Full QVI-HJB Formulation with Stochastic Volatility

Gemini

July 21, 2025

1 Overview

This document outlines the full, time-dependent Quasi-Variational Inequality (QVI) for an algorithmic trading agent operating in an environment with stochastic volatility. This formulation does not use asymptotic analysis and represents the complete, unsimplified problem. The model incorporates a stochastic variance process, allowing the agent's strategy to adapt to changing market regimes.

2 State Space and Dynamics

The state vector is expanded to include v_t, the market variance.

$$X_t = (x, s, q, a, b, z, v, t, a_{pos}, b_{pos})$$

The core stochastic dynamics are:

$$\begin{aligned} ds_t &= z_t dt + \sqrt{v_t} dW_t^s \\ dz_t &= \mu_z(z,t) dt + \eta dW_t^z \\ dv_t &= \kappa_v(\theta_v - v_t) dt + \eta_v \sqrt{v_t} dW_t^v \end{aligned}$$

The drift of the signal, $\mu_z(z,t)$, is time-dependent. The variance v_t follows a mean-reverting square-root process. We assume the Wiener processes can be correlated: $d\langle W^s, W^z \rangle = \rho_{sz} dt$, $d\langle W^s, W^v \rangle = \rho_{sv} dt$, and $d\langle W^z, W^v \rangle = \rho_{zv} dt$.

3 The Time-Dependent HJB-QVI

The value function $V(t, X_t)$ must solve the following equation:

$$\min\left(\frac{\partial V}{\partial t} + \rho V - \left(\mathcal{L}V - \phi(q - q_{target}(z,t))^2\right), \quad V - \mathcal{M}V\right) = 0$$

This is solved over the funding cycle $t \in [0, T]$ subject to a terminal condition. The inventory penalty is centered around a dynamic, time-dependent target $q_{target}(z,t)$.

4 Continuation Operator ($\mathcal{L}V$)

The continuation operator $\mathcal{L}V$ describes the passive evolution of the system, now including the dynamics of variance.

$$\begin{aligned} \mathcal{L}V = &\left(z\frac{\partial V}{\partial s} + \mu_z(z,t)\frac{\partial V}{\partial z} + \kappa_v(\theta_v - v_t)\frac{\partial V}{\partial v}\right) \\ &+ \frac{1}{2}v_t\frac{\partial^2 V}{\partial s^2} + \frac{1}{2}\eta^2\frac{\partial^2 V}{\partial z^2} + \frac{1}{2}\eta_v^2 v_t\frac{\partial^2 V}{\partial v^2} \\ &+ \rho_{sz}\sqrt{v_t}\eta\frac{\partial^2 V}{\partial s \partial z} + \rho_{sv} v_t \eta_v \frac{\partial^2 V}{\partial s \partial v} + \rho_{zv}\eta\eta_v\sqrt{v_t}\frac{\partial^2 V}{\partial z \partial v} \\ &+ Aae^{-Ba_{pos}}\left[V(\ldots, q-1, a-1, \ldots) - V\right] \\ &+ Abe^{-Bb_{pos}}\left[V(\ldots, q+1, b-1, \ldots) - V\right] \end{aligned}$$

5 Impulse Operator ($\mathcal{M}V$)

The impulse operator $\mathcal{M}V$ is the value of the best possible discrete action from the available menu A. Its structure is unchanged, but it operates on the new, higher-dimensional value function $V(t, ..., v, ...)$.

$$\mathcal{M}V = \sup_{\alpha \in A, Q \in \text{Sizes}} \{V(\text{state after action } \alpha(Q)) - \text{Cost}(\alpha(Q))\}$$

The menu of actions A includes simple actions (e.g., Post Bid) and compound actions (e.g., Post Symmetric Quote).

- **Market Buy:** $V_{MB}(Q) = V(t, x - Qs_{ask} - K_{market}, s, q + Q, \dots)$
- **Post Symmetric Quote:** $V_{PSQ}(Q_a, Q_b) = V(t, x - 2K_{post}, s, q, a + Q_a, b + Q_b, \dots)$

The model chooses to act if $\mathcal{M}V > V$, triggering the optimal impulse α^*.