# Robotics Practical Example: From Theory to Simulation

## 1. Introduction to Robotic Systems

This module presents a structured approach to understanding and implementing robotic system simulations. We'll progress from theoretical foundations to practical simulation implementation, covering mobile robots.

**Learning Objectives:**

- Understand the fundamental differences between kinematic and dynamic models
- Derive and implement equations of motion for robotic systems
- Translate theoretical models into simulation code
- Analyze simulated robot behaviour and validate against theoretical predictions

## 2. Mobile Robot Systems: Theory to Simulation

### 2.1 Theoretical Foundations

#### 2.1.1 Kinematic Model of Differential Drive Robot

Kinematics deals with the motion of bodies without considering the forces that cause them. For mobile robots, especially wheeled ones, kinematic modeling involves understanding how wheel motions translate to robot movement.

A differential drive robot uses two independently controlled wheels to achieve motion. The kinematic model describes the geometrical relationships without considering forces and inertia.

**State Variables:**

- Position: $(x, y)$ in world frame
- Orientation: $\theta$ (heading angle)
- Wheel angular velocities: $\omega_L$ (left wheel), $\omega_R$ (right wheel)

**Forward Kinematics Equations:** The robot's linear velocity $v$ and angular velocity $\omega$ can be computed from wheel velocities:

$$v = \frac{v_L + v_R}{2} \tag{1}$$

$$\omega = \frac{v_R - v_L}{L} \tag{2}$$

With the linear velocities of the wheels are:

$$v_L = r \cdot \omega_L \tag{3.1}$$

$$v_R = r \cdot \omega_R \tag{3.2}$$

where:

- $r$ is the wheel radius
- $L$ is the distance between wheels (wheel base)

**Motion Equations:** The position and orientation change according to:

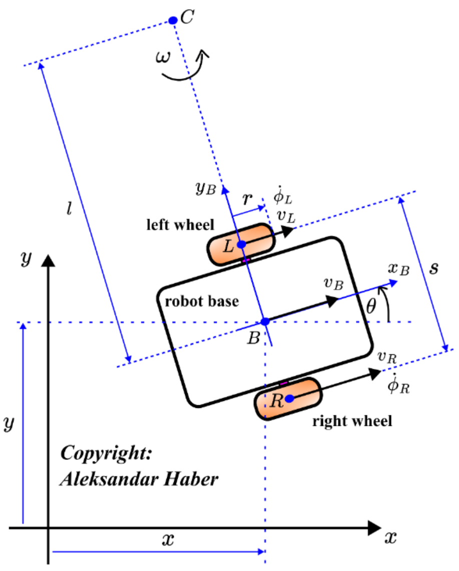
$$\dot{x} = v \cdot \cos\theta \tag{4.1}$$
$$\dot{y} = v \cdot \sin\theta \tag{4.2}$$
$$\dot{\theta} = \omega \tag{4.3}$$

These equations describe how the robot's position and orientation change based on wheel velocities.
A rotation matrix-based formulation would typically include a full transformation from body-frame velocities $[v_x \quad v_y \quad \omega]^T$ to world-frame velocities:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix}$$

But the differential drive robot has **non-holonomic constraints**—it cannot move sideways ($v_y = 0$). Therefore, this general transformation is not applicable because:

- The lateral velocity component $v_y$ is always zero.
- The system is underactuated and constrained to planar trajectories that follow the heading direction only.



**Inverse Kinematics:** Converting desired robot velocities to wheel velocities:

$$\omega_L = \frac{2v-\omega L}{2r} \tag{5.1}$$
$$\omega_R = \frac{2v+\omega L}{2r} \tag{5.2}$$

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**Proof**

To find $\omega_L$ and $\omega_R$ in terms of the desired robot velocities ($v$ and $\omega$), we need to solve the forward kinematics equations for these variables.
Step 1: Set up the equations
We have:

$$v = \frac{r\cdot\omega_L + r\cdot\omega_R}{2} \tag{P1.1}$$

$$\omega = \frac{r\cdot\omega_R - r\cdot\omega_L}{L} \tag{P1.2}$$

Step 2: Rearrange Eqn. (P1.1)
Multiply both sides by 2:

$$2v = r\cdot(\omega_R + \omega_L) \tag{P1.3}$$

Step 3: Rearrange Eqn. (P1.2)
Multiply both sides by L:

$$\omega L = r\cdot(\omega_R - \omega_L) \tag{P1.4}$$

Step 4: Create a system of two equations
We now have:

$$\omega_L + \omega_R = \frac{2v}{r} \tag{P1.5}$$

$$(\omega_R - \omega_L) = \frac{\omega L}{r} \tag{P1.6}$$

Step 5: Solve for $\omega_L + \omega_R$ and $\omega_R - \omega_L$

Step 6: Solve for $\omega_R$
Add the two equations ((P1.5) and (P1.6)):

$$2\omega_R = \frac{2v}{r} + \frac{\omega L}{r} \tag{P1.7}$$

$$\omega_R = \left(\frac{2v+\omega L}{2r}\right) \tag{P1.8}$$

Step 7: Solve for $\omega_L$
Substitute $\omega_R$ in Eqn. (P1.8) into Eqn. (P1.5)

$$\omega_L + \left(\frac{2v+\omega L}{2r}\right) = \frac{2v}{r} \tag{P1.9.1}$$

$$\omega_L = \frac{2v}{r} - \left(\frac{2v+\omega L}{2r}\right) \tag{P1.9.2}$$

$$\omega_L = \frac{4v}{2r} - \left(\frac{2v+\omega L}{2r}\right) \tag{P1.9.3}$$

$$\omega_L = \left(\frac{2v-\omega L}{2r}\right) \tag{P1.10}$$

**End of Proof**

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Key Assumptions

1. **Ideal Rolling Contact**: The wheels roll without slipping on the surface. This is a critical assumption that allows us to relate wheel rotation directly to robot motion.
2. **Rigid Body**: The robot is assumed to be a rigid body with no flexibility between components.
3. **Planar Motion**: The robot moves on a flat, horizontal surface with no inclines or irregularities.

4. **Point Contact**: The wheels make point contact with the ground.
5. **No Lateral Slip:** The wheels cannot slide sideways (non-holonomic constraint).
6. **Constant Parameters**: The wheel radius (r) and wheel base (L) are constant.
7. **Symmetric Design**: The robot is assumed to be symmetric around its center axis.
8. **No Friction Model**: The equations assume perfect traction without modeling complex friction.
9. **Instantaneous Velocity Changes**: The kinematic model allows for instantaneous changes in velocity, unlike a dynamic model which would account for acceleration limits.

### 2.1.2 Dynamic Model of Differential Drive Robot

The dynamic model considers forces, torques, and inertial properties.

**Additional Parameters:**

- Mass: $m$ of the robot
- Moment of inertia: $I$around the center
- Friction coefficients: $B_R$, $B_L$ for wheels

**Dynamic Equations via Lagrangian Method**

The Lagrangian $L = T - V$, where $T$ is kinetic energy and $V$ is potential energy.

For a differential drive robot on a flat plane (ignoring potential energy):

$$T = \frac{1}{2}mv^2 + \frac{1}{2}I_z\omega^2 \tag{6}$$

The equations of motion are derived using:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \tag{7}$$

where $q_i$ are the generalized coordinates $(x, y, \theta)$ and $Q_i$ are the generalized forces.

For a differential drive robot:

- Calculate the total kinetic energy:
$$T = \frac{1}{2}mv^2 + \frac{1}{2}I_z\omega^2 \tag{8}$$
- Express in terms of wheel velocities:
$$v = \frac{r\cdot(\omega_R+\omega_L)}{2} \tag{9.1}$$
$$\omega = \frac{r\cdot(\omega_R-\omega_L)}{L} \tag{9.2}$$
- Substituting Eqn. (9.1) and (9.2) into Eqn. (8)
$$T = \frac{1}{2}m\left(\frac{r\cdot(\omega_R+\omega_L)}{2}\right)^2 + \frac{1}{2}I_z\left(\frac{r\cdot(\omega_R-\omega_L)}{L}\right)^2 \tag{10}$$
- Apply Lagrangian equations for wheel rotations: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\omega}_R}\right) = \tau_R$(torque on right wheel)$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\omega}_L}\right) = \tau_L$ (torque on left wheel)
- The final dynamic equations relate wheel torques to accelerations:

$$\tau_R = \left(\frac{mr^2}{4} + \frac{I_z r^2}{L^2}\right)\dot{\omega}_R + \left(\frac{mr^2}{4} - \frac{I_z r^2}{L^2}\right)\dot{\omega}_L \tag{11.1}$$
$$\tau_L = \left(\frac{mr^2}{4} - \frac{I_z r^2}{L^2}\right)\dot{\omega}_R + \left(\frac{mr^2}{4} + \frac{I_z r^2}{L^2}\right)\dot{\omega}_L \tag{11.2}$$

$$\begin{bmatrix}\tau_R \\ \tau_L\end{bmatrix} = \begin{bmatrix}\frac{mr^2}{4} + \frac{I_z r^2}{L^2} & \frac{mr^2}{4} - \frac{I_z r^2}{L^2} \\ \frac{mr^2}{4} - \frac{I_z r^2}{L^2} & \frac{mr^2}{4} + \frac{I_z r^2}{L^2}\end{bmatrix}\begin{bmatrix}\dot{\omega}_R \\ \dot{\omega}_L\end{bmatrix} + \begin{bmatrix}B_R & 0 \\ 0 & B_L\end{bmatrix}\begin{bmatrix}\omega_R \\ \omega_L\end{bmatrix} \tag{12}$$

**Dynamic Equations:** Relating torque inputs to wheel acceleration:

$$\begin{bmatrix}\tau_R \\ \tau_L\end{bmatrix} = \begin{bmatrix}J_R & J_{RL} \\ J_{RL} & J_L\end{bmatrix}\begin{bmatrix}\dot{\omega}_R \\ \dot{\omega}_L\end{bmatrix} + \begin{bmatrix}B_R & 0 \\ 0 & B_L\end{bmatrix}\begin{bmatrix}\omega_R \\ \omega_L\end{bmatrix} \tag{13}$$

Where:

- $J_R = \frac{mr^2}{4} + \frac{I_z r^2}{L^2} = J_L$re the effective inertias for each wheel
- $J_{RL} = \frac{mr^2}{4} - \frac{I_z r^2}{L^2}$ is the coupling inertia
- $\tau_R, \tau_L$ are the applied torques
- $B_R, B_L$ are the viscous friction coefficients

### 2.2 Simulation Implementation (simulation_mobile_robot_with_obstacles.m)

#### 2.2.1 State Initialization

```
% Initialize state variables
x = 0;        % initial x position [m]
y = 0;        % initial y position [m]
theta = 0;    % initial orientation [rad]
omega_r = 0;  % initial right wheel angular velocity [rad/s]
omega_l = 0;  % initial left wheel angular velocity [rad/s]
```

#### 2.2.2 Dynamic Update Function

The dynamics equations are implemented as:

```
function [omega_r, omega_l] = update_dynamics(omega_r, omega_l, tau_r, tau_l, robot, dt)
  % Compute the determinant of the inertia matrix
  det_J = robot.J_r * robot.J_l - robot.J_rl^2;

  % Compute wheel accelerations
  omega_dot_r = ((robot.J_l * (tau_r - robot.B_r * omega_r) - ...
          robot.J_rl * (tau_l - robot.B_l * omega_l)) / det_J);

  omega_dot_l = ((robot.J_r * (tau_l - robot.B_l * omega_l) - ...
          robot.J_rl * (tau_r - robot.B_r * omega_r)) / det_J);
  % Update wheel velocities
  omega_r = omega_r + omega_dot_r * dt;
  omega_l = omega_l + omega_dot_l * dt;
end
```

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### 2.2.3 Forward Kinematics Implementation

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```
function [v, omega] = forward_kinematics(omega_r, omega_l, robot)
  % Linear and angular velocity of the robot
  v = robot.wheel_radius * (omega_r + omega_l) / 2;
  omega = robot.wheel_radius * (omega_r - omega_l) / robot.wheel_base;
end
```

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### 2.2.4 Kinematic Update Function

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```
function [x, y, theta] = update_kinematics(x, y, theta, v, omega, dt)
  % Update robot position and orientation
  x = x + v * cos(theta) * dt;
  y = y + v * sin(theta) * dt;
  theta = theta + omega * dt;

  % Normalize theta to [-pi, pi]
  theta = atan2(sin(theta), cos(theta));
end
```

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### 2.2.5 Simulation Loop Structure

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```
% Main simulation loop
for i = 1:num_steps
  % 1. Calculate control inputs (torques)

  % 2. Update dynamics (wheel velocities)
  [omega_r, omega_l] = update_dynamics(omega_r, omega_l, tau_r, tau_l, robot, dt);

  % 3. Calculate robot velocities from wheel velocities
  [v, omega] = forward_kinematics(omega_r, omega_l, robot);

  % 4. Update robot position and orientation
  [x, y, theta] = update_kinematics(x, y, theta, v, omega, dt);

  % 5. Store states for later analysis
  history.x(i+1) = x;
  history.y(i+1) = y;
  history.theta(i+1) = theta;
end
```

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## 3. Discussion and Exercises

### 3.1 Theoretical Questions

1. What are the fundamental differences between a kinematic and dynamic model?
2. Explain how the coupling term $J_{RL}$ affects the motion of a differential drive robot.
3. What are the limitations of both simulation models? How could they be improved?

### 3.2 Programming Exercises

1. Modify the differential drive simulation to include wheel slip

## 4. Conclusion

This module has demonstrated the progression from theoretical equations to practical simulations for robotic systems. By understanding both the mathematical foundations and their implementation, students gain insight into the behavior of complex robotic systems and develop the skills to design, analyze, and control them.

The relationship between theory and simulation is bidirectional - theoretical models inform simulation design, while simulation results can validate or challenge our theoretical understanding. This iterative process is central to the advancement of robotics and control systems.