Financing Innovative Assets: Endogenous Concentration in Startup Innovation*

Hanjoon Ryu†

October 29, 2025

Preliminary and incomplete
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Abstract

This paper develops a model of startup innovation where financing frictions in implementing successful innovations concentrate prior innovation activity. Because funds are more valuable ex-post upon success of innovation than failure, there exists within-firm complementarity ex-ante between innovation and savings. Consequently, startup funds have increasing returns to scale, concentrating innovation towards well-funded startups that additionally raise costly entrepreneurial financing. Non-innovators exhibit perfect inelasticitiy to financing costs – but only locally. Utilizing a model of financing frictions that enables tractable aggregation, I show that time-sensitiveness in implementation increases concentration while ease of 'killer acquisitions' decreases it. Concentration is amplified in general equilibrium.

JEL Classification: D21, E22, E23, G31, G34, O16, O30
Keywords: Innovation; financial frictions; concentration; technology adoption; venture capital financing; creative destruction; killer acquisitions.

*The first version was circulated in September 2025. I have greatly benefited from discussions with Gregor Jarosch, Andrea Lanteri, Adriano Rampini, S. "Vish" Viswanathan, Melanie Wallskog, and seminar participants at Duke University (Economics Department). All errors are my own.

†Duke University, Department of Economics. Email: hanjoon.ryu@duke.edu.

1 Introduction

Innovation by new firms is crucial for sustained economic growth. But there exists tremendous heterogeneity among startups in their innovation intensity. In particular, innovation tends to be disproportionately concentrated towards a tiny fraction of these new firms. Where is the concentration coming from, and what are its implications on the efficiency of innovation in the macroeconomy?

To raise the stakes, I think of innovation as a broad concept with applications in several contexts. It may refer to startup firms' research and development where breakthroughs occur at the frontier of scientific technology. Innovation is also at work in smaller scales; for example, restaurant owners may invest into branding and interior designs to expand their customer base and grow into a successful franchise. Even on a household level, parents may arrange private tutoring for their children so that they can get into prestigious universities. In sum, this paper refers to any resource-spending activity that increases the chance of success leading to a greater payoff, as 'innovation.'

Returning to the question, I present a model of 'startup innovation' where frictions to financing the *implementation* of success ex-post concentrate innovation among startups ex-ante. The core model has a simple setup: A startup chooses how much of its funds to spend on innovation whose success probability is increasing and concave in innovation spending. The only additional assumption in this static problem is that upon success, the startup remains subject to financing frictions in implementing the innovative technology into productive assets.

In reference to the three examples above, a successful R&D often requires subsequent investment, starting a franchise requires hefty upfront expenses, and enrollment at prestigious universities requires expensive tuition. The assumption at hand posits that each 'startup' in the respective context – firm, restaurant owner, children (with parents) – cannot raise external financing frictionlessly to fund such 'implementation.'

The assumption implies that there exists complementarity, statically and within each startup, between innovation spending and *savings*: the former increases the chance of success and the latter the payoff upon success. As such, startup value function is locally strictly convex, indicating *increasing* returns to scale in startup funding.

Consequently, only startups initially endowed with enough internal funds choose to innovate because they can afford to keep savings for implementation. In contrast, poorly funded startups may simply forgo innovation. If startups have access to costly external financing (e.g., venture-capital funds) before undertaking innovation, only the well-funded startups raise even more funds externally to scale up both innovation spending and savings for implementation, while the rest of the innovators – that is, of the startups that do *not* forgo innovation – self-select into financial autarky, inducing them to forgo innovation altogether or innovate only a little. As such, innovation becomes endogenously concentrated.

Despite the absence of any fixed cost of innovation, these 'non/little-innovators' under self-selected financial autarky exhibit perfect inelasticity in their innovation choice with respect to the financing costs, but only *locally*. A marginal decrease in the cost of pre-innovation financing does not lead to a marginal increase in these autarkic startups' innovation intensity. But if the decrease is sufficiently large, such startups opt out from financial autarky, thereby *discretely* scaling up their innovation intensity and their savings for implementation.

But in the aggregate, this endogenous separation among startups – which increases when financing costs rise – may *increase* the elasticity of innovation in the economy to financing costs. Given a higher cost of 'entrepreneurial' financing prior to innovation, startups with the greatest needs for external funding self-select first into autarky, which severely constrains their innovation choice. Consequently, extensive margin in entrepreneurial financing has first-order effects. Therefore, when innovation becomes more concentrated, this model predicts that (i) the proportion of autarkic startups with local perfect inelasticity to financing costs increases, and yet (ii) the elasticity of aggregate innovation may increase.

These findings have immediate efficiency implications. The model features no heterogeneity between startups ex-ante in terms of their innovation 'technology' (e.g., R&D capabilities, business skills, children's academic potential, etc.) or preferences of their entrepreneurs; given diminishing marginal probability of success, efficiency is thus attained only if the marginal probability is equalized across all startups. Nevertheless, due to the complementarity between innovation and savings, only a few startups undertake a disproportionately large amount of innovation in equilibrium, resulting in a sizable wedge in marginal success rates of innovation across startups.

In relation to economic growth, this paper thus brings to light the fundamental complementarity between technology and financing. It suggests that, in contrast to the conventional views that entrepreneurs choose between transformative innovation and "subsistence/lifestyle" businesses based on their skills or preferences (e.g., Schoar, 2010, Hurst and Pugsley, 2011), the choice may be based on their *financial* assets. Put differently, this paper argues that macroeconomic concentration of innovation across entrepreneurs may be a phenomenon due to the household asset distribution.

Application. This model predicts that factors that worsen or mitigate financing frictions during implementation also worsen or mitigate inefficient concentration in innovation. To sharpen the focus, I frame this paper's applications on startup *firms* that spend on R&D to obtain an innovative technology, upon which they can raise financing for its implementation from financial investors (e.g., an initial public offering) but subject to frictions. It would also be interesting to apply this model for small businesses (where entrepreneurs face the self-financing constraint even after success), or households' intergenerational choice on education (potentially with long-term welfare implications).

I consider two such factors for startup firms in the extended model. The first is when a successful innovation is time-sensitive in that if implementation is delayed, it yields reduced returns. This applies if either industries are technologically evolving fast or incumbent firms may quickly imitate successful but yet-to-be-implemented innovations in order to usurp the first-mover advantage. The severer the frictions to financing ex-post upon a successful innovation, the more valuable it is to keep large funds that are readily deployable for timely implementation. Ex-ante, this increases the complementarity between innovation and savings, thereby concentrating innovation.

The second factor concerns acquisitions of startup firms by incumbents. I allow incumbent firms to acquire a startup firm whose successful innovation is about to replace their existing operations. I assume that incumbents cannot operate the new technology as efficiently themselves; given large inefficiency, the term "killer acquisitions" (as formalized by Cunningham, Ederer and Ma, 2021) would be warranted. If upon a successful innovation startup firms are able to more easily liquidate the technology through such acquisitions, they face less frictions in raising financing to implement the technology themselves. This weakens the complementarity between innovation and savings and thus reduces concentration in innovation.

Furthermore, when such acquisitions involve greater inefficiency (i.e., they become more 'killing'), the above effect becomes relatively stronger for startup firms with small endowed funds. With large inefficiency in acquisitions, it is only those startup firms who cannot save sufficiently towards implementation of success that would ever consider pursuing being acquired by well-financed incumbent firms. Thus, poorly funded startup firms benefit the most when 'killer acquisitions' become easy.

In the context of general equilibrium, this paper identifies an important amplification channel for aggregate innovation. When aggregate startup innovation falls due to concentration, the economy becomes less productive because there are less of more productive firms. Consequently, the distribution of household assets shifts down in general equilibrium. If startup firms' initial endowments of internal funds follow the same distribution as household assets (since entrepreneurs come from households), then more startup firms opt into financial autarky and curtail innovation, thereby increasing concentration in innovation and further reducing aggregate innovation.

On a methodological note, tractability of aggregation is often crucial in growth literature (e.g., Klette and Kortum, 2004). To retain it, I adopt a simplified version of the financing model with bargaining à la Ryu (2025) when modeling financing at the implementation stage in this paper. While the core analytic results do not depend on it, this particular setup induces – in the model – all successful startups to become identical on the equilibrium path; frictions that give rise to the concentration in innovation ex-ante come from the scenario of failing to raise financing upon successful innovation. This is off the equilibrium path, and *therefore*, it determines startup firms' outside options when bargaining with prospective investors. Thus, startup firms' payoff upon success increases in their savings that reduce rents that investors extract through bargaining, even though savings do not affect the path of play for successful innovators. Leveraging this handy feature, I aggregate the production side of the economy in closed form.

Related literature. The key contribution of this paper is to endogenize the concentration in innovation across new firms through financial frictions. Papers such as Schoar (2010) and Hurst and Pugsley (2011) document cross-sectional skewness in innovation capacity by young firms, in that only a small fraction of such entrepreneurs aim for transformative innovations. Decker, Haltiwanger, Jarmin and Miranda (2016) – and more recently, Kim, Choi, Goldschlag and Haltiwanger (2024) – show that such skewness has fluctuated over time. This paper suggests that entrepreneurs' asset endowment may disproportionately affect their incentives to pursue a larger-scale innovation that requires costlier implementation upon success.

This insight has immediate implications for the vast literature on growth and innovation. In canonical models such as Romer (1990), Grossman and Helpman (1991), Aghion and Howitt (1992), and Klette and Kortum (2004), firm entrants face homogeneous innovation problems and the cross-section of innovation choice is not central. Acemoglu and Cao (2015) as well as Akcigit and Kerr (2018) introduce heterogeneous innovation into the framework where new firms focus on more radical innovation relative to incumbent firms. Recently, Ottonello and Winberry (2024) study how financial frictions may affect innovation choice among incumbent firms. In relation, this paper shows that payoffs to becoming an incumbent firm, subject to financing frictions, may induce a first-order effect on the extensive margin of entry through innovation.

This paper expands the literature on technology adoption, e.g., Comin and Hobijn (2007), Anzoategui, Comin, Gertler and Martinez (2019), Buera and Trachter (2024), by highlighting its cross-sectional effects due to frictions in financing the adoption. Furthermore, it identifies an amplification channel in general equilibrium for aggregate innovation via household asset distribution, and thus relates to works such as Comin and Gertler (2006) and Buera, Hopenhayn, Shin and Trachter (2021) discussing macroeconomic effects of technology adoption beyond short-term business cycles.

This paper also contributes to the literature on business dynamism. Akcigit and Ates (2023) argue that the secular decline in the business dynamism in the U.S. is mainly due to a drop in knowledge diffusion from frontier firms, decreasing the returns from becoming a laggard firm expecting such a diffusion. Fons-Rosen, Roldan-Blanco and Schmitz (2024) address how “killer acquisitions” may incentivize startup creation. This paper suggests that such factors may also have important distributional implications – specifically, skewness and concentration – in startups’ innovation capacity.

Lastly, the local convexity result in this paper that arises due to the complementarity between innovation and savings strengthens and also generalizes the mechanism by Vereshchagina and Hopenhayn (2009), where the entrepreneur’s discrete choice between starting a business and working as an employee gives rise to local nonconcavity.

Overview. The rest of the paper is organized as follows. Section 2 discusses empirical patterns on startup concentration. Section 3 presents the main model of endogenous concentration due to financing frictions, which is then embedded through Section 4 into a general equilibrium framework with a balanced-growth path. The extended model is then calibrated in Section 5 for quantitative analysis. Section 6 concludes.

2 Stylized Facts

Forthcoming.

3 Model

In this section, I present a static model of entrepreneurship to demonstrate how financing frictions during implementation concentrate innovation among startups.

First, I fix the terminology and the timeline. There is the ‘pre-innovation’ stage where entrepreneurs raise ‘entrepreneurial financing’ before establishing a startup. It is followed by the ‘innovation’ stage where the startup chooses how much to spend on innovation and how much to save. Then, there is, conditional on successful innovation, the ‘implementation’ stage where the startup faces financing frictions in implementing the innovative technology into a productive asset.

Following the timeline backwards, the present section is composed of two parts. The first part in Section 3.1 characterizes startups’ optimal policy on innovation and savings in anticipation of the subsequent implementation stage upon success. It analyzes the core mechanism that financing frictions during implementation may create (locally) increasing returns to scale in startup funding. The second part in Section 3.2 illustrates how this effect may concentrate entrepreneurial financing and hence innovation activity, and increase the elasticity of aggregate innovation to financing costs through the extensive margin in entrepreneurial financing. Assumptions for the implementation stage are generic; a concrete setup will be presented in Section 4.1.

3.1 Innovation and savings

Setup. There are two dates $t \in \{0, 1\}$. There is a startup with initial funds $a \geq 0$: I intentionally omit date indexing for this variable on date 0. It is owned by a group risk-neutral and penniless insiders – which includes the entrepreneur – with a rate of time preference on (nonnegative) dividends given by $\beta \in (0, 1)$. There is a non-contingent lending market at interest rate $R \equiv \beta^{-1}$. Since the returns from lending R exactly compensate for the discount factor β, the startup is assumed without loss to pay zero dividends to the insiders on the first date $t = 0$, which simplifies analysis.

The startup has access to a technology to produce innovative assets over the two dates. On the first date, it spends funds on innovation $h_0 \geq 0$. At the beginning of date $t = 1$, the prior innovation spending h_0 leads to a successful development of an innovative technology with probability $\Lambda(h_0)$. The success function $\Lambda : \mathbb{R}_+ \to [0, 1]$ is twice continuously differentiable, nondecreasing and weakly concave, $\Lambda' \geq 0$, $\Lambda'' \leq 0$, and satisfies $\Lambda(0) = 0 < \Lambda'(0)$. For now, assume that the startup cannot finance innovation externally, $h \in [0, a]$; external financing will be addressed shortly in Section 3.2 in the context of entrepreneurial financing.

Upon successful development, the startup may implement the innovative technology into productive assets on the second date. Importantly, it remains subject to financing frictions, and hence the returns from the implementation depend on the amount of its savings $a_1 \equiv R(a - h_0)$ post success; specifically, the marginal value of savings exceeds unity. In the present part, I take a generic value function conditional on successful development $V_s(a_1)$, which is twice continuously differentiable and satisfies $V_s' \geq 1$, $V_s'' \leq 0$, $V_s(0) \geq 0$ and the Inada condition $V_s'(0) = \infty$; Section 4.1 provides a model with financing frictions that gives rise to these properties.[1] If innovation fails, the startup engages in a non-innovative business. For tractability and as a normalization, I assume that the startup's conditional value function upon failure is simply $V_f(a_1) \equiv a_1$; the crucial part is that saved funds are more valuable upon success of innovation than upon its failure, $V_s'(a_1) - V_f'(a_1) > 0$ for $a_1 < \overline{a}_1 \equiv \inf\{a_1 \geq 0 \mid V_s'(a_1) = 1\} \in (0, \infty]$.

Practically, $\overline{a}_1$, defined above, is the amount of savings required to overcome financing frictions during implementation. If it is finite, the assumed conditions on V_s imply that $V_s' = 1$ and $V_s'' = 0$ on $(\overline{a}_1, \infty)$.

Innovation problem. The startup with funds $a \geq 0$ solves

$$V(a) \equiv \max_{h \in [0,a]} \beta\Big(\Lambda(h) V_s\big(R(a-h)\big) + \big(1 - \Lambda(h)\big) V_f\big(R(a-h)\big)\Big). \tag{1}$$

Letting μ_0 and μ_a Lagrange multiplier for the nonnegativity constraint on innovation

[1] The second derivative V_s'' may have discontinuity at $\overline{a}_1 > 0$, which is defined shortly.

$h \geq 0$ and self-financing constraint $a \geq h$, respectively, the first-order condition gives

$$\Lambda'(h)\beta\Big(V_s\big(R(a-h)\big) - R(a-h)\Big) + \mu_0 = 1 + \Lambda(h)\Big(V_s'\big(R(a-h)\big) - 1\Big) + \mu_a, \quad (2)$$

and the second-order condition is satisfied.

The left-hand side, without the multiplier, is the marginal returns from increasing the chance of successful innovation. The right-hand side, without the multiplier, is the marginal cost from saving less due to innovation spending. As seen, innovation not only raises the chance of success, but also lowers the payoff from success.

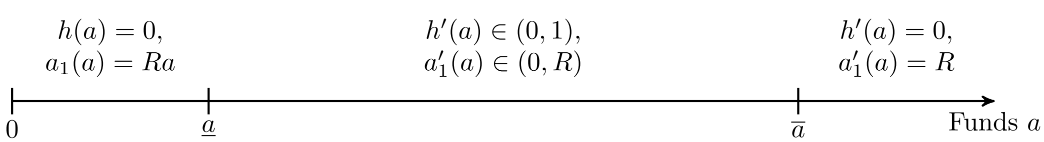
Denote $h(a)$ as the startup's optimal innovation and $a_1(a) \equiv R\big(a - h(a)\big)$ as its optimal savings – given funds a. Define $\overline{a} \equiv a_1^{(-1)}(\overline{a}_1)$ – that is, $a_1(\overline{a}) = \overline{a}_1$ – as the least amount of funds $a = \overline{a}$ such that the startup, in equilibrium, does not face financing frictions conditional on success. Accordingly, I refer to $\overline{a}$ as the 'relevance threshold.'

Solution. Given the assumption of profitability, the startup's optimal policy obtains.

Assumption 1 (Profitability of innovation)**.** $\Lambda'(0)\beta\big(V_s(\overline{a}_1) - \overline{a}_1\big) \in (1, \infty)$.

Lemma 1 (Monotonicity of innovation and savings)**.** *Optimal innovation spending $h(a)$ is nondecreasing in the startup's initial funds a whereas optimal savings $a_1(a) \equiv R\big(a - h(a)\big)$ are strictly increasing; thus, $h' \in [0, 1)$. There is an innovation threshold in initial funds, $\underline{a} \in [0, \infty)$, such that $h(a) = 0$ if and only if $a \leq \underline{a}$. The threshold is positive, $\underline{a} > 0$, if and only if $\Lambda'(0)\beta V_s(0) < 1$, in which case it solves $\Lambda'(0)\beta\big(V_s(R\underline{a}) - R\underline{a}\big) = 1$. Lastly, $h'(a) \in (0, 1)$ if and only if $a \in (\underline{a}, \overline{a})$.*

Figure 1: Monotonicity of innovation spending and savings



Innovation threshold is positive, $\underline{a} > 0$, if and only if $\Lambda'(0)\beta V_s(0) < 1$. Relevance threshold is finite, $\overline{a} \in (\underline{a}, \infty)$, if and only if $\overline{a}_1 \equiv \inf\{a_1 \geq 0 \mid V_s'(a_1) = 1\} < \infty$.

If $a \geq \overline{a}$, the startup has enough funds to save itself out from the financing frictions during the implementation stage $V_s'\big(a_1(a)\big) = 1$. Therefore, the net payoff from successful innovation, $V_s\big(a_1(a)\big) - a_1(a)$, is maximized at $\overline{a}$ and constant above it. As such, optimal innovation spending, $h(a)$, is also constant above it. On the other side, if the payoff from successful innovation is too low without enough savings, $\underline{a} > 0$, and the startup has little funds, $a \in (0, \underline{a}]$, it optimally forgoes innovation, $h(a) = 0$.

Below the innovation threshold, the startup endogenously exits, and above the relevance threshold, additional savings yield zero net gains. Either way, the marginal value of funds ex-ante, V', is constant at unity outside the interval $[\underline{a}, \overline{a}]$.

Corollary 1. *For $a \notin [\underline{a}, \overline{a}]$, $V'(a) = 1$ and $V''(a) = 0$.*

But the key part of Lemma 1 is that on $[\underline{a}, \overline{a}]$, *both* innovation spending, $h(a)$, and savings, $a_1(a)$, are strictly increasing in the startup's initial funds, a. The following result shows what this finding implies about the startup's value function ex-ante, V.

Proposition 1 (Increasing returns to scale)**.** *Frictions in financing innovative assets, when they marginally affect the startup's innovation problem, add strict convexity to its payoff ex-ante with respect to the initial funds prior to innovation: for $a \in (\underline{a}, \overline{a})$,*

$$V''(a) = \Lambda\big(h(a)\big)V_s''\big(a_1(a)\big)R\big(1 - h'(a)\big) + \Lambda'\big(h(a)\big)h'(a)\Big(V_s'\big(a_1(a)\big) - 1\Big). \quad (3)$$

Proof. Immediate from the envelope condition and the chain rule. □

Corollary 2. *If $a \in (\underline{a}, \overline{a})$ and $V_s''\big(a_1(a)\big) = 0$, then $V''(a) > 0$.*

In Equation (3), the first term on the right-hand side indicates that the curvature from V_s, that is, $\frac{d}{da}V_s'\big(a_1(a)\big) = V_s''\big(a_1(a)\big)R\big(1 - h'(a)\big)$, is reduced by the success probability $\Lambda\big(h(a)\big)$. But this only dampens the concavity of V_s''. It is the second term, strictly positive on $a \in (\underline{a}, \overline{a})$, that adds 'strict' convexity and gives rise to strictly increasing returns to scale, as Corollary 2 illustrates.

What drives the convexity effect on the value function ex-ante, V, is financing frictions upon successful innovation that requires costly implementation. Because funds are thus more valuable upon success of innovation than upon its failure $V_s'(a_1) > 1$, a startup with more internal funds a optimally increases not only innovation spending $h(a)$ but also savings for implementation $a_1(a)$, as Lemma 1 shows. This indicates that there is *complementarity* between innovation spending and savings. That is, a greater chance of success makes savings more valuable in expectation, and a greater return from success makes innovation spending more profitable; this is shown via the term $\beta\Lambda(h)\big[V_s\big(R(a-h)\big) - R(a-h)\big]$ in the startup's objective function from Equation (1). By expanding the startup's feasibility set, an increase in initial funds improves these two multiplicative margins, thereby generating the increasing returns to scale.

This effect obtains despite diminishing marginal success probability of innovation, $\Lambda'' \leq 0$, and diminishing returns to savings conditional on success, $V_s'' \leq 0$. Given $a \in [\underline{a}, \overline{a}]$, the startup optimizes innovation spending against these two diminishing – yet *multiplicative* – margins. Implicitly differentiate Equation (2) on $(\underline{a}, \overline{a})$ to get

$$h'(a) = \frac{\Lambda'(h)\big(V_s'(a_1) - 1\big) - \Lambda(h)RV_s''(a_1)}{2\Lambda'(h)\big(V_s'(a_1) - 1\big) - \Lambda(h)RV_s''(a_1) - \Lambda''(h)\beta\big(V_s(a_1) - a_1\big)} \in (0, 1). \quad (4)$$

As shown, concavities in Λ and V_s reduce the slope of optimal innovation spending, h'. This dampens the convexity effect as Equation (3) shows, but never entirely offsets it.

Characterization. Generally, Proposition 1 fails to imply actual convexity of V on the entire $(\underline{a}, \overline{a})$, because the concavity of V_s not only directly affects V in expectation but also diminishes the net marginal value of savings ex-post $V_s' - 1$, thereby weakening the complementarity. To characterize when and where V exhibits actual increasing returns to scale, I assume a sufficient regularity condition.

Assumption 2 (Sufficient condition for single crossing)**.** *Both* $h \mapsto \frac{-\Lambda''(h)}{\Lambda'(h)}$ *and*

$$a_1 \mapsto \left(\frac{-V_s''(a_1)}{V_s'(a_1) - 1}\right)\left(\frac{V_s(a_1) - a_1}{V_s'(a_1) - 1}\right)^{1+R}$$

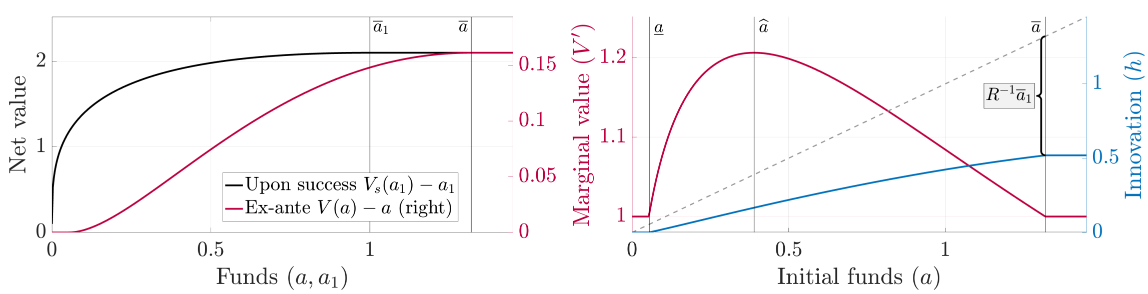
are nondecreasing, and one of them strictly increasing, wherever well-defined.

The condition states that the underlying concavity, in success probability Λ and payoff upon success $V_s - a_1$, does not diminish too rapidly relative to the slope. The looser condition on V_s allows for the Inada condition $V_s'(0) = \infty$, whereas the tighter one on Λ enforces a finite $\Lambda'(0) \in (0, \infty)$.[2] The condition on Λ is satisfied by the standard form $\Lambda(h) \equiv 1 - \exp(-\lambda h)$ or a quadratic form $\Lambda(h) \equiv 1 - \frac{\lambda^2}{4} \max\left\{\frac{2}{\lambda} - h, 0\right\}^2$, given $\lambda > 0$; the one on V_s is satisfied by $V_s(a_1) = a_1^\alpha + a_1$ given $\alpha \in (0, 1)$, for example.

Given the reasonable regularity condition, a simple characterization obtains.

Proposition 2 (More increasing returns for less funds)**.** *There is* $\widehat{a} \in [\underline{a}, \overline{a})$ *such that the startup's value function ex-ante,* V*, is strictly convex on* $[\underline{a}, \widehat{a}]$ *and strictly concave on* $[\widehat{a}, \overline{a}]$*. If the innovation threshold is positive,* $\underline{a} > 0$*, then convexity dominates in a neighborhood above it,* $\widehat{a} > \underline{a}$*.*

Figure 2: Slack in innovation and increasing returns to scale



(a) Value functions, ex-post and ex-ante

(b) Optimal innovation and savings

The curve in black in the left plot is an exogenous value function conditional upon success net of remaining funds $V_s(a_1) - a_1$. I let $\Lambda(h) \equiv 1 - \exp(-\lambda h)$. Parameters are $\beta = R^{-1} = 0.8$, $\lambda = \Lambda'(0) = 1$, and $\Lambda'(0)\beta V_s(0) = 0.08 < 1$.

Illustration. Figure 2 displays the solution to the innovation problem with initial funds a, given an exogenous value function conditional on successful innovation $V_s(a_1)$, with

[2] Characterization (that is, Proposition 2) works as long as at least one of Λ and V_s satisfies the tighter condition, which implies that at most one of the two may satisfy the Inada condition.

normalized $\overline{a}_1 = 1$. As Figure 2b shows, marginal value of initial funds $V'(a)$ is strictly increasing only when (though not always when) both innovation spending h and savings $R(a-h)$ are strictly increasing – in the present example, between $\underline{a} \approx 0.06$ and $\widehat{a} \approx 0.39$. This is despite the fact that both Λ and V_s exhibit strictly diminishing marginal success probability $\Lambda'' < 0$ and strictly diminishing marginal returns to savings $V_s'' < 0$, below $\overline{a}_1 = 1$, respectively.

Admittedly, convexity does not globally dominate on $(\underline{a}, \overline{a})$. As $a \to \overline{a} \approx 1.32$, both marginal success probability $\Lambda'\big(h(a)\big)$ and net marginal returns to savings $V_s'\big(a_1(a)\big) - 1$ diminish – entirely for the latter. These two changes weaken the complementarity between innovation spending and savings, and hence also the convexity effect. Simultaneously, concavity of V_s is less dampened because success probability $\Lambda\big(h(a)\big)$ goes up. As such, ex-ante payoff becomes strictly concave between $\widehat{a} \approx 0.39$ and $\overline{a} \approx 1.32$.

Nevertheless, this 'locally' increasing returns to scale are still highly relevant when startups and their entrepreneurs face limited funding. Conceptually, $\overline{a}_1 = 1$ corresponds to the amount of funds that a successful startup firm would have at its disposal after completing an initial public offering ('IPO'). The fact that an IPO is generally much larger than venture capital financing for startup firms suggests that, in light of the severe financing frictions that startups tend to face before successful innovation, returns to scale in startup funding may be predominantly *increasing* in practice.

3.2 Entrepreneurial financing

Given the result from Section 3.1 of increasing returns to scale in startup funding, I now address the question of external financing for startups in the pre-innovation stage.

Setup. Consider the initial date $t = 0$ but prior to the innovation problem. There is an entrepreneur with a rate of time preference β and endowed with funds $a_0 \geq 0$; I call a_0 entrepreneurs' 'endowed' funds and a, from Section 3.1 startups' 'initial' funds. In anticipation of the innovation problem described in Section 3.1, the entrepreneur may, before establishing a startup, raise additional funds $e \geq 0$ from venture capitalists through a Walrasian market at a unit price p_e, which is determined in equilibrium. I assume that $p_e > 1$, which endogenously holds in the general equilibrium model in Section 4. This reflects the fact that financing at such an early stage often involves significant frictions, such as agency problems. Crucially, and consistent with the interpretation, entrepreneurs cannot supply their own endowed funds a_0 to other participants in the market at price p_e, that is, $e \geq 0$.

For simplicity, venture capitalists are assumed to receive proportional ownership stakes in the startup – which equal $p_e e$ in value – in return for entrepreneurial financing; as such, they become its fellow insiders along with the entrepreneur.

The entrepreneur endowed with funds $a_0 \geq 0$ solves

$$\max_{e \geq 0} V(a_0 + e) - p_e e.$$

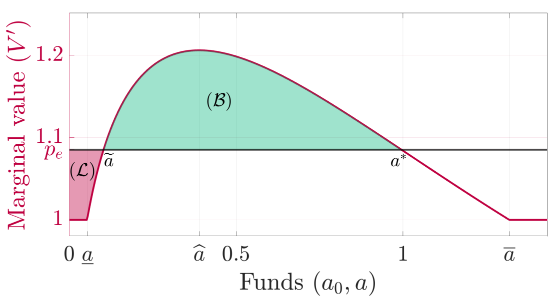
Let μ_e be Lagrange multiplier for the nonnegativity constraint. If the value function for innovation problem V is locally convex $\widehat{a} > \underline{a}$ as per Proposition 2, the typical optimality conditions

$$V'(a_0 + e) + \mu_e = p_e, \quad V''(a_0 + e) \leq 0 \tag{5}$$

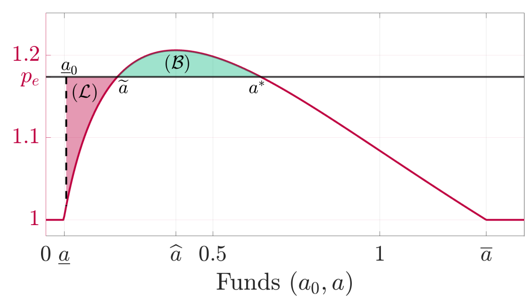
may not be sufficient for determining optimal financing. If $a_0 < \widehat{a}$ and $V'(a_0) < p_e < V'(\widehat{a})$, then it may be optimal to forgo financing, $e = 0$, even though there exists a positive amount of financing $e > \widehat{a} - a_0$ – such that $\mu_e = 0$ – that satisfies the above.

Characterization. If $\widehat{a} > \underline{a}$ and $p_e \in (V'(\underline{a}), V'(\widehat{a}))$, Proposition 2, with Corollary 1, implies that the price p_e crosses the marginal value V' twice. Let $a^*(p_e) > \widehat{a}$ solve $V'(a^*) = p_e$ and $V''(a^*) < 0$, and $\widetilde{a}(p_e) < \widehat{a}$ solve $V'(\widetilde{a}) = p_e$ but $V''(\widetilde{a}) > 0$. Note that, as functions of p_e, these objects $a^* = a^*(p_e)$ and $\widetilde{a} = \widetilde{a}(p_e)$ satisfy the following: $a^{*\prime} = 1/V''(a^*) < 0$, $\widetilde{a}' = 1/V''(\widetilde{a}) > 0$, and both converge to $\widehat{a}$ as $p_e \to V'(\widehat{a})$.

Figure 3: Demand for entrepreneurial financing



(a) Low price of external financing



(b) High price of external financing

Lemma 2 (Demand under local convexity)**.** *Suppose $\widehat{a} > \underline{a}$. Given $p_e \in (V'(\underline{a}), V'(\widehat{a}))$, there exists $\underline{a}_0(p_e) \in [0, \widetilde{a}(p_e))$ such that optimal demand for entrepreneurial financing $e(a_0, p_e)$ satisfies*

$$e(a_0, p_e) = \begin{cases} 0, & \text{if } a_0 < \underline{a}_0(p_e), \\ (a^*(p_e) - a_0)^+, & \text{if } a_0 > \underline{a}_0(p_e). \end{cases}$$

There is $\widehat{p}_e \in (V'(\underline{a}), V'(\widehat{a}))$ such that the financing threshold $\underline{a}_0(p_e)$ is zero if and only if $p_e \in (1, \widehat{p}_e]$ and positive and strictly increasing if and only if $p_e \in (\widehat{p}_e, V'(\widehat{a}))$. As p_e converges to $V'(\widehat{a})$ from below, both $\underline{a}_0(p_e) < \widehat{a}$ and $a^*(p_e) > \widehat{a}$ converge to $\widehat{a}$.

Figure 3 illustrates the demand for entrepreneurial financing under local convexity $\widehat{a} > \underline{a}$, which is Lemma 2. In Figure 3a with a relatively low price of external financing, the inframarginal loss area, $\mathcal{L}$, even when computed from $a_0 = 0$, is smaller than the inframarginal benefit area, $\mathcal{B}$. Thus, entrepreneurs endowed with $a_0 < a^*$ raise financing up to a^*. In Figure 3b featuring a higher price, $\mathcal{L}$ equals $\mathcal{B}$ when computed to the right of the financing threshold $\underline{a}_0 \geq 0$; the threshold is given as the solution in $a_0 \geq 0$ for

$$\underbrace{\int_{a_0}^{\widetilde{a}(p_e)} \big(p_e - V'(a)\big)\, da}_{\equiv \mathcal{L}(a_0, p_e)} = \underbrace{\int_{\widetilde{a}(p_e)}^{a^*(p_e)} \big(V'(a) - p_e\big)\, da}_{\equiv \mathcal{B}(p_e)},$$

and equals zero in case of no solution. Thus, entrepreneurs with $a_0 < \underline{a}_0$ optimally forgo financing, as the loss from financing up to a^*, $\mathcal{L}(a_0, p_e)$, exceeds its benefit, $\mathcal{B}(p_e)$.

The key assumption that gives rise to the positive financing threshold $\underline{a}_0 > 0$ under a high price p_e is that entrepreneurs cannot sell their own endowed funds a_0 at that price, that is, $e \geq 0$. If this is relaxed, then all entrepreneurs will sell their entire endowed funds. The assumption is consistent with the interpretation for $p_e > 1$, which is that financing at such an early stage of enterprise involves significant frictions including agency problems; entrepreneurs' own internal funds are free from such frictions only to the extent it is their own enterprise – not others' – that the funds are being used for.

Aggregate implications. As Lemma 2 states, there is a price $\widehat{p}_e \in (V'(\underline{a}), V'(\widehat{a}))$ solving $\mathcal{L}(0, \widehat{p}_e) = \mathcal{B}(\widehat{p}_e)$, above which the financing threshold becomes strictly positive. Below this level $\widehat{p}_e$, the price of external financing only affects, up to first order, entrepreneurs' intensive margin: when $p_e < \widehat{p}_e$ marginally increases, all entrepreneurs uniformly reduce the amount of financing subject to nonnegativity. Above the threshold, it affects, even in first order, their extensive margin as well: when $p_e > \widehat{p}_e$ marginally increases, the entrepreneurs with the least endowed funds $a_0 = \underline{a}_0$ among the ones that used to raise positive financing – that is, those who, without the price increase, would have financed the *most*, $e = a^* - \underline{a}_0$ – now forgo financing.

Consequently, the demand curve for entrepreneurial financing exhibits discontinuously greater price elasticity above the price threshold $\widehat{p}_e$ than below.

Proposition 3 (Kinked demand)**.** *Suppose entrepreneurs' endowed funds a_0 are distributed according to $G(a_0)$ admitting continuous density $g(a_0)$ with $g(0) > 0$. Given local convexity $\widehat{a} > \underline{a}$, the aggregate demand function for entrepreneurial financing*

$$E^d(p_e) \equiv \int_{\underline{a}_0(p_e)}^{a^*(p_e)} \big(a^*(p_e) - a_0\big)\, dG(a_0)$$

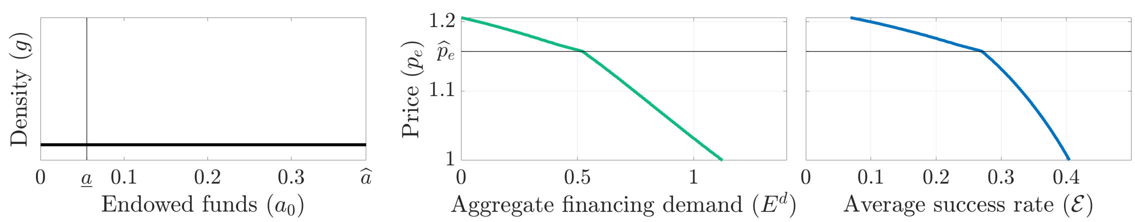
is kinked at $\widehat{p}_e$: its price elasticity satisfies

$$\frac{d\log E^d(p_e)}{d\log p_e} = \begin{cases} \frac{p_e}{E^d(p_e)} \frac{G(a^*(p_e))}{V''(a^*(p_e))}, & \text{if } p_e < \widehat{p}_e, \\ \frac{p_e}{E^d(p_e)} \left(\frac{G(a^*(p_e))-G(\underline{a}_0(p_e))}{V''(a^*(p_e))} - \frac{(a^*(p_e)-\underline{a}_0(p_e))^2}{p_e-V'(\underline{a}_0(p_e))} g\big(\underline{a}_0(p_e)\big) \right), & \text{if } p_e > \widehat{p}_e. \end{cases}$$

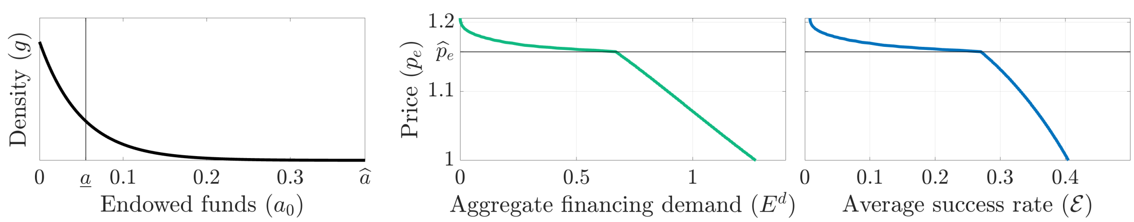
Proof. Immediate from the Leibniz rule and implicit function theorem. □

The second term in the price elasticity of demand when the price exceeds the kink point represents the extensive margin with a first-order effect: a marginally higher price increases the financing threshold by $\underline{a}_0'(p_e) = \frac{a^*(p_e)-\underline{a}(p_e)}{p_e-V'(\underline{a}_0(p_e))}$ and so entrepreneurs with endowed funds between $\underline{a}_0(p_e)$ and $\underline{a}_0(p_e) + \underline{a}_0'(p_e)\ dp_e$, with local density $g(\underline{a}_0(p_e))$, who used to raise financing in the amount $a^*(p_e) - \underline{a}_0(p_e)$, now forgo financing.

Figure 4: Aggregate demand for financing and success rate of innovation



(a) Uniform distribution of endowments



(b) Exponential distribution in endowments

When the extensive margin in entrepreneurial financing becomes first-order such that concentration arises, it increases the elasticity of aggregate innovation to financing costs. With a higher cost, more entrepreneurs endowed with small internal funds self-select into financial autarky, thereby constraining their subsequent innovation.

Corollary 3 (Kinked innovation)**.** *The unconditional average success rate of innovation, or firm entry rate equivalently,*

$$\mathcal{E}(p_e) \equiv \int_0^\infty \Lambda\Big(h\big(a_0 + e(a_0, p_e)\big)\Big)\ dG(a_0)$$

is kinked at $\widehat{p}_e$.

Furthermore, under concentration, *financial* heterogeneity across entrepreneurs ex-ante becomes a key factor in aggregate innovation. As Figure 4 illustrates, average success rate of innovation, unconditional on entrepreneurial financing or innovation spending, exhibits starkly different elasticity for $p_e > \widehat{p}_e$, depending on the distribution of endowed funds $G(a_0)$; this is because as price rises, entrepreneurs with the least funds self-select first into autarky.

Lastly, the above result has an immediate business-cycle implication. Supposing that the cost of entrepreneurial financing is countercyclical, aggregate innovation by startups may be more procyclical in the presence of concentration due to high price $p_e > \widehat{p}_e$ than without $p_e < \widehat{p}_e$.

4 Equilibrium

This section presents an equilibrium setup with a balanced growth path that embeds the main model in Section 3. In Section 4.1, I propose a model of financing frictions at the implementation stage that (i) gives rise to the properties of conditional value function upon success V_s as discussed in Section 3.1, (ii) allows for comparative statics in time sensitiveness of innovation, and (iii) still enables tractable aggregation. Utilizing these features, Sections 4.2 and 4.3 present a simple balanced-growth framework in general equilibrium with creative destruction.

4.1 Implementation and financing

In this part, I present the setup for startups upon successful development of an innovative technology. Time is discrete and, as alluded, infinite $t = 1, 2, \ldots$. Initial date $t = 0$ is for entrepreneurial financing and innovation.

A key goal of the present setup is to ensure that although incumbent firms are all identical and essentially static in equilibrium, remaining financial slack upon successful development of innovative technology a_1 still matters, that is, $V_s'(0) > 1$. The feature of homogeneous incumbents in equilibrium will greatly simplify analysis of how the complementarity between innovation and savings, as discussed in Section 3.1, affects aggregate efficiency in production.

Setup. A startup possessing an innovative technology must first invest into building productive assets that embodies the technology. To finance the investment, the successful startup may raise financing from an investor through an 'initial public offering.' I assume that upon an IPO, a firm is no longer subject to financing frictions; financing can be raised at any time, in any amount, and at a constant marginal cost of unity. But the IPO itself involves Nash bargaining with investors, where the startup's outside option is exclusion from financing.[3] I call a firm an incumbent if it has completed an IPO. I also call a firm that has succeeded in innovation, regardless of completing an IPO, an 'operating' firm.

Let $\{i_t\}_{t=1,2,\ldots}$ a sequence of investment by an operating firm. Capital stock evolves as

$$k_t = k_{t-1} + i_t,$$

where $k_0 \equiv 0$. There is no depreciation in capital.

I assume that the immediate date upon successful development $t = 1$ may be special, in the sense that it may be the only date when capital stock can be scaled without convex adjustment cost of investment. For $t = 2, 3, \ldots$, investment incurs a convex adjustment cost $\Phi(i_t/k_{t-1})k_{t-1}$, where $\Phi : \mathbb{R} \to \mathbb{R}_+$ satisfies $\Phi(0) = \Phi'(0) = 0 \leq \Phi''$; absence of such convex cost is nested by $\Phi'' = 0$ globally. The extent of this subsequent convex cost represents in a reduced-form how time-sensitive innovative technology is.

Operating firms have access to a Walrasian labor market at wage w, determined in equilibrium. Given capital stock that it owns k_t and labor that it hires l_t, an operating firm produces

$$y_t \equiv Zf(k_t, l_t),$$

where $Z > 0$ and $f : \mathbb{R}^2_+ \to \mathbb{R}_+$ is strictly increasing and satisfies decreasing returns to scale and Inada conditions. Output at date t can be spent on wage expense wl_t, but not on investment or convex adjustment cost at date t, which is irrelevant for incumbents due to frictionless financing.

At the end of each date $t = 1, 2, \ldots$, an operating firm's technology along with the associated capital stock may exogenously become obsolete with probability $\xi \in [0, 1)$. Obsolescence terminates a firm's operation, and its capital stock is scrapped with zero salvage value.

Characterizing an incumbent. By assumption, an incumbent firm enjoys frictionless financing. Given existing capital $\widetilde{k} > 0$, its value function when investment is subject to Φ, that is, for $t = 2, 3, \ldots$, solves

$$\widehat{V}_i(\widetilde{k}) \equiv \max_{k,l\geq 0} Zf(k,l) - (k - \widetilde{k}) - \Phi\Big(\frac{k - \widetilde{k}}{\widetilde{k}}\Big)\widetilde{k} - wl + \beta(1-\xi)\widehat{V}(k). \tag{6}$$

Since f has decreasing returns to scale, there exists a unique steady state. Steady-state

[3] For consistency, I may also assume that entrepreneurial financing in Section 3.2 involves bargaining with venture capitalists. Since Nash bargaining is cooperative, this would only affect payoffs and not allocation, unless there is prior entry choice by entrepreneurs involving a sunk cost. I do not directly model such prior entrepreneur entry choice in this paper.

inputs $(\overline{k}, \overline{l})$ and the associated value $\overline{V}_i$ without financing frictions solve

$$Zf_k(\overline{k}, \overline{l}) = 1 - \beta(1-\xi), \quad Zf_l(\overline{k}, \overline{l}) = w, \quad \text{and } \overline{V}_i = \frac{Zf(\overline{k}, \overline{l}) - w\overline{l}}{1 - \beta(1-\xi)}. \tag{7}$$

On date 1, absence of convex adjustment cost implies that optimal investment for a new incumbent is independent of funds a_1 and satisfies $i_1 = \overline{k}$. Accordingly, optimal labor demand also immediately attains the steady-state level, $l_1 = \overline{l}$, so that

$$V_i(a_1) = \overline{V}_i - (\overline{k} - a_1). \tag{8}$$

Therefore, $V_i' = 1$ globally.

A convenient feature of the setup is that, in equilibrium, all operating firms immediately attain and indefinitely remain at the same steady state until obsolescence, due to frictionless financing – and also to the absence of adjustment cost at date 1 and of depreciation in capital stock.[4] Note that Φ does not show up in incumbents' value in equilibrium. This is because incumbent firms are assumed to enjoy frictionless financing, which allows them to build innovative assets in a timely fashion.

Initial public offering and outside options. A successful startup with funds $a_1 \geq 0$ is assumed without loss to immediately gain frictionless access to financing from investors via an initial public offering. IPO involves bargaining: given $\theta \in (0, 1)$, it solves

$$\begin{aligned} &\max_{x_s \in [0,1]} \Big(x_s V_i(a_1) - V_o(a_1) \Big)^{\theta} \Big((1 - x_s) V_i(a_1) - 0 \Big)^{1-\theta} \\ \implies &V_s(a_1) = V_o(a_1) + \theta\Big(V_i(a_1) - V_o(a_1) \Big) = \theta V_i(a_1) + (1-\theta) V_o(a_1). \end{aligned} \tag{9}$$

Here, V_o is the startup's value function at date 1 without access to external financing:

$$V_o(a_1) \equiv \max_{k,l,d \geq 0} d + \beta\Big[(1-\xi)\widehat{V}_o\big(k, a'\big) + \xi a'\Big], \quad \text{such that} \tag{10}$$

$$a_1 \geq k,$$

$$a_1 + Zf(k, l) - k \geq wl + d,$$

[4] Positive depreciation would incentivize incumbents at $t = 1$ to invest beyond the steady state level free of adjustment costs and let the excess capital stock gradually depreciate.

and $a' = R(a_1 + Zf(k,l) - k - wl - d)$, where $\widehat{V}_o$ solves

$$\widehat{V}_o(\widetilde{k}, a) \equiv \max_{k,l,d \geq 0} d + \beta\Big[(1-\xi)\widehat{V}_o\big(k, a'\big) + \xi a'\Big], \quad \text{such that} \tag{11}$$

$$a \geq (k - \widetilde{k}) + \Phi\Big(\frac{k-\widetilde{k}}{\widetilde{k}}\Big)\widetilde{k},$$

$$a + Zf(k,l) - (k - \widetilde{k}) - \Phi\Big(\frac{k-\widetilde{k}}{\widetilde{k}}\Big)\widetilde{k} \geq wl + d,$$

and $a' = R\big(a + Zf(k,l) - (k-\widetilde{k}) - \Phi\big((k-\widetilde{k})/\widetilde{k}\big)\widetilde{k} - wl - d\big)$. This is the startup's reservation value against the failure of an IPO. Although an IPO never fails in equilibrium since $V_i(a) \geq V_o(a)$, what happens if it does affects the payoffs from bargaining – and hence the optimal innovation choice ex-ante.

Since Equation (9) is an identity, $V_s' = \theta + (1-\theta)V_o'$ and $V_s'' = (1-\theta)V_o''$ by Equation (8). That is, the slope and the curvature of V_s are tied to those of V_o.

Lemma 3 (Delayed installment)**.** *Optimal investment on date* 1 *without access to financing,* k_{o1}*, equals* $\min\{a_1, \overline{k}\}$*, and* $V_o'' < 0$ *on* $[0, \overline{k})$ *and* $V_o' = 1$ *on* $[\overline{k}, \infty)$*.*

Corollary 4 (Value of savings)**.** $V_s'' < 0$ *on* $[0, \overline{k})$ *and* $V_s' = 1$ *on* $[\overline{k}, \infty)$*.*

In reference to Section 3.1, therefore, $\overline{a}_1 \equiv \overline{k}$. Even though all successful startups immediately attain the long-term optimal capital stock in equilibrium due to cooperative bargaining, financial slack upon successful development of an innovative technology a_1 are still valuable $V_s'(a_1) > 1$ because it improves the startup's outside options vis-à-vis investors if the IPO were to fail. That is, a successful startup with more savings could better implement the innovative technology into operations without frictionless financing, such that investors can extract less in rents by offering frictionless financing to it. With $a_1 \geq \overline{a}_1 = \overline{k}$, the firm needs no external financing, so that $V_s'(a_1) = 1$.

4.2 Environment

In this part, I present a general equilibrium environment with a balanced-growth path that embeds the previous setups in Sections 3 and 4.1. Since the distribution of entrepreneurs' endowed funds $G(a_0)$ is a key factor for the elasticity of aggregate innovation (see Figure 4), I endogenize the distribution by entertaining household heterogeneity subject to canonical incomplete markets with persistent idiosyncratic labor endowment shocks. I entertain overlapping generations of entrepreneurs where each cohort of new entrepreneurs inherits the concurrent distribution of household assets.

I use $\tau \in \{0\} \cup \mathbb{N}$ to index the time horizon, or 'date,' in the economy, and reinterpret $t \in \{0\} \cup \mathbb{N}$ as, consistent with Sections 3 and 4.1, indexing the 'age' of individual entrepreneurs and their firms. Date τ is inside parenthesis, and age t is in subscript.

Households. There are a unit-mass continuum of infinitely-lived and ex-ante identical households. Each household has preference over stochastic consumption streams $\{c(\tau)\}_{\tau=0,1,\dots}$ and orders them according to

$$\mathbb{E}\left[\sum_{\tau=0}^{\infty} \widehat{\beta}^{\tau} u\big(c(\tau)\big)\right],$$

where $\widehat{\beta} \in (0,1)$ describes their time preference and u is a standard CRRA flow utility

$$u(c) \equiv \begin{cases} \frac{c^{1-\sigma}-1}{1-\sigma}, & \sigma \in (0,\infty) \setminus \{1\} \\ \log c, & \sigma = 1. \end{cases}$$

Each household is endowed with labor hours that they supply inelastically in a Walrasian market at wage $w(\tau)$. Labor endowment both grows over time due to labor-augmenting technology and fluctuates persistently due to idiosyncratic shocks: a household's labor endowment on τ is $l^h(\tau) = \gamma^{\tau} \widehat{l}_{s(\tau)}$ where $\gamma > 1$ is the intrinsic growth rate of the economy along a balanced growth path. The idiosyncratic component $\widehat{l}_{s(\tau)}$ is modeled as: given $0 \leq \widehat{l}_d < \widehat{l}_u$, $s(\tau) \in \{d, u\}$ follows a Markov chain such that for $\tau \in \mathbb{N}$ and $s \in \{d, u\}$,

$$\mathbb{P}\big(s(\tau) = s \mid s(\tau-1) \neq s\big) \equiv \pi_s \in (0,1).$$

I assume that $\mathbb{P}\big(s(0) = u\big) = \frac{\pi_u}{\pi_u + \pi_d}$, such that the aggregate labor supply at any date τ is $L(\tau) = \gamma^{\tau} L$ where

$$L \equiv \frac{\pi_u}{\pi_u + \pi_d} \widehat{l}_u + \frac{\pi_d}{\pi_u + \pi_d} \widehat{l}_d > 0.$$

Households are subject to standard incomplete markets in that they cannot insure against their idiosyncratic labor endowment shock. Without loss, households may only trade a single joint security that pools equity claims on all firms in the economy through intermediate entities including entrepreneurs, venture capitalists and investors. Households cannot borrow against the security. Let $a^h(\tau) \geq 0$ denote a household's holding of the security on date τ. On the next date, it yields a gross return $R(\tau+1)$, determined in equilibrium. Define aggregate household asset holdings

$$A^h(\tau) \equiv \int_0^{\infty} a \; dG(a \mid \tau),$$

where $G(\,\cdot \mid \tau)$ is the cumulative household asset distribution on date τ in equilibrium.

Entrepreneurs. On each date, a continuum of entrepreneurs in mass $\nu \in (0,1)$ are created. Entrepreneurs are intermediate 'entities,' who do not have their intrinsic preferences and instead serve as a pass-through between households and firms. A fraction ν of each household's assets are transferred to each new entrepreneur as their 'endowed funds' in a way that preserves the household asset distribution. These entrepreneurs issue equity claims in return, which are all pooled into the joint security and then distributed back to each household in the same amount as the transfer.

Assuming entrepreneurs as a pass-through rather than as agents that consume goods allows tractable aggregation of production even in the general equilibrium framework. Heterogeneity in new entrepreneurs' endowed funds subsequently induces heterogeneity in entrepreneurs' ownership stakes in incumbent firms. But this latter heterogeneity is inconsequential for market clearing since dividends paid to entrepreneurs are pooled into the joint security along with dividends paid to venture capitalist and investor, and then distributed across households proportional to their asset positions.

When a startup's innovation fails or an operating firm becomes obsolete, the entrepreneur that has established the firm immediately pays out their remaining funds to their equity claimants and gets terminated.

Venture capitalist and investor. There are a representative venture capitalist and a representative investor, who are, like entrepreneurs, intermediate entities serving as a pass-through and, unlike them, infinitely lived. Both provide additional funding to firms – venture capitalist to non-operating (i.e., pre-innovation) startups and investor to operating firms.

Venture capitalist incurs, in goods, a convex cost of aggregate entrepreneurial financing $\Psi\big(\nu E(\tau)/A^h(\tau)\big)A^h(\tau)$, where $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $\Psi(0) = \Psi'(0) = 0 < \Psi''$, and supplies funding to individual entrepreneurs in the pre-innovation stage through a Walrasian market, taking the price $p_e(\tau)$ as given. By the first-order condition,

$$p_e(\tau) = 1 + \Psi'\left(\frac{\nu E(\tau)}{A^h(\tau)}\right) A^h(\tau) \implies E(\tau) = \frac{1}{\nu}\Psi'^{(-1)}\left(\frac{p_e(\tau)-1}{A^h(\tau)}\right) A^h(\tau). \tag{12}$$

Investor, on the other hand, incurs no such cost and bargains with operating firms over granting permanent frictionless financing, but cannot finance entrepreneurs in the pre-innovation stage.

Technology. Production function for operating firms is Cobb-Douglas $f(k, l) \equiv k^{\alpha_k} l^{\alpha_l}$ with $\alpha_k, \alpha_l \in (0, 1)$ and $\alpha \equiv \alpha_k + \alpha_l \in (0, 1)$. Total factor productivity, while constant for each cohort of operating firms (whose entrepreneurs were born on date τ) over their productive life $\{\tau + t\}_{t=1,2,\ldots}$, grows across cohorts:

$$Z(\tau) \equiv \gamma^{(1-\alpha)\tau} Z,$$

given $Z > 0$. That is, a younger cohort of operating firms has a superior production technology. To ensure a balanced-growth path, assume that the success probability of startup innovation also evolves across cohorts as

$$\Lambda(h \mid \tau) \equiv 1 - \exp\left(-\frac{\lambda}{\gamma^\tau} h\right),$$

where $\lambda > 0$.

Relative convex adjustment cost of investment post initial operation Φ and cost function for entrepreneurial financing Ψ are quadratic:

$$\Phi(i) \equiv \phi \frac{i^2}{2}, \ \Psi(E) \equiv \psi \frac{E^2}{2},$$

where $\phi \geq 0$ and $\psi > 0$.

Creative destruction. All operating firms produce homogeneous output goods that households consume, with their price normalized to unity. The mass of operating firms is fixed at unity. On any date, startups that succeed in innovation replace operating firms of earlier cohorts at random. Given the average firm entry rate of startups $\mathcal{E} \in [0, 1]$ as defined in Corollary 3, the rate of obsolescence in Section 4.1 thus satisfies

$$\xi \equiv \nu \mathcal{E} \in [0, 1).$$

Startup acquisitions. An incumbent firm about to be replaced by a startup following a successful innovation may, with probability $\zeta \in [0, 1]$, offer to acquire the startup's new technology through bargaining. The incumbent firm cannot operate the new technology as efficiently as the startup: given the startup is of cohort τ and the incumbent of cohort $\widetilde{\tau} < \tau$, the acquisition reduces the total factor productivity of the new technology down to $\omega Z(\tau) + (1 - \omega) Z(\widetilde{\tau})$, $\omega \in (0, 1)$ and allows the incumbent's existing capital stock to avoid obsolescence up to a fraction $1 - \omega$ by incurring a one-time upgrading cost of $\left(\gamma^{\tau - \widetilde{\tau}} - 1\right)\kappa$, $\kappa > 0$, per unit of the retained capital stock. Upon acquisition, the merged firm enjoys permanent frictionless financing and, just on that date, does not incur convex adjustment cost of investment.

For tractability, I ensure that there is no gain in the model from any incumbent firm acquiring a startup that has completed an IPO and is about to replace it.[5]

[5]This is conservative with respect to the effects of startup acquisitions. When technological gap is small and replacement requires scrapping of existing capital, it is plausible that replacement destroys value, such that literal 'killer' acquisitions, with $\omega = \kappa = 0$, may be efficient.

Assumption 3 (No gains from startup acquisitions)**.** *Parameters* $(\alpha_k, \alpha_l, \gamma, \kappa)$ *satisfy*

$$\frac{1-\alpha_l}{\alpha_k} \geq \frac{\gamma}{\gamma-1} - \kappa.$$

Given the assumption, acquisitions are relevant only for a startup that has failed in IPO – and hence loses financing access – following a successful innovation. Suppose that such a startup at age $t = 1$ and belonging to a cohort τ has funds $a_1 \geq 0$, and the incumbent firm about to be replaced by it belongs to a cohort $\widetilde{\tau} < \tau$. Acquisition represents a positive gain if and only if $V_o^{acq}(\tau \mid \widetilde{\tau}) + a_1 > V_o(a_1 \mid \tau)$ where

$$V_o^{acq}(\tau \mid \widetilde{\tau}) \equiv \omega \overline{V}_i(\tau) + (1-\omega)\overline{V}_i(\widetilde{\tau}) - \omega \overline{k}(\tau) - (1-\omega)\big(\gamma^{\tau-\widetilde{\tau}} - 1\big)\kappa \overline{k}(\widetilde{\tau}),$$

in which case, given $\widetilde{\theta} \in (0, 1]$, bargaining for it solves

$$\max_{x\in[0,1]} \Big(x\big(V_o^{acq}(\tau \mid \widetilde{\tau}) + a_1\big) - V_o(a_1 \mid \tau)\Big)^{\widetilde{\theta}} \Big((1-x)\big(V_o^{acq}(\tau \mid \widetilde{\tau}) + a_1\big) - 0\Big)^{1-\widetilde{\theta}}.$$

Therefore, a startup of cohort τ with funds $a_1 \geq 0$, following a successful innovation replacing an incumbent firm of cohort $\widetilde{\tau} < \tau$ but upon an unsuccessful IPO, has value

$$\widetilde{V}_o(a_1 \mid \tau, \widetilde{\tau}) = V_o(a_1 \mid \tau) + \zeta\widetilde{\theta}\Big(V_o^{acq}(\tau \mid \widetilde{\tau}) - \big(V_o(a_1 \mid \tau) - a_1\big)\Big)^{+}.$$

Since startup acquisitions do not occur in equilibrium, I set $\widetilde{\theta} = 1$ without loss. Then, the conditional value function upon successful innovation is

$$\begin{aligned} V_s(a_1 \mid \tau) = {} & \theta\Big(\overline{V}_o(\tau) - \big(\overline{k}(\tau) - a_1\big)\Big) + (1-\theta)V_o(a_1 \mid \tau) \\ & + (1-\theta)\zeta \sum_{t=1}^{\infty} \nu\mathcal{E}(1-\nu\mathcal{E})^{t-1}\Big(V_o^{acq}(\tau \mid \widetilde{\tau}) - \big(V_o(a_1 \mid \tau) - a_1\big)\Big)^{+} \end{aligned}$$

along a balanced-growth path. In the baseline analysis of general equilibrium, I suppress startup acquisitions $\zeta = 0$; the above then coincides with Equation (9).[6]

Balanced growth. Forthcoming.

4.3 Aggregation

Because each cohort of incumbent firms remain at an identical steady state and the production function is homothetic, aggregation of production is straightforward. By Equation (7), the steady-state production inputs for each cohort $\{(\overline{k}(\tau), \overline{l}(\tau))\}_{\tau=0,1,\ldots}$

[6]Given $\zeta > 0$, $V_s(\,\cdot \mid \tau)$ does not satisfy Assumption 2 due to the kinks within the summation.

grow by a factor of γ across cohorts. Therefore, aggregate output is

$$\begin{aligned}
Y(\tau) &= \sum_{t=1}^{\infty} \nu\mathcal{E}(1-\nu\mathcal{E})^{t-1} Z(\tau-t)\bar{k}(\tau-t)^{\alpha_k}\bar{l}(\tau-t)^{\alpha_l} \\
&= \gamma^{\tau} Z \left(\frac{\nu\mathcal{E}}{\nu\mathcal{E}+(\gamma-1)}\right) \bar{k}(0)^{\alpha_k}\bar{l}(0)^{\alpha_l}.
\end{aligned}$$

Aggregate labor demand is $L^d(\tau) = \frac{\alpha_l}{w} Y(\tau)$.

Substituting the cohort-specific steady-state conditions in Equation (7) and wage w that clears labor market $L^d(\tau) = L(\tau)$, incumbent firms' production inputs are

$$\begin{aligned}
\bar{k}(\tau) &= \gamma^{\tau} Z^{\frac{1}{1-\alpha_k}} \left(\frac{\alpha_k}{1-\beta(1-\nu\mathcal{E})}\right)^{\frac{1}{1-\alpha_k}} \left(\frac{\nu\mathcal{E}+(\gamma-1)}{\nu\mathcal{E}} L\right)^{\frac{\alpha_l}{1-\alpha_k}}, \\
\bar{l}(\tau) &= \gamma^{\tau} \left(\frac{\nu\mathcal{E}+(\gamma-1)}{\nu\mathcal{E}} L\right);
\end{aligned}$$

both fractions above decrease in $\mathcal{E}$. Aggregate production and investment are

$$Y(\tau) = \gamma^{\tau} Z^{\frac{1}{1-\alpha_k}} L^{\frac{\alpha_l}{1-\alpha_k}} \left(\frac{\alpha_k}{1-\beta(1-\nu\mathcal{E})}\right)^{\frac{\alpha_k}{1-\alpha_k}} \left(\frac{\nu\mathcal{E}}{\nu\mathcal{E}+(\gamma-1)}\right)^{\frac{1-\alpha}{1-\alpha_k}}, \tag{13}$$

and $I(\tau) = \nu\mathcal{E}\bar{k}(\tau-1)$. Lastly, wage is

$$w = \alpha_l Z^{\frac{1}{1-\alpha_k}} L^{-\frac{1-\alpha}{1-\alpha_k}} \left(\frac{\alpha_k}{1-\beta(1-\nu\mathcal{E})}\right)^{\frac{\alpha_k}{1-\alpha_k}} \left(\frac{\nu\mathcal{E}}{\nu\mathcal{E}+(\gamma-1)}\right)^{\frac{1-\alpha}{1-\alpha_k}}.$$

As seen, the average firm entry rate $\mathcal{E}$ determines aggregate production in equilibrium through three channels. First, more entry implies faster replacement of legacy capital, and so individual incumbents invest less into capital in anticipation of the creative destruction. Second, more entry implies more productive firms. This increases aggregate labor demand, but the supply is inelastic. Hence, wage rises in equilibrium, which lowers labor employed by each surviving incumbent firm of a given cohort. Since capital and labor are complements by Cobb-Douglas production, this also reduces individual firms' capital stock. Third, there are more of more productive firms and less of less productive firms. This compositional effect dominates the second effect and increases the aggregate production.[7]

Heterogeneous innovation. Endogenous heterogeneity matters in the aggregation of innovation spending and its financing. Define aggregate innovation spending

$$H(\tau) \equiv \int h\Big(a(\gamma^{\tau} a_0 \mid \tau) \mid \tau\Big)\, dG(a_0),$$

[7] The last exponent in Equation (13) equals $1 - \frac{\alpha_l}{1-\alpha_k}$; the first term is the composition effect.

and time-invariant average success rate

$$\mathcal{E} \equiv \int 1 - \exp\left(-\frac{\lambda}{\gamma^\tau} h\big(a(\gamma^\tau a_0 \mid \tau) \mid \tau\big)\right) \, dG(a_0).$$

Since success probability is strictly concave in innovation spending, Jensen's inequality implies that any heterogeneity in innovation spending implies aggregate inefficiency in innovation. Define excess innovation spending

$$\Delta(\tau) \equiv H(\tau) - \Lambda^{(-1)}(\mathcal{E} \mid \tau) \geq 0,$$

which is strictly positive if and only if the distribution of innovation spending across startups is nondegenerate.

Rest forthcoming.

5 Quantitative Analysis

Forthcoming.

6 Conclusion

This paper presents a model of startup innovation and financing where financing frictions in the implementation stage concentrate innovation across startups. Financing frictions give rise to a complementarity, statically and within the individual startup, between innovation spending and savings, which induces locally *increasing* returns to scale in startup funding. The simplicity of the core model suggests that this mechanism is likely ubiquitous.

This fundamental complementarity between technology and financing endogenously gives rise to concentration in innovation. The key intermediate channel is costs of entrepreneurial financing that induce only well-funded entrepreneurs to leverage the increasing returns to scale through additional financing. This paper finds that the extensive margin in entrepreneurial financing has a first-order effect up to the distribution of entrepreneurs' financial positions, making aggregate innovation highly elastic to financing conditions. The paper proceeds to general equilibrium analysis based on the insight, explaining the secular rise in concentration through time-sensitiveness of innovative technology, highlighting the negative distributional consequences of prohibiting "killer acquisitions," and also identifying a key amplification channel through household asset distribution.

Entrepreneurs' seemingly discrete choice of whether to aim for a transformative innovation or to remain at a small business has often been viewed as mainly representing ex-ante heterogeneity in preferences or innate business skills. This paper suggests that the choice may also be substantially influenced by their financial positions ex-ante that do not have much to do with such primitive factors. As explored, this change in perspective may have far-reaching implications on how to understand innovation and growth in the macroeconomy.

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Appendix A Omitted Proofs

A.1 Lemma 1 (Monotonicity of innovation and savings)

Proof. First, $h(a) < a$ for $a > 0$. This is because, by Equation (2), $h(a) = a$ if and only if $\Lambda'(a)\beta V_s(0) \geq 1 + \Lambda(a)\Big(V_s'(0) - 1\Big)$, contradicting the Inada condition on V_s.

By Equation (2), it holds that for $a > 0$, $h(a) = 0$ if and only if

$$\Lambda'(0)\beta\Big(V_s(Ra) - Ra\Big) \leq 1.$$

The left-hand side is increasing in $a \leq R^{-1}\overline{a}_1$ and, by Assumption 1, strictly exceeds unity on the right-hand side for a high enough a. This establishes the claim on $\underline{a} \geq 0$.

From Equation (4), $h'(a) \in [0, 1)$ on $(\underline{a}, \infty)$, so that $a_1' \in (0, R]$ on $[\underline{a}, \infty)$. If $a_1(a) < \overline{a}_1$, then, by the definition of $\overline{a}_1$, the numerator in Equation (4) is positive, so that $h'(a) > 0$. Otherwise, $V_s'\big(a_1(a)\big) - 1 = V_s''\big(a_1(a)\big) = 0$ so that $h'(a) = 0$.[8] $\square$

A.2 Proposition 2 (More increasing returns for less funds)

Proof. The proof proceeds as follows. I first obtain conditions that must be satisfied when $V'' = 0$. Utilizing these, I then show that under Assumption 2, $V''' < 0$ when $V'' = 0$. Thus, V''crosses zero on $(\underline{a}, \overline{a})$ at most once and only from above, and hence establishing the first claim (except that $\widehat{a} < \overline{a}$). The second claim will be addressed next. The remaining part $\widehat{a} < \overline{a}$ is shown at the end.

To ease notation, define $W(a_1) \equiv V_s(a_1) - a_1$ and omit policy functions $h(a)$ and $a_1(a)$ inside Λ and W. Note that

$$\begin{aligned} V'' &= \Lambda W'' R(1 - h') + \Lambda' h' W' \\ &= \Lambda W'' R \frac{\Lambda' W' - \beta \Lambda'' W}{2\Lambda' W' - R\Lambda W'' - \beta\Lambda'' W} + \Lambda' W' \frac{\Lambda' W' - R\Lambda W''}{2\Lambda' W' - R\Lambda W'' - \beta\Lambda'' W} \end{aligned}$$

by Equation (4). Multiply by $2\Lambda' W' - R\Lambda W'' - \beta\Lambda'' W > 0$ and rearrange to get: $V'' < 0$ if and only if

$$\frac{-\Lambda''}{\Lambda'}\frac{W}{W'} > \frac{\Lambda'}{\Lambda}\frac{W'}{-W''}. \tag{A.1}$$

Therefore, $V'' = 0$ on $(\underline{a}, \overline{a})$ if and only if

$$\frac{-\Lambda''}{\Lambda'}\frac{W}{W'} = \frac{\Lambda'}{\Lambda}\frac{W'}{-W''} \equiv \chi > 0. \tag{A.2}$$

[8] In case $\overline{a}_1 < \infty$ and $\Lambda''\big(h(\overline{a})\big) = 0$, the expression for $h'(a)$ in Equation (4) is undefined because the first-order condition in Equation (2) fails to determine h so that implicit function theorem cannot be invoked. h is still constant above $\overline{a}$ because it has zero marginal benefit.

Suppose that $V''(\widehat{a}) = 0$ for some $\widehat{a} \in (\underline{a}, \overline{a})$. From Inequality (A.1) which holds as equality at $\widehat{a}$, $V'''(\widehat{a}) < 0$ if and only if

$$0 < \frac{d}{da}\left(\frac{-\Lambda''}{\Lambda'}\frac{W}{W'} - \frac{\Lambda'}{\Lambda}\frac{W'}{-W''}\right)\Big|_{a=\widehat{a}}.$$

Differentiate and multiply by $2 + R\frac{\Lambda}{\Lambda'}\frac{-W''}{W'} + \beta\frac{-\Lambda''}{\Lambda'}\frac{W}{W'} > 0$: $V'''(\widehat{a}) < 0$ if and only if

$$\begin{aligned}
0 &< \frac{\Lambda''^2 - \Lambda'''\Lambda'}{\Lambda'^2}\frac{W}{W'}\left(1 + R\frac{\Lambda}{\Lambda'}\frac{-W''}{W'}\right) + \frac{-\Lambda''}{\Lambda'}\left(1 + \frac{(-W'')W}{W'^2}\right)\left(1 + \beta\frac{-\Lambda''}{\Lambda'}\frac{W}{W'}\right) \\
&\quad + \left(\frac{-\Lambda''}{\Lambda} + \frac{\Lambda'^2}{\Lambda^2}\right)\frac{W'}{-W''}\left(1 + R\frac{\Lambda}{\Lambda'}\frac{-W''}{W'}\right) - \frac{\Lambda'}{\Lambda}\left(\frac{W'''W'}{W''^2} - 1\right)\left(1 + \beta\frac{-\Lambda''}{\Lambda'}\frac{W}{W'}\right) \\
&= \left(1 + \frac{R}{\chi}\right)\frac{\Lambda''^2 - \Lambda'''\Lambda'}{\Lambda'^2} + (1 + \beta\chi)\chi\left((1 + R)\frac{W'}{W} - (2 + R)\frac{W''}{W'} + \frac{-W'''}{-W''}\right) \\
&= \left(1 + \frac{R}{\chi}\right)\frac{d}{dh}\frac{-\Lambda''}{\Lambda'}\Big|_{h=h(\widehat{a})} + (1 + \beta\chi)\chi\frac{d}{da_1}\log\left(\frac{-W''}{W'}\right)\left(\frac{W}{W'}\right)^{1+R}\Big|_{a_1=a_1(\widehat{a})},
\end{aligned}$$

where the first equality utilizes Equation (A.2) to substitute out all Λ except for the first fraction. Assumption 2 then ensures that the above inequality indeed holds. This proves the single-crossing from above.

If $\underline{a} > 0$, then the right-hand side of Inequality (A.1) diverges to infinity as $a \to \underline{a}^-$ since $\Lambda\big(h(\underline{a})\big) = \Lambda(0) = 0$. Hence, $\widehat{a} > \underline{a}$.

Lastly regarding the claim that $\widehat{a} < \overline{a}$, Inequality (A.1) can be rearranged to: $V'' < 0$ if and only if

$$1 < \frac{-\Lambda''\Lambda}{\Lambda'^2}\frac{-W''W}{W'^2}.$$

The first fraction is increasing by Assumption 2 and $\Lambda'(0) > 0$. Note that Assumption 1 implies that W is a bounded function. As an intermediate step, I prove the following: if a twice differentiable function $F : \mathbb{R}_+ \to \mathbb{R}_+$ with $F' \geq 0 \geq F''$ and $F'(0) > 0$ is bounded from above, then $D \equiv \frac{-F''F}{F'^2} \geq 0$ is unbounded in the tail. If this holds, there exists a, sufficiently high towards, $\overline{a}$ where the above inequality holds, such that $\widehat{a} < \overline{a}$.

Suppose by way of contraposition that $D \geq 0$ is bounded in the tail: there exist $\varepsilon > 0$ and $M > 0$ such that $D(x) \leq M$ for $x \geq \varepsilon$. Letting $d(x) \equiv 1/F'(x)$, for $x \geq \varepsilon$,

$$d'(x) = \frac{-F''(x)}{F'(x)^2} = \frac{D(x)}{F(x)} \leq \frac{M}{F(\varepsilon)} \equiv C \in (0, \infty).$$

Therefore, $1/F'(x) = d(x) \leq d(\varepsilon) + C(x - \varepsilon)$ for $x \geq \varepsilon$, and hence

$$\begin{aligned}
F(x) &= F(\varepsilon) + \int_\varepsilon^x F'(t)\, dt \geq F(\varepsilon) + \int_\varepsilon^x \frac{1}{Ct + d(\varepsilon) - C\varepsilon}\, dt \\
&= F(\varepsilon) + \frac{1}{C}\Big(\log(Cx + d(\varepsilon) - C\varepsilon) - \log d(\varepsilon)\Big) \to \infty
\end{aligned}$$

as $x \to \infty$. Hence, the claim is proved. □

A.3 Lemma 2 (Demand under convexity)

Proof. Let $p_e \in (V'(\underline{a}), V'(\widehat{a}))$. For $a_0 \geq \widetilde{a}(p_e)$, $e(a_0, p_e) = \max\{a^*(p_e) - a_0, 0\}$ since $V''\big(\widetilde{a}(p_e)\big) > 0$. Consider $a_0 < \widetilde{a}(p_e)$. Then,

$$V\big(a^*(p_e)\big) - p_e\big(a_u(p_e) - a_0\big) - V\big(a_0\big) = \int_{a_0}^{a^*(p_e)} \big(V'(a) - p_e\big)\, da$$
$$= \int_{\widetilde{a}(p_e)}^{a^*(p_e)} \big(V'(a) - p_e\big)\, da - \int_{a_0}^{\widetilde{a}(p_e)} \big(p_e - V'(a)\big)\, da \equiv \mathcal{B}(p_e) - \mathcal{L}(a, p_e).$$

Since $p_e > V'$ on $[0, \widetilde{a}(p_e))$ and $V' > p_e$ on $(\widetilde{a}(p_e), a^*(p_e))$, the integrands of both $\mathcal{B}$ and $\mathcal{L}$ from the second line are positive on their respective interiors.

Since $\mathcal{L}$ is decreasing in $a_0 \in [0, \widetilde{a}(p_e))$, $V\big(a^*(p_e)\big) - p_e\big(a^*(p_e) - a_0\big) \geq V\big(a_0\big)$ implies that $V\big(a^*(p_e)\big) - p_e\big(a^*(p_e) - a_0'\big) > V\big(a_0'\big)$ for $a_0' \in (a_0, \widetilde{a}(p_e))$. This proves the existence of $\widehat{a}_0(p_e)$. Since $V'' > 0$ on $(\underline{a}, \widehat{a})$, $\mathcal{B}'(p_e) < 0$ and $\mathcal{L}_{p_e}(a_0, p_e) > 0$. In particular, $\mathcal{B}(p_e) \to 0$ as $p_e \to V'(\widehat{a})$, while $\mathcal{L}\big(0, p_e\big) \to 0$ as $p_e \to V'(\underline{a})$. This proves the existence of $\widehat{p}_e$.

Monotonicity of $\widehat{a}_0(p_e)$ is from implicitly differentiating $\mathcal{B}(p_e) = \mathcal{L}\big(\widehat{a}_0(p_e), p_e\big)$ since $\mathcal{L}_a(a_0, p_e) < 0$, and its limit is due to the limit of $\mathcal{B}$. Monotonicity and limit of $a^*(p_e)$ are obvious. □

A.4 Lemma 3 (Delayed installment)

Proof. With respect to Equation (11), let $\widehat{\mu}_d$, $\widehat{\mu}_k$ Lagrange multipliers for the self-financing constraint for $d \geq 0$ and the investment-financing constraint $a \geq (k - \widetilde{k}) + \Phi\big((k - \widetilde{k})/\widetilde{k}\big)\widetilde{k}$, respectively; the budget constraint $a + Zf(k, l) - (k - \widetilde{k}) - \Phi\big((k - \widetilde{k})/\widetilde{k}\big)\widetilde{k} - wl \geq d$ does not bind since (i) $Zf(k, l) > wl$ from $Zf_l(k, l) = w$ and $f_{ll} < 0$, and (ii) $R = \beta^{-1}$. It is easily verified that $\widehat{V}_o$ admits the same steady-state capital stock $\overline{k}$ as under frictionless financing $\widehat{V}_i$. The first-order condition for dividends d and the envelope conditions give

$$\widehat{V}_{ok}(\widetilde{k}, a) = \widehat{V}_{oa}(\widetilde{k}, a) \left(1 + \Phi'\Big(\frac{k - \widetilde{k}}{\widetilde{k}}\Big)\frac{k}{\widetilde{k}} - \Phi\Big(\frac{k - \widetilde{k}}{\widetilde{k}}\Big)\right). \tag{A.3}$$

The proof is structured as follows: (i) $k_{o1} = \max\{a_1, \overline{k}\}$; (ii) $\widehat{V}_o$ is weakly concave; (iii) $V_o''(a_1) < 0$ if and only if $a_1 < \overline{k}$.

Claim (i): $k_{o1} = \max\{a_1, \overline{k}\}$. In Equation (10), let μ_d, μ_k Lagrange multipliers for

$d \geq 0$ and $a_1 \geq k$, respectively. The first-order condition for dividends is

$$1 + \mu_d = (1 - \xi)\widehat{V}_{oa}(k, a') + \xi.$$

Substituting Equation (A.3) along with the above into the Euler equation give

$$\mu_k = \beta\big(1 - \xi + \mu_d\big)\left(1 + \Phi'\Big(\frac{k' - k}{k}\Big)\frac{k'}{k} - \Phi\Big(\frac{k' - k}{k}\Big)\right) + \big(1 + \mu_d\big)\big(Zf_k(k, l) - 1\big).$$

Suppose $a_1 < \overline{k}$ so that $k_{o1} = k < \overline{k}$. By convergence, $k' > k$. The assumed properties of Φ then imply that

$$\Phi'\Big(\frac{k' - k}{k}\Big)\frac{k'}{k} \geq \Phi'\Big(\frac{k' - k}{k}\Big)\frac{k' - k}{k} \geq \Phi\Big(\frac{k' - k}{k}\Big).$$

Also, $Zf_k(k, l) - 1 > -\beta(1 - \xi)$ by Equation (7) since, letting $l(k)$ solve $Zf_l(k, l) = w$,

$$\begin{aligned} f_k(\overline{k}, l(\overline{k})) - f_k(k, l(k)) &= \int_k^{\overline{k}} f_{kk}(\widehat{k}, l(\widehat{k})) + f_{kl}(\widehat{k}, l(\widehat{k}))l'(\widehat{k})\ d\widehat{k} \\ &= \int_k^{\overline{k}} f_{kk}(\widehat{k}, l(\widehat{k})) - \frac{f_{kl}^2}{f_{ll}}(\widehat{k}, l(\widehat{k}))\ d\widehat{k} < 0, \end{aligned}$$

where $l' = -f_{kl}/f_{ll}$ by implicit function theorem. The last inequality is because f is decreasing returns to scale, which implies that its Hessian matrix has a positive determinant. Therefore,

$$\mu_k \geq \beta\Big(1 - \xi + \mu_d - (1 - \xi)(1 + \mu_d)\Big) = \beta\xi\mu_d \geq 0,$$

and so $k_{o1} = a_1$. If $a_1 \geq \overline{k}$, frictionless steady state is immediately attained $k_{o1} = \overline{k}$.

Claim (ii)*: $\widehat{V}_o$ is weakly concave.* Take two distinct state vectors $S_o^0 \equiv (\widetilde{k}^0, a^0) \neq (\widetilde{k}^1, a^1) \equiv S_o^1$. Given weight $\eta \in (0, 1)$, let $(\widetilde{k}^\eta, a^\eta) \equiv S_o^\eta \equiv (1 - \eta)S_o^0 + \eta S_o^1$. Let $k_o(S_o)$ optimal capital stock given state S_o; this pins down labor choice by $Zf_l(k, l(k)) = w$, and dividends are set to zero without loss since $R = \beta^{-1}$. Also, let $a_o'(S_o)$ optimal choice of funds in the next date subject to $R\big(a + Zf(k, l) - (k - \widetilde{k}) - \Phi(k - \widetilde{k}) - wl\big) \geq a_o'$, which trivially binds. That is, given state S_o today, the state vector next date is $(k_o(S_o), a_o'(S_o))$.

Define $k_o^\eta \equiv (1 - \eta)k_o(S_o^0) + \eta k_o(S_o^1)$ and $a_o'^\eta \equiv (1 - \eta)a_o'(S_o^0) + \eta a_o'(S_o^1)$; these do not equal $k_o(S_o^\eta)$ and $a_o'(S_o^\eta)$ in general. Given S_o^η, the weighted capital stock k_o^η is feasible:

$$
\begin{aligned}
k_o^\eta - \widetilde{k}^\eta + \Phi\Big(\frac{k_o^\eta - \widetilde{k}^\eta}{\widetilde{k}^\eta}\Big)\widetilde{k}^\eta &= (1-\eta)\big(k_o(S_o^0) - \widetilde{k}^0\big) + \eta\big(k_o(S_o^1) - \widetilde{k}^1\big) \\
&\quad + \Phi\left(\frac{(1-\eta)\big(k_o(S_o^0) - \widetilde{k}^0\big) + \eta\big(k_o(S_o^1) - \widetilde{k}^1\big)}{(1-\eta)\widetilde{k}^0 + \eta\widetilde{k}^1}\right)\big((1-\eta)\widetilde{k}^0 + \eta\widetilde{k}^1\big) \\
\leq (1-\eta)\left(k_o(S_o^0) - \widetilde{k}^0 + \Phi\Big(\frac{k_o(S_o^0) - \widetilde{k}^0}{\widetilde{k}^0}\Big)\widetilde{k}^0\right) &+ \eta\left(k_o(S_o^1) - \widetilde{k}^1 + \Phi\Big(\frac{k_o(S_o^1) - \widetilde{k}^1}{\widetilde{k}^1}\Big)\widetilde{k}^1\right) \\
\leq (1-\eta)a^0 + \eta a^1 = a_\eta,
\end{aligned}
\tag{A.4}
$$

because $(i, \widetilde{k}) \mapsto \Phi(i/\widetilde{k})\widetilde{k}$ is convex. The weighted funds $a_o'^\eta$ are also feasible since

$$
\begin{aligned}
a_o'^\eta &= (1-\eta)a_o'(S_o^0) + \eta a_o'(S_o^1) \\
&= (1-\eta)R\Bigg(a^0 + Zf\big(k_o(S_o^0), l(k_o(S_o^0))\big) - wl(k_o(S_o^0)) \\
&\qquad\qquad - \big(k_o(S_o^0) - \widetilde{k}^0\big) - \Phi\Big(\frac{k_o(S_o^0) - \widetilde{k}^0}{\widetilde{k}^0}\Big)\widetilde{k}^0\Bigg) \\
&\quad + \eta R\Bigg(a^1 + Zf\big(k_o(S_o^1), l(k_o(S_o^1))\big) - wl(k_o(S_o^1)) \\
&\qquad\qquad - \big(k_o(S_o^1) - \widetilde{k}^1\big) - \Phi\Big(\frac{k_o(S_o^1) - \widetilde{k}^1}{\widetilde{k}^1}\Big)\widetilde{k}^1\Bigg) \\
&< R\Bigg(a^\eta + Zf\big(k_o^\eta, l(k_o^\eta)\big) - wl(k_o^\eta) - \big(k_o^\eta - \widetilde{k}^\eta\big) - \Phi\Big(\frac{k_o^\eta - \widetilde{k}^\eta}{\widetilde{k}^\eta}\Big)\widetilde{k}^\eta\Bigg).
\end{aligned}
\tag{A.5}
$$

The second equality is since the constraint on a_o' trivially binds. The inequality is because (i) $k \mapsto Zf(k, l(k)) - wl(k)$ is strictly concave in k from the envelope condition with respect to choosing optimal l given k and from f having decreasing returns to scale, and (ii) $(i, \widetilde{k}) \mapsto -\Phi(i/\widetilde{k})\widetilde{k}$ is concave. By a standard reasoning in dynamic programming, therefore, $\widehat{V}_o$ is weakly concave.[9]

Claim (iii): *$V_o''(a_1) < 0$ if and only if $a_1 < \overline{k}$.* Let $a_1 < \overline{k}$. By Claim (ii),

$$
V_o(a_1) = \beta\Big((1-\xi)\widehat{V}_o\big(a_1, R\big(Zf(a_1, l(a_1)) - wl(a_1)\big)\big) + \xi R\big(Zf(a_1, l(a_1)) - wl(a_1)\big)\Big),
$$

[9]The reasoning involves defining a standard 'maximizing' contraction map T on the space of continuous bounded functions and showing that if f is weakly concave, then Tf is also weakly concave. If it can also be shown that Tf is always strictly concave, then value function, as a unique fixed point of T, is globally strictly concave; this is not the case in the present setup.

and thus, omitting $\big(a_1, R(Zf(a_1, l(a_1)) - wl(a_1))\big)$ in $\widehat{V}_o$ and $(a_1, l(a_1))$ in f,

$$\begin{aligned} V_o'' &= \beta(1-\xi)\Big(\widehat{V}_{okk} + 2\widehat{V}_{oka}RZf_k + \widehat{V}_{oaa}(RZf_k)^2\Big) + \Big((1-\xi)\widehat{V}_{oa} + \xi\Big)Z\left(f_{kk} - \frac{f_{kl}^2}{f_{ll}}\right) \\ &< \beta(1-\xi)\Big(\widehat{V}_{okk} + 2\widehat{V}_{oka}RZf_k + \widehat{V}_{oaa}(RZf_k)^2\Big). \end{aligned} \tag{A.6}$$

The strict inequality above is again due to f having decreasing returns to scale.

Since $\widehat{V}_o$ is weakly concave, its Hessian matrix has a nonnegative determinant:

$$\widehat{V}_{okk}\widehat{V}_{oaa} - \widehat{V}_{oka}^2 \geq 0,$$

and therefore, $\widehat{V}_{oka} \leq \sqrt{\widehat{V}_{okk}\widehat{V}_{oaa}}$. Substituting it into Inequality (A.6),

$$\begin{aligned} V_o'' &< \beta(1-\xi)\left(\widehat{V}_{okk} + 2RZf_k\sqrt{\widehat{V}_{okk}\widehat{V}_{oaa}} + (RZf_k)^2\widehat{V}_{oaa}\right) \\ &= -\beta(1-\xi)\left(\sqrt{-\widehat{V}_{okk}} - RZf_k\sqrt{-\widehat{V}_{oaa}}\right)^2 \leq 0. \end{aligned}$$

If $a_1 \geq \overline{k}$, then $V_o(a_1) = \overline{V}_i + (a_1 - \overline{k})$ and so $V_o'(a_1) = 1$. The proof is complete. $\square$