Dilutive Financing*

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Abstract

This paper presents a dynamic model of firm financing where firms use financial slack to reduce rent extraction by financiers possessing bargaining power. Financing is lumpy because it is optimal to bargain infrequently. Moreover, firms typically finance 'early' before exhausting internal funds to bargain when their outside options are better. Firms with better prospects maintain greater financial slack. Firms with good financing alternatives always keep funds that exceed investment needs, whereas firms lacking such alternatives delay financing until funds are depleted – and occasionally forgo investment – to avoid paying excessive rents. Investment irreversibility magnifies financing rents for unproductive firms.

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1 Introduction

Slack is a pervasive feature of firm financing. Firms generally – and even large, established and creditworthy ones – preserve funds internally instead of fully deploying them productively such as for investment or distributing them to financial stakeholders such as in dividends.[1] Prima facie, this constitutes both opportunity costs for firms, because keeping funds yields lower returns than cost of funds, and macroeconomic inefficiency, because funds are not allocated to more financially constrained firms. Given its ubiquity, it is important to understand the origin and nature of 'financial' slack.

The canonical framework on financial slack, dating to Baumol (1952) and Tobin (1956) who posited fixed transaction costs, mainly focuses on frictions in liquidity.[2] Accordingly, firms with great access to financing, such that financing is liquid, must have little incentives to preserve funds, which is inconsistent with empirical evidence.

In this paper, I present a new theory of dynamic firm financing and financial slack based on financier bargaining power. It explains two key aspects of financial slack: first, firms raise financing infrequently and in a lumpy fashion; second, they typically do so "early," that is, well before running out of funds. Both features emerge in my model as part of firms' optimal financing strategy to reduce rents that financiers are able to extract. The first feature of 'lumpy financing' microfounds fixed transaction costs, and the second feature of 'early financing' fills the gap in the existing literature.

I consider an environment where a firm regularly needs financing, due to negative shocks or investment needs. While it may maintain internal funds in anticipation, these involve a carry cost. The firm can raise funds from external financiers at any point, and there is no transaction cost of financing. It is thus feasible, and first-best, to hold zero internal funds and raise external financing incrementally whenever needed.

The key element that gives rise to financial slack is a bargaining friction. A firm can bargain with financiers at any point to raise funds, but while it is bargaining with one financier, the firm cannot immediately locate *another* financier if the bargaining were to fail. This assumption of financiers' 'local monopoly' captures the idea that finding alternative sources of financing can prove challenging in practice because financiers are often rather specialized. For example, startups rely on venture capital funds that have expertise for a specific industry and stage of the venture. Similarly, large and established firms resort to a handful of investment banks that address these firms' sizable funding needs with syndicated financing. In either case, it plausibly takes time to switch to another financier in response to a failed financing from one financier.[3]

[1] 'Funds' may refer only to cash reserves, but also include, more broadly, a remaining capacity for borrowing at low interest rate, such as lines of credit or short-term debt.

[2] Baumol and Tobin (1989) note that Allais (1947) first proposed the same basic mechanism.

[3] For modeling, one may consider search/informational frictions arising upon failed bargaining.

As a result, successful financing creates a positive surplus relative to a failure of bargaining, and financiers extract a portion of the surplus as rents, thereby diluting firm value ex-ante. In response, firms choose to raise financing in excess of immediate funding needs, and maintain internal funds, in order to avoid needing to bargain with financiers and incur dilution too frequently. Thus, financing is optimally lumpy.

Moreover, firms may choose to raise financing early, before exhausting internal funds, in order to bargain when their outside options are better. With financial slack retained at bargaining, firms may avoid having their "back against the wall": even if financing were to fail, they could use these funds to cover losses, temporarily forestalling business failure while seeking alternative sources of financing. By raising funds before financing is really needed, firms can thus improve their outside options, thereby reducing the surplus from financing – and hence the rents that financiers can extract.

In sum, financial slack reduces dilution of firm value by improving firms' bargaining with financiers. Lumpy financing reduces the frequency of dilution, and early financing reduces its size. This, I argue, is what determines the dynamics of firm financing.

On one hand, this theory microfounds fixed transaction costs such that they are determined not only by structural parameters but also by firms' endogenous financing strategy. With respect to structural parameters, this bargaining-based framework predicts that firms with better future prospects, ceteris paribus, have more financial slack, consistent with empirical literature. In addition, this model shows that in equilibrium, firms strategically *choose* between financing costs that are proportional to either financing amount (as if it were a 'variable transaction cost') – when they optimally raise financing early – or net firm value – when they optimally do so after running out of funds. It thus explains two commonly assumed forms of transaction costs (fixed and 'variable') as arising from firms' optimal strategy given bargaining frictions.

On the other hand, this theory has a key point of departure with comparative statics. In liquidity-based frameworks, better access to financing should directly reduce financing costs and hence lead to less financial slack, but this monotonicity is at odds with empirical findings. In this paper with bargaining frictions, firms with great access to financing can benefit greatly from raising financing before running out of funds. This is because if financing were to fail, off the equilibrium path, such firms can quickly find alternative sources of financing but only if they have not already run out of funds. As such, if these firms choose to keep funds when they bargain, their bargaining position is greatly improved, which greatly reduces financiers' rents on the equilibrium path. Put differently, it is firms' access to alternative financing *coupled* with financial slack kept at financing – that is, early financing – that reduces financing costs. Therefore, firms with better access to financing do not necessarily have less financial slack.

Furthermore, this model of firm financing has rich macroeconomic implications.

First, because it is a model of financial frictions, it naturally predicts underinvestment. More importantly, because the frictions consist in bargaining, the real economy also affects firms' financial frictions, and unevenly across firms. Suppose that capital stock is illiquid in the economy. Then, firms have worse bargaining positions vis-à-vis financiers because if financing were to fail, they cannot easily divest their capital to other firms to obtain extra funds and avoid running out of funds. Thus, financing costs rise for all firms. But its extent is endogenously heterogeneous. Productive firms invest more and so, if financing were to fail, can obtain more funds by reducing their investment; as such, their bargaining positions deteriorate less than those for unproductive firms. In short, given that liquidity of capital stock is procyclical, this model predicts that both the average and the dispersion of firms' financing costs are countercyclical.

Methodologically, the framework I propose is tractable, and allows general characterization of optimal financial slack, including in environments with stochastic parameters (e.g., a stochastic discount factor) and investment. I model the key bargaining friction – the time it takes to access alternative sources of financing – as a Poisson process. This results in a tractable recursive structure between equilibrium value function and reservation value function which captures firms' endogenous outside options. Leveraging it, I summarize the unique Markov-perfect dynamic bargaining solution with simple (s, S) bounds, characterize firms' optimal financing strategy, and obtain comparative statics for financial slack in parameters of the bargaining frictions.[4]

The rest of the paper is organized as follows. Section 2 describes related literature. Section 3 presents the core mechanism with a stylized two-period model. Section 4 builds the main dynamic model in infinite time horizon, and Section 5 analyzes it. Section 6 extends the framework with an investment choice. Section 7 concludes.

2 Related Literature

This paper makes contributions to four branches of the literature. The first two parts, I and II, discuss in greater detail the contributions already outlined, while the last two, III and IV, present additional contributions.

I. Illiquidity as an endogenous phenomenon due to bargaining. As mentioned, the classic theory of household money demand by Baumol (1952) and Tobin (1956) has been vastly influential. They posit illiquidity in trading an asset (i.e., bank deposit) due to a fixed transaction cost. The theory has been applied by Décamps, Mariotti, Rochet and Villeneuve (2011) and Bolton, Chen and Wang (2011) to the firm context, generating lumpy financing. Bolton, Chen and Wang (2013) explain early financing as

[4] The numerical algorithm uses a standard linear solver to jointly determine both equilibrium and reservation value functions, making numerical solutions easy to obtain for a general setup.

'market timing' given stochastic fixed costs.[5] As a different form of liquidity frictions, Hugonnier, Malamud and Morellec (2014) study search frictions in finding a financier.

Whereas existing theories assume illiquidity in financing as a primitive from the environment, my theory explains it as an endogenous phenomenon due to bargaining.[6]

Empirically, this paper coherently explains observed patterns of firm financial slack. Albertus, Glover and Levine (2025) examine how multinational firms responded to a positive liquidity shock, concluding that existing theories cannot explain the observed high retention of funds – even among well-governed and financially unconstrained samples. Similarly, Graham (2022) reports that firms, small or large, regard financial flexibility as the single most important factor in capital structure and investment decisions. In addition, this paper agrees with Opler, Pinkowitz, Stulz and Williamson (1999), Bates, Kahle and Stulz (2009), Graham and Leary (2018) and Begenau and Palazzo (2021), who document that firms with steep growth have large financial slack.

II. Capital reallocation, productivity and financing. This paper's framework, without featuring secured debt, allows financing frictions to worsen when capital becomes difficult to divest. In doing so, it extends the overall insights from the vast literature on borrowing constraints emphasizing the role of a resale price of capital – including Kiyotaki and Moore (1997), Lorenzoni (2008), Rampini and Viswanathan (2013), etc. – to general forms of firm financing, such as equity. The paper also highlights, consistent with Caggese (2007), Kurlat (2013), Lanteri (2018) and Cui (2022), the role of firm productivity in how difficulty of divestment affects firms.

III. Endogenous timing of trades and endogenous dynamic outside options. The seminal work by Rubinstein and Wolinsky (1985) extends the foundational bargaining game of Rubinstein (1982) to endogenize agents' outside options through search frictions. In this framework, agents can trade only upon an exogenous match, and they bargain over the surplus created by the match. Recently, McClellan (2024) endogenizes the timing of a trade, when the agent's outside option evolves exogenously.

In relation, this paper builds a tractable framework that jointly endogenizes timing of trades and dynamic evolution of outside options. Also, by modeling *alternative* trading opportunities as a Poisson arrival, it enables a clarity of analysis on marginal costs and benefits in choosing when to trade.

Concretely, this theory parsimoniously unifies two disparate mechanisms studied by Kiyotaki, Moore and Zhang (2024), with respect to how repeated rounds of bargaining dilute agents' payoffs and shorten their 'horizons,' and Lagos and Zhang (2022), with respect to how outside options off-path may limit the counterparty's bargaining power on-path. Moreover, this paper endogenizes agents' outside options in terms of their

[5] The same mechanism is used by Alvarez and Lippi (2009) to study household money demand.

[6] Supplemental Appendix SA.1 presents a comparison chart across models for financial slack.

dynamic bargaining strategy, leading to a distinct prediction of early bargaining.

This paper also relates to the literature on holdup such as Hart and Moore (1990). Commonality is in the underlying incentives: a first-best choice worsens agents' subsequent bargaining position. The point of departure – other than the context – is that this paper focuses on repeated rounds of optimally-timed holdup. As such, the frictions are preserved under dynamics unlike in Che and Sákovics (2004), for example.

IV. Complementing the literature on dynamic firm financing. In a sense, this paper's dynamics is driven by an incompleteness of long-term contracts; the model restricts to one-off trades for simplicity. Naturally, it generalizes debt dilution literature studying how specific forms of incomplete financing contracts affect firm dynamics. Myers (1977) shows that debt overhang dilutes equity value and suppresses investment. Admati, DeMarzo, Hellwig and Pfleiderer (2018) and DeMarzo and He (2021) study the 'leverage ratchet' effect where deleveraging benefits senior debt at the cost to equity. In Donaldson, Koont, Piacentino and Vanasco (2024), unused credit lines threaten new creditors with potential dilution, thus preventing dilution of existing creditors.

More broadly, this paper agrees with DeMarzo and Fishman (2007a, 2007b) and DeMarzo, Fishman, He and Wang (2012), but without entertaining agency frictions. It also builds on the literature on risk management, such as Froot, Scharfstein and Stein (1993), Rampini and Viswanathan (2010) and Mian and Santos (2018), by rationalizing financial slack even in substantial excess of firms' investment needs. Lastly, it shows that ease of liquidation may induce early refinancing of debt, instead of renegotiation upon default as in Bolton and Scharfstein (1996) and Hart and Moore (1998).

3 Core Mechanism

This section presents a stylized model that captures the core economic insights that due to financiers' bargaining power, firms may optimally raise financing infrequently and hence in a lumpy fashion, and also early – that is, before internal funds are exhausted.

Setup. There are two periods with three dates $t \in \{0, 1, 2\}$. There is no discounting. There is a project that lasts for the two periods. In each period, the project requires a unit of input; with any less input, it fails. The project yields a payoff $v > 2$ on terminal date $t = 2$. An agent called entrepreneur owns a firm that runs the project.[7]

The firm cannot produce the input. There are two suppliers, one visiting the firm on date 0 and the other on date 1, who can produce it at a constant marginal cost normalized to unity. Each of them has bargaining power and, if asked by the firm to

[7]The present setup gives a minimal environment that allows for multiple rounds of bargaining so that the core mechanism can be captured. The main model in Section 4 will entertain generic business profiles that allow for operating profits subject to occasional losses due to volatility.

produce the input, extracts a fraction $1-\theta$, $\theta \in (0,1)$, of the surplus from bargaining in addition to being compensated for the cost of producing the input. A supplier cannot bargain with the firm or produce the input except on the date of his visit. The input is durable, but it requires a marginal carry cost $\varphi > 0$ to store the input for one period.[8]

Denote h_t as the firm's inventory holding of stored input on date t; '(financial) slack' is when h_t is positive. By backward induction, the firm purchases from the second supplier $\max\{0, 1-h_1\}$ units of the input on date 1. If $h_1 \geq 1$, then there is no need to bargain with the second supplier. If $h_1 < 1$, the project fails without more input. The firm thus bargains with the second supplier and pays him the production cost $1-h_1$ plus his rent, which is a fraction $1-\theta$ of the surplus from bargaining $v-(1-h_1)$. Due to the carry cost $\varphi > 0$, $h_1 \in \{0,1\}$ in equilibrium as long as $h_0 \leq 1$. Firm value on date 1 is, then, v if $h_1 = 1$, and $v-1-(1-\theta)(v-1) = \theta(v-1)$ if $h_1 = 0$. Bargaining thus 'dilutes' firm value ex-ante: $\widehat{v} \equiv \theta(v-1) < v-1$ since $\theta < 1$.

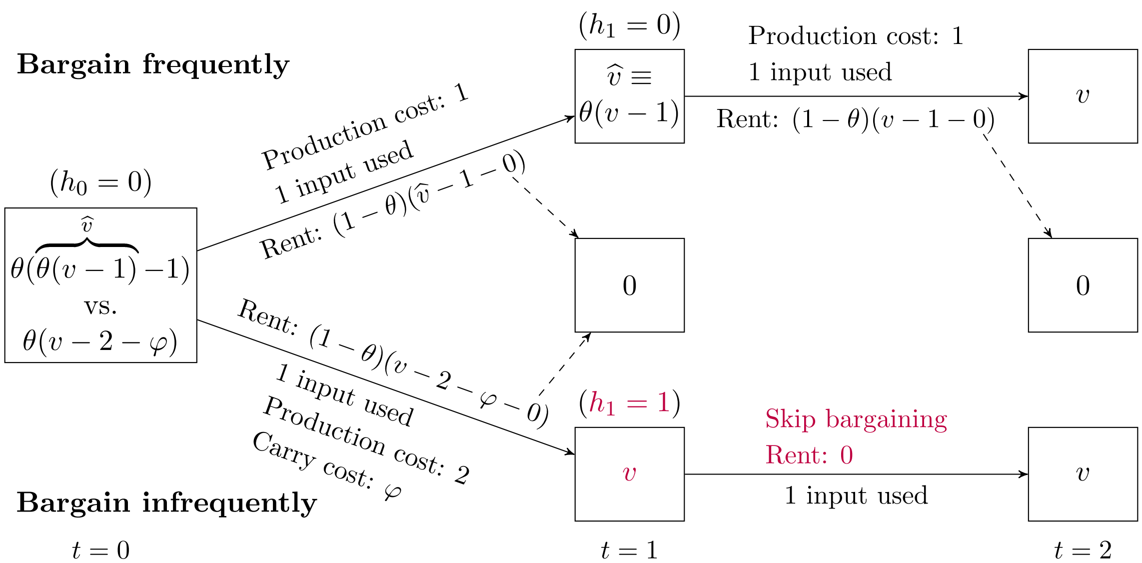


Figure 1: Frequency of bargaining

Rectangles show firm value at each node. The firm's input inventory on date t is h_t. Dashed arrows represent the firm's outside options – i.e., what happens if bargaining fails. Firm values and rents are determined as:

$$\text{Firm value}_t = \underbrace{\text{Firm value}_{t+1} - \text{Costs}_t}_{=\text{Continuation value}_t} - \text{Rent}_t, \quad \text{Rent}_t = (1-\theta)(\underbrace{\text{Continuation value}_t - \text{Outside option}_t}_{=\text{Bargaining surplus}_t}).$$

Frequency of bargaining and lumpy purchase. Suppose that the firm starts with no input inventory $h_0 = 0$. The entrepreneur chooses whether to bargain twice on both dates $t \in \{0,1\}$ or just once on date $t = 0$. If she bargains frequently on both dates, the entrepreneur purchases a unit of the input from the first supplier on date 0. No

[8]This can represent either the entrepreneur's utility cost or decay by a factor of $\frac{\varphi}{1+\varphi} \in (0,1)$.

input is stored into the next date $h_1 = 0$, so that firm value on date 1 is diluted $\widehat{v}$ from bargaining with the second supplier. If she instead bargains infrequently (that is, only on date 0), she buys two units from the first supplier. One of them is used in the first period, and the other is stored for the next period $h_1 = 1$ at the carry cost φ. Since it is unnecessary to bargain with the second supplier, firm value on date 1 is v.

Either way, the first supplier, when bargaining with the firm on date 0, extracts as his rent a fraction $1-\theta$ of the difference between firm value on date 1 and the total costs on date 0, which include production cost and, if applicable, carry cost.

The frequent bargaining maximizes net project value $v-2 > v-2-\varphi$, since the other choice involves a carry cost $\varphi > 0$. But the entrepreneur maximizes her own payoff, which is diluted firm value on date 0. Figure 1 describes the comparison. She optimally bargains infrequently if and only if $\theta\big(v-2-\varphi\big) \geq \theta\big(\theta(v-1)-1\big)$, or, equivalently,

$$(1-\theta)(v-1) \geq \varphi; \tag{1}$$

that is, if the rent that the second supplier would extract $(1-\theta)(v-1)$ upon frequent bargaining exceeds the carry cost φ upon infrequent bargaining.[9]

Even though the carry cost φ makes the project less valuable, the entrepreneur may optimally bargain infrequently; by avoiding the need to bargain with the second supplier, she ensures that her payoff from the project is diluted once rather than twice.

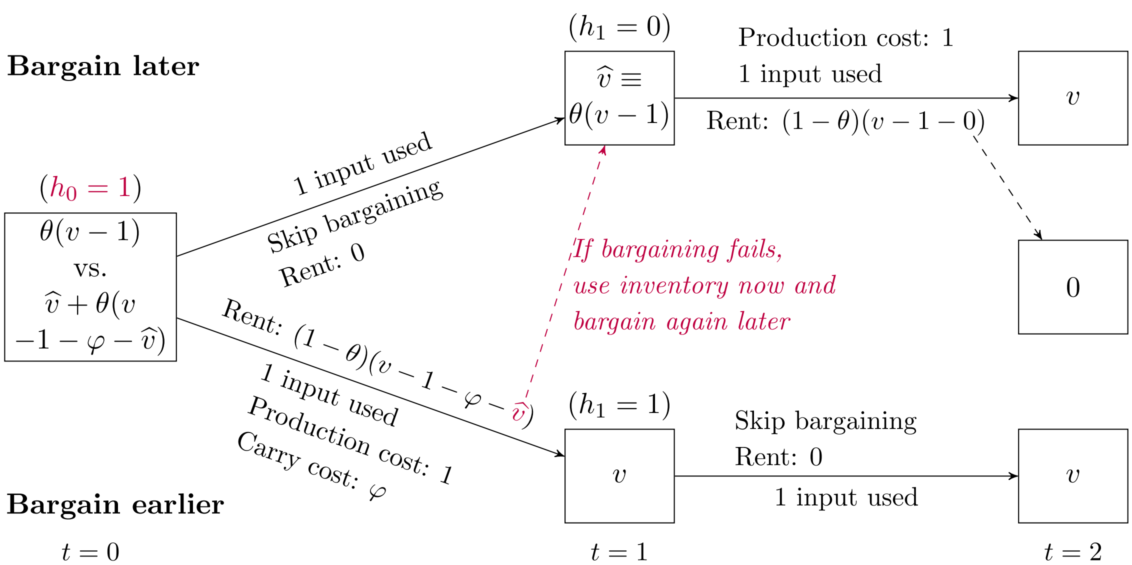


Figure 2: Timing of bargaining

Timing of bargaining and outside options. Next, suppose instead that the firm initially has one unit of the input $h_0 = 1$, so that it only needs to buy one unit for

[9]The entrepreneur may also optimally abandon the project if $\theta(v-1) \leq 1$ and $v-2-\varphi \leq 0$.

the second period. The entrepreneur needs to bargain once in any case, but must now choose when to bargain. If she bargains on date 1 with the second supplier, then her payoff is $\widehat{v}$. Note that while bargaining with the second supplier, the entrepreneur is at the brink of losing the project since there is no 'slack' $h_1 = 0$. If instead she bargains earlier on date 0 with the first supplier, she is no longer at the brink; even if bargaining were to fail on date 0, the entrepreneur can use the inventory $h_0 = 1$ now and bargain later with the second supplier. Thus, the value of her outside option is $\widehat{v}$, instead of 0.

How does an improvement in the outside option affect the firm's value? The firm bargains with a supplier to split the continuation value from bargaining, which is tomorrow's firm value minus today's production and carry costs. But the firm has already secured the outside option value even if the bargaining were to fail. It is thus only the remainder – 'bargaining surplus' – that actually gets split according to the bargaining weights $(\theta, 1-\theta)$ with the supplier. That is,

$$\begin{aligned}\text{Firm value}_t &= \text{Continuation value}_t - \text{Rent}_t \\ &= \text{Continuation value}_t - (1-\theta)\big(\underbrace{\text{Continuation value}_t - \text{Outside option}_t}_{=\text{Bargaining surplus}_t}\big) \\ &= \text{Outside option}_t + \theta\big(\text{Continuation value}_t - \text{Outside option}_t\big).\end{aligned}$$

Therefore, an increase in the firm's outside option when it bargains earlier, if continuation value is fixed, raises firm value by a fraction $1-\theta$. Of course, earlier bargaining reduces continuation value due to carry cost φ, reducing firm value by a fraction θ.

As Figure 2 outlines, the entrepreneur optimally bargains earlier if and only if $\widehat{v} + \theta(v - 1 - \varphi - \widehat{v}) \geq \theta(v-1)$, or, equivalently,

$$(1-\theta)\widehat{v} \geq \theta\varphi; \tag{2}$$

i.e., gain from a better outside option $(1-\theta)\widehat{v}$ outweighs her loss from carry cost $\theta\varphi$.

Lastly, suppose instead that if bargaining were to fail on date 0, the second supplier cancels his visit on date 1; for example, failed bargaining is received as a negative signal. Then, earlier bargaining does not improve the firm's outside option value relative to later bargaining – it is zero as the project fails. Since the left-hand side of Inequality (2) is zero, earlier bargaining is never optimal, and hence there is never slack on date 1.

Discussion. Slack – that is, positive inventory – may endogenously emerge on date 1, despite the carry cost, if it is optimal to bargain infrequently rather than frequently or earlier rather than later. The entrepreneur bargains infrequently by purchasing enough at a time to sustain multiple periods without needing to bargain and pay rents. She may also optimally purchase earlier to bargain when her outside options are better, which requires that she has the ability to bargain again later even upon a failed bargaining.

One comparative statics that already obtains is that the conditions for slack – Inequalities (1) and (2) – are easier to satisfy when terminal payoff v is larger. The result is general. A supplier extracts a fraction $1-\theta$ of bargaining surplus. When the firm's future value is higher, the surplus is, ceteris paribus, larger, and so is suppliers' rent. The entrepreneur thus employs slack more to mitigate rent extraction.

These results arise because of bargaining frictions. Bargaining weight θ does not affect the set of feasible allocations or the fact that zero slack is uniquely first-best, but slack may still be optimal if $\theta < 1$. There is both an underlying friction of suppliers' local monopoly – in that the firm cannot immediately trade with another supplier if trading with one supplier fails – and a 'technology' to overcome it – in that in each period a supplier visits the firm.[10] The firm unilaterally chooses when to trade, but does not fully internalize the gains from a trade due to suppliers' bargaining power $1-\theta > 0$. Therefore, its optimal choice of when to trade may fail to be first-best.

This stylized model thus captures the paper's core insight that bargaining frictions may make it optimal to bargain infrequently and early. Of course, the analysis is incomplete. First, the initial inventory $h_0 \in \{0,1\}$ should reflect the firm's preceding choice. Second, the standard intuition that strengthening firms' access should move firms towards the absence of frictions with zero slack cannot be entirely false. Therefore, I now transition to a fully dynamic setup in infinite horizon for more complete analysis.

4 Model

This section sets up the main model of firm financing subject to bargaining frictions.

Environment. Time is continuous and infinite $t \in [0, \infty)$. All agents are risk neutral and have a common time discount rate $\rho > 0$. Agents called insiders own a firm to run a business. The business has an exogenous cash flow profile, given in Section 4.1. The firm holds internal funds (or 'funds') $h_t \geq 0$ to which cash flow accrues. Internal funds yield a return $r < \rho$.[11] The spread $\varphi \equiv \rho - r > 0$ is the carry cost of internal funds.

Insiders may frictionlessly receive a nonnegative dividend. If internal funds are depleted without immediate financing, the business fails with zero salvage value.[12]

Financing bargaining. To avoid business failure, the firm needs regular financing. Insiders are assumed to be penniless and hence must raise funds for the firm from homogeneous agents called financiers. Insiders can choose when to bargain à la Nash

[10] Supplemental Appendix SA.2 discusses setups where multiple suppliers visit each period.

[11] Internal funds can, via a simple reformulation of cash flow in Section 4.1, represent slack against borrowing constraint in a risk-free instantaneous lending market with interest rate $r < \rho$.

[12] Sections 6.3 and 6.4 entertain richer liquidation technologies in context of divestment.

with financiers for funds.[13] Bargaining weights for insiders and financiers are $\theta \in (0,1)$ and $1-\theta$, respectively. In return for providing funds, financiers receive a proportional ownership stake in the firm; due to homogeneous preferences, this is without loss.

To focus on the bargaining-driven dynamics of financing, I do not separate the timing of bargaining from the timing of financing: at any $t \geq 0$, financiers cannot commit to a contract for future financing at $t' > t$. For simplicity, I assume that financiers are deep-pocketed but upon financing, they become penniless insiders.[14]

Outside options. If insiders can walk away from bargaining and immediately find alternative financiers, bargaining is trivial because they can induce perfect competition among financiers. I therefore assume financier local monopoly: if bargaining with one group of financiers were to fail, it takes time for insiders to find another group of financiers to bargain with and raise funds from. I call this time lag 'exclusion,' which can be permanent or temporary. If permanent, excluded firm insiders continue the business until funds are depleted, at which point the business fails. Generally, excluded insiders are 're-included' into the financing market at a Poisson arrival rate $\gamma \geq 0$ that parametrizes the accessibility of alternative financing. Insiders thus face a stochastic time lag of finding another financing counterparty. Re-inclusion means regaining the ability to finance, and insiders may choose not to promptly finance upon re-inclusion.

The setup can be interpreted as an environment where there is a search friction but only off the equilibrium path. This is, arguably, rather plausible. Corporate managers can forecast whether they will need funds soon and start engaging with financial institutions in advance. At the same time, firms typically cannot engage with multiple institutions separately to make them compete à la Bertrand. One may regard the present assumption as a stylized version of a search friction where a match is both durable until actual financing and exclusive.

4.1 Cash flow

The business has an exogenous cash flow profile. Consider two concrete examples:

1. The business is a 'startup' that incurs a constant flow expense $\kappa\, dt$ with $\kappa > 0$, until succeeding upon a stopping time τ at a Poisson rate $\lambda > 0$. Upon success, the business earns a terminal payoff $\Pi > \frac{\kappa}{\lambda}$ and ends.
2. The business is an 'operating firm' that indefinitely earns an average flow profit $\pi\, dt$, where $\pi > 0$, but subject to occasional losses due to volatility $\sigma\, dB_t$, where $\sigma > 0$ and B_t is a standard Brownian motion.

[13]The framework à la Kalai (1977) gives identical results given deep-pocketed financiers.

[14]An alternative setup would be to assume that financiers can commit to funding the firm but only up to a capacity constraint. This would require the model to take a stand on the capacity.

Generically, time-t cash inflow is $\mu\ dt + \sigma\ dB_t$, where $\mu \in \mathbb{R},\ \sigma \geq 0$; for startups, $\mu \equiv -\kappa < 0$ and for operating firms, $\mu \equiv \pi > 0$. The business may 'succeed' upon a stopping time τ at a Poisson rate $\lambda \geq 0$, upon which insiders receive a terminal dividend $\Pi + h_\tau$, $\Pi \in \mathbb{R}$.

Assumption 1. *$\mu + \lambda\Pi > 0$. If $\sigma = 0$, then $\mu < 0$ and either $r \leq 0$ or $\frac{\mu+\lambda\Pi}{\rho+\lambda} < \frac{|\mu|}{r}$.*

The first part gives a positive net present value of the business $\frac{\mu+\lambda\Pi}{\rho+\lambda} > 0$ in the absence of frictions. The second part ensures that external financing is regularly needed.[15]

4.2 Dividend payout and bargaining for financing

By linear preference, optimal Markov-perfect dividend policy is $\max\{0, h_t - \overline{h}\}$ above a threshold $\overline{h} \geq 0$. Letting $V(h)$ denote firm value given funds h, $\overline{h}$ solves dividend optimality $V'(\overline{h}) = 1$, i.e., smooth pasting, and super contact $\sigma > 0 \implies V''(\overline{h}) = 0$.

Suppose that at a given point, firm insiders have chosen to bargain with financiers for funds. Let $V_o(h)$ denote insiders' reservation value ('outside option if bargaining fails') given funds h, and $x \in [0,1]$ their retained ownership. Nash bargaining solves

$$\begin{aligned} &\max_{x\in[0,1],\ \overline{h}\geq 0} \Big(xV(\overline{h}) - V_o(h)\Big)^{\theta} \Big((1-x)V(\overline{h}) - (\overline{h}-h)\Big)^{1-\theta} \\ \implies\quad &\overline{h} \in \arg\max_{\widetilde{h}}\ V(\widetilde{h}) - \widetilde{h} \quad \text{(which implies } V'(\overline{h}) = 1\text{), and} \\ &x(h)V(\overline{h}) = V_o(h) + \theta\Big(V(\overline{h}) - (\overline{h}-h) - V_o(h)\Big) \qquad (3) \end{aligned}$$

The funding target $\overline{h}$ maximizes net firm value $V(\overline{h}) - \overline{h}$.[16] Ownership is split in a way that insiders' retained value from bargaining $x(h)V(\overline{h})$ given h equals their reservation value $V_o(h)$ plus a fraction θ of the bargaining surplus $V(\overline{h}) - (\overline{h} - h) - V_o(h)$.

Insiders' reservation value $V_o(h)$ does not, conditional on bargaining and specifically within the above bargaining problem, affect firm value post financing $V(\overline{h})$. But it still affects insiders' payoff from bargaining. Thus, reservation value affects insiders' optimal choice of *when* to bargain for financing, and hence, ex-ante, firm value as well.

Algebraically, financing is optimal at h if and only if firm value is given by $V(h) = x(h)V(\overline{h})$. The presence of $\theta < 1$ in $x(h)V(\overline{h})$ shows that the firm's value 'function' V is diluted in anticipation of financiers' rent extraction from optimally-timed financing.

[15] Even without volatility $\sigma = 0$, loss occurs with positive probability $\mu < 0$ (i.e., certainty), and fully covering losses with returns on internal funds $h_t \geq |\mu|/r$ is suboptimal.

[16] The funding target, taken as the infimum of the arg max, coincides with the dividend payout threshold because both equalize marginal value of funds with marginal cost of financing. This identity may fail given transaction costs of dividend payout, but this paper's results are robust.

5 Analysis

This section presents the analysis of the equilibrium – which is shown to exist uniquely.

Lemma 1 (Existence and uniqueness)**.** *There exist unique V and V_o, the equilibrium value function of insiders and their reservation value function.*

As an aside, the proof in Appendix A.1 leverages the two-step recursive structure: (Markov-perfect) equilibrium value function V depends on itself because (i) V depends on V_o through bargaining θ, and (ii) V_o depends on V through re-inclusion γ. The expected discounting at rate $\rho > 0$ due to the time delay in re-inclusion $\gamma < \infty$ makes this recursion strictly contractionary. Accordingly, V exists as a unique fixed point by contraction mapping theorem. The second step of the recursion then determines V_o.[17]

Sections 5.1 through 5.2 discuss lumpy financing and early financing. Section 5.3 analyzes how financial slack reduces dilution. Section 5.4 explores comparative statics.

5.1 Lumpy financing

Let $\overline{h} = \inf\{\arg\max_h \; V(h) - h\}$.[18] Define $B \subset [0, \overline{h}]$ as the set of internal funds h at which it is optimal in equilibrium to bargain with financiers for funds: $h \in B$ if and only if

$$V(h) = x(h)V(\overline{h}) = V_o(h) + \theta\big(V(\overline{h}) - (\overline{h} - h) - V_o(h)\big). \tag{4}$$

Obviously, $0 \in B$, since $\theta > 0$. Equation (4) gives an expression for financiers' rents as

$$(1-\theta)\Bigg[\underbrace{\Big(V(\overline{h}) - (\overline{h} - h) - V(h)\Big)}_{\equiv J(h)} + \underbrace{\Big(V(h) - V_o(h)\Big)}_{\equiv E(h)}\Bigg]. \tag{5}$$

Above, $J(h)$ is the joint surplus from financing when insiders choose to enter into bargaining with financiers, and $E(h)$ is insiders' loss due to exclusion: if bargaining fails once they have entered into it, insiders get $V_o(h)$ instead of $V(h)$. Financiers receive as their rents a fraction $1-\theta > 0$ of not just $J(h)$ but also $E(h)$. Since $\gamma < \infty$, exclusion always involves a nonzero loss, that is, $E > 0$ globally (which will be shown shortly). Therefore, insiders cannot entirely eliminate financing rents even when raising financing at h arbitrarily close to $\overline{h}$. At the same time, $h \to \overline{h}^-$ implies infinite financing frequency, that is, paying $(1-\theta)(J(h) + E(h)) \to (1-\theta)E(\overline{h}) > 0$ infinitely often. Incremental financing, therefore, is never optimal in continuous time.

[17] Supplemental Appendix SA.3 graphically illustrates the proof in Appendix A.1.

[18] When $h_t > \overline{h}$, the firm immediately pays out dividends $h_t - \overline{h}$ to insiders. Thus, $h \mapsto V(h) - h$ is constant above $\overline{h}$, and therefore choosing the funding target $\overline{h}$ as the infimum is without loss.

Note that financiers cannot extract rents if either $\theta = 1$ or $\gamma \to \infty$. It is obvious from Equation (5) that $\theta = 1$ eliminates financiers' rents. Fix $\theta \in (0, 1)$ but let $\gamma \to \infty$ so that $E(h) \to 0$ pointwise for $h > 0$. Note that for any $\gamma < \infty$, $h \to \overline{h}$ gives $J(h) \to J(\overline{h}) = 0$.[19] Therefore, when $\gamma \to \infty$, financiers' rents $(1 - \theta)\big(J(h) + E(h)\big)$ vanish as financing occurs infinitely close to the funding target $\overline{h}$. As expected (and demonstrated shortly in Section 5.4), there is no financial slack in the absence of rent extraction by financiers, that is, either $\theta = 1$ or $\gamma \to \infty$.

To summarize, the present theory predicts 'lumpy' and infrequent financing, which indicates the emergence of an endogenous financing friction, if and only if bargaining is nontrivial. This feature is demonstrated by the first main result below and its proof.

Proposition 1. *Financing is lumpy and intermittent,* $\sup B < \overline{h}$.

Proof. Suppose not, that is, for any sufficiently small $\varepsilon > 0$, financing at $h = \overline{h} - \varepsilon$ in the amount ε is optimal. By Equation (4),

$$V(\overline{h} - \varepsilon) = x(\overline{h} - \varepsilon)V(\overline{h}) = \theta\big(V(\overline{h}) - \varepsilon\big) + (1 - \theta)V_o(\overline{h} - \varepsilon).$$

Letting $\varepsilon \to 0^+$ gives $V(\overline{h}) = \theta V(\overline{h}) + (1 - \theta)V_o(\overline{h})$ by continuity (see Appendix A.1). Since $\theta < 1$, this is equivalent to $V(\overline{h}) = V_o(\overline{h})$. But by Assumption 1, there is a finite time interval over which, without financing, internal funds $\overline{h}$ get depleted with nonzero probability. Also, over any finite time interval, re-inclusion fails to occur with nonzero probability since $\gamma < \infty$. In sum, there is nonzero probability that today's exclusion causes business failure within a finite time interval. Without exclusion, positive surplus is retained at depletion since $\theta > 0$. Hence, $V(\overline{h}) > V_o(\overline{h})$, a contradiction. □

5.2 Early financing

Next, let us analyze insiders' financing strategy $B \supset \{0\}$. The following lemma establishes that it is an interval from zero: that is, the optimal financing strategy is monotone in internal funds h. While this is a technical result, I include a partial proof in the article because its derivation points at the core of the economic mechanism.

Lemma 2 (Monotone financing strategy)**.** *If* $h \in B$*, then* $[0, h] \subset B$*.*

Proof. Suppose $B \setminus \{0\}$ is nonempty. Immediate financing is optimal on this set, and hence is preferred to an instantaneously delayed financing: for $h \in B \setminus \{0\}$,

$$\rho V(h) - rhV'(h) \geq \mathcal{H}(V)(h), \tag{6}$$

[19] It is easily shown that as $\gamma \to \infty$, $J(h) \to 0$ for $h > 0$ conditional on financing being optimal.

where

$$\mathcal{H}(V)(h) \equiv \mu V'(h) + \frac{1}{2}\sigma^2 V''(h) + \lambda\big(\Pi + h - V(h)\big) \tag{7}$$

is the infinitesimal generator for the exogenous cash flow in Section 4.1. Note that

$$\rho V(\overline{h}) - r\overline{h} = \mathcal{H}(V)(\overline{h}) \quad \text{and} \tag{8}$$

$$\rho V_o(h) - rhV_o'(h) = \mathcal{H}(V_o)(h) + \gamma\big(V(h) - V_o(h)\big), \tag{9}$$

where $V'(\overline{h}) = 1$. The HJB under exclusion (9) has a term for re-inclusion at rate γ.

Equation (4), being an identity on B, implies that $V'(h) = \theta + (1-\theta)V_o'(h)$ and $V''(h) = (1-\theta)V_o''(h)$ on B, with a potential exception at $h = 0$. Thus, for $h \in B \setminus \{0\}$,

$$\mathcal{H}(V)(h) = \theta\mathcal{H}(V)(\overline{h}) + (1-\theta)\mathcal{H}(V_o)(h). \tag{10}$$

Substituting (4), (8), (9), (10) into (6) cancels out $\mathcal{H}(V)(h)$ and gives: for $h \in B \setminus \{0\}$,

$$(1-\theta)\gamma\big(V(h) - V_o(h)\big) \geq \theta\varphi\big(\overline{h} - h\big). \tag{11}$$

This inequality is single-crossing so that $h \in B$ implies $[0, h] \subset B$. Showing the single-crossing property requires a technical proof for concavity of V_o; see Appendix A.2. □

Suppose that financing is optimal at $h_t = h > 0$. This implies that when insiders compare bargaining earlier now against delaying bargaining by an instant dt as a one-shot deviation, they prefer the former. In this comparison, the risk of running out of funds due to the dt delay is negligible because $h_t > 0$. Also, exogenous cash inflow during the instant $(t, t+dt]$, represented by $\mathcal{H}(V)(h)$ in the proof above, simply shifts continuation value $V(\overline{h}) - (\overline{h} - h_t)$ and insiders' reservation value $V_o(h_t)$ in parallel, and so does not affect the surplus from bargaining (which is the difference) – and hence, financiers' rents. The cancellation of $\mathcal{H}(V)(h)$ in the proof reflects this invariance.[20]

Earlier financing creates two material changes compared to instantaneous delay. On one hand, there is an additional carry cost, because funds $\overline{h} - h_t$ come in at t instead of $t + dt$. This reduces continuation value from bargaining $V(\overline{h}) - (\overline{h} - h_t)$ by $\varphi(\overline{h} - h_t)\ dt$. Due to bargaining (4), insiders do not fully internalize this loss since financiers bear a fraction $1 - \theta$ as a reduction in their rents. This is the right-hand side of Inequality (11). On the other hand, insiders' outside options are better when they finance earlier, because of the chance of instantaneously finding alternative financiers during $(t, t+dt]$. Put differently, financiers at t face competition from alternative financiers over $(t, t+dt]$ up to probability $\gamma\ dt$, which is irrelevant for financiers at $t + dt$. Earlier financing thus raises insiders' reservation value $V_o(h_t)$ by $\gamma(V(h_t) - V_o(h_t))\ dt$. Again due to bargaining

[20]The axiom of invariance to positive affine transformation, formalized by Nash (1950), holds.

(4), firm value rises by a fraction $1-\theta$ since financiers' rents are reduced by as much, giving the left-hand side of Inequality (11).[21]

Lemma 2 allows the equilibrium to be fully characterized by '(s,S)' bounds $(\underline{h}, \overline{h})$, where $\underline{h} \equiv \sup B$. When $\lim_{s\to t^-} h_s = \underline{h}$, insiders raise financing in the amount $\Delta h \equiv \overline{h} - \underline{h} > 0$ so that $h_t = \overline{h}$. Henceforth, I refer to $\underline{h}$ both as 'financing threshold' – to address the dynamics of financing – and 'funding reserve' – to focus on the amount of financial slack when financing is raised. I say that insiders engage in 'early financing' if $\underline{h} > 0$, that is, if insiders raise financing before running out of funds. I also refer to $\overline{h}$ and Δh as 'funding target' and 'financing amount,' respectively.

Corollary 1 (Monotonicity of early financing)**.** *Given the other parameters, there exists* $\underline{\gamma} \in (0,\infty)$ *such that early financing is optimal* $\underline{h} > 0$ *if and only if* $\gamma > \underline{\gamma}$*.*

Corollary 1 formalizes the logical implication from the discussion of Lemma 2 that the only reason that insiders would choose to finance early, $\underline{h} > 0$, is that if financing were to fail, off the equilibrium path, insiders could use this funding reserve to cover losses while seeking alternative sources of financing to prevent business failure. When this is sufficiently difficult, $\gamma \le \underline{\gamma}$, insiders delay financing until funds are depleted, because they cannot sufficiently improve their outside options by financing early.

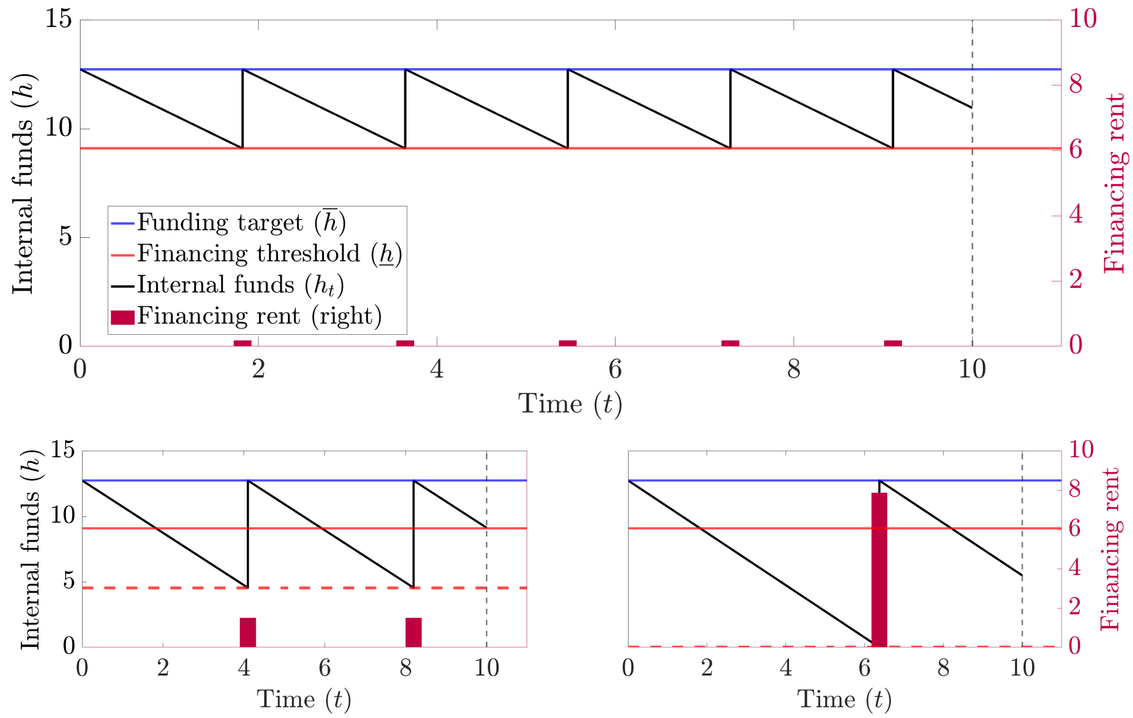
5.3 Financial slack and dilution

I now analyze how financial slack reduces dilution. For illustration, consider a startup that incurs a fixed expense $\kappa\, dt$ until success at Poisson rate λ with a terminal payoff Π. Baseline parameters are: $(\rho, r) = (0.05, 0)$, $(\theta,\gamma) = (0.5, 1)$ and $(\kappa, \lambda, \Pi) = (2, 0.1, 50)$.

Size-frequency tradeoff. Figure 3 illustrates the relationship between financial slack and dilution. In the main plot that shows optimal financing strategy, insiders raise financing once every $\frac{\Delta h}{\kappa} \approx 1.8$ periods (until success) and 'very early' with a large funding reserve $\underline{h} \approx 9.1$. At each financing, insiders incur small dilution: 0.18 in value. If this cost were 'fixed,' the strategy that the main plot illustrates would be strictly dominated by further delaying financing with lower thresholds $\widetilde{h} < \underline{h}$, as the cost would be less frequently incurred. The bottom two subplots, Figures 3a and 3b, show that these deviations are indeed not optimal. Although frequency decreases, the size of dilution endogenously magnifies, to 1.5 with $\widetilde{h} = 0.5\underline{h}$ and even up to 7.88 with $\widetilde{h} = 0$.

Why is dilution magnified when insiders raise financing with smaller financial slack? The answer lies in insiders' outside options. In Figure 4, I decompose valuation on the vertical axis, assuming a one-shot strategy of immediate financing at each h across

[21] Inequality (11) coincides with Inequality (2) in the stylized model of Section 3, with one innocuous generalization: if bargaining fails on $t = 0$, the second supplier cancels his visit with probability $1-\gamma$, $\gamma \in [0,1]$. Earlier bargaining raises the firm's reservation value by $\gamma\big(\widehat{v} - 0\big)$.



(a) Halved funding reserve $\widetilde{h} = 0.5\underline{h}$

(b) Finance only at depletion $\widetilde{h} = 0$

Figure 3: Financial slack and dilution

The plots describe a startup with parameters $(\rho, r) = (0.05, 0)$, $(\theta, \gamma) = (0.5, 1)$, $(\kappa, \lambda, \Pi) = (2, 0.1, 50)$. The graphs in solid black tracks the history of internal funds h_t until the business succeeds – at $t = 10$ in this simulation. Each financing, which is when h_t jumps from financing threshold $\underline{h}$ to funding target $\overline{h}$, involves dilution given in magenta bar. The bottom two subplots depict suboptimally lower financing thresholds.

$[0, \overline{h}]$ on the horizontal axis. The rightmost edge in blue is the funding target $\overline{h}$, and the red vertical line the financing threshold $\underline{h}$.[22]

The solid black line at the top of Figure 4 is firm value upon financing $V(\overline{h})$. To attain this 'post-money' value, financiers must provide financing $\overline{h} - h$, represented as the height of the light gray area right below it; this amount decreases in h at a unit slope. Financing surplus, however, is not simply the difference between post-money value $V(\overline{h})$ and 'money' $\overline{h} - h$. Insiders possess outside options $V_o(h)$ at bargaining, represented as the height of the dark gray area. Even if financing were to fail, insiders that have kept financial slack may still seek alternative funding sources $\gamma = 1$ to prevent business failure. But if financing were to fail without financial slack, the business would promptly fail. As such, value of insiders' outside options rises steeply in financial slack.

Therefore, the financing surplus, which is the two colored areas in the middle, decreases in financial slack. This is what gets divided, according to the ratio $(1-\theta : \theta)$, into financiers' rents in magenta and insiders' surplus retention in light blue.

In short, financial slack monotonically reduces the size of dilution. But then, why

[22] As can be inferred, immediate financing being considered is, on $(\underline{h}, \overline{h}]$, a (one-shot) deviation.

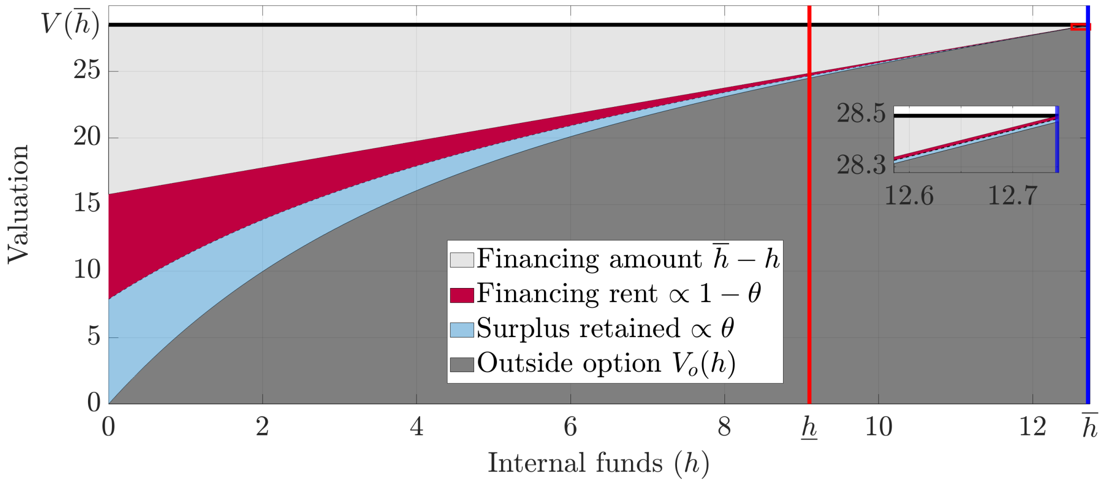


Figure 4: Financial slack, outside options, and the size of dilution

This plot addresses the same startup example as in Figure 3. It shows value decomposition when insiders with funds $h \in [0, \overline{h}]$ choose to raise financing – suboptimally on $(\underline{h}, \overline{h}]$. Continuation value from bargaining is $V(\overline{h}) - (\overline{h} - h)$, which is the height of the bottom of the light gray area, and outside option value is the height of the dark gray area. The surplus from financing is in between the two, which is split, by ratio $(1-\theta : \theta)$, to financiers' rents in magenta and insiders' retained surplus in light blue.

is the optimal funding reserve $\underline{h}$ not even higher? The answer is the frequency. Given funding target $\overline{h}$, a higher financing threshold $\underline{h}$ decreases financing amount Δh and thus increases the frequency of dilution, while its size no longer decreases as steeply.[23]

Access to financing. When is early financing not optimal? As Corollary 1 shows, it is not optimal when insiders do not have good access to alternative financing $\gamma \leq \underline{\gamma}$. Then, financial slack does not improve insiders' outside options enough to justify the increases in carry cost and financing frequency.

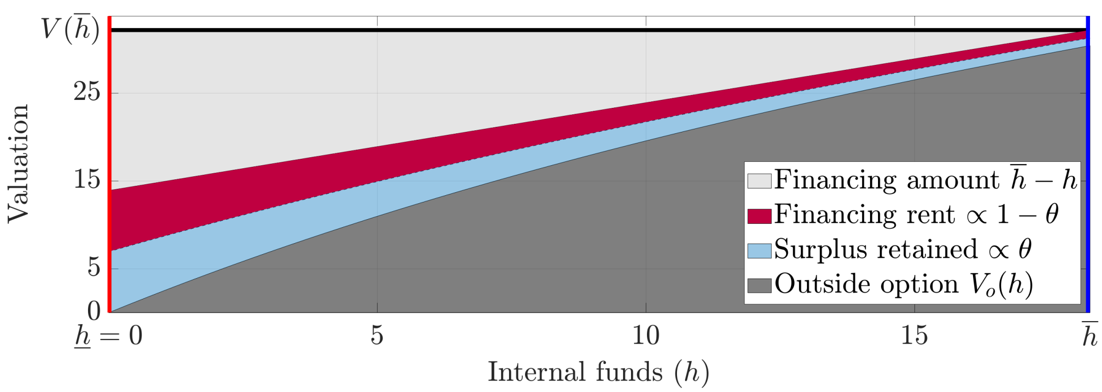


Figure 5: Poor access to alternative financing $\gamma = 0$

The height of the plot is adjusted so that slopes are visually comparable to those in Figure 4.

[23] There is a gap even at the top $V(\overline{h}) - V_o(\overline{h}) > 0$, which is a mere 0.022 but certainly nonzero. As Proposition 1 shows, this non-vanishing loss from exclusion induces lumpy financing.

Figure 5 illustrates such an instance with $\gamma = 0$ for an otherwise identical startup. Financial slack increases insiders' outside options at a much lower slope, since there is no chance of re-inclusion $\gamma = 0$.[24] Consequently, the size of dilution is never reduced by financial slack enough to compensate for the increased frequency of dilution. These insiders instead delay financing until funds are depleted and, when they do raise funds, raise a much larger amount $\Delta h \approx 18.27 > 3.64$ to reduce the frequency of dilution.

Then, what does the optimality of early financing imply about dilution?

Proposition 2 (Early financing and dilution)**.** *In equilibrium, financiers extract rents from insiders through financing in the amount of:* $(1-\theta)\big(V(\overline{h}) - \overline{h}\big)$ *if insiders delay financing until funds are depleted* $\underline{h} = 0$*; and*

$$\frac{\varphi}{\gamma}\Delta h \tag{12}$$

if insiders finance with financial slack $\underline{h} > 0$*. Financing is early* $\underline{h} > 0$ *if and only if*

$$(1-\theta)\gamma > \frac{\varphi\overline{h}}{V(\overline{h}) - \overline{h}}. \tag{13}$$

Proof. Financiers' rent when $\underline{h} = 0$ is from (3). For (13), evaluate (11) at $h = 0$. For (12), enforce equality on (11) at $h = \underline{h} > 0$ given $(1-\theta)\big(V(\underline{h}) - V_o(\underline{h})\big) = \theta\big(V(\overline{h}) - V(\underline{h}) - \Delta h\big)$ from (4): $V(\overline{h}) - V(\underline{h}) - \Delta h = (1-\underline{x})V(\overline{h}) - \Delta h$ is financiers' rent. □

Proposition 2 gives a compact and explicit expression (12) capturing the dynamic tradeoff between the size and frequency of dilution. From the discussion of Lemma 2, earlier financing at $h_t > 0$ reduces financiers' rents by the amount $(1-\theta)\gamma\big(V(h_t) - V_o(h_t)\big)\,dt$. If $h_t \leq \underline{h}$, then $V(h_t) - V_o(h_t) = x(h_t)V(\overline{h}) - V_o(h_t)$ is insiders' surplus from financing at h_t. Also, by a standard property of bargaining, Equation (3) implies that $\theta : 1-\theta$ equals the ratio of insiders' surplus to financiers' rents. Thus,

$$(1-\theta)\gamma\big(V(h_t) - V_o(h_t)\big) = (1-\theta)\gamma\big(x(h_t)V(\overline{h}) - V_o(h_t)\big) = \theta\gamma\Big(\big(1-x(h_t)\big)V(\overline{h}) - \big(\overline{h} - h_t\big)\Big).$$

In short, earlier financing, wherever on $(0, \overline{h}]$ it is optimal, reduces financiers' rents by a factor of $\theta\gamma\,dt$. On the other hand, it involves a carry cost of $\varphi(\overline{h} - h_t)\,dt$, but insiders bear a fraction θ of it due to bargaining. Thus, θ cancels out from the comparison, and the optimal interior financing threshold $\underline{h} > 0$ equalizes marginal costs and benefits of earlier financing by equalizing γ times the size of dilution to φ times financing amount Δh, which reduces the frequency of dilution. This dynamic tradeoff between the size and the frequency of dilution is compactly captured in the expression (12).

[24] The slope exceeds unity $V_o'(h_t) > 1$ only due to the chance of business success upon τ at Poisson rate λ before funds are depleted: $\mathbb{P}\big(\tau \leq t + h_t/\kappa \mid \tau > t\big)$. Since τ is independent of success or failure of bargaining, it does not affect optimal timing of bargaining; see Section 5.2.

Observe from (12) that the size of dilution given early financing is, in equilibrium, proportional to financing amount. Even though there is no variable transaction cost of financing, the size of dilution may resemble one in equilibrium. And this expression holds pointwise: conditional on early financing, if parameters $(\rho, r, \theta, \gamma, \mu, \sigma, \lambda, \Pi)$ exogenously fluctuate but with a constant ratio of carry cost of financial slack $\varphi \equiv \rho - r$ to availability of financing alternatives γ, then the firm is seen as though incurring a constant marginal cost of financing.[25] Inequality (13) in Proposition 2 thus shows that firms strategically *choose* between financing costs that are proportional to either net firm value $V(\overline{h}) - \overline{h}$ or financing amount Δh – via their choice on early financing.

5.4 Comparative statics of financial slack

The workhorse proof of Lemma 2 shows that the infinitesimal generator $\mathcal{H}(V)$ for exogenous cash flow in Section 4.1 is canceled on the financing interval $B = [0, \underline{h}]$ when $\underline{h} > 0$. Using this feature, I establish comparative statics in bargaining parameters.

For clarity, suppose $r \geq 0$; a negative yield (very) slightly complicates exposition.

Proposition 3 (Comparative statics in θ and γ)**.** *The equilibrium $(\underline{h}, \overline{h})$ is invariant to γ when $\gamma \leq \underline{\gamma}$. **(i) Funding target** $\overline{h}$ is decreasing in θ, and decreasing in γ when $\gamma \geq \underline{\gamma}$.*[26] ***(ii) Financing threshold** $\underline{h}$ is decreasing in θ when $\underline{h} > 0$; $\underline{h} = 0$ is constant in θ above some $\underline{\theta} < 1$. **(iii) Financing amount** Δh – in case $\underline{h} > 0$ (i.e., $\Delta h < \overline{h}$) – is constant (increasing) in θ given $r = 0$ ($r > 0$), and decreasing in γ given $r = 0$. **(iv) Financial slack vanishes** $\overline{h}$, $\underline{h}$, $\Delta h \to 0$ as either $\theta \to 1$ or $\gamma \to +\infty$.*

First, an increase in either bargaining weight θ or access to alternative financing γ ($\geq \underline{\gamma}$) lowers insiders' incentives to keep financial slack, and hence decreases funding target $\overline{h}$ – down to zero in the limit along with $\underline{h}$ and Δh. Second, funding reserve $\underline{h}$ is decreasing in θ as well. As Proposition 2 shows, insiders reduce the size of dilution by financing early $\underline{h} > 0$. When θ is higher, financiers can extract a smaller fraction of financing surplus as rents, and so insiders do not keep as large funding reserves.

Funding reserve $\underline{h}$ is, however, non-monotonic in access to alternative financing γ. With γ above but near $\underline{\gamma}$, insiders finance early to improve their outside options. Alternative financiers are, however, not readily available, so that large financial slack is necessary to make financiers with whom the insiders bargain compete against many potential rivals. As alternative financing becomes increasingly accessible, a smaller and smaller $\underline{h}$ is enough to make financiers compete against as many – or more – rivals.

[25] When parameters $(\rho, r, \theta, \gamma, \mu, \sigma, \lambda, \Pi)$ exogenously fluctuate, financing strategy $(\underline{h}, \overline{h})$ also fluctuates. Nevertheless, formal results presented so far, including Proposition 2, hold *pointwise*; the proof of Proposition 5 in Appendix A.6 shows how such variation is accounted for.

[26] Within this proposition, 'decreasing'/'increasing' indicates strict monotonicity only.

Lastly, financing amount Δh determines the frequency of financing, and hence of dilution. Proposition 2 shows that given early financing, the size of dilution is $\frac{\varphi}{\gamma}\Delta h$, which is constant in θ and decreasing in γ. Therefore, its optimal frequency, which is decreasing in financing amount Δh, is constant in θ and increasing in γ.[27]

Proposition 3 is significant in two regards. First, note that θ affects neither the set of feasible allocations nor which of them is uniquely first-best, and yet gives first-best allocation if and only if $\theta \to 1$. This shows that the core friction in the present mechanism is not in liquidity: (i) there are both a latent friction $\gamma < \infty$ and a costless technology to overcome it, but (ii) the agent that controls when to exercise the technology does not fully internalize the gains from exercising it $\theta < 1$. Due to this positive externality, there is endogenous underprovision in frequency of financing transactions. In short, illiquidity arises from this *dynamic* positive externality due to bargaining.[28]

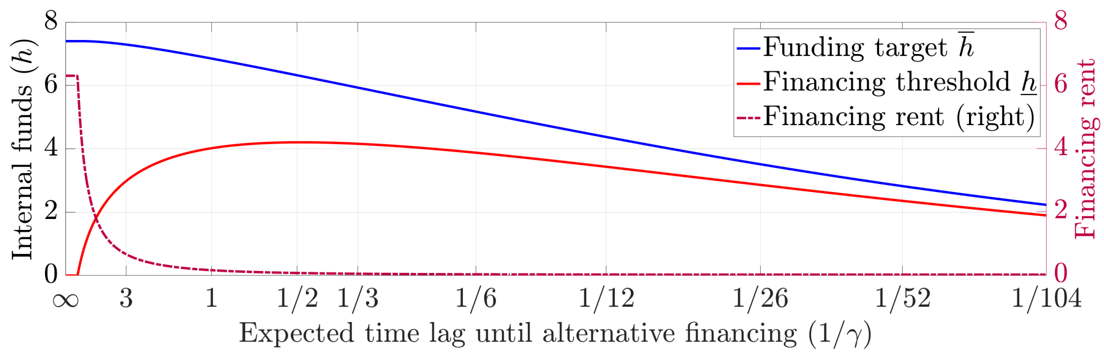


Figure 6: Comparative statics in access to alternative financing γ

'Operating firm' given $\rho = 0.05,\ r = 0,\ \theta = 0.5,\ \pi \equiv \mu = 1,\ \sigma = 2,\ \lambda = \Pi = 0$. Financing rent is never zero.

Second, the non-monotonicity of financing threshold $\underline{h}$ in γ, as shown in Figure 6, differentiates this paper from existing theories. In such models, early financing is rationalized through exogenous fluctuations in fixed transaction costs (e.g., Bolton et al., 2013), search frictions (e.g., Hugonnier et al., 2014), or risk management in anticipation of investment opportunities (e.g., Froot et al., 1993). Accordingly, firms with superior and more reliable access to financing face less financing frictions, and thus have smaller incentives to keep funding reserves. In relation, the present theory adds a richer mechanism where the extent to which firms' access to financing can mitigate financing frictions depends on the amount of financial slack that firms choose to keep. Under this framework, therefore, the seemingly excessive amount of financial slack

[27] If $r > 0$, a higher $\overline{h}$, holding financing amount Δh fixed, implies less frequent financing due to higher yields $rh_t\, dt$ on $h_t \in (\overline{h} - \Delta h, \overline{h}]$; since $\frac{\partial \overline{h}}{\partial \theta} < 0$, Δh is thus increasing in θ when $\underline{h} > 0$.

[28] If $\theta = 0$, any allocation is an equilibrium since the agent that chooses when to trade (insiders) is indifferent. Thus, the game is not upper-hemicontinuous as $\theta \to 0$. Still, gains from financing are *fully* 'externalized' to another agent (financiers). Thus, there exists a first-best equilibrium where the agent with choice adopts the best strategy for the other agent with benefit.

often seen with large, established and creditworthy firms should not be a puzzle.[29]

6 Investment Extensions

As I now introduce capital investment, one question that might arise is 'why consider investment?' A more precise question, though, would be 'why consider investment *explicitly*?' It is because in the main model through Sections 4 and 5 as well as in the stylized model in Section 3, investment has been implicitly assumed. This is less implicit in case of the stylized model with a two-period project and also for the example of a startup: both incur constant 'losses' – or fixed investment expenses – until future payoff materializes. Investment is also there in the other example of an operating firm: occasional losses need to be covered in order to keep earning profits in the future.

Nevertheless, introducing investment explicitly is integral in this paper's overall analysis, precisely because the frictions consist in bargaining. Specifically, investment, when modeled explicitly, is firms' direct *control* over flow of funds, and therefore, it must jointly affect both firms' outside options off the equilibrium path and the size of dilution on the path. That is, firms that can obtain extra funds by reducing investment or divesting capital to other firms have good outside options vis-à-vis external financiers; if external financing were to fail, such firms can do so in order to avoid running out of funds. From such firms, financiers can extract small rents in equilibrium.

In this paper, reducing nonnegative investment and divestment (i.e., negative investment) are thus analytically equivalent off-path responses to reduce on-path dilution. But they *are* distinct, at least macroeconomically; as discussed at the introduction in Section 1, divestment relates to investment irreversibility, and reducing investment is tied to firm productivity. In short, this paper, by direct implications from its core mechanism, positions capital reallocation and firm productivity as two separate conduits of the single underlying economic force – that firm financing involves bargaining.

This section explores how firms' optimal financing strategy shapes this rich interaction between investment and dilution. For clarity of exposition, I continue addressing a single firm in the backdrop of financiers, but, consistent with the above discussion, the implications extend to macroeconomics. Section 6.1 extends the setup with investment choice, and Section 6.2 shows that bargaining frictions induce underinvestment. Lastly, Sections 6.3 and 6.4 explain how firms' *strategic* underinvestment, including divestment and reducing investment, interacts with dilution from external financing.[30]

[29] In the presence of investment opportunities, Supplemental Appendix SA.6 shows that this model may predict funding reserves even in large excess of investment needs – see Figure SA.4.

[30] Supplemental Appendices SA.5 and SA.6 motivate and conduct illustrative analysis on another aspect of the interaction between investment and dilution that builds upon the previous discussion of comparative statics with respect to access to external financing from Section 5.4.

6.1 Formal extended setup

I modify the cash flow profile with a standard investment model à la Hayashi (1982).

Production and investment. A firm owns capital K_t and employs it to produce

$$\big(A\ dt + \sigma\ dB_t\big)\ K_t$$

in cash flow, with $A, \sigma > 0$. The firm can invest $I_t \in \mathbb{R}$ into its capital stock, subject to adjustment cost. Given flow investment $I_t\ dt$, the firm incurs an additional flow cost in funds $\Psi(I_t/K_t)K_t\ dt$; Ψ satisfies $\Psi(0) = \Psi'(0) = 0 < \Psi''$. When the firm is neither financing nor paying dividends, internal funds $H_t \geq 0$ evolve as: writing $i_t \equiv I_t/K_t$,

$$dH_t = \Big(A - i_t - \Psi(i_t)\Big)K_t\ dt + \sigma K_t\ dB_t.$$

For simplicity, assume a quadratic cost: $\Psi(i) \equiv \psi\frac{i^2}{2}$ for some $\psi > 0$. Given depreciation rate $\delta \geq 0$, the firm's capital stock K_t evolves as $dK_t/K_t = \big(i_t - \delta\big)\ dt$. While there is no explicit capital trade, $i_t < 0$ represents divestment of capital into funds.

Stochastic parameters. I allow the parameters (ρ, r), (θ, γ), (A, σ), and (ψ, δ) to be functions of an underlying state variable $s_t \in S \subset \mathbb{R}$. Let s_t evolve continuously as $ds_t = \mu_s(s_t)\ dt + \sigma_s(s_t)\ dZ_t$, where Z_t is a standard Brownian motion independent of B_t and both μ_s and σ_s satisfy standard regularity conditions. In baseline, $s_t = s$ is constant: $\mu_s = \sigma_s = 0$. Its law of motion is summarized by the infinitesimal generator

$$\mathcal{S}(V) \equiv \mu_s \cdot V_s + \frac{1}{2}\sigma_s^2 \cdot V_{ss}. \tag{14}$$

I omit notations for s in parameters: 'ρ' is a stochastic discount rate $\rho(s_t)$, for example.

Investment optimization. Define $W(s, K, H)$ as firm value given state s, capital $K > 0$, and funds $H \geq 0$. Letting $h \equiv H/K$ and $V(s, h) \equiv W(s, 1, h)$, homogeneity in (K, H) gives $W(s, K, H) = KV(s, h)$. Omitting (s, h), per-capital HJB equation is

$$\rho V - rhV_h = \max_{i \in \mathbb{R}} \left\{ \big(A - i - \Psi(i)\big)V_h + \frac{1}{2}\sigma^2 V_{hh} + \big(i - \delta\big)\underbrace{\big(V - hV_h\big)}_{=W_K} + \mathcal{S}(V) \right\}.$$

The first-order condition gives insiders' optimal investment rate as

$$i(s, h) = \frac{1}{\psi}\left(\frac{V(s, h)}{V_h(s, h)} - h - 1\right). \tag{15}$$

6.2 Underinvestment

Let $\overline{h}(s) \equiv \inf\{h \mid V_h(s,h) = 1\}$ denote funding target given state s. I first show that insiders with less internal funds invest less. Equation (15) yields $\partial i(s,h)/\partial h = -\frac{1}{\psi}\frac{V(s,h)}{V_h(s,h)^2}V_{hh}(s,h)$. It thus reduces to a claim for concavity of V in internal funds.

Lemma 3 (Funds-driven underinvestment)**.** $V''(h) < 0$ *for all* $h < \overline{h}$.[31]

Lemma 3 does not yet imply global underinvestment, because one might wonder whether investment is first-best at least at $h = \overline{h}(s)$. The answer is a definitive no.

Proposition 4 (Underinvestment at the top)**.** *Let* $i^*(s)$ *be first-best investment and* $\bar{\imath}(s)$ *optimal investment at state* s *with* $h = \overline{h}(s)$*. Then,* $\bar{\imath}(s) < i^*(s)$ *for all* s*.*

Proof. Let $V^*(s)$ first-best value per capital: $\theta^* = 1$. First-best investment rate is

$$i^*(s) = \frac{1}{\psi}\Big(V^*(s) - 1\Big).$$

Letting $\overline{V}(s) \equiv V(s, \overline{h}(s))$, optimal investment rate at funding target is

$$\bar{\imath}(s) = \frac{1}{\psi}\Big(\overline{V}(s) - \overline{h}(s) - 1\Big),$$

since $\overline{V}_h = 1$. It must be that $V^*(s) > \overline{V}(s) - \overline{h}(s)$ so that $i^*(s) > \bar{\imath}(s)$. If first-best insiders with value $V^*(s)$ are somehow endowed with funds $h = \overline{h}(s)$, the entirety is optimally paid out promptly so that their value is $V^*(s) + \overline{h}(s)$. If they instead use the funds to mimic the optimal strategy under $\theta < 1$, they achieve a strictly higher value than $\overline{V}(s)$ due to the absence of dilution. Therefore, letting $\widetilde{V}^*$ value of this deviation strategy, it follows that $\overline{V}(s) < \widetilde{V}^*(s, \overline{h}(s)) \le V^*(s) + \overline{h}(s)$, as claimed. $\square$

Investment reduces funds. As long as insiders are incentivized to preserve funds $\overline{h} > 0$ despite the carry cost, they invest strictly less than without such incentives. With a lower h, they are more incentivized so, and hence further reduce investment.

While this mechanism is well-known in the literature on liquidity frictions, it is still noteworthy that here, underinvestment arises purely due to bargaining frictions.

6.3 Strategic underinvestment

In the main model from Sections 4 through 5, insiders only had financial slack as their strategic choice. Here, they can optimize with both investment and financial slack. How does this added choice affect firms' optimal financial slack?

[31] Just for this lemma, I let s constant to avoid the complexities of PDE. While not proven, concavity in funds seems plausible when parameters stochastically vary independently of funds.

But first, let us abbreviate notations. Letting $\overline{h}(s)$, $\underline{h}(s)$ denote funding target and financing threshold given state s, respectively, define $\overline{V}(s) \equiv V(s, \overline{h}(s))$, $\underline{V}(s) \equiv V(s, \underline{h}(s))$, and $\underline{V}^o(s) \equiv V^o(s, \underline{h}(s))$. The exclusion symbol o is in superscript for compatibility with partial derivatives; e.g., $\underline{V}^o_h(s) \equiv V^o_h(s, \underline{h}(s))$. Let $\bar{\imath}(s)$, $\underline{i}(s)$, $\underline{i}^o(s)$ denote optimal investment rates at corresponding funds $\overline{h}(s)$, $\underline{h}(s)$ and per exclusion status. I also omit s in the optimal investment policy function: $i(h)$ is optimal investment rate at h with access to financing, which also depends on notationally suppressed state s.

Proposition 5 (Underinvestment and dilution)**.** *With notations for exogenous state s suppressed where appropriate, the following hold, in equilibrium, pointwise for each $s \in S$. (i) Financiers extract rents from insiders through financing in the amount of: $(1-\theta)\big(\overline{V} - \overline{h}\big)$ if insiders delay financing until funds are depleted $\underline{h}(s) = 0$; and*

$$\frac{1}{\gamma + \frac{1}{2}(\bar{\imath} - \underline{i})}\left[\varphi \Delta h - \frac{1}{2}\Big(\bar{\imath} - \underline{i}\Big)\Big(\underline{V} - \underline{h}\Big) + \frac{1}{2}\left(\frac{1-\theta}{\theta}\right)\Big(\underline{i} - \underline{i}^o\Big)\Big(\underline{V}^o - \underline{h}\underline{V}^o_h\Big)\right] \tag{16}$$

if insiders finance with financial slack $\underline{h}(s) > 0$. (ii) Financing is early $\underline{h}(s) > 0$ if

$$(1-\theta)\gamma + \frac{1}{2}\Big(\bar{\imath} - i(0)\Big) > \frac{\varphi \overline{h}}{\overline{V} - \overline{h}}. \tag{17}$$

Inequality (17) implies that with investment choice, insiders may raise financing early $\underline{h} > 0$ even when alternative external financiers are not available $\gamma = 0$, as illustrated by Figure 7. This is due to the term $\frac{1}{2}\big(\bar{\imath} - i(0)\big)$, which is absent in Inequality (13) from Proposition 2. It represents how much insiders underinvest because of funding depletion $\bar{\imath} - i(0)$, net of the quadratic adjustment cost $\frac{1}{2}$.

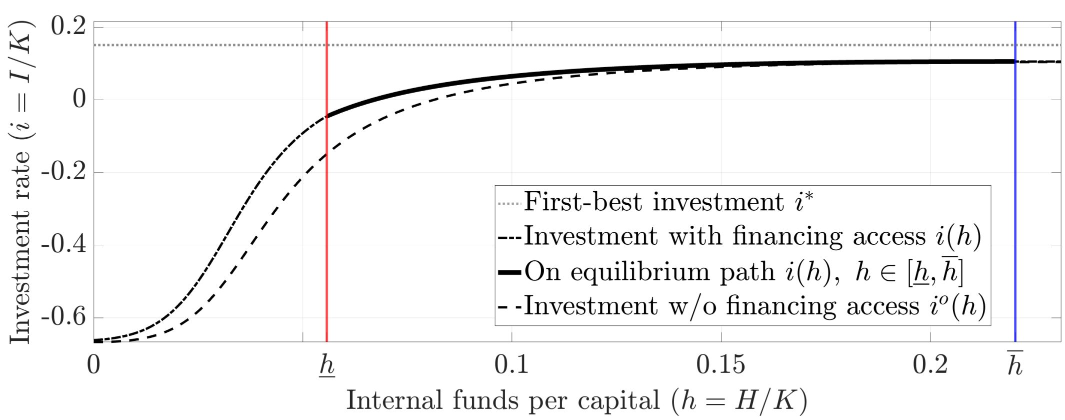


Figure 7: Underinvestment and financing dynamics

Bargaining parameters are $(\theta, \gamma) = (0.5, 0)$, and other parameters are adopted from Bolton et al. (2011): $(\rho, r) = (0.06, 0.05)$, $(A, \sigma) = (0.18, 0.09)$ and $(\psi, \delta) = (1.5, 0.1007)$. All parameters are constant.

One may conjecture that if $\bar{\imath} - i(0)$ is large, insiders raise financing early to reduce the underinvestment that arises due to low funds $i(h)$ on $h \in [0, \underline{h}]$. This, however, is not precise. Consider a canonical 'fixed' transaction cost of financing. Regardless of where a financing threshold is, insiders reduce investment by the same magnitude to avoid reaching it and paying the given cost; zero threshold thus strictly dominates (see Bolton et al., 2011). In my theory, early financing $\underline{h} > 0$ even with $\gamma = 0$ still arises because, as (16) shows, the size of dilution is endogenous in financing strategy.

Note first that funds-driven underinvestment $\bar{\imath} - \underline{i} > 0$ increases surplus from financing, and hence dilution, because financing eliminates losses from it. At the same time, dilution is, from insiders' perspective, a (endogenous) 'fixed' transaction cost of financing. Therefore, if it is large in equilibrium, insiders try to avoid financing by underinvesting more when funds fall towards financing threshold. In short, there is complementarity between funds-driven underinvestment and dilution.

With flexible investment, there is another factor. If financing were to fail, insiders could underinvest even further in response to low funds $\underline{i} - \underline{i}^o > 0$. As the fraction $\frac{1-\theta}{\theta}$ in (16) indicates, this 'fallback' underinvestment improves insiders' outside options vis-à-vis financiers: even when permanently excluded from external financing $\gamma = 0$, insiders could mitigate the risk of business failure by underinvesting even more when funds are low. This would be feasible only if business failure is not imminent $\underline{h} > 0$.

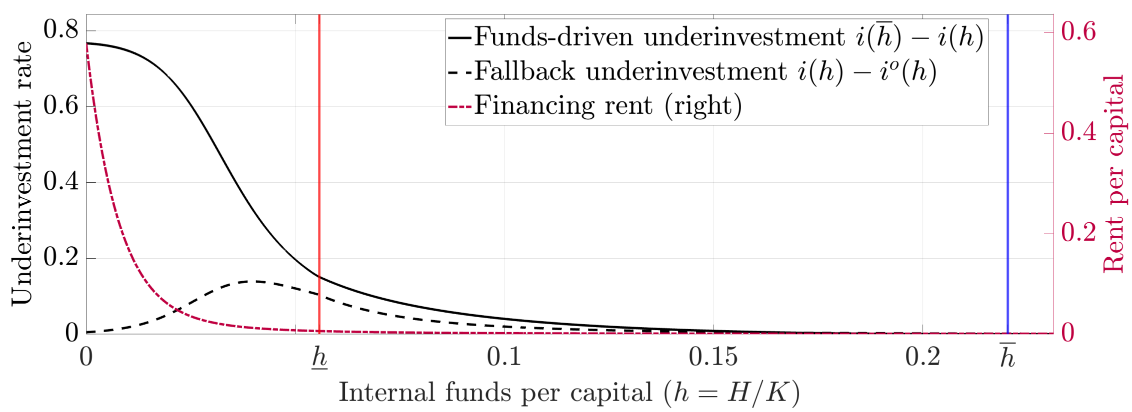


Figure 8: Strategic underinvestment and dilution

Financing rent is computed given a one-shot choice of financing at each h, which is suboptimal for $h > \underline{h}$. Due to exclusion, financing rent does not completely vanish $(1-\theta)\big(\overline{V} - V^o(\overline{h})\big) \approx 0.001 > 0$; see Section 5.1.

Put differently, fallback underinvestment is a form of alternative *self*-financing, reducing the size of dilution when insiders raise funds with financial slack. By the aforementioned complementarity, then, insiders in equilibrium underinvest less when close to financing threshold, which further reduces dilution..., and so forth. As Figure 8 shows, funding reserve $\underline{h} \approx 0.056$ makes dilution negligible – from 0.799 at $h = 0$ down to 0.006 at $h = \underline{h}$ – even without access to alternative *external* financing $\gamma = 0$.

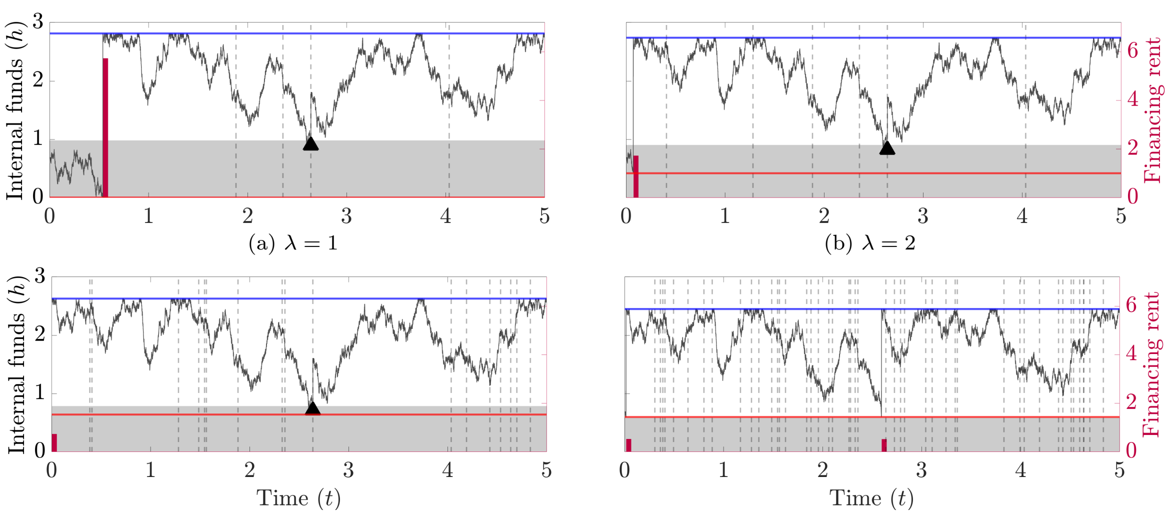
What determines exact early financing threshold $\underline{h} > 0$? Expression (16) is from

$$\theta\gamma\Big((1-\underline{x})\overline{V} - \Delta h\Big) + \frac{\theta}{2}\Big(\bar{\imath} - \underline{i}\Big)\underbrace{\Big(\overline{V} - \overline{h}\Big)}_{=\overline{W}_K} = \theta\varphi\Delta h + \frac{1-\theta}{2}\Big(\underline{i} - \underline{i}^o\Big)\underbrace{\Big(\underline{V}^o - \underline{h}\underline{V}_h^o\Big)}_{=\underline{W}_K^o}, \quad (18)$$

where $(1-\underline{x})\overline{V} - \Delta h$ is the size of dilution. The threshold equalizes marginal benefits (on the left-hand side) of earlier financing at $h_t = \underline{h}$ over instantaneous delay to its marginal costs (on the right-hand side). Here, I only discuss the second term on each side – the first terms are already addressed in the discussion following Proposition 2.

On the one hand, earlier financing raises continuation value by removing losses from funds-driven underinvestment $\frac{1}{2}\big(\bar{\imath} - \underline{i}\big)\overline{W}_K\, dt$. But insiders do not fully internalize this gain since financiers extract a fraction $1-\theta$ in rents. On the other hand, delaying financing by dt increases insiders' reservation value at bargaining by $\frac{1}{2}(\underline{i} - \underline{i}^o)\underline{W}_K^o\, dt$; if financing were to fail anyway, insiders would have accumulated more capital at $t + dt$ by delaying the bargaining. By financing earlier at t instead, insiders' reservation value is lower by as much, which increases financiers' rents by a fraction $1-\theta$.

Illustration. To illustrate the analysis, I modify the main model from Section 4 with lumpy divestment opportunities. The business has 'normalized' running cash flow $\pi\, dt + \sigma\, dB_t$. At a Poisson arrival rate of $\lambda > 0$, the business receives opportunities to downsize future cash flow by a factor of $\eta \in (0, 1)$ in return for receiving funds $\xi > 0$. I let $(\rho, r) = (0.07, 0)$, $(\pi, \sigma) = (1, 1)$ and $(\eta, \xi) = (0.9, 0.7)$ so that divestment is not first-best. Lastly, $\theta = 0.5$ and there are no alternative external financiers $\gamma = 0$.



(a) $\lambda = 1$ (b) $\lambda = 2$ (c) $\lambda = 4$ (d) $\lambda = 10$

Figure 9: Ease of divestment opportunities λ, given $\gamma = 0$

Plots track internal funds, dilution and divestment given stochastic arrivals of divestment opportunities (vertical dashed lines in gray). In the gray area, divestment is optimal; the black marker indicates actual divestment.

As Figure 9 shows, firms that are able to easily underinvest – in this case, divest – incur small dilution from financing. Consequently, they have small incentives on the path of equilibrium to actually divest in order to avoid financing. Indeed, with a frequent opportunity to divest $\lambda = 10$ in Figure 9d, firms never divest in equilibrium.

6.4 Downward scalability of investment

Section 6.3 shows that fallback underinvestment is a means of alternative self-financing. But underinvestment is not confined to divestment. To explore the effects of downward scalability of investment, I introduce a nonnegativity constraint to the setup in Section 6.1, $i \geq 0$. Investment is thus perfectly irreversible, although it can be flexibly scaled.

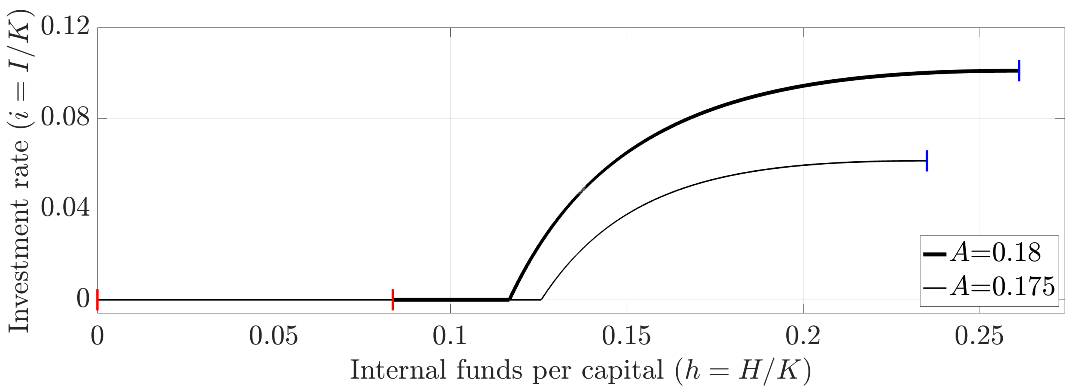


Figure 10: No alternative external financiers $\gamma = 0$ and no divestment $i \geq 0$

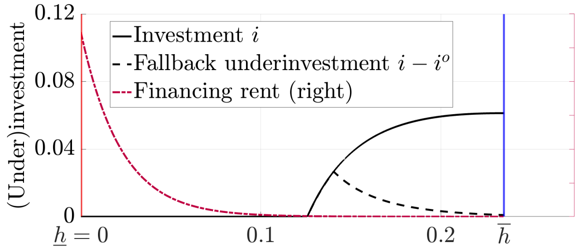
The black curves, in different thickness, represent optimal investment policy given internal funds on the equilibrium path, bounded by funding target $\overline{h}$ in the blue line segments and financing threshold $\underline{h}$ in the red ones.

For the purpose of present analysis, upward scalability of investment is not directly relevant. Therefore, I fix convex adjustment cost $\psi = 1.5$ that governs upward scalability, and instead vary productivity $A \in \{0.175, 0.18\}$ for simple comparative statics. Other parameters are as in Figure 7, including no alternative external financiers $\gamma = 0$.

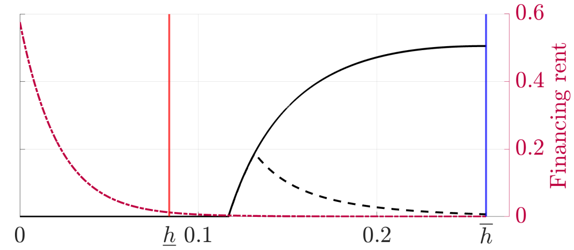
Figure 10 compares the two scenarios. The result that $\underline{h}_1 = 0$ given $A = A_1 \equiv 0.175$, on its own, appears consistent with the preceding analysis throughout this paper. If insiders cannot find alternative external financiers or divest capital in response to a financing failure, it would be difficult to improve their outside options vis-à-vis financiers by having financial slack because, supposedly, there is nothing they can do to avoid running out of funds. Insiders, therefore, delay financing as much as possible.

But then why do insiders finance early $\underline{h}_2 > 0$ with $A = A_2 \equiv 0.18$? The answer has to do with the ability to reduce positive investment in response to a financing failure. Figure 11 adds graphs for fallback underinvestment and financing rent to each case. In Figure 11a with $A = 0.175$, fallback underinvestment is concentrated towards higher funds. Because divestment is impossible, insiders that are already investing none

$i(h) = 0$ cannot underinvest more in response to a financing failure $i(h) - i^o(h) = 0$. Since fallback underinvestment is constrained by irreversibility, insiders' outside options vis-à-vis financiers are not sufficiently improved with financial slack, so that $\underline{h}_1 = 0$.



(a) $A = 0.175$



(b) $A = 0.18$

Figure 11: Upward-concentrated fallback underinvestment

In Figure 11b with a higher $A = 0.18$, fallback underinvestment is still concentrated upwards, but there are two differences. First, because insiders invest more on path $i(h)$ than with a lower $A = 0.175$, they can underinvest more off path if financing were to fail; that is, $i(h)-i^o(h)$ is larger given $i(h) > 0$. Second, if financing were to fail, insiders would have a better chance for fallback underinvestment because higher productivity $A_2 > A_1$ increases the upward drift in funds h. Hence, irreversibility constraint is less binding, and insiders raise funds early to bargain with better outside options. The size of dilution is 0.013, which is, although more than twice that with divestment (see Figure 8), smaller by an order of magnitude than 0.546 with $A = 0.175$.

In sum, when alternative external financing is difficult to find $\gamma = 0$, firms with good *self*-financing alternatives – that is, those that (i) can easily divest (see Section 6.3), or (ii) are productive and sustain large investment with flexibility to scale it down – incur small dilution. In equilibrium, they may not actually divest or drastically underinvest. But the ability to do so if financing were to fail improves firms' outside options, and hence the payoffs from bargaining, when they raise financing with financial slack.

7 Conclusion

I present a dynamic theory of firm financial slack based on financier bargaining power. It predicts that firms choose to raise financing in a lumpy fashion and maintain internal funds in order to bargain with financiers – and pay rents – infrequently. Moreover, firms may choose to raise financing early, before running out of internal funds, in order to bargain when their outside options are better due to the ability to pursue financing alternatives. Thus, lumpy financing reduces the frequency of financiers' rent extraction and early financing its size. In short, I argue that it is how financial slack improves firms' bargaining with financiers that determines the dynamics of firm financing.

By jointly rationalizing lumpy financing and early financing through bargaining, this theory explains illiquidity in firm financing – in the form of financial slack – not as an irreducible primitive, but rather as a phenomenon due to firms' (second-best) optimal financing strategy. Because this mechanism captures firms' economic incentives at a deeper level, it yields richer implications. Macroeconomically in particular, this theory positions capital reallocation and firm productivity as two separate conduits of the single underlying economic force – that firm financing involves bargaining.

Looking forward, this paper suggests further research on questions such as how financial frictions and the real economy dynamically interact, and how market structure in financial sector may affect firm financing and macroeconomic outcomes.

Fundamentally, this paper delivers a conceptual takeaway. The core of its theory is: What might have happened determines what does happen – even macroeconomically. Bargaining is just one such context, in which firms' outside options if financing were to fail ('what might have happened') determine how much firm value is diluted in anticipation of financiers' rent extraction ('what does happen'). The richness of this theory's predictions reaffirms that there are great returns to bringing to light such *strategic* incentives of individual agents for studying economic outcomes at larger scales.

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Appendix A Omitted Proofs

A.1 Lemma 1 (Existence and uniqueness).

Proof. Let $\mathcal{W}$ be the space of continuous nonnegative bounded functions on $\mathbf{H} \equiv [0, H]$ with an arbitrary $H \gg 0$, and complete it with L^∞ supremum metric. Keep cash flow as in Section 4.1, and also the assumptions on dividends and business failure upon funding depletion without prompt financing.[32] Define $T, T_o : \mathcal{W} \to \mathcal{W}$ as: for $W \in \mathcal{W}$,

1. $h \mapsto T_o W(h)$ is the value function of 'excluded' insiders lacking the ability to raise financing (so that the boundary condition is $T_o W(0) = 0$), who, upon 're-inclusion' that occurs at the stopping time τ_γ given by a Poisson rate $\gamma \in [0, \infty)$, receive a terminal payoff of $W(h_{\tau_\gamma})$; and
2. $h \mapsto TW(h)$ is the value function of non-excluded insiders who can, at any $t \geq 0$, either (i) forgo financing, or (ii) raise financing through bargaining with financiers where insiders' outside option from bargaining is given by $W(h_t)$:
   - if $W(h_t)$ is sufficiently high, 'bargaining' involves zero financing and zero rents since (iii) insiders *take* the outside option such that $TW(h_t) = W(h_t)$.

By the Theorem of the Maximum, the above transformations are well-defined as self-maps given the continuity restriction in $\mathcal{W}$.

An equilibrium value function V is a fixed point of the *concatenated* map $\widehat{T} \equiv T \circ T_o$, and vice versa; if V exists, then $V_o \equiv T_o V$. If $\widehat{T}$ is a contraction on $\mathcal{W}$, by contraction mapping theorem there exists a unique fixed point $V \in \mathcal{W}$ such that $V = \widehat{T}V$.

The claim on $\widehat{T}$ follows from Blackwell's Lemma. First, obviously $\widehat{T}$ is nondecreasing: $W' \geq W$ on $\mathbf{H}$ implies $\widehat{T}W' \geq \widehat{T}W$ on $\mathbf{H}$. Second, let us show that there exists some $\zeta \in [0, 1)$ such that for any $W \in \mathcal{W}$ and $w \in \mathbb{R}_{++}$, $\widehat{T}(W + w) \leq \widehat{T}W + \zeta w$.

Recall that τ_γ occurs at a Poisson rate $\gamma \in [0, \infty)$. Then,

$$T_o(W + w) \leq T_o W + \zeta^\rho_\gamma w, \tag{A.1}$$

where $\zeta^\rho_\gamma \equiv \mathbb{E}\left[e^{-\rho\tau_\gamma}\right] = \frac{\gamma}{\gamma+\rho} < 1$.[33] As an aside, T_o is a contraction on $\mathcal{W}$, and its unique fixed point is invariant to $\gamma \in [0, \infty)$. Next, for any $w \in \mathbb{R}_{++}$,

$$T(W + w) \leq TW + w. \tag{A.2}$$

[32] Also enforce that whenever $h_t > H$, dividend payout of at least $h_t - H$ is mandatory.

[33] The Poisson structure is not essential; one can, for example, assume that re-inclusion deterministically occurs at $\tau_\gamma \equiv 1/\gamma > 0$, in which case $T_o W(h)$ is excluded insiders' value at $t = 0$ with $h_0 = h$ given the otherwise identical environment. As long as re-inclusion takes some time with nonzero probability $\mathbb{P}(\tau_\gamma > 0) > 0$, the bonus dividend $w \in \mathbb{R}_{++}$ that is received only upon re-inclusion is, from $t = 0$, discounted in expectation (at least) by a factor of $\mathbb{E}[e^{-\rho\tau_\gamma}] < 1$.

This is because marginal benefit from uniform global improvement in reservation value function does not exceed unity. The upper bound is attained wherever on $h_t \in \mathbf{H}$ (iii) the outside option $W(h_t)$ is so good that insiders take it, $TW(h_t) = W(h_t)$. Otherwise, marginal benefit is $1-\theta$ wherever on $\mathbf{H}$ insiders optimally (ii) raise positive funds through bargaining, and zero wherever on $\mathbf{H}$ insiders optimally (i) forgo financing.[34]

T is not strictly a contraction, but it is not 'expansionary': its contraction coefficient is unity. Therefore, $\widehat{T}$ inherits the coefficient $\zeta_\gamma^\rho \in [0,1)$ from T_o: for any $w \in \mathbb{R}_{++}$,

$$\widehat{T}(W+w) = T\big(T_o(W+w)\big) \le T\big(T_oW + \zeta_\gamma^\rho w\big) \le T\big(T_oW\big) + \zeta_\gamma^\rho w = \widehat{T}W + \zeta_\gamma^\rho w.$$

The first inequality is from Inequality (A.1) and monotonicity of T, and the second from Inequality (A.2). By Blackwell's Lemma, $\widehat{T}$ is a contraction on $\mathcal{W}$, as desired.[35][36] $\square$

A.2 Lemma 2 (Monotone financing strategy)

Proof. (Continued from Section 5.2) Rewrite Inequality (11) as:

$$\begin{aligned} G(h) &\equiv (1-\theta)\gamma\Big(V(h) - V_o(h)\Big) - \theta\varphi\Big(\overline{h} - h\Big) \ge 0 \\ \implies \quad G'(h) &= \theta\Big(\varphi - (1-\theta)\gamma\big(V_o'(h) - 1\big)\Big), \end{aligned} \tag{A.3}$$

since $V'(h) = \theta + (1-\theta)V_o'(h)$ from Equation (4). Since $\underline{h} \equiv \sup B > 0$, insiders at $h_t = \underline{h}$ are indifferent between financing and instantaneous delay $G(\underline{h}) = 0$. Since delay is optimal above $\underline{h}$, $G'(\underline{h}) \le 0$. If V_o is concave in h on $[0, \overline{h}]$, then Equation (A.3) implies that $G'(h) \le 0$ for $h < \underline{h}$ so that $G(h) \ge 0$ for $h < \underline{h}$. Therefore, $[0, \underline{h}] \subset B$.

On the (strict) concavity of V_o, consider the associated dividend threshold $\overline{h}_o > 0$. The proof proceeds as (i) $\overline{h}_o > \overline{h}$, (ii) $V_o'' < 0$ on $[\underline{h}, \overline{h}_o)$, and (iii) $V_o'' < 0$ on $[0, \underline{h})$.

(i) $\overline{h}_o > \overline{h}$. Suppose not. Smooth pasting and super contact at $h = \overline{h}_o$ yield

$$\begin{aligned} \rho V_o(\overline{h}_o) - r\overline{h}_o &= \mu + \lambda\Big(\Pi + \overline{h}_o - V_o(\overline{h}_o)\Big) + \gamma\Big(V(\overline{h}_o) - V_o(\overline{h}_o)\Big) \\ &\ge \mu + \lambda\Big(\Pi + \overline{h}_o - V_o(\overline{h}_o)\Big), \end{aligned}$$

because $\gamma \ge 0$ and $V(\overline{h}_o) > V_o(\overline{h}_o)$ from exclusion. Solve for $V_o(\overline{h}_o)$ to obtain

$$V_o(\overline{h}_o) \ge \frac{\mu + \lambda\Pi}{\rho + \lambda} + \Big(\frac{r+\lambda}{\rho+\lambda}\Big)\overline{h}_o.$$

[34] These are pointwise effects of a rise in $W(h)$ on $TW(h)$. When W globally shifts up uniformly, marginal effect on any $TW(h)$ is a weighted average of 1, $1-\theta$ and 0, and hence unity at most.

[35] Supplemental Appendix SA.3 graphically illustrates the proof.

[36] Using $\widehat{T}$ for numerical algorithm is inefficient: for any $W \in \mathcal{W}$, T_oW and TW are themselves fixed points of the respective contractions specific to W. Algorithm in Supplemental Appendix SB.1 uses Poisson re-inclusion to jointly obtain (V, V_o) by a single standard HJB iteration.

Since $\overline{h}_o$ is the dividend payout threshold under exclusion and $\overline{h} \geq \overline{h}_o$ by assumption,

$$\begin{aligned} V_o(\overline{h}) &= V_o(\overline{h}_o) + (\overline{h} - \overline{h}_o) \geq \frac{\mu + \lambda \Pi}{\rho + \lambda} + \Big(\frac{r+\lambda}{\rho+\lambda}\Big)\overline{h}_o + (\overline{h} - \overline{h}_o) \\ &= \frac{\mu + \lambda \Pi}{\rho + \lambda} + \overline{h} - \Big(1 - \frac{r+\lambda}{\rho+\lambda}\Big)\overline{h}_o \geq \frac{\mu + \lambda \Pi}{\rho + \lambda} + \Big(\frac{r+\lambda}{\rho+\lambda}\Big)\overline{h} = V(\overline{h}), \end{aligned}$$

where the last inequality is since $\frac{r+\lambda}{\rho+\lambda} < 1$ and $\overline{h} \geq \overline{h}_o$. This contradicts Proposition 1.

(ii) $V_o'' < 0$ on $[\underline{h}, \overline{h}_o)$. First, suppose $\sigma > 0$. Differentiate (9) at $h = \overline{h}_o$ while substituting $V_o'(\overline{h}_o) = 1$, $V_o''(\overline{h}_o) = 0$ to obtain a third (left) derivative as

$$V_o'''(\overline{h}_o) = \frac{2}{\sigma^2}\Big(\varphi - \gamma\big(V'(\overline{h}_o) - 1\big)\Big) = \frac{2}{\sigma^2}\varphi > 0,$$

because $\overline{h}_o > \overline{h}$ and so $V'(\overline{h}_o) = 1$. Therefore, there exists some neighborhood below $\overline{h}_o$ on which $V_o'' < 0$. Suppose by way of contradiction that there exists $\widehat{h} \in [\underline{h}, \overline{h}_o)$ such that $V_o'' < 0$ on $(\widehat{h}, \overline{h}_o)$ but $V_o''(\widehat{h}) = 0$. Then, $V_o'''(\widehat{h}) \leq 0$. Differentiating the HJB at $h = \widehat{h}$ gives a third (left) derivative as

$$0 \geq V_o'''(\widehat{h}) = \frac{2}{\sigma^2}\Big(\varphi V_o'(\widehat{h}) + \lambda\big(V_o'(\widehat{h}) - 1\big) + \gamma\big(V_o'(\widehat{h}) - V'(\widehat{h})\big)\Big).$$

Since $V_o'(\widehat{h}) > 1$ from $V_o'(\overline{h}_o) = 1$ and $V_o'' < 0$ on $(\widehat{h}, \overline{h}_o)$, the above implies that $\gamma > 0$ (that is, contradiction is reached if $\gamma = 0$) and $V_o'(\widehat{h}) - V'(\widehat{h}) < 0$. Then $\widehat{h} < \overline{h}$, because $h \geq \overline{h}$ implies $V'(h) = 1$. Since $V_o'(\overline{h}) - V'(\overline{h}) > 0$ due to $\overline{h} \in (\widehat{h}, \overline{h}_o)$, intermediate value theorem implies that there exists $\widetilde{h} \in (\widehat{h}, \overline{h})$ such that $V_o'(\widetilde{h}) - V'(\widetilde{h}) = 0$. But

$$\begin{aligned} V_o''(\widetilde{h}) - V''(\widetilde{h}) &= \frac{2}{\sigma^2}\Bigg(\Big((\rho + \lambda + \gamma)V_o(\widetilde{h}) - r\widetilde{h}V'(\widetilde{h}) - \lambda(\Pi + \widetilde{h}) - \mu V'(\widetilde{h}) - \gamma V(\widetilde{h})\Big) \\ &\qquad - \Big((\rho + \lambda)V(\widetilde{h}) - r\widetilde{h}V'(\widetilde{h}) - \lambda(\Pi + \widetilde{h}) - \mu V'(\widetilde{h})\Big)\Bigg) \\ &= \frac{2}{\sigma^2}(\rho + \lambda + \gamma)\Big(V_o(\widetilde{h}) - V(\widetilde{h})\Big) < 0; \end{aligned}$$

that is, the graph of $V_o' - V'$ on $[\widehat{h}, \overline{h}]$ can never cross zero from below, which contradicts $V_o'(\widehat{h}) - V'(\widehat{h}) < 0 < V_o'(\overline{h}) - V'(\overline{h})$ and $\widehat{h} < \overline{h}$.

Next, let $\sigma = 0$. By Assumption 1, $\mu < 0$, $\lambda > 0$ and $\Pi > -\frac{\mu}{\lambda} > 0$. Since this is essentially the startup example from Section 4.1, relabel $\kappa \equiv -\mu > 0$. Differentiating (9) at $h = \overline{h}_o$ gives a second (left) derivative as

$$V_o''(\overline{h}_o) = -\frac{\varphi}{\kappa - r\overline{h}_o},$$

because of smooth pasting $V_o'(\overline{h}_o) = 1$ and $\overline{h}_o > \overline{h}$ giving $V'(\overline{h}_o) = 1$. Since the last part of Assumption 1 ensures that $\kappa > r\overline{h}_o$, it follows that $V_o''(\overline{h}_o) < 0$ (as a left derivative). Again suppose that there exists $\widehat{h} \in [\underline{h}, \overline{h}_o)$ such that $V_o'' < 0$ on $(\widehat{h}, \overline{h}_o]$ and $V_o''(\widehat{h}) = 0$. Differentiating (9) at $\widehat{h}$ gives

$$\varphi V_o'(\widehat{h}) + \lambda\Big(V_o'(\widehat{h}) - 1\Big) + \gamma\Big(V_o'(\widehat{h}) - V'(\widehat{h})\Big) = 0.$$

Since $V_o'(\widehat{h}) > 1$, this again implies that $\gamma > 0$ and $V_o'(\widehat{h}) - V'(\widehat{h}) < 0$. The same reasoning as with $\sigma > 0$ implies $\widehat{h} < \overline{h}$. Since $V_o'(\overline{h}) - V'(\overline{h}) > 0$, intermediate value theorem again implies that there exists $\widetilde{h} \in (\widehat{h}, \overline{h})$ such that $V_o'(\widetilde{h}) = V'(\widetilde{h})$. Then,

$$\begin{aligned}
\rho V_o(\widetilde{h}) - r\widetilde{h}V'(\widetilde{h}) &= \lambda\Big(\Pi + \widetilde{h} - V_o(\widetilde{h})\Big) - \kappa V'(\widetilde{h}) + \gamma\Big(V(\widetilde{h}) - V_o(\widetilde{h})\Big), \\
\rho V(\widetilde{h}) - r\widetilde{h}V'(\widetilde{h}) &= \lambda\Big(\Pi + \widetilde{h} - V(\widetilde{h})\Big) - \kappa V'(\widetilde{h}),
\end{aligned}$$

and therefore, $V_o(\widetilde{h}) = V(\widetilde{h})$, contradicting exclusion $\gamma < \infty$.

(iii) $V_o'' < 0$ on $[0, \underline{h})$. First, suppose $\sigma > 0$. Because $V_o''(\underline{h}) < 0$, there exists a neighborhood below $\underline{h}$ on which $V_o'' < 0$. The aforementioned observation on G' implies that if $V_o'' < 0$ on a neighborhood below $\underline{h}$, then its closure is a subset of B. Substituting (4) that holds on B, the HJB equation for V_o on this interval is

$$\rho V_o(h) - rhV_o'(h) = \mathcal{H}(V_o)(h) + \theta\gamma\Big(\big(V(\overline{h}) - \overline{h}\big) + h - V_o(h)\Big),$$

where $\mathcal{H}$ is defined in (7). Suppose by way of contradiction that there is $\widetilde{h} \in [0, \underline{h})$ such that $V_o'' < 0$ on $(\widetilde{h}, \underline{h})$ but $V_o''(\widetilde{h}) = 0$. Note that $V_o'(\widetilde{h}) > 1$, because $V_o'(\overline{h}_o) = 1$ and $V_o'' < 0$ on $(\widetilde{h}, \overline{h}_o)$. Differentiating the above HJB at $\widetilde{h}$ gives a third derivative as

$$V_o'''(\widetilde{h}) = \frac{2}{\sigma^2}\Big((\varphi + \lambda + \theta\gamma)V_o'(\widetilde{h}) - (\lambda + \theta\gamma)\Big) > \frac{2}{\sigma^2}\varphi > 0,$$

which contradicts $V_o''(\widetilde{h}) = 0$ and $V_o'' < 0$ on $(\widetilde{h}, \underline{h}]$.

Next, let $\sigma = 0$ and relabel $\kappa \equiv -\mu > 0$. Since $V_o''(\underline{h}) < 0$, there is a neighborhood below $\underline{h}$ which is a subset of B and on which $V_o'' < 0$. Differentiating the HJB gives

$$\varphi V_o'(h) + (\lambda + \theta\gamma)\Big(V_o'(h) - 1\Big) + (k - rh)V_o''(h) = 0.$$

Suppose by way of contradiction that there exists $\widehat{h} < \overline{h}$ such that $V_o'' < 0$ on $(\widehat{h}, \overline{h}]$ but $V_o''(\widehat{h}) = 0$. Then, $[\widehat{h}, \overline{h}] \subset B$ by the property of G'; so the above holds at $h = \widehat{h}$. But it contradicts $V_o'(\widehat{h}) > 1$, which follows from $V_o'(\overline{h}_o) = 1$ and $V_o'' < 0$ on $(\widehat{h}, \overline{h}_o]$.

Auxiliary claim: $V'' < 0$ on $[0, \overline{h})$. First consider $[\underline{h}, \overline{h})$. If $\sigma > 0$, strict concavity holds by the same reasoning (ii) above but with $\gamma = 0$ so that a third (left) derivative at

$\widehat{h}$ gives contradiction. Suppose $\sigma = 0$ and adopt the startup relabeling. Differentiating the HJB (7) gives a second (left for $h = \overline{h}$, right for $h = \underline{h}$) derivative on $[\underline{h}, \overline{h}]$ as

$$V''(h) = -\frac{\varphi V'(h) + \lambda\big(V'(h) - 1\big)}{\kappa - rh}.$$

By Assumption 1, $r\overline{h} < \kappa$. By smooth pasting $V'(\overline{h}) = 1$, $V''(\overline{h}) < 0$. For any $\widehat{h} \in [\underline{h}, \overline{h})$ such that $V'' < 0$ on $(\widehat{h}, \overline{h}]$, $V'(\widehat{h}) > 1$ and so $V''(\widehat{h}) < 0$. As such, $V'' < 0$ on $[\underline{h}, \overline{h}]$.

Strict concavity of V on $[0, \underline{h}]$ is immediate because $V'' = (1-\theta)V_o'' < 0$ on it. □

A.3 Corollary 1 (Monotonicity of early financing)

Proof. Evaluate Inequality (11) at $h = 0$ and reverse it: $B = \{0\}$ if and only if

$$(1-\theta)\gamma V(0) = (1-\theta)\gamma\theta\big(V(\overline{h}) - \overline{h}\big) \leq \theta\varphi\overline{h} \implies (1-\theta)\gamma \leq \frac{\varphi\overline{h}}{V(\overline{h}) - \overline{h}}.$$

The claim then follows because the right-hand side is positive and nonincreasing in γ, as implied by the reasoning for Part 1 in the proof of Proposition 3.[37] □

A.4 Proposition 3 (Comparative statics in θ and γ)

Proof. Constancy of $(\underline{h}, \overline{h})$ in $\gamma \in [0, \underline{\gamma}]$ is because $\gamma \leq \underline{\gamma}$ implies $V_o(\underline{h}) = V_o(0) = 0$.

Part 1. Take $\gamma_2 > \gamma_1 \geq \underline{\gamma}$ and consider the equilibrium with $\gamma = \gamma_2$. When insiders bargain with financiers, they choose $\overline{h}_2$ to maximize $V(\overline{h}_2; \gamma_2) - \overline{h}_2$. Suppose that they agree, as a one-shot deviation, to choose $\overline{h}_1$ instead and then mimic the optimal financing strategy under $\gamma = \gamma_1$ (i.e., refinance at $\underline{h}_1$, pay out above $\overline{h}_1$) until next financing. Denote the payoff function associated with this strategy as $\widetilde{V}$. Note that $\widetilde{V}(\overline{h}_1; \gamma_2) > V(\overline{h}_1; \gamma_1)$, as the reservation value at $h_t = \underline{h}_1 > 0$ is strictly higher with $\gamma = \gamma_2 > \gamma_1$. Since $\overline{h}_2$ without the one-shot deviation is optimal, $V(\overline{h}_2; \gamma_2) - \overline{h}_2 \geq \widetilde{V}(\overline{h}_1; \gamma_2) - \overline{h}_1 > V(\overline{h}_1; \gamma_1) - \overline{h}_1$. Finally, since

$$V(\overline{h}; \gamma) - \overline{h} = \frac{1}{\rho + \lambda}\Big(\mu + \lambda\Pi - \varphi\overline{h}\Big), \tag{A.4}$$

from evaluating Equation (8) with $V'(\overline{h}) = 1$ and $\frac{1}{2}\sigma^2 V''(\overline{h}) = 0$, we have $\overline{h}_2 < \overline{h}_1$. Global strict monotonicity in θ is established by a similar reasoning.

Part 2. Strict monotonicity of $\underline{h} = \overline{h} - \Delta h > 0$ in θ is immediate from decreasing $\overline{h}$ in Part 1 and nondecreasing Δh in Part 3. The claim on existence of $\underline{\theta}$ is from Parts 3 and 4 since $\Delta h > 0$ is nondecreasing in θ when $\underline{h} > 0$ but $\Delta h \to 0$ when $\theta \to 1$.

[37] Only *strict* monotonicity in the reasoning for Part 1 of Proposition 3 relies on Corollary 1.

Part 3. Strict monotonicity of Δh in θ when $\underline{h} = 0$ follows from that of $\overline{h}$ in θ in Part 1. Now, suppose that $\underline{h} > 0$. Rearranging (12) which equals $V(\overline{h}) - V(\underline{h}) - \Delta h$ gives

$$1 + \frac{\varphi}{\gamma} = \frac{V(\overline{h}) - V(\underline{h})}{\Delta h} = \frac{1}{\Delta h} \int_{\overline{h}-\Delta h}^{\overline{h}} V'(h)\, dh. \tag{A.5}$$

That is, early financing specifies the *average* of the first derivative of V over $[\overline{h} - \Delta h, \overline{h}]$. Recall that dividend optimality stipulates that $V'(\overline{h}) = 1$ and $\frac{1}{2}\sigma^2 V''(\overline{h}) = 0$. Therefore, given $\overline{h} > 0$, one can equivalently think of V as satisfying the HJB equation globally below $\overline{h}$ and Δh as simply choosing when the first derivative on the interval meets the desired average $1 + \frac{\varphi}{\gamma}$. By strict concavity of V below $\overline{h}$ (see the end of Appendix A.2), $V'(h) > 1$ if and only if $h < \overline{h}$.

Since the first derivative is concerned, $V(\overline{h})$ is irrelevant. Differentiate the HJB:

$$\sigma > 0 \implies V'''(h) = \frac{2}{\sigma^2}\Big[-(\mu + rh)V''(h) + (\varphi + \lambda)V'(h) - \lambda \Big],$$
$$\sigma = 0 \implies V''(h) = -\frac{1}{\kappa - rh}\Big[(\varphi + \lambda)V'(h) - \lambda \Big].$$

If $r = 0$, the above equation that determines the evolution of V' below $\overline{h}$ is independent of (θ, γ). Since the desired average is $1 + \frac{\varphi}{\gamma}$, the claims when $r = 0$ are established.

Suppose $r > 0$. First, let $\sigma > 0$. Given an arbitrary $\widetilde{\Delta h} > 0$, a higher $\overline{h}$ means that for any $\delta h \in (0, \widetilde{\Delta h}]$, $V'''(\overline{h} - \delta h)$ is pointwise higher (recall that V'' is negative below $\overline{h}$). Therefore, $V''(\overline{h} - \delta h) = -\int_{\overline{h}-\delta h}^{\overline{h}} V'''(h)\, dh$ is pointwise lower. Next, let $\sigma = 0$. Because $V'(h) > 1$ for $h < \overline{h}$ and $\kappa > r\overline{h}$ by Assumption 1, a higher $\overline{h}$ means that, again because $V'(h) > 1$ for $h < \overline{h}$, $V''(\overline{h} - \delta h)$ is pointwise lower. In either case, then,

$$V'(\overline{h} - \delta h) = 1 - \int_{\overline{h}-\delta h}^{\overline{h}} V''(h)\, dh$$

is pointwise higher, and so a higher $\overline{h}$, ceteris paribus, reduces Δh. The claim then follows from Part 1.

Part 4. Since $\overline{h} \geq 0$ decreases in θ, it converges as $\theta \to 1^-$ by monotone convergence theorem. Suppose by way of contradiction that $\overline{h} \to \widetilde{h} > 0$ as $\theta \to 1^-$. Inequality (13) implies that there exists[38] some $\underline{\theta} \in (0, 1)$ such that $\underline{h} = 0$ for any $\theta \in [\underline{\theta}, 1)$, because $\overline{h} \geq \widetilde{h} > 0$ and $V(\overline{h}) - \overline{h}$ is bounded above by the frictionless net present value.

Consider $\theta > \underline{\theta}$. Since $\underline{h} = 0$, Proposition 2 implies that the size of dilution is $(1-\theta)\big(V(\overline{h}) - \overline{h}\big)$, which vanishes as $\theta \to 1^-$ because $V(\overline{h}) - \overline{h}$ is bounded above. At the same time, the buffer interval $[\underline{h}, \overline{h}]$ converges, in a two-dimensional sense, to $[0, \widetilde{h}]$ with a strictly positive length. For any $\theta \in [\underline{\theta}, 1)$, insiders incur a carry cost in

[38]This holds in itself by Part 2. But Part 2 relies on Part 4. So I reason independently here.

equilibrium that does not vanish with $\theta \to 1^-$ exactly because $\widetilde{h} > 0$. But dilution from financing vanishes. As such, given a sufficiently high $\theta \in [\overline{\theta}, 1)$, it must be strictly profitable to lower the funding target $\overline{h} < \widetilde{h}$ and reduce the non-vanishing carry cost while raising the frequency of the vanishingly small dilution, a contradiction.

Next, Inequality (13) implies that $\underline{h}$ is positive for γ sufficiently high, since net value $V(\overline{h}) - \overline{h}$ is increasing in γ and $\overline{h}$ is decreasing. From the proof for Part 3 above, $\Delta h \to 0$ as $\gamma \to \infty$. Therefore, it suffices to show that $\underline{h} = \overline{h} - \Delta h \to 0$ as well. Suppose by way of contradiction that $\underline{h} \to \widetilde{h} > 0$.[39] For any fixed small $\varepsilon \in (0, \widetilde{h})$, $V - V_o \to 0$ uniformly on $[\varepsilon, \widetilde{h}]$ as $\gamma \to \infty$, and so does $V' - V_o' \to 0$. Nash bargaining (4) then implies that $V_o'(h) \to 1$ uniformly on $[\varepsilon, \widetilde{h}]$. Therefore, across $h \in [\varepsilon, \widetilde{h}]$, marginal reduction in financing rent $(1-\theta)\big(V_o'(h) - 1\big)$ from a higher h vanishes. Hence, the marginal benefit of $\underline{h} \to \widetilde{h}$, taken as a single-dimensional Markov strategy, vanishes. But its marginal cost is constant at $\frac{\varphi}{\rho+\lambda} > 0$ because $\varphi\widetilde{h}\ dt$ is a constant carry cost. Therefore, it is strictly profitable, asymptotically, to set $\underline{h} = \varepsilon < \widetilde{h}$, a contradiction. □

A.5 Lemma 3 (Funds-driven underinvestment)

Proof. I use a slightly different notation just for this proof. Let $\mathbf{B}$ (not B) the set of internal fund levels $h \leq \overline{h}$ where financing is optimal, and let $\underline{h} = \sup \mathbf{B}$. Similar to the proof of Lemma 2, the argument proceeds as (i) $\overline{h}_o > \overline{h}$, (ii) $V_o''(h),\ V''(h) < 0$ on $[0, \overline{h}) \setminus \mathbf{B}$, and (iii) $V_o''(h),\ V''(h) < 0$ on $\mathbf{B}$. Unlike it, investment choice makes it elusive to prove that $\mathbf{B}$, which contains zero, is an interval from zero. As such, I do not impose the structure of $\mathbf{B}$, except that $\underline{h} < \overline{h}$ for which Proposition 1 holds, in proving the strict concavity of V.[40] Therefore, Parts (ii) and (iii) are iteratively proven.

(i) $\overline{h}_o > \overline{h}$. Suppose not. First, rearrange the main HJB as

$$\left(\rho + \delta + \frac{1}{\psi}\right) V - \left(A + \frac{1}{2\psi} + \left(r + \delta + \frac{1}{\psi}\right) h\right) V' - \frac{1}{2\psi}\left(\frac{V^2}{V'} - 2hV - h^2 V'\right)$$
$$- \frac{1}{2}\sigma^2 V'' \equiv \widetilde{\rho} V - \left(\widetilde{\mu} + \widetilde{r} h\right) V' - \frac{1}{2\psi}\left(\frac{V^2}{V'} - 2hV - h^2 V'\right) - \frac{1}{2}\sigma^2 V'' = 0.$$

Rearrange for $\overline{V} \equiv V(\overline{h})$ and $\overline{V}_o \equiv V_o(\overline{h}_o)$ using smooth pasting and super contact:

$$\Omega(\overline{V}|\overline{h}) = 0,\ \Omega(\overline{V}_o|\overline{h}_o) + \gamma\left(V(\overline{h}_o) - \overline{V}_o\right) = 0,$$

where $\Omega(v|h) \equiv \frac{1}{2\psi} v^2 - \left(\widetilde{\rho} + \frac{h}{\psi}\right) v + \left(\widetilde{\mu} + \widetilde{r} h - \frac{h^2}{2\psi}\right)$. Then, $\Omega_v(\overline{V}|\overline{h}) < 0$, because,

[39] By Bolzano-Weierstrass theorem, such an increasing sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ ($\gamma_n \to \infty$) exists.

[40] This is why Inequality (17) in Proposition 5 might potentially not be a necessary condition for early financing – although numerical algorithm in Supplemental Appendix SB.1, which does not rely on $\mathbf{B}$ being an interval, seems to always generate an interval for it.

letting V^* first-best value, both roots solving $\Omega(v^*|0) = 0$ are strictly between $v^- - \overline{h}$ and $v^+ - \overline{h}$, where $v^\pm$ are the two roots of $\Omega(v|\overline{h}) = 0$, even as $\overline{V} - \overline{h} < V^*$; i.e., $\overline{V} = v^-$.[41] Since $V_o(\overline{h}) = \overline{V}_o + (\overline{h} - \overline{h}_o) < \overline{V}$ by $\overline{h} \geq \overline{h}_o$, it holds that $0 < \Omega\big(\overline{V}_o + (\overline{h} - \overline{h}_o)|\overline{h}\big)$, i.e.,

$$
\begin{aligned}
0 &< \frac{1}{2\psi}\overline{V}_o^2 - \left(\widetilde{\rho} + \frac{\overline{h}_o}{\psi}\right)\overline{V}_o + \left(\widetilde{\mu} + \widetilde{r}\overline{h}_o\right) - \varphi(\overline{h} - \overline{h}_o) + \frac{\overline{h}_o^2}{2\psi} - \frac{\overline{h}^2}{\psi} \\
&\leq \frac{1}{2\psi}\overline{V}_o^2 - \left(\widetilde{\rho} + \frac{\overline{h}_o}{\psi}\right)\overline{V}_o + \left(\widetilde{\mu} + \widetilde{r}\overline{h}_o - \frac{\overline{h}_o^2}{2\psi}\right) = -\gamma\Big(V(\overline{h}_o) - \overline{V}_o\Big),
\end{aligned}
$$

where the second inequality is from $\overline{h} \geq \overline{h}_o$. This contradicts exclusion $V(\overline{h}_o) > \overline{V}_o$.

Parts (ii) and (iii) need some notations. Let $\widetilde{\mathbf{B}}$ be $\mathbf{B}$ without its isolated points. Let $\mathcal{B}$ a collection of disjoint closed intervals with nonempty interiors whose union gives $\widetilde{\mathbf{B}}$. Let $\mathcal{C}$ a collection of disjoint intervals with nonempty interiors whose union gives the closure of $[0, \overline{h}] \setminus \widetilde{\mathbf{B}}$ but excluding $\overline{h}$. Index each interval by its supremum:

$$
\begin{aligned}
&\mathcal{B} =: \{B_{h_j} \mid j \in \mathcal{J}_B\}, \quad \mathcal{C} =: \{C_{h_j} \mid j \in \mathcal{J}_C\}, \quad \text{where} \\
&\{h_j \mid j \in \mathcal{J}_B\} \equiv \{\sup B \mid B \in \mathcal{B}\} \text{ and } \{h_j \mid j \in \mathcal{J}_C\} \equiv \{\sup C \mid C \in \mathcal{C}\}.
\end{aligned}
$$

Given the requirements of nonempty interiors, both $\mathcal{J}_B$ and $\mathcal{J}_C$ are at most countable.

(ii) V_o'', $V'' < 0$ on C_{h_j} for all $j \in \mathcal{J}_C$. First, consider $C_{\overline{h}} = [\underline{h}, \overline{h})$. I address V_o first. Differentiate the HJB for V_o at $h = \overline{h}_o > \overline{h}$ for a third (right) derivative:

$$
\overline{V}_o''' = \frac{2}{\sigma^2}\left(\varphi + \frac{2}{\psi}\overline{h}_o\right) > 0,
$$

where $\overline{h}_o > \overline{h}$ gives $V'(\overline{h}_o) = \overline{V}_o'$. Thus, $V_o'' < 0$ on a neighborhood below $\overline{h}_o$. Suppose BYOC that there exists $\widehat{h} \in [\underline{h}, \overline{h}_o)$ such that $V_o'' < 0$ on $(\widehat{h}, \overline{h}_o)$ but $V_o''(\widehat{h}) = 0$. Then,

$$
0 \geq V_o'''(\widehat{h}) = \frac{2}{\sigma^2}\left(\varphi V_o'(\widehat{h}) + \frac{2}{\psi}\widehat{h}V_o'(\widehat{h}) - \gamma\Big(V'(\widehat{h}) - V_o'(\widehat{h})\Big)\right).
$$

Therefore, $\gamma > 0$ and $V'(\widehat{h}) - V_o'(\widehat{h}) > 0$. Since $V'(\overline{h}) - V_o'(\overline{h}) < 0$ from $\overline{h}_o > \overline{h}$, intermediate value theorem implies that there is $\widetilde{h} \in (\widehat{h}, \overline{h})$ such that $V'(\widetilde{h}) - V_o'(\widetilde{h}) = 0$.

$$
\begin{aligned}
V''(\widetilde{h}) - V_o''(\widetilde{h}) &= \frac{2}{\sigma^2}\left(\left(\widetilde{\rho} + \gamma + \frac{\widetilde{h}}{\psi}\right)\Big(V(\widetilde{h}) - V_o(\widetilde{h})\Big) - \frac{1}{2\psi}\frac{V(\widetilde{h})^2 - V_o(\widetilde{h})^2}{V'(\widetilde{h})}\right) \\
&= \frac{2}{\sigma^2}\left(\widetilde{\rho} + \gamma - \frac{1}{\psi}\underbrace{\left(\frac{V(\widetilde{h}) + V_o(\widetilde{h})}{2V'(\widetilde{h})} - \widetilde{h}\right)}_{\equiv(a)}\right)\underbrace{\Big(V(\widetilde{h}) - V_o(\widetilde{h})\Big)}_{>0}.
\end{aligned}
$$

[41] To have a well-defined value of V^*, assume that $A \leq (\rho + \delta)\left(\frac{(\rho+\delta)\psi}{2} + 1\right)$.

Note that $(a) < \psi\widetilde{\rho}$. This is established by the following reasoning. First,

$$\frac{V(\widetilde{h}) + V_o(\widetilde{h})}{2V'(\widetilde{h})} - \widetilde{h} < \frac{V(\widetilde{h})}{V'(\widetilde{h})} - \widetilde{h} < \overline{V} - \overline{h},$$

because $V'' < 0$ on $[\underline{h}, \overline{h})$ – which holds by the identical reasoning (up to imposing a nonpositive third derivative of V at $h = V''^{(-1)}(0)$) but with $\gamma = 0$ – and so

$$\frac{\partial}{\partial h}\left(\frac{V(h)}{V'(h)} - h\right) = 1 - \frac{V(h)}{V'(h)^2}V''(h) - 1 > 0.$$

Next, $\overline{V} - \overline{h} < \psi\widetilde{\rho}$, since $\overline{V}$ is, as shown when proving (i), the lower root of $\Phi(v|\overline{h}) = 0$:

$$\overline{V} = \overline{h} + \psi\left(\widetilde{\rho} - \sqrt{\left(\widetilde{\rho} + \frac{\overline{h}}{\psi}\right)^2 - 2\frac{\widetilde{\mu} + \widetilde{r}\overline{h}}{\psi} + \frac{\overline{h}^2}{\psi^2}}\right) < \overline{h} + \psi\widetilde{\rho}.$$

Thus, $(a) < \psi\widetilde{\rho}$. And therefore, $V''(\widetilde{h}) - V_o''(\widetilde{h}) > 0$, contradicting $\overline{h} > \widehat{h}$ and $V'(\widehat{h}) - V_o'(\widehat{h}) > 0 > V'(\overline{h}) - V_o'(\overline{h})$.

As for V on $C_{\overline{h}}$, the same reasoning as above but with $\gamma = 0$ gives contradiction more directly since $0 < V'''(\widehat{h})$. Next, for other C_{h_j}s with $h_j < \overline{h}$, assume that $V_o'' < 0$ on $[h_j, \overline{h}_o)$ and $V'' < 0$ on $[h_j, \overline{h})$ and proceed with the above reasoning starting at the contradiction-inducing assumption of the existence of $\widehat{h} \in C_{h_j}$ where the second derivative becomes zero for the first time (with h going down). Regarding applying intermediate value theorem to the existence of $\widetilde{h}$, which is relevant for V_o, use the fact that such $\widetilde{h}$ can only exist on one of the C_{h_j}s; on any $B_h \in \mathcal{B}$, $V' = \theta + (1-\theta)V_o' < V_o'$ since $V_o' > 1$ by the assumed strict concavity of V_o on $[h_j, \overline{h}^o]$ and $\overline{V}_o' = 1$. Jointly with Part (iii), the initial assumption of strict concavity above h_j will hold.

(iii) V_o'', $V'' < 0$ on B_{h_j} for all $j \in \mathcal{J}_B$. Start by assuming that $V_o'' < 0$ on $[h_j, \overline{h}_o)$, which will be iteratively validated jointly with Part (ii). Suppose by way of contradiction that there exists $\widehat{h} \in B_{h_j} \setminus \{h_j\}$ such that $V_o'' < 0$ on $(\widehat{h}, h_j)$ but $V_o''(\widehat{h}) = 0$. Since $B_{h_j} \subset \mathbf{B}$, substitute the identity $V = \theta\big(\overline{V} - (\overline{h} - h)\big) + (1-\theta)V_o$ on B_{h_j} and differentiate the HJB for V_o to obtain the third derivative at $h = \widehat{h}$ as

$$V_o'''(\widehat{h}) = \frac{2}{\sigma^2}\left(\left(\varphi + 2\frac{\widehat{h}}{\psi}\right)V_o'(\widehat{h}) + \theta\gamma\Big(V_o'(\widehat{h}) - 1\Big)\right) > 0,$$

since $V_o'(\widehat{h}) > 1$ from $V_o'' < 0$ on $(\widehat{h}, \overline{h}_o)$ and $\overline{V}_o' = 1$. This contradicts $V_o''(\widehat{h}) = 0$ and $V_o'' < 0$ on a neighborhood above $\widehat{h}$. Hence, $V_o'' < 0$ and $V'' = (1-\theta)V_o'' < 0$ on B_{h_j}.

Iteratively from above, Parts (ii) and (iii) prove that $V'' < 0$ on $[0, \overline{h})$. $\square$

A.6 Proposition 5 (Underinvestment and dilution)

Proof. Financing is preferred to an instantaneous delay at (s,h) $(h>0)$ if and only if

$$\rho V(s,h) - rhV_h(s,h) \geq \mathcal{H}(V)(s,h) + \mathcal{K}(V)(s,h) + \mathcal{S}(V)(s,h), \quad \text{(A.6)}$$

$$\text{where } \mathcal{H}(V) \equiv \left(A + \frac{1}{2\psi}\right) V_h + \frac{1}{2}\sigma^2 \cdot V_{hh} - \left(\delta + \frac{1}{2\psi}\right)\left(V - hV_h\right),$$

$$\mathcal{K}(V) \equiv \frac{1}{2} i\left(V - hV_h\right) \quad \text{given optimal } i \text{ from Equation (15)},$$

and $\mathcal{S}$ is given in (14). The right-hand side obtains by substituting (15). When financing is optimal, the bargaining identity (4) applies, that is,

$$V(s,h) = \theta\left(V(s,\overline{h}(s)) - \overline{h}(s) + h\right) + (1-\theta)V^o(s,h). \quad \text{(A.7)}$$

Last, V^o satisfies $\rho V^o - rhV^o_h = \mathcal{H}(V^o) + \mathcal{K}(V^o) + \mathcal{S}(V^o) + \gamma\left(V - V^o\right)$.

Since $V_h(s,\overline{h}(s)) = 1$ and $V_{hh}(s,\overline{h}(s)) = 0$ by smooth pasting and super contact,

$$V_h(s,h) = \theta + (1-\theta)V^o_h(s,h), \quad V_{hh}(s,h) = (1-\theta)V^o_{hh}(s,h),$$

$$V(s,h) - hV_h(s,h) = \theta\Big(V(s,\overline{h}(s)) - \overline{h}(s)\Big) + (1-\theta)\Big(V^o(s,h) - hV^o_h(s,h)\Big).$$

Consequently, $\mathcal{H}(V)(s,h) = \theta\mathcal{H}(V)(s,\overline{h}(s)) + (1-\theta)\mathcal{H}(V^o)(s,h)$.

$\overline{h}'(s)$ is defined by implicit function theorem on $V_{hh}(s,\overline{h}(s)) = 0$ since $V_{hhh} > 0$ at $(s,\overline{h}(s))$. Totally differentiating $V_h(s,\overline{h}(s)) = 1$ in s gives $V_{sh}(s,\overline{h}(s)) = 0$. Then,

$$\frac{d}{ds}\Big(V(s,\overline{h}(s)) - \overline{h}(s)\Big) = V_s(s,\overline{h}(s)) + V_h(s,\overline{h}(s))\overline{h}'(s) - \overline{h}'(s) = V_s(s,\overline{h}(s)),$$

$$\frac{d^2}{ds^2}\Big(V(s,\overline{h}(s)) - \overline{h}(s)\Big) = V_{ss}(s,\overline{h}(s)) + V_{sh}(s,\overline{h}(s))\overline{h}'(s) = V_{ss}(s,\overline{h}(s)).$$

As such, $\mathcal{S}(V)(s,h) = \theta\mathcal{S}(V)(s,\overline{h}(s)) + (1-\theta)\mathcal{S}(V^o)(s,h)$.

$\mathcal{H}$ and $\mathcal{S}$ are thus canceled (see Section 5.2), so that Inequality (A.6) becomes

$$(1-\theta)\gamma\Big(V(s,h) - V^o(s,h)\Big) \geq \theta\varphi\left(\overline{h}(s) - h\right)$$
$$+ \mathcal{K}(V)(s,h) - \theta\mathcal{K}(V)(s,\overline{h}(s)) - (1-\theta)\mathcal{K}(V^o)(s,h). \quad \text{(A.8)}$$

Equation (A.7) implies that if bargaining is optimal at h, then $V - hV_h = \theta\left(\overline{V} - \overline{h}\right) + (1-\theta)\left(V^o - hV^o_h\right)$; substitute it into $\mathcal{K}(V)(s,h)$ in (A.8). Evaluate at $h = 0$ to obtain (17). Given $h = \underline{h}(s)$ where (A.8) holds with equality, substitute $(1-\theta)(\underline{V} - \underline{V}^o) = \theta\left((1-\underline{x})\overline{V} - \Delta h\right)$ (see discussion of Proposition 2 in Section 5.3) to obtain (18). Add and subtract $\frac{\theta}{2}\left(\bar{\imath} - \underline{i}\right)\left(\underline{V} - \underline{h}\right)$ on the right-hand side of (A.8) at $h = \underline{h}(s)$, substitute $\left(\overline{V} - \overline{h}\right) - \left(\underline{V} - \underline{h}\right) = (1-\underline{x})\overline{V} - \Delta h$, and solve for $(1-x)\overline{V} - \Delta h$ to obtain (16). $\square$

Supplemental Appendix A Other Discussions

SA.1 Comparison with existing models

Comparison		Models for financial slack: Fixed costs	Search frictions	This paper
I. Mechanism				
Zero financial slack		Not first-best	Not feasible	First-best/feasible
Lumpy financing	Arises?	Yes	Yes	Yes
	Rationale	Optimality	Feasibility	Optimality
Early financing	Arises?	Not in itself	Yes	Yes
	Rationale	'Market timing'	Feasibility	Optimality
II. Implications				
Future prospect $\uparrow$		No change	Slack $\uparrow$	Slack $\uparrow$
Observed financing costs		As assumed	$\propto$ value	If financing is early, $\propto$ funds raised; if not, $\propto$ value
Access to financing $\uparrow$		Slack $\downarrow$	Slack $\downarrow$	Not necessarily
Macro connection		Financing $\rightarrow$ real	Financing $\longleftrightarrow$ real	Financing $\longleftrightarrow$ real

SA.2 Robustness to within-period bargaining

Here, I discuss how θ can accommodate various static bargaining environments. I illustrate it using the stylized setup in Section 3, but the implications extend to the main model. Let c_t continuation value and v_t^o the firm's reservation value on date t.

Multiple suppliers (financiers) on each date. Suppose that $n \in \mathbb{N}$ suppliers visit the firm on each date. I assume that the firm bargains with one supplier at a time. If bargaining were to fail, the firm can bargain with another supplier on the same date. Each supplier commits to refusing to bargain with the firm conditional on his previous bargaining failure with it. Let v_t^m firm value from bargaining with the m^{th} supplier on date t, $m \leq n$. Then,

$$\begin{aligned}
v_t^n &= \theta c_t + (1-\theta) v_t^o, \\
v_t^{n-1} &= \theta c_t + (1-\theta) v_t^n = \big(1-(1-\theta)^2\big) c_t + (1-\theta)^2 v_t^o, \\
v_t^{n-2} &= \theta c_t + (1-\theta) v_t^{n-1} = \big(1-(1-\theta)^3\big) c_t + (1-\theta)^3 v_t^o \\
&\dots \\
v_t^1 &= \theta c_t + (1-\theta) v_t^2 = \big(1-(1-\theta)^n\big) c_t + (1-\theta)^n v_t^o.
\end{aligned}$$

Therefore, it is equivalent to $\theta_n \equiv \big(1-(1-\theta)^n\big) \in [\theta, 1)$; expectedly, $\theta_n \to 1$ as $n \to \infty$.

Nesting Bertrand competition. Let $n = 2$ above, and suppose that each supplier can fully commit to refusing to bargain with the firm consecutively, and otherwise can commit to refusing to bargain with an i.i.d. probability $\chi \in [0, 1]$ in case the firm has failed in bargaining beforehand. Once the firm is denied bargaining by either supplier, no more bargaining is allowed on that date. Because bargaining does not fail on the path of play in any subgame, the value of each supplier's outside options when he is bargaining with the firm is zero. Letting v_t^B firm's value from bargaining,

$$\begin{aligned} v_t^B &= \theta c_t + (1-\theta)\Big(\chi v_t^o + (1-\chi) v_t^B\Big) \\ \implies v_t^B &= \frac{\theta}{1-(1-\theta)(1-\chi)} c_t + \left(1 - \frac{\theta}{1-(1-\theta)(1-\chi)}\right) v_t^o. \end{aligned}$$

Therefore, this is equivalent to $\theta_\chi \equiv \frac{\theta}{1-(1-\theta)(1-\chi)} \in [\theta, 1]$. Note that $\theta_\chi = 1$ if and only if $\chi = 0$, that is, suppliers cannot commit at all except upon immediate bargaining failure. This essentially describes Bertrand competition. In sum, when the firm must bargain with each supplier bilaterally, perfect competition à la Bertrand between two suppliers obtains if and only if suppliers lack commitment technology altogether $\chi = 0$.

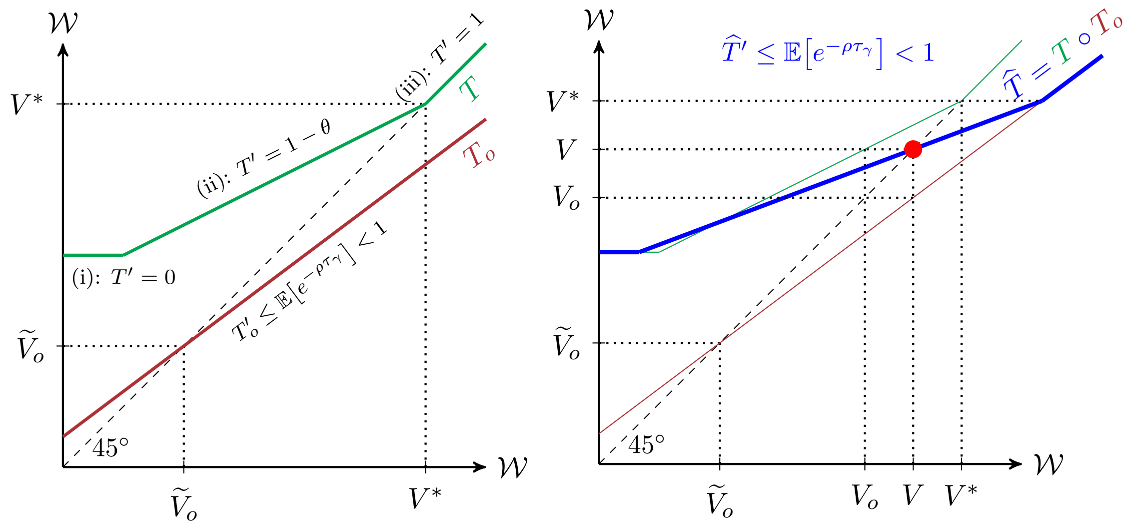
SA.3 Contraction from two-step recursion: illustration

Take the space $\mathcal{W}$ and the three self-maps $\widehat{T}, T, T_o : \mathcal{W} \to \mathcal{W}$ as defined in Appendix A.1. Here, I illustrate the proof that $\widehat{T}$ is a contraction, such that contraction mapping theorem can be applied. Let $V^* \in \mathcal{W}$ be first-best value function, that is, without financing frictions and $\widetilde{V}_o \in \mathcal{W}$ value function under complete autarky, that is, $\gamma = 0$.

By way of analogy, visualize the infinite-dimensional $\mathcal{W}$ as a simple $\mathbb{R}_+$. A continuous, nondecreasing, and piece-wise differentiable self-map on $\mathbb{R}_+$ is a contraction if and only if its derivative, wherever defined, is below ζ for some $\zeta \in [0, 1)$.

Figure SA.1a describes T and T_o. First consider T. If insiders' outside option is preferred to the first-best $W \geq V^*$, they are going to simply take it – the embedded choice (iii) in the definition of T – see Appendix A.1. Therefore, its slope above V^* is unity. If $W < V^*$, then insiders do not take the option, but this 'outside' option is still beneficial to the extent that insiders bargain with financiers. Since better outside option reduces financiers' rents by a fraction $1 - \theta$ subject to bargaining taking place, the slope of T below V^* is (ii) $1 - \theta$ given bargaining and (i) zero given no bargaining. As shown, T is not a contraction: any point above V^* is one of its fixed points.

Next, consider T_o: insiders are on autarky until the stopping time τ_γ when the business gives W as a terminal payoff. A higher terminal payoff benefits insiders but only up to a discounting due to time delay $\mathbb{E}\big[e^{-\rho\tau_\gamma}\big] \in [0, 1)$, which thus bounds the slope of T_o. Therefore, T_o is a contraction and has a unique fixed point.



(a) Before composition T, T_o

(b) Composition $\widehat{T}$ and unique fixed point

Figure SA.1: Two-step recursion as a contraction map

But T_o, in itself, is a particularly *un*interesting contraction. Regardless of $\gamma \in [0, \infty)$, its fixed point is always $\widetilde{V}_o$. If at τ_γ insiders receive their own value as a terminal payoff, they are exactly compensated for the termination anyway. Accordingly, the graph of T_o rotates as γ changes but centered on the invariant fixed point $(\widetilde{V}_o, \widetilde{V}_o)$.

Even though T_o by itself is a trivial contraction, its time discounting due to $\gamma < \infty$ still makes the composition map $\widehat{T} = T \circ T_o$ contractionary, because, as Figure SA.1b illustrates, the slope of T never exceeds unity. Hence, there exists a unique fixed point $V = \widehat{T}(V)$ by contraction mapping theorem.

SA.4 Costs and benefits of financial slack

Given a general cash flow profile from Section 4.1, posit the equilibrium $(\underline{h}, \overline{h})$ along with the implied value functions (V, V_o). First, define dilution ratio as

$$D(h) \equiv (1-\theta)\left(1 - \frac{V_o(h) - h}{V(\overline{h}) - \overline{h}}\right),$$

i.e., financiers' rent $(1-\theta)\big(V(\overline{h}) - (\overline{h} - h) - V_o(h)\big)$ over net firm value post financing $V(\overline{h}) - \overline{h}$.[42] It is decreasing in h, and satisfies $1-\theta = D(0) > D(\overline{h}) > 0$. Let $\underline{D} \equiv D(\underline{h})$.

To analyze lifetime firm value without a recursive HJB formulation, a couple of stochastic processes must be specified. Define a counting process $\{n_t\}_{t\geq 0}$ by $n_0 = 0$ and $dn_t = \mathbb{1}\left(\lim_{s\to t^-} h_s = \underline{h}\right)$. The process n_t tracks how many times financing, and

[42] D and ownership retention x in Section 4.2 satisfy $D(h)\big(V(\overline{h}) - \overline{h}\big) = \big(1 - x(h)\big)V(\overline{h}) - (\overline{h} - h)$.

hence dilution $\underline{D}$, has occurred over the time interval $(0,t]$. Also define $\{\tau_m\}_{m\in\mathbb{N}}$ as

$$\tau_m \equiv \inf\{t \geq 0 \mid n_t = m\},$$

which is the associated increasing sequence of stopping times for m^{th} financing; that is, the first financing occurs at $t = \tau_1 > 0$, and so forth. Lastly, let τ be the stopping time for terminal success arriving at a Poisson rate λ with payoff Π. Note that n_τ counts the entire lifetime financing rounds; in particular, $\lambda > (=)\ 0$ if and only if $n_\tau \overset{a.s.}{<} (\overset{a.s.}{=})\ \infty$.

Firm value $V(h)$ can be decomposed as follows. Think of it equivalently as $h + \big(V(h) - h\big)$ where insiders, counterfactually, first receive h as one-time dividends and, going forward, recognize both cash flow $\mu\ dt + \sigma\ dB_t$ (including terminal Π at λ) and carry cost flow $-\varphi h_t\ dt$ as immediate utility flow scaled by their undiluted ownership $(1-\underline{D})^{n_t}$. Given $h_0 = \overline{h}$,

$$\begin{aligned} V(\overline{h}) - \overline{h} &= \mathbb{E}_0\left[\int_0^\tau e^{-\rho t}(1-\underline{D})^{n_t}\Big(\big(\mu - \varphi h_t\big)\ dt + \sigma\ dB_t\Big) + e^{-\rho\tau}(1-\underline{D})^{n_\tau}\Pi\right] \\ &= \mathbb{E}_0\left[\int_0^\tau e^{-\rho t}\Big(\mu - \varphi h_t\Big)\ dt + e^{-\rho\tau}\Pi\right] - \mathbb{E}_0\left[\sum_{m=1}^{n_\tau} e^{-\rho\tau_m}\underline{D}\Big(V(\overline{h}) - \overline{h}\Big)\right]. \end{aligned}$$

Inside the second expectation on the last line, both $\underline{D}$ and $V(\overline{h}) - \overline{h}$ are constant in equilibrium for all $\{\tau_m\}_{m\in\mathbb{N}}$ due to the time-invariant financing threshold $\underline{h}$ and funding target $\overline{h}$, and thus can be brought outside the unconditional expectation. Rearranging yields

$$V(\overline{h}) - \overline{h} = \frac{\text{NPV} - \mathcal{C}}{1 + \mathcal{D}}, \tag{SA.1}$$

where

$$\text{NPV} \equiv \mathbb{E}_0\left[\int_0^\tau e^{-\rho t}\mu\ dt + e^{-\rho\tau}\Pi\right] = \frac{\mu + \lambda\Pi}{\rho + \lambda}, \quad \mathcal{C} \equiv \mathbb{E}_0\left[\int_0^\tau e^{-\rho t}\varphi h_t\ dt\right] \equiv \underline{\mathcal{C}} + \mathcal{C}_\Delta$$

with $\underline{\mathcal{C}} \equiv \frac{\varphi}{\rho+\lambda}\underline{h}$ and $\mathcal{C}_\Delta \equiv \varphi\mathbb{E}_0\left[\int_0^\tau e^{-\rho t}(h_t - \underline{h})\ dt\right]$, and

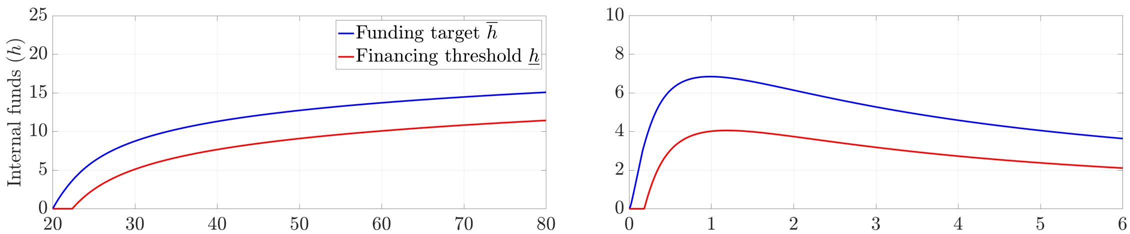
$$\mathcal{D} \equiv \underline{D}\mathbb{E}_0\left[\sum_{m=1}^{n_\tau} e^{-\rho\tau_m}\right].$$

As Equation (SA.1) shows, firm value in net $V(h) - h$, even at the target funding capacity, is lower than the frictionless net present value of the business due to the carry cost of financial slack $\mathcal{C}$ and dilution $\mathcal{D}$. Insiders choose a financing strategy $(\underline{h}, \Delta h)$ to maximize net firm value, balancing the reduction of dilution $\mathcal{D}$ against the carry cost $\mathcal{C}$. On one hand, both funding reserve $\underline{h}$ and financing amount $\Delta h = \overline{h} - \underline{h}$ lower dilution $\mathcal{D}$, by reducing its size $\underline{D}$ and frequency $\mathbb{E}_0\left[\sum_{m=1}^{n_\tau} e^{-\rho\tau_m}\right]$, respectively. On the other hand, financial slack involves carry costs $\mathcal{C} = \underline{\mathcal{C}} + \mathcal{C}_\Delta$, with $\underline{\mathcal{C}}$ from the funding reserve $\underline{h}$ and $\mathcal{C}_\Delta$ from the financing amount Δh.

Financing is always optimally lumpy $\Delta h > 0$ because zero financing amount $\Delta h \to 0$ fails to drive the size of dilution $\underline{D}$ down to zero, as Proposition 1 shows, and yet blows its frequency $\mathbb{E}_0\left[\sum_{m=1}^{n_\tau} e^{-\rho\tau_m}\right]$ up to infinity. In contrast, financing is not always optimally early, as Corollary 1 shows. The funding reserve $\underline{h}$, viewed as insiders' one-dimensional Markov strategy, involves a greater marginal carry cost $\frac{\varphi}{\rho+\lambda}$ than financing amount Δh, because $\varphi\underline{h}\ dt$ is a fixed flow cost whereas $\varphi(h_t - \underline{h})\ dt < \varphi\Delta h\ dt$ almost always. As such, insiders do not employ a funding reserve if it fails to sufficiently reduce the size of dilution $\underline{D}$.

SA.5 Comparative statics in business parameters (Π, π)

To provide an alternative segue into investment, let us start by comparing the two stylized examples from Section 4.1. Startups incur a constant loss $-\kappa\ dt$ until success arrives at Poisson rate $\lambda > 0$ giving a terminal payoff $\Pi > \frac{\kappa}{\lambda}$. Operating firms receive volatile cash inflow $\pi\ dt + \sigma\ dB_t$. I vary Π in startups and π in operating firms.



(a) SU: future payoff (Π)

(b) OF: average profit (π)

Figure SA.2: Future payoff versus current profitability

Horizontal axis: parameter being varied. Vertical line: equilibrium $(\underline{h}, \overline{h})$. SU: startup, OF: operating firm. Fixed parameters are: common – $(\rho, r) = (0.05, 0)$, $(\theta, \gamma) = (0.5, 1)$; SU – $(\kappa, \lambda) = (2, 0.1)$; OF – $\sigma = 2$. Formal results are presented in Supplemental Appendix C.

Figure SA.2 contrasts the comparative statics in future payoff Π versus profitability π. Consider Figure SA.2a. Although startups' cash flow $-\kappa\ dt$ is fixed, insiders increase financial slack when future payoff Π is higher. This is due to bargaining: a higher future value increases the surplus from financing, and hence financiers can extract more rents. Insiders, therefore, respond by increasing financial slack to mitigate dilution.

In contrast, Figure SA.2b shows that financial slack is non-monotonic in profitability π. On one hand, it has the same effect as Π for startups in increasing firm value and hence the size of dilution. This is manifested over the domain of π where financial slack is upward-sloped. On the other hand, a higher profitability improves current cash flow, and hence – given fixed volatility σ – makes losses less likely. Since dilution becomes less and less likely to be triggered as π goes up, insiders decrease financial slack.

These results give another motivation for adding investment, which reduces current cash flow to improve future payoff. Investment gives rise to rich interactions between dilution, current cash flow, and future prospect: (i) current cash flow decreases the frequency of dilution, (ii) future payoff increases the size of dilution, and also (iii) dilution – both in frequency and size – reduces insiders' gains from investment.

SA.6 Financial slack under investment opportunities

For the present stylized exercise, I slightly modify the cash flow profile in Section 4.1.

Cash flow and investment opportunities. The business has 'normalized' cash inflow $\pi\ dt + \sigma\ dB_t$, with $\pi,\ \sigma > 0$. Opportunities to scale up cash inflow to $\eta\big(\pi\ dt + \sigma\ dB_t\big)$, $\eta > 1$, arrive at a Poisson rate $\lambda > 0$, but it requires a normalized upfront investment *expense* of $\xi > 0$.[43] Firm value $W(k,h)$ given $k \in \mathbb{N} \cup \{0\}$ rounds of past investment and funds h is homogeneous so that $W(k,h) = \eta^k W\left(1, \frac{h}{\eta^k}\right) \equiv \eta^k V\left(\frac{h}{\eta^k}\right)$.

To ensure that investment is first-best, I assume the following.

Assumption SA.6.1. *$\pi > \lambda\xi$ and $\frac{(\eta-1)\pi}{\rho} \geq \xi$.*

Investment choice. Upon receiving an opportunity to invest, a firm with funds h can (i) fund the investment internally, with value $\eta V\left(\frac{h-\xi}{\eta}\right)$, (ii) forgo the investment, with value $V(h)$, or (iii) finance the investment externally, with value given by bargaining as

$$\widehat{V}_o(h) + \theta\Big(\eta V(\overline{h}) - \big(\xi + \eta\overline{h} - h\big) - \widehat{V}_o(h)\Big),$$

where the firm' reservation value is $\widehat{V}_o(h) \equiv \max\left\{V_o(h), \eta V_o\left(\frac{h-\xi}{\eta}\right)\right\}$:

- When $\widehat{V}_o(h) = V_o(h)$, the firm optimally forgoes investment if financing fails.
- When $\widehat{V}_o(h) = \eta V_o\left(\frac{h-\xi}{\eta}\right)$, the firm optimally invests even if financing fails. I term this 'financing the investment with (credible) commitment to execution.'

The term 'commitment' is simply a shorthand for when, given that choice (iii) is optimal, the optimality of investment itself does not depend on successful financing.[44]

Financing access. I study how access to alternative financing γ affects financial slack and investment, given $(\rho, r) = (0.07, 0)$, $\theta = 0.5$, $(\pi, \sigma) = (1, 2)$ and $(\lambda, \xi) = (0.5, 0.7)$.

Let $\gamma = 0.3$. As Figure SA.3 illustrates, firms may sometimes forgo – as shown in black square markers – investment opportunities that increase frictionless net present value. For these firms, financing involves large rent extraction as their access to alternative financing is not so good. Therefore, they optimally forgo investment when

[43] In the illustration part from Section 6.3 at Figure 9, ξ was fund proceeds from divestment.

[44] By subgame perfection, this is equivalent to the firm first funding the investment internally – per choice (i) – and then promptly raising financing – per Section 4.2.

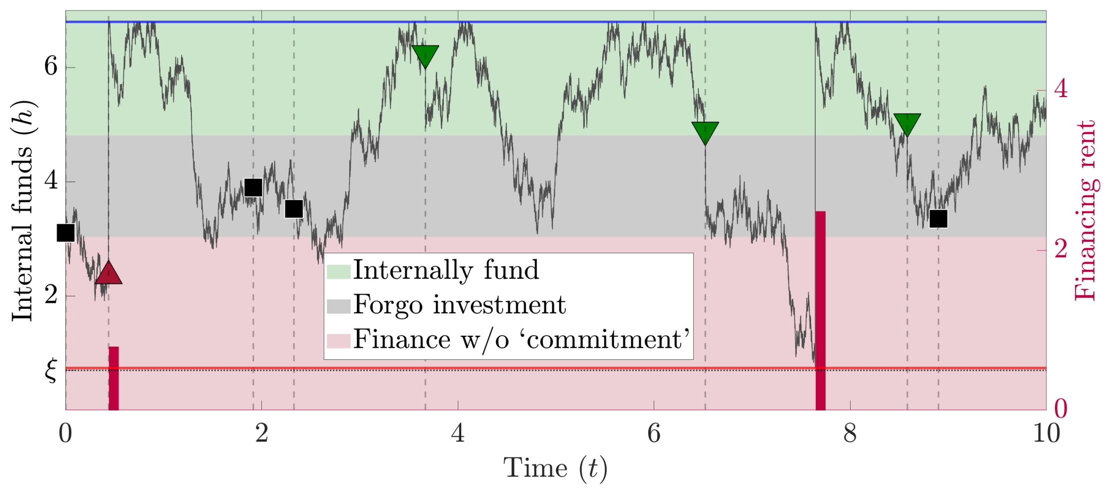


Figure SA.3: Optimal investment policy under dilution from financing

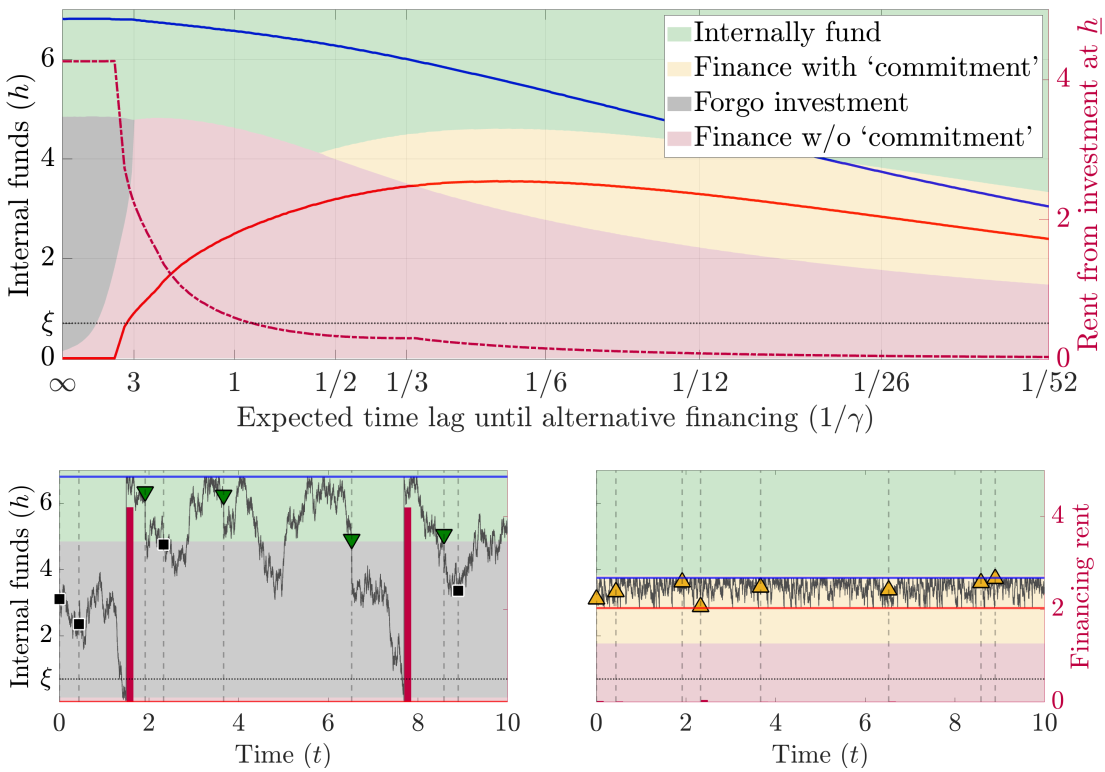
The black curve is a simulated history of normalized internal funds along the left axis. The horizontal lines in blue and red are funding target $\overline{h}$ and financing threshold $\underline{h}$, respectively. Investment opportunities arrive at the vertical dashed lines, each requiring an upfront normalized expense ξ whose size is indicated on the left axis. The markers in each shaded region indicate the choice made upon each opportunity in accordance with investment policy described in the legend. The magenta bar is normalized financing rent: one at investment financing ($t \approx 0.44$), and one at non-investment financing ($t \approx 7.64$). Access to alternative financing is $\gamma = 0.3$.

the lumpy expense would greatly dilute firm value ex-ante by making the sizable rent extraction by financiers much likelier to be triggered soon. When rent extraction has already become highly likely soon due to low funds, however, firm value is already diluted enough ex-ante so that it is optimal to increase diluted firm value by bargaining with financiers over financing the investment.

Figure SA.4 illustrates optimal financing and investment policies as the strength of financing access varies $\gamma \in [0, 52]$. The subplots describe $\gamma \in \{0, 26\}$. In Figure SA.4a with $\gamma = 0$, firms face substantial dilution from financing because they cannot access alternative financiers. As a result, they delay financing as much as possible and often forgo investment in order to avoid large dilution from financing.

In contrast, Figure SA.4b describes a firm that can expect to find alternative financiers very quickly $\gamma = 26$. It finances in small lumps $\Delta h \approx 0.91$ and thus extremely frequently – 53 times over $t \in [0, 10]$ in the present simulation – because of the extremely negligible size of dilution around 0.004 at the financing threshold. Nevertheless, it maintains a sizable funding reserve $\underline{h} \approx 2.84$. As discussed in Section 5, this financial slack is what reduces dilution to such a negligible size.

'Commitment' to investment execution and the size of dilution. Moreover, firms with such good access to financing $\gamma = 26$ always finance lumpy investments despite keeping funding reserves $\underline{h} \approx 2.84$ in large excess of the expense $\xi = 0.7$. For



(a) Poor access to financing $\gamma = 0$

(b) Good access to financing $\gamma = 26$

Figure SA.4: Financing access and investment policy

In the main plot with comparative statics in γ, colored areas show optimal investment policy given funds h per the legend. The semi-dashed line in magenta is the size of dilution from financing the investment given the opportunity arrives when $h_t = \underline{h}$. The dilution graph kinks downward when 'commitment' starts to hold – i.e., where the red curve enters the yellow region. For explanation of the subplots, see Figure SA.3.

any $h_t \in [\underline{h}, \overline{h}]$, these firms are willing to invest even if financing fails: even at $h_t = \underline{h}$,

$$\eta V_o\left(\frac{\underline{h}-\xi}{\eta}\right) \approx 23.228 > 22.102 \approx V_o(\underline{h}). \tag{SA.2}$$

This is because both γ and $\underline{h}$ are high. Because $\underline{h} \approx 2.84$ is high, firms can fund the investment internally and still have a substantial amount of normalized remaining funds $\frac{\underline{h}-\xi}{\eta} \approx 1.95$. Because γ is high, firms' normalized reservation value falls only slightly as a result $V_o\left(\frac{\underline{h}-\xi}{\eta}\right) \approx 21.116 < 22.102 \approx V_o(\underline{h})$. Investment thus raises the actual reservation value, as Inequality (SA.2) shows.

This is how such firms incur negligible dilution even when financing the investment. Even if financing were to fail, investment would still be optimally undertaken. Because investment does not depend on financing, returns on investment do not constitute a surplus from financing. Dilution thus becomes quite negligible (approximately 0.054), because this credible 'commitment' reduces it by $(1-\theta)\left(\eta V_o\left(\frac{\underline{h}-\xi}{\eta}\right) - V_o(h)\right) \approx 0.563$.

SA.7 Stochastic investment returns

Consider stochastic returns on investment. Concretely, consider a firm fluctuating through a discrete Markov chain $s \in \{1, 2, 3, 4\}$ across normal times $A^2 = A^3 = 0.18$, a boom $A^1 = 1.2A^2$ and a bust $A^4 = 0.8A^3$. The two normal times are distinguished in terms of their prospects. At A^2, it is likelier to enter a boom soon, whereas at A^3, it is likelier to enter a bust. The following matrix summarizes the Markov chain:

From\To	A^1	A^2	A^3	A^4
$A^1 = 0.216$	$\cdot$	0.3	0	0
$A^2 = 0.180$	0.3	$\cdot$	0.3	0
$A^3 = 0.180$	0	0.3	$\cdot$	0.3
$A^4 = 0.144$	0	0	0.3	$\cdot$

(SA.3)

where elements are Poisson rates of transitioning from a state (in row) to another (in column); the stationary distribution over state space $\{1, 2, 3, 4\}$ is uniform. This setup isolates the effect of future investment returns. Both A^2 and A^3 have the same current revenue, but A^2 merits increased investment due to greater expected returns.

To have a well-defined first-best solution, I increase ψ from 1.5 to 2 so that investment is harder to scale.[45] I maintain the other baseline parameters, including $\gamma = 1$.

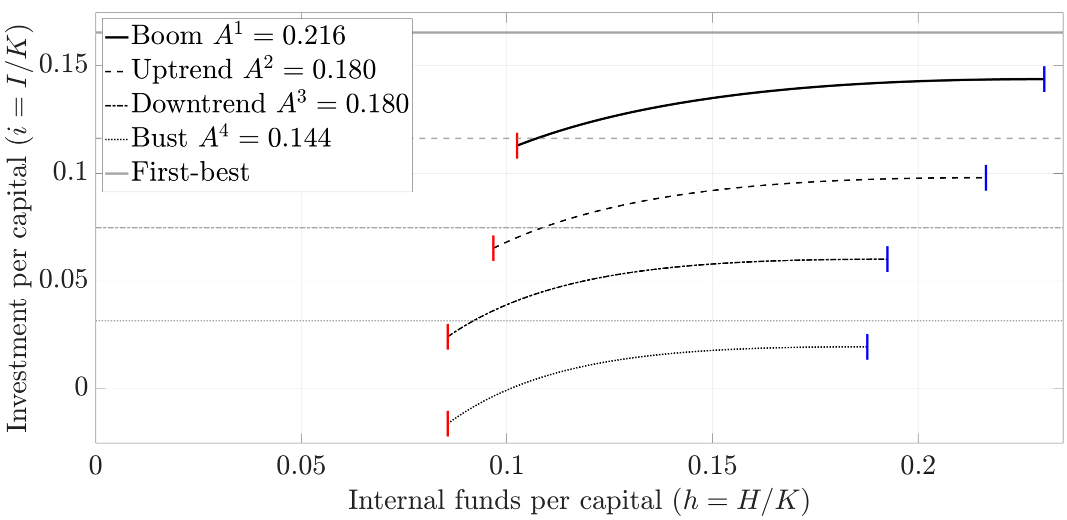


Figure SA.5: Expected investment returns increase financial slack the most

The black curves, in different styles, represent the optimal investment policy given internal funds in each state that occur on the equilibrium path, bounded in domain by funding target $\overline{h}^s$ in the blue line segment and financing threshold $\underline{h}^s$ in the red line segment. The flat lines in gray are the first-best investment under each state. The other parameters are $\rho = 0.06,\ r = 0.05,\ \theta = 0.5,\ \gamma = 1,\ \delta = 0.1007,\ \sigma = 0.09,\ \psi = 2.$

[45]Compared to the baseline case of constant $A = 0.18$, there is much greater upward growth potential $A^1 = 0.216$. If ψ is low, then A^1 is an inordinately great time to substantially scale up investment with small inefficiency, so that the first-best value blows up.

Figure SA.5 plots insider-optimal financing and investment strategies in states $s \in \{1,2,3,4\}$. I discuss underinvestment and demonstrate how it is mainly fluctuations in expected – rather than realized – investment returns that shift financial slack.[46]

(1) Underinvestment. The average underinvestment at A^2 relative to the first-best is approximately 2.06%, higher than that at A^3 of approximately 1.53% despite the same current revenue. With lucrative expected returns, insiders underinvest more to avoid dilution because firm value is higher and so is the size of dilution. With lower expected returns, they underinvest less because firm value is lower and so is the size of dilution. Dilution concerns induce more underinvestment when firms expect higher capital productivity in a near future, despite the convex adjustment cost that incentivizes anticipatory investment smoothing.

(2) Financial slack. Both the financing threshold and funding target $(\underline{h}^s, \overline{h}^s)$ expand most noticeably as future investment returns fluctuate. Insiders' equilibrium value rises – and so does the size of dilution – in anticipation of a boom, even as the current net cash inflow $\big(A^s - i_t - \Psi(i_t)\big)K_t\, dt$ declines due to greater investment. This is exactly analogous to the combination, from the startup example in Section 5.4, of a higher upside potential Π and a higher cash burn rate κ such that flow business value $\lambda\Pi - \kappa$ has not decreased (see Figure SA.2a). Each of these changes increases financial slack.

In contrast, realized changes in productivity, i.e., $A^2 \to A^1$ and $A^3 \to A^4$, lead to smaller adjustments in financial slack. With $A^2 \to A^1$, continuation value rises but so does the current gross cash inflow $A^s\, dt$, so that net cash inflow does not change substantially despite the increased investment. In addition, the rise in value is dampened because there is no more upside potential in the future, only downside. The combined effect is that dilution gets somewhat larger, but cash inflow net of investment is similar. With $A^3 \to A^4$, the same effects obtain in the opposite direction.[47]

[46] In what follows, averages are computed given the ergodic distribution on $h \in [\underline{h}^s, \overline{h}^s]$ for each $s = 1,2,3,4$, conditional on no Markov shift having occurred in the past – i.e., 'timeless.'

[47] Indeed, a more standard AR(1)-type process for stationary productivity A_t generates more stable financial slack under a fixed law of motion. Maintain the other parameters and let the log of A_t be driven by persistent Brownian shocks with mean reversion:

$$d\log A_t = \frac{1}{\nu}\left(\mu_a - \log A_t\right)\, dt + \sqrt{2/\nu}\;\sigma_a\, dZ_t,$$

where dZ_t is a standard Brownian motion independent of dB_t. $\{A_t\}_{t\geq 0}$ satisfies $\log A_t \sim \mathcal{N}(\mu_a, \sigma_a^2)$ and $\mathrm{Corr}(\log A_t, \log A_{t+s}) = e^{-\frac{s}{\nu}}$. Set $\sigma_a = 0.1437$, $\nu = 3.8772$ to preserve the autocovariance of the discrete Markov chain (SA.3), and $\mu_a \approx -1.743$ such that the median of A_t is $0.175 < 0.18$, adjusted against the greater upside potential $\mathbb{P}(A_t > 0.216) \approx 0.07$. Across a 95% confidence interval $[\underline{A}, \overline{A}]$ with $\underline{A} \equiv e^{\mu_a - 1.96\sigma_a} \approx 0.133 < A^4$ and $\overline{A} \equiv e^{\mu_a + 1.96\sigma_a} \approx 0.230 > A^1$, maximum variation in $\overline{h}(\cdot)$, attained with a 73% rise in realized productivity $\underline{A} \to \overline{A}$, is 0.0267. This is only 11% greater than how much it changes with a relatively moderate shift in prospect. Financing threshold $\underline{h}(\cdot)$ is even more stable, with maximum variation of $\underline{h}(\overline{A}) - \underline{h}(\underline{A}) \approx 0.0089$, a mere 80% of the change with $A^2 \leftrightarrow A^3$.

In summary, with fluctuations in realized productivity, the effect of changing firm value on the size of dilution is partially offset by the variation in current revenue that helps lower the frequency of dilution, and also dampened by mean reversion. With fluctuations in expected productivity, net cash flow varies directly and in the opposite direction due to variation in investment despite the same revenue, and mean reversion has less bite. Financial slack, therefore, is more sensitive to future investment returns.

SA.8 Anticipatory effects of financing outlook

How do firms react when their financing outlook fluctuates? In models with stochastic 'fixed' transaction costs of financing, e.g., Bolton et al. (2013), firms would raise funds earlier when they expect a rise in fixed cost, than when they expect a drop. Thus, a pessimistic shift in financing outlook should prompt a temporary surge in financing.

In this paper, financing threshold may decrease when firms expect alternative financing γ to become less accessible, $\underline{h}^2 \approx 0.0908 > 0.0867 \approx \underline{h}^3$, as Figure SA.6 shows.

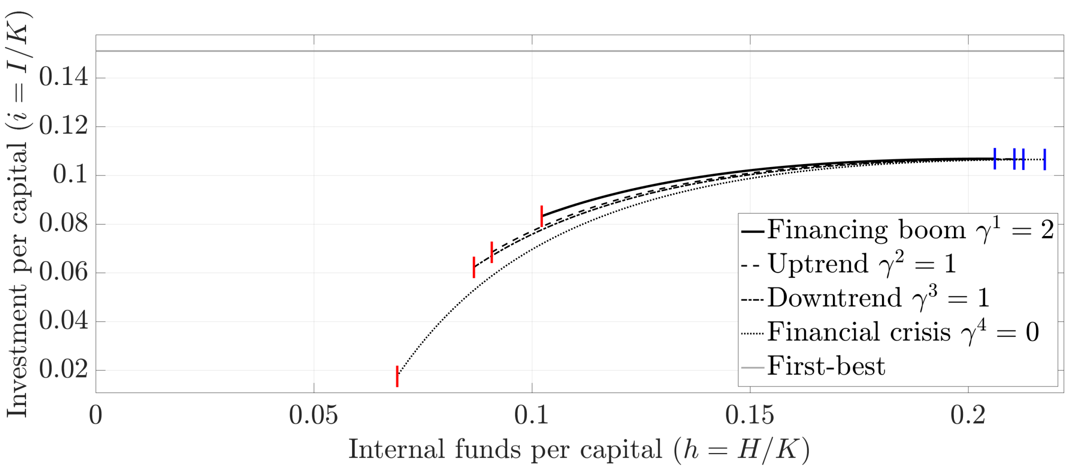


Figure SA.6: Financial slack and underinvestment across financial cycle

Other parameters are $\rho = 0.06,\ r = 0.05,\ \theta = 0.5,\ A = 0.18,\ \delta = 0.1007,\ \sigma = 0.09,\ \psi = 1.5$. The flat line in gray is the first-best. The Markov chain is again given by (SA.3), with γ^s replacing A^s.

SA.9 Productivity during crisis

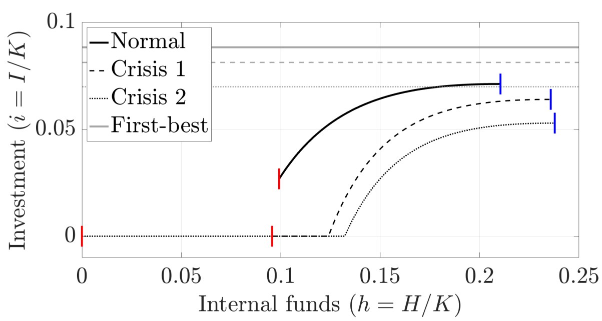
Motivated by Section 6.4, I conduct a business fluctuation exercise co-varying (γ, ϕ, A), where $\phi \geq 0$ is the reversibility parameter: $\Psi(i) \equiv \frac{\psi}{\mathbb{1}_{i\geq 0}+\phi\mathbb{1}_{i<0}}\frac{i^2}{2}$. There are three states

$s \in \{0, 1, 2\}$ with the Markov chain in Poisson rates:

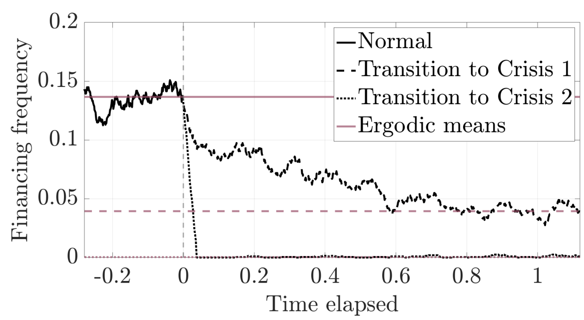
From\To	(γ^s, ϕ^s, A^s)	$s = 0$	$s = 1$	$s = 2$
Normal: $s = 0$	$(1, 0.5, 0.18)$	$\cdot$	0.1	0.1
Crisis 1: $s = 1$	$(0, 0, 0.17)$	0.5	$\cdot$	0
Crisis 2: $s = 2$	$(0, 0, 0.16)$	0.5	0	$\cdot$

During the 'normal' time $s = 0$, $(\gamma^0, \phi^0, A^0) = (1, 0.5, 0.18)$. Both crises $s \in \{1, 2\}$ exhibit perfect irreversibility and an absence of alternative financing $\gamma^s = \phi^s = 0$ as well as a differential drop in productivity $A^0 = 0.18 > A^1 = 0.17 > A^2 = 0.16$.

A crisis featuring both investment irreversibility and lack of alternative financing has vastly different effects depending on productivity. With low productivity $A^2 = 0.16$, excluded insiders cannot easily implement fallback underinvestment, and hence firms delay financing until funds are depleted $\underline{h}^2 = 0$. Dilution thus amplifies from 0.001 in $s = 0$ to 0.553 in $s = 2$, and financing gets much lumpier $\Delta h^2 = \overline{h}^2 \approx 0.238 > 0.111 \approx \Delta h^0$. Financing all but freezes, with one in 2,000 firms raising funds in a unit period.



(a) Financial slack and investment



(b) Transitional financing

Figure SA.7: Equilibrium and crisis transition

Higher productivity $A^1 = 0.17 > 0.6 = A^2$ induces a different pattern. The upward-concentrated fallback underinvestment becomes easier for excluded insiders to implement. Firms, therefore, keep financing early $\underline{h}^1 \approx 0.096$. Dilution remains similarly negligible 0.0017, and financing amount rises only moderately $\Delta h^1 \approx 0.140 > 0.111 \approx \Delta h^0$. Financing occurs with moderate (though reduced) frequency; on average, one in 25 firms (down from one in seven) finances in a unit period.

In sum, small variations in productivity may induce drastically different financing dynamics when financial market depth dries up and investment irreversibility peaks.

Supplemental Appendix B Algorithms

SB.1 Numerical algorithm for flexible investment

I explain the numerical algorithm for the setup in Section 6 with convex adjustment cost of investment – for the main analysis in Section 5, I use analytic solution outlined in Supplemental Appendix SB.2. Supplemental Appendix SB.1.1 sets up the main algorithm for the case without fluctuations in parameters. I then introduce Markov chains, both discrete and continuous, in Supplemental Appendix SB.1.2. Last, Supplemental Appendix SB.1.3 describes how to solve the model under investment irreversibility, as introduced in Section 6.4 and Supplemental Appendix SA.9.

SB.1.1 Main algorithm

Formulation. Start by setting some $H > 0$. It should be higher than $\overline{h}^o$, the funding target under exclusion, which is higher than $\overline{h}$ but typically by a slight margin. V on $[0, H]$ satisfies:

$$h \geq \overline{h} \implies 0 = V_{hh}(h) \quad (\because V_h = 1 \text{ on } [\overline{h}, \infty)) \tag{SB.1}$$

$$\begin{aligned} h \in [\underline{h}, \overline{h}] \implies \rho V(h) - rhV_h(h) &= \max_i \big(A - i - \Psi(i)\big)V_h + \big(i - \delta\big)\big(V - hV_h\big) + \frac{1}{2}\sigma^2 V_{hh} \\ &= \left(A + \frac{1}{2\psi} + \left(\delta + \frac{1}{\psi}\right) h\right) V_h - \left(\delta + \frac{1}{\psi}\right) V + \frac{1}{2\psi}\frac{(V - hV_h)^2}{V_h} + \frac{1}{2}\sigma^2 V_{hh} \end{aligned} \tag{SB.2}$$

$$h \leq \underline{h} \implies V(h) = \theta\Big(V(H) - H + h\Big) + (1 - \theta)V^o(h). \tag{SB.3}$$

Note that (SB.1) implies $V(H) - H = V(\overline{h}) - \overline{h}$, which is being substituted into (SB.3). Next, V^o on $[0, H]$ satisfies:

$$h \geq \overline{h}^o \implies 0 = V^o_{hh}(h) \quad (\because V^o_h = 1 \text{ on } [\overline{h}^o, \infty)) \tag{SB.4}$$

$$\begin{aligned} h \in [0, \overline{h}^o] \implies & \rho V^o(h) - rhV^o_h(h) \\ &= \left(A + \frac{1}{2\psi} + \left(\delta + \frac{1}{\psi}\right) h\right) V^o_h - \left(\delta + \frac{1}{\psi}\right) V^o + \frac{1}{2\psi}\frac{(V^o - hV^o_h)^2}{V^o_h} + \frac{1}{2}\sigma^2 V^o_{hh} \\ &\quad + \gamma\big(V(h) - V^o(h)\big). \end{aligned} \tag{SB.5}$$

For ease of notation, define

$$\alpha \equiv \rho + \delta + \frac{1}{\psi}, \quad \beta(h) \equiv A + \frac{1}{2\psi} + \left(r + \delta + \frac{1}{\psi}\right) h, \quad \xi(v, v_h, h) \equiv \frac{1}{2\psi}\frac{(v - hv_h)^2}{v_h}.$$

The five piecewise equalities above – (SB.1) through (SB.5) – switch to strict inequalities when evaluated outside the respective intervals, with left-hand sides being higher. Therefore, these can be summarized as follows: for $h \in [0, H]$,

$$\alpha V(h) - \beta(h)V_h(h) - \frac{1}{2}\sigma^2 V_{hh}(h) = \mathrm{NL}(V, V^o, h)$$
$$\equiv \max\Big\{\alpha V(h) - \beta(h)V_h(h), \xi\big(V(h), V_h(h), h\big),$$
$$\alpha\Big(\theta\big(V(H) - H + h\big) + (1-\theta)V^o(h)\Big) - \beta(h)V_h(h) - \frac{1}{2}\sigma^2 V_{hh}(h)\Big\}, \quad \text{(SB.6)}$$
$$(\alpha+\gamma)V^o(h) - \beta(h)V^o_h(h) - \frac{1}{2}\sigma^2 V^o_{hh}(h) = \mathrm{NL}^o(V, V^o, h)$$
$$\equiv \max\Big\{(\alpha+\gamma)V^o(h) - \beta(h)V^o_h(h), \xi\big(V^o(h), V^o_h(h), h\big) + \gamma V(h)\Big\}. \quad \text{(SB.7)}$$

Both NL and NL^o capture the nonlinear components of the pair of differential equations. In the expression for NL, the first element gives the maximum on $[\overline{h}, H]$, the second on $[\underline{h}, \overline{h}]$ and the last on $[0, \underline{h}]$, and similarly in NL^o given $\underline{h}^o = 0$.[48] Last, the boundary conditions are:

$$V(0) = \theta\Big(V(H) - H\Big),\ V_h(H) = 1 \quad \text{(SB.8)}$$
$$V^o(0) = 0,\ V^o_h(H) = 1. \quad \text{(SB.9)}$$

Discretization and linearization. Let us discretize the fund space $[0, H]$ into N_h evenly spaced grids, and let $\Delta H \equiv \frac{H}{N_h - 1}$ the grid size. For now, let $i \in \{1, 2, \ldots, N_h\}$ index $[0, H]$ increasingly, and denote $h \in [0, H]^{N_h}$ as the column vector discretizing $[0, H]$, such that $h(0) = 0$, $h(N_h) = H$. Posit $V_0, V^o_0 \in \mathbb{R}^{N_h}$ as column vectors representing conjectured approximate value functions under inclusion and exclusion, respectively. Let $W_0 \in \mathbb{R}^{2N_h}$ with

$$W_0 \equiv \begin{pmatrix} V_0 \\ V^o_0 \end{pmatrix},$$

represent the stacked value functions. $i \in \{N_h + 1, \ldots, 2N_h\}$ represents funds under exclusion.

The core of the algorithm is to summarize the left-hand sides of the combined HJB equations (SB.6), (SB.7) as well as the boundary conditions (SB.8), (SB.9) into a single

[48] The first elements of NL and NL^o make use of the fact that $V_{hh}, V^o_{hh} \le 0$ with the equality if and only if $h \ge \overline{h},\ h \ge \overline{h}^o$, respectively. The second elements are the standard HJB equations. The third element in NL derives from $V \ge \theta\big(V(H) - H + h\big) + (1-\theta)V^o$, with the equality if and only if $h \le \underline{h}$.

$2N_h \times 2N_h$ sparse matrix $M(W)$, which depends on the true stacked value function W, such that $M(W) \cdot W = \mathbf{NL}(W)$, where $\mathbf{NL}$ is essentially a stack of NL and NL^o but with some additional adjustments at the respective end rows, to be specified soon. M should depend on W exclusively due to the upwinding method described shortly. I will start with some initial W_0 and use a linear solver to obtain W_1 such that

$$\frac{W_1 - W_0}{\Delta t} + M(W_0) \cdot W_1 = \mathbf{NL}(W_0).$$

Above, the first term on the left-hand side is the pseudo-time derivative to enable efficient convergence, with $\Delta t \in \mathbb{R}_{++}$. Until W_1 is sufficiently close to W_0, update W_0 and repeat.

In principle, $M(\cdot)$ is a mapping from $\mathbb{R}^{2N_h}$ to $\mathbb{R}^{2N_h \times 2N_h}$, but the sparsity substantially reduces its rank. This mapping should capture everything linear in both the combined HJB equations, including the linear differential terms, and the boundary conditions.

Upwinding the derivatives. I will approximate the combined HJB equations (SB.6), (SB.7) for the interior rows of h only, i.e., $i = 2, 3, \ldots, N_h - 1$. The endpoints $i \in \{1, N_h\}$ will be reserved for the boundary conditions (SB.8), (SB.9). For each of the interior rows $i \in \{2, 3, \ldots, N_h - 1\}$, I approximate the first and second derivatives of a given approximated function V as follows: define

$$\Delta_h^f \equiv \left(0, -\frac{1}{\Delta H}, \frac{1}{\Delta H}\right), \quad \Delta_h^b \equiv \left(-\frac{1}{\Delta H}, \frac{1}{\Delta H}, 0\right),$$
$$\Delta_h^c \equiv \left(-\frac{1}{2\Delta H}, 0, \frac{1}{2\Delta H}\right), \ \Delta_h^2 \equiv \left(\frac{1}{\Delta H^2}, -\frac{2}{\Delta H^2}, \frac{1}{\Delta H^2}\right).$$

Then, letting $\widehat{V}(i) \equiv \Big(V(i-1), V(i), V(i+1)\Big)'$, $\Delta_h^f \cdot \widehat{V}(i)$ denotes the forward first difference, $\Delta_h^b \cdot \widehat{V}(i)$ the backward first difference, $\Delta_h^c \cdot \widehat{V}(i)$ the centered first difference, and $\Delta_h^2 \cdot \widehat{V}(i)$ the second difference, all in h.

I follow the standard numerical method of 'upwinding' where the forward/backward difference is used in approximating the first derivative with a positive/negative drift. Determining the sign of the drift in cash flow, however, is somewhat tricky since it depends on which region – $[0, \underline{h}], (\underline{h}, \overline{h}]$ or $(\overline{h}, H]$ under inclusion and $[0, \overline{h}^o]$ or $(\overline{h}^o, H]$ under exclusion – the current internal funds level belongs to. Determining the region is equivalent to determining which element is the maximum in NL and NL^o in Equations (SB.6) and (SB.7), respectievly, whose elements all involve first derivatives.

Therefore, I employ the centered first difference to determine the regions and then use them to implement the upwinding. Given V_0 and V_0^o, define ten row indicator vectors – $f_m, b_m \in \{0,1\}^{N_h}$ for $m = 1,2,3$ and $f_m^o, b_m^o \in \{0,1\}^{N_h}$ for $m = 1,2$ –

such that $i \in \{1, N_h\} \implies \forall m,\ f_m(i) = b_m(i) = f_m^o(i) = b_m^o(i) = 0$ and for $i \in \{2, 3, \ldots, N_h - 1\}$, $f_1(i) = f_1^o(i) = 1$,

$$f_2(i) \equiv \mathbb{1}\Bigg(A - \mathrm{E}\Big(V_0(i), \Delta_h^c \cdot \widehat{V}_0(i), h(i)\Big) + \Big(r + \delta - \mathrm{I}\Big(V_0(i), \Delta_h^c \cdot \widehat{V}_0(i), h(i)\Big)\Big)h(i) > 0\Bigg),$$

$$f_2^o(i) \equiv \mathbb{1}\Bigg(A - \mathrm{E}\Big(V_0^o(i), \Delta_h^c \cdot \widehat{V}_0^o(i), h(i)\Big) + \Big(r + \delta - \mathrm{I}\Big(V_0^o(i), \Delta_h^c \cdot \widehat{V}_0^o(i), h(i)\Big)\Big)h(i) > 0\Bigg),$$

$$\begin{aligned} f_3(i) \equiv \mathbb{1}\Bigg(&A - \mathrm{E}\Big(V_0^o(i) - \theta\big(V_0(N_h) - H + h(i)\big), \Delta_h^c \cdot \widehat{V}_0(i) - \theta, h(i)\Big) \\ &+ \Big(r + \delta - \mathrm{I}\Big(V_0^o(i) - \theta\big(V_0(N_h) - H + h(i)\big), \Delta_h^c \cdot \widehat{V}_0(i) - \theta, h(i)\Big)h(i)\Big) > 0\Bigg), \end{aligned}$$

and $b_m(i) = 1 - f_m(i)$, $b_m^o(i) = 1 - f_m^o(i)$ for all m. Above, I and E denote optimized values of gross investment i[49] and total investment expense $i + \Psi(i)$, respectively, given as

$$\mathrm{I}(v, v_h, h) = \frac{1}{\psi}\left(\frac{v}{v_h} - h - 1\right), \quad \mathrm{E}(v, v_h, h) = \frac{1}{2\psi}\left(\left(\frac{v - hv_h}{v_h}\right)^2 - 1\right).$$

As can be inferred from $f_1 = f_1^o = 1$, I use forward differences on $(\overline{h}, H]$ and $(\overline{h}^o, H]$. Let

$$\Delta_{hm}(i) \equiv f_m(i)\Delta_h^f + b_m(i)\Delta_h^b, \quad \Delta_{hmo}(i) \equiv f_m^o(i)\Delta_h^f + b_m^o(i)\Delta_h^b.$$

Let $m^*(i) \in \{1, 2, 3\}$ and $m^{o*}(i) \in \{1, 2\}$ index the maximum, respectively in the sets

$$\begin{aligned} \Bigg\{&\alpha V_0(i) - \beta(h(i))\Delta_{h1}(i) \cdot \widehat{V}_0(i), \xi\big(V_0(i), \Delta_{h2}(i) \cdot \widehat{V}_0(i), h(i)\big), \alpha\Big(\theta(V_0(N_h) - H + h(i)) \\ &+ (1 - \theta)V_0^o(i)\Big) - \beta(h(i))\Delta_{h3}(i) \cdot \widehat{V}_0(i) - \frac{1}{2}\sigma^2\Delta_h^2 \cdot \widehat{V}_0(i)\Bigg\}, \end{aligned} \tag{SB.10}$$

$$\Big\{(\alpha + \gamma)V_0^o(i) - \beta(h(i))\Delta_{h1o}(i) \cdot \widehat{V}_0^o(i), \xi\big(V_0^o(i), \Delta_{h2o}(i) \cdot \widehat{V}_0^o(i), h(i)\big) + \gamma V_0(i)\Big\}. \tag{SB.11}$$

Last, let $f(i) \equiv f_{m^*(i)}(i)$, $f^o(i) \equiv f_{m^{o*}(i)}^o(i)$, $b(i) \equiv 1 - f(i)$, $b^o(i) \equiv 1 - f^o(i)$, and

$$\Delta_h(i) \equiv f(i)\Delta_h^f + b(i)\Delta_h^b, \quad \Delta_h^o(i) \equiv f^o(i)\Delta_h^f + b^o(i)\Delta_h^b.$$

These three-dimensional row vectors $\Delta_h(i),\ \Delta_h^o(i) \in \mathbb{R}^3$ implement the upwinding for first differences in the construction of $M(\cdot)$, to which I now transition.

Constructing the matrix. As a reminder, $M(W_0)$ is a $2N_h \times 2N_h$ sparse matrix,

[49] I and δ enter the drift in cash flow because h is internal funds per capital.

because W_0 is a stacked vector of V_0, V_0^o. For each of $i = 2, 3, \ldots, N_h - 1$ rows, let

$$M(i, i-1 : i+1 \mid W_0) \equiv (0, \alpha, 0) - \beta(h(i))\Delta_h(i) - \frac{1}{2}\sigma^2\Delta_h^2,$$

where $M(i, i-1 : i+1)$ denotes the i^{th} row from the $(i-1)^{\text{th}}$ column to the $(i+1)^{\text{th}}$ column, to account for the left-hand side of Equation (SB.6). Similarly, for each of $(N_h + i)^{\text{th}}$ rows with $i = 2, \ldots, N_h - 1$, implement the left-hand side of (SB.7) by

$$M(N_h + i, N_h + i - 1 : N_h + i + 1 \mid W_0) \equiv (0, \alpha + \gamma, 0) - \beta(h(i))\Delta_h^o(i) - \frac{1}{2}\sigma^2\Delta_h^2.$$

Next, construct the 'nonlinear' column vector $\mathbf{NL}(W_0) \in \mathbb{R}^{2N_h}$ as follows: for each of $i = 2, 3, \ldots, N_h - 1$ rows, $\mathbf{NL}(i \mid W_0)$ is the maximum in the set (SB.10) and $\mathbf{NL}(N_h + i \mid W_0)$ in the set (SB.11). The rows $i = 1, N_h, N_h + 1, 2N_h$ will be separately specified right below.

Last, the boundary conditions (SB.8) and (SB.9) are implemented as:

$$\begin{aligned}
&M(1, 1 \mid W_0) \equiv \alpha,\ M(1, N_h \mid W_0) \equiv -\theta\left(\frac{1}{\Delta t} + \alpha\right),\ \mathbf{NL}(1 \mid W_0) \equiv -\theta\left(\frac{1}{\Delta t} + \alpha\right)H,\\
&M(N_h, N_h - 1 \mid W_0) \equiv -\left(\frac{1}{\Delta t} + \alpha\right),\ M(N_h, N_h \mid W_0) \equiv \alpha,\\
&\qquad\qquad\qquad\qquad\qquad\qquad \mathbf{NL}(N_h \mid W_0) \equiv \left(\frac{1}{\Delta t} + \alpha\right)\Delta H,\\
&M(N_h + 1, N_h + 1 \mid W_0) \equiv 0,\ \mathbf{NL}(N_h + 1 \mid W_0) \equiv 0,\\
&M(2N_h, 2N_h - 1 \mid W_0) \equiv -\left(\frac{1}{\Delta t} + \alpha + \gamma\right),\ M(2N_h, 2N_h \mid W_0) \equiv \alpha + \gamma,\\
&\qquad\qquad\qquad\qquad\qquad\qquad \mathbf{NL}(2N_h \mid W_0) \equiv \left(\frac{1}{\Delta t} + \alpha + \gamma\right)\Delta H.
\end{aligned}$$

Any unspecified element of $M(W_0)$ is set to zero, making it highly sparse.

Iteration to solution. Posit H. Start with some initial guess V_0, V_0^o, and stack them into W_0. Obtain W_1 that solves

$$\left(\frac{1}{\Delta t}\mathbb{I}_{2N_h} + M(W_0)\right) \cdot W_1 = \frac{1}{\Delta t}W_0 + \mathbf{NL}(W_0),$$

where $\mathbb{I}_{2N_h}$ is the $2N_h \times 2N_h$ identity matrix, also highly sparse. If W_1 is close enough to W_0, stop; $V \equiv W_1(1 : N_h)$ and $V^o \equiv W_1(N_h + 1 : 2N_h)$. Otherwise, update $W_0 \equiv aW_1 + (1 - a)W_0$ for some weight $a \in (0, 1]$ and repeat.

If H is set too high, the algorithm might converge too slowly and the solution becomes unnecessarily coarse on $[0, \overline{h}]$. On the other hand, if H is too low, then possibly $H < \overline{h}^o$, in which case the algorithm fails. Therefore, I run the algorithm twice, an initializer and a verifier. During initialization, I use an adequate fraction (say 0.2) of the first-best value V^* as the initial H and use a high error tolerance. If convergence fails, I raise the initial H. If it succeeds, I choose a new H to be only slightly higher than $\overline{h}^o$ and run the verifier with a low error tolerance.

SB.1.2 Fluctuations in parameters

Stacked value. Let N_s denote the number of Markov states. For a continuous Markov chain, N_s is the number of grids in state-space discretization. I use $s \in \{1, 2, \ldots, N_s\}$ to index the Markov states. Given $\big(V_0(i_h, s), V_0^o(i_h, s)\big)$ for $(i_h, s) \in \{1, \ldots, N_h\} \times \{1, \ldots, N_s\}$, define

$$W_0 \equiv \Big(V_0(:,1)' \quad \ldots \quad V_0(:,N_s)' \quad V_0^o(:,1)' \quad \ldots \quad V_0^o(:,N_s)'\Big)' \in \mathbb{R}^{2N_hN_s}$$

as their stacked column vector. $i \in \{1, \ldots, 2N_hN_s\}$ now jointly indexes (i_h, s) and inclusion/exclusion. The mapping $M : \mathbb{R}^{2N_hN_s} \to \mathbb{R}^{2N_hN_s \times 2N_hN_s}$ will be defined in a fashion overall identical to Section SB.1.1 for each of the $N_h \times N_h$ blocks corresponding to $s \in \{1, 2, \ldots, N_s\}$ along the main diagonal $M\big((s-1)N_h + 1 : sN_h, (s-1)N_h + 1 : sN_h \mid W_0\big)$. There will be, however, an additional sparse matrix for the Markov chain and some changes to NL, NL^o.

Discrete Markov chain. Consider a Markov chain in Poisson arrival rates of transition given as

$$\begin{pmatrix} -\lambda^1 & \lambda^1_2 & \ldots & \lambda^1_{N_s} \\ \lambda^2_1 & -\lambda^2 & \ldots & \lambda^2_{N_s} \\ \ldots & \ldots & \ldots & \ldots \\ \lambda^{N_s}_1 & \lambda^{N_s}_2 & \ldots & -\lambda^{N_s} \end{pmatrix},$$

where $\lambda^s_{s'} \geq 0$ is the Poisson rate of transition at s to $s' \neq s$, and $\lambda^s \equiv \sum_{s' \neq s} \lambda^s_{s'}$.

Modify the combined HJB equations (SB.6) and (SB.7): for $s \in \{1, \ldots, N_s\}$,

$$\begin{aligned}
&(\alpha(s) + \lambda^s)V(h,s) - \beta(h,s)V_h(h,s) - \frac{1}{2}\sigma^2 V_{hh}(h,s) - \sum_{s' \neq s} \lambda^s_{s'} V(h,s') = \mathrm{NL}(V, V^o, h, s) \\
&\equiv \max\Big\{(\alpha(s) + \lambda^s)V(h,s) - \beta(h,s)V_h(h,s) - \sum_{s' \neq s} \lambda^s_{s'} V(h,s'), \xi\big(V(h,s), V_h(h,s), h, s\big), \\
&\quad (\alpha(s) + \lambda^s)\Big(\theta(s)\big(V(H,s) - H + h\big) + (1 - \theta(s))V^o(h,s)\Big) - \beta(h,s)V_h(h,s) \\
&\qquad\qquad - \frac{1}{2}\sigma^2 V_{hh}(h,s) - \sum_{s' \neq s} \lambda^s_{s'} V(h,s')\Big\},
\end{aligned} \tag{SB.12}$$

$$
\begin{aligned}
&(\alpha(s)+\lambda^s+\gamma(s))V^o(h,s)-\beta(h,s)V_h^o(h,s)-\frac{1}{2}\sigma^2V_{hh}^o(h)-\sum_{s'\neq s}\lambda_{s'}^sV^o(h,s')\\
&=\mathrm{NL}^o(V,V^o,h,s)\equiv\max\Big\{(\alpha(s)+\lambda^s+\gamma(s))V^o(h,s)-\beta(h,s)V_h^o(h,s)\\
&\qquad\qquad-\sum_{s'\neq s}\lambda_{s'}^sV^o(h,s'),\xi\big(V^o(h,s),V_h^o(h,s),h,s\big)+\gamma(s)V(h)\Big\}. \qquad \text{(SB.13)}
\end{aligned}
$$

The dependence of $\alpha,\beta,\theta,\gamma,\xi$ on s captures the fluctuations in parameters.

Define a Markov chain matrix for the entire (i_h,s) space by

$$
\Lambda\equiv\begin{pmatrix}
-\lambda^1\widetilde{\mathbb{I}}_{N_h} & \lambda_2^1\widetilde{\mathbb{I}}_{N_h} & \dots & \lambda_{N_s}^1\widetilde{\mathbb{I}}_{N_h}\\
\lambda_1^2\widetilde{\mathbb{I}}_{N_h} & -\lambda^2\widetilde{\mathbb{I}}_{N_h} & \dots & \lambda_{N_s}^2\widetilde{\mathbb{I}}_{N_h}\\
\dots & \dots & \dots & \dots\\
\lambda_1^{N_s}\widetilde{\mathbb{I}}_{N_h} & \lambda_2^{N_s}\widetilde{\mathbb{I}}_{N_h} & \dots & -\lambda^{N_s}\widetilde{\mathbb{I}}_{N_h}
\end{pmatrix}\in\mathbb{R}^{N_hN_s\times N_hN_s},
$$

where $\widetilde{\mathbb{I}}_{N_h}$ is the $N_h\times N_h$ identity matrix but with the first and last main diagonal elements replaced with zero; the first and the last rows in each block are preserved for the boundary conditions. Proceed to extend Λ to both inclusion and exclusion by defining

$$
\boldsymbol{\Lambda}\equiv\begin{pmatrix}\Lambda & \mathbf{0}\\ \mathbf{0} & \Lambda\end{pmatrix}\in\mathbb{R}^{2N_hN_s\times 2N_hN_s}.
$$

The zero off-diagonal blocks indicate that Λ is orthogonal to re-inclusion upon γ.

The construction of each main diagonal block of $M(W_0)$ – i.e., $M\big((s-1)N_h+1:sN_h,(s-1)N_h+1:sN_h\mid W_0\big)$ – is unchanged, both for the interior rows $i\in\{(s-1)N_h+2,\dots,sN_h-1\}$ that implement the combined HJB (SB.12), (SB.13)[50] and for the boundaries $i\in\{(s-1)N_h+1,sN_h\}$ that implement the same boundary conditions. $\mathbf{NL}(W_0)$ is adjusted slightly for the interior, following the modified definition of $\mathrm{NL},\mathrm{NL}^o$ in (SB.12), (SB.13). Once $M(\cdot)$, $\mathbf{NL}(\cdot)$ are constructed, iteratively solve

$$
\left(\frac{1}{\Delta t}\mathbb{I}_{2N_hN_s}+M(W_0)-\boldsymbol{\Lambda}\right)\cdot W_1=\frac{1}{\Delta t}W_0+\mathbf{NL}(W_0).
$$

Continuous Markov chain. Let s_t follow $ds_t=\mu_s(s_t)\ dt+\sigma_s(s_t)\ dZ_t$. Discretize the state space into N_s grids with size ΔS. Let $s\in\{1,2,\dots,N_s\}$ index the state space increasingly. The above law of motion is 'discretized' as a Markov chain:

$$
\begin{aligned}
\mu_s(s)\geq 0 &\implies \lambda_{s-1}^s=\frac{\sigma_s(s)^2}{2\Delta S^2},\quad \lambda_{s+1}^s=\frac{\mu_s(s)}{\Delta S}+\frac{\sigma_s(s)^2}{2\Delta S^2},\\
\mu_s(s)<0 &\implies \lambda_{s-1}^s=-\frac{\mu_s(s)}{\Delta S}+\frac{\sigma_s(s)^2}{2\Delta S^2},\quad \lambda_{s+1}^s=\frac{\sigma_s(s)^2}{2\Delta S^2}.
\end{aligned}
$$

[50] The maximizing indices m^*,m^{o*} are based on the modified NL and NL^o in (SB.12), (SB.13).

As for the endpoints $s \in \{1, N_s\}$, mean reversion will generally allow upwinding of the first-order terms $\pm \frac{\mu_s(s)}{\Delta S}$. The second-order terms, however, cannot be correctly computed, as they go outside the grid. I therefore use W_0 to compute second as well as third finite differences at $s \in \{2, N_s - 1\}$ and use them to linearly approximate the endpoint second derivatives.[51]

Once the discretized Markov chain is set up, follow the same procedure as above.

SB.1.3 Investment irreversibility

The algorithm remains mostly the same and is modified only as follows. First, whenever $\frac{1}{\psi}$ appears in the above algorithm, multiply it by $1 + (\phi - 1) \cdot \mathbb{1}(\frac{V_0}{\Delta_h \cdot V_0} - h < 1)$; use V_0^o, Δ_h^o instead of V_0, Δ_h when appropriate. The indicator function tracks whether the firm divests. Second, move all terms involving this modified expression to the inside of **NL** instead of M, as investment versus divestment can make the system highly nonlinear.

SB.2 Closed-form solution for exogenous cash flow

Sections 5.2 and 5.3 inform a general procedure for analytically solving the equilibrium for exogenous cash flow models in Section 4 when the HJB equation (7) admits an analytic solution for value functions, which requires $r = 0$. By Lemma 2, the equilibrium is fully characterized by a pair $(\underline{h}, \overline{h})$, $0 \leq \underline{h} < \overline{h}$. The procedure is as below:

1. In all cases, $V'(\overline{h}) = 1$, and if $\sigma > 0$, then $V''(\overline{h}) = 0$.
2. Solve the model with $\gamma = 0$. By Corollary 1, $\underline{h} = 0$, and $\overline{h}$ is implicitly defined by $V(0) = \theta\big(V(\overline{h}) - \overline{h}\big)$.
3. For $\gamma > 0$, first determine whether $\underline{h} = 0$ or $\underline{h} > 0$. This can be done as follows: (i) posit the value of $\overline{h}$ obtained in Step 2, and (ii) evaluate Inequality (13). If $\underline{h} = 0$, assign to $\overline{h}$ the value obtained in Step 2.
4. If $\underline{h} > 0$, then use the following conditions to implicitly determine $(\underline{h}, \overline{h})$:
   (a) Stationary Recursion: $V(\overline{h}) - V(\underline{h}) = \left(1 + \frac{\varphi}{\gamma}\right) \Delta h$,
   (b) Threshold Indifference: for $h \in [0, \underline{h}]$,

   $$\rho V_o(h) - rhV_o'(h) = \gamma\theta\big(V(\overline{h}) - \overline{h} + \underline{h} - V_o(\overline{h})\big) + \Lambda(V_o)(h) + \mathcal{H}(V_o)(h),$$

   with boundary conditions $V_o(0) = 0$, $G(\underline{h}) = 0$, where G is defined in Appendix A.2, and
   (c) Smooth Pasting: $V'(\underline{h}) = \theta + (1 - \theta)V_o'(\underline{h})$.[52]

[51] At the endpoints, the 'discrete jump' interpretation might not hold: the algorithm still works.
[52] If $\sigma = 0$, then one of the three conditions are redundant.

Supplemental Appendix C Other Derivations

Here, I go through analytic solutions to the startup and operating firm examples. The main purpose is to formally prove comparative statics in business parameters as illustrated in Supplemental Appendix SA.5. An analytic solution to the HJB equation requires that the internal yield be zero $r = 0$, which is presently assumed. Note, as an aside, that regardless of the existence of an analytic solution for the value function, all the formal results in the main article are valid.

Proofs are in Supplemental Appendices SC.3 and SC.4.

SC.1 Solving startup equilibrium

The business incurs a fixed flow expense $\kappa\, dt$, $\kappa > 0$, until success arrives at Poisson rate $\lambda > 0$ upon which the business terminates with one-time payoff $\Pi > 0$. As discussed, assume $r = 0$. Let us reiterate the first part of Assumption 1 as a reference.

Assumption SC.1.1 (Positive net present value)**.** *$\lambda\Pi > \kappa$.*

Then, V on $(\underline{h}, \overline{h})$ satisfies the following ODE:

$$\begin{aligned} \rho V(h) &= \lambda\big(\Pi + h - V(h)\big) - \kappa V'(h) \\ \implies V(h) &= -ce^{-\frac{\rho+\lambda}{\kappa}h} + \frac{\lambda}{\rho+\lambda}\left(\Pi + h - \frac{\kappa}{\rho+\lambda}\right), \end{aligned}$$

for some $c \in \mathbb{R}$. In addition, since $V'(\overline{h}) = 1$, we have

$$V(h) = \frac{1}{\rho+\lambda}\left(\lambda\Big(\Pi + h\Big) - \frac{\kappa}{\rho+\lambda}\left(\lambda + \rho e^{\frac{\rho+\lambda}{\kappa}(\overline{h}-h)}\right)\right), \tag{SC.1}$$

$$V(\overline{h}) = \frac{1}{\rho+\lambda}\left(\lambda\Big(\Pi + \overline{h}\Big) - \kappa\right). \tag{SC.2}$$

SC.1.1 Baseline: no re-inclusion

First consider $\gamma = 0$. Since $\underline{h} = 0$ by Corollary 1, the equilibrium – just $\overline{h}$ in this case – is implicitly defined by the stationary recursion as follows:

$$\begin{aligned} &V(0) = x(0)V(\overline{h}) = \theta(V(\overline{h}) - \overline{h}) \\ \iff &\theta\rho\big((\rho+\lambda)\overline{h} + \kappa\big) + (1-\theta)\lambda\big((\rho+\lambda)\Pi - \kappa\big) = \rho\kappa \exp\left(\frac{\rho+\lambda}{\kappa}\overline{h}\right). \end{aligned} \tag{SC.3}$$

In the first line, $V_o(0) = 0$ is used. Note that the solution to Equation SC.3 is positive if and only if Assumption SC.1.1 holds.

With Equation (SC.3), comparative statics is straightforward.

Proposition SC.1.1 (Comparative statics – Startups without re-inclusion)**.**

$$\frac{\partial \overline{h}}{\partial \theta} = -\frac{1}{\rho}\frac{\lambda\Pi - \kappa - \rho\overline{h}}{\exp\left(\frac{\rho+\lambda}{\kappa}\overline{h}\right) - \theta} < 0,\ \lim_{\theta\to 0}\overline{h} < \frac{\lambda\Pi - \kappa}{\rho},\ \lim_{\theta\to 1}\overline{h} = 0, \quad \textit{and} \ \lim_{\theta\to 1}\frac{\partial \overline{h}}{\partial \theta} = -\infty, \tag{SC.4}$$

$$\frac{\partial \overline{h}}{\partial \Pi} = \frac{\lambda}{\rho}\frac{1-\theta}{\exp\left(\frac{\rho+\lambda}{\kappa}\overline{h}\right) - \theta} > 0, \tag{SC.5}$$

$$\lim_{\lambda\to\kappa/\Pi}\overline{h} = \lim_{\lambda\to\infty}\overline{h} = 0. \tag{SC.6}$$

SC.1.2 General comparative statics

Consider the general case of $\gamma \geq 0$. Inequality (13) translates into: $\underline{h} > 0$ if and only if

$$\rho\overline{h} < \frac{(1-\theta)\gamma}{\rho+\lambda+(1-\theta)\gamma}(\lambda\Pi - \kappa). \tag{SC.7}$$

By Section SB.2, the following result is obtained.

Proposition SC.1.2 (Startup financing)**.** *Denote*

$$\eta \equiv \frac{(1-\theta)\gamma}{\rho+\lambda+(1-\theta)\gamma}, \quad \xi \equiv \frac{\lambda\Pi - \kappa}{\rho}.$$

The equilibrium is characterized by $\underline{h} = 0$ and $\overline{h}$ implicitly defined by Equation (SC.3) if

$$\theta\rho\big((\rho+\lambda)\eta\xi + \kappa\big) + (1-\theta)\lambda\big((\rho+\lambda)\Pi - \kappa\big) \geq \rho\kappa\exp\left(\frac{\rho+\lambda}{\kappa}\eta\xi\right). \tag{SC.8}$$

If the inequality is strictly reversed, then $\underline{h} = \overline{h} - \Delta h > 0$ and $\overline{h} > \Delta h$ is implicitly defined by

$$\begin{aligned}\frac{1-\theta}{\rho+\lambda}&\left((\rho+\lambda+\theta\gamma)\lambda\Pi - \theta\rho\gamma\left(\overline{h} + \frac{\kappa}{\rho+\lambda+\theta\gamma}\right) - (\lambda+\theta\gamma)\kappa\right) \\ &= \rho\left(\frac{\rho+\lambda+\gamma}{\gamma}\Delta h + (1-\theta)\frac{\kappa}{\rho+\lambda+\theta\gamma}\right)\exp\left(\frac{\rho+\lambda+\theta\gamma}{\kappa}(\overline{h} - \Delta h)\right),\end{aligned} \tag{SC.9}$$

where $\Delta h = \overline{h} - \underline{h} > 0$ is given by

$$1 + \frac{\rho+\lambda}{\kappa}\left(1 + \frac{\rho+\lambda}{\gamma}\right)\Delta h = \exp\left(\frac{\rho+\lambda}{\kappa}\Delta h\right). \tag{SC.10}$$

Proposition SC.1.3 (Comparative statics – startups). *$\overline{h}$ strictly increases in Π. When $\underline{h} > 0$, $\underline{h}$ strictly increases in Π and Δh is constant in Π and strictly decreasing in λ. Last, $\overline{h}$, $\underline{h}$ and Δh converge to zero as either (i) Π goes to κ/λ or (ii) λ goes to either κ/Π or ∞.*

Proof. Immediate from Proposition SC.1.2. □

Proposition SC.1.4 (Breakeven re-inclusion – Startups). *$\underline{\gamma}$ strictly decreases in Π, and converges to zero as Π goes to ∞. It goes to ∞ as Π goes down to κ/λ, the lower bound in Assumption SC.1.1.*

SC.2 Solving operating firm equilibrium

The second example involves a fixed average profit but with volatility. That is, the underlying cash flow of the business is captured by

$$\pi \ dt + \sigma \ dB_t,$$

with $\pi, \sigma^2 > 0$, where B_t is a standard Brownian motion. Again, assume $r = 0$ for simplicity.

Note that V on $(\underline{h}, \overline{h})$ satisfies the following ODE:

$$\rho V(h) = \pi V'(h) + \frac{1}{2}\sigma^2 V''(h).$$

In addition, since insiders will receive dividends at $h_t = \overline{h}$ such that $\overline{h}$ becomes a reflection boundary, both smooth pasting and super contact conditions must hold at $\overline{h}$, i.e., $V'(\overline{h}) = 1,\ V''(\overline{h}) = 0$. Therefore,

$$V(h) = \frac{1}{\Phi + \phi}\left(\frac{\Phi}{\phi}e^{-\phi(\overline{h}-h)} - \frac{\phi}{\Phi}e^{\Phi(\overline{h}-h)}\right), \tag{SC.11}$$

$$V(\overline{h}) = \frac{\pi}{\rho}, \tag{SC.12}$$

where $\Phi \equiv (\sqrt{\pi^2 + 2\rho\sigma^2} + \pi)/\sigma^2$ and $\phi \equiv (\sqrt{\pi^2 + 2\rho\sigma^2} - \pi)/\sigma^2$.

SC.2.1 Baseline: no re-inclusion

First suppose that $\gamma = 0$. Then, again by Corollary 1, $\underline{h} = 0$ and $\overline{h}$ is implicitly defined by $V(0) = \theta\left(V(\overline{h}) - \overline{h}\right)$, which is simply

$$\theta\left(\frac{\pi}{\rho} - \overline{h}\right) = \frac{1}{\Phi + \phi}\left(\frac{\Phi}{\phi}e^{-\phi\overline{h}} - \frac{\phi}{\Phi}e^{\Phi\overline{h}}\right). \tag{SC.13}$$

From the above, we can derive comparative statics.

Proposition SC.2.1 (Comparative statics – Operating firms without re-inclusion)**.**

$$
\frac{\partial \overline{h}}{\partial \theta} = -\frac{1}{\rho}\frac{\pi - \rho\overline{h}}{\frac{\Phi}{\Phi+\phi}e^{-\phi\overline{h}} + \frac{\phi}{\Phi+\phi}e^{\Phi\overline{h}} - \theta} < 0,
$$

$$
\lim_{\theta\to 0}\overline{h} < \frac{\pi}{\rho},\ \lim_{\theta\to 1}\overline{h} = 0,\ \textit{and}\ \lim_{\theta\to 1}\frac{\partial \overline{h}}{\partial \theta} = -\infty, \tag{SC.14}
$$

$$
\frac{\partial \overline{h}}{\partial \sigma^2} = \frac{\rho}{\pi^2 + 2\rho\sigma^2}\frac{\overline{h}\sqrt{\pi^2+2\rho\sigma^2}\left(e^{\Phi\overline{h}} + e^{-\phi\overline{h}}\right) - \sigma^2\left(e^{\Phi\overline{h}} - e^{-\phi\overline{h}}\right)}{\sqrt{\pi^2+2\rho\sigma^2}\left(e^{\Phi\overline{h}} + e^{-\phi\overline{h}} - 2\theta\right) - \pi\left(e^{\Phi\overline{h}} - e^{-\phi\overline{h}}\right)} > 0,
$$

$$
\lim_{\sigma^2\to 0}\overline{h} = 0,\ \lim_{\sigma^2\to\infty}\overline{h} = \frac{\pi}{\rho},\ \lim_{\sigma^2\to 0}\frac{\partial \overline{h}}{\partial \sigma^2} = \infty,\ \textit{and}\ \lim_{\sigma^2\to\infty}\frac{\partial \overline{h}}{\partial \sigma^2} = 0, \tag{SC.15}
$$

$$
\lim_{\pi\to 0}\overline{h} = \lim_{\pi\to\infty}\overline{h} = 0. \tag{SC.16}
$$

SC.2.2 General comparative statics

Now consider $\gamma \geq 0$. Let us first evaluate Inequality (13): $\underline{h} > 0$ if and only if

$$
\rho\overline{h} < \frac{(1-\theta)\gamma}{\rho + (1-\theta)\gamma}\pi. \tag{SC.17}
$$

Proposition SC.2.2 (Operating firm financing)**.** *Denote*

$$
\eta \equiv \frac{(1-\theta)\gamma}{\rho+(1-\theta)\gamma}, \quad \xi \equiv \frac{\pi}{\rho}.
$$

The equilibrium is characterized by $\underline{h} = 0$ *and* $\overline{h}$ *implicitly defined by Equation (SC.13) if*

$$
\theta(1-\eta)\xi \leq \frac{1}{\Phi+\phi}\left(\frac{\Phi}{\phi}\exp\left(-\phi\eta\xi\right) - \frac{\phi}{\Phi}\exp\left(\Phi\eta\xi\right)\right). \tag{SC.18}
$$

If the inequality is strictly reversed, then $\underline{h} = \overline{h} - \Delta h > 0$ *and* $\overline{h} > \Delta h$ *is implicitly defined by*

$$
\begin{aligned}
&\left(\frac{\pi}{\rho+\theta\gamma} - \frac{1}{1-\theta}\left(1+\frac{\rho}{\gamma}\right)\Delta h\right)\frac{\phi_o\exp\big(\phi_o(\overline{h}-\Delta h)\big) + \Phi_o\exp\big(-\Phi_o(\overline{h}-\Delta h)\big)}{\exp\big(\phi_o(\overline{h}-\Delta h)\big) - \exp\big(-\Phi_o(\overline{h}-\Delta h)\big)} \\
&\qquad + \theta\left(\Phi_o+\phi_o\right)\frac{\gamma}{\rho}\left(\left(1+\frac{\rho}{\rho+\theta\gamma}\right)\frac{\pi}{\rho} - \overline{h}\right)\frac{\exp\left(-2\pi(\overline{h}-\Delta h)/\sigma^2\right)}{\exp\big(\phi_o(\overline{h}-\Delta h)\big) - \exp\big(-\Phi_o(\overline{h}-\Delta h)\big)} \\
&\qquad = \frac{\rho+\theta\gamma}{(1-\theta)\rho}\left(\phi\left(\frac{\pi}{\rho} - \left(1+\frac{\rho}{\gamma}\right)\Delta h\right) + \frac{\phi}{\Phi}\exp\big(\Phi\Delta h\big)\right) - \frac{\theta}{1-\theta}\left(1+\frac{\gamma}{\rho}\right),
\end{aligned} \tag{SC.19}
$$

where

$$\Phi_o \equiv (\sqrt{\pi^2 + 2(\rho+\theta\gamma)\sigma^2} + \pi)/\sigma^2, \quad \phi_o \equiv (\sqrt{\pi^2 + 2(\rho+\theta\gamma)\sigma^2} - \pi)/\sigma^2$$

and $\Delta h = \overline{h} - \underline{h} > 0$ *is given by*

$$\frac{\pi}{\rho} - \left(1 + \frac{\rho}{\gamma}\right)\Delta h = \frac{1}{\Phi + \phi}\left(\frac{\Phi}{\phi}e^{-\phi\Delta h} - \frac{\phi}{\Phi}e^{\Phi\Delta h}\right). \tag{SC.20}$$

Proposition SC.2.3 (Comparative statics – Operating firms)**.** *Both* $\overline{h}$ *and* Δh *strictly increase in* σ^2*. There exists* $\overline{\sigma}^2 > 0$ *such that* $\sigma^2 \geq \overline{\sigma}^2$ *if and only if* $\underline{h} = 0$*. Above it,* $\overline{h} = \Delta h$ *converge to* $\frac{\pi}{\rho}$ *as* $\sigma^2 \to \infty$*.* $\overline{h}$*,* $\underline{h}$*,* Δh *converge to zero as either* σ^2 *goes to zero or* π *goes to zero.* Δh *converges to zero as* π *goes to* ∞*. Last, there exists* $\underline{\pi} > 0$ *such that* $\pi \leq \underline{\pi}$ *if and only if* $\underline{h} = 0$*.*

Proposition SC.2.4 (Breakeven re-inclusion – Operating firms)**.** $\underline{\gamma}$ *strictly increases in* σ^2*, strictly decreases in* π*, and diverges to* ∞ *as either* σ^2 *goes to* ∞ *or* π *goes to zero. It converges to zero as either* σ^2 *goes to zero or* π *goes to* ∞*.*

SC.3 Proofs for startup derivation

Proposition SC.1.1 (Comparative statics – Startups without re-inclusion).

Proof. Most results above are straightforward. For the first inequality in (SC.4), it is sufficient to show that $\lambda\Pi - \kappa - \rho\overline{h}(0) > 0$, where $\overline{h}(\theta)$ is $\overline{h}$ expressed as a function of $\theta \in [0, 1]$.

To prove the claim, note that

$$\begin{aligned}
& \lambda\Pi - \kappa > \rho\overline{h}(0) \\
\iff & \exp\left(\frac{\rho+\lambda}{\kappa}\frac{\lambda\Pi - \kappa}{\rho}\right) > \exp\left(\frac{\rho+\lambda}{\kappa}\overline{h}(0)\right) \\
& \qquad = \frac{\lambda}{\rho}\left((\rho+\lambda)\frac{\Pi}{\kappa} - 1\right) \quad \because (\mathit{SC}.3) \text{ with } \theta \equiv 0 \\
& \qquad = 1 + \frac{\rho+\lambda}{\kappa}\frac{\lambda\Pi - \kappa}{\rho} \\
\iff & \frac{\rho+\lambda}{\kappa}\frac{\lambda\Pi - \kappa}{\rho} > 0,
\end{aligned}$$

which is equivalent to Assumption SC.1.1. □

Proposition SC.1.2 (Startup financing).

Proof. If (SC.7) holds, then Inequality (SC.7) also holds with $\overline{h}$ defined by (SC.3).

Therefore, $\underline{h} = 0$. Now, consider the case where (SC.7) fails. Let us use Smooth Pasting and Stationary Recursion to determine $(\underline{h}, \overline{h})$.

Smooth Pasting. For ease of notation, denote $\underline{V}_o \equiv V_o(\underline{h})$ and $\underline{V} \equiv \theta(V(\overline{h}) - \overline{h} + \underline{h}) + (1-\theta)\underline{V}_o$. The insiders' non-excluded and excluded value functions for $h \in [\underline{h}, \overline{h}]$ are given by

$$\begin{aligned}
V(h) &= \int_0^{(h-\underline{h})/\kappa} \lambda e^{-(\rho+\lambda)t}\big(\Pi + h - \kappa t\big)\, dt + e^{-\frac{\rho+\lambda}{\kappa}(h-\underline{h})}\underline{V} \\
&= \frac{\lambda}{\rho+\lambda}\left(\Pi + h - \frac{\kappa}{\rho+\lambda}\right) - \left(\frac{\lambda}{\rho+\lambda}\left(\Pi + \underline{h} - \frac{\kappa}{\rho+\lambda}\right) - \underline{V}\right) e^{-\frac{\rho+\lambda}{\kappa}(h-\underline{h})}, \\
V_o(h) &= V(h) - e^{-\frac{\rho+\lambda+\gamma}{\kappa}(h-\underline{h})}\big(\underline{V} - \underline{V}_o\big).
\end{aligned}$$

$V_o(h)$ is derived based on the observation that, given the strategy of waiting on $(\underline{h}, h)$ regardless of market access, the only difference that exclusion creates is that one has $\underline{V}_o$ instead of $\underline{V}$ at $h = \underline{h}$ if neither success nor re-inclusion occurs while the internal funds h run down to $\underline{h}$.

Note that

$$\begin{aligned}
V(\overline{h}) &= \frac{1}{\rho+\lambda}\big(\lambda(\Pi + \overline{h}) - \kappa\big) \\
\implies \frac{\rho\kappa}{(\rho+\lambda)^2} &= \left(\frac{\lambda}{\rho+\lambda}\left(\Pi + \underline{h} - \frac{\kappa}{\rho+\lambda}\right) - \underline{V}\right)\exp\left(-\frac{\rho+\lambda}{\kappa}(\overline{h} - \underline{h})\right) \\
\implies V(h) &= \frac{\lambda}{\rho+\lambda}\left(\Pi + h - \frac{\kappa}{\rho+\lambda}\right) - \frac{\rho\kappa}{(\rho+\lambda)^2}\exp\left(\frac{\rho+\lambda}{\kappa}(\overline{h} - h)\right), \text{ and} \\
\underline{V} - \underline{V}_o &= \frac{\theta}{1-\theta}\left(-\frac{\rho}{\rho+\lambda}\left(\overline{h} - \underline{h} + \frac{1}{\rho+\lambda}\kappa\right) + \frac{\rho\kappa}{(\rho+\lambda)^2}\exp\left(\frac{\rho+\lambda}{\kappa}(\overline{h} - \underline{h})\right)\right).
\end{aligned}$$

Next, denote by V_d a payoff function on $(\underline{h}, \overline{h}]$ for the deviation strategy of immediate financing. That is, for $h \in (\underline{h}, \overline{h}]$,

$$V_d(h) \equiv \theta\big(V(\overline{h}) - \overline{h} + h\big) + (1-\theta)V_o(h).$$

Smooth pasting condition is

$$V'(\underline{h}) = V_d'(\underline{h}) = \theta + (1-\theta)\left(V'(\underline{h}) + \frac{\rho+\lambda+\gamma}{\kappa}(\underline{V} - \underline{V}_o)\right),$$

which, after some algebra, is equivalent to Equation (SC.10).

Stationary Recursion. This time, start by deriving V_o on $[0, \underline{h}]$, which satisfies:

$$\begin{aligned}
&\rho V_o(h) = \gamma\Big(\theta\big(V(\overline{h}) - \overline{h} + h\big) + (1-\theta)V_o(h) - V_o(h)\Big) + \lambda\big(\Pi + h - V_o(h)\big) - \kappa V_o'(h), \\
&V_o(0) = 0 \\
\implies &V_o(h) = \frac{1}{\rho+\lambda+\theta\gamma}\Bigg[\lambda\Pi + \theta\gamma\big(V(\overline{h}) - \overline{h}\big) - \frac{\lambda+\theta\gamma}{\rho+\lambda+\theta\gamma}\kappa + (\lambda+\theta\gamma)h \\
&\qquad - \left(\lambda\Pi + \theta\gamma\big(V(\overline{h}) - \overline{h}\big) - \frac{\lambda+\theta\gamma}{\rho+\lambda+\theta\gamma}\kappa\right)\exp\left(-\frac{\rho+\lambda+\theta\gamma}{\kappa}h\right)\Bigg].
\end{aligned}$$

Since $\Delta h \equiv \overline{h} - \underline{h} > 0$ has been determined by Equation (SC.10), $\overline{h}$ is obtained by the recursion:

$$V(\overline{h} - \Delta h) = \theta\Big(V(\overline{h}) - \Delta h\Big) + (1-\theta)V_o(\overline{h} - \Delta h).$$

Simplifying and substituting (SC.10) give Equation (SC.9). □

Proposition SC.1.4 (Breakeven re-inclusion – Startups).

Proof. $\underline{\gamma}$ is defined by $\underline{\eta}\xi = \overline{h}^*$, where $(\underline{h}^*, \overline{h}^*)$ is the equilibrium associated with $\gamma = \underline{\gamma}$ and $(\underline{\eta}, \xi)$ given by Proposition SC.1.2. Since $\gamma = \underline{\gamma}$ implies $\underline{h}^* = 0$, Proposition SC.1.1, in particular Equation (SC.5), holds with $\overline{h}$ replaced with $\underline{\eta}\xi$. Note that

$$\begin{aligned}
&\frac{\lambda}{\rho}\frac{1-\theta}{\exp\left(\frac{\rho+\lambda}{\kappa}\underline{\eta}\xi\right) - \theta} = \frac{\partial(\underline{\eta}\xi)}{\partial\Pi} = \underline{\eta}\frac{\partial\xi}{\partial\Pi} + \xi\frac{\partial\underline{\eta}}{\partial\Pi}, \quad \frac{\partial\xi}{\partial\Pi} = \frac{\lambda}{\rho} \\
\implies &\frac{\partial\underline{\eta}}{\partial\Pi} = \frac{\lambda}{\lambda\Pi - \kappa}\left(\frac{1-\theta}{\exp\left(\frac{\rho+\lambda}{\kappa}\underline{\eta}\xi\right) - \theta} - \frac{1-\theta}{1 + \frac{\rho+\lambda}{\underline{\gamma}} - \theta}\right).
\end{aligned}$$

Since $\overline{h}^* = \underline{\eta}\xi$, smooth pasting holds at $\underline{h}^* = 0$. Therefore, from Equation (SC.10),

$$\exp\left(\frac{\rho+\lambda}{\kappa}\underline{\eta}\xi\right) = 1 + \frac{\rho+\lambda}{\gamma}\frac{\rho+\lambda+\gamma}{\kappa}\underline{\eta}\xi.$$

Next, assume for now that $\underline{\eta}\xi = \overline{h}^* > \frac{\kappa}{\rho+\lambda+\underline{\gamma}}$, which will be established at the end. Then,

$$\exp\left(\frac{\rho+\lambda}{\kappa}\underline{\eta}\xi\right) > 1 + \frac{\rho+\lambda}{\underline{\gamma}},$$

Thus, $\partial\underline{\eta}/\partial\Pi < 0$. Since $\underline{\eta} = \frac{(1-\theta)\underline{\gamma}}{\rho+\lambda+(1-\theta)\underline{\gamma}}$, we have $\partial\underline{\gamma}/\partial\Pi < 0$.

Next is the convergence claim. Since $\overline{h}^*$ satisfies Equation (SC.3), it goes to ∞ as Π does. Note that $\Delta h^* \equiv \overline{h}^* - \underline{h}^* = \overline{h}^*$ satisfies Equation (SC.10) with $\gamma = \underline{\gamma}$. Since ρ, λ, κ are fixed, the only way for the solution of Equation (SC.10) to be satisfied by a

Δh that diverges to infinity is by having the linear coefficient on the LHS also diverge to infinity. This can only be achieved if $\underline{\gamma}$ goes to zero, as claimed.

The divergence claim is straightforward from $\underline{\eta}\xi > \frac{\kappa}{\rho+\lambda+\underline{\gamma}}$. Since $\xi = (\lambda\Pi - \kappa)/\rho$ and $\underline{\eta}$ is in the unit interval, the LHS vanishes as Π goes down to κ/λ. Therefore, the RHS also vanishes, i.e., $\underline{\gamma} \to \infty$.

Finally, as for the intermediate claim on the strict lower bound on $\underline{\eta}\xi$, first rearrange the Smooth Pasting condition – i.e., Equation (SC.10) – into the following:

$$\frac{\gamma}{\rho+\lambda}\left(\exp\left(\frac{\rho+\lambda}{\kappa}\Delta h\right)-1\right) = \frac{\rho+\lambda+\gamma}{\kappa}\Delta h.$$

Denote the LHS and RHS above as functions of Δh. Note that $\text{LHS}(0) = \text{RHS}(0)$ and $\text{LHS}'(0) < \text{RHS}'(0)$. Therefore, the LHS crosses the RHS only once and from below on $\mathbb{R}_{++}$. Note that

$$\text{LHS}\left(\frac{\kappa}{\rho+\lambda+\gamma}\right) = \frac{\gamma}{\rho+\lambda}\left(\exp\left(\frac{\rho+\lambda}{\rho+\lambda+\gamma}\right)-1\right) < 1 = \text{RHS}\left(\frac{\kappa}{\rho+\lambda+\gamma}\right).$$

This holds for any set of parameters because, letting $f(x) \equiv x\left(\exp\left(\frac{1}{1+x}\right)-1\right)$, we have

$$\forall\ x > 0,\ f'(x) > 0,\ \text{and}\ \lim_{x\to\infty} f(x) = 1.$$

Therefore, $\text{LHS}(\Delta h) < \text{RHS}(\Delta h)$ for any $\Delta h \in (0, \frac{\kappa}{\rho+\lambda+\gamma}]$. That is, if Smooth Pasting holds at $\underline{h}$, then it must be that $\Delta h = \overline{h} - \underline{h} > \frac{\kappa}{\rho+\lambda+\gamma}$. Since $\gamma = \underline{\gamma}$ means that Smooth Pasting holds at $\underline{h}^* = 0$, it must be that $\underline{\gamma}\xi = \overline{h}^* = \Delta h^* > \frac{\kappa}{\rho+\lambda+\underline{\gamma}}$, as claimed. □

SC.4 Proofs for operating firms derivations

Proposition SC.2.1 (Comparative statics – Operating firms without re-inclusion).

Proof. Most results are straightforward. As for the sign of $\partial\overline{h}/\partial\sigma^2$, first write the denominator of the second fraction as $e^{-\phi\overline{h}}DN(\overline{h})$ where

$$DN(z) \equiv \sqrt{\pi^2 + 2\rho\sigma^2}\left(e^{(\Phi+\phi)z} + 1 - 2\theta e^{\phi z}\right) - \pi\left(e^{(\Phi+\phi)z} - 1\right).$$

Then, it is easily verified that $DN(0) > 0,\ DN'(z) > 0$. Therefore, the denominator is positive. Next, write the numerator as $e^{-\phi\overline{h}}NM(\overline{h})$ where

$$NM(z) \equiv z\sqrt{\pi^2 + 2\rho\sigma^2}\left(e^{(\Phi+\phi)z} + 1\right) - \sigma^2\left(e^{(\Phi+\phi)z} - 1\right).$$

Then, it is easily verified that $NM(0) = NM'(0) = NM''(0) = 0 < NM'''(z)$ for all $z \geq 0$. Therefore, for any positive z, NM is positive as well. Since $\overline{h} > 0$, positivity is established.

Last, the limit of $\partial\overline{h}/\partial\sigma^2$ as $\sigma^2 \to 0$ is established as follows. First, note that as $\sigma^2 \to 0$,

$$\frac{\Phi/\phi}{\Phi+\rho} \to \frac{\pi}{\rho}, \quad \frac{\phi/\Phi}{\Phi+\phi} \to 0.$$

Since $\phi \to \rho/\pi$ and $\overline{h} \to 0$, the first term on the right-hand side of (SC.13) goes to π/ρ. Therefore,

$$\frac{\phi/\Phi}{\Phi+\phi} e^{\Phi\overline{h}} \to (1-\theta)\frac{\pi}{\rho} > 0,$$

implying that $e^{\Phi\overline{h}} \to +\infty$. Since $\Phi\overline{h} = \left(\sqrt{\pi^2+2\rho\sigma^2}+\pi\right)\frac{\overline{h}}{\sigma^2}$, it follows that $\overline{h}/\sigma^2 \to +\infty$. Since $\overline{h} \to 0$ as $\sigma^2 \to 0$, L'hospital's rule establishes that

$$+\infty = \lim_{\sigma^2\to 0}\frac{\overline{h}}{\sigma^2} = \lim_{\sigma^2}\frac{\partial\overline{h}/\partial\sigma^2}{\partial\sigma^2/\partial\sigma^2} = \lim_{\sigma^2\to 0}\frac{\partial\overline{h}}{\partial\sigma^2},$$

as claimed. □

Proposition SC.2.2 (Operating firm financing).

Proof. First, Inequality (SC.18) is simply Inequality (SC.17) reformulated through Equation (SC.13). Therefore, the equilibrium claim when $\underline{h} = 0$ is straightforward. Suppose now that Inequality (SC.18) fails.

Threshold Indifference. V_o on $[0, \underline{h}]$ satisfies

$$\begin{aligned}
&\rho V_o(h) = \gamma\theta\left(\frac{\pi}{\rho} - \overline{h} + h - V_o(h)\right) + \pi V_o'(h) + \frac{1}{2}\sigma^2 V_o''(h), \\
&V_o(0) = 0, \quad V_o(\underline{h}) = \frac{\pi}{\rho} - \left(1 + \frac{\rho}{(1-\theta)\gamma}\right)\Delta h \equiv \underline{V}_o \\
\implies\; & V_o(h) = \frac{\theta\gamma}{\rho+\theta\gamma}\left[\left(1+\frac{\rho}{\rho+\theta\gamma}\right)\frac{\pi}{\rho} + h - \overline{h}\right] \\
&\quad + \frac{\rho}{\rho+\theta\gamma}\left[\left\{\frac{\pi}{\rho+\theta\gamma} - \frac{1}{1-\theta}\left(1+\frac{\rho}{\gamma}\right)\Delta h\right.\right. \\
&\qquad\left. + \frac{\theta\gamma}{\rho}\left(\left(1+\frac{\rho}{\rho+\theta\gamma}\right)\frac{\pi}{\rho} - \overline{h}\right)e^{-\Phi_o\underline{h}}\right\}\frac{e^{\phi_o h} - e^{-\Phi_o h}}{e^{\phi_o\underline{h}} - e^{-\Phi_o\underline{h}}} \\
&\qquad \left. - \frac{\theta\gamma}{\rho}\left(\left(1+\frac{\rho}{\rho+\theta\gamma}\right)\frac{\pi}{\rho} - \overline{h}\right)e^{-\Phi_o h}\right].
\end{aligned} \tag{SC.21}$$

The boundary condition at $\underline{h}$ is given by Threshold Indifference ($G(\underline{h}) = 0$).

Stationary Recursion. As h goes down to $\underline{h}$ from above, V on $[\underline{h}, \overline{h}]$ must converge to the financing value based on $\underline{V}_o$. Substituting $\underline{h}$ into Equation (SC.11), denoting $\Delta h \equiv \overline{h} - \underline{h}$ and equating it to $\theta\left(V(\overline{h}) - \Delta h\right) + (1-\theta)\underline{V}_o$ give (SC.20).

Smooth Pasting. On $[0, \underline{h}]$, V is characterized by immediate financing. Therefore, for $h \in [0, \underline{h}]$,

$$V(h) = \theta \left(\frac{\pi}{\rho} - \overline{h} + h \right) + (1 - \theta) V_o(h).$$

Its derivative at $h = \underline{h} = \overline{h} - \Delta h$, with V_o given by Equation (SC.21), must agree with the derivative of Equation (SC.11) evaluated at the same point. Some algebra with substituting (SC.20) gives (SC.19). □

Proposition SC.2.3 (Comparative statics – Operating firms).

Proof. First, on σ^2. A higher σ^2 is less desirable due to forcing more frequent dilution; hence, $V(\overline{h}) - \overline{h} = \frac{\pi}{\rho} - \overline{h}$ must be decreasing in σ^2. Monotonicity of Δh and existence of $\overline{\sigma}^2$ are since

$$\frac{\partial}{\partial \sigma^2} \left[\frac{1}{\Phi + \phi} \left(\frac{\Phi}{\phi} e^{-\phi z} - \frac{\phi}{\Phi} e^{\Phi z} \right) \right] < 0, \quad \lim_{\sigma^2 \to 0} \left[\frac{1}{\Phi + \phi} \left(\frac{\Phi}{\phi} e^{-\phi z} - \frac{\phi}{\Phi} e^{\Phi z} \right) \right] = -\infty.$$

When $\sigma^2 > \overline{\sigma}^2$, Proposition SC.2.1 applies. As $\sigma^2 \to 0$, the business becomes a constant perpetuity stream. Hence, $\overline{h} \to 0$ and so do Δh, $\underline{h}$ since they add up to $\overline{h}$.

Next, on π. Since $V(\overline{h}) - \overline{h} = \frac{\pi}{\rho} - \overline{h} \geq 0$, $\pi \to 0$ implies $\overline{h} \to 0$. The existence of $\underline{\pi}$ is immediate from that of $\overline{\sigma}^2$ since an equilibrium with (π, σ) is isomorphic to that with $(b\pi, b\sigma)$ for any $b > 0$. As $\pi \to \infty$, the left- and right-hand sides of (SC.20) go to $+\infty$, $-\infty$ with any fixed $\Delta h > 0$. Therefore, $\Delta h \to 0$. □

Proposition SC.2.4 (Breakeven re-inclusion – Operating firms).

Proof. $\underline{\gamma}$ is defined by $\underline{\eta}\xi = \overline{h}^*$ where $(\underline{h}^*, \overline{h}^*)$ is the equilibrium associated with $\gamma = \underline{\gamma}$ and $(\underline{\eta}, \xi)$ defined by Proposition SC.2.2. Since $\gamma = \underline{\gamma}$ implies $\underline{h}^* = 0$, Proposition SC.2.3, in particular Equation (SC.15), holds with $\overline{h}$ replaced with $\underline{\eta}\xi$, that is, $\partial \underline{\eta}\xi / \partial \sigma^2 > 0$. Since $\partial \eta / \partial \gamma > 0$, $\partial \xi / \partial \gamma = 0$, it follows that $\partial \underline{\gamma} / \partial \sigma^2 > 0$. The limit claims follow from the existence of $\overline{\sigma}^2$ for any γ in Proposition SC.2.3. The remaining claims on π follow from the isomorphism stated in the proof of Proposition SC.2.3. □