Relating two combinatorial models in the representation theory of the symplectic group

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Abstract

Representation theory is a basic tool for understanding group symmetry using linear algebra, namely group elements are represented as invertible matrices. The symplectic groups $Sp(2n, \mathbb{C})$ are an important class of infinite groups, also known as simple Lie groups of type C. An irreducible representation of $Sp(2n, \mathbb{C})$ is indexed by partitions with at most n parts, or Young diagrams with at most n rows. A basis of such a representation is indexed by two types of fillings of the mentioned Young diagrams with integers ranging from $-n$ to n except 0, known as King tableaux and De concini tableaux. *I give an implementation of an algorithm which constructs a bijection between these two sets of tableaux.* This bijection has many applications to the study of representations of the symplectic group.

Introduction

Representation Theory

Group

In mathematics, Groups is defined as the algebraic structure which satisfies three conditions

(1) All elements in a group are closed under operation. $\forall a, b \in G$ such that $a * b \in G$

(2) Associativity $a * (b * c) = (a * b) * c$

(3) It has unique identity element such that $e * a = a$

(4) Each elements inside of group has their inverse such that $a * (a^{-1}) = e$

Representation theory

Representation theory is a branch of mathematics to study the groups embedded in vector space.

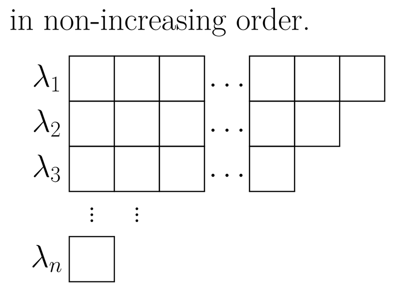
$$\phi : G \to GL(V)$$

Symplectic group

Semistandard Young Tableaux

Young diagram

Definition. A Young diagram is a finite collection of boxes, or cells, arranged in left-justified rows, with the row lengths in non-increasing order.

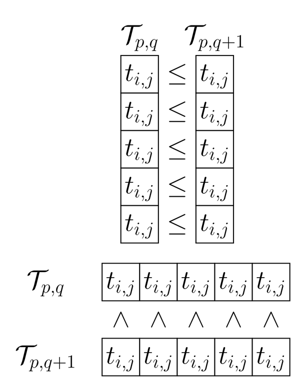


$$\{\lambda_i \geq \lambda_j : i, j \in \mathbb{N}, i < j\}$$

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n), l(\lambda) = |\lambda|$$

Semistandard young tableaux

Definition. A tableau is called semistandard if the entries inside of young diagram weakly increase along each row and strictly increase down each column.



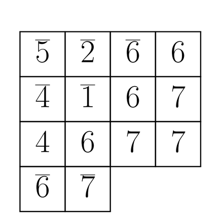
Example

1	2	3	3	3	4
2	4	5	6	6	
3	5	6	7		
6					

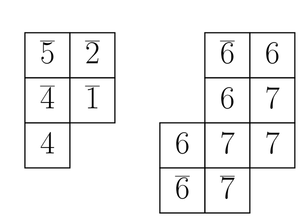
$$\mathcal{T}_\lambda = \{t_{i,j} : 1 \leq i \leq l(\lambda), 1 \leq j \leq \lambda_i\}$$

Intermediate "hybrid" tableaux

Definition. A tableau $\mathcal{T}$ is in $\mathcal{M}^d(\lambda, n)$ if and only if (i) it is d-semistandard, (ii) has admissible columns, and (iii) the subtableau of $\mathcal{T}$, say of shape μ, consisting of these entries $[[\mathbf{d}]]_d$ such that $\mathcal{D}(\mu, d)$.



$\mathcal{M}^5((4^3, 2), 7)$



De Concini part(left)
King part(right)

Note. $\mathcal{D}(\lambda, n) = \mathcal{M}^n(\lambda, n)$ and $\mathcal{K}(\lambda, n) = \mathcal{M}^1(\lambda, n)$

Bijection problem

Suppose that $\Phi^n_\lambda : \mathcal{M}^n(\lambda, n) \to \mathcal{M}^{n-1}(\lambda, n)$ is weight preserving bijection. Define the map $\Psi^{n,d}_\lambda : \mathcal{M}^d(\lambda, n) \to \mathcal{M}^{d-1}(\lambda, n)$, for each d, $n \geq d \geq 1$ as follows. Let $\mathcal{T}_\lambda \in \mathcal{M}^d(\lambda, n)$ and Let $\mathcal{T}_\mu$ be the De Concini part of $\mathcal{T}_\lambda$ of shape μ. $\mathcal{T}_\mu$ is an element of shape $\mathcal{M}^d(\mu, d)$. Define $\Psi^{n,d}_\lambda(\mathcal{T}_\lambda)$ to be the tableau formed by replacing $\mathcal{T}$ with $\Phi^d_\mu(\mathcal{T}_\mu)$. Thus, the bijection process between De Concini to King tableaux is solved by

$$\Psi^{n,2}_\lambda \circ \Psi^{n,3}_\lambda \circ \ldots \Psi^{n,n}_\lambda$$

De Concini and King Tableaux

Different ordered sets

The difference between the two sets $\mathcal{D}(\lambda, n)$ and $\mathcal{K}(\lambda, n)$ are stems primarily from the fact that different total orders on $[[\mathbf{n}]]$ are used.

$$\text{Let } [[\mathbf{n}]] = \{\bar{1}, 1, \bar{2}, 2, \bar{3}, 3, \ldots \bar{n}, n\}$$

- Order set of $\mathcal{D}(\lambda, n)$

$$\cdots <_n \bar{3} <_n \bar{2} <_n \bar{1} <_n 1 <_n 2 <_n 3 <_n \ldots$$

$$[[\mathbf{n}]]_n = \{\bar{n}, \overline{n-1}, \overline{n-2}, \ldots, \bar{1}, 1, 2, \ldots, n\}.$$

- Order set of $\mathcal{K}(\lambda, n)$

$$\bar{1} <_1 1 <_1 \bar{2} <_1 2 <_1 \bar{3} <_1 3 <_1 \ldots$$

$$[[\mathbf{n}]]_1 = \{\bar{1}, 1, \bar{2}, 2, \bar{3}, 3, \ldots, \overline{n-1}, n-1, \bar{n}, n\}.$$

Definition. $<_d$ on $[[\mathbf{n}]]$ is given by

$$\bar{d} <_d \overline{d-1} <_d \cdots <_d \bar{1} <_d 1 <_d 2 <_d \cdots <_d d$$
$$<_d \overline{d+1} <_d d+1 <_d \overline{d+2} <_d d+2 <_d \ldots$$

De Concini tableaux

$\leq_3$ (rows), $\wedge_3$ (columns)

$\bar{3}$	$\bar{3}$	1
$\bar{2}$	$\bar{1}$	2
2	3	

$\mathcal{D}(\lambda, 3),\ \lambda = \{3, 3, 2\}$

King tableaux

$\leq_1$ (rows), $\wedge_1$ (columns)

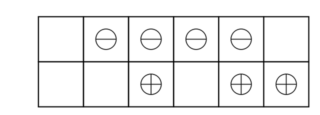
$\bar{1}$	$\bar{1}$	$\bar{2}$
2	2	3
$\bar{3}$	$\bar{3}$	

$\mathcal{K}(\lambda, 3),\ \lambda = \{3, 3, 2\}$

Admissible Columns

Circle diagram

A circle diagram is a method of viewing a subset of $[[\mathbf{n}]]$. It is constructed on a 2 x n grid. The squares in the top (bottom) row correspond to the barred (unbarred) elements.

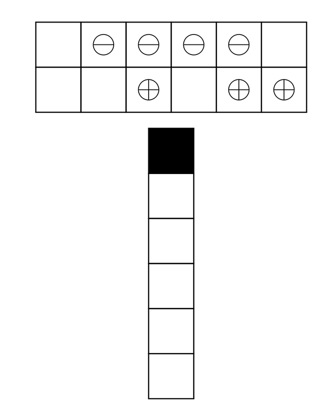


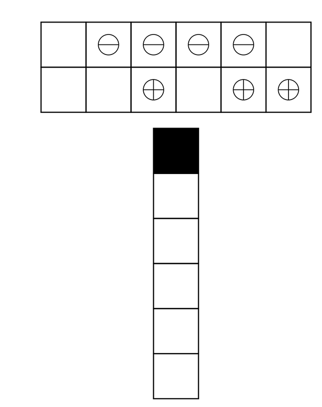
$$\{\bar{2}, \bar{3}, \bar{4}, \bar{5}, 3, 5, 6\}$$

Admissible

Definition. if $2 \times n$ circle diagram $\mathcal{P}$ is admissible if for each m, $1 \leq m \leq n$, the first m slots contain no more than m circles.

$$|\{x \in \mathcal{P} : ||x|| \leq m\}| \leq m$$





Implementing Bijection Between De Concini and King tableaux

bijection problem

bijection problem

bijection problem

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