Discrete Fundamental Theorem of Surfaces

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Motivation

- Local rigid motion invariant representations
  - Geometry processing, shape analysis
  - Shape deformation
  - Thin shell simulation
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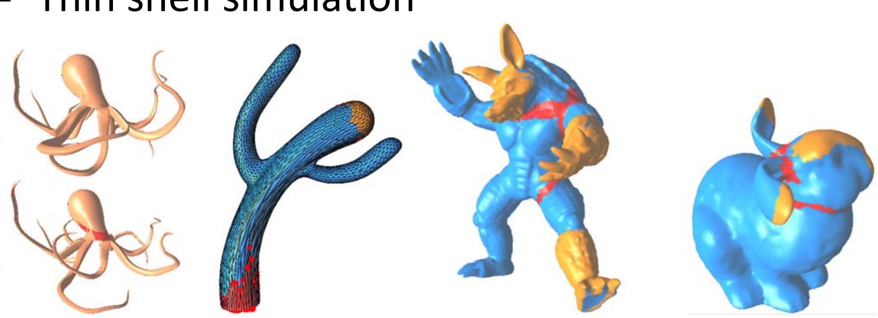
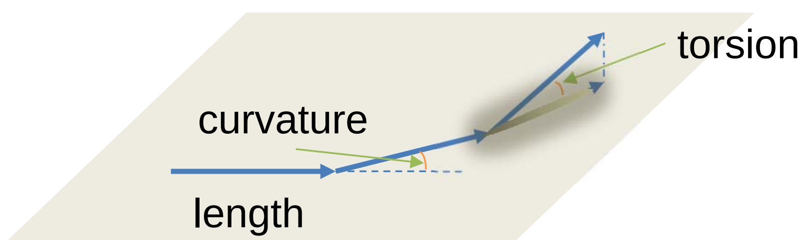


Differential Invariants

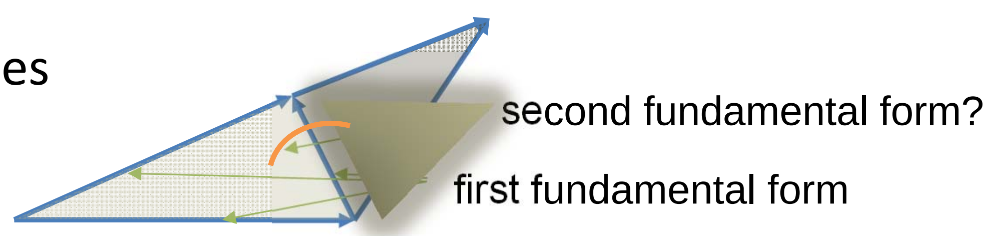
- 1D Curve embedded in 3D
  - First derivative : length
  - Second derivative : curvature
  - Third derivative: torsion
- 2D surface embedded in 3D
  - First derivatives: length (first fundamental form)
  - Second derivatives:
    curvatures (second fundamental form)

Shape from Discrete Local Rep.

- Curves



- Surfaces

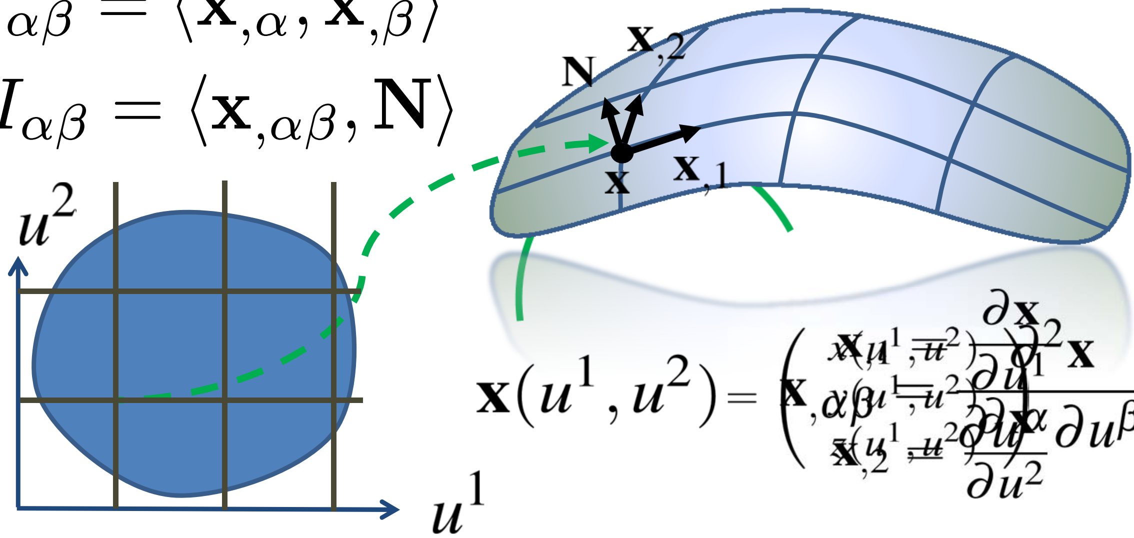


How does it correspond to the continuous theory?
What if there are conflicts?

Fundamental Theorem of Surfaces

$$I_{\alpha\beta} = \langle \mathbf{x}_{,\alpha}, \mathbf{x}_{,\beta} \rangle$$

$$II_{\alpha\beta} = \langle \mathbf{x}_{,\alpha\beta}, \mathbf{N} \rangle$$



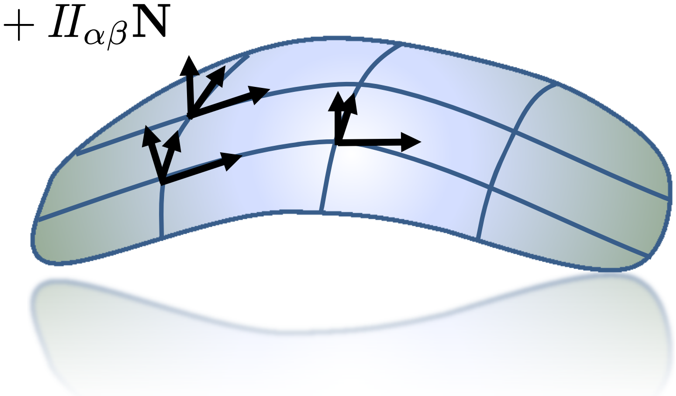
$$\mathbf{x}(u^1, u^2) = \begin{pmatrix} x(u^1, u^2) \\ y(u^1, u^2) \\ z(u^1, u^2) \end{pmatrix}$$

$$\mathbf{x}_{,1} = \frac{\partial \mathbf{x}}{\partial u^1} \qquad \mathbf{x}_{,2} = \frac{\partial \mathbf{x}}{\partial u^2} \qquad \mathbf{x}_{,\alpha\beta} = \frac{\partial^2 \mathbf{x}}{\partial u^\alpha \partial u^\beta}$$

Fundamental Theorem of Surfaces

$$\mathbf{x}_{,\alpha\beta} = \Gamma^1_{\alpha\beta}\mathbf{x}_{,1} + \Gamma^2_{\alpha\beta}\mathbf{x}_{,2} + II_{\alpha\beta}\mathbf{N}$$

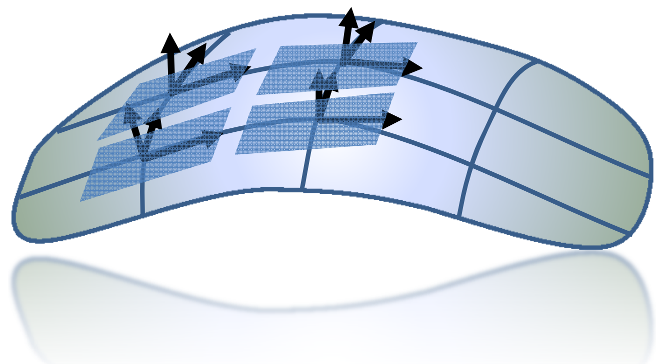
$$\mathbf{x}_{,\alpha\beta} = \Gamma^\gamma_{\alpha\beta}\mathbf{x}_{,\gamma} + II_{\alpha\beta}\mathbf{N}$$



Fundamental Theorem of Surfaces

$$\mathbf{x}_{,\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta}\mathbf{x}_{,\gamma} + II_{\alpha\beta}\mathbf{N}$$

Gauss's surface equations



$$0 = \mathbf{x}_{,\alpha\beta\gamma} - \mathbf{x}_{,\alpha\gamma\beta}$$

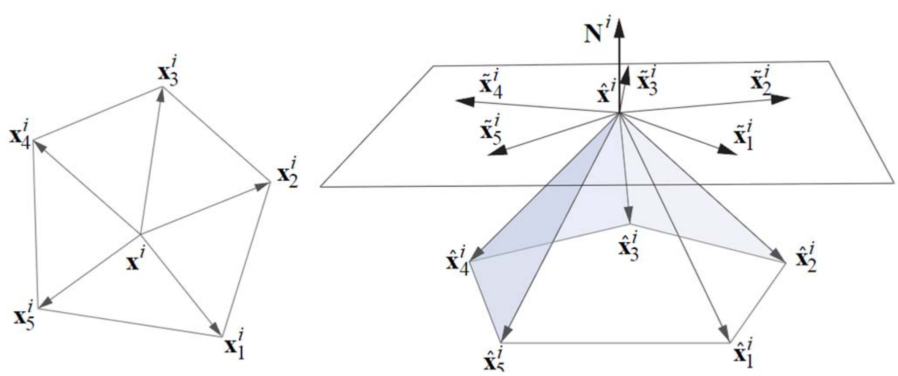
$$= \underbrace{(R^{\tau}_{\alpha\gamma\beta} - (II^{\tau}_{\gamma}II_{\alpha\beta} - II^{\tau}_{\beta}II_{\alpha\gamma}))}_{=\,0}\mathbf{x}_{,\tau} + \underbrace{(\nabla_{\gamma}II_{\alpha\beta} - \nabla_{\beta}II_{\alpha\gamma})}_{=\,0}\mathbf{N}$$

Gauss's Equation

Mainardi-Codazzi Equations

Linear Rotation-Invariant Coords.

- one-ring of a vertex



[LSLCO05] LIPMAN, Y., SORKINE, O., LEVIN, D., COHEN-OR, D.: Linear rotation-invariant coordinates for meshes. ACM Trans. Graph. (SIGGRAPH) 2005

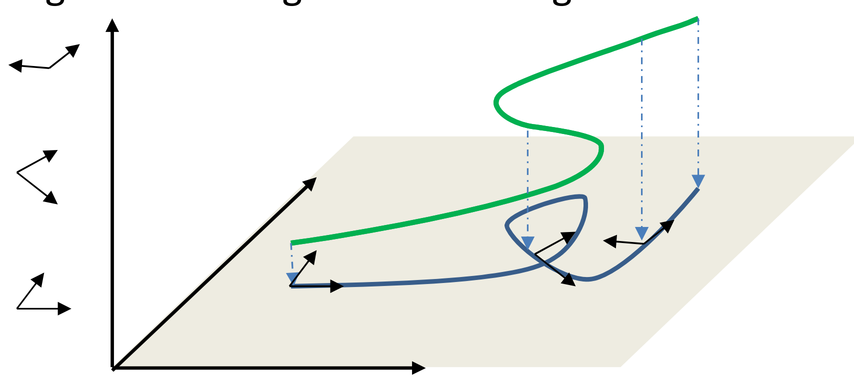
Separate Frames & Coordinates

- Insensitive to translational constraints

The bump plane model from [BS08] BOTSCH M., SORKINE O.: On linear variational surface deformation methods. 2008

Lift of an Immersion

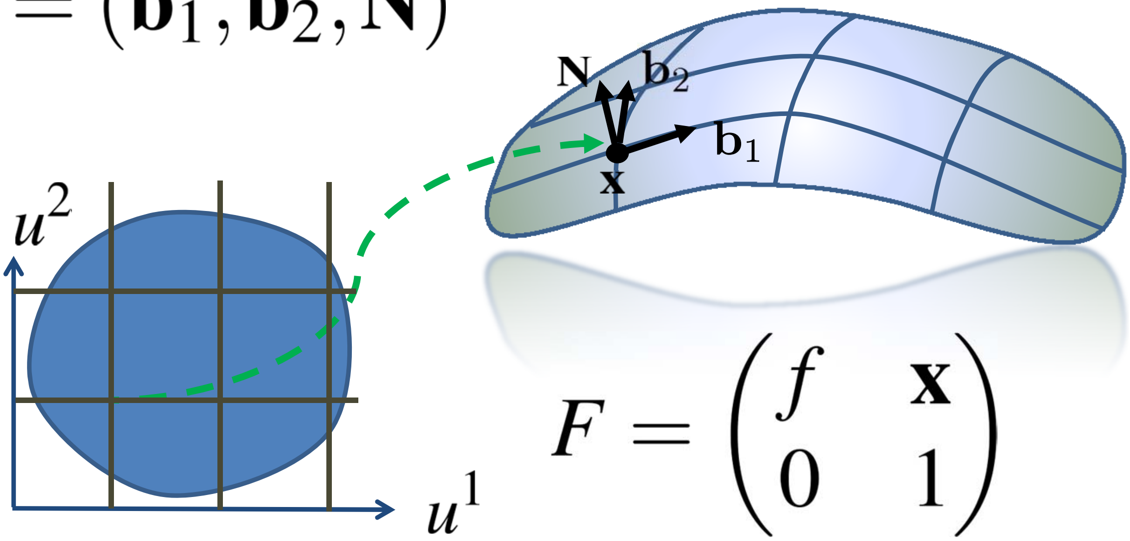
- Piecing together line segments or triangles



- Planar curve lifted to 3D
  - Introducing intermediate variables: frame is the 3rd dimension.

Lift of a Surface Patch (3D to 6D)

$$f = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{N})$$



$$F = \begin{pmatrix} f & \mathbf{x} \\ 0 & 1 \end{pmatrix}$$

Local Surface Description

$$\Omega = F^{-1} dF$$

$$= \begin{pmatrix} f^T df & f^T d\mathbf{x} \\ 0 & 0 \end{pmatrix}$$

Modified Surface Equations

$$\begin{array}{c} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \Omega_R \qquad\qquad\quad \Omega_x \\ \Omega = F^{-1}dF = \begin{pmatrix} f^T df & f^T d\mathbf{x} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\omega_1^2 & -\omega_1^3 & \omega^1 \\ \omega_1^2 & 0 & -\omega_2^3 & \omega^2 \\ \omega_1^3 & \omega_2^3 & 0 & \omega^3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$d\mathbf{x} = f\,\Omega_x = \omega^1 \mathbf{b}_1 + \omega^2 \mathbf{b}_2$$

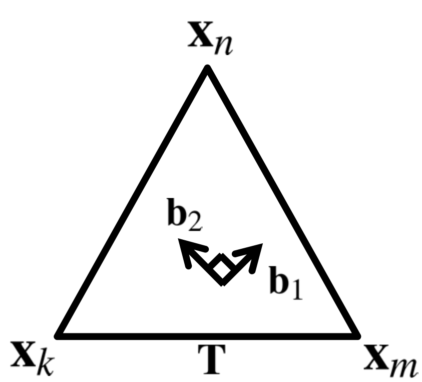
$$df = f\,\Omega_R = f \begin{pmatrix} 0 & -\omega_1^2 & -\omega_1^3 \\ \omega_1^2 & 0 & -\omega_2^3 \\ \omega_1^3 & \omega_2^3 & 0 \end{pmatrix}$$

Position Equation

- Local Frame $(\mathbf{b}_1, \mathbf{b}_2)$

$$\mathbf{x}_n - \mathbf{x}_m = a^1_{mn,T}\mathbf{b}_1 + a^2_{mn,T}\mathbf{b}_2$$

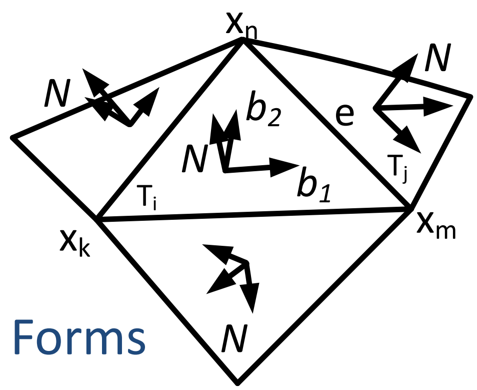
- derived from 1st fund. forms
  - edge lengths



Frame Equations

- Transition rotation

$$f_j - f_i = f_i R_{ij} - f_i$$

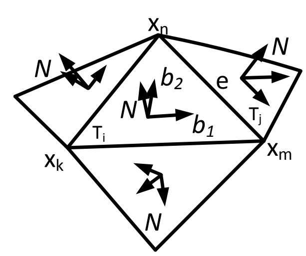


- Based on 1st and 2nd fund. Forms
  - dihedral angles

Discrete Surface Equations

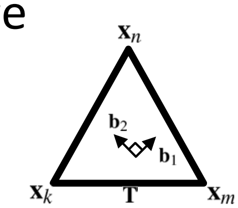
- Frame Equation across each edge

$$\boxed{\forall e_{ij} \in E, \;\; f_j - f_i R_{ij} = 0}$$



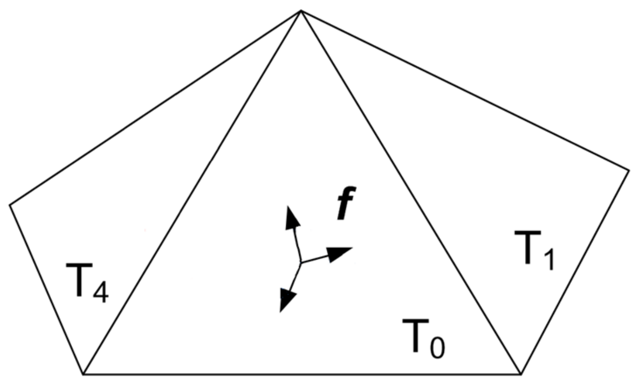
- Position equation along each (half-)edge

$$\boxed{\mathbf{x}_n - \mathbf{x}_m - (a^1_{mn,T}\mathbf{b}_{1,T} + a^2_{mn,T}\mathbf{b}_{2,T}) = 0}$$



Discrete Fundamental Theorems

- Local discrete fundamental theorem of surfaces



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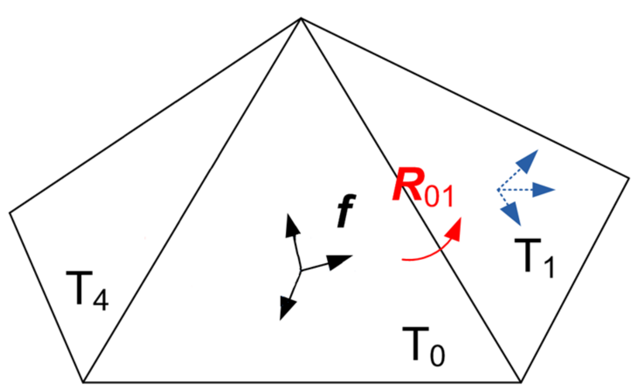

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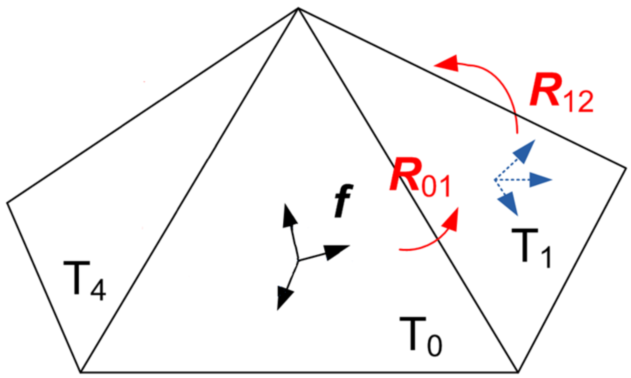

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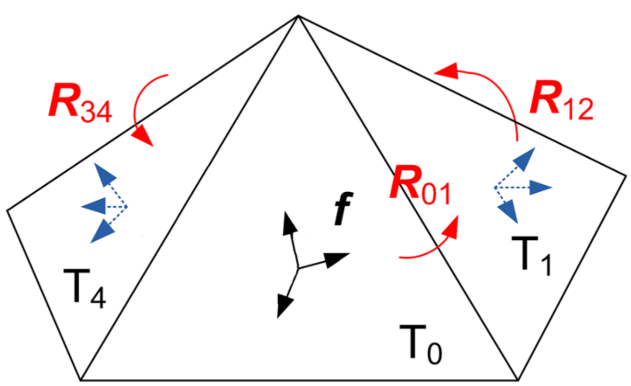

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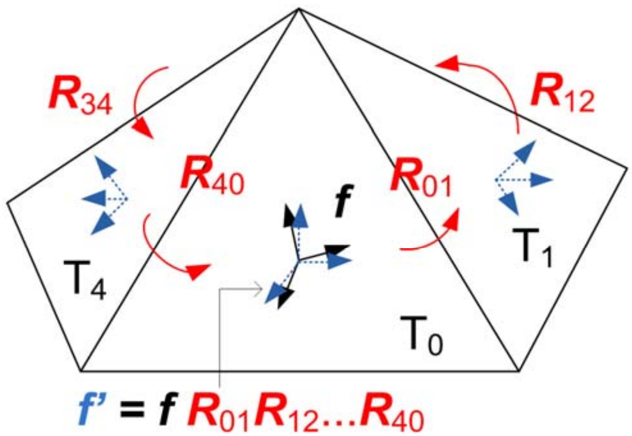
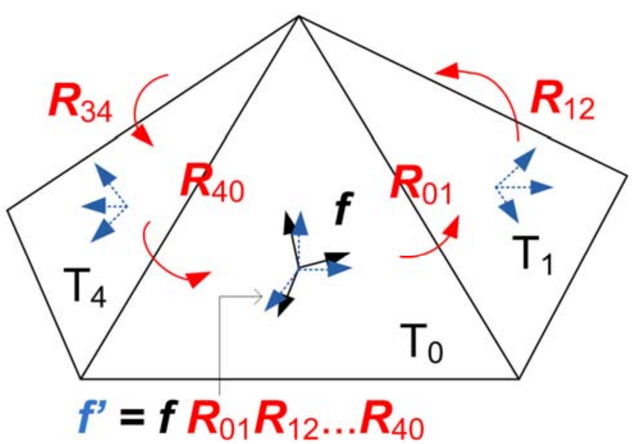


Discrete Fundamental Theorems

$$\boxed{\Pi_{i\in 0,\ldots,n-1} R_{i,(i+1) \bmod n} - Id = 0}$$

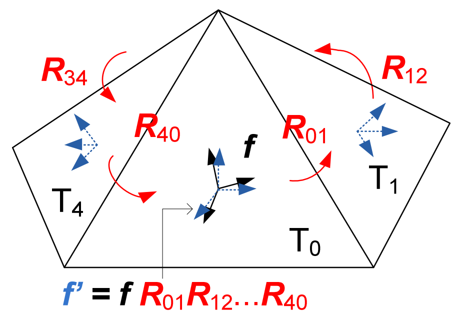


Discrete Fundamental Theorems

- Local discrete fundamental theorem of surfaces
  - Discrete integrability condition:

$$\Pi_{i\in 0,\ldots,n-1} R_{i,(i+1)\,\mathrm{mod}\,n} - Id = 0$$

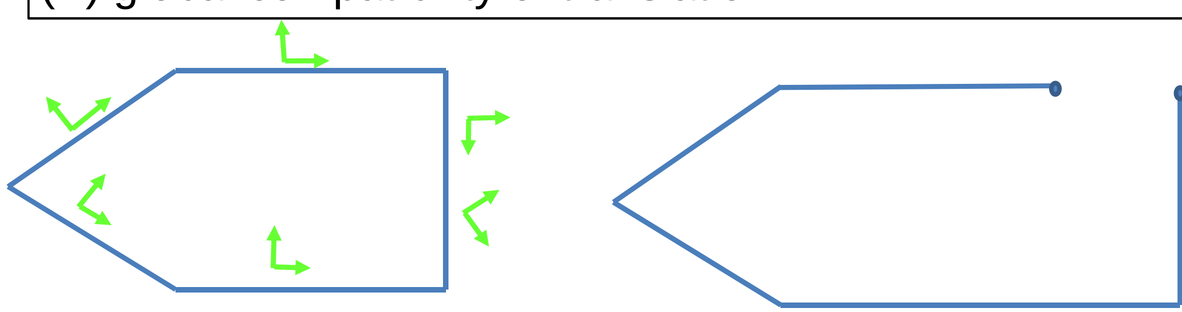
- Alignment of normal
  - The Codazzi equations
- Alignment of tangents
  - Gauss's equation



Global Version

Global discrete fundamental theorem of surfaces

Closed mesh with genus g (i) local integrability condition (omit 2 vertices) (ii) global compatibility of rotation (iii) global compatibility of translation



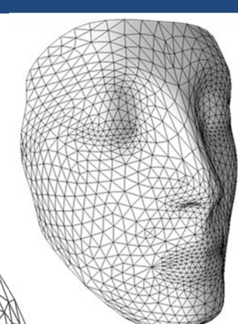
Application

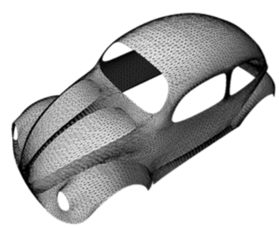
- Surface Reconstruction
- Mesh Deformation
- Quasi-isometric Parameterization
- Implementation: minimize

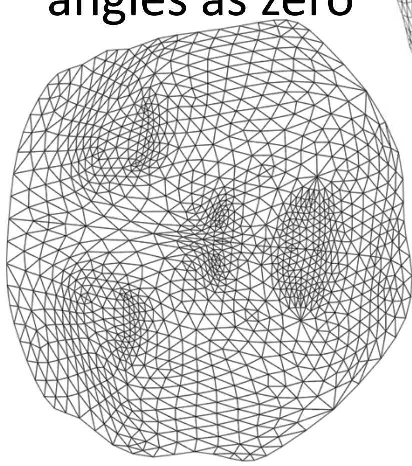
$$\int_M \|dF - F\Omega\|^2 \qquad \Omega = F^{-1}dF$$

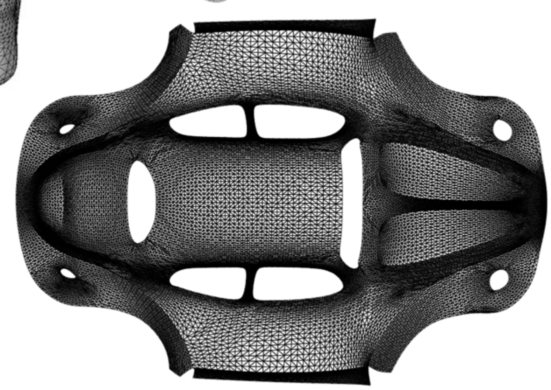
Quasi-isometric Parameterization

- Set all dihedral angles as zero

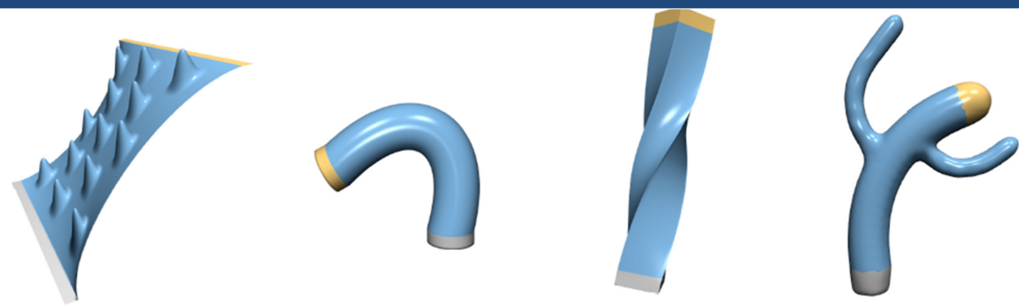








Comparison with previous work



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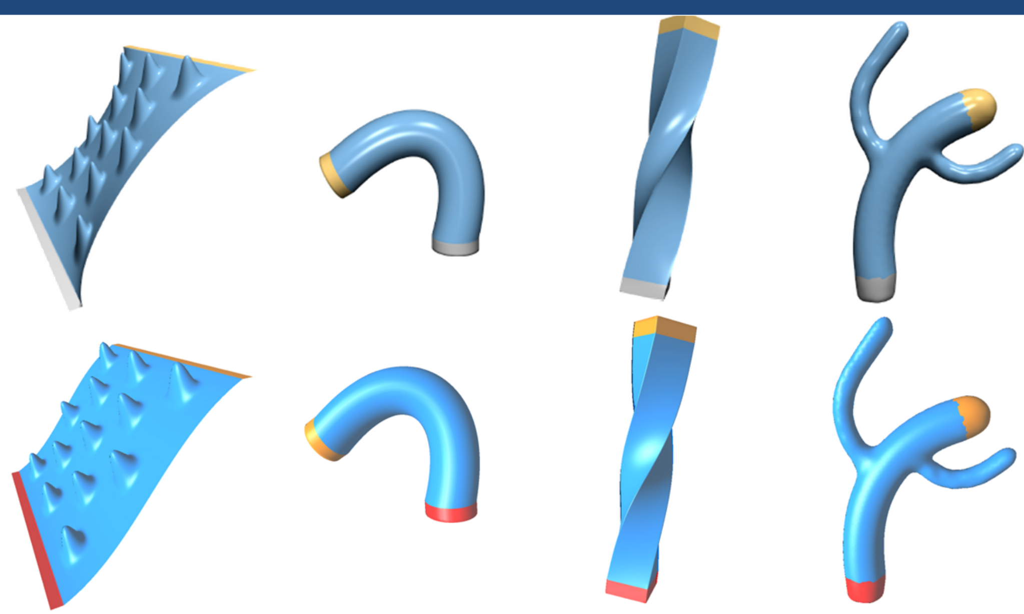
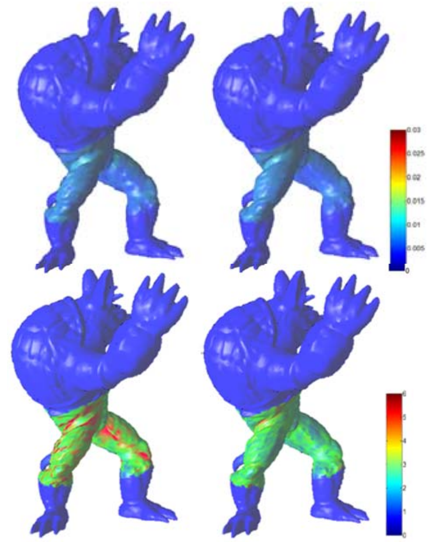


Comparison with [LSLCO05]

- Distortion
  - edge length
- Distortion
  - dihedral angles



Conclusion

- Discrete fundamental theorem of surfaces
  - I : edge lengths
  - II : dihedral angles
- A simple sparse linear system
  - from an arbitrary set of I and II
- Deformation with position & orientation constraints
- Limitations: nonlinear integrability condition, orthonormality requirements, self-intersection.

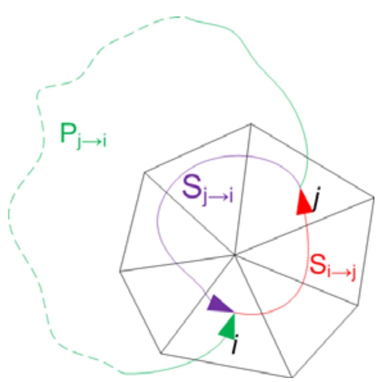
Thank you!

Questions?

– Acknowledgements: NSF IIS-0953-96, CCF-0936830, CCF-0811313, and CMMI-0757123

Sketch of the Proof

- Uniquely determined frames for faces



- Uniquely determined positions for vertices