State Estimation - Assignment 1

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1 Question 1

$\mathbf{x}$ is a random variable of length K :

$$\mathbf{x} = \mathcal{N}(\mathbf{0}, \mathbf{1})$$

a) What type of random variable is the following random variable?

$$\mathbf{y} = \mathbf{x}^\top \mathbf{x}$$

Answer: x is a random variable of length k. y is a summation of k random variables.

$$y = x_1^\top x_1 + x_2^\top x_2 + \cdots + x_k^\top x_k = \sum_{i=1}^{k} x_i^\top x_i \tag{1}$$

b) Calculate the mean and variance of $\mathbf{y}$.

Answer: Mean is equal to k and variance is 2k.

$$\begin{aligned} E[y] &= E\left[\sum_{i=1}^{k} \mathbf{x}_i^T \mathbf{x}_i\right] \\ &= E\left[x_1^\top x_1 + x_2^\top x_2 + \cdots + x_k^\top x_k\right] \\ &= E\left[x_1^\top x_1\right] + E\left[x_2^\top x_2\right] + \cdots + E\left[x_k^\top x_k\right] \end{aligned} \tag{2}$$

$$\Sigma = E\left[x_1^\top x_1\right] = 1 \quad x \sim N(0,1) \rightarrow \mu = 0, \Sigma = 1$$

$$\mu_y \ = E[y] = \mathbf{1} + 1 + \cdots + 1 = k$$

$$
\begin{aligned}
\operatorname{var}[y] &= E\left[(y-\mu_y)(y-\mu_y)^\top\right] = E\left[(y-\mu_y)^2\right] \\
&= E\left(y^2\right) - 2E[y]\mu_y + \mu_y^2 = E\left[y^2\right] - k^2 \\
E\left[y^2\right] &= E\left[\left(x_1^\top x_1 + x_2^\top x_2 + \cdots + x_k^\top x_k\right)^2\right] \\
&= E[\left(x_1^\top x_1\right)^2 + \left(x_2^\top x_2\right)^2 + \cdots + \left(x_k^\top x_k\right)^2 + \\
&x_1^\top x_1 x_2^\top x_2 + \cdots + x_j^\top x_j x_k^\top x_k] \\
&= \sum_{i=1}^{k} 3 + \sum_{i=1}^{k(k-1)} E\underbrace{\left[x_1^\top x_1\right]}_{1} E\underbrace{\left[x_2^\top x_1\right]}_{1} + \cdots \\
&= 3k + k^2 - k \\
\operatorname{var}[y] &= 2k
\end{aligned} \tag{3}
$$

c) Using Python, plot the PDF of $\mathbf{y}$ for $\mathrm{K} = 1, 2, 3, 10, 100$.

Answer: In the Python code in my Github, I have plotted the PDF of y for different K. Here I am attaching the figures.

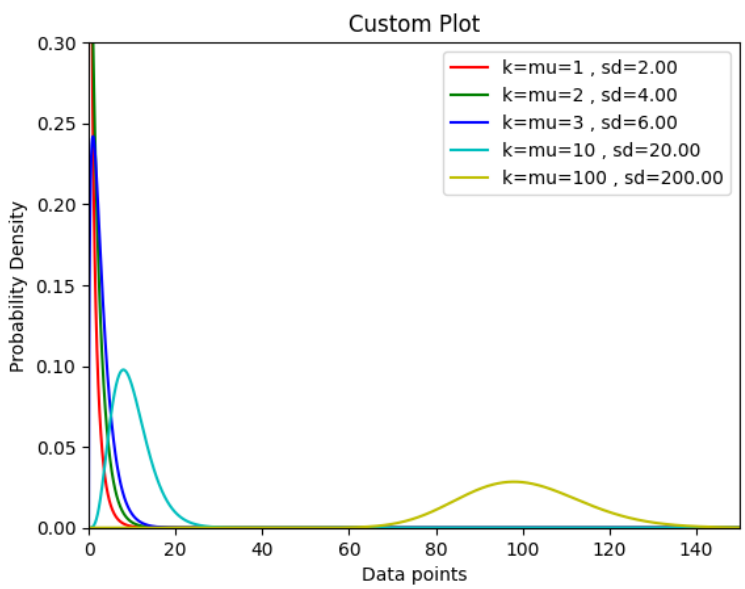

Figure 1: PDF of y for k= 1, 2, 3, 10, and 100.

2 Question 2

x is a random variable of length N :

$$\mathbf{x} = \mathcal{N}(\mu, \mathbf{\Sigma})$$

a) Assume $\mathbf{x}$ is transformed linearly, i.e. $\mathbf{y} = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is an $N \times N$ matrix. Calculate the mean and covariance of $\mathbf{y}$. Show the derivations.

Answer: Here, I have calculated the mean and covariance of y.

$E[y] = \mu_y \rightarrow E[Ax] = AE[x] = A\mu_x$

$$\begin{aligned}
&\operatorname{cov} y = E\left[(y-\mu)(y-\mu)^\top\right] \\
&= E\left[(Ax-\mu)(Ax-\mu)^\top\right] \\
&= E\left[(Ax-\mu)\left(x^\top A^\top - \mu\right)^\top\right] = E\left[A(x-\mu)(x-\mu)^\top\right] \\
&= AE\left[(x-\mu)(x-\mu)^\top\right]A^\top = A\Sigma A^\top
\end{aligned} \tag{4}$$

b) Repeat a), when $\mathbf{y} = \mathbf{A_1}\mathbf{x} + \mathbf{A_2}\mathbf{x}$.

Answer: I have calculated the mean and covariance of y when $y = A_1x + A_2x$.

$$\begin{aligned}
&\quad y = A_1x + A_2x \\
&E[y] = E\left[A_1x + A_2x\right] \quad = E\left[A_1x\right] + E\left[A_2x\right] \; = A_1E[x] + A_2E[x] \quad = (A_1 + A_2)\mu \\
&\operatorname{Cov} y = E\left[(y-\mu)(y-\mu)^\top\right] = E\left[\left(A_1x + A_2x - \mu\right)\left(A_1x + A_2x - \mu\right)^\top\right] \\
&\\
&\; = E\left[\left(A_1x + A_2 - \left(A_1 + A_2\right)\mu\right)\left(x^\top A_1^\top + x^\top A_2^\top - \left(A_1^\top A_2^\top\right)\mu\right)\right] \\
&\; = E\left[\left(A_1 + A_2\right)(x-\mu)\left(x^\top - \mu\right)\left(A_1^\top + A_2^\top\right)\right] = \left(A_1 + A_2\right)E\left[(x-\mu)\left(x^\top - \mu\right)\right] \\
&= \left(A_1 + A_2\right)\sum\left(A_1 + A_2\right)^\top
\end{aligned} \tag{5}$$

c) If $\mathbf{x}$ is transformed by a nonlinear differentiable function, i.e. $\mathbf{y} = \mathbf{f}(\mathbf{x})$, compute the covariance matrix of $\mathbf{y}$. Show the derivation.

Answer: In this part, I have derived the covariance matrix of y, when a nonlinear differentiable function transforms x.

$$y = f(x)$$

$$f(x) \approx \mu_y + G(x - \mu_x),$$

$$G = \left.\frac{\partial f(x)}{\partial x}\right|_{x=\mu_x}$$

$$\mu_y = f(\mu_x)$$

$$\begin{aligned}
&E\left[(y-\mu_y)(y-\mu_y))^\top\right] \\
&E\left[(\mu_y + G\,(x-\mu_x) - \mu_y)(\mu_y + G(x-\mu_x) - \mu_y)^\top\right] \\
&E\left[G\,(x-\mu_x)\,(G\,(x-\mu_x))^T\right] \\
&E\left[G\,(x-\mu_x)\,(x-\mu_x)^\top G^\top\right] = G\Sigma G^\top
\end{aligned}$$

d) Apply c) when

$$\mathbf{x} = \begin{bmatrix} \rho \\ \theta \end{bmatrix}, \quad \mathbf{\Sigma} = \begin{bmatrix} \sigma^2_{\rho\rho} & \sigma^2_{\rho\theta} \\ \sigma^2_{\rho\theta} & \sigma^2_{\theta\theta} \end{bmatrix}, \quad \text{and } \mathbf{y} = \begin{bmatrix} \rho\cos\theta \\ \rho\sin\theta \end{bmatrix}.$$

Compute the covariance of $\mathbf{y}$ analytically. This models how range-bearing measurements in the polar coordinate frame are converted to a Cartesian coordinate frame.

Answer: In this part, I have computed the covariance of y analytically.

$$x = \begin{bmatrix} \rho \\ \theta \end{bmatrix}$$

$$y = \begin{bmatrix} \rho\cos\theta \\ p\sin\theta \end{bmatrix}$$

$$G = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{bmatrix}$$

$$G\Sigma G^\top = \begin{bmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{bmatrix} \begin{bmatrix} \sigma^2_{\rho\rho} & \sigma^2_{\rho\theta} \\ \sigma^2_{\rho\theta} & \sigma^2_{\theta\theta} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\rho\sin\theta & \rho\cos\theta \end{bmatrix}$$

e) Simulate d) using the Monte Carlo simulation, i.e. assume

$$\mathbf{x} = \begin{bmatrix} 1\text{ m} \\ 0.5^\circ \end{bmatrix}, \quad \mathbf{\Sigma} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.005 \end{bmatrix}.$$

Sample 1000 points from this distribution and plot the transformed results on $\mathrm{x} - \mathrm{y}$ coordinates. Plot the uncertainty ellipse, calculated from part **d**). Overlay the ellipse on the point samples.

Answer: First, I derived the covariance in the x-y coordinates analytically. I used it in a Python program to plot 1000 points from this distribution and plot the uncertainty ellipse on the point samples. I found information about the uncertainty ellipse from the following website:
https://www.visiondummy.com/2014/04/draw-error-ellipse-representing-covariance-matrix/

$$\begin{bmatrix} \cos 0.5 & -\sin 0.5 \\ \sin 0.5 & \cos 0.5 \end{bmatrix} \begin{bmatrix} 0.01 & 0 \\ 0 & 0.005 \end{bmatrix} \begin{bmatrix} \cos 0.5 & \sin 0.5 \\ -\sin 0.5 & \cos 0.5 \end{bmatrix}$$

$$\begin{bmatrix} 0.01\cos 0.5 & -0.005 & \sin 0.5 \\ 0.01\sin 0.5 & 0.005 & \cos 0.5 \end{bmatrix} \begin{bmatrix} \cos 0.5 & \sin 0.5 \\ -\sin 0.5 & \cos 0.5 \end{bmatrix}$$

$$\begin{bmatrix} 0.01\cos^2 0.5 + 0.005\sin^2 0.5 & 0.005\sin 0.5\cos 0.5 \\ 0.005\sin 0.5\cos 0.5 & 0.01\sin^2 0.5 + 0.005\cos^2 0.5 \end{bmatrix}$$

$$\begin{bmatrix} 9.9996 \times 10^{-3} & 4.3631 \times 10^{-3} \\ 4.3631 \times 10^{-3} & 5.0004 \times 10^{-3} \end{bmatrix}$$

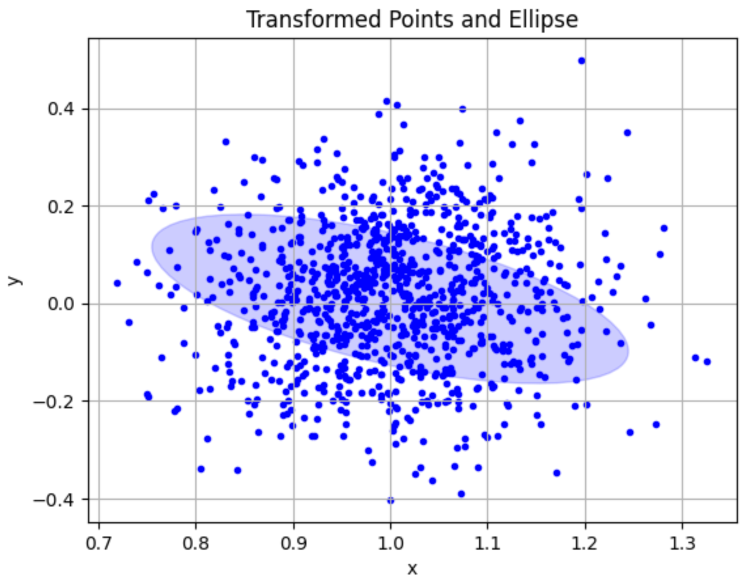

Figure 2: Uncertainty ellipse and the sample points