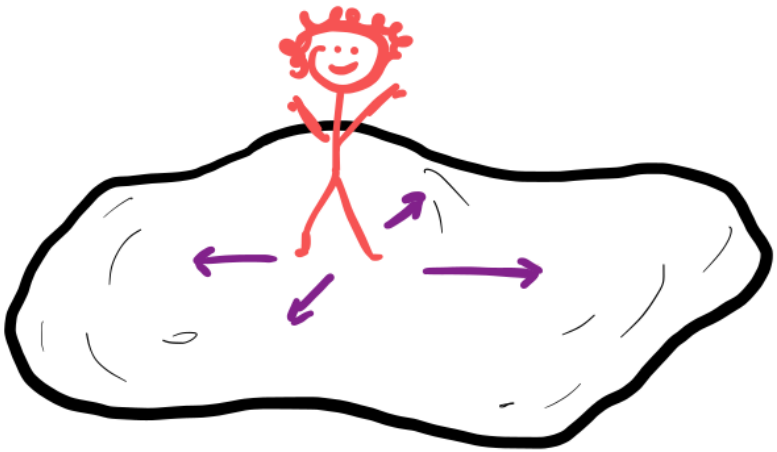
What is a manifold?



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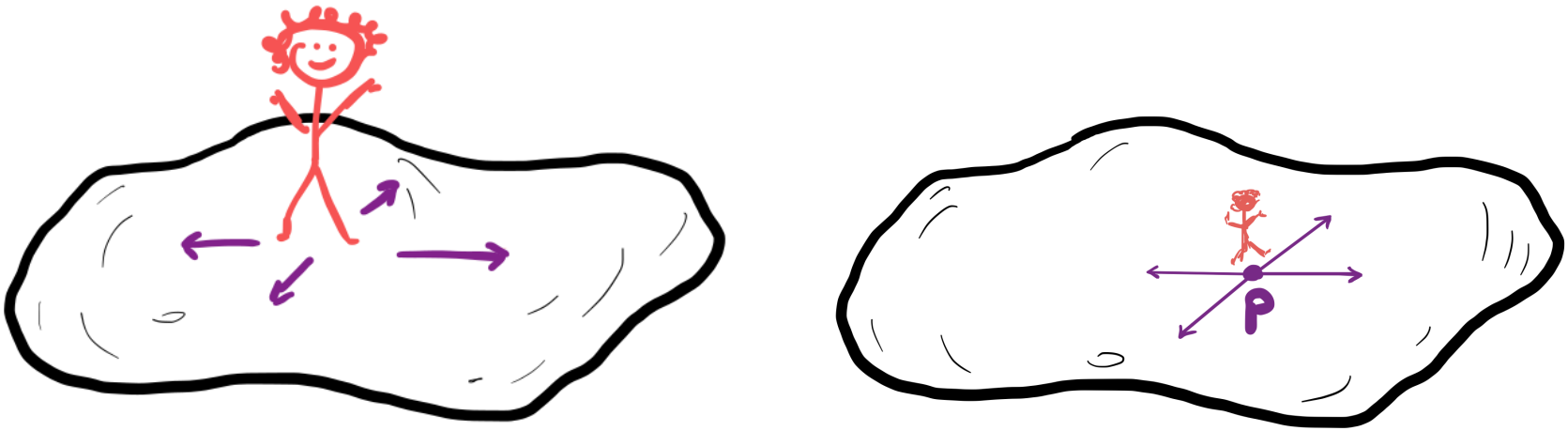
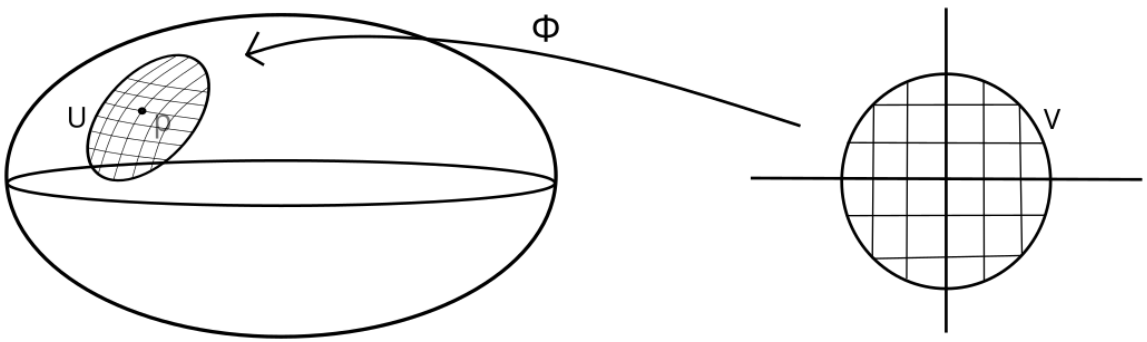


surface of a potato is a manifold: locally looks like a disk

Smooth manifolds

- topological **manifold**: second countable Hausdorff topological space M locally homeomorphic to open ball in $\mathbb{R}^n$
- every $p \in M$ has a coordinate chart: $p \in U \subset M$ open, homeomorphism $\phi\colon V \to U$ for $V \subset \mathbb{R}^n$ open ball
- **smooth manifold**: all coordinate transformations from overlapping charts are smooth



Picture courtesy of Dominik Gutwein.

Examples of smooth manifolds

- empty set (of any dimension)
- 0-dimensional: isolated points
- 1-dimensional: $\mathbb{R}$, $\mathbb{S}^1$
- n-dimensional: open disc $\mathbb{D} \subset \mathbb{R}^n$
- $n = 2$: $\mathbb{R}^2$, $\mathbb{S}^2$, $\mathbb{T}^2$, Σ_g for $g \geq 1$

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- $n \geq 3$: complicated; classification for $n \geq 4$ impossible
- $\mathbb{R}^n$, $\mathbb{S}^n$, $\mathbb{T}^n$, $\mathbb{RP}^n$, $\mathbb{CP}^n$,
  $\{[z_0 : z_1 : z_2 : z_3 : z_4] \in \mathbb{CP}^4 \mid z_0^5 + \cdots + z_4^5 = 0\}$
  configuration spaces in physics and engineering
- **not** a manifold: letter "X"

Manifolds with boundary or corners

- $\overline{\mathbb{D}} \subset \mathbb{R}^n$ is a manifold with **boundary**
- interior points locally look like open ball in $\mathbb{R}^n$,
  boundary points look like open ball in upper half of $\mathbb{R}^n$
- manifolds with **corners**: local model is Euclidean quadrant
  $[0,1]^n \subset \mathbb{R}^n$ has corners

Precise technical definition

See blackboard.

Defining manifolds in mathlib

- allow different smoothness
- allow boundary and corners
- allow different base field: $\mathbb{R}$, $\mathbb{C}$ or p-adic numbers $\mathbb{Q}_p$
- infinite-dimensional manifolds (e.g. $C^k(M, N)$)

Formalising manifolds with boundary

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- charts on M (one preferred chart at each point)
- compatibility condition: transition maps lie in structure groupoid