Project Check Point

Hanjun Li
910083045

April 5, 2017

1 Proposed Project

The proposed project is to examine the method described by [Y. Wang, B. Liu, and Y. Tong 2012] to reconstruct surfaces from a discrete representation of the first and second fundamental forms by solving a linear system. And the proposed plan includes: First carefully following the mathematical derivation of the discrete fundamental theorem of surface in the paper, and understanding the advantages of the methods over existed ones; Second, implementing the reconstruction algorithm and understanding the engineering decisions included; Finally, experimenting with possible modification of the algorithm, and exploring applications of the algorithm. this project.

2 Report of Progress

Following the proposed plan, the first step is to understand the discrete fundamental theorem of surface. In plain words, the theorem says that given the connectivity of a set of triangles, the edge lengths, as the discrete first fundamental form, and the dihedral angles between adjacent triangles, as the discrete second fundamental form, we can uniquely determine a surface mesh, up to translation and rotation. The theorem can be illustrated and understood by the following steps to reconstruct the surface.

First, we can use the law of cosines to compute the angles of each triangle face of the surface, from its edge lengths. Second, we can assume a local frame of each triangle by using the normalized first edge as e_1, the face normal as e_3, and $e_3 \times e_1$ as e_2. The first edge can be consistently determined by the triangle connectivity data. (i.e. use the first vertex to second vertex of each triangle as the first edge.) Although we don't have a concrete description of each local frame, we can compute the coordinate of the edge vectors with respect to each local frame. (The first edge always has $(edge\_len, 0, 0)$ as its coordinate. The coordinates of the other 2 edges can be calculated as cosine and sine of the inner angles computed in step one.) Third, we can calculate the rotation matrices that convert one local frame to the ones of its adjacent triangles, as a product of three rotations. The leftmost rotation rotates the local frame f_A of triangle A around z axis, the normal direction, so that e_1 aligns with the edge it shares with triangle B, e_{AB}. The middle rotation rotate the frame around x axis, now aligned with the shared edge, so that e_3 aligns with the normal direction of triangle B. The right most rotation rotate the frame around z axis, now aligned with the normal of B, so that the frame aligns with the local frame of triangle B, f_B. Finally, fixing any orthonormal basis for one triangle, and a position for one of its vertex, we can construct the whole surface. The other 2 vertices are determined by their coordinates and the fixed local frame. We can further determine the local frames for the adjacent faces using the rotation matrices, and then, their vertices.

The second step is to implement the algorithm of surface reconstruction, from the discrete fundamental forms. The algorithm is very similar to the above steps to theoretically reconstruct the surface, with the last step changed to minimizing the weighted least square energy function with respect to $\mathbf{x}$, the vertex positions, and $\mathbf{f}$, the local frames. The energies are defined in the paper as:

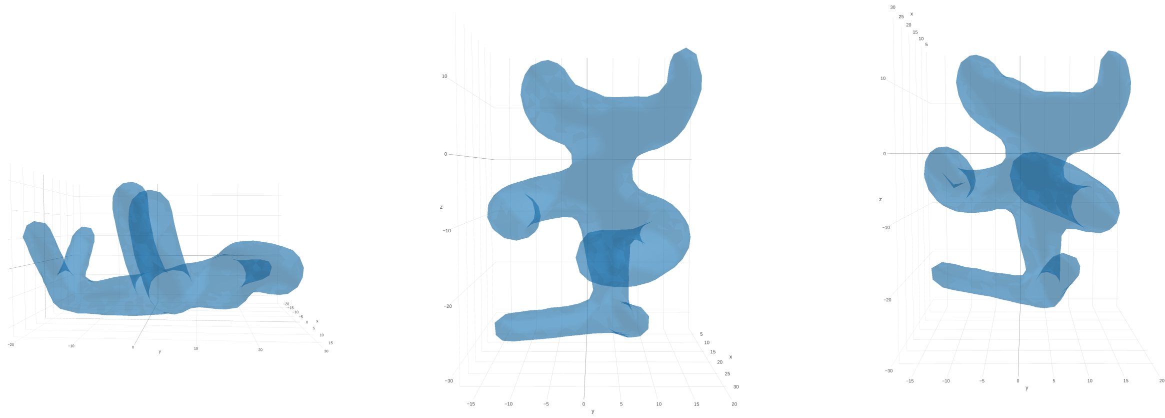
$$E_f(\mathbf{f}) = \frac{1}{2} \sum_{e \in E, e = T_i \cap T_j} w_e^{-1} ||f_j - f_i * R_{ij}||^2$$

$$E_x(\mathbf{x}, \mathbf{f}) = \frac{1}{2} \sum_{e = v_m V_n} \sum_{T \ni e} w_{e,T} ||\mathbf{x_n} - \mathbf{x_m} - f_T * (a^1_{e,T}, a^2_{e,T}, 0)^T||^2$$

$$E_M(\mathbf{x}, \mathbf{f}) = wE_f + E_x$$

Where the R_{ij}'s are the rotation matrices, and $(a^1_{e,T}, a^2_{e,T}, 0)^T$'s are the coordinates, and the w_e's are the cotangent edge weights. The linear system I derived to minimize it, (and the derivation) is given by:

$$E_f = \frac{1}{2} ||W_f^{1/2} M_3 \mathbf{f}'||^2$$



(a) The original mesh. (b) The rotated mesh. (c) The stretched mesh.

Figure 1: The figures are produced using the provided data file for homework 2. Figure 1a is simply a plot for the provided mesh. Figure 1b is the result of solving the linear system derived above, with an arbitrary $\mathbf{f_0}$ and the original $\mathbf{x_0}$ as the constraint (seed). Figure 1c is an experiment result of using the same seed as before, with additional constraint of normally rotated positions ($\mathbf{x}$'s) around the left arm, and one position at the left hand moved by 5 in the z direction. The result is not ideal probably since the bad choices of constraints. But it still illustrates somewhat how the algorithm behaves (or possible implementation bugs).

$$E_x = \frac{1}{2}||W_{x,l}^{1/2}(M_1\mathbf{x} - M_{2,l}\mathbf{f}')||^2 + \frac{1}{2}||W_{x,r}^{1/2}(M_1\mathbf{x} - M_{2,r}\mathbf{f}')||^2$$

$$(w * R_3^T W_f R_3 - R_{2,l}^T W_{x,l} R_{2,l} - R_{2,r}^T W_{x,r} R_{2,r})\mathbf{f}' + (R_{2,l}^T W_{x,l} R_1 + R_{2,r}^T W_{x,r} R_1)\mathbf{x} = 0$$

$$(-R_1^T W_{x,l} R_{2,l} - R_1^T W_{x,r} R_{2,r})\mathbf{f}' + (R_1^T W_{x,l} R_1 + R_1^T W_{x,r} R_1)\mathbf{x} = 0$$

Where R's and W's are the corresponding (sparse block) matrices according the the energy definition, $\mathbf{f}'$ are constructed by transposing every frame block in $\mathbf{f}$. The sparse linear equation can be solved with a given vertex position and a given local frame. One example that shows it works is given by figure 1.

3 Related Works

I have went through the related papers mentioned in the paper. [U. Pinkall, K. Polthier 1993] describes the construction of discrete Laplace, whose cotangent weights are used in the energy functions (I don't fully understand why, yet). [M. Desbrun, E. Kanso, Y. Tong 2006] describes a useful definition of the discrete differential forms, which are used in the derivation of the energy function. I will carefully read the survey paper on reconstruction methods [M. Botsch, O. Sorkine 2008] and the mentioned paper that describes a similar but different methods [Y. Lipman, O. Sorkine, D. Levin 2005], to have a better understanding of the subject, and to come up with ideas to add my own contribution for this project.

References

[1] Y. Wang, B. Liu, and Y. Tong. *Linear Surface Reconstruction from Discrete Fundamental Forms on Triangle Meshes.* Computer Graphics Forum, 31(8), pp. 2277-2287 2012.

[2] Y. Lipman, O. Sorkine, D. Levin, and D. Cohen-Or. *Linear Rotation-invariant Coordinates for Meshes.* ACM Trans. Graph. 24(July 2005), pp. 479-487, 2005.

[3] Pinkall, U., Polthier, K. *Computing discrete minimal surfaces and their conjugates.* Experimental Mathematics 2 (1993), 15-36, 1993.

[4] Desbrun, M., Kanso, E., Tong, Y. *Discrete differential forms for computational modeling.* SIGGRAPH'06, ACM, pp. 39-54, 2006.

[5] Botsch, M., Sorkine, O. *On linear variational surface deformation methods.* IEEE Transactions on Visualization and Computer Graphics 14, 1 (2008), 213-230, 2008.