Exercises of Chapter 4, 5 of Analysis and Data-Based Reconstruction of Complex Nonlinear Dynamical Systems

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Contents

1 Question 5.3

Question: Statistical moments of the Wiener process
Compute (a) $\langle W(t) \mid W(s) = W_s \rangle$
(b) $\left\langle W^2(s)W(t) \right\rangle$ for $t > s$.

1.1 a

To solve this problem, we need to understand the properties of the Wiener process (Brownian motion) and its statistical moments.
The Wiener process, denoted by $W(t)$, is a continuous-time stochastic process with the following properties:
1-$W(0) = 0$ (the process starts at zero)
2-$W(t) - W(s) \sim N(0, t - s)$ for $t > s$ (the increments are normally distributed with mean 0 and variance $t - s$)
3-$W(t) - W(s)$ is independent of the past values of the process (independent increments) Now, let's compute the conditional expectation $\langle W(t) \mid W(s) = W_s \rangle$.
First, we need to understand the notation:
$\langle \cdot \mid \cdot \rangle$ denotes the conditional expectation $W(t)$ and $W(s)$ are the values of the Wiener process at times t and s, respectively W_s is the given value of the Wiener process at time s To find the conditional expectation, we can use the property of independent increments and the fact that the Wiener process has mean 0 at any time.

$$\langle W(t) \mid W(s) = W_s \rangle = \langle W(t) - W(s) + W(s) \mid W(s) = W_s \rangle$$

$$= \langle W(t) - W(s) \mid W(s) = W_s \rangle + \langle W(s) \mid W(s) = W_s \rangle$$

$$= \langle W(t) - W(s) \mid W(s) = W_s \rangle + W_s$$

(since $W(s)$ is a constant given $W(s) = W_s$)

Now, we need to find $\langle W(t) - W(s) \mid W(s) = W_s \rangle$. Since the increment $W(t) - W(s)$ is independent of the past values of the process (property 3), it is also independent of the value $W(s) = W_s$. Therefore, the conditional expectation is equal to the unconditional expectation:

$$\langle W(t) - W(s) \mid W(s) = W_s \rangle = \langle W(t) - W(s) \rangle = 0$$

(by property 2)
Substituting this result back into the original equation, we get:

$$\langle W(t) \mid W(s) = W_s \rangle = 0 + W_s = W_s$$

Therefore, the conditional expectation of $W(t)$ given $W(s) = W_s$ is simply W_s.

1.2 b

To compute $\langle W^2(s)W(t) \rangle$ for $t > s$, we need to use other properties of the Wiener process and the moments of the normal distribution. First, let's express $W^2(s)W(t)$ in terms of the increments of the Wiener process:

$$W^2(s)W(t) = [W(s)]^2[W(t) - W(s) + W(s)]$$
$$= [W(s)]^2[W(t) - W(s)] + [W(s)]^3$$

Now, we can take the expectation of both sides:

$$\langle W^2(s)W(t) \rangle = \langle [W(s)]^2[W(t) - W(s)] \rangle + \langle [W(s)]^3 \rangle$$

Since the increment $W(t) - W(s)$ is independent of $W(s)$ (property 3 of the Wiener process), we can separate the expectations:

$$\langle W^2(s)W(t)\rangle = \langle [W(s)]^2\rangle\langle W(t) - W(s)\rangle + \langle [W(s)]^3\rangle$$

Using the fact that $\langle W(t) - W(s)\rangle = 0$ (property 2 of the Wiener process), the first term on the right-hand side becomes 0. To evaluate $\langle [W(s)]^2\rangle$ and $\langle [W(s)]^3\rangle$, we can use the moments of the normal distribution. Since $W(s) \sim N(0, s)$, we have:

$$\langle [W(s)]^2\rangle = \mathrm{Var}[W(s)] = s$$

$$\langle [W(s)]^3\rangle = 0$$

(odd moments of a normal distribution are 0)
Therefore, we have:

$$\langle W^2(s)W(t)\rangle = s \cdot 0 + 0 = 0$$

Thus, the expectation $\langle W^2(s)W(t)\rangle$ for $t > s$ is equal to 0.

2 Question 5.4

Question: Statistical moments of the integral of the Wiener process
Define $Z(t) = \int_0^t W(s)ds$ and show that
(a) $\langle Z(t)\rangle = 0$
(b) $\mathrm{var}(Z(t)) = \frac{t^3}{3}$.

2.1 a

To show that $\langle Z(t)\rangle = 0$, we need to understand the properties of the Wiener process and the stochastic integral. First, let's define $Z(t)$ as stated:

$$Z(t) = \int_0^t W(s)ds$$

where $W(s)$ is the Wiener process. Now, we need to find the expectation of $Z(t)$, denoted by $\langle Z(t)\rangle$. We can express the expectation of $Z(t)$ using the definition of the stochastic integral:

$$\langle Z(t)\rangle = \left\langle \int_0^t W(s)ds \right\rangle$$

Since the Wiener process has zero mean, i.e., $\langle W(s)\rangle = 0$ for all s, we can use the linearity property of the expectation and the stochastic integral:

$$\begin{aligned}\langle Z(t)\rangle &= \left\langle \int_0^t \langle W(s)\rangle ds \right\rangle \\ &= \int_0^t \langle W(s)\rangle ds \\ &= \int_0^t 0 ds \\ &= 0\end{aligned}$$

Therefore, we have shown that $\langle Z(t)\rangle = 0$. This result is expected because the Wiener process has zero mean, and the stochastic integral is a linear operation that preserves the mean of the integrand.

2.2 b

To compute the variance of $Z(t)$, we need to evaluate $\operatorname{Var}(Z(t)) = \left\langle Z(t)^2 \right\rangle - \langle Z(t) \rangle^2$. First, let's compute $\left\langle Z(t)^2 \right\rangle$:

$$\left\langle Z(t)^2 \right\rangle = \left\langle \left(\int_0^t W(s) ds \right)^2 \right\rangle$$

Using the properties of the Wiener process, we can expand this expression as follows:

$$\left\langle Z(t)^2 \right\rangle = \left\langle \int_0^t \int_0^t W(s) W(u) ds du \right\rangle$$

Since the Wiener increments are independent, we have:

$$\left\langle Z(t)^2 \right\rangle = \int_0^t \int_0^t \langle W(s) W(u) \rangle ds du$$

For $s < u$, the covariance $\langle W(s) W(u) \rangle = \min(s, u)$. For $s > u$, $\langle W(s) W(u) \rangle = \min(s, u)$ as well, since $W(s) W(u)$ is symmetric. So, we can write:

$$\langle W(s) W(u) \rangle = \min(s, u)$$

Therefore:

$$\left\langle Z(t)^2 \right\rangle = \int_0^t \int_0^t \min(s, u) ds du$$

Evaluating this integral gives:

$$\begin{aligned}
\left\langle Z(t)^2 \right\rangle &= \int_0^t \left(\int_0^s u du + \int_s^t s du \right) ds \\
&= \int_0^t \left(\frac{s^2}{2} + s(t-s) \right) ds \\
&= \int_0^t \left(\frac{s^2}{2} + st - s^2 \right) ds \\
&= \left[\frac{s^3}{6} + \frac{st^2}{2} - \frac{s^3}{3} \right]_0^t \\
&= \frac{t^3}{6} + \frac{t^3}{2} - \frac{t^3}{3} \\
&= \frac{t^3}{3}
\end{aligned}$$

Now, we have $\operatorname{Var}(Z(t)) = \left\langle Z(t)^2 \right\rangle - \langle Z(t) \rangle^2$:

$$\operatorname{Var}(Z(t)) = \frac{t^3}{3} - 0^2 = \frac{t^3}{3}$$

Therefore, $\operatorname{Var}(Z(t)) = \frac{t^3}{3}$.