Exercises of Chapter 2, 3 of Analysis and Data-Based Reconstruction of Complex Nonlinear Dynamical Systems

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2.1 Exercise 2.1

2.1.1 Part a

the moment generator function is given by $Z_x(u) = \langle e^{u^T x}\rangle$ Now, let's consider the taylor's expansion of $e^{u^T x}$

$$e^{u^T x} = \sum_{i=0}^{\infty} \frac{(u^T x)^i}{i!} = 1 + u^T x + \frac{(u^T x)^2}{2!} + \cdots$$

Mean operator and Σ are interchangeable. also, every $u^T x$ is:

$$u^T x = u_1 x_1 + \cdots + u_n x_n$$

now, we want to calculate $\frac{\partial^k}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} z_x(u)$

$$\frac{\partial^k}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} Z_x(u)|_{u=0} = \frac{\partial^k}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \sum_{i=0}^{\infty} \langle \frac{(u^\top x)^i}{i!} \rangle|_{u=0}$$

$$= D^{(\vec{k})} (\sum_{i=0}^{k-1} \langle \frac{(u^T x)^i}{i!} \rangle + \frac{1}{k!} \langle (u^T x)^i \rangle + \sum_{i=k+1}^{\infty} \frac{1}{i!} \langle (u^T x)^i \rangle)|_{u=0}$$

$$= D^{(\vec{k})}(x) \sum_{i=0}^{k-1} \langle \frac{(u^T x)^i}{i!} + \langle \frac{1}{k!} (x_1 u_1 + \cdots + x_n u_n)^k \rangle + D^{(\vec{k})}(x) \sum_{i=k+1}^{(k)} \frac{1}{i!} \langle (u^T x)^i \rangle|_{u=0}$$

$$= \frac{1}{k!} (k)(k-1) \cdots (1) \langle x_1^{k_1} \cdots x_n^{k_n} \rangle|_{u=0} = \langle x_1^{k_1} \cdots x_n^{k_n} \rangle$$

2.1.2 Part b

We have:

$$Z_x(t) = E(e^{xt})$$

$$= \int e^{xt} \frac{1}{\sqrt{(2\pi\sigma^2)}} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2} dx$$

Define $z = \frac{x-\mu}{\sigma}$, which implies $x = z\sigma + \mu$.
We also have:

$$Z_z(t) = E(e^{zt}) = \int e^{zt} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$= e^{\frac{1}{2}t^2}.$$

Then, using the change-of-variable technique, we get

$$Z_x(t) = e^{\mu t} \int e^{z\sigma t} \frac{1}{\sqrt{(2\pi\sigma^2)}} e^{-\frac{1}{2}z^2} |\frac{dx}{dz}| dz$$

$$= e^{\mu t} \int e^{z\sigma t} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$= e^{\mu t} e^{\frac{1}{2}\sigma^2 t^2}$$

2.2 Exercise 2.2

The conditional distribution of any subset vector x, given the complement vector y, is a multivariate normal distribution

$$x \mid y \sim \mathcal{N}(\mu_{x|y}, \Sigma_{x|y})$$

Lets assume that by construction, the joint distribution of x and y are:

$$x, y \sim \mathcal{N}(\mu, \Sigma).$$

Where:

$$\mu = \left(\begin{array}{c} \mu_x \\ \mu_y \end{array} \right)$$

$$\Sigma = \left(\begin{array}{cc} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{array} \right) = \left(\begin{array}{cc} \mathrm{Var}(x) & \mathrm{Cov}(x,y) \\ \mathrm{Cov}(y,x) & \mathrm{Var}(y) \end{array} \right) = \left(\begin{array}{cc} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{array} \right)$$

The marginal distribution of y is

$$y \sim \mathcal{N}(\mu_y, \sigma_{yy}).$$

According to the law of conditional probability, it holds that

$$p(x \mid y) = \frac{p(x,y)}{p(y)}$$

By the definition, we have:

$$p(x \mid y) = \frac{\mathcal{N}(X; \mu, \Sigma)}{\mathcal{N}(y; \mu_y, \sigma_{yy})}.$$

Where:

$$X = \left(\begin{array}{c} x \\ y \end{array} \right)$$

Using the probability density function of the p-variate normal distribution, this becomes:

$$\begin{aligned} p(x \mid y) &= \frac{1/\sqrt{(2\pi)^n |\Sigma|} \cdot \exp[-\frac{1}{2}(X-\mu)^{\mathrm{T}}\Sigma^{-1}(X-\mu)]}{1/\sqrt{(2\pi)^{n_y} |\sigma_{yy}|} \cdot \exp[-\frac{1}{2}(y-\mu_y)^{\mathrm{T}}\sigma_{yy}^{-1}(y-\mu_y)]} \\ &= \frac{1}{\sqrt{(2\pi)^{n-n_y}}} \cdot \sqrt{\frac{|\sigma_{yy}|}{|\Sigma|}} \cdot \exp[-\frac{1}{2}(X-\mu)^{\mathrm{T}}\Sigma^{-1}(X-\mu) + \frac{1}{2}(y-\mu_y)^{\mathrm{T}}\sigma_{yy}^{-1}(y-\mu_y)]. \end{aligned}$$

Writing the inverse of Σ as

$$\Sigma^{-1} = \left[\begin{array}{cc} \sigma^{xx} & \sigma^{xy} \\ \sigma^{yx} & \sigma^{yy} \end{array} \right] = \left[\begin{array}{cc} (\sigma_{xx} - \sigma_{xy}\sigma_{yy}^{-1}\sigma_{yx})^{-1} & -(\sigma_{xx} - \sigma_{xy}\sigma_{yy}^{-1}\sigma_{yx})^{-1}\sigma_{xy}\sigma_{yy}^{-1} \\ -\sigma_{yy}^{-1}\sigma_{yx}(\sigma_{xx} - \sigma_{xy}\sigma_{yy}^{-1}\sigma_{yx})^{-1} & \sigma_{yy}^{-1} + \sigma_{yy}^{-1}\sigma_{yx}(\sigma_{xx} - \sigma_{xy}\sigma_{yy}^{-1}\sigma_{yx})^{-1}\sigma_{xy}\sigma_{yy}^{-1} \end{array} \right]$$

In the matrix form, we get:

$$\begin{aligned} p(x \mid y) = & \frac{1}{\sqrt{(2\pi)^{n-n_y}}} \cdot \sqrt{\frac{|\sigma_{yy}|}{|\Sigma|}} \cdot \\ & \exp[-\frac{1}{2}(\left[\begin{array}{c} x \\ y \end{array} \right] - \left[\begin{array}{c} \mu_x \\ \mu_y \end{array} \right])^{\mathrm{T}} \left[\begin{array}{cc} \sigma^{xx} & \sigma^{xy} \\ \sigma^{yx} & \sigma^{yy} \end{array} \right] (\left[\begin{array}{c} x \\ y \end{array} \right] - \left[\begin{array}{c} \mu_x \\ \mu_y \end{array} \right]). \\ & . \quad + \frac{1}{2}(y-\mu_y)^{\mathrm{T}}\sigma_{yy}^{-1}(y-\mu_y)]. \end{aligned}$$

By inserting the inverse and keeping in mind that both matrices are symmetric, we get:

$$
\begin{aligned}
p(x \mid y) = & \frac{1}{\sqrt{(2\pi)^{n-n_y}}} \cdot \sqrt{\frac{|\sigma_{yy}|}{|\Sigma|}} \cdot \\
& \exp[-\frac{1}{2}((x-\mu_x)^{\mathrm{T}}(\sigma_{xx}-\sigma_{xy}\sigma_{yy}^{-1}\sigma_{yx})^{-1}(x-\mu_x) - .. \\
& 2(x-\mu_x)^{\mathrm{T}}(\sigma_{xx}-\sigma_{xy}\sigma_{yy}^{-1}\sigma_{yx})^{-1}\sigma_{xy}\sigma_{yy}^{-1}(y-\mu_y)+ \\
& .(y-\mu_y)^{\mathrm{T}}[\sigma_{yy}^{-1}+\sigma_{yy}^{-1}\sigma_{yx}(\sigma_{xx}-\sigma_{xy}\sigma_{yy}^{-1}\sigma_{yx})^{-1}\sigma_{xy}\sigma_{yy}^{-1}](y-\mu_y)) \\
& .+\frac{1}{2}((y-\mu_y)^{\mathrm{T}}\sigma_{yy}^{-1}(y-\mu_y))].
\end{aligned}
$$

Eliminating some terms, we have:

$$
\begin{aligned}
p(x \mid y) = & \frac{1}{\sqrt{(2\pi)^{n-n_y}}} \cdot \sqrt{\frac{|\sigma_{yy}|}{|\Sigma|}} \cdot \\
& \exp[-\frac{1}{2}((x-\mu_x)^{\mathrm{T}}(\sigma_{xx}-\sigma_{xy}\sigma_{yy}^{-1}\sigma_{yx})^{-1}(x-\mu_x) - .. \\
& 2(x-\mu_x)^{\mathrm{T}}(\sigma_{xx}-\sigma_{xy}\sigma_{yy}^{-1}\sigma_{yx})^{-1}\sigma_{xy}\sigma_{yy}^{-1}(y-\mu_y)+ \\
& ..(y-\mu_y)^{\mathrm{T}}\sigma_{yy}^{-1}\sigma_{yx}(\sigma_{xx}-\sigma_{xy}\sigma_{yy}^{-1}\sigma_{yx})^{-1}\sigma_{xy}\sigma_{yy}^{-1}(y-\mu_y))].
\end{aligned}
$$

Rearranging the terms, we have

$$
\begin{aligned}
p(x \mid y) = & \frac{1}{\sqrt{(2\pi)^{n-n_y}}} \cdot \sqrt{\frac{|\sigma_{yy}|}{|\Sigma|}} \cdot \exp[-\frac{1}{2} \cdot . \\
& .[(x-\mu_x)-\sigma_{xy}\sigma_{yy}^{-1}(y-\mu_y)]^{\mathrm{T}}(\sigma_{xx}-\sigma_{xy}\sigma_{yy}^{-1}\sigma_{yx})^{-1}[(x-\mu_x)-\sigma_{xy}\sigma_{yy}^{-1}(y-\mu_y)]] \\
= & \frac{1}{\sqrt{(2\pi)^{n-n_y}}} \cdot \sqrt{\frac{|\sigma_{yy}|}{|\Sigma|}} \cdot \exp[-\frac{1}{2} \cdot . \\
& .[x-(\mu_x+\sigma_{xy}\sigma_{yy}^{-1}(y-\mu_y))]^{\mathrm{T}}(\sigma_{xx}-\sigma_{xy}\sigma_{yy}^{-1}\sigma_{yx})^{-1}[x-(\mu_x+\sigma_{xy}\sigma_{yy}^{-1}(y-\mu_y))]]
\end{aligned}
$$

where we have used the fact that $\sigma_{yx} = \sigma_{xy}^{\mathrm{T}}$, because Σ is a covariance matrix.
The determinant of a block matrix is

$$
\left| \begin{matrix} A & B \\ C & D \end{matrix} \right| = |D| \cdot |A - BD^{-1}C|,
$$

such that we have for Σ that

$$
\left| \begin{matrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{matrix} \right| = |\sigma_{yy}| \cdot |\sigma_{xx} - \sigma_{xy}\sigma_{yy}^{-1}\sigma_{yx}|.
$$

With this and $n - n_y = n_x$, we finally arrive at

$$
\begin{aligned}
p(x \mid y) = & \frac{1}{\sqrt{(2\pi)^{n_1}|\sigma_{xx}-\sigma_{xy}\sigma_{yy}^{-1}\sigma_{yx}|}} \cdot \exp[-\frac{1}{2}.. \\
& .[x-(\mu_x+\sigma_{xy}\sigma_{yy}^{-1}(y-\mu_y))]^{\mathrm{T}}(\sigma_{xx}-\sigma_{xy}\sigma_{yy}^{-1}\sigma_{yx})^{-1}[x-(\mu_x+\sigma_{xy}\sigma_{yy}^{-1}(y-\mu_y))]]
\end{aligned}
$$

which is the probability density function of a multivariate normal distribution

$$
p(x \mid y) = \mathcal{N}(x; \mu_{x|y}, \Sigma_{x|y})
$$

with the mean $\mu_{x|y}$ and covariance $\Sigma_{x|y}$ given by:

$$
\begin{aligned}
\mu_{x|y} &= \mu_x + \sigma_{xy}\sigma_{yy}^{-1}(y - \mu_y) \\
\Sigma_{x|y} &= \sigma_{xx} - \sigma_{xy}\sigma_{yy}^{-1}\sigma_{yx}
\end{aligned}
$$

By plugging this equation in to the probability, we get:

$$p(x,y) = \frac{\exp\{-\frac{1}{2(1-\rho^2)}[(\frac{x-\mu_x}{\sigma_x})^2 - 2\rho(\frac{x-\mu_x}{\sigma_x})(\frac{y-\mu_y}{\sigma_y}) + (\frac{y-\mu_y}{\sigma_y})^2]\}}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}$$

and

$$\langle y \mid x \rangle = \mu_y + \rho\frac{\sigma_y}{\sigma_x}(x - \mu_x)$$
$$\sigma^2_{y|x} = \sigma^2_y(1-\rho^2)$$

2.3 Exercise 2.3

2.3.1 Part a

Suppose $X \sim N_k(\mu, g)$ where g is full-rank variance covariance matrix, thus it is positive definite.
By definition of p-variate normal distribution, any linear combination of X has a univariate normal distribution. That is, $t^T X \sim N(t^T\mu, t^T\Sigma t)$ for any vector $t \in R^p$.
Moment generating function of some $Z \sim N(\mu, \sigma^2)$ is

$$M_Z(s) = E[e^{sZ}] = e^{\mu s + \sigma^2 s^2/2} \quad , s \in R$$

By cholesky decomposition, we can decompose $g = ll^T$ for some nonsingular matrix l since g is positive definite. Transform $X \mapsto Y$ such that $Y = l^{-1}(X - \mu)$, i.e. $X = \mu + lY$ Then it follows that $Y = (Y_1, \ldots, Y_p)^T \sim N_p(0, I_p)$. This is the same as a change of variables such that X is from normal distribution with zero mean a unit variance.
In other words $Y_1, \ldots, Y_p$ are independent standard normal.
Therefore using MGF of standard normal distribution,

$$\begin{aligned}
M_X(t) &= E[e^{t^T X}] \\
&= E[e^{t^T(\mu + lY)}] \\
&= e^{t^T\mu}E[e^{\ell^T Y}] \quad , \ell^T = t^T l \\
&= e^{t^T\mu}E[e^{\sum_{i=1}^p \ell_i Y_i}] \quad , \ell = (\ell_1, \ldots, \ell_p) \\
&= e^{t^T\mu}\prod_{i=1}^p E[e^{l_i Y_i}] \\
&= e^{t^T\mu}\prod_{i=1}^p e^{\ell_i^2/2} \\
&= \exp(\mu^T t + \frac{1}{2}\ell^T\ell) \\
&= \exp(\mu^T t + \frac{1}{2}t^T g t)
\end{aligned}$$

2.3.2 Part b

In exercise 2.1, we proved that:

$$.\frac{\partial^k}{\partial u_1^{k_1}\cdots\partial u_n^{k_n}} Z_{\mathbf{x}}(\mathbf{u})|_{\mathbf{u}=\mathbf{0}} = \langle x_1^{k_1}\cdots x_n^{k_n}\rangle$$

Now, consider the case of 4 variates of n-variational Gaussian MGF. first of all, we can zero-mean all the X by a n-dimensional constant shift vector. Then:

$$.\frac{\partial^4}{\partial u_i \partial u_j \partial u_k \partial u_l} Z_{\mathbf{x}}(\mathbf{u})|_{\mathbf{u}=\mathbf{0}} = \langle x_i x_j x_k x_l\rangle$$

Also, we calculated MGF as below:

$$\exp(\mu^T t + \frac{1}{2} t^T g t)$$

By differentiating the same as Exercise 2.1, we get to

$$\langle x_1^{k_1} \cdots x_n^{k_n} \rangle = \frac{1}{4!} (\prod_{i',j',k',l' \in i,j,k,l} g_{i'j'} g_{k'l'}) = g_{ij} g_{kl} + g_{ik} g_{jl} + g_{il} g_{jk}$$

The last equality is true because of symmetry of Matrix g.

2.4 Exercise 2.4

2.4.1 Brownian motion

The Chapman-Kolmogorov equation for Brownian motion with setting initial condition at $0, 0$

$$p(x, t \mid 0, 0) = \int_{-\infty}^{\infty} p(x, t \mid x', t') p(x', t' \mid 0, 0) dx'$$

where $p(x, t \mid x', t')$ is the transitional probability of Brownian motion. Since

$$p(x, t \mid x', t') = P(X(t) = x \mid X(t') = x') = \frac{1}{\sqrt{2\pi(t - t')}} e^{-\frac{(x-x')^2}{2(t-t')}}$$

In other words, We need to show

$$\frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(t - t')}} e^{-\frac{(x-x')^2}{2(t-t')}} \frac{1}{\sqrt{2\pi t'}} e^{-\frac{x'^2}{2s}} dx'$$

By simplifying the integral:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(t - t')}} e^{-\frac{(x-x')^2}{2(t-t')}} \frac{1}{\sqrt{2\pi t'}} e^{-\frac{x'^2}{2t'}} dx' &= \frac{1}{2\pi \sqrt{t'(t - t')}} \int_{-\infty}^{\infty} e^{-\frac{(x-x')^2}{2(t-t')}} e^{-\frac{x'^2}{2t'}} dx' \\
&= \frac{1}{2\pi \sqrt{t'(t - t')}} \int_{-\infty}^{\infty} \exp(-\frac{x^2}{2t}) \exp(\frac{t}{2t'(t' - t)} (x' - \frac{xt'}{t})^2) dx' \\
&= \frac{1}{\sqrt{2\pi t}} \exp(-\frac{x^2}{2t})
\end{aligned}$$

Which shows that Brownian motion satisfies Chapman-Kolmogorov equation.

2.4.2 Cauchy Process

We need to show that:

$$p(x, t \mid 0, 0) = \int_{-\infty}^{\infty} p(x, t \mid x', t') p(x', t' \mid 0, 0) dx'$$

By putting the Cauchy process into the transitional probability, we are getting:

$$p(x, t \mid x', t') = P(X(t) = x \mid X(t') = x') = \frac{t' \cdot (t - t')}{\pi^2 \cdot ((x - x')^2 + (t - t')^2)(x'^2 + t'^2)}$$

Let's put them into the integral:

$$p(x, t \mid 0, 0) = \int_{-\infty}^{\infty} \frac{t' \cdot (t - t')}{\pi^2 \cdot ((x - x')^2 + (t - t')^2)(x'^2 + t'^2)} dx'$$

By using Mathematica to solve this integral, the resault is:

$$\begin{aligned}
&= \frac{1}{[\pi^2 \cdot (x^2 + (t - 2t')^2)(x^2 + t^2)|t - t'|} ((t - t') \\
&(|t - t'|(t' x \ln((x' - x)^2 + (t - t')^2) - t' x \ln(x'^2 + t'^2) + (-x^2 - t \cdot (t - 2t')) \arctan(\frac{x'}{t'})) \\
&t' \cdot (x^2 - t \cdot (t - 2t')) \arctan(\frac{x' - x}{|t - t'|})))
\end{aligned}$$

By putting the upper and lower limit of integral, this can be rewrote in this form:

$$\frac{(t-t')(2tx^2+2t^3-8t't^2+8t'^2t)}{\pi\cdot(2t-2t')(x^4+(2t^2-4t't+4t'^2)x^2+t^4-4t't^3+4t'^2t^2)}$$

And by simplifying one gets to:

$$\frac{t}{\pi\cdot(x^2+t^2)}$$

Which is the desired solution to Chapman-Kolmogorov equation.

3 Chapter 3 exercises

3.1 Exercise 3.1

Let's start with Kramers-Moyal expantion.

$$\frac{\partial P(x,t)}{\partial t}\sum_{n=1}^{\infty}(-\frac{\partial}{\partial x})^n[D^{(n)}(x,t)P(x,t)\mid .$$

$$\int x^n\frac{\partial P'(x,t)}{\partial t}dx=\frac{\partial}{\partial t}\int x^nP(x,t)dx=\frac{\partial}{\partial t}\langle x^n\rangle$$

$$\int x^n\sum_{k=1}^{\infty}(-\frac{\partial}{\partial x})^k[D^{(k)}(x,t)P(x,t)]dx$$

$$=\sum_{k=1}^{\infty}\int x^n(-\frac{\partial}{\partial x})^k[D^{(k)}(x,t)P(x,t)]dx$$

$$=\sum_{k=1}^{\infty}\int(-\frac{\partial}{\partial x})[x^n(-\frac{\partial}{\partial x})^{k-1}[D^{(k)}(x,t)P(x,t)]]dx$$

$$+\sum_{k=1}^{\infty}\int nx^{n-1}(-\frac{\partial}{\partial x})^{k-1}[D^{(k)}(x,t)P(x,t)]dx$$

Now, with recursion:

$$=-\int\frac{\partial}{\partial x}\{\sum_{k=1}^{\infty}[x^n(-\frac{\partial}{\partial x})^{k-1}+nx^n(-\frac{\partial}{\partial x})^{k-2}][D^{(k)}(x,t)P(x,t)]\}dx$$

$$+\sum_{k=1}^{\infty}\int n(n-1)x^{n-2}(-\frac{\partial}{\partial x})^{k-2}[D^{(k)}(x,t)P(x,t)]dx$$

Now, this can be rewritten in the form:

$$-\int\frac{\partial}{\partial x}\{\sum_{k=1}^{\infty}[x^n(-\frac{\partial}{\partial x})^{k-1}..+nx^{n-1}(-\frac{\partial}{\partial x})^{k-2}+\cdots$$

$$...+n(n-1)\cdots(n-k+2)x^{n-k+1}][D^{(k)}(x,t)p(x,t)]\}dx$$

$$+\sum_{k=1}^{\infty}n(n-1)\cdots(n-k+1)\int x^{n-k}D^{(k)}(x,t)P(x,t)dx$$

The first term, is always zero. And the second term, is equivalent to:

$$\sum_{k=1}^{n}\frac{n!}{(n-k)!}\langle x^{n-k}D^{(k)}(x,t)\rangle$$

Which proves:

$$\frac{\partial}{\partial t}\langle x^n\rangle=\sum_{k=1}^{n}\frac{n!}{(n-k)!}\langle x^{n-k}D^{(k)}(x,t)\rangle$$

3.2 Exercise 3.2

3.2.1 Part a

We are going to demonstrate one step, then by induction, it will be true for all steps.

$$P(x,t) = \int P(x,t;x',t')dx'$$
$$P(x,t;x',t') = P(x,t \mid x',t')P(x',t')$$

By chapman-kolmogorov equation we can write:

$$P(x,t \mid x',t') = \int P(x,t \mid x'',t'')P(x'',t'', \mid x',t')dx'' \quad (t > t'' > t')$$
$$\longrightarrow P(x,t) = \int P(x,t \mid x',t')P(x',t')dx'$$
$$= \int dx' \int dx'' P(x,t \mid x'',t'')P(x'',t'' \mid x',t') \quad (t > t'' > t')$$

now by defining the $\tau = (t - t_0)/N$ and $t_n = t_0 + n\tau$, we can do this for any given N.

$$p(x,t) = \lim_{Narrow\infty} \int dx_{N-1} \cdots \int dx_0 p(x,t \mid x_{N-1},t_{N-1}) \cdots p(x_1,t_1 \mid x_0,t_0)p(x_0,t_0).$$

3.2.2 Part b

Let's begin by The Fokker-Plank Equation and calculate for small τ :

$$\frac{\partial}{\partial t}P(x,t \mid x',t') = [-\frac{\partial}{\partial x}D^{(\prime)}(x,t) + \frac{\partial^2}{\partial x^2}D^{(2)}(x,t)]P(x,t \mid x',t')$$

Use the definition of $\mathcal{L}_{FP}$ and write for small τ :

$$P(x,t+\tau \mid x',t) = e^{\tau\mathcal{L}_{FP}}P(x,t \mid x',t') = e^{\tau\mathcal{L}_{FP}}\delta(x-x'), \delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{iu(x-x')}du$$

Now using Kramers-Moyal Coefficients:

$$P(x,t+\tau \mid x',t) = \{1 - \tau\frac{\partial}{\partial x}D^{(1)}(x,t) + \tau\frac{\partial^2}{\partial x^2}D^{(2)}(x,t) + O(\tau^2)\}\delta(x-x')$$

In the integral form:

$$arrow P(x,t+\tau \mid x',t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{iu(x-x')}[1 - iu\tau D^{(1)}(x',t) - u^2\tau D^{(2)}(x',t), O(\tau^2)]du$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{iu(x-x')-iu\tau D^{(1)}(x',t)-u^2D^{(2)}(x',t)\tau^2}du$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} exp\{\tau D^{(2)}(x',t)[-(u - \frac{i(x-x'-\tau D^{(1)}(x',t).}{2\tau D^{(2)}(x',t)})^2 - (\frac{x-x'-\tau D^{(1)}(x',t)}{2\tau D^{(2)}(x',t)})^2]\}du$$

Then we get to:

$$p(x,t) = \lim_{Narrow\infty} \int \underbrace{\cdots}_{\text{N times}} \int \{\Pi_{i=0}^{N-1}(4\pi D^{(2)}(x_i,t_i)\tau)^{-1/2}dx_i\} \exp(-\sum_{i=0}^{N-1}\frac{[x_{i+1}-x_i-D^{(1)}(x_i,t_i)]^2}{4D^{(2)}(x_i,t_i)\tau})p(x_0,t_0)$$

Now, for the derivative:

$$\lim_{Narrow\infty}\sum_{i=1}^{N-1}\frac{(x_{i+1}-x_i-\tau D^{(1)}(x_i,t_i))}{4\tau D^{(2)}(x_i,t_i)} = \lim_{Narrow\infty}\sum_{i=1}^{N-1}\frac{1}{4D^{(2)}(x_i,t_i)}\lfloor\frac{x_{i.1}-x_i}{\tau} - D^{(1)}(x_i,t_i)]$$
$$= \int_{t_0}^{t}\frac{\dot{x}(t') - D^{(1)}(x(t'),t'.}{4D^{(2)}(x(t',t').}dt'$$
$$\lim_{Narrow\infty}\sum_{i=0}^{N-1}\frac{[x_{i+1}-x_i-D^{(1)}(x_i,t_i)\tau]^2}{4D^{(2)}(x_i,t_i)\tau} = \int_{t_0}^{t}\frac{[\dot{x}(t') - D^{(1)}(x(t'),t')]^2}{4D^{(2)}(x(t'),t')}dt'$$

3.2.3 Part c

Both exponential and KM-coefficients part are positive, The integral must remain positive for all t.

3.3 Exercise 3.3

3.3.1 Part a

From CK equation:

$$P(x,t \mid x',t') = \int P(x,t|x'',t'+\tau)P(x'',t'+\tau \mid x',t')dx'' = \int dx'' P(x,t \mid x'',t'+\tau) \int dy \delta(y-x'')P(y,t'+\tau \mid x',t')$$

Now, by using $P(x,t+\tau \mid x',t) = \int \delta(y-x)P(y,t+\tau \mid x',t)dy$ and Taylor expansion of delta distribution:

$$\delta(y-x'') = \delta(x'-x''+y-x') = \sum_{n=0}^{\infty} \frac{(y-x')^n}{n!} (\frac{\partial}{\partial x'})^n \delta(x'-x'')$$

We get to:

$$P(x,t \mid x',t') = \int dx'' P(x,t \mid x'',t'+\tau) \int dy \delta(y-x'')P(y,t'+\tau \mid x',t')$$

$$= \int dx'' P(x,t \mid x'',t'+\tau)[\sum_{n=1}^{\infty} \frac{1}{n!} \int dy (y-x')^n P(y,t'+\tau \mid x',t')(\frac{\partial}{\partial x'})^n \delta(x'-x'')]$$

$$= \int dx'' P(x,t \mid x'',t'+\tau)[\int dy P(y,t'+\tau \mid x',t') + \sum_{n=1}^{\infty} \frac{1}{n!}(\int dy (y-x')^n P(y,t'+\tau \mid x',t'))(\frac{\partial}{\partial x'})^n]\delta(x'-x'')$$

Also, $K^{(n)}(x',t',\tau) = \int dy (y-x')^n P(y,t'+\tau \mid x',t')$. Then we can rewrite:

$$= [1 + \sum_{n=1}^{\infty} \frac{K^{(n)}(x',t',\tau)}{n!}(\frac{\partial}{\partial x'})^n] \int dx'' P(x,t \mid x'',t'+\tau)\delta(x'-x'')$$

$$= P(x,t \mid x',t'+\tau) + (\sum_{n=1}^{\infty} \frac{K^{(n)}(x',t',\tau)}{n!}(\frac{\partial}{\partial x'})^n)P(x,t \mid x',t'+\tau)$$

For difference of probability:

$$P(x,t \mid x',t') - P(x,t \mid x',t'+\tau) = (\sum_{n=1}^{\infty} \frac{K^{(n)}(x',t',\tau)}{n!}(\frac{\partial}{\partial x'})^n)P(x,t \mid x',t'+\tau)$$

As for the limit:

$$\lim_{\tau arrow 0} \frac{P(x,t \mid x',t') - P(x,t \mid x',t'+\tau)}{\tau} = -\frac{\partial}{\partial t}P(x,t \mid x',t') = \sum_{n=1}^{\infty} \frac{1}{n!} \lim_{\tau arrow 0} \frac{K^{(n)}(x',t',\tau)}{\tau}(\frac{\partial}{\partial x'})^n P(x,t \mid x',t'+\tau)$$

By putting $D^{(n)}(x',t') = \frac{1}{n!} \lim_{\tau arrow 0} \frac{K^{(n)}(x',t',\tau)}{\tau}$ We get to the desired relation:

$$\frac{\partial}{\partial t}P(x,t \mid x',t') = -\sum_{n=1}^{\infty} D^{(n)}(x',t')(\frac{\partial}{\partial x'})^n P(x,t \mid x',t')$$

3.3.2 Part b

Let's construct The adjoint operator, directly.

$$\int h(x')\mathcal{L}_{KM}f(x')dx' = \sum_{n=1}^{\infty}\int h(x')(-\frac{\partial}{\partial x})^n D^{(n)}(x',t')f(x')dx'$$
$$= \sum_{n=1}^{\infty}[-h(x')(-\frac{\partial}{\partial x})^{n-1}D^{(n)}(x't')f(x') + \int(\frac{\partial}{\partial x'}h(x'))(-\frac{\partial}{\partial x'})D^{(n)}(x',t')f(x')dx']$$
$$= \sum_{n=1}^{\infty}\int D^{(n)}(x',t')((\frac{\partial}{\partial x'})^n h(x'))f(x')dx' =$$
$$\int f(x')[\sum_{n=1}^{\infty}(\frac{\partial}{\partial x'})^n D^{(n)}(x',t')]h(x')dx' = \int f(x')\mathcal{L}_{KM}^{+}h(x')dx'$$

By defining $\mathcal{L}_{KM}^{+} = \sum_{n=1}^{\infty}(\frac{\partial}{\partial x'})^n D^{(n)}(x',t')$ we get to the desired equation.

3.4 Exercise 3.4

3.4.1 Part a

Let's start by Master equation:

$$\frac{\partial}{\partial t}P(n,t) = \sum_{n'}(\omega_{n,n'}P_{n'}(t) - \omega_{n',n}P_n(t))$$

With $\omega_{n,n'} = \frac{1}{2\Delta t}$, we get to:

$$(\frac{1}{\Delta t})(P(n,N)-P(n,N-1)) = \omega_{n,n+1}P(n+1,N-1)+\omega_{n,n-1}P(n-1,N-1)-\omega_{n+1,n}P(n,N-1)-\omega_{n-1,n}P(n,N-1)$$

Transitions of random walk are from neighbours, so that:

$$P(i,N) - P(i,N-1) = \frac{1}{2}(P(i+1,N-1)P(i-1,N-1)) - P(i,N-1)$$

And we get to the desired answer:

$$P(i,N) = \frac{1}{2}[P(i+1,N-1) + P(i-1,N-1)]$$

As the question asked for.

3.4.2 Part b

Above equation in terms of continuity is written as:

$$P(x,t) = \frac{1}{2}(P(x+a,t-\tau) + P(x-a,t-\tau))$$

Now, let's calculate the derivatives:

$$\tau \lim_{\tau arrow 0}\frac{P(x,t)-P(x,t-\tau)}{\tau} = \frac{a^2}{2}\lim_{a arrow 0}\frac{1}{a^2}[P(x+a,t)+P(x-a,t)-2P(x,t)]$$

By making parameter $D = \frac{a^2}{2\tau}$ we get to the partial deviates with respect to units of space and time, we get to the equation.

$$\frac{\partial}{\partial t}P = D\frac{\partial^2 P}{\partial x^2}$$

3.4.3 Part c

We got to the diffusion equation:

$$\frac{\partial}{\partial t}P(x,t) = D\frac{\partial^2}{\partial x^2}P(x,t)$$

The diffusion equation is a partial differential equation(Also known as heat equation). The unknown quantity is a function $P(x,t)$. To complete the problem statement we need to specify an initial condition (at $t = 0$) and boundary conditions. boundary conditions are at infinity, so we take

$$P(x,t) \to 0, x \to \pm\infty.$$

We take a delta function initial condition:

$$P(x,0) = \delta(x).$$

The equation can be solved by using the Fourier transform:

$$P(x,t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \mathrm{e}^{ikx} P_k(t) dk$$

The inverse transform is

$$P_k(t) = \int_{-\infty}^{\infty} \mathrm{e}^{-ikx} P(x,t) dx$$

So the transform of the initial condition is

$$P_k(0) = 1$$

Substituting $P(x,t)$ in the diffusion equation gives

$$\int_{-\infty}^{\infty} \frac{1}{2\pi} \mathrm{e}^{ikx} (\dot{P}_k(t) + Dk^2 P_k(t)) dk = 0$$

This simplifies to

$$\dot{P}_k(t) + Dk^2 P_k(t) = 0$$

With the solution

$$P_k(t) = P_k(0)\mathrm{e}^{-Dk^2 t} = \mathrm{e}^{-Dk^2 t}$$

Putting it all together

$$P(x,t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \mathrm{e}^{ikx} \mathrm{e}^{-Dk^2 t} dk$$

And all that's left is to do the k integral. Note that the k integral is a Gaussian:

$$\int_{-\infty}^{\infty} \mathrm{e}^{-y^2} dy = \sqrt{\pi}.$$

We should get

$$P(x,t) = \frac{1}{\sqrt{4\pi Dt}} \mathrm{e}^{-\frac{x^2}{4Dt}}$$

3.4.4 Part d

We should do the same calculations as above.

$$P(x', t+\tau | x, t) = \delta(x' - x)$$

Transformation of initial condition is still same. now we should substitute it in the diffusion equation:

$$\int_{-\infty}^{\infty} \frac{1}{2\pi} \mathrm{e}^{ik(x'-x)} (\dot{P}_k(\tau) + Dk^2 P_k(\tau)) dk = 0$$

Which simplifies to:

$$\dot{P}_k(\tau) + Dk^2 P_k(\tau) = 0$$

This has the same solution as above, but keep in mind that this time, $t \to \tau$ and k is adjoint Fourier variable of $x' - x$ instead of x.
So we get to the answer as:

$$P(x', t+\tau | x, t) = \frac{1}{\sqrt{4\pi D\tau}} \mathrm{e}^{-\frac{(x'-x)^2}{4D\tau}}$$

3.4.5 Part e

Starting by:

$$P(x,t) = \frac{1}{\sqrt{4\pi Dt}} \mathrm{e}^{-\frac{x^2}{4Dt}}$$

Notice that this is similar to a Gaussian distribution. So let's prove a more general case of this distribution that initial condition is not as $x_0 = 0$, instead, it is starting at μ which we know that mean of a random walk, is the position of start. so this is a solution for the conditional part aswell. Change of variables $\sigma = \sqrt{2D\tau}$

$$\begin{aligned}
\langle x^2 \rangle &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp(-\frac{(x-\mu)^2}{2\sigma^2})\mathrm{d}x - \mu^2 \\
&= \frac{\sqrt{2}\sigma}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sqrt{2}\sigma t + \mu)^2 \exp(-t^2)\mathrm{d}t - \mu^2 \\
&= \frac{1}{\sqrt{\pi}}(2\sigma^2 \int_{-\infty}^{\infty} t^2 \exp(-t^2)\mathrm{d}t + 2\sqrt{2}\sigma\mu \int_{-\infty}^{\infty} t\exp(-t^2)\mathrm{d}t + \mu^2 \int_{-\infty}^{\infty} \exp(-t^2)\mathrm{d}t) - \mu^2 \\
&= \frac{1}{\sqrt{\pi}}(2\sigma^2 \int_{-\infty}^{\infty} t^2 \exp(-t^2)\mathrm{d}t + 2\sqrt{2}\sigma\mu[-\frac{1}{2}\exp(-t^2)]_{-\infty}^{\infty} + \mu^2\sqrt{\pi}) - \mu^2 \\
&= \frac{1}{\sqrt{\pi}}(2\sigma^2 \int_{-\infty}^{\infty} t^2 \exp(-t^2)\mathrm{d}t + 2\sqrt{2}\sigma\mu \cdot 0) + \mu^2 - \mu^2 \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^2 \exp(-t^2)\mathrm{d}t \\
&= \frac{2\sigma^2}{\sqrt{\pi}}([-\frac{t}{2}\exp(-t^2)]_{-\infty}^{\infty} + \frac{1}{2}\int_{-\infty}^{\infty} \exp(-t^2)\mathrm{d}t) \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2} \int_{-\infty}^{\infty} \exp(-t^2)\mathrm{d}t \\
&= \frac{2\sigma^2\sqrt{\pi}}{2\sqrt{\pi}} \\
&= \sigma^2 = 2D\tau
\end{aligned}$$

3.5 Exercise 3.5

3.5.1 Part a

The master equation:

$$\frac{\partial}{\partial t} P_n(t) = \sum_{n'} (\omega_{n,n'} P_{n'}(t) - \omega_{n',n} P_n(t))$$

Question's hypothesis:

$$\omega_{n,n'} = \lambda \delta_{n,n'+1}$$

Which λ is > 0.
Putting $\omega_{n,n'}$ in master equation:

$$\frac{\partial}{\partial t} P_n(t) = \sum_{n'} (\lambda\delta_{n,n'+1} P_{n'}(t) - \lambda\delta_{n'+1,n} P_n(t))$$

We get to:

$$\dot{P}_n(t) = \lambda P_{n-1}(t) - \lambda P_n(t).$$

As the question needed.

3.5.2 Part b

We put the given equation in question in the equation we got in last part:

$$\dot{P}_n = \frac{d}{dt}[\frac{(\lambda t)^n}{n!}\exp(-\lambda t)] = \frac{n\lambda(\lambda t)^{n-1}}{n!}\exp(-\lambda t) - \frac{\lambda(\lambda t)^n}{n!}\exp(-\lambda t) = \lambda P_{n-1} - \lambda P_n$$

3.5.3 Part c

for $\langle n \rangle$ we can write:

$$\begin{aligned}
\langle n \rangle &= \sum n \frac{(\lambda t)^n}{n!} e^{-\lambda t} \\
&= \lambda \sum \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t} \\
&= \lambda e^{-\lambda t} \sum \frac{(\lambda t)^{n-1}}{(n-1)!} \\
&= \lambda e^{-\lambda t} \sum \frac{(\lambda t)^m}{m!} \\
&= \lambda e^{-\lambda t} (\lambda t)^n e^{-\lambda t} \\
&= \lambda e^{-\lambda t} e^{\lambda t} \\
&= \lambda
\end{aligned}$$

And also for $\langle n^2 \rangle$ we can write:

$$\begin{aligned}
\langle n^2 \rangle &= \sum n^2 \frac{(\lambda t)^n}{n!} e^{-\lambda t} \\
&= \sum \frac{n}{(n-1)!} (\lambda t) e^{-\lambda t} \\
&= \sum \frac{n-1+1}{n-1} \frac{1}{(n-2)!} (\lambda t)^n e^{-\lambda t} \\
&= \lambda^2 \sum \frac{(\lambda t)^{n-2}}{(n-2)!} e^{-\lambda t} + \lambda \sum \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t} \\
&= \lambda^2 + \lambda
\end{aligned}$$

So variance is the same as mean $\sigma^2 = \langle n^2 \rangle - (\langle n \rangle)^2 = \lambda$.