CS 577 - Discrete Primer

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WISCONSIN
UNIVERSITY OF WISCONSIN–MADISON

Discrete Mathematics

Definition

Rigorous mathematical study of discrete structures.

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Key Discrete Concepts for CS 577

Core

- Logic
- Sets
- Recurrences
- Relations and Function
- Graphs and Trees
- Counting

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Applied in CS 577

- Proofs esp. Induction
- Invariants
- Program Correctness

Logic

Propositions

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A statement that is either true or false.

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True proposition:

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False proposition:

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Operations

- And: $\wedge$, &, &&
- Or: $\vee$, |, ||
- Negation: $\neg$, !
- Implies: $\implies$
- If and only if (iff): $\iff$
  $P \iff Q \equiv P \implies Q \wedge Q \implies P$

Truth Tables

a	b	$a \wedge b$	$a \vee b$	$a \implies b$	$\neg a$
F	F				
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Logical Equivalence

TopHat 1: Is $P \implies Q$ equivalent to $\neg P \implies \neg Q$?

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Exercise: Prove it!

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Quantifiers

- For all: $\forall$. $\forall x \in \mathbb{Z}, Even(x) \iff Odd(x+1)$
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- Order matters when combining quantifiers!

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Logical Equivalence

TopHat 2: What is the logical equivalence of $\neg(\forall x S(x))$?

Sets

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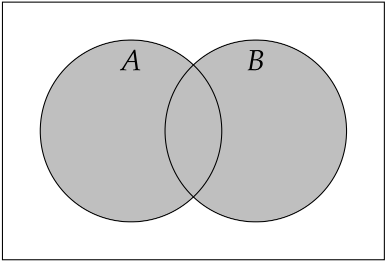
Basic Notations

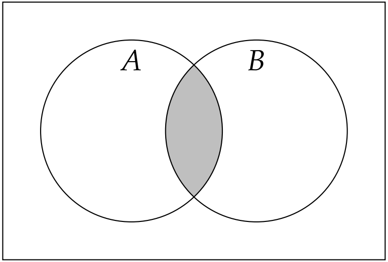
$A \subset B$	A is a proper subset of B, meaning that A contains some (or none) of the elements of B but not all.
$A \subseteq B$	A is subset of B and A may contain all of the elements of B.
$\lvert A \rvert$	The cardinality of A is the number of elements in the set.

Set Operations

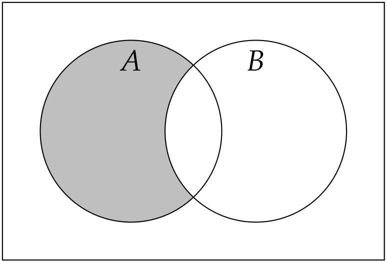
Union: $A \cup B$



Intersection: $A \cap B$



Set difference: $A \setminus B$ or $A - B$



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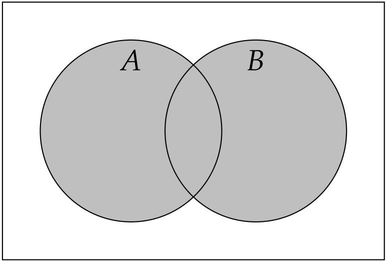

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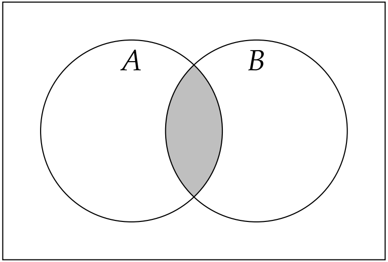

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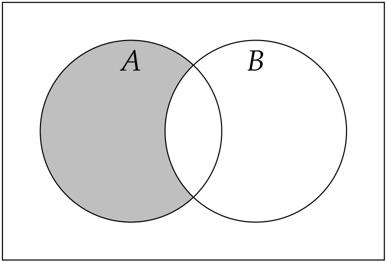

Other Notions

$\emptyset$ **or** $\{\}$	The null or empty set.
$\mathcal{P}(A)$	Power set of A. A set of all possible subsets of A (including $\emptyset$).

TopHats

TopHat 3

What is $\{a, b, c\} \setminus \{c, d, e\}$?

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TopHat 4

What is the size of $\mathcal{P}(A)$ for some set A?

Relations and Functions

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Properties of Relations

Reflexive If $\forall a \in A, R(a, a)$. (*antireflexive*: $\forall a \in A, \neg R(a, a)$)

Symmetric If $\forall a, b \in A, R(a, b) \iff R(b, a)$. (*antisymmetric*: $\forall a, b \in A, R(a, b) \cap R(b, a) \implies a = b$)

Transitive If $\forall a, b, c \in A, R(a, b) \cap R(b, c) \implies R(a, c)$.

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Types of Relations

- Equivalence Relations: reflexive, symmetric, and transitive.
- Order Relations: antisymmetric and transitive.
- Functions

Functions

Definition

$f : A \to B$ is a function from A to B. That is for every $a \in A$ there is at most one $b \in B$.
Ex. $f(x) = y + 1$ for $x, y \in \mathbb{R}$.

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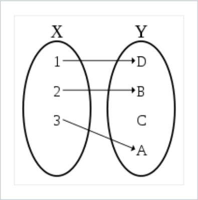
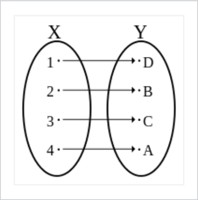
Terminology

- **Domain**: The values of A.
- **Range / Codomain**: The values of B

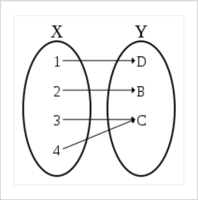
Functions



An injective non-surjective function (injection, not a **bijection**)



An injective surjective function (**bijection**)



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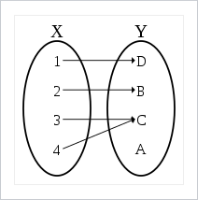

A non-injective non-surjective function (also not a **bijection**)

Types of Functions

- one-to-one / injective
- onto / surjective
- bijection (both onto and one-to-one)

Induction

Proof by Induction

What is induction?

- The most important proof technique in discrete math and CS.
- It proves that $P(n)$ holds for every natural number n, i.e., $n = 0, 1, 2, 3, \ldots$.

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Step 1 State the induction hypothesis.

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Special Types of Induction

- Strong induction: we assume true for 1 to k instead of just k.
- Structural induction: we are reasoning about a structure that we map to the natural numbers.

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Induction Exercises

- Show $\sum_1^n 2^n = 2^{n+1} - 2$.

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- Show $\sum_1^n 2^n = 2^{n+1} - 2$.
- Show, for $n \geq 5$, $4n < 2^n$.

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Proof by Picture Actually not valid!

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Proof by Cases (Brute Force / Exhaustion) Split into cases and prove separately for each case.

Counting

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Basic Techniques

- k-to-1 Rule: Is there a k to 1 ratio between 2 sets?
- Sum Rule: Combine disjoint sets; add cardinality.
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Perms and Comb

- **k-Permutation**: $k!$
- **r-Permutation of n items**: ${}_nP_r = P(n,r) = \frac{n!}{(n-r)!}$
- **r-Combination of n items**: ${}_nC_r = C(n,r) = \frac{n!}{r!(n-r)!}$

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Pigeonhole Principal

If n pigeons are placed into m holes, and $n > m$, then at least one hole has more than one pigeon.

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Robot Exercise

Suppose we have a robot which walks on a 2-dimensional grid. The rows and columns of the grid are labelled by integers. Our robot starts at position $(0, 0)$, and can only move diagonally, one square at a time. Can we get to $(8, 9)$? Why or why not?

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Proving correctness

- Requires 2 proofs (one for soundness and one for completeness).
- Often requires identifying invariants and induction.

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Exercises

Assume $T(1) = 1$ for all.

- $T(n) = T(n/2) + 1$
- $T(n) = T(n/2) + n$
- $T(n) = 3T(n/3) + n$

Graphs and Trees

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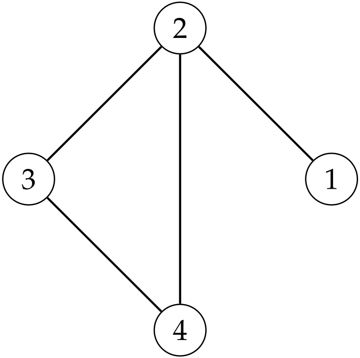
- Forests

Graph Encodings



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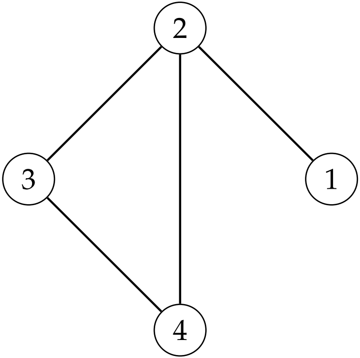

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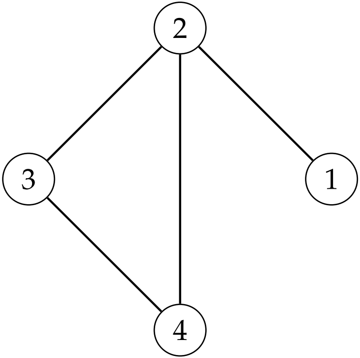

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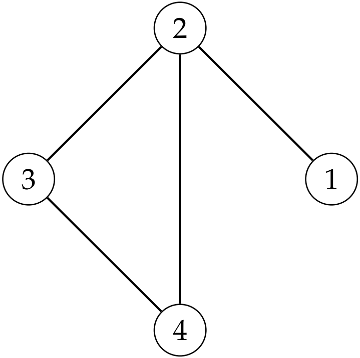
- **Edge list**: List of all node pairs representing the edges.

Graph Encodings



Representations

- **Adjacency matrix**: $|V|$ by $|V|$ matrix with a 1 if nodes are adjacent.
- **Adjacency list**: For each node, list adjacent nodes.
- **Edge list**: List of all node pairs representing the edges.
- **Incidence matrix**: $|V|$ by $|E|$ matrix with a 1 if node is incident to the edge.

Trees

Definition

- A connected graph without cycles.
- A single node may be designated as the *root* of the tree.
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TopHat 6

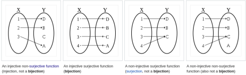
Is P_{10} a tree?

Appendix

References

Image Sources I

https://brand.wisc.edu/web/logos/



https://en.wikipedia.org/wiki/Bijection