ADVANCED ECONOMIC THEORY - GAME THEORY

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1 Strategic Game

1.1 Startegic game with Ordinal preferences

Definition 1.1.1 (Strategic game with ordinal preferences)**.** A strategic game consists of:

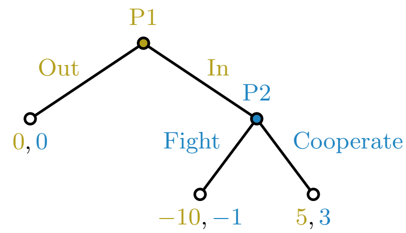
(1) a set of players,
(2) for each player,
  - a set of actions,
  - preferences over action profiles.

1.1.1 From Dynamic game to Static game

- Ignore timing of moves
- Ignore history
- Treat all actions as if chosen simultaneously before the game starts.

The game becomes STATIC over STRATEGIES.

But NASH EQUILIBRIUM only exists in the strategic game and not optimal at subgame as it IGNORES CREDIBLE THREAT.



Eg. Entry game

(1) NE: (Cooperate, In), (Out, Fight)
(2) SPE: (Cooperate, In)

2 Extensive game

2.1 Extensive game with perfect information

Definition 2.1.1 (Extensive game with perfect information)**.** An *extensive game with perfect information* consists of:

- a set of players;
- a set of sequences (terminal histories) (=the set of all sequences of actions that may occur) with the property that no sequence is a proper subhistory of any other sequence
- a *player function* that assigns a player to every sequence that is a proper subhistory of some terminal history;
- for each player, preferences over the set of terminal histories.

2.1.1 Subgame of an Extensive Game with perfect information

Definition 2.1.2. Let Γ be an extensive game with perfect information, with player function P. For any nonterminal history h of Γ, the *subgame* $\Gamma(h)$ following the history h is the extensive game defined as follows.

- **Players:** the players in Γ.
- **Terminal histories:** the set of all sequences h' of actions such that (h, h') is a terminal history of Γ.
- **Player function:** the player at history h' in the subgame is

$$P(h, h') \quad \text{for each proper subhistory } h' \text{ of a terminal history.}$$

- **Preferences:** each player prefers h' to h'' in the subgame if and only if she prefers (h, h') to (h, h'') in Γ.

2.1.2 Strategic form of an Extensive Game

Definition 2.1.3 (Strategic form of an Extensive game with perfect information)**.** Given an extensive game with perfect information, its *strategic form* is the strategic game defined as follows:

- **Players:** the same players as in the extensive game.
- **Actions:** each player's set of actions is her set of strategies in the extensive game.
- **Payoffs:** for any strategy profile s, each player's payoff is the payoff of the terminal history generated by s in the extensive game.

In this strategic form,

$$\boxed{\text{Nash equilibria of the extensive game} \ = \ \text{Nash equilibria of its strategic form.}}$$

2.2 Strategy

Definition 2.2.1 (Strategy)**.** A strategy of player i in an extensive game with perfect information is a function

$$s_i : \{\, h \in H : P(h) = i \,\} \ \longrightarrow \ A(h),$$

that assigns to each history h with $P(h) = i$ an action $s_i(h) \in A(h)$.

Note. Strategy vs Action:

(1) Strategic-form (normal form / payoff matrix): A single action (e.g, $S_1 = \{C, D\}$)
(2) Extensive-form (tree): a complete contingent plan, specifying an action at every possible information set (e.g, $(S_1 = \{(CC), (CD), (DC), (DD)\})$

Note.
- Normal form = actions.
- Strategic form = strategies.

2.3 Outcome

Definition 2.3.1 (Outcome of a Strategy profile)**.** The terminal history generated when we start at the root and let each player take the action prescribed by her strategy whenever it is her turn. It is the actual path of play.

Let $s = (s_1, \ldots, s_n)$ be a strategy profile in an extensive game with perfect information. Define the outcome $O(s)$ recursively as follows.

(1) Start at the empty history: $h_0 = \emptyset$.
(2) At step k, let player $i = P(h_{k-1})$ move.
(3) That player's strategy specifies the action

$$a_k = s_i(h_{k-1}).$$

(4) Update the history:

$$h_k = (h_{k-1}, a_k).$$

Stop when h_K is a terminal history. The *outcome* of the strategy profile s is then

$$O(s) = h_K.$$

Remark. Outcome is a realized path (what actually happens) among all paths of the game.

2.4 Subgame Perfect Equilibrium

Definition 2.4.1 (Subgame Perfect Equilibrium)**.** A strategy profile s^* in an extensive game with perfect information is a subgame perfect equilibrium if starting from every history h in the game, no player can gain from deviating.

Formally, for every player i, every history h with $P(h) = i$, and every alternative strategy r_i of player i,

$$u_i\big(O_h(s^*)\big) \;\geq\; u_i\big(O_h(r_i, s^*_{-i})\big),$$

Note. There can be multiple SP Equilibria s^*, but each of them must give the same payoff $O_h(s^*) = aconstant$.

2.4.1 NE of Subgame

Proposition 2.4.2 (SPE and NE)**.** Let s^* be a strategy profile in an extensive game with perfect information.

(i) Since $O_\emptyset(s) = O(s)$, the definition of subgame perfection implies

$$s^* \text{ is SPE} \implies s^* \text{ is NE}.$$

(ii) If s^* is an SPE, then for every history h and every player i,

$$u_i\big(O_h(s^*)\big) \;\geq\; u_i\big(O_h(r_i, s^*_{-i})\big) \qquad \text{for all } r_i,$$

so the continuation strategies induced by s^* form a Nash equilibrium in every subgame.

(iii) Conversely, if a strategy profile induces a Nash equilibrium in every subgame, then

$$\forall i, \forall h: \quad u_i\big(O_h(s^*)\big) \;\geq\; u_i\big(O_h(r_i, s^*_{-i})\big) \quad \forall r_i,$$

hence the strategy profile is a subgame perfect equilibrium.

Proposition 2.4.3 (SPE and Backward Induction)**.** For any finite-horizon extensive game with perfect information, the set of subgame perfect equilibria is exactly the set of strategy profiles obtained by backward induction. Formally,

$$\{\text{SPE}\} \;=\; \{\text{strategy profiles generated by backward induction}\}.$$

Proposition 2.4.4 (Existence of SPE)**.** Every finite extensive game with perfect information has at least one subgame perfect equilibrium. Formally,

$$\Gamma \text{ finite} \implies \exists\, s^* \in \text{SPE}(\Gamma).$$

2.5 Other comments

Note (Backward induction)**.** A tiny change at the end of the game can dramatically change the beginning, because backward induction propagates that change all the way back.

2.6 Ultimatum game

Definition 2.6.1 (Ultimatum game)**.** Game setup:

(1) Players: Two players divide a pie of size c.
(2) Strategies:
- Player 1: choose an offer x.
- Player 2: choose Y or N for *every* possible x.
- Payoffs:
  - If Yes, $(c - x,\ x)$
  - If No, $(0,\ 0)$

2.6.1 1-stage

Backward Induction

- If $x > 0$: Player 2's best response is Y.
- If $x = 0$: both Y and N are optimal; one optimal strategy is "always accept."

Given "always accept," Player 1 maximizes $c - x$ by choosing $x = 0$.

SPE

$$s_1^* = 0, \qquad s_2^*(x) = Y \quad \text{for all } x.$$

Outcome

Player 1 keeps the entire pie; Player 2 accepts.

Note. P2 has no future bargaining power, so P1 extracts the entire pie.

2.6.2 2-stage (with Discounting)

- players take turns making offers
- each round of delay makes both players worse off (because of discounting)

Let each player i discount the future by δ_i, with

$$0 < \delta_i < 1.$$

- More patient = larger δ_i.
- More impatient = smaller δ_i.

Full SPE includes

(1) Period 1: P2 offers $(\delta_1,\, 1 - \delta_1)$; P1 accepts
(2) Period 2 subgame:
  - P1 would offer $(1, 0)$
  - P2 would accept

SPE Outcome:

- P2 offers $(\delta_1,\, 1 - \delta_1)$
- P1 accepts
- Game ends

(all these strategies must form NEs in every subgame)

Period	**Prop.**	**If reject →**	**Cont. value**	**Offer must give**	**Keeps**
3 (last)	P1	Game ends; P1 gets all	P2: 0	$(1, 0)$	1
2	P2	P1 rejects → go to 3; P1 gets 1 discounted by δ_1	P1: δ_1	$(\delta_1,\ 1 - \delta_1)$	$1 - \delta_1$
1	P1	P2 rejects → go to 2; P2 gets $(1 - \delta_1)$ discounted by δ_2	P2: $\delta_2(1 - \delta_1)$	$\big(1 - \delta_2(1 - \delta_1),\ \delta_2(1 - \delta_1)\big)$	$1 - \delta_2 + \delta_1\delta_2$

3 Repeated Game

3.1 Game is repeated

- The *same players* interact over and over.
- Each player *observes all past actions*.
- They can *reward or punish* based on history (reputation effects).
- Future payoffs matter, but are *discounted* by the factor $\delta \in (0, 1)$.

3.1.1 Comparison

(1) One-shot game: static, NE
(2) Repeated game: dynamic
- Infinitely
- Finitely

3.2 Extensive Game: Repeated Game

Definition 3.2.1. Let G be a strategic game. Denote the set of players by N, and the action set and payoff function of each player i by A_i and u_i respectively. The infinitely repeated game of G with discount factor δ is the extensive game with perfect information and simultaneous moves in which:

- the set of players is N,
- the set of terminal histories is the set of infinite sequences $(a_1, a_2, \ldots)$ of action profiles in G,
- the player function assigns the set of all players to every proper subhistory of every terminal history,
- the set of actions available to player i after any history is A_i,
- each player i evaluates each terminal history $(a_1, a_2, \ldots)$ according to its discounted average

$$(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}u_i(a_t).$$

3.3 Preferences

In repeated games, the outcome is not just one payoff, but **a sequence of payoffs across many periods**. Preferences are over a sequence of outcomes.

3.3.1 Outcome

Definition 3.3.1. A sequence of outcomes in a repeated game:

$$(u(a_1),\, u(a_2),\, u(a_3), \ldots), \qquad 0<\delta<1$$

3.3.2 Equivalent Payoff Functions

Definition 3.3.2. Given outcomes $a_1, a_2, \ldots, a_T$, player i evaluates the stream by

$$u_i(a_1) + \delta\, u_i(a_2) + \delta^2 u_i(a_3) + \cdots$$

Lemma 3.3.3. The discounted sum using payoff functions u and v represents the same preferences over infinite streams if and only if

$$v(x) = \alpha + \beta\, u(x), \qquad \beta > 0.$$

3.4 Discounting

Remark. In a finite game with no discount and no cap on effort, continuation values can escalate without bound.

- $\delta < 1 \;\Rightarrow\;$ bounded values
- $\delta = 1 \;\Rightarrow\;$ unbounded values

3.5 Outcome

For a path of actions

$$a_1,\, a_2,\, a_3,\, \ldots,$$

(1) Different actions: Discounted payoff to player i is

$$u_i(a_1) \;+\; \delta\, u_i(a_2) \;+\; \delta^2 u_i(a_3) \;+\; \cdots .$$

(2) Same actions: Discounted payoff to player i is:

$$u_i(a) + \delta\, u_i(a) + \delta^2 u_i(a) + \cdots = \frac{u_i(a)}{1-\delta}.$$

3.5.1 Constant stream

Definition 3.5.1 (Discounted average formula)**.** If a player receives the same payoff every period, then the weighted average is c. The discounted average formula is

$$(1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} c.$$

Plug in the constant value c:

$$(1-\delta)(c + \delta c + \delta^2 c + \cdots).$$

Factor out c:

$$c(1-\delta)(1 + \delta + \delta^2 + \cdots).$$

Use the geometric series identity $\sum_{t=0}^{\infty} \delta^t = \frac{1}{1-\delta}$:

$$c(1-\delta) \cdot \frac{1}{1-\delta} = c.$$

3.6 One deviation property

Proposition 3.6.1 (One deviation property)**.** In an infinitely repeated game, a strategy profile is a Subgame Perfect Equilibrium (SPE) if and only if no player can gain by deviating **in just one period** after any history.

Remark. If the strategy is not SPE, then there mist exist:

- some history h
- some player i
- and a one-period deviation at h

that gives player i strictly higher payoff.

3.7 Strategies

A strategy is now a function of the entire history.

3.7.1 Cooperate every period

history = {(C,C), (C,C), (C,C),...}

3.7.2 Defect every period

history = {(D,D), (D,D), (D,D),...}

3.7.3 Unrelenting punishment (Grim Trigger)

- Start with C
- One player deviates to D
- The opponent punishes forever. In period 1, choose C. In every future period $t \geq 2$, choose

$$a_t = \begin{cases} C, & \text{if the other player chose } C \text{ in every period } 1, \ldots, t-1, \\ D, & \text{if the other player chose } D \text{ in any period } 1, \ldots, t-1. \end{cases}$$

- Strategy pair: (Grim, Grim)

history = {(C,C), (D,C), (D,D), (D,D),...}

3.7.4 k-period Punishment

- If deviation occurs: punish with D for k periods, then return to C.
- Strategy pair: (k-punishment, k-punishment)

3.7.5 Tit-for-Tat

- Strategy pair: (TFT, TFT)
- Both players start with C, then each plays whatever the opponent played last period.

3.8 Nash Equilibrium

(1) (All-D, All-D) is a NE
  - Outcome path: (D,D), (D,D), (D,D),...
  - Payoff: each gets: $\frac{1}{1-\delta}$

(2) Grin trigger: (UP, UP) is a NE
  - If no one deviates: (C, C), (C, C), (C, C), ...
  - If someone deviates once: (C, C), then (D, D) forever
  - Condition for no profitable deviation: $\delta \geq \frac{1}{2}$

(3) k=punishment: (k-punishment, k-punishment) is NE
  - If no one deviates: (C, C), (C, C), (C, C), ...
  - One shot deviation: (D,C), then k periods of (D,D), then back to (C, C)

3.8.1 Comparisons

Strategy	Punishment	Strength	Needed δ
Grim Trigger	D FOREVER	Very strong	Small-to-medium δ is enough
TFT	Punish 1 period only	Very weak	δ must be very large

3.9 PUnishment

Definition 3.9.1. Punishing is a best response in the continuation game (credible). If it is not, the strategy is not sequentially rational and the entire equilibrium is not SPE.

4 Extended Model

Given actions A, B, C, D,...

- History is a sequence (ordered list of action profiles) $h = (a^1, a^2, \ldots)$, each a^t is a single action/ an action profile taken at time t.
  (1) Sequential move game: Each a^t is a single action. eg, $a^t = A$
  (2) Simultaneous move game: Each a^t is an action profile of actions chosen together by players who move simultaneously eg, $a^t = (A, B)$
- A Strategy is a function, but we can represent a strategy as a list of actions

4.1 With Simultaneous moves (Repeated game in One period)

4.1.1 Definition

Definition 4.1.1. An extensive game with simultaneous moves consists of:

(1) a set of Players N = {1, 2}
(2) a set of actions available to each player at each history $A_i(h)$
(3) a set of Terminal histories, h is a Terminal history iff either:
- $h = (a^1, a^2, ..., a^k)$ is finite and $P(h)$ is undefined
- $h = (a^1, a^2, ...)$ is infinite and $a_t \in \prod_{i \in P(h_{t-1})} A_i(h_{t-1})$, where $h_{t-1} = (a_1, \ldots, a_{t-1})$.

(4) Player function is a set of players $P(h) = \{1, 2\}$ (1 and 2 move simultaneously)
(5) Preferences over terminal histories

4.2 With Sequential moves

4.2.1 Stackelberg Duopoly

Remark. Fewer options → stronger commitment → better outcome.

Definition 4.2.1. Stackelberg is a model of sequential quantity competition between two firms:
- Firm 1 (Leader) chooses quantity first
- Firm 2 (Follower) observes q_1, and then chooses its own quantity q_2

1. **Follower's best response:**
$$\hat{q}_2^*(q_1) = \arg\max_{q_2} \; q_2 P(q_1 + q_2) - C_2(q_2).$$
2. **Leader's problem:**
$$q_1^* = \arg\max_{q_1} \; q_1 \, P\big(q_1 + \hat{q}_2^*(q_1)\big) - C_1(q_1).$$
3. **Follower's equilibrium output:**
$$q_2^* = \hat{q}_2^*(q_1^*).$$
4. **Stackelberg equilibrium:**
$$(q_1^*, \, q_2^*).$$

- Leader $q_1 = \frac{\alpha - c}{2}$
- Follower $q_2 = \frac{\alpha - c}{4}$

Remark. Leader produces MORE. Follower produces LESS. Leader earns MORE profit. Follower earns LESS profit.

5 Strategic Game with Imperfect Information

5.1 Bayesian game

Definition 5.1.1 (Bayesian Game)**.** A Bayesian game consists of:

(1) a set of players;
- a set of actions;
- a set of signals she may receive, and a **signal function** that assigns a signal to each state;
- for every signal, a **belief** over the states consistent with that signal (a probability distribution over the associated states);

- a **Bernoulli payoff function** over pairs (a, ω), where a is an action profile and ω is a state; the expected value of this function represents the player's preferences over lotteries on such pairs.

(2) a set of states.

5.2 State

Definition 5.2.1. A **state** is a realized value at the start of the game which has complete description of:

$$\omega \in \Omega$$

- players' preferences,
- players' information,
- anything relevant for payoffs.

5.3 Player

- Each type t_i of each player i is treated as a separate player.
- Players = all pairs (i, t_i), each type chooses one action.

5.3.1 Signals

Remark. From states to signals:

- A state of the world $\omega \in \Omega$
- Each state generates a signal $\tau_i(\omega)$
- Player i observes the signal, not the state
- The mapping from states to signals determines how much information the signal carries

Definition 5.3.1 (Signal from 1 state)**.** Player i's signal in state ω is

$$\tau_i(\omega),$$

- τ_i is the **signal function** for player i.
- If the true state is ω, then player i observes the signal $\tau_i(\omega)$.

The signal is deterministic — each state ALWAYS produces the same signal for a given player.

Definition 5.3.2 (Signal from multiple states)**.** Define the set of states consistent with signal t_i, After receiving t_i, player i cannot distinguish which state it is.

$$\{\omega \in \Omega : \tau_i(\omega) = t_i\},$$

5.3.2 Type

Definition 5.3.3. We refer to player i as player typer t_i if they receives signal t_i.

5.3.3 Action

Let $a(j, t_j)$ be the action of type t_j of player j. In state ω, player j's type is $\tau_j(\omega)$, so her action is

$$\hat{a}_j(\omega) = a(j, \tau_j(\omega)).$$

5.3.4 Action Profile

If type t_i chooses a_i, then the action profile is below, where a_i is i's action and $\hat{a}_{-i}(\omega)$ are the actions of all other players' types.

$$(a_i,\ \hat{a}_{-i}(\omega)),$$

5.3.5 Belief

Definition 5.3.4 (Belief)**.** Each type of each player i holds a belief about the likelihood of states consistent with their signals.

$$\mu(t \mid \tau_i(\omega))$$

5.3.6 How players update belief = information types

Information type tells you:

- Whether Bayes' rule changes beliefs
- How much beliefs change
- How precise beliefs become

Info Type	**Signal Map**	**Posterior**	**Bayes**
Perfect	$\tau(\omega_i) \neq \tau(\omega_j)$	$\mu(\omega_i \vert s) = \mathbf{1}\{\tau(\omega_i) = s\}$	Degenerate
None	$\tau(\omega_1) = \cdots = \tau(\omega_K)$	$\mu(\omega_i \vert s) = \mu(\omega_i)$	No update
Partial	Mixed equality / inequality	$\mu(\omega_i \vert s) = \frac{\Pr(s \vert \omega_i)\mu(\omega_i)}{\sum_{k:\tau(\omega_k)=s} \Pr(s \vert \omega_k)\mu(\omega_k)}$ where denumerator is $p(s)$	$0 < \mu(\omega_i \vert s) < 1$

5.4 Bernoulli payoff function

Definition 5.4.1. Each player i has a payoff function

$$u_i(a, \omega),$$

which gives the payoff when the action profile is a and the state is ω.

5.5 Expected Payoff

Definition 5.5.1 (Expected payoff of type t_i)**.** The expected payoff of type t_i of player i, when choosing action a_i, is

$$\sum_{\omega \in \Omega} \Pr(\omega \mid t_i)\; u_i((a_i,\ \hat{a}_{-i}(\omega)),\ \omega)\,.$$

It is the belief-weighted average over all states ω of the Bernoulli payoff, given type t_i's beliefs and the actions that other players take in each state.

5.6 Bayesian Nash Equilirbium

Definition 5.6.1 (Bayesian Nash Equilibrium)**.** An **action profile** $a(i, t_i)$ for every player i and type t_i is a **Bayesian Nash equilibrium** if for every i and every type t_i,

$$a(i, t_i)\ \in\ \arg\max_{a_i \in A_i} \sum_{\omega \in \Omega} \Pr(\omega \mid t_i)\; u_i((a_i,\ \hat{a}_{-i}(\omega)),\ \omega)\,.$$

In words: each type chooses the action that maximizes its expected payoff, given its beliefs about the state and the actions taken by the other players' types.(Best response to beliefs)

5.7 Example games

5.7.1 Bayesian Cournot Duopoly

Remark. From any one firm's point of view, the other firm's cost type is unknown.

- It might be high cost.
- It might be low cost.
- The only thing you know is the probability of each.

Remark. Think about it,

- Opponent's type is random variable.
- You hold beliefs about their action.
- Their output becomes probabilistic.
- You best-respond to the expected output.

Setup

(1) **Players:** two firms
$$N = \{1, 2\}.$$
(2) **Types:** high or low cost
$$T_i = \{H, L\}, \qquad c_L < c_H.$$
(3) **Beliefs:** common prior
$$\Pr(c_j = c_H) = p, \qquad \Pr(c_j = c_L) = 1 - p.$$
(4) **Actions:** each type chooses a quantity
$$\sigma_i = (q_i^H, q_i^L), \qquad q_i^t \geq 0.$$
(5) **Payoff:** for firm i with cost c_i
$$\pi_i = q_i\big(a - c_i - (q_i + q_{-i})\big).$$

Best responses

(1) **Expected payoff of type c_i:**
$$\begin{aligned} E[\pi_i] &= q_i(a - c_i - q_i - E[q_{-i}]) \\ &= p\, q_i(a - c_i - q_i - q_{-i}^H) + (1-p)\, q_i(a - c_i - q_i - q_{-i}^L) \end{aligned}$$
with $E[q_{-i}]$ is the expected quantity of other player
$$E[q_{-i}] = p\, q_{-i}^H + (1-p)\, q_{-i}^L.$$
(2) **Best-response (from firm's own view)**
$$a - c_i - 2q_i - E[q_{-i}] = 0$$
$$\implies \quad q_i^{BR}(c_i) = \frac{a - c_i - E[q_{-i}]}{2}.$$
(3) Expected Best-response quantity (from opponent firm's view):
$$E[q_i^{BR}] \;=\; p \cdot q_i^{H,BR} \;+\; (1-p) \cdot q_i^{L,BR}.$$
- $E[q_i^{BR}] = \frac{a - Ec - Eq_{-i}}{2}$
- Expected cost of opponent: $E[c] = p\, c^H + (1-p)c^L$.

(4) **Bayesian Nash Equilibrium:** a profile
$$(q_1^H, q_1^L;\; q_2^H, q_2^L)$$
such that every q_i^t solves its best-response condition given $E[q_{-i}]$.

Note. Condition: $q_i^*(c_i) > 0$ for each c.

5.7.2 Auction Model

Setup

(1) **Players & States:** two firms; true oil amount $\omega \in \{1, \ldots, 1000\}$, uniform.
(2) **Signals (types):** with assymetric information
- $\tau_1 : \Omega \longrightarrow \{L, H\}$

$$\tau_1(\omega) = \begin{cases} L, & \omega < 500, \\ H, & \omega \geq 500. \end{cases}$$

- $\tau_2 : \Omega \longrightarrow \{NH, VH\}$

$$\tau_2(\omega) = \begin{cases} NH, & \omega < 750, \\ VH, & \omega \geq 750. \end{cases}$$

(3) **Beliefs:**
- For firm 1:
  - P(Firm 2: High | Firm 1: Low) = 0
  - P(Firm 2: Low | Firm 1: Low) = $\frac{1}{2}$
  - P(Firm 2: High| Firm 1: High) = $\frac{1}{2}$
  - P(Firm 2: Low | Firm 1: High) = $\frac{1}{2}$

1's types \ 2's types	**not high**	**very high**
low	$1,\ E_{L,NH}[\omega] = 250$	0
high	$\frac{1}{2},\ E_{H,NH}[\omega] = 625$	$\frac{1}{2},\ E_{H,VH}[\omega] = 875$

- For firm 2:

2's types \ 1's types	**high**	**low**
not high	$\frac{1}{3}$, $E_{NH,H}[\omega] = 625$	$\frac{2}{3}$, $E_{H,VH}[\omega] = 250$
very high	1, $E_{H,VH}[\omega] = 875$	0

(4) **Actions:** Bid / Not Bid.
(5) **Payoffs:** Given the bidding price, the payoff depends on number of firms who bid.

$$(E(\omega \mid \text{Type of firm}) \text{ - Price}) \cdot \text{Wining Prob}$$

- If no bid: 0.
- If bid alone: $(E[\omega] - 275)$, probability of winning = 1.
- If both bid: $\frac{1}{2}(E[\omega] - 550)$, probability of winning = $\frac{1}{2}$

Strategies

(1) **Dominance:**
- Firm 1 Low: $E[\omega] = 250 \rightarrow$ bidding always negative $\rightarrow$ never bid.
- Firm 2 Very High: $E[\omega] = 875 \rightarrow$ bidding always positive $\rightarrow$ always bid.
- Firm 1 High: $E[\omega] \geq 625 \rightarrow$ bidding profitable $\rightarrow$ always bid.

(2) **Firm 2 Not High:** posterior mixes $\omega \in [1, 500]$ and $\omega \in [500, 750]$; expected profit from bidding $< 0 \rightarrow$ do not bid.

(3) **Bayesian Nash Equilibrium:**

$$\begin{cases} \text{Firm 1: Low}\to\text{NotBid},\ \ \text{High}\to\text{Bid}, \\ \text{Firm 2: NotHigh}\to\text{NotBid},\ \ \text{VeryHigh}\to\text{Bid}. \end{cases}$$

6 Signaling Game

6.1 Extensive game with Imperfect Information

Definition 6.1.1 (Extensive Game)**.** An *extensive game* consists of:

- A set N of players.
- A set H of histories (sequences of actions), with each *terminal history* representing a completed game.
- A *player function* that specifies the player, or chance, who moves after each non-terminal history.
- A specification of the probabilities governing the moves of chance.
- A specification of the *information sets*, i.e. sets of histories among which each player cannot distinguish at each point at which she moves.
- Preferences (typically payoff functions) over terminal histories for each player.

6.1.1 Mixed strategy in extensive game

Definition 6.1.2 (Mixed strategy in extensive game)**.** A mixed strategy in an extensive game is a probability distribution over a player's pure strategies. A pure strategy can be viewed as a degenerate mixed strategy that assigns probability 1 to it; hence pure strategies are contained in the set of mixed strategies.

6.1.2 Nash equilibrium

Definition 6.1.3 (Nash equilibrium of an extensive game)**.** A mixed strategy profile a^* is a Nash equilibrium if, for each player i and every mixed strategy a_i of player i,

$$\mathbb{E}[u_i(a^*)] \ \geq \ \mathbb{E}[u_i(a_i, a^*_{-i})],$$

where expected payoffs are evaluated according to a payoff function representing player i's preferences over lotteries.

6.2 Sequential game

Equilibrium in extensive games with imperfect information must involve:

(1) Strategies (what players do)
(2) Beliefs (what players think happened)

6.2.1 Beliefs

Definition 6.2.1 (Belief system)**.** a rule that assigns, at each information set, a probability distribution over over the nodes (histories) in information set I_i

$$\mu = \{\mu_i(I_i) \text{ for each player } i\}$$

6.2.2 Behavioral startegies

Definition 6.2.2. A function that assigns to each information set I_i a probability distribution over its available actions at set I_i.

$$\beta_i : I_i \to \Delta(A(I_i))$$

6.2.3 Assessment

Definition 6.2.3 (Assessment)**.** A pair, where β is a profile of behavioral strategies and μ is a belief system.

$$(\beta, \mu)$$

6.2.4 Equilibrium conditions

Remark. In Signaling game, an "equilibrium" must specify:

- Strategies (what players do): $\beta_1 = \sigma^*, \beta_2 = \alpha^*$
- Beliefs (what players think is happening): μ^*
- Optimality (given beliefs) Sender + Receiver best-response conditions + Bayesian Updating

Conditions

(1) Sequential rationality
For each player i and information set I_i, the action chosen must maximize relative to all alternative actions available at I_i.

$$\mathbb{E}[\, u_i \mid I_i, \beta, \mu \,]$$

(2) Consistency of beliefs with strategies: $\mu_i(h) = \dfrac{\Pr(h \mid \beta)}{\sum_{h' \in I_i} \Pr(h' \mid \beta)}$, for all $h \in I_i$.

6.3 Investment Game

Setup

(1) **Players:** $N = \{\text{CEO}, \text{Investor}\}$.
(2) **Histories:** $H = \{h_0 = \emptyset,\ h_1(t) = (t),\ h_2(t,m) = (t,m),\ h_3(t,m,e) = (t,m,e)\}$ with terminal histories $H_T = \{(t,m,e) : t \in [0,1],\ m \in M,\ e \in \mathbb{R}_+\}$.
(3) **Player function:** $P(h_0) = \text{Nature},\ P(h_1) = \text{CEO},\ P(h_2) = \text{Investor}$.
(4) **Chance moves:** $t \sim U[0,1]$, with $\Pr(t \in A) = \int_A 1\, dt$.
(5) **Information sets:** CEO observes t so $I_{\text{CEO}}(t) = \{(t)\}$; Investor observes m but not t, so $I_{\text{Inv}}(m) = \{(t,m) : t \in [0,1]\}$.
(6) **Payoffs:** For (t,m,e): Investor $u_I(t,m,e) = te - \frac{1}{2}e^2$; CEO $u_{\text{CEO}}(t,m,e) = (t+b)e - \frac{1}{2}e^2 = u_I(t,m,e) + be$.

6.3.1 Equilibrium

(1) Pooling/uninformative equilibrium
(2) Mixed/partially revealing equilibrium
(3) Fully informative equilibrium

7 Past exams

7.1 Bayesian Game

7.1.1 F2021, Q3

Players X and Y play a Matching Pennies game with incomplete information. The payoff matrix is:

	$Y:H$	$Y:T$
$X:H$	$(1, -1+c)$	$(-1, 1)$
$X:T$	$(-1, 1+c)$	$(1, -1)$

This is a version of the standard game where player X is the *Matcher*, player Y is the *Mismatcher*, and player Y receives an additional payoff c if she plays H.

The additional payoff c can take one of two values: $c_L = 0$ or $c_H = 0.5$. Player Y knows the true value of c. Player X does not. Player X believes that $c = 0.5$ with probability $p = \frac{3}{4}$.

3(a) Does the game have a pure strategy Bayesian Nash Equilibrium (BNE)?

3(b) Find a mixed strategy BNE when player X and type c_H are mixing. Is the BNE unique?

7.1.2 F2018, Q3

Each player $i \in \{1, 2\}$ privately observes her type (value)

$$v_i \in \{1, 2, \ldots, 100\}.$$

The action set of each player is $\{I, N\}$, where I denotes *Invest* and N denotes *Not invest*.

The payoff of player i from investing, given her type v_i and the action a_{-i} of the other player, is

$$u_i(I, a_{-i}; v_i) = \begin{cases} v_i - 10, & \text{if } a_{-i} = I, \\ v_i - 60, & \text{if } a_{-i} = N, \end{cases}$$

while the payoff from not investing is

$$u_i(N, a_{-i}; v_i) = 0 \quad \text{for both } a_{-i} \in \{I, N\}.$$

3a: Show that there are types of each player who have a strictly dominated action. Find all such types and describe which actions are strictly dominated.

3b: Consider a game, where all strictly dominated actions (i.e., all actions that you identified in the previous question) are eliminated. Show that there are types for whom some action become strictly dominated in this new game. In other words, show that there are actions that are strictly dominated at the second stage of elimination even if they were not strictly dominated at the first stage.

3c Find all types for whom there is only one action that survives the iterated elimination of strictly dominated actions.(If you cannot solve this question, go to the next one.)

3d: Find a Bayesian equilibrium in this game. (Hint: Either use the answer to the previous question, or, explain that equilibrium strategies must have a threshold form: player i Invests only if her or his value v_i is above certain threshold.)

7.2 Extensive game with Perfect information

7.2.1 F2021, Q2