Relating two combinatorial models in the representation theory of the symplectic group

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Abstract

Representation theory is a basic tool for understanding group symmetry using linear algebra, namely group elements are represented as invertible matrices. The symplectic groups $Sp(2n,\mathbb{C})$ are an important class of infinite groups, also known as simple Lie groups of type C. An irreducible representation of $Sp(2n,\mathbb{C})$ is indexed by partitions with at most n parts, or Young diagrams with at most n rows. A basis of such a representation is indexed by two types of fillings of the mentioned Young diagrams with integers ranging from $-n$ to n except 0, known as King tableaux and De concini tableaux. I give an implementation of an algorithm which constructs a bijection between these two sets of tableaux. This bijection has many applications to the study of representations of the symplectic group.

Symplectic Group

Group

Group is a algebraic structure that is closed under multiplication and having identity element and inverses of each element in a group.

Symplectic group

Symplectic group is the group that satisfies following condition.

$$Sp_{2n} = \{A : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}^{2n}\}$$

Q is symplectic form :

$$Q : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}^{2n}$$

$$(Q(e_i, e_j))_{1 \geq i,j \geq 2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

Representation Theory

Representation theory is a branch of mathematics to study the groups based on the way they act on vector space.

$$Z_4 = (0,1,2,3)$$
$$\downarrow$$
$$\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&-1\\1&0\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix}, \begin{pmatrix}0&1\\-1&0\end{pmatrix}\right)$$

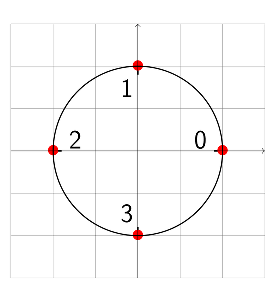


figure 1

Z_4 acting on $\mathbb{R}^2$

$$\phi : G \to GL(V)$$

Irreducible Representation

If representation of G (or $GL(V)$) is built up out of other representations by direct sum, then it is called reducible representation. otherwise, it is called irreducible representation.

Every representation is the direct sum of irreducible representations.

$$V = V_1 \oplus V_2 \oplus V_3 \oplus \cdots \oplus V_n$$

where V_i are distinct irreducible representations.

Young Tableaux

Irreducible representation of Symplectic group is indexed by Young diagram.

Definition. A Young diagram is a finite collection of boxes, or cells, arranged in left-justified rows, with the row lengths in non-increasing order.

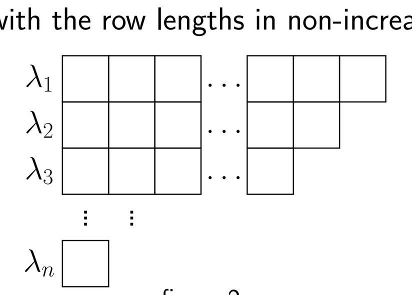


figure 2

$\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n), \mathrm{l}(\lambda) = |n|$

Semistandard Young Tableaux

Definition. A tableau is called semistandard if the entries inside of Young diagram weakly increase along each row and strictly increase down each column.

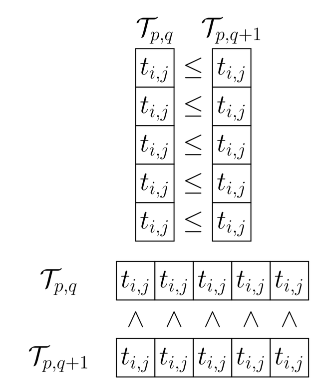


figure 3

Example

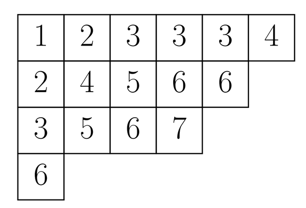


figure 4

$$\mathcal{T}_\lambda = \{t_{i,j} : 1 \leq i \leq l(\lambda),\ 1 \leq j \leq \lambda_i\}$$

Ordered set

Semistandard Young diagram is filling from the set which is given by

$$\{\bar{1}, 1, \bar{2}, 2, \bar{3}, 3, \ldots\ \bar{n}, n\}$$

Ordered set

This set is ordered by the rule described below

$$\bar{d} <_d \overline{d-1} <_d \cdots <_d \bar{1} <_d 1 <_d 2 <_d \cdots <_d d$$
$$<_d \overline{d+1} <_d d+1 <_d \overline{d+2} <_d d+2 <_d \ldots$$

$$[[\mathbf{n}]]_d = (\bar{d}, \overline{d-1}, \overline{d-2}, \ldots \bar{1}, 1, 2, 3, \ldots, d, \overline{d+1}, d+1, \overline{d+2}, d+2, \ldots).$$

and this ordered set is filling d-Semistandard Young Tableaux
note. $||n||$ indicate $\overline{n}$ or n

Circle diagram

One column of Semistandard Young tableaux can be expressed as the form of Circle diagram

Circle diagram is a method of viewing a subset of [[**n**]]. It is constructed on a 2 x n grid. The squares in the top (bottom) row correspond to the barred (unbarred) elements.

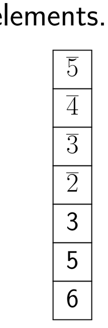


figure 7
One column n-Semistandard Young tableaux

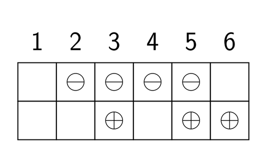
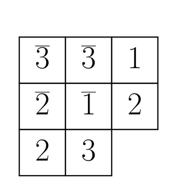


figure 8
Circle diagram of figure 7

Weight of Tableaux

weight of tableaux denoted by $wt(\mathcal{T})$ is n-tuple $(j_1, j_2, \ldots, j_n)$,where j_i is the number of $i's$ minus the number of $(\bar{i})'s$ occurring in $\mathcal{T}$



from figure 5

$wt(\mathcal{D}((3,3,2),3)) = (0,1,1)$

King Tableaux

King Tableaux has following properties

• Having 1-semistandard ordered set

$$\cdots <_n 3 <_n 2 <_n \bar{1} <_n 1 <_n 2 <_n 3 <_n \ldots$$

$$[[\mathbf{n}]]_n = \{\bar{n}, \overline{n-1}, \overline{n-2}, \ldots\ , \bar{1}, 1, 2, \ldots, n\}.$$

• First columns of each rows are larger than $||n||$

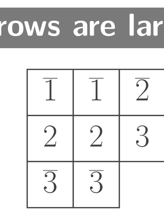


figure 6

$\mathcal{K}(\lambda, 3),\ \lambda = (3,3,2)$

De Concini Tableaux

De Concini Tableaux has following properties

• Having n-semistandard ordered set

$$\cdots <_n 3 <_n 2 <_n \bar{1} <_n 1 <_n 2 <_n 3 <_n \ldots$$

$$[[\mathbf{n}]]_n = \{n, \overline{n-1}, \overline{n-2}, \ldots\ , \bar{1}, 1, 2, \ldots, n\}.$$

• Each columns are admissible

Column is expressed as $2 \times n$ circle diagram, and it is admissible when the first m slots of circle diagram contain no more than m circles.

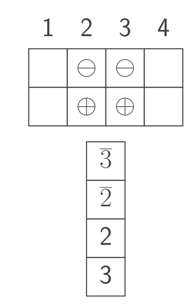


figure 8
Non-admissible column

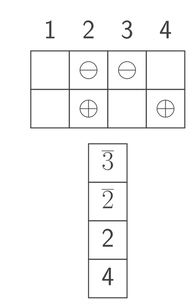
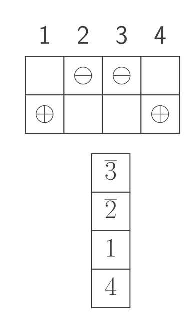
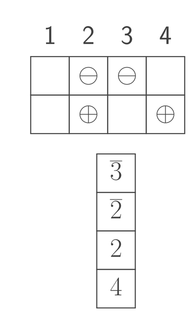


figure 9
Admissible column

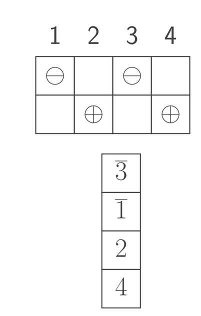
• Each columns can be splitted into left and right part



left part



Original column
from figure 9



right part

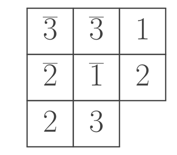


figure 10
Orignal unsplitted De Concini Tableaux

$\mathcal{D}(\lambda, 3),\ \lambda = \{3,3,2\}$

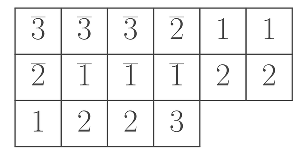


figure 11
Splitted De Concini Tableaux

• Splitted form follows the definition of Semistandard Young Tableaux

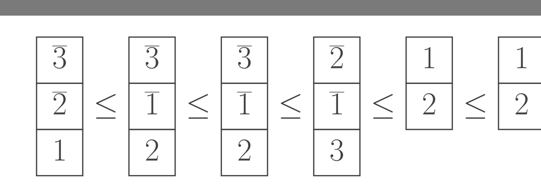


figure 12

Splitted form of De Concini Tableaux weakly increasing along each row and strickly increase down each column

Hybrid Tableaux

Hybrid Tableaux consist of sub-tableaux part and skew-tableaux part, and each tableaux are corresponding to De Concini Tableaux and King Tableaux

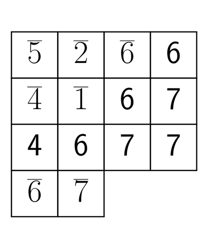


figure 10

$\mathcal{M}^5((4^3, 2), 7)$

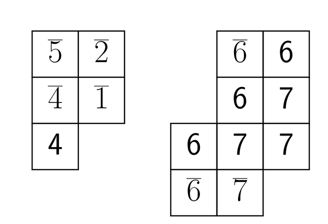


figure 11
De Concini part(left)
King part(right)

Bijection between $\mathcal{D}$ and $\mathcal{K}$

Definition. A tableau $\mathcal{T}$ is in $\mathcal{M}^d(\lambda, n)$ if and only if (i) it is d-semistandard, (ii) has admissible columns, and (iii) the subtableau of $\mathcal{T}$, say of shape μ, consisting of these entries $[[\mathbf{d}]]_d$ such that $\mathcal{D}(\mu, d)$.

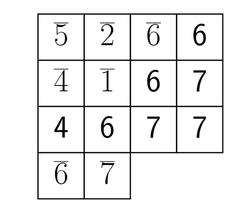


figure 10

$\mathcal{M}^5((4^3, 2), 7)$

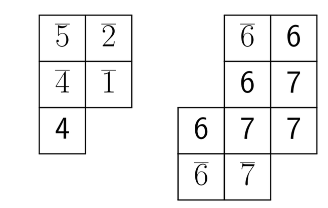


figure 11
De Concini part(left)
King part(right)

Bijection process

Suppose that $\Phi^n_\lambda : \mathcal{M}^n(\lambda, n) \to \mathcal{M}^{n-1}(\lambda, n)$ is weight preserving bijection. Define the map $\Psi^{n,d}_\lambda : \mathcal{M}^d(\lambda, n) \to \mathcal{M}^{d-1}(\lambda, n)$, for each d, $n \geq d \geq 1$ as follows. Let $\mathcal{T}_\lambda \in \mathcal{M}^d(\lambda, n)$ and Let $\mathcal{T}_\mu$ be the De Concini part of $\mathcal{T}_\lambda$ of shape μ. $\mathcal{T}_\mu$ is an element of shape $\mathcal{M}^d(\mu, d)$. Define $\Psi^{n,d}_\lambda(\mathcal{T}_\lambda)$ to be the tableau formed by replacing $\mathcal{T}$ with $\Phi^d_\mu(\mathcal{T}_\mu)$.
Thus, the bijection process between De Concini to King tableaux is solved by

$$\Psi^{n,2}_\lambda \circ \Psi^{n,3}_\lambda \circ \ldots \Psi^{n,n}_\lambda : \mathcal{M}^n(\lambda, n) \to \mathcal{M}^1(\lambda, n)$$

Note. $\mathcal{D}(\lambda, n) = \mathcal{M}^n(\lambda, n)$ and $\mathcal{K}(\lambda, n) = \mathcal{M}^1(\lambda, n)$

Blank

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