Relating two combinatorial models in the representation theory of the symplectic group

Hanjoon Choe(Presenter), Cristian Lenart(Advisor)

Department of Mathematics and Statistics, University at Albany,SUNY

Abstract

Representation theory is a basic tool for understanding group symmetry using linear algebra, namely group elements are represented as invertible matrices. The symplectic groups $Sp(2n,\mathbb{C})$ are an important class of infinite groups, also known as simple Lie groups of type C. An irreducible representation of $Sp(2n,\mathbb{C})$ is indexed by partitions with at most n parts, or Young diagrams with at most n rows. A basis of such a representation is indexed by two types of fillings of the mentioned Young diagrams with integers ranging from $-n$ to n except 0, known as King tableaux and De concini tableaux. I give an implementation of an algorithm which constructs a bijection between these two sets of tableaux. This bijection has many applications to the study of representations of the symplectic group.

Symplectic Group

Group

Group is a algebraic structure that is closed under operation and having identity element and inverses of each element in a group.

Symplectic group

Symplectic group is the group that satisfies following condition.

$$Sp_{2n} = \{A : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}^{2n}\}$$

Q is symplectic form :

$$Q : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}^{2n}$$

$$(Q(e_i, e_j))_{1\geq i,j\geq 2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

Representation Theory

Representation theory is a branch of mathematics to study the groups based on the way they act on vector space.

$$Z_4 = (0,1,2,3)$$
$$\downarrow$$
$$\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&-1\\1&0\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix},\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right)$$

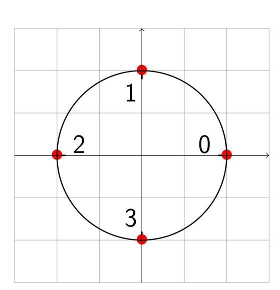


figure 1

Z_4 acting on $\mathbb{R}^2$

$$\phi : G \to GL(V)$$

Irreducible Representation

If representation of G (or $GL(V)$) is built up out of other representations by direct sum, then it is called reducible representation. otherwise, it is called irreducible representation.

Every representation is the direct sum of irreducible representations.

$$V = V_1 \oplus V_2 \oplus V_3 \oplus \cdots \oplus V_i$$

where $V_i's$ are distinct irreducible representations.

Young Tableaux

Irreducible representation of Symplectic group is indexed by Young diagram.

Definition. A Young diagram is a finite collection of boxes, or cells, arranged in left-justified rows, with the row lengths in non-increasing order.

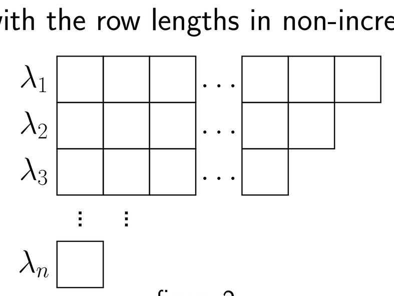


figure 2

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n), l(\lambda) = |n|$$

Semistandard Young Tableaux

Definition. A tableau is called semistandard if the entries inside of Young diagram weakly increase along each row and strictly increase down each column.

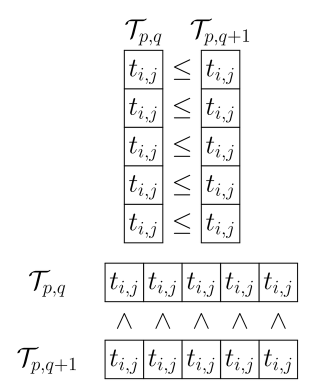


figure 3

Example

1	2	3	3	3	4
2	4	5	6	6	
3	5	6	7		
6					

figure 4

$$\mathcal{T}_\lambda = \{t_{i,j} : 1 \leq i \leq l(\lambda),\ 1 \leq j \leq \lambda_i\}$$

Ordered set

Semistandard Young Tableaux is filling the diagram with the set which is given by

$$\{\bar{1}, 1, \bar{2}, 2, \bar{3}, 3, \ldots \bar{n}, n\}$$

Ordered set

This set is ordered by the rule described below

$$\bar{d} <_d \overline{d-1} <_d \cdots <_d \bar{1} <_d 1 <_d 2 <_d \cdots <_d d$$
$$<_d \overline{d+1} <_d d+1 <_d \overline{d+2} <_d d+2 <_d \ldots$$

$$[[\mathbf{n}]]_d = (\bar{d}, \overline{d-1}, \overline{d-2}, \ldots \bar{1}, 1, 2, 3, \ldots, d, \overline{d+1}, d+1, \overline{d+2}, d+2, \ldots).$$

and this ordered set is filling d-Semistandard Young Tableaux

note. $||n||$ indicate $\overline{n}$ or n

Circle diagram

One column of Semistandard Young tableaux can be expressed as the form of Circle diagram

Circle diagram is a method of viewing a subset of [[n]]. It is constructed on a 2 x n grid. The squares in the top (bottom) row correspond to the barred (unbarred) elements.

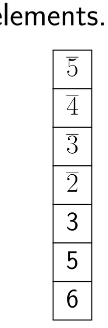


figure 7

One column n-Semistandard Young tableaux

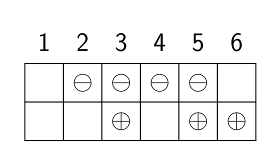


figure 8

Circle diagram of figure 7

Weight of Tableaux

weight of tableaux denoted by $wt(\mathcal{T})$ is n-tuple $(j_1, j_2, \ldots, j_n)$, where j_i is the number of $i's$ minus the number of $(\bar{i})'s$ occurring in $\mathcal{T}$

$\bar{3}$	$\bar{3}$	1
$\bar{2}$	$\bar{1}$	2
2	3	

from figure 5

$$wt(\mathcal{D}((3,3,2),3)) = (0,1,1)$$

King Tableaux

King Tableaux has following properties

• Having 1-semistandard ordered set

$$\bar{1} <_1 1 <_1 \bar{2} <_1 2 <_1 \bar{3} <_1 3 <_1 \bar{4} <_n \ldots$$

$$[[\mathbf{n}]]_1 = (\bar{1}, 1, \bar{2}, 2, \bar{3}, 3, \bar{4}, \ldots)$$

• First columns of each rows are larger than $||n||$

$\bar{1}$	$\bar{1}$	$\bar{2}$
2	2	3
$\bar{3}$	$\bar{3}$	

figure 6

$$\mathcal{K}(\lambda, 3),\ \lambda = (3,3,2)$$

De Concini Tableaux

De Concini Tableaux has following properties

• Having n-semistandard ordered set

$$\cdots <_n \bar{3} <_n \bar{2} <_n \bar{1} <_n 1 <_n 2 <_n 3 <_n \ldots$$

$$[[\mathbf{n}]]_n = \{\overline{n}, \overline{n-1}, \overline{n-2}, \ldots\ , \bar{1}, 1, 2, \ldots, n\}.$$

• Each columns are admissible

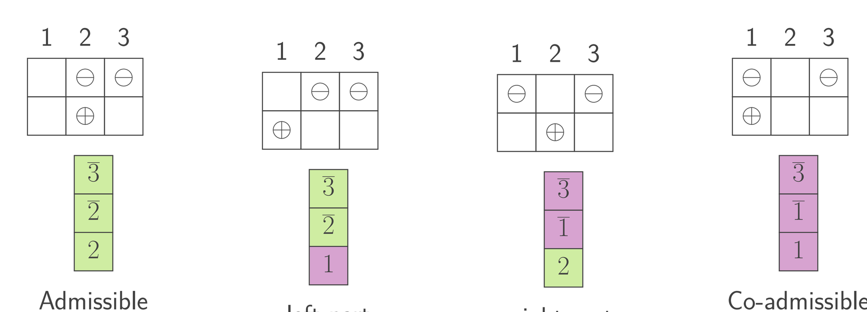
Column is expressed as $2 \times n$ circle diagram, and it is admissible when the first m slots of circle diagram contain no more than m circles.

figure 8
Non-admissible column

figure 9
Admissible column

Each slots of circle diagram corresponding to elements of one column Young tableaux. First and second row represent barred and unbarred elements of Young tableaux. if nth slot has two circle, then it is called full slot, if having one circle, then called half slot, and if neither of them, then called empty slot.

• Each columns can be splitted into left and right part



from figure 9

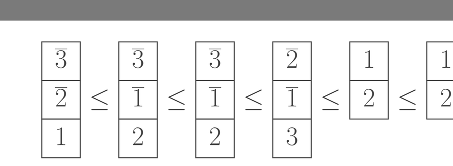
One-admissible column consists of barred and unbarred parts which are called A and D parts respectively. Left and right parts are consist of (A,C) and (B,D). B and C are made by moving full slots of circle diagram to the left empty slot, then barred part and unbarred part become B and C

figure 10
Orignal unsplitted De Concini Tableaux

$\mathcal{D}(\lambda, 3),\ \lambda = \{3,3,2\}$

figure 11
Splitted De Concini Tableaux

• Splitted form follows the definition of Semistandard Young Tableaux



from figure 11

Splitted form of De Concini Tableaux weakly increasing along each row and strickly increase down each column

Hybrid Tableaux

Hybrid Tableaux consist of sub part and skew part, and each parts are corresponding to De Concini Tableaux and King Tableaux

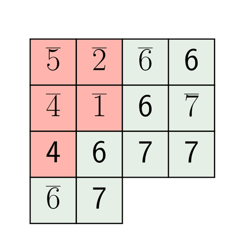


figure 10
$\mathcal{M}^5((4^3, 2), 7)$
Intermediate Hybrid Tableaux

Otherwise, Hybrid Tableaux can be called d-Semistandard Tableaux denoted by $\mathcal{M}^d(\lambda, n)$. Since its ordered set is d-Semistandard ordered set, 1-Semistandard Tableaux is King Tableaux and n-Semistandard Tableaux is De Concini Tableaux

$$\mathcal{K}(\lambda, n) = \mathcal{M}^1(\lambda, n) \text{ and } \mathcal{D}(\lambda, n) = \mathcal{M}^n(\lambda, n)$$

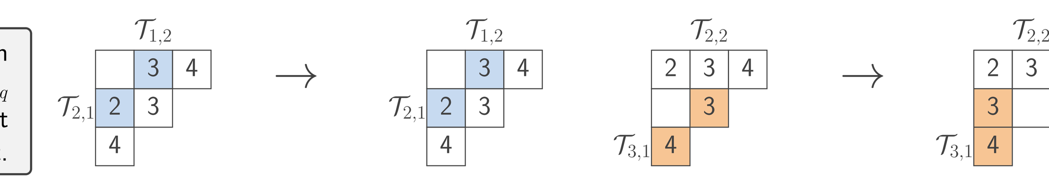
Bijection from De Concini to King Tableaux

Jeu De Taquin

For the bijection, Symplectic jeu de taquin is applied to hybrid tableaux. sjdt is an extention of Schutzenberger's original jeu de taquin. sjdt is used to construct weight preserving bijection between $\mathcal{D}$ and $\mathcal{K}$

Original Jeu De Taquin

It is easy to comprehend with sliding puzzle in which the pieces are moved by comparing $\mathcal{T}_{p+1,q}$ with $\mathcal{T}_{p,q+1}$, and there are two possible movement called horizontal movement and vertical movement.



Vertical Movement(only ↑)

Horizontal Movement(only ←)

$\mathcal{T}_{p+1,q} \geq \mathcal{T}_{p,q+1}$

$\mathcal{T}_{p+1,q} < \mathcal{T}_{p,q+1}$

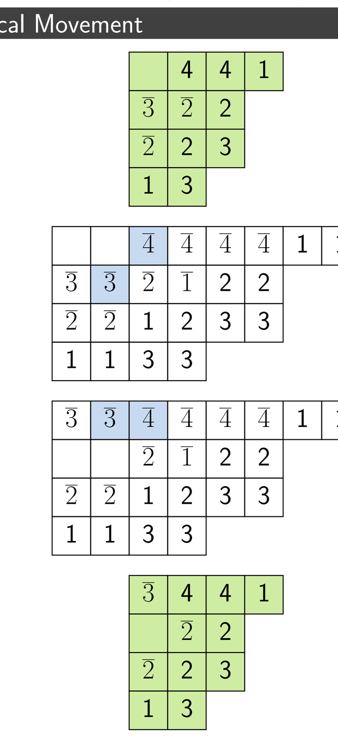
Symplectic Jeu De Taquin

sjdt is a bit complicated since it has to preserve admissible condition(also weight of tableaux) when applying to hybrid tableaux($\mathcal{M}$).

Vertical Movement and Horizontal Movement in Symplectic Jeu De Taquin

Two movements are operated on splitted form of tableaux.

Vertical Movement

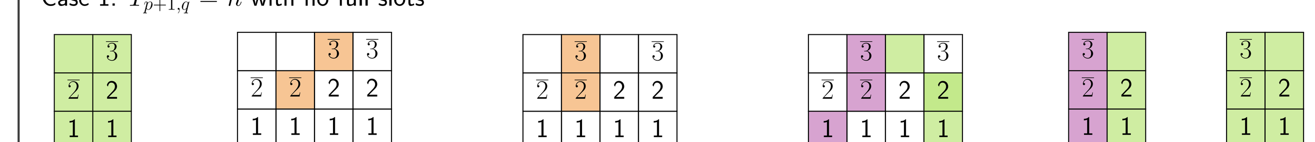


If $r\mathcal{T}_{p+1,q}$ is less than $l\mathcal{T}_{p,q+1}$, then move $\mathcal{T}_{p+1}$ to $\mathcal{T}_p$. Movement itself is the same as ojdt-algorithm but comparing right and left part of each columns

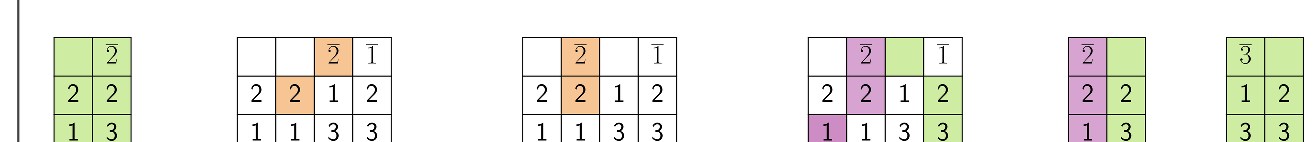
Horizontal Movement

If classifying each cases,

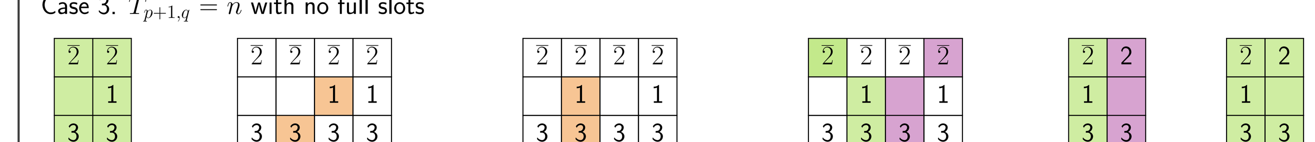
Case 1. $T_{p+1,q} = n$ with no full slots



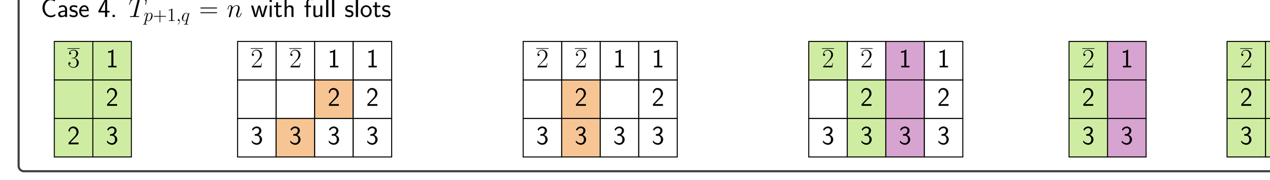
Case 2. $T_{p+1,q} = \bar{n}$ with full slots



Case 3. $T_{p+1,q} = n$ with no full slots

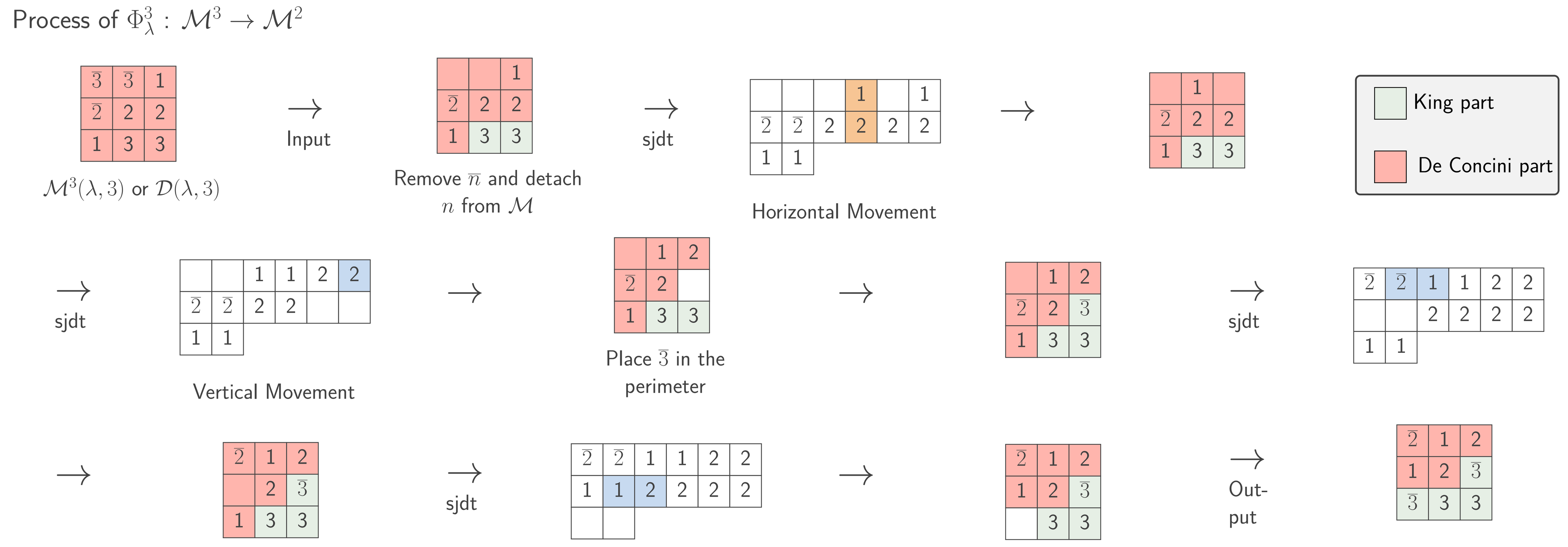


Case 4. $T_{p+1,q} = n$ with full slots



Example of Bijection Process

Process of $\Phi^3_\lambda : \mathcal{M}^3 \to \mathcal{M}^2$



$\mathcal{M}^3(\lambda, 3)$ or $\mathcal{D}(\lambda, 3)$

Remove $\overline{n}$ and detach n from $\mathcal{M}$

Horizontal Movement

Vertical Movement

Place $\bar{3}$ in the perimeter

Vertical Movement

$\mathcal{M}^2(\lambda, 3)$

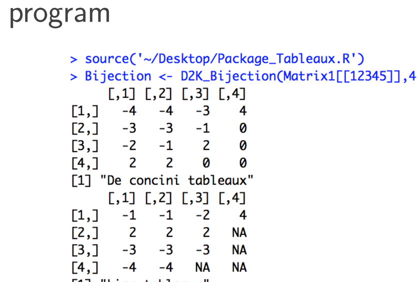
Thus, Bijection from D to K is applying

$$\Phi^2_\lambda \circ \cdots \circ \Phi^{n-2}_\lambda \circ \Phi^{n-1}_\lambda \circ \Phi^n_\lambda : \mathcal{M}^n \to \mathcal{M}^1$$

Implementation

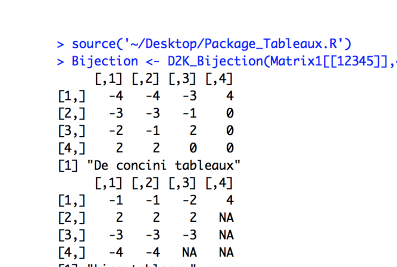
I implemented bijection algorithm into R program

R program



```
> source('~/Desktop/Package_Tableaux.R')
> Bijection <- D2K_Bijection(Matrix1[[12345]],4,4)
     [,1] [,2] [,3] [,4]
[1,]   -4   -4   -3    4
[2,]   -3   -3   -1    0
[3,]   -2   -1    2    0
[4,]    2    2    0    0
[1] "De concini tableaux"
     [,1] [,2] [,3] [,4]
[1,]   -1   -1   -2    4
[2,]    2    2    2   NA
[3,]   -3   -3   -3   NA
[4,]   -4   -4   NA   NA
[1] "king tableaux"
>
```

Demonstration



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[1,]   -4   -4   -3    4
[2,]   -3   -3   -1    0
[3,]   -2   -1    2    0
[4,]    2    2    0    0
[1] "De concini tableaux"
     [,1] [,2] [,3] [,4]
[1,]   -1   -1   -2    4
[2,]    2    2    2   NA
[3,]   -3   -3   -3   NA
[4,]   -4   -4   NA   NA
[1] "king tableaux"
>
```

Demonstration

Reference

Jefferey T.S.(1999) A Symplectic jeu de taquin bijection between the tableaux of King and of De Concini.