Relating two models in the representation theory of the symplectic group

Hanjoon Choe, Christian Lenart

Department of Mathematics and Statistics, University at Albany,SUNY

Abstract

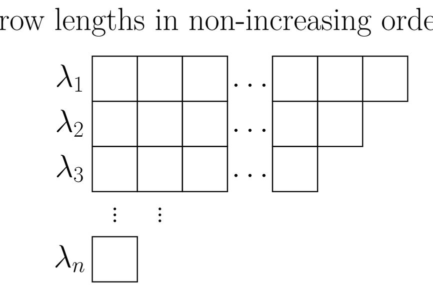
Representation theory is a basic tool for understanding group symmetry using linear algebra, namely group elements are represented as invertible matrices. The symplectic groups $Sp(2n, \mathbb{C})$ are an important class of infinite groups, also known as simple Lie groups of type C. An irreducible representation of $Sp(2n, \mathbb{C})$ is indexed by partitions with at most n parts, or Young diagrams with at most n rows. A basis of such a representation is indexed by two types of fillings of the mentioned Young diagrams with integers ranging from $-n$ to n except 0, known as King tableaux and De concini tableaux. *I give an implementation of an algorithm which constructs a bijection between these two sets of tableaux.* This bijection has many applications to the study of representations of the symplectic group.

Introduction

Semistandard Young Tableaux

Young diagram

Definition. A Young diagram is a finite collection of boxes, or cells, arranged in left-justified rows, with the row lengths in non-increasing order.

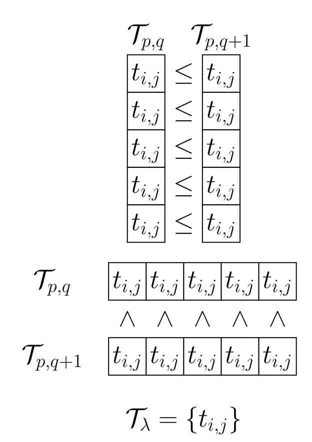


$$\{\lambda_i \geq \lambda_j : i, j \in \mathbb{N}, i < j\}$$

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n), \natural(\lambda) = |\lambda|$$

Semistandard young tableaux

Definition. A tableau is called semistandard if the entries inside of young diagram weakly increase along each row and strictly increase down each column.



$$\mathcal{T}_\lambda = \{t_{i,j}\}$$

Example

1	2	3	3	3	4
2	4	5	6	6	
3	5	6	7		
6					

$$(i, j) \in \lambda \iff 1 \leq i \leq l(\lambda),\ 1 \leq j \leq \lambda_i$$

De Concini and King Tableaux

Different ordered sets

The difference between the two sets $\mathcal{D}(\lambda$,n) and $\mathcal{K}(\lambda$,n) are stems primarily from the fact that different total orders on $[[\mathbf{n}]]$ are used.

$$\text{Let } [[\mathbf{n}]] = \{\bar{1}, 1, \bar{2}, 2, \bar{3}, 3, \dots \bar{n}, n\}.$$

Order set of $\mathcal{D}(\lambda$,n)

$$\dots <_n \overline{3} <_n \overline{2} <_n \overline{1} <_n 1 <_n 2 <_n 3 <_n \dots$$

$$[[\mathbf{n}]]_n = \{\overline{n}, \overline{n-1}, \overline{n-2}, \dots\ , \overline{1}, 1, 2, \dots, n\}.$$

Order set of $\mathcal{K}(\lambda$,n)

$$\overline{1} <_1 1 <_1 \overline{2} <_1 2 <_1 \overline{3} <_1 3 <_1 \dots$$

$$[[\mathbf{n}]]_1 = \{\overline{1}, 1, \overline{2}, 2, \overline{3}, 3, \dots\ , \overline{n-1}, n-1, \overline{n}, n\}.$$

Definition. $<_d$ on $[[\mathbf{n}]]$ is given by

$$\bar{d} <_d \overline{d-1} <_d \dots <_d \overline{1} <_d 1 <_d 2 <_d \dots <_d d$$
$$<_d \overline{d+1} <_d d+1 <_d \overline{d+2} <_d d+2 <_d \dots$$

De Concini tableaux

$$\begin{array}{c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{3}{c}{\leq_3} \\ \cline{2-4} & \overline{3} & \overline{3} & 1 \\ \cline{2-4} \wedge_3 & \overline{2} & \overline{1} & 2 \\ \cline{2-4} & 2 & 3 & \multicolumn{1}{c}{} \\ \cline{2-3} \end{array}$$

$$\mathcal{D}(\lambda, 3),\ \lambda = \{3, 3, 2\}$$

King tableaux

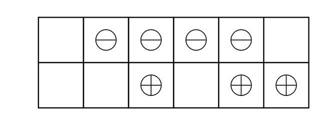
$$\begin{array}{c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{3}{c}{\leq_1} \\ \cline{2-4} & \overline{1} & \overline{1} & \overline{2} \\ \cline{2-4} \wedge_1 & 2 & 2 & 3 \\ \cline{2-4} & \overline{3} & \overline{3} & \multicolumn{1}{c}{} \\ \cline{2-3} \end{array}$$

$$\mathcal{K}(\lambda, 3),\ \lambda = \{3, 3, 2\}$$

Admissible Columns

$\mathcal{D}((1^k), n)$ and $\mathcal{D}((1^k), n)$ of one column tableau can be expressed in terms of "circle diagram"

Circle diagram



$$\{\overline{2}, \overline{3}, \overline{4}, \overline{5}, 3, 5, 6\}$$

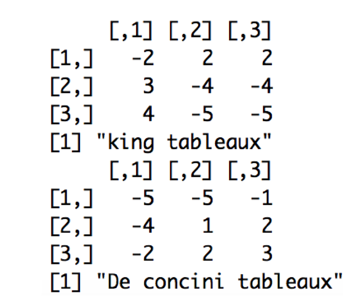
Admissible

Definition. if $2 \times n$ circle diagram $\mathcal{P}$ is admissible if for each $m,\ 1 \leq m \leq n$, the first m slots contain no more than m circles.

$$|\{x \in \mathcal{P} : ||x|| \leq m\}| \leq m$$

Implementation

Results



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     [,1] [,2] [,3]
[1,]   -2    2    2
[2,]    3   -4   -4
[3,]    4   -5   -5
[1] "king tableaux"
     [,1] [,2] [,3]
[1,]   -5   -5   -1
[2,]   -4    1    2
[3,]   -2    2    3
[1] "De concini tableaux"
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Figure 1: Figure Demonstaration

Additional Information

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References

Acknowledgements

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Contact Information

- Web: 11
- Email: 11
- Phone: +1 (000) 111 1111