Relating two combinatorial models in the representation theory of the symplectic group

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Abstract

Representation theory is a basic tool for understanding group symmetry using linear algebra, namely group elements are represented as invertible matrices. The symplectic groups $Sp(2n,\mathbb{C})$ are an important class of infinite groups, also known as simple Lie groups of type C. An irreducible representation of $Sp(2n,\mathbb{C})$ is indexed by partitions with at most n parts, or Young diagrams with at most n rows. A basis of such a representation is indexed by two types of fillings of the mentioned Young diagrams with integers ranging from $-n$ to n except 0, known as King tableaux and De Concini tableaux. I give an implementation of an algorithm which constructs a bijection between these two sets of tableaux. This bijection has many applications to the study of representations of the symplectic group.

Motivation

In this project, I have studied three kinds of tableaux: De Concini, King, and Kashiwara-Nakashima tableaux. These tableaux can be transformed to one another by certain bijections, which are available in the literature. I made a code generating each class of tableaux, and I implemented these bijections. I believe that researchers who are working in this area will use my package. This project was a very useful learning experience for me.

Symplectic Group

Group

A group is an algebraic structure with following properties:
(1) it is closed under a group operation (multiplication).
(2) there is an identity element.
(3) every element has an inverse.

Symplectic group

The group $Sp(2n,\mathbb{C})$ is the group of $2n\times 2n$ complex matrices A satisfying

$$M = A^t M A$$

where $M=\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}$

The complex matrix $A=\begin{pmatrix}0&1\\1&1\end{pmatrix}$ is in the symplectic group $Sp(2,\mathbb{C})$ Since

$$\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

Representation Theory

Representation theory

Representation theory is a branch of mathematics which studies how a group acts on a vector space.

$$\phi : G \to GL(V)$$

$Z_4 = (0,1,2,3)$

$\downarrow$

$$\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&-1\\1&0\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix},\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right)$$

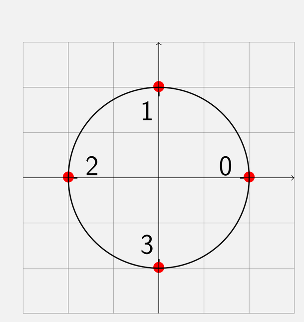


figure 1

Z_4 acting on $\mathbb{R}^2$

Irreducible representation

If a representation of G is built up out of other representations by direct sum, then it is called a reducible representation. otherwise, it is called an irreducible representation.

Every representation is the direct sum of irreducible representations.

$$V = V_1 \oplus V_2 \oplus V_3 \oplus \cdots \oplus V_n$$

where $V_i's$ are irreducible representations.

$$\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}\oplus\begin{pmatrix}b_{11}&b_{12}&b_{13}\\b_{21}&b_{22}&b_{23}\end{pmatrix}=\begin{pmatrix}a_{11}&a_{12}&0&0&0\\a_{21}&a_{22}&0&0&0\\0&0&b_{11}&b_{12}&b_{13}\\0&0&b_{21}&b_{22}&b_{23}\end{pmatrix}$$

Young diagram

The irreducible representations of $Sp(2n,\mathbb{C})$ are indexed by partitions with at most n parts, or Young diagrams with at most n row. A basis of such a representation is indexed by two types of fillings of the mentioned Young diagrams with integers ranging from $-n$ to n except 0, known as King tableaux and De Concini tableaux.

Definition of Young diagram

A Young diagram is a finite collection of boxes, or cells, arranged in left-justified rows, with the row lengths in non-increasing order.

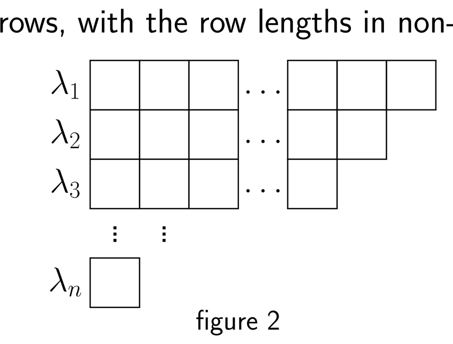


figure 2

$\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n), l(\lambda) = |n|$

Semistandard Young Tableaux

A tableau is called semistandard if the entries inside of Young diagram weakly increase along each row and strictly increase down each column.

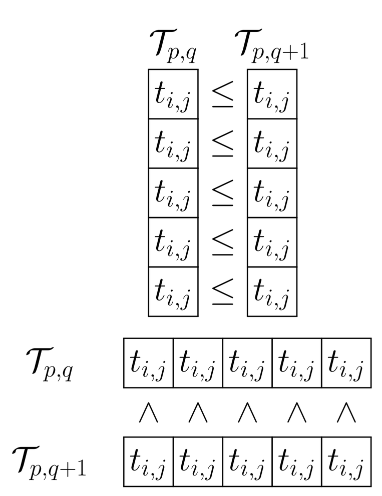


figure 3

$\mathcal{T}_\lambda = \{t_{i,j} : 1 \leq i \leq l(\lambda),\ 1 \leq j \leq \lambda_i\}$

Example

1	2	3	3	3	4
2	4	5	6	6	
3	5	6	7		
6					

figure 4

$\lambda = (6,5,4,1)$

The set used to fill the boxes in a Young diagram is given below

$$[[n]] = \{\overline{n}, \overline{n-1}, \ldots, \overline{1}, 1, \ldots, n-1, n\}$$

Here we allow negative values $-k = \overline{k}$.

King Tableaux

King tableaux have the following properties:

- **filling with a given set below:**

$$[[n]]_{\mathcal{K}} = \{\overline{1} <_{\mathcal{K}} 1 <_{\mathcal{K}} \overline{2} <_{\mathcal{K}} 2 <_{\mathcal{K}} \overline{3} <_{\mathcal{K}} 3 <_{\mathcal{K}} \overline{4} <_{\mathcal{K}} \cdots <_{\mathcal{K}} n\}$$

- **First entry in kth row is greater or equal to $\overline{k}$**

$\overline{1} \leq$	$\overline{1}$	$\overline{1}$	2
$\overline{2} \leq$	2	2	3
$\overline{3} \leq$	3	3	

figure 8

$\mathcal{K}(\lambda,3),\ \lambda=(3,3,2)$

De Concini Tableaux

De Concini tableaux have the following properties:

- **filling with a given set below:**

$$[[n]]_{\mathcal{D}} = \{n <_{\mathcal{D}} \overline{n-1} <_{\mathcal{D}} \overline{n-2} <_{\mathcal{D}} \cdots <_{\mathcal{D}} n-1 <_{\mathcal{D}} n\}.$$

- **Each columns are admissible**

A column is expressed as a $2\times n$ circle diagram, and it is admissible when the first m slots contain no more than m circles.

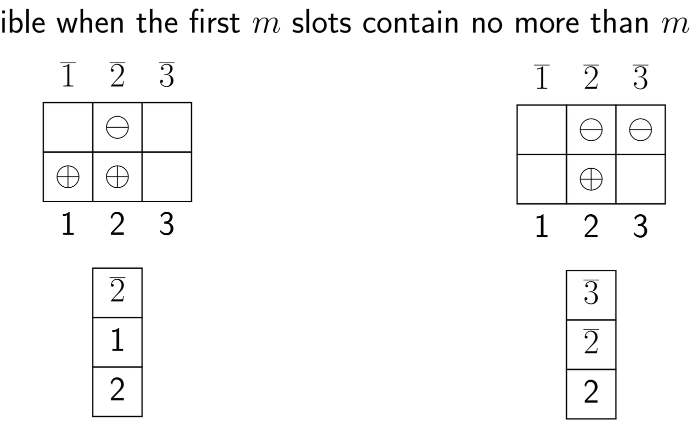
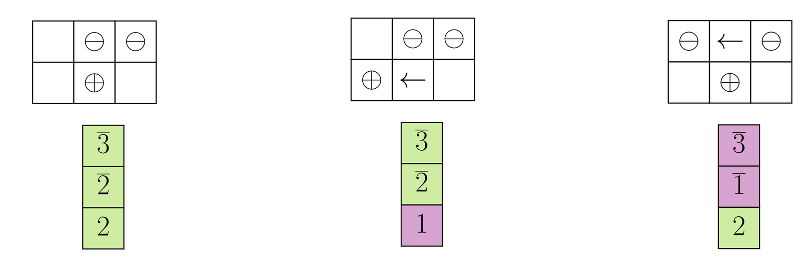


figure 9 Non-admissible column

figure 10 Admissible column

If the nth slot has two circles, then it is called a full slot, if it has one circle, then it is called a half slot, and if it has no circle, then it is called an empty slot.

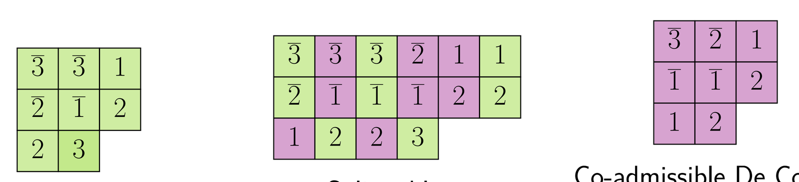
- **Each columns can be split into left and right part**



from figure 10

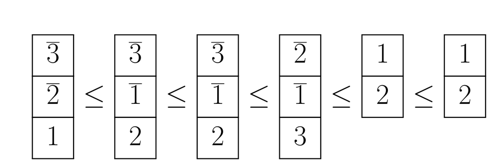
To make the left and right parts from an admissible column, move each mth full slot to the leftmost empty slot less than m. If moving unbarred entries only, then we obtain the left part. If moving barred entries only, then we obtain the right part.

A co-admissible column is obtained by moving each mth full slot, both unbarred and barred, to the leftmost empty slot less than m.



$\mathcal{D}(\lambda,3),\ \lambda=(3,3,2)$

- **Split form follows the definition of semistandard Young tableaux**



from figure 11

Hybrid Tableaux

A hybrid tableau consists of a De Concini tableau in its top left corner, and a skew King tableau in the bottom right corner.

- **The set of hybrid tableaux uses the order $[[n]]_d$ below:**

$$[[n]]_d = \{\overline{d} <_{\mathcal{D}} \overline{d-1} <_{\mathcal{D}} \cdots <_{\mathcal{D}} d-1 <_{\mathcal{D}} d <_{\mathcal{K}} \overline{d+1} <_{\mathcal{K}} d+1 <_{\mathcal{K}} \cdots <_{\mathcal{K}} \overline{n} <_{\mathcal{K}} n\}.$$

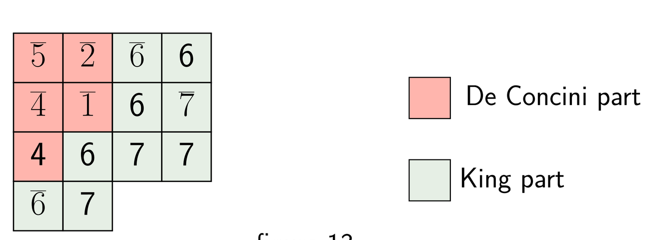


figure 13

$\mathcal{M}^5((4^3,2),7)$

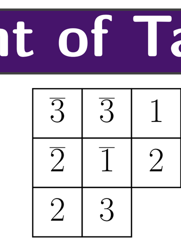
Intermediate Hybrid Tableau

This hybrid tableau is also called 5-semistandard Young tableau, as it uses the following ordered set:

$$[[7]]_5 = \{5 <_{\mathcal{D}} \overline{4} <_{\mathcal{D}} \overline{3} <_{\mathcal{D}} \overline{2} <_{\mathcal{D}} \overline{1} <_{\mathcal{D}} 1 <_{\mathcal{D}} 2 <_{\mathcal{D}} 3 <_{\mathcal{D}} 4 <_{\mathcal{D}} 5 <_{\mathcal{K}} \overline{6} <_{\mathcal{K}} 6 <_{\mathcal{K}} \overline{7} <_{\mathcal{K}} 7\}.$$

Note. The set of k-semistandard tableaux of shape λ is denoted by $\mathcal{M}^k(\lambda,n)$, so $\mathcal{M}^n(\lambda,n)=\mathcal{D}$ is the set of De Concini tableaux, and $\mathcal{M}^1(\lambda,n)=\mathcal{K}$ is the set of King tableaux.

Weight of Tableaux



$\boxed{1}+\boxed{\overline{1}} = 0$

$\boxed{3}+\boxed{\overline{3}}\,\boxed{3} = -1$

$\boxed{2}\,\boxed{2}+\boxed{\overline{2}} = 1$

from figure 7

$wt(\mathcal{D}((3,3,2),3)) = (0,1,-1)$

The weight of a tableau $\mathcal{T}$, denoted by $wt(\mathcal{T})$, is an n-tuple $(j_1, j_2, \ldots, j_n)$, where j_i is the number of i's minus the number of $\overline{i}$'s occurring in $\mathcal{T}$.

Bijection from De Concini to King Tableaux

For the bijection, *symplectic jeu de taquin* (sjdt) is applied to hybrid tableaux; sjdt is an extention of Schützenberger's original jeu de taquin. sjdt is used to construct a weight preserving bijection between $\mathcal{D}$ and $\mathcal{K}$.

Original Jeu de Taquin

It is easy to comprehend by sliding puzzle pieces. These are moved by comparing $\mathcal{T}_{p+1,q}$ with $\mathcal{T}_{p,q+1}$, and there are two possible moves called horizontal and vertical.

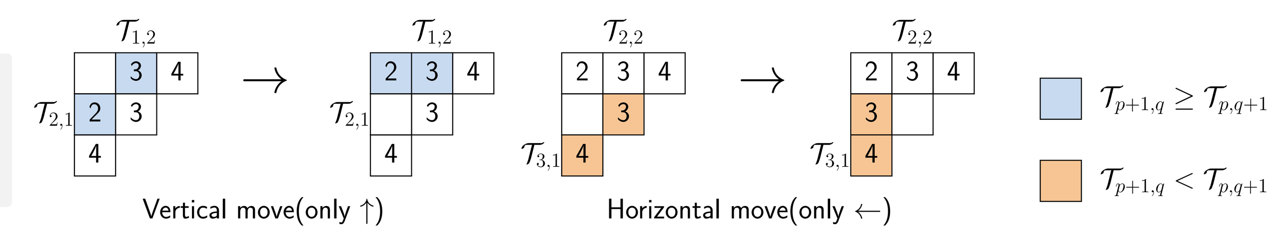


figure 14

Symplectic Jeu de Taquin

sjdt is a bit complicated since it has to preserve the admissible column condition when applied to hybrid tableaux.

Vertical move and horizontal move in symplectic jeu de taquin

These moves are performed on the split form of tableaux.

Vertical move

When $r\mathcal{T}_{p+1,q} \leq l\mathcal{T}_{p,q+1}$, move $\mathcal{T}_{p+1}$ to $\mathcal{T}_p$. The move itself is the same as the original jeu de taquin algorithm, but we compare the right and left parts of each column.

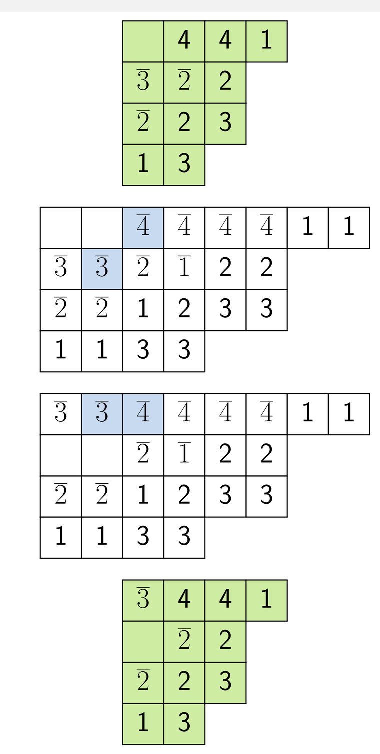


figure 15

Horizontal move

When $r\mathcal{T}_{p+1,q} > l\mathcal{T}_{p,q+1}$, move $\mathcal{T}_{p,q+1}$ to $\mathcal{T}_{p,q}$.
We have the following cases:

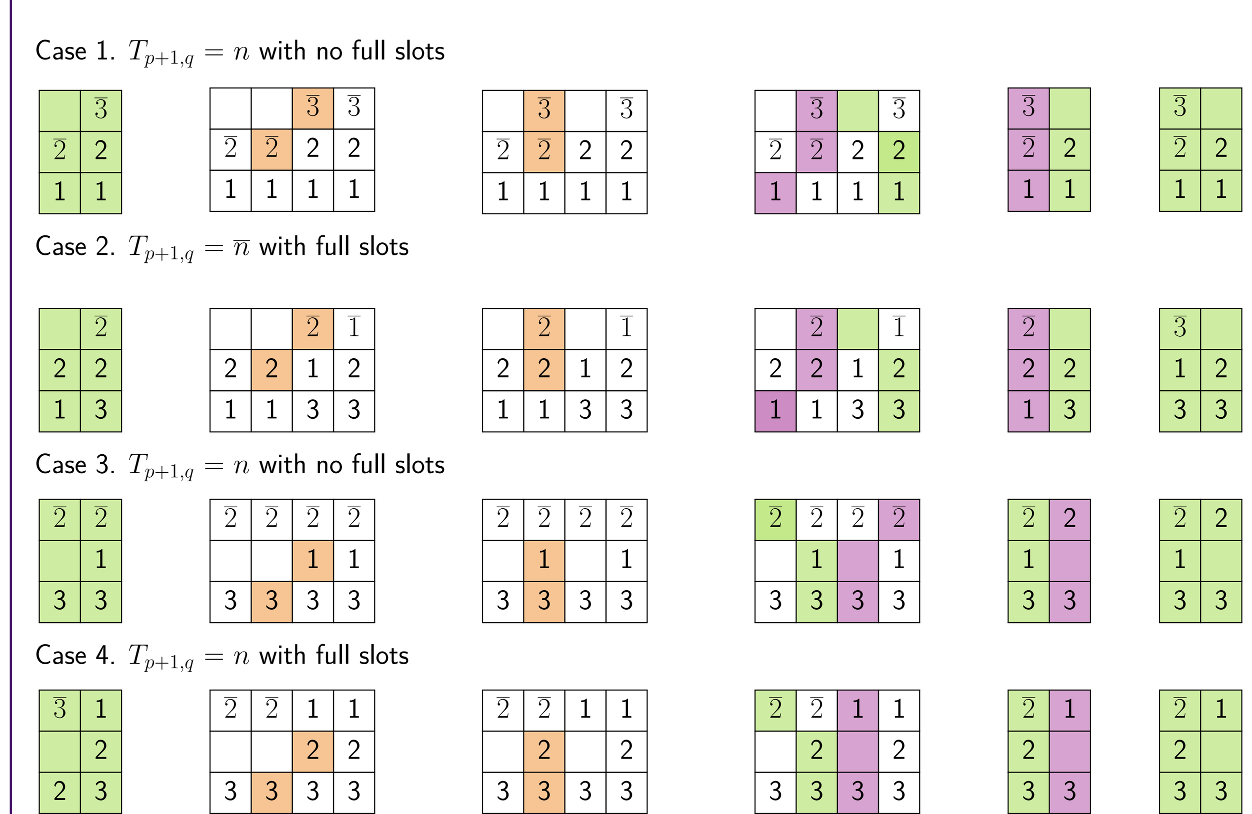


figure 16

Example for the Bijection Algorithm

$$\mathcal{D}(\lambda,3) \xrightarrow{\Phi_\lambda^3} \mathcal{M}^2(\lambda,3) \xrightarrow{\Phi_\lambda^2} \mathcal{K}(\lambda,3)$$

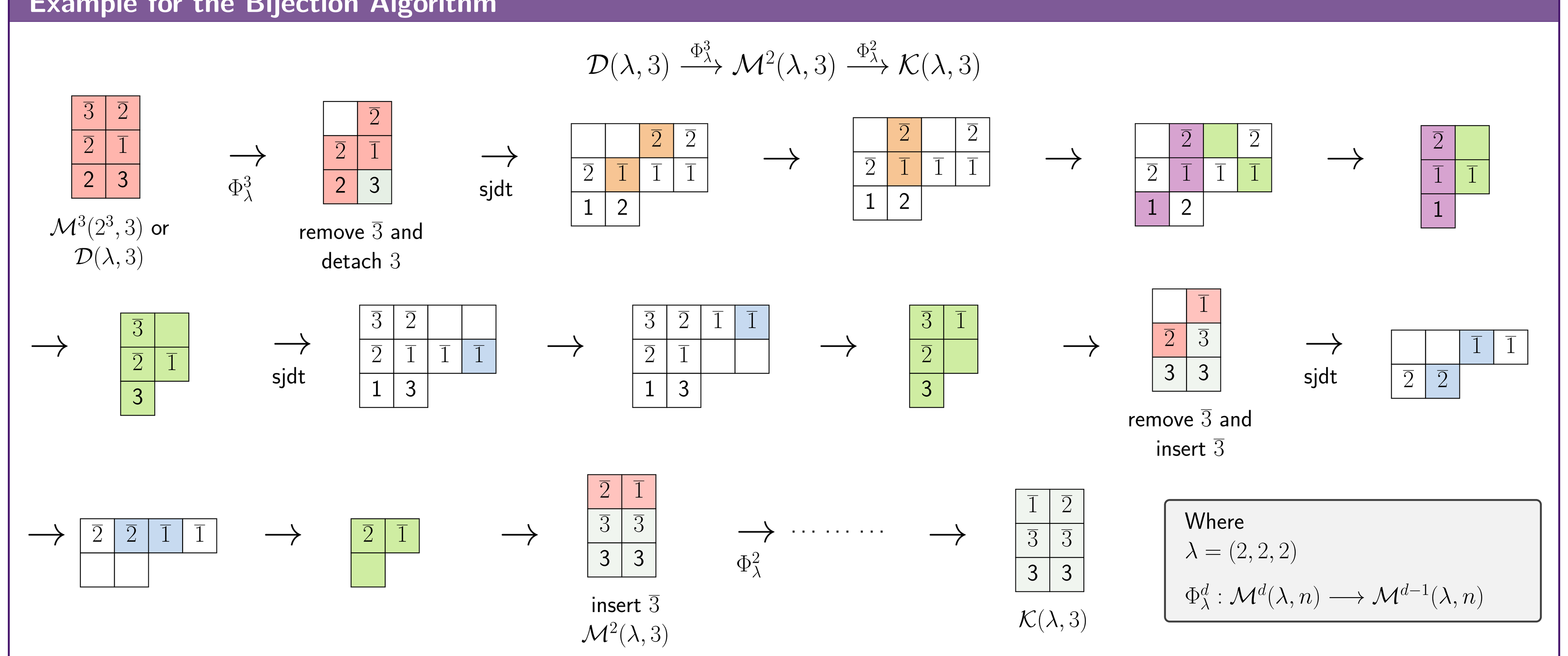


figure 17

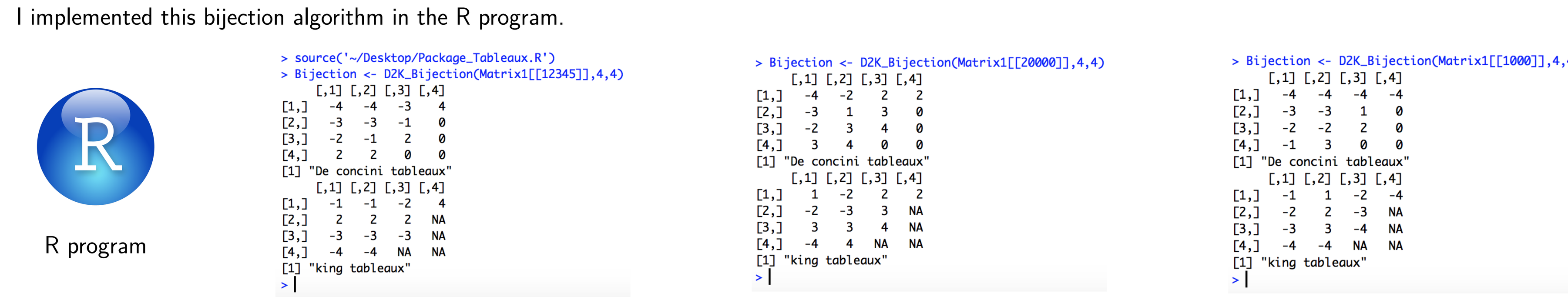
The bijection consists of successively applying the maps $\Phi_\lambda^k : \mathcal{M}^k(\lambda,n)\to\mathcal{M}^{k-1}(\lambda,n)$, for $k=n,n-1,\ldots,2$.

$$\mathcal{D}(\lambda,n) \xrightarrow{\Phi_\lambda^n} \mathcal{M}^{n-1}(\lambda,n) \xrightarrow{\Phi_\lambda^{n-1}} \mathcal{M}^{n-2}(\lambda,n) \xrightarrow{\Phi_\lambda^{n-2}} \ldots \xrightarrow{\Phi_\lambda^3} \mathcal{M}^2(\lambda,n) \xrightarrow{\Phi_\lambda^2} \mathcal{K}(\lambda,n)$$

Implementation

I implemented this bijection algorithm in the R program.

R program



Reference

J. Sheats. A symplectic jeu de taquin bijection between the tableaux of King and of De Concini. *Trans. Amer. Math. Soc.* 351:3569-3607, 1999.