forever. In period 1, choose C. In every future period $t \geq 2$, choose

$$a_t = \begin{cases} C, & \text{if the other player chose } C \text{ in every period } 1, \ldots, t-1, \\ D, & \text{if the other player chose } D \text{ in any period } 1, \ldots, t-1. \end{cases}$$

- Strategy pair: (Grim, Grim)

history = {(C,C), (D,C), (D,D), (D,D),...}

7.3.4 k-period Punishment

- If deviation occurs: punish with D for k periods, then return to C.
- Strategy pair: (k-punishment, k-punishment)

7.3.5 Tit-for-Tat

- Strategy pair: (TFT, TFT)
- Both players start with C, then each plays whatever the opponent played last period.

7.4 Nash Equilibrium

(1) (All-D, All-D) is a NE
- Outcome path: (D,D), (D,D), (D,D),...
- Payoff: each gets: $\frac{1}{1-\delta}$

(2) Grin trigger: (UP, UP) is a NE
- If no one deviates: (C, C), (C, C), (C, C), ...
- If someone deviates once: (C, C), then (D, D) forever
- Condition for no profitable deviation: $\delta \geq \frac{1}{2}$

(3) k=punishment: (k-punishment, k-punishment) is NE
- If no one deviates: (C, C), (C, C), (C, C), ...
- One shot deviation: (D,C), then k periods of (D,D), then back to (C, C)

8 Extensive Games with Perfect information - Extended Model

Given actions A, B, C, D,...

- History is a sequence (ordered list of action profiles) $h = (a^1, a^2, ...)$, each a^t is a single action/ an action profile taken at time t.
  (1) Sequential move game: Each a^t is a single action. eg, $a^t = A$
  (2) Simultaneous move game: Each a^t is an action profile of actions chosen together by players who move simultaneously eg, $a^t = (A, B)$
- A Strategy is a function, but we can represent a strategy as a list of actions

8.1 With Simultaneous moves

8.1.1 Definition

Definition 8.1.1. An extensive game with simultaneous moves consists of:

(1) a set of Players N = {1, 2}
(2) a set of actions available to each player at each history $A_i(h)$
(3) a set of Terminal histories, h is a Terminal history iff either:
- $h = (a^1, a^2, ..., a^k)$ is finite and $P(h)$ is undefined
- $h = (a^1, a^2, ...)$ is infinite and $a_t \in \prod_{i \in P(h_{t-1})} A_i(h_{t-1}), \qquad \text{where } h_{t-1} = (a_1, \ldots, a_{t-1})$.

(4) Player function is a set of players $P(h) = \{1, 2\}$ (1 and 2 move simultaneously)
(5) Preferences over terminal histories

8.1.2 Strategy

8.1.3 Subgame NE

Remark 8.1.2.