MAT337: Introduction to Real Analysis - Prof. Fried Tong

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Contents

Part I
Real Numbers

1 An overview of Real Numbers

2 Real numbers and their arithmetic

Remark. We can represent a real number as:
1. An Infinite decimals expansion
2. An Finite decimals expansion
3. A class of equivalent expansion

2.1 Infinite Decimals

Definition 2.1. An infinite decimal expansion is a sequence (function)

$$x(n) = a_n, \quad a_n \in \{0, 1, \ldots, 9\},\ n \geq 1.$$

It can be written as

$$x = a_0.a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}a_{17}a_{18}\ldots$$

2.2 Finite Decimals

Definition 2.2. From infinite decimals, we can place all **finite decimals** on the number line by dividing each small interval of length starting from n to $n+1$ into tenths, hundredths,...

$$a_0.a_1a_2\ldots a_k \;\leq\; x \;\leq\; a_0.a_1a_2\ldots a_k + 10^{-k}$$

2.3 Equivalence Classes

Definition 2.3. We define an equivalence relation on infinite decimals by

$$a_0a_1a_2\ldots a_{k-1}a_k000\ldots = a_0a_1a_2\ldots a_{k-1}(a_k - 1)999\ldots, \quad a_k \neq 0$$

Each real number is an equivalence class of infinite decimal expansions given by this identification. The set of all real numbers is denoted by $\mathbb{R}$.

Example 1. Number $1 = \{1.000\ldots, 0.999\ldots\}$

2.4 Eventually Periodic Decimals

Definition 2.4. A function F that takes a fraction $\frac{a}{b}$ and turns it into a decimal form.

Theorem 2.4.1.
- Every rational number (a fraction $\frac{a}{b}$) gives an eventually periodic decimal.

$$\forall i > n, \exists n, k \in \mathbb{Z}^+, a_{i+k} = a_i \implies x \in \mathbb{R}, x = a_0.a_1a_2\ldots \text{ is eventuallly periodic.}$$

- Every eventually periodic decimal comes from a rational number.

Therefore,

$$\text{Fraction} \iff \text{Eventually repeating decimal.}$$

2.5 Ordering

Definition 2.5 (Order of Infinite Decimal Expansions). Let

$$x = a_0.a_1a_2a_3\ldots, \quad y = b_0.b_1b_2b_3\ldots$$

be two distinct real numbers. For two real numbers x and y, either $x < y, x = y, x > y$.

$$x \,?\, y = \begin{cases} x = y & \text{if } a_i = b_i \text{ for all } i \geq 0, \\ x < y & \text{if there exists } k \geq 0 \text{ such that } a_i = b_i \text{ for } i < k \text{ and } a_k < b_k, \\ x > y & \text{if there exists } k \geq 0 \text{ such that } a_i = b_i \text{ for } i < k \text{ and } a_k > b_k. \end{cases}$$

2.6 Archimedean Property

For any $x, y > 0$, there exists some $n \in \mathbb{N}$ such that:

$$nx > y$$

Equivalent Form: For any $z > 0$, there exists some integer $k \geq 0$ such that

$$10^{-k} < z$$

Proof. Any real number z has a decimal expansion of the form

$$z = z_0 \cdot z_1z_2z_3\ldots$$

Since $\exists k, z_k$ is a first non-zero digit, all digits before this index are zeroes. □

2.7 Distance on Real line

Definition 2.6. Absolute value:

$$|x| = \max\{x, -x\}$$

Definition 2.7. Distance between two points x and y:

$$d(x,y) = |x - y|$$

Some key properties on distance:

- $d(x,y) \geq 0$ for all $x, y \in \mathbb{R}$.
- $d(x,y) = 0 \iff x = y$.

3 Ordered Field and Bounds

An ordered field F is a set with addition, multiplication, and an order $\leq$ compatible with these operations.

Definition 3.1. A subset $S \subseteq F$ is **bounded above** if

$$\exists M \in F \quad \text{such that} \quad \forall x \in S,\ x \leq M$$

Then M is called an **upper bound** of S.

Definition 3.2. A subset $S \subseteq F$ is bounded below if there exists an element $K \in F$ such that

$$\forall x \in S, \quad x \geq K.$$

Here, K is called a **lower bound** of S.

3.1 Supremum

Definition 3.3. Given S is a non-empty subset of R and bounded above, denote supS = L such that:
- L is an upper bound of $S : \forall s \in S, s \leq L$.
- If M is any other upper bound then $L \leq M$.

3.1.1 Characterization of the Supremum

$$\sup S = L \iff \begin{cases} 1.\ L \text{ is an upper bound of } S, \\ 2.\ \forall K < L,\ \exists x \in S \ \text{ with } K < x < L \end{cases}$$

Definition 3.4. Let $S \subseteq \mathbb{R}$ be bounded below. A number $a \in \mathbb{R}$ is called the **infimum** (greatest lower bound) of S, denoted $a = \inf S$, if and only if:
1. a is a lower bound: $\forall s \in S, a \leq s$.
2. a is the greatest lower bound: $\forall \ell \in \mathbb{R}, (\forall s \in S,\ \ell \leq s) \implies \ell \leq a$

3.2 Infimum

Definition 3.5. Given S is a non-empty subset of R and bounded below, denote infS = L such that:
- L is an lower bound of $S : \forall s \in S, s \geq L$.
- If M is any other upper bound then $L \geq M$.

3.3 Maximum/Minimum

Definition 3.6. A real number a_0 is called the **maximum** of A, if and only if $a_0 \in A$ and $\forall a \in A, a \leq a_0$.

Definition 3.7. A real number a_1 is called the **minimum** of A, if and only if $a_1 \in A$ and $\forall a \in A,\ a_1 \leq a$.

Remark. "The supremum is not necessarily an element of the set. But if the set does have a maximum, then that max element is the supremum."

3.4 Uniquness

- Completeness separates $\mathbb{R}$ from $\mathbb{Q}$.

Remark. Supremum and infimum are unique.

3.5 Special Cases:

- If a set is not bounded above, we sometimes write $\sup S = +\infty$.
- If a set is not bounded below, we write $\inf S = -\infty$.
- By convention, for the empty set:

$$\sup \emptyset = -\infty, \quad \inf \emptyset = +\infty.$$

3.6 Least Upper Bound Principle

Theorem 3.7.1. Every nonempty subset S of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$. Similarly, every nonempty subset S of $\mathbb{R}$ that is bounded below has an infimum in $\mathbb{R}$.

Remark. Intuition: even if a set does not have a min or max, the LUB Principle helps to approximate the best upper/lower bound of that set (which are LUP or GLB).

Remark. LUB Principle work for real numbers and the definition of the real numbers as all infinite decimals is essential. Because $\mathbb{R}$ is dense, the infinite decimal expansion allower to describe real numbers as limit of sequences of rational numbers.

Part II
Sequence

4 Sequence

Definition 4.1 (Sequence)**.** A *sequence* is a function $f : \mathbb{N} \to S$, and $f(n) = a_n$ is the nth term.

$$a_1, a_2, a_3, \ldots$$

- Individual term: a_n
- Whole sequence (a_n)

Remark. Sequence is an **ordered** list of components. Unlike a set, order of sequence matters and repetition is allowed.

5 Limits

A sequence (a_n) converges to a limit L is as n increases, the terms a_n get closer and closer to L.

Definition 5.1. The real number L is the **limit** of a sequence (a_n) if for every $\varepsilon > 0$ (where ε represents the desired accuracy), there exists an integer N such that for all $n \geq N$, the distance between a_n and L is smaller than ε.

In mathematical terms:

$$|a_n - L| < \varepsilon \quad \text{for all } n \geq N$$

This is written as:

$$\lim_{n\to\infty} a_n = L$$

This means that as n gets larger, a_n gets arbitrarily close to L.

Remark. Relate to the Decimal Expansion of real numbers. If $|a_n - L| < \varepsilon$, then the sequence a_n approximates L to at least some number of decimal places.

Example 2. For example, if

$$\varepsilon = \frac{1}{2} \times 10^{-3} = 0.0005,$$

then the sequence must be within 0.0005 of L after some point N.

5.1 Squeeze Theorem

Definition 5.2. Suppose that three sequences (a_n), (b_n), and (c_n) satisfy $a_n \leq b_n \leq c_n$ for all $n \geq 1$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$. Then $\lim_{n\to\infty} b_n = L$.

6 Basic Properties of Limits

Definition 6.1 (Bounded Sequence)**.** A sequence (a_n) of real numbers is called **bounded** if there exists a real number $M \in \mathbb{R}$ such that, for all $n \in \mathbb{N}$, we have:

$$|a_n| \leq M.$$

Proposition. If $(a_n)_{n=1}^{\infty}$ is a convergent sequence of real numbers, then the set $\{a_n : n \in \mathbb{N}\}$ is bounded. In words, every convergent sequence is bounded.

Theorem 6.1.1. If $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$, and $\alpha \in \mathbb{R}$, then

1. $\lim\limits_{n\to\infty}(a_n + b_n) = L + M$,
2. $\lim\limits_{n\to\infty}(\alpha a_n) = \alpha L$,
3. $\lim\limits_{n\to\infty}(a_n b_n) = LM$,
4. $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \dfrac{L}{M}$ if $M \neq 0$.

In the sequence $\left(\frac{a_n}{b_n}\right)_{n=1}^{\infty}$, we ignore terms with $b_n = 0$. There is no problem doing this because $M \neq 0$ implies that $b_n \neq 0$ for all sufficiently large n. (We use "for all sufficiently large n" as shorthand for saying there is some N so that this holds for all $n \geq N$.)

7 Monotone Sequences

Definition 7.1 (Monotone Increasing Sequence)**.** A sequence (a_n) is called **monotone increasing** if:

$$a_n \leq a_{n+1} \quad \text{for all } n \geq 1.$$

That is, each term is less than or equal to the next.

Definition 7.2 (Strictly Monotone Increasing Sequence)**.** A sequence (a_n) is called **strictly monotone increasing** if:

$$a_n < a_{n+1} \quad \text{for all } n \geq 1.$$

That is, every term is strictly less than the next one; the sequence is always going up and never staying the same.

Theorem 7.2.1 (Monotone Convergence Theorem)**.** Let $(a_n)_{n\in\mathbb{N}}$ be a monotone sequence of real numbers, either

- $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ (Monotone increasing)
- $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$(Monotone decreasing)

Then the following are equivalent (biconditinal):

1. The sequence (a_n) has a finite limit in $\mathbb{R}$. (Convergence)
   - $\lim_{n\to\infty} a_n = \sup\{a_n\}$ if a_n is increasing
   - $\lim_{n\to\infty} a_n = \inf\{a_n\}$ if a_n is decreasing
2. The sequence (a_n) is bounded (above OR below).

7.1 Limit Superior and Limit Inferior

Remark. From any sequence, we can construct monotone sequences to define the *limit superior* and *limit inferior*.

- These limits always exist for bounded sequences.
- If (x_n) is bounded, then for each $n \in \mathbb{N}$, the subset $\{x_k : k \geq n\}$ is also bounded.

Original Sequence (a_n)	$x_n = \inf\{a_k : k \geq n\}$	$y_n = \sup\{a_k : k \geq n\}$
Increasing	$x_n = a_n$ (increasing)	$y_n = \lim_{k\to\infty} a_k$ (decreasing or constant)
Decreasing	$x_n = \lim_{k\to\infty} a_k$ (increasing or constant)	$y_n = a_n$ (decreasing)

Definition 7.3. Let (a_n) be a sequence. Define the sequences:

$$x_n = \inf\{a_k : k \geq n\}, \quad y_n = \sup\{a_k : k \geq n\}$$

Then, the **limit inferior** and **limit superior** of (a_n), if exist, are defined as:

$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty} x_n = x, \quad \limsup_{n\to\infty} a_n := \lim_{n\to\infty} y_n = y$$

7.2 Tail of a Sequence

Definition 7.4 (K-Tail of a Sequence)**.** For a sequence (a_n), the N-tail (where $N \in \mathbb{N}$), or just the tail, of (a_n) is the sequence starting at N, usually written as

$$\{a_{n+K}\}_{n=1}^{\infty} \quad \text{or} \quad \{a_n\}_{n=K+1}^{\infty}$$

8 Subsequence

Think of a sequence like a list of terms labeled by their indices k where $k = 1, 2, 3, \ldots$. Each term a_n in the sequence is an output of the sequence (by definition, sequence is a function taking natural numbers $n \in \mathbb{N}$).

Definition 8.1. Let (a_n) be a sequence of real numbers. Define the **subsequence** of (a_n) and denote it as (a_{n_k}):

$$(a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5}, \ldots)$$

- $n \in \mathbb{N}$ is the inputs as well as indexes of the sequence (a_n)
- $k \in \mathbb{N}$ is the indexes of the subsequence (a_{n_k})
- n_k is the corresponding indexes of the original sequence and are strictly increasing $n_1 < n_2 < n_3 < \ldots$

Proposition. If (a_n) is a convergent sequence, then every subsequence (a_{n_k}) is also convergent, and

$$\lim_{k\to\infty} a_{n_k} = \lim_{n\to\infty} a_n$$

Theorem 8.1.1 (Bolzano-Weierstrass Theorem)**.** Every bounded sequence contains a convergent subsequence.

Part III
Series

9 Infinite Series

9.1 From Sequences to Series

Definition 9.1 (Partial Sums)**.** Suppose (a_n) is a sequence of real numbers $a_1, a_2, \ldots$. Define partial sums s_m as summation of the first n terms in the sequence:

$$s_n = \sum_{k=1}^{n} a_k$$

Definition 9.2 (Sequence of Partial Sums)**.**

$$(s_n) = s_1, s_2, s_3, \ldots$$

where each term s_n is a partial sum of the first n terms in the sequence.

Definition 9.3 (Infinite Series)**.** An **infinite series** is the **"limit" of a sequence of partial sums**. The limit is written as

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$$

9.2 Properties

Theorem 9.3.1 (Linearity of Series)**.** Suppose that $\sum a_n$ and $\sum b_n$ are convergent series, and let c be a constant. Then:

1. **Scaling by a constant:**
$$\sum ca_n \text{ converges, and } \sum ca_n = c \sum a_n.$$
2. **Sum of series:**
$$\sum (a_n + b_n) \text{ converges, and } \sum (a_n + b_n) = \sum a_n + \sum b_n.$$

10 Convergence of a Series

Definition 10.1 (Convergence of a Series)**.** A series $\sum a_n$ converges iff the sequence of partial sums (s_n) converges.

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \lim_{n\to\infty} s_n \text{ exists and is finite.}$$

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = L$$

In this case, we call $\sum_{n=1}^{\infty} a_n$ a **convergent series**. If the series does not converge, then it is said to diverge.

10.1 Cauchy Criterion for Series

Theorem 10.1.1 (Cauchy sequence)**.** Equivalent conditions for convergence of $\sum a_n$:

1. The series converges.
2. For every $\varepsilon > 0$, there exists N such that for all $n \geq N$,

$$\left| \sum_{k=n+1}^{m} a_k \right| < \varepsilon \quad \text{for all } m \geq n.$$

3. The sequence of partial sums (s_n) is a Cauchy sequence.

Note. The intuition here is tails of the series must vanish if the series is convergent.

10.2 Divergence Test

Theorem 10.1.2. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$. The CONVERSE is NOT true.

10.2.1 n-th Term Test for Divergence

Theorem 10.1.3 (n-th Term test for Divergence)**.** Let $\sum_{n=1}^{\infty} a_n$ be an infinite series.

$$\lim_{n\to\infty} a_n = \begin{cases} \neq 0 & \implies \text{the series } \sum a_n \text{ diverges,} \\ 0 & \implies \text{no conclusion can be drawn. The series may converge or diverge.} \end{cases}$$

10.3 Necessary and Sufficient condition with Partial sums

Theorem 10.1.4.

$$\text{If } a_k \geq 0 \;\; k \geq 1, \quad s_n = \sum_{k=1}^{n} a_k \text{ is monotone increasing}$$

Then $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence of partial sums (s_n) is bounded above. That is, there exists a number $M \in \mathbb{R}$ such that

$$s_n < M \quad \text{for all } n \in \mathbb{N}$$

Note. Every convergent series is boundedn.

Problem 1 (3.2D)**.** Let $(a_n)_{n\geq 1}$ be a monotone decreasing sequence and $a_n > 0$. Then

$$\sum_{n=1}^{\infty} a_n \quad \text{converges} \quad \iff \quad \sum_{k=0}^{\infty} 2^k a_{2^k} \quad \text{converges.}$$

10.4 Absolute Convergence Test

Note. Some series behave differently depending on the order of their terms.

Definition 10.2. A series $\sum_{n=1}^{\infty} a_n$ is called *absolutely convergent* if

$$\sum_{n=1}^{\infty} |a_n| < \infty.$$

A series that converges but is not absolutely convergent is called *conditionally convergent*.

Problem 2 (3.2B)**.** If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. (Absolute convergence test)

Proof. Note that for all n:

$$0 \le a_n + |a_n| \le 2|a_n|.$$

Since $\sum |a_n|$ converges, by the Comparison Test,

$$\sum_{n=0}^{\infty} (a_n + |a_n|) \text{ converges.}$$

Now observe:

$$\sum_{n=0}^{\infty} (a_n + |a_n|) - \sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{\infty} a_n.$$

By Linearity of Series,

$$\sum_{n=0}^{\infty} a_n \text{ converges.}$$

□

10.4.1 Conditional convergence

Definition 10.3. A series $\sum_{n=1}^{\infty} a_n$ is called *conditionally convergent* if

$$\sum_{n=1}^{\infty} a_n \text{ converges, but } \sum_{n=1}^{\infty} |a_n| \text{ diverges.}$$

Conditional convergence is sensitive to term order.

10.5 Independence of starting index

Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers, and let $p \ge 1$ be any fixed integer. Then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=p}^{\infty} a_n \text{ converges.}$$

10.6 Root Test

Theorem 10.3.1 (Root Test)**.** Consider a series $\sum_{n=1}^{\infty} a_n$ and define

$$l = \limsup_{n \to \infty} \sqrt[n]{a_n}.$$

- If $l < 1$, the series converges.
- If $l > 1$, the series diverges.
- If $l = 1$, the test is inconclusive and another test is needed.

Idea: This test looks at the average growth rate per step.

10.7 Comparison Test

Theorem 10.3.2 (Comparison Test)**.** Suppose $|a_n| \le b_n$ for all $n \ge 1$.
If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges, and moreover

$$\left| \sum_{n=1}^{\infty} a_n \right| \le \sum_{n=1}^{\infty} b_n.$$

Contrapositive: if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ must also diverge.

Problem 3 (3.2M). Let (a_n) be a sequence with $a_n \geq 0$ for all $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} \frac{a_n}{1+a_n} \text{ converges.}$$

10.7.1 Limit Comparison Test

Theorem 10.3.3 (Limit Comparison Test — Bounded-ratio form). Let (a_n) and (b_n) be sequences with $b_n \geq 0$ for all n. If

$$L := \limsup_{n\to\infty} \frac{|a_n|}{b_n} < \infty$$

- If $\sum_{n=1}^{\infty} b_n < \infty$ converges, then $\sum_{n=1}^{\infty} a_n$ converges absolutely (hence converges).
- If $\sum_{n=1}^{\infty} b_n < \infty$ diverges, it tells us nothing.

This is the weaker version.

10.8 Ratio Test

Theorem 10.3.4 (Ratio test).

$$L \equiv \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$\begin{cases} L < 1 & \implies \sum a_n \text{ converges,} \\ L > 1 \text{ or } L = \infty & \implies \sum a_n \text{ diverges,} \\ L = 1 & \implies \text{test is inconclusive (find another way).} \end{cases}$$

Note. A sequence or a series converges only if the terms shrink fast enough. The idea is to look at the ratio of the next term to the current one.

10.9 p-Series Test

Theorem 10.3.5 (p-Series Test). Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad p > 0.$$

$$\begin{cases} p > 1 & \implies \text{the series converges,} \\ p \leq 1 & \implies \text{the series diverges.} \end{cases}$$

11 Alternate Series

Alternating Series:

A sequence is called *alternating* if it has the form

$$a_n = (-1)^n b_n \quad \text{or} \quad a_n = (-1)^{n+1} b_n, \quad \text{with } b_n \geq 0.$$

Theorem 11.0.1. Leibniz Test

Let (a_n) be a (decreasing) sequence such that

$$a_1 \geq a_2 \geq a_3 \geq \dots \quad \text{and} \quad a_n \to 0 \text{ as } n \to \infty.$$

Then the alternating series converges.

$$\sum_{n=1}^{\infty}(-1)^n a_n$$

12 Geometric Series

A sequence is geometric if

$$a_n = ar^n \quad (n \geq 0).$$

Theorem 12.0.1.

$$\sum_{n=0}^{\infty} ar^n = \begin{cases} \dfrac{a}{1-r}, & |r| < 1, \\ \text{diverges}, & |r| \geq 1. \end{cases}$$

Note. A benchmark for convergence: We often bound a series by a geometric series.

13 Rearrangement

Definition 13.1 (Rearrangement)**.** A *rearrangement* of a series $\sum_{n=1}^{\infty} a_n$ is a series containing the same terms in a different order, given by a permutation π of the natural numbers:

$$\sum_{n=1}^{\infty} a_{\pi(n)}.$$

Theorem 13.1.1. Every rearrangement of an absolutely convergent series converges to the same limit.

Theorem 13.1.2. If $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent series, then for every real number L, there exists a rearrangement of the series that converges to L.

14 Telescoping Series

Definition 14.1.

$$\sum_{n=1}^{\infty}\left(b_n - b_{n+1}\right) = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \cdots$$

Part IV
Topological Space

15 The Space $\mathbb{R}^n$ (Vector structures)

Definition 15.1 (Vector Space $\mathbb{R}^\ltimes$)**.** $\mathbb{R}^n$ is the set of all ordered n-tuples of real numbers:

$$x = (x_1, x_2, \ldots, x_n), \quad x_i \in \mathbb{R}$$

.

15.1 Algebraic structure

Basic vector space operations:
1. Addition: $x + y = (x_1 + y_1,\, x_2 + y_2,\, \ldots,\, x_n + y_n)$
2. Scalar multiplication $tx = (tx_1,\, tx_2,\, \ldots,\, tx_n)$
3. The zero vector is (0,0,...,0).

15.2 Geometric structure

15.2.1 Norm (length of a vector):

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}.$$

This is the Euclidean norm, derived from Pythagoras' theorem.

15.2.2 Distance between points:

$$\|x - y\| = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2}.$$

15.2.3 Triangle inequality:

The triangle inequality holds for the Euclidean norm on $\mathbb{R}^n$:

$$\|x + y\| \le \|x\| + \|y\| \quad \text{for all } x, y \in \mathbb{R}^n.$$

Moreover, equality holds if and only if either $x = 0$ or $y = cx$ with $c \ge 0$.

15.3 Analytic structure

15.3.1 Dot product:

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i.$$

- Dot product of a vector with itself:
$$\langle x, x \rangle = \|x\|^2.$$
- Dot product between sum and difference of 2 vectors:

$$\langle x + y, x + y \rangle = \|x + y\|^2 - \|x\|^2 - \|y\|^2$$

15.3.2 Linearity:

$$\langle rx + sy, z\rangle = r\langle x, z\rangle + s\langle y, z\rangle,$$

$$\langle x, sy + tz\rangle = s\langle x, y\rangle + t\langle x, z\rangle.$$

This gives the **inner product structure**, which ties geometry (angles, orthogonality) to algebra.

15.3.3 Cauchy-Schwarz Inequality

For all $x, y \in \mathbb{R}^n$:

$$|\langle x, y\rangle| \le \|x\|\,\|y\|.$$

Equality holds if and only if x, y are collinear (that is, one is a scalar multiple of the other).

15.3.4 Orthonomal set

Lemma 1. Let $\{v_1, \ldots, v_m\}$ be an orthonormal set in $\mathbb{R}^n$. Then

$$\left\| \sum_{i=1}^{m} a_i v_i \right\| = \left(\sum_{i=1}^{m} |a_i|^2 \right)^{1/2}.$$

An orthonormal set in $\mathbb{R}^n$ is linearly independent. Thus, an orthonormal basis for $\mathbb{R}^n$ is a basis and has exactly n elements.

Each v_i is a "unit direction" pointing in a different orthogonal axis. That's why the norm of the combination only depends on the squares of the coefficients a_i. Since none of the v_i can be written as a combination of the others, they are automatically linearly independent.

16 Convergence and Completeness in $\mathbb{R}^n$

Note. In $\mathbb{R}^n$, distance between points is measured using the Euclidean norm, which plays the same role as absolute value in $\mathbb{R}$.

16.1 Convergence in $\mathbb{R}^n$

Definition 16.1 (Convergence in $\mathbb{R}^n$)**.** A sequence of points $(\mathbf{x}_k)$ in $\mathbb{R}^n$ **converges** to a point $\mathbf{a}$ if for every $\varepsilon > 0$, there exists an integer $N = N(\varepsilon)$ such that

$$\|\mathbf{x}_k - \mathbf{a}\| < \varepsilon \quad \text{for all } k \ge N.$$

In this case, we write

$$\lim_{k\to\infty} \mathbf{x}_k = \mathbf{a}.$$

Lemma 2. Let $(\mathbf{x}_k)$ be a sequence in $\mathbb{R}^n$. Then

$$\lim_{k\to\infty} \mathbf{x}_k = \mathbf{a} \quad \text{if and only if} \quad \lim_{k\to\infty} \|\mathbf{x}_k - \mathbf{a}\| = 0.$$

Note. The points x_k get closer and closer to a no matter which direction they come from.

Lemma 3. A sequence $\mathbf{x}_k = (x_{k,1}, \ldots, x_{k,n})$ in $\mathbb{R}^n$ converges to a point $\mathbf{a} = (a_1, \ldots, a_n)$ if and only if each component converges:

$$\lim_{k\to\infty} \mathbf{x}_k = \mathbf{a} \quad \text{if and only if} \quad \lim_{k\to\infty} x_{k,i} = a_i \quad \text{for } 1 \le i \le n.$$

Note. Clarifications:
- A point x_k is treated as a vector
- A sequence of points contains multiple points $(x_1, x_2, \ldots, x_n)$

17 Cauchy sequence in $\mathbb{R}^n$

A sequence is Cauchy when its terms eventually get arbitrarily close to each other.

Definition 17.1. A sequence $\mathbf{x}_k$ in $\mathbb{R}^n$ is called **Cauchy** if for every $\varepsilon > 0$, there exists an integer N such that

$$\|\mathbf{x}_k - \mathbf{x}_\ell\| < \varepsilon \quad \text{for all } k, \ell \geq N.$$

A set $S \subset \mathbb{R}^n$ is **complete** if every Cauchy sequence of points in S converges to a point in S.

Theorem 17.1.1 (Completeness Theorem for $\mathbb{R}^n$)**.**

Every Cauchy sequence in $\mathbb{R}^n$ converges. Thus, $\mathbb{R}^n$ is complete.

Part V
Useful expressions

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Sequence	**Formula**	$\lim_{n\to\infty} a_n$	**Series Convergence**	$\lim_{N\to\infty} \sum_{n=1}^{N} a_n$	**Notes**
(a_n)	$\frac{1}{n}$	0	Diverges	∞	Harmonic series
(b_n)	$\frac{(-1)^n}{n}$	0	Converges	Finite ($\approx \ln 2$)	Alternating series test
(c_n)	$\frac{1}{n^2}$	0	Converges	$\frac{\pi^2}{6}$	Basel problem result