Green function and its applications

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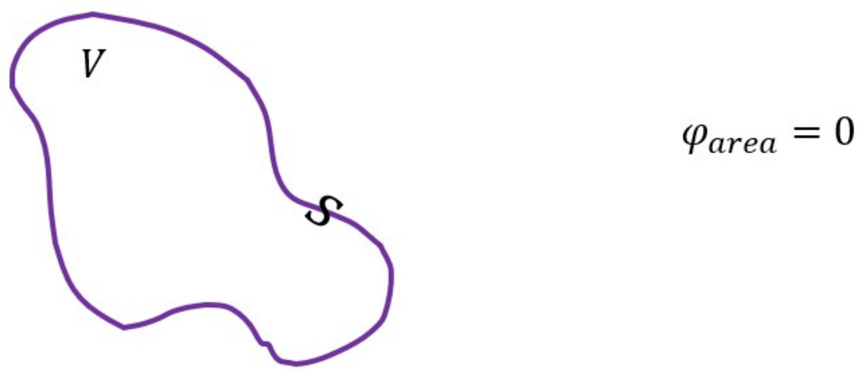
According to Poisson equation, We have:

$$\vec{\nabla}^2\varphi = -4\pi\rho$$

Where there is no charge, the equation changes to the Laplace equation:

$$\vec{\nabla}^2\varphi = 0$$

Suppose we have a charge distribution which boundary conditions are clear. We



provide the following answer to solve:

$$\phi(\vec{\mathbf{x}}) = \int \rho(\vec{\mathbf{x}}')G(\vec{\mathbf{x}},\vec{\mathbf{x}}')\, d^3x$$

That $G(\vec{\mathbf{x}},\vec{\mathbf{x}}')$ is a Green function. Logically ϕ should work in the Poisson equation.

$$\vec{\nabla}^2\phi(\vec{\mathbf{x}}) = \int \rho(\vec{\mathbf{x}}')\vec{\nabla}^2 G(\vec{\mathbf{x}},\vec{\mathbf{x}}')\, d^3x \ = -4\pi\rho(\vec{\mathbf{x}})$$

We know that:

$$\int f(x')\delta(x-x')\, d^3x^prime \ = f(x)$$

According to this:

$$\vec{\nabla}^2 G(\vec{\mathbf{x}},\vec{\mathbf{x}}') = -\delta(x-x')/\epsilon_0, G(\vec{\mathbf{x}},\vec{\mathbf{x}}')_{area} = 0$$

But this answer doesn't work in general. For the general answer, we use the Green theorem, which is a subset of the divergence theorem. Green's theorem states:

$$\vec{\mathbf{F}} = \varphi\vec{\nabla}\psi - \psi\vec{\nabla}\varphi$$

$$\Rightarrow \int_V (\varphi(x)\vec{\nabla}^2\psi(x) - \psi(x)\vec{\nabla}^2\varphi(x))\, d^3x \; = \oint_s (\varphi(x)\frac{\partial\psi}{\partial n} - \psi(x)\frac{\partial\varphi}{\partial n})\, ds$$

If we consider ψ as G:

$$\int (\rho(\vec{\mathbf{x}}')\vec{\nabla}'^2 G(x,x') - G(x,x')\vec{\nabla}'^2\rho(\vec{\mathbf{x}}'))\, d^3x' \; =$$

$$\oint_s (\varphi(x')\frac{\partial G(x')}{\partial n'} - G(x')\frac{\partial\varphi}{\partial n'})\, ds' - 4\pi\rho(x) + 4\pi\int_V \rho(x')G(x,x')\, d^3x' \; =$$

$$\varphi(x) = \int_V \rho(\mathbf{x}')G(x,x')\, d^3x \; - 1/4\pi \oint_s (\varphi(x')\frac{\partial G(x,x')}{\partial n'} - G(x,x')\frac{\partial\varphi(x')}{\partial n'}\, ds$$

If φ and G are on the zero surface, the surface integral is zero.But the freedom to define G creates different situations

1 Dirichelt's Boundary Condition

$$G_D(x,x') = 0 \qquad\qquad x' \; On \; the \; surface$$

As a result, the sentence containing G in the surface integral becomes zero. So we will have:

$$\varphi(x) = \int_V \rho(x')G(x,x')\, d^3x \; - \frac{1}{4\pi}\oint_s (\varphi(x')\frac{\partial G(x,x')}{\partial n'}\, ds'$$

2 Neumann boundary conditions

$$\frac{\partial G_N}{\partial N} = \frac{-4\pi}{s} \qquad\qquad on \; the \; surface$$

Here s is the area, which the integral is taken on

$$\Rightarrow \varphi(x) = \int_V \rho(x')G(x,x')\, d^3x' \; + \frac{1}{4\pi}\oint_s G_N(x,x')\frac{\partial\varphi(x')}{\partial n'}\, ds + \frac{1}{s}\oint_s (\varphi(x')\, ds$$

Now we want to look at The Green Contradiction Theorem:
We know that $E = -\boldsymbol{\nabla} V$, $\boldsymbol{\nabla}^2 V = -\rho/\epsilon_0$
If we have ρ_1, ρ_2 to create E_1, E_2:

$$\int_{space} (\vec{\mathbf{E}}_1.\vec{\mathbf{E}}_2)\, d\tau \ = \int_{space} (\vec{\boldsymbol{\nabla}} V_1).(\vec{\boldsymbol{\nabla}} V_2)\, d\tau$$

We know that : $\vec{\boldsymbol{\nabla}}.(V_1 \vec{\boldsymbol{\nabla}} V_2) = (\vec{\boldsymbol{\nabla}} V_1).(\vec{\boldsymbol{\nabla}} V_2) + V_1(\boldsymbol{\nabla}^2 V_2)$

$$\Rightarrow \int (\vec{\mathbf{E}}_1.\vec{\mathbf{E}}_2)\, d\tau \ = \int \boldsymbol{\nabla}.(V_1 \vec{\boldsymbol{\nabla}} V_2)\, d\tau \ - \int V_1(\boldsymbol{\nabla}^2 V_2)\, d\tau$$

We know that: $\int (\vec{\boldsymbol{\nabla}}.\vec{\mathbf{A}})\, d\tau \ = \oint_s \vec{\mathbf{A}}.\, \vec{\mathbf{d}} a$

$$\Rightarrow \int (\vec{\mathbf{E}}_1.\vec{\mathbf{E}}_2)\, d\tau \ = \oint_s V_1(\boldsymbol{\nabla}^2 V_2)\, da \ + 1/\epsilon_0 \int V_1 \rho_2\, d\tau$$

If we take the term surface integral on a large sphere that covers all space and according to the assumptions there V_1 and V_2 tend to zero, the integral becomes zero, so:

$$\int (\vec{\mathbf{E}}_1.\vec{\mathbf{E}}_2)\, d\tau \ = \frac{1}{\epsilon_0} \int V_1 \rho_2\, d\tau$$

Now by using $\vec{\boldsymbol{\nabla}}.(V_1 \vec{\boldsymbol{\nabla}} V_2) = (\vec{\boldsymbol{\nabla}} V_1).(\vec{\boldsymbol{\nabla}} V_2) + V_1(\boldsymbol{\nabla}^2 V_2)$ we can prove (like the previous part)

$$\int (\vec{\mathbf{E}}_1.\vec{\mathbf{E}}_2)\, d\tau \ = \frac{1}{\epsilon_0} \int V_2 \rho_1\, d\tau$$

By equating the last two equations, we have:

$$\int V_1 \rho_2\, d\tau \ = \int V_2 \rho_1\, d\tau$$

This is not just a matter of bulk density, but more generally, if we have two Surface and volumetric load distribution of ρ, σ which create the potential for φ, And two Surface and volumetric load distribution of ρ, σ' which create the potential for φ', we will have:

$$\int_V \varphi \rho'\, d\tau \ + \int_s \varphi \sigma'\, ds \ = \int_V \varphi' \rho\, d\tau \ + \int_s \varphi' \sigma\, ds$$

3 Green's theorem in conductors

If ρ_1, ρ_2 and that are scattered on conductors in space, we know that the surface potential is constant for the conductors and the surface conductor is the equipotential surface. As a result, V can be subtracted from the integral. The remainder of the integral is simply the charge of each conductor.

Assuming we have N conductor in space:

$$\int_{space} \rho_1 V_2 \, d\tau \ = \Sigma_{j=1}^{N} Q_{1j} V_{2j}$$

Now if in second case we have charge Q_{2j} on these conductors:

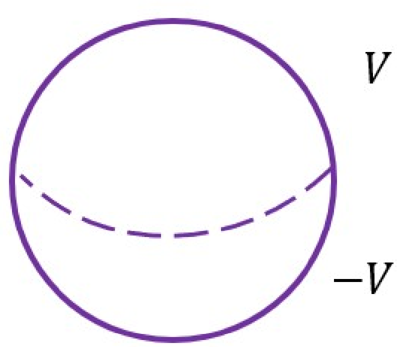
$$\int_{space} \rho_2 V_1 \, d\tau \ = \Sigma_{j=1}^{N} Q_{2j} V_{1j}$$

According to the equation obtained earlier, we have:

$$\Sigma_{j=1}^{N} Q_{1j} V_{2j} = \Sigma_{j=1}^{N} Q_{2j} V_{1j}$$

4 Now we consider another use of the Green function:

Suppose a spherical metal shell with the northern hemisphere at potential 1 and the southern hemisphere at potential 2. We have two Green functions, one for the potential outside the sphere and the other for the potential inside the sphere.
Here we have the Dirichelt's boundary conditions (Because the value of the function is known at the boundary.).



$$\phi(\vec{\mathbf{x}}) = \int_V \rho(\vec{\mathbf{x}}) G(\vec{\mathbf{x}}, \vec{\mathbf{x}}') \, d^3x' \ - \frac{1}{4\pi} \oint_s \varphi(\vec{\mathbf{x}}') \frac{\partial G}{\partial n'} \, da'$$

Here we have no charge density on the volume. So:

$$\int_V \rho(\vec{\mathbf{x}})G(\vec{\mathbf{x}},\vec{\mathbf{x}}')\,d^3x' \;=0$$

$$\Rightarrow \phi(\vec{\mathbf{x}}) = -\frac{1}{4\pi}\oint_s \varphi(\vec{\mathbf{x}}')\frac{\partial G}{\partial n'}\,da$$

Here we need to get the Green function for a sphere.

5 Solve the problem of electrostatic's boundary value with Green function.

$$\nabla'^2(\frac{1}{\left|x-x'\right|}) = -4\pi\delta(x-x')$$

Function $\left|x-x'\right|$ is one of the functions that depends on x and x'. This function is called Green.

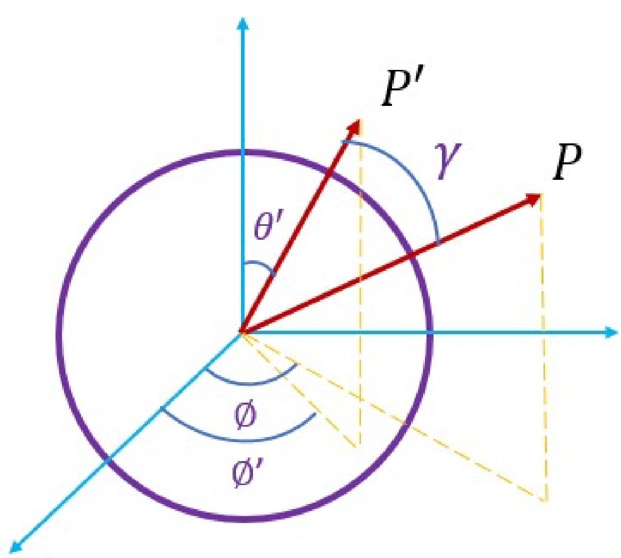
$$\nabla'^2 = -4\pi\delta(x-x')$$

$$G(x,x') = (\frac{1}{\left|x-x'\right|}) + F(x-x')$$

Logically $$\nabla'^2 F(x,x') = 0$$

Now we get the Green function for a sphere in the presence of a point charge. And write the function as:

$$G(x,x') = (\frac{1}{|x-x'|}) - \frac{a}{x(|x-(\frac{a'^2}{x'^2})x|)}$$



We can write this function based on spherical coordinates.

$$G(x,x') = (\frac{1}{(x^2+x'^2-2xx'\cos(\gamma)^{\frac{1}{2}}}) - \frac{1}{((x^2\frac{x'^2}{a^2})+a^2-2xx'\cos(\gamma))^{\frac{1}{2}}}$$

$$\Phi(\vec{\mathbf{x}}) = \frac{1}{4\pi}\int_0^{\frac{\pi}{2}}\int_0^{2\pi}(\frac{(\frac{r^2}{a})-a)}{(r^2+a^2-2ar\cos(\theta)^{\frac{3}{2}}}Va^2\sin(\theta')\,d\theta' d\phi'$$

$$+\frac{1}{4\pi}\int_0^{\frac{\pi}{2}}\int_0^{2\pi}(\frac{(\frac{r^2}{a})-a)}{(r^2+a^2-2ar\cos(\theta')^{\frac{3}{2}}} - Va^2\sin'(\theta)\,d\theta' d\phi$$

$$\varphi = \frac{Va^2((r^2/a)-a)}{2}\int_0^{\pi/2}\frac{\sin(\theta')\,d\theta'}{(r^2+a^2-2ar\cos(\theta')^{\frac{3}{2}}}$$

$$-\frac{Va^2((\frac{r^2}{a})-a)}{2}\int_{\frac{\pi}{2}}^{2\pi}\frac{\sin(\theta')\,d\theta'}{(r^2+a^2-2ar\cos(\theta')^{\frac{3}{2}}}$$

$$= \frac{V(r^2-a^2)}{2r}[\frac{-2}{\sqrt{r^2+a^2}} + \frac{1}{r-a} + \frac{1}{r+a}]$$

$$= V(1-\frac{(r^2-a^2)}{r\sqrt{r^2+a^2}})$$

Here the potential is calculated on axis z.