CSE 100: HUFFMAN ALGORITHM

Announcements

- HW4
  - Homework 4 - Due tomorrow 11/15 @ 11:59PM
- PA3
  - Will be released tomorrow
- Monday – 11/19
  - Midterm – in class – Covers till Friday’s class

Goals for Today

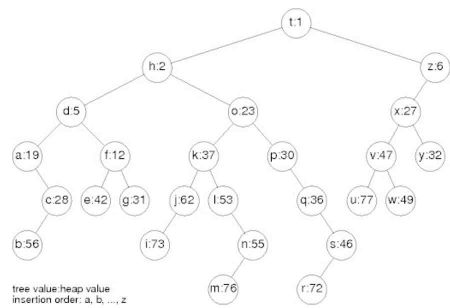
- Apply Huffman's algorithm to build coding trees
- Explain how heaps work
- Use the C++ priority queue class

PA3 (Data Compression)…

- Data compression: Represent digital media using the fewest number of bits!

We will do this using trees!

Text, video, audio



11001110	01110000	01100111	00000011	11110000	11111111	01010001	11100011
00000101	11101111	00001000	01001011	11011110	00010000	10010111	10111100
00100001	00101111	01111000	01000010	01011110	11110000	01000110	01111100
00001111	10110111	00000011	01111110	10100001	00011100	01100100	11001100
00011011	11001111	11111110	11100011	11000001	00001011	11011011	01011101
11001010	11101111	10010010	10010101	11011111	00100100	00101001	00011010
01010011	00110000	01010100	01010011	11101100	01100111	10111010	01011001
01000011	11101111	00010111	11010111	01001011	10111010	00011111	11110000
10001100	10010110	00001010	00010000	10011000	00111011	00100000	11011100
00011010	01101101	00000011	11001100	11001100	11001110	01111001	01101101
10010100	10001101	01100001	01011010	01010010	10111011	10010101	11011100

All data is bits!

Fixed length encoding

- Fixed length: each symbol is represented using a fixed number of bits
- For example for the symbols ‘s’, ‘p’, ’a’, ‘m’ one possible encoding is:

spamspamspam spamspamspam

Text file

000110110001101100011011 000110110001101100011011 000110110001101100011011

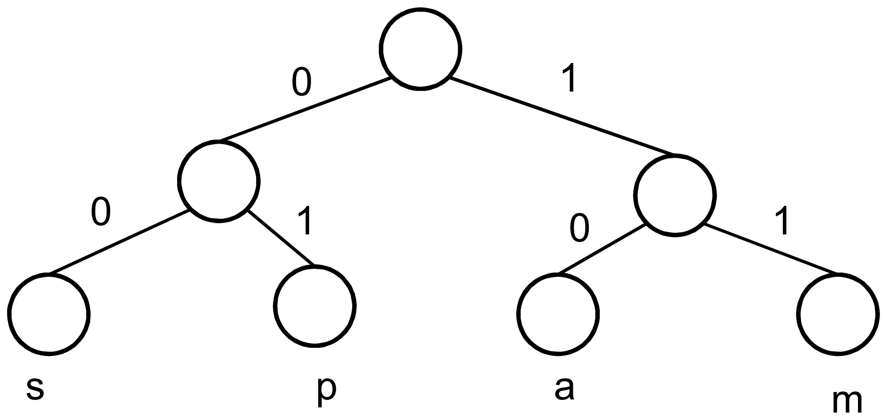
Encoded file

Symbol	Code word
s	00
p	01
a	10
m	11

For a dictionary consisting of M symbols, what is the minimum number of bits needed to encode each symbol (assume fixed length binary codes) ?

A. 2^M B. M C. M/2 D. ceil($\log_2$ M) E. None of these

Binary codes as Binary Trees



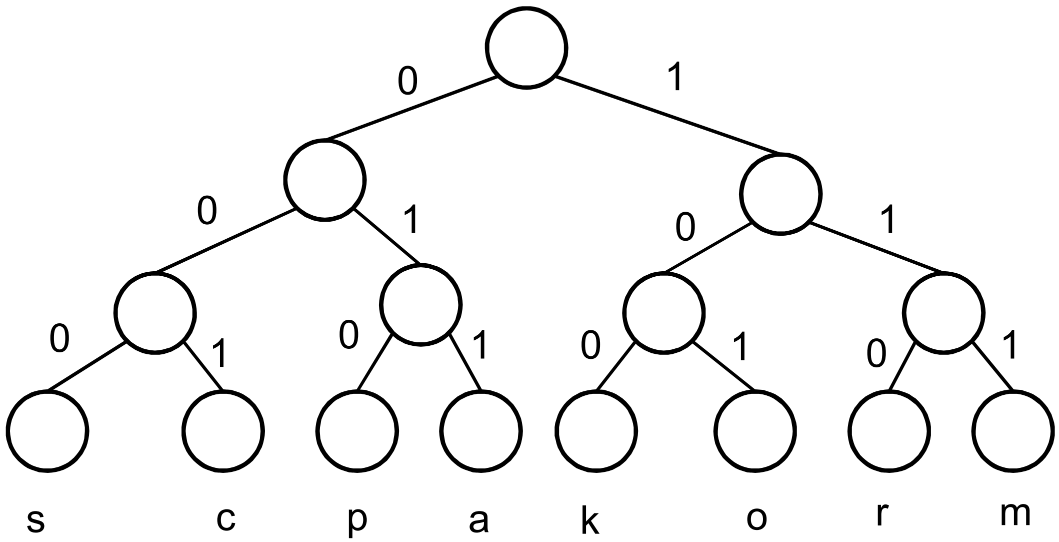
Code A

Symbol	Codeword
s	00
p	01
a	10
m	11

- Do we need to be constrained to fixed length encoding?
- What if certain symbols appeared more often than others?

- Symbols are leaf nodes
- Root to leaf node gives the codeword for each symbol
- Once we have the tree we can encode and decode data
- Given the tree
  - Encode the string ‘papa’
  - Decode the binary sequence ‘01101100’

Decoding on binary trees, another example



Decode the bitstream 110101001100 using the given binary tree

A. scam
B. rork
C. rock
D. korp

- Do we need to be constrained to fixed length encoding?
- What if certain symbols appeared more often than others?

Variable length codes

ssssssssssssssss
sspppppaampamm

Text file

Symbol	Counts
s	18
p	6
a	3
m	3

Symbol	Frequency
s	0.6
p	0.2
a	0.1
m	0.1

Code A

Symbol	Codeword
s	00
p	01
a	10
m	11

Code B

Symbol	Codeword
s	0
p	1
a	10
m	11

Average length (code A) = 2 bits/symbol

Average length (code B) = 0.6 *1 +0.2 *1 + 0.1* 2+ 0.1*2
= 1.2 bits/symbol

Comparing encoding schemes

ssssssssssssssss
sspppppaampamm

Text file

Symbol	Counts
s	18
p	6
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Symbol	Frequency
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Code A

Symbol	Codeword
s	00
p	01
a	10
m	11

Code B

Symbol	Codeword
s	0
p	1
a	10
m	11

Is code B better than code A?

A. Yes
B. No
C. Depends

Variable length codes

Variable length codes have to necessarily be prefix codes for correct decoding

A *prefix code* is one where no symbol's codeword is a prefix of another

Code A

Symbol	Codeword
s	00
p	01
a	10
m	11

Code B

Symbol	Codeword
s	0
p	1
a	10
m	11

Code B is not a prefix code

Use Huffman's algorithm to produce the minimal average-length code!

ssssssssssssssss
ssppppaampamm

Text file

Symbol	Counts
s	18
p	6
a	3
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Code A

Symbol	Codeword
s	00
p	01
a	10
m	11

Your turn: Apply Huffman's algorithm to the following symbols with the given frequencies

A: 6; B: 4; C: 4; D: 0; E: 0; F: 0; G: 1; H: 2

PA3: encoding/decoding

ENCODING:

1. **Scan text file to compute frequencies**
2. **Build Huffman Tree**
3. **Find code for every symbol (letter)**
4. **Create new compressed file by saving the entire code at the top of the file followed by the code for each symbol (letter) in the file**

DECODING:

1. **Read the file header (which contains the code) to recreate the tree**
2. **Decode each letter by reading the file and using the tree**

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DECODING:

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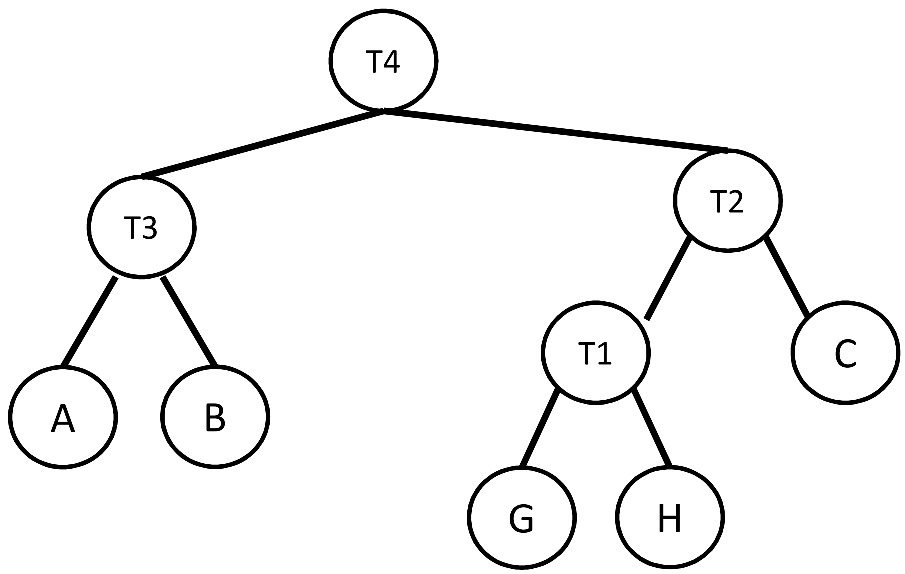
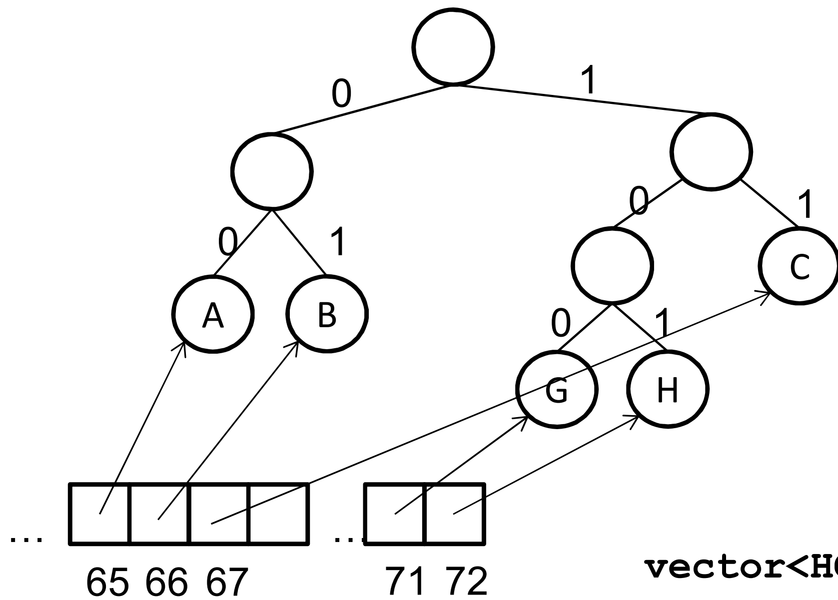
Encoding a symbol: let's think implementation!



- Compression using trees:
  - Devise a “good” code/tree
  - Encode symbols using this tree

A very bad way is to start at the root and search down the tree until you find the symbol you are trying to encode, why?

Encoding a symbol



A much better way is to maintain a list of leaves and then to traverse the tree to the root (and then reverse the code)

```
vector<HCNode*> leaves;
…
leaves = vector<HCNode*>(256, (HCNode*)0);
```

PA3: encoding/decoding

ENCODING:

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DECODING:

1. **Read the file header (which contains the code) to recreate the tree**
2. **Decode each letter by reading the file and using the tree**

Building the tree: Huffman’s algorithm

0. Determine the count of each symbol in the input message.

1. Create a forest of single-node trees containing symbols and counts for each non-zero-count symbol.
2. Loop while there is more than 1 tree in the forest:

   2a. Remove the two lowest count trees

   2b. Combine these two trees into a new tree (summing their counts). 2c. Insert this new tree in the forest, and go to 2.
3. Return the one tree in the forest as the Huffman code tree.

Building the tree: Huffman's algorithm

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You know how to create a tree. But how do you maintain the forest? Choose the best data structure/ADT:

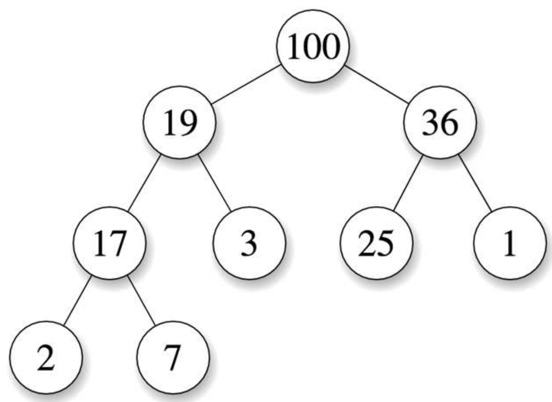
A. A list
B. A BST
C. A priority queue (heap)

Aside: Heaps

Have you seen a heap? A. Yes B. No C. Yes, but I don't remember them

Aside: Heaps

Have you seen a heap? A. Yes B. No C. Yes, but I don't remember them



if P is a parent node of C, then the *key* (the *value*) of P is either greater than or equal to (*in a max heap*) or less than or equal to (*in a min heap*) the key of C. The node at the "top" of the heap (with no parents) is called the *root* node

https://en.wikipedia.org/wiki/Heap_(data_structure)

Priority Queues in C++

A C++ **`priority_queue`** is a generic container, and can hold any kind of thing as specified with a template parameter when it is created: for example **`HCNode`**s, or pointers to **`HCNode`**s, etc.

```
#include <queue>
std::priority_queue<HCNode> p;
```

By default, a **`priority_queue<T>`** uses **`operator<`** defined for objects of type T:

- **if `a < b`, `b` is taken to have higher priority than `a` and `b` will come out before `a`**

Priority Queues in C++

```
#ifndef HCNODE_H
#define HCNODE_H
class HCNode {

public:
  HCNode* parent; // pointer to parent; null if root
  HCNode* child0; // pointer to "0" child; null if leaf
  HCNode* child1; // pointer to "1" child; null if leaf
  unsigned char symb; // symbol
  int count; // count/frequency of symbols in subtree

  // for less-than comparisons between HCNodes
  bool operator<(HCNode const &) const;
};

#endif
```

In HCNode.cpp:

```
#include HCNODE_HPP
/** Compare this HCNode and other for priority
ordering.
 * Smaller count means higher priority.
 * Use node symbol for deterministic tiebreaking
 */
bool HCNode::operator<(HCNode const & other) const {
  // if counts are different, just compare counts
  if(count != other.count) return count > other.count;

  // counts are equal. use symbol value to break tie.
  // (for this to work, internal HCNodes
  // must have symb set.)
  return symb < other.symb;
};

#endif
```

Is this implementation of operator< correct to use with the C++ priority queue (which uses a MAX-heap)?

A. Yes
B. No

Using std::priority_queue in Huffman's algorithm

- If you create an STL container such as priority_queue to hold HCNode objects:

```
#include <queue>
std::priority_queue<HCNode> pq;
```

- ... then adding an HCNode object to the priority_queue:

```
HCNode n;
pq.push(n);
```

- ... actually creates a copy of the HCNode, and adds the copy to the queue. You probably don't want that. Instead, set up the container to hold pointers to HCNode objects:

```
std::priority_queue<HCNode*> pq;
HCNode* p = new HCNode();
pq.push(p);
```

Using std::priority_queue in Huffman's

Instead, set up the container to hold pointers to HCNode objects:

```
std::priority_queue<HCNode*> pq;
HCNode* p = new HCNode();
pq.push(p);
```

What is the problem with the above approach?

A. Since the priority queue is storing copies of HCNode objects, we have a memory leak

B. The nodes in the priority queue cannot be correctly compared

C. Adds a copy of the pointer to the node into the priority queue

D. The node is created on the run time stack rather than the heap

Using std::priority_queue in Huffman's algorithm

Instead, set up the container to hold pointers to HCNode objects:

```
std::priority_queue<HCNode*> pq;
HCNode* p = new HCNode();
pq.push(p);
```

What is the problem with the above approach?

- our operator< is a member function of the HCNode class. It is not defined for pointers to HCNodes. What to do?

std::priority_queue template arguments

The template for priority_queue takes 3 arguments:

```
template < class T, class Container = vector<T>,
           class Compare = less<typename Container::value_type> > class priority_queue;
```

- The first is the type of the elements contained in the queue.
- If it is the only template argument used, the remaining 2 get their default values:
  - a **vector<T>** is used as the internal store for the queue,
  - **less** a class that provides priority comparisons
- Okay to use vector container , but we want to tell the priority_queue to first dereference the HCNode pointers it contains, and then apply operator<
- How to do that? We need to provide the priority_queue with a Compare class

Defining a “comparison class”

- The documentation says of the third template argument:
- Compare: Comparison class: A class such that the expression comp(a,b), where comp is an object of this class and a and b are elements of the container, **returns true if a is to be placed earlier than b** in a strict weak ordering operation. This can be a class implementing a function call operator...

Here’s how to define a class implementing the function call operator() that performs the required comparison:

comp(a, b) returns True if priority of a < priority of b (hence, ‘b’ will be ahead of ‘a’ in Queue)

```
class HCNodePtrComp {
  bool operator()(HCNode* & lhs, HCNode* & rhs) const {
    // dereference the pointers and use operator<
    return *lhs < *rhs;
  }
};
```

Now, create the priority_queue as:

```
std::priority_queue<HCNode*,std::vector<HCNode*>,HCNodePtrComp> pq;
```

and priority comparisons will be done as appropriate.

PA3: encoding/decoding

ENCODING:

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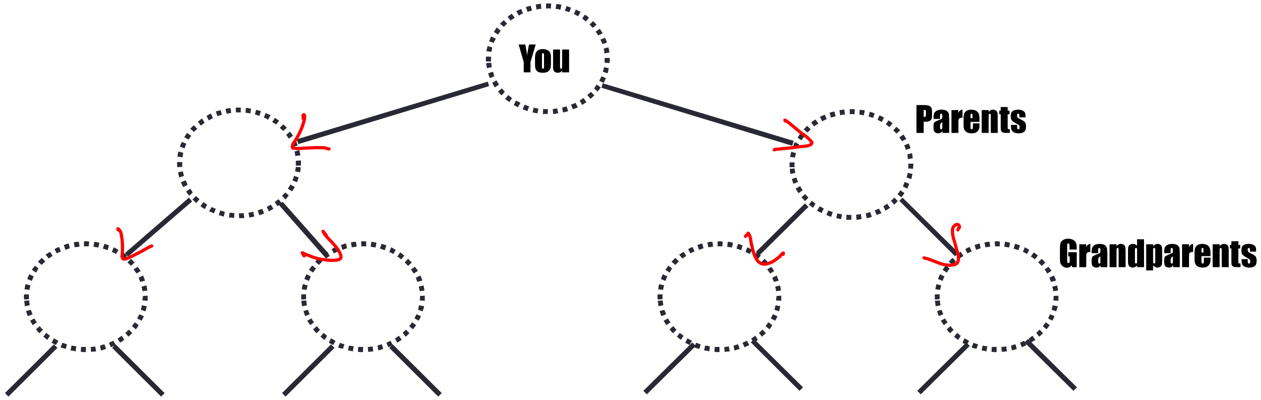
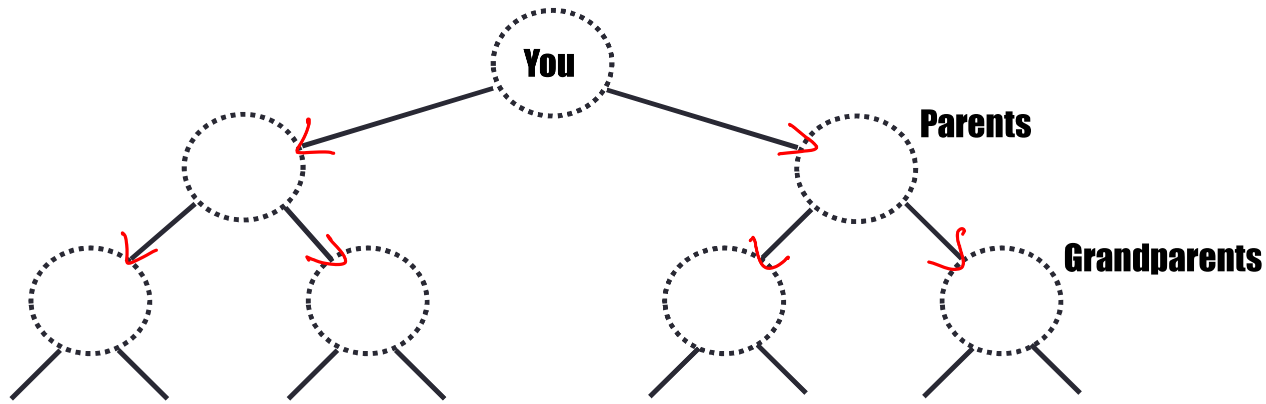
From Trees to Graphs



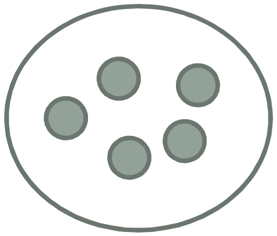
Is this a tree, or…?

From Trees to Graphs

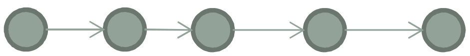


Is this a tree, or…?
1 generation = 30 years ➔ 100 generations over the last 3000 years
2^100 = 1.267x10^30   (How many people are on earth?)

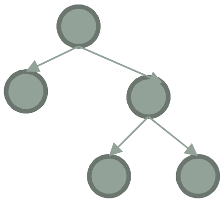
Kinds of Data Structures



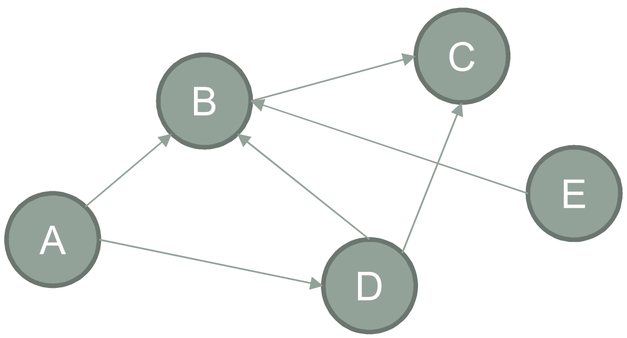
Unstructured structures
(sets)



Sequential, linear structures
(arrays, linked lists)



Hierarchical structures
(trees)

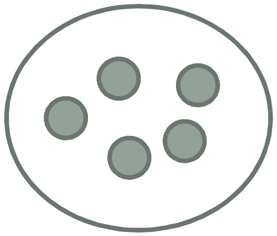


<u>Graphs</u>

Which of the following is NOT true about graphs?

A. They consist of both vertices and edges
B. They have an inherent order
C. Edges may be weighed or unweighted
D. Edges may be directed or undirected
E. They may contain cycles

Kinds of Data Structures



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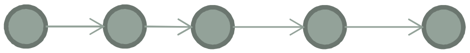

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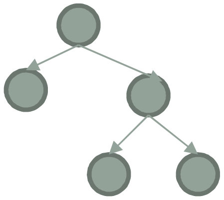

Hierarchical structures
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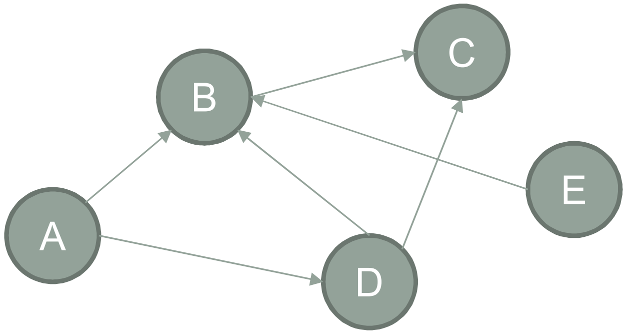
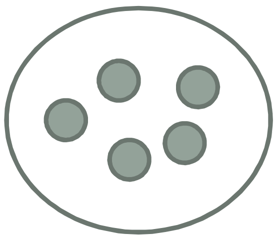


Graphs

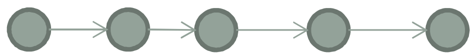
Which of the following is ALWAYS a graph:

A. A list
B. A tree
C. Both
D. Neither

Kinds of Data Structures



Unstructured structures
(sets)



Sequential, linear structures
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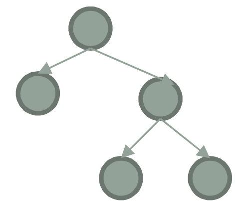
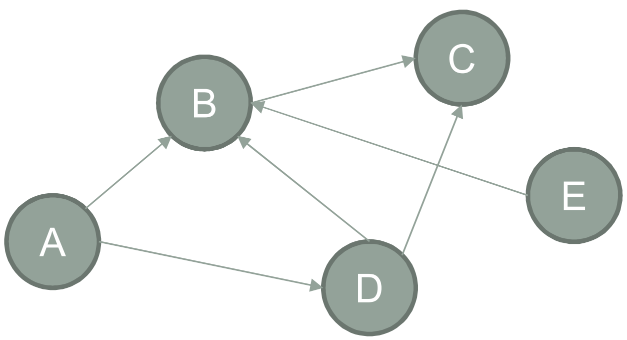


Hierarchical structures
(trees)



<u>Graphs</u>

Consist of:

- A collection of elements (“nodes” or “vertices”)
- A set of connections (“edges” or “links” or “arcs”) between pairs of nodes.
  - Edges may be directed or undirected
  - Edges may have weight associated with them

Graphs are not hierarchical or sequential,
no requirements for a “root” or “parent/child”
relationships between nodes

Note that trees are special cases of graphs; lists are special cases of trees.

Graphs

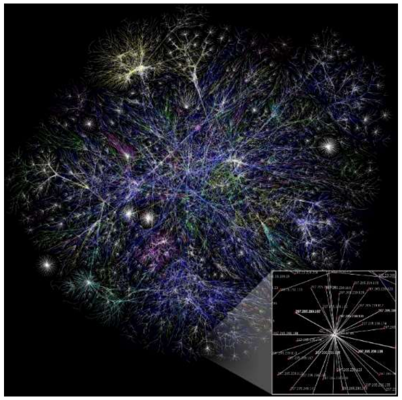
Basic objects : **vertices, nodes**

Relationships between them : **edges, arcs, links**

Graphs

Basic objects : websites

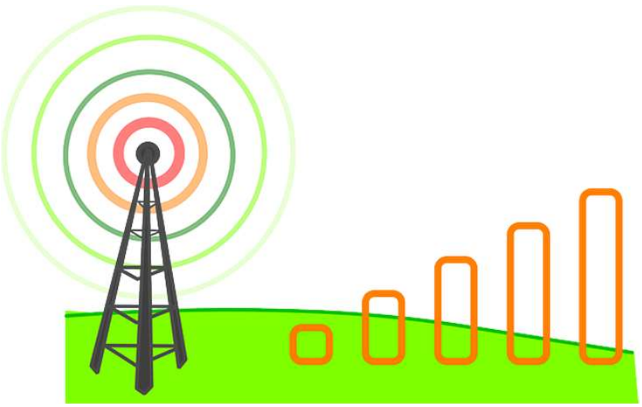
Relationships between them : hyperlinks



Graphs

Basic objects : **cell phone towers**

Relationships between them : **coverage area overlaps**



Graphs

Basic objects : game units

Relationships between them : paths on map

Graphs

Basic objects : people

Relationships between them : friends

Graphs

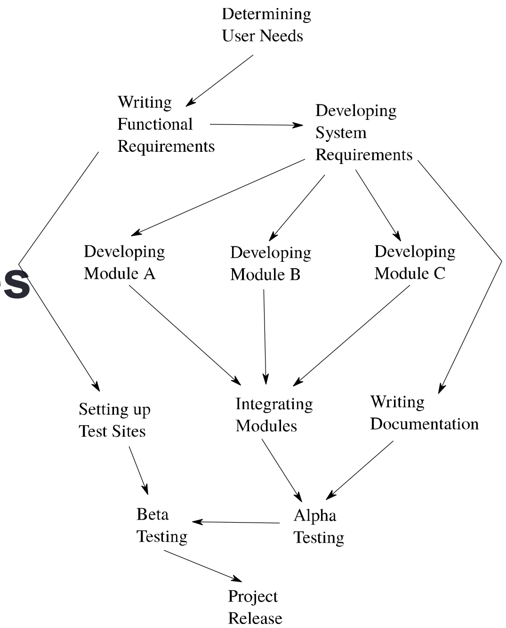
Basic objects : **cities**

Relationships between them : **nonstop flights OR roads**

Graphs

Basic objects : tasks

Relationships between them : dependencies



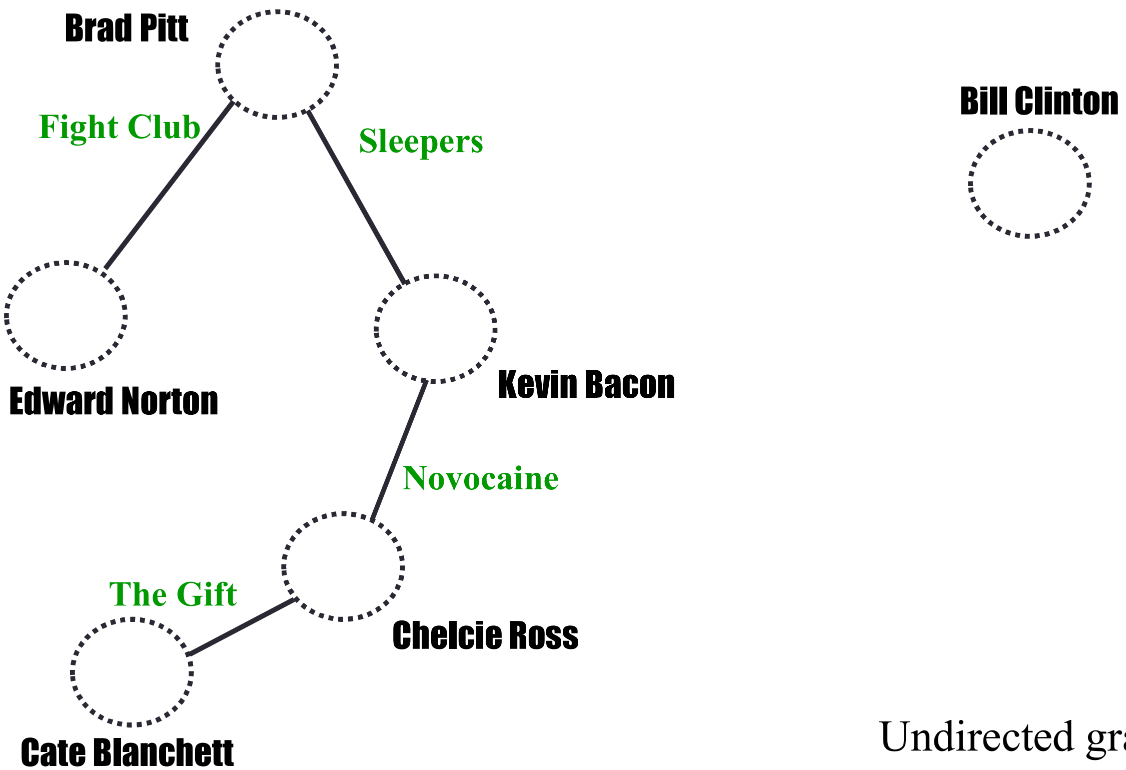
Why Graphs?

But don't just take my word for it…

https://www.coursera.org/learn/advanced-data-structures/lecture/3ovpb/in-the-real-world-graphs-at-google

https://www.coursera.org/learn/advanced-data-structures/lecture/ACQAt/in-the-real-world-more-graphs-at-google

Another (Important?) Application of Graphs

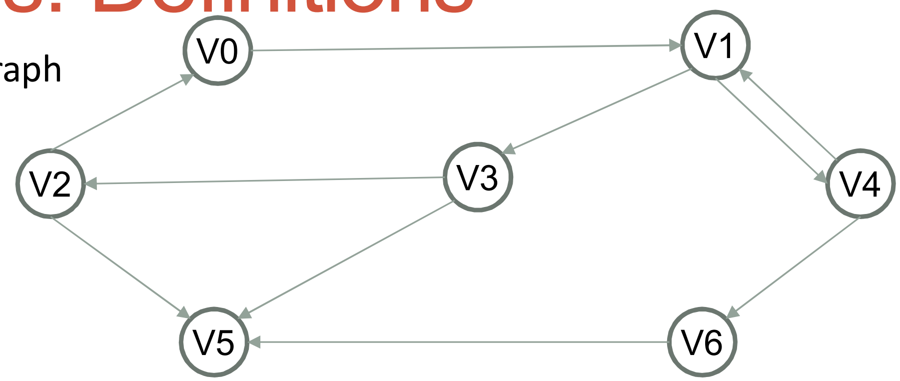


The "Oracle of Bacon" at **oracleofbacon.org/**

Undirected graphs model relationships in which all connections are two-way.

Graphs: Definitions

A directed graph

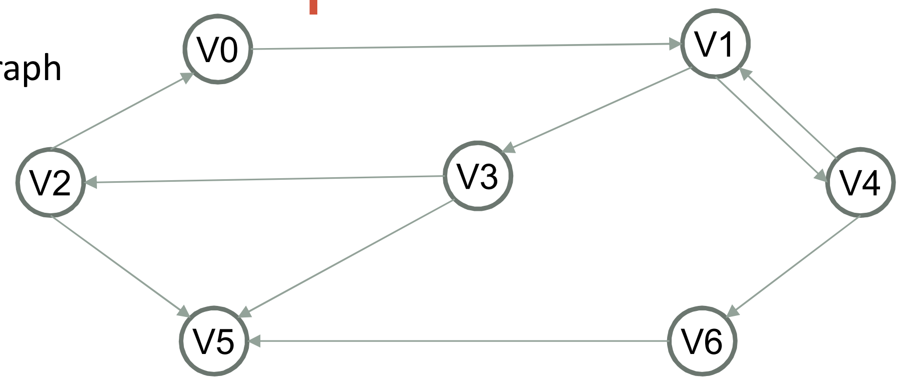


A graph G = (V,E) consists of a set of vertices V and a set of edges E

- Each edge in E is a pair (v,w) such that v and w are in V.
- If G is an *undirected* graph, (v,w) in E means vertices v and w are connected by an edge in G. This (v,w) is an unordered pair
- If G is a *directed* graph, (v,w) in E means there is an edge going from vertex v to vertex w in G. This (v,w) is an ordered pair; there may or may not also be an edge (w,v) in E
- In a *weighted* graph, each edge also has a “weight” or “cost” c, and an edge in E is a triple (v,w,c)
- When talking about the size of a problem involving a graph, the number of vertices |V| and the number of edges |E| will be relevant

Graphs: Example

A directed graph



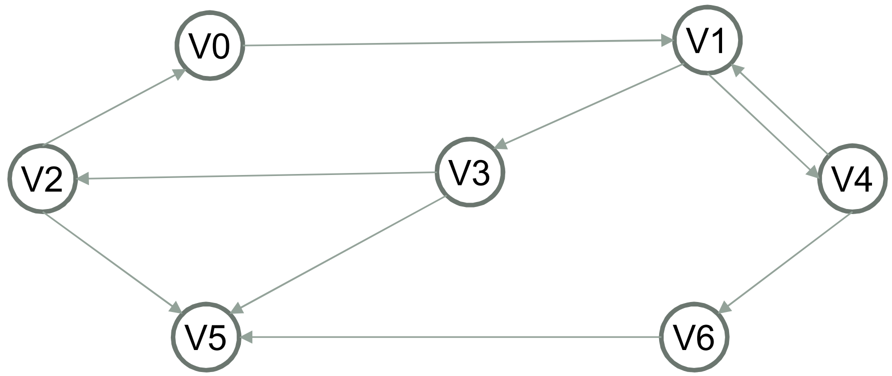
V = {

|V| =

E = {

|E|

Representing Graphs: Adjacency Matrix

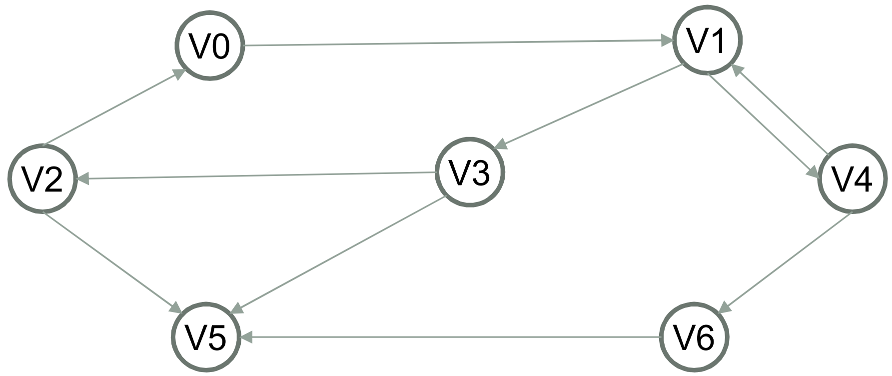


	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

A 2D array where each entry [i][j] encodes connectivity information between i and j

- For an unweighted graph, the entry is 1 if there is an edge from i to j, 0 otherwise
- For a weighted graph, the entry is the weight of the edge from i to j, or “infinity” if there is no edge
- Note an undirected graph’s adjacency matrix will be symmetrical

Representing Graphs: Adjacency Matrix



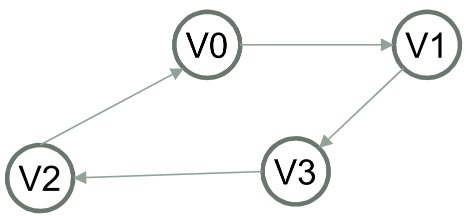
	0	1	2	3	4	5	6
0		1					
1				1	1		
2	1					1	
3			1			1	
4		1					1
5							
6						1	

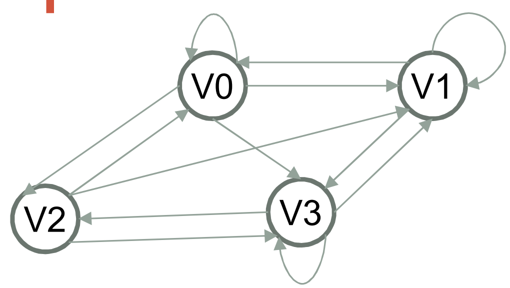
How big is an adjacency matrix in terms of the number of nodes and edges (BigO, tightest bound)?

A. |V|
B. |V|+|E|
C. |V|^2
D. |E|^2
E. Other

When is that OK? When is it a problem?

Sparse vs. Dense Graphs





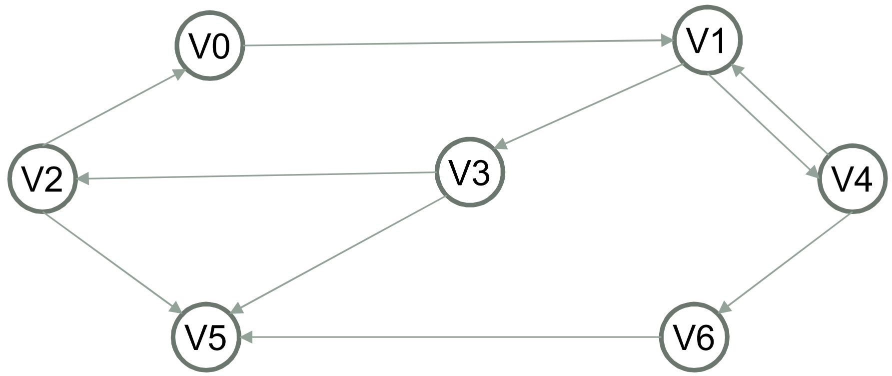
	0	1	2	3
0	0	1	0	0
1	0	0	0	1
2	1	0	0	0
3	0	0	1	0

	0	1	2	3
0	1	1	1	1
1	1	1	0	1
2	1	1	0	1
3	0	1	1	1

A dense graph is one where $|E|$ is “close to” $|V|^2$.
A sparse graph is one where $|E|$ is “closer to” $|V|$.

Adjacency matrices are space inefficient for sparse graphs

Representing Graphs: Adjacency Lists



V0:

V1:

V2:

V3:

V4:

V5:

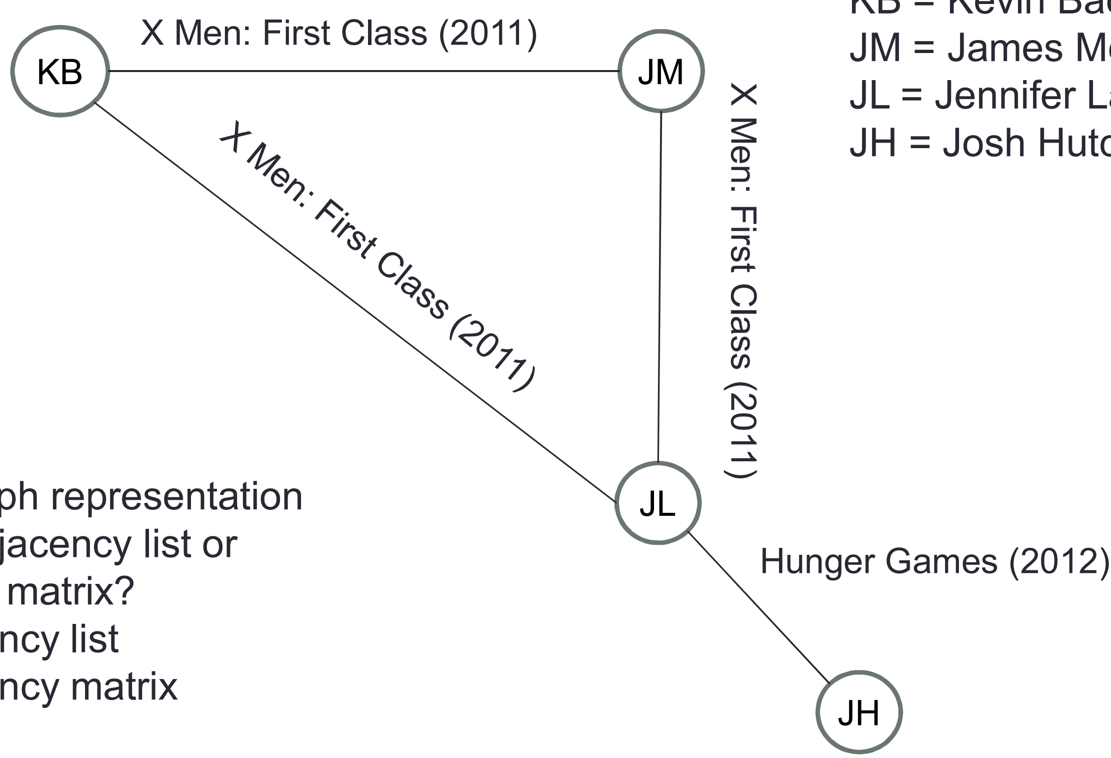
V6:

Each vertex has a list with the vertices adjacent to it.
In a weighted graph this list will include weights.

How much storage does this representation need?
(BigO, tightest bound)

A. |V|
B. |E|
C. |V|+|E|
D. |V|^2
E. |E|^2

Movie graphs: Matrix vs Lists

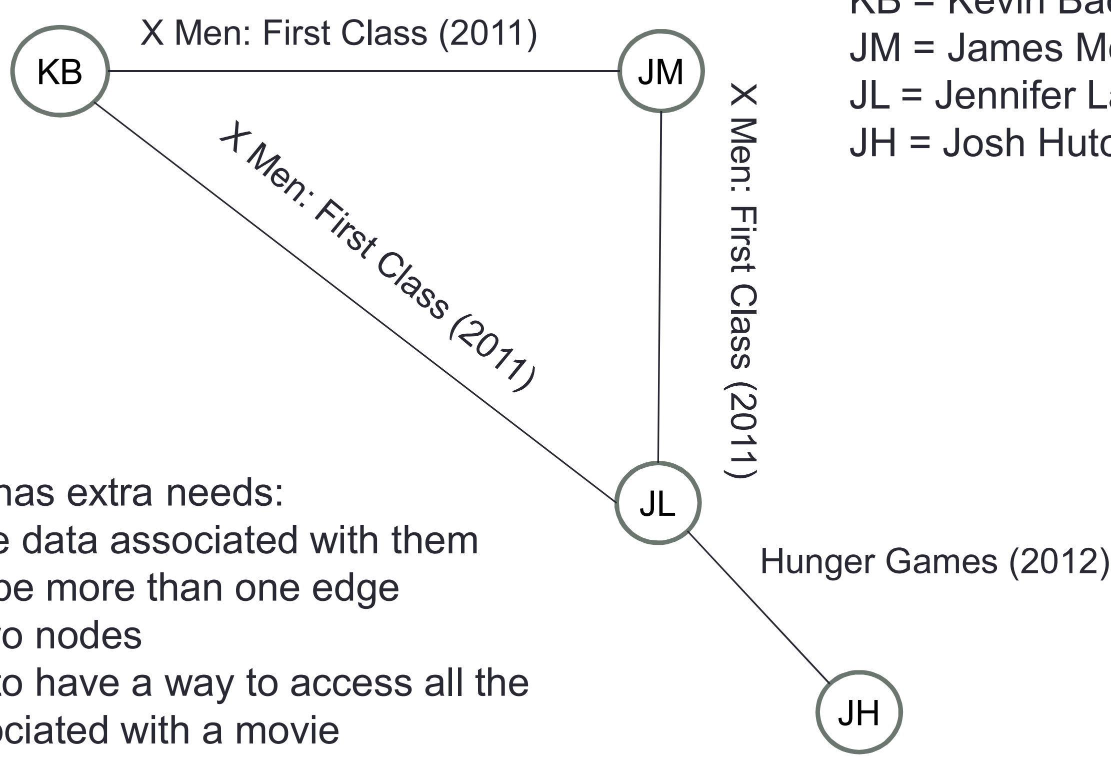


KB = Kevin Bacon
JM = James McAvoy
JL = Jennifer Lawrence
JH = Josh Hutcherson

Which graph representation is best: adjacency list or adjacency matrix?

A. Adjacency list
B. Adjacency matrix

Movie graphs: Representation hints

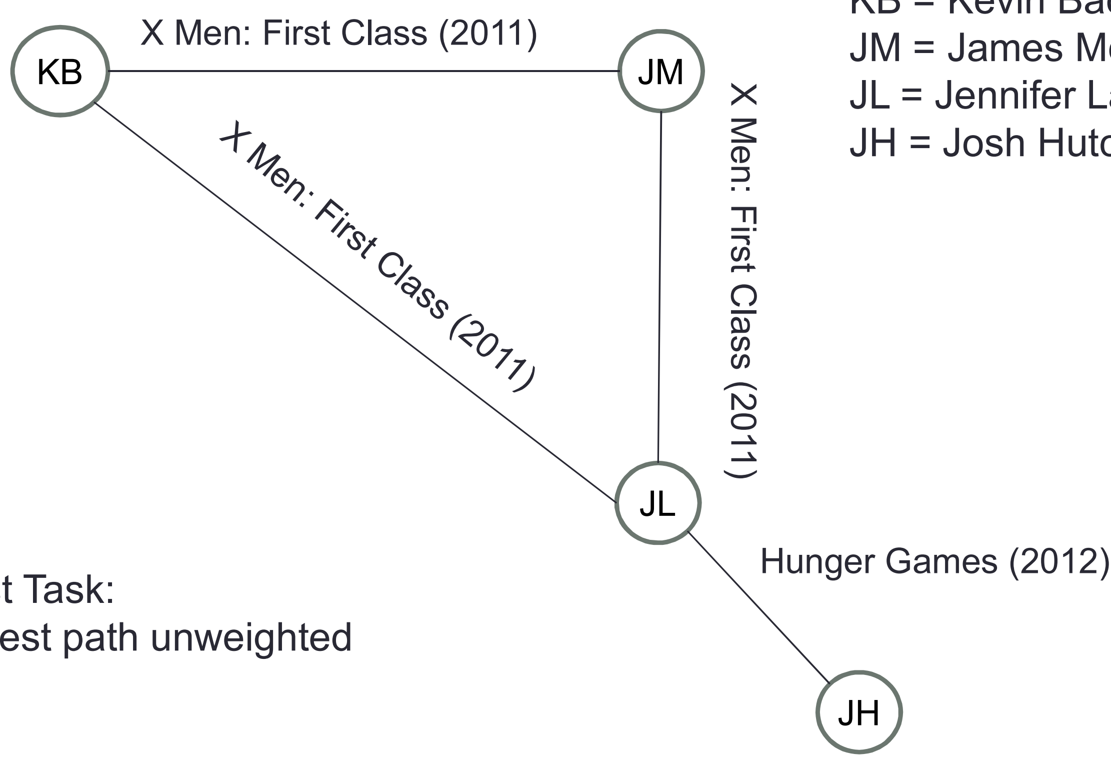


KB = Kevin Bacon
JM = James McAvoy
JL = Jennifer Lawrence
JH = Josh Hutcherson

This problem has extra needs:

- Edges have data associated with them
- There can be more than one edge between two nodes
- It is useful to have a way to access all the actors associated with a movie
- You will need to adapt the data structures we look at in class to meet your needs for PA3

Movie graphs: Representation hints



KB = Kevin Bacon
JM = James McAvoy
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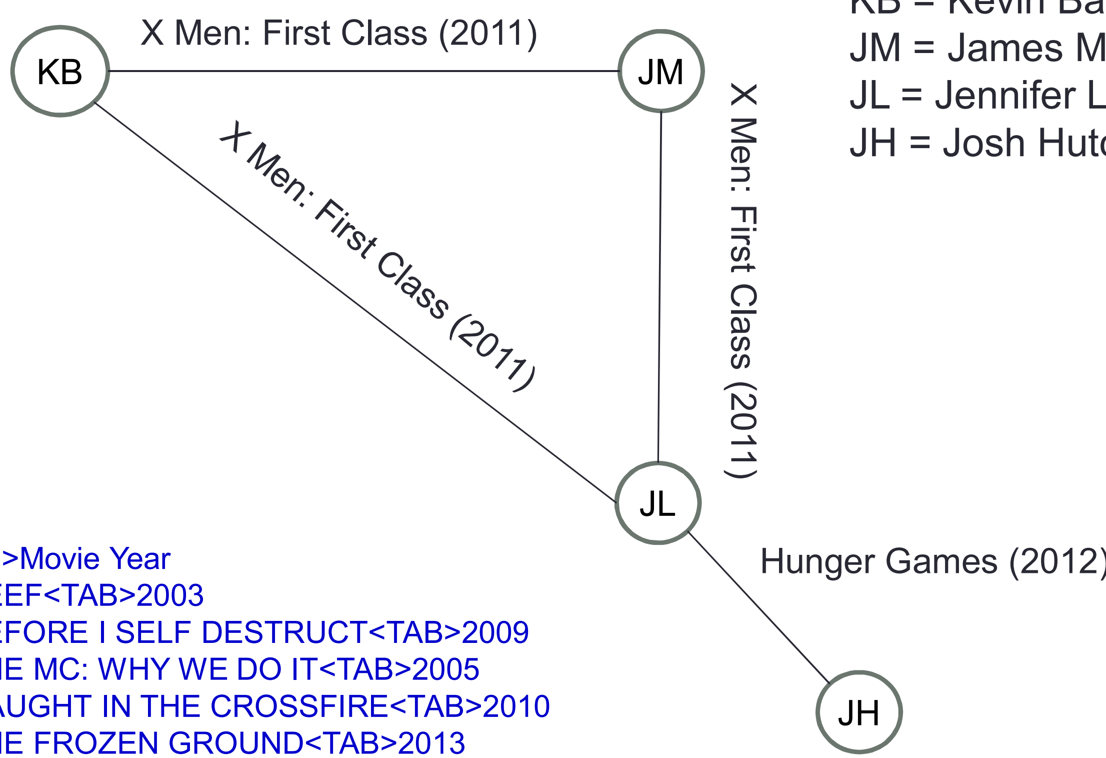
- PA3’s First Task:
  - Shortest path unweighted

Movie graphs: Representation hints



KB = Kevin Bacon
JM = James McAvoy
JL = Jennifer Lawrence
JH = Josh Hutcherson

Actor/Actress<TAB>Movie Year
50 CENT<TAB>BEEF<TAB>2003
50 CENT<TAB>BEFORE I SELF DESTRUCT<TAB>2009
50 CENT<TAB>THE MC: WHY WE DO IT<TAB>2005
50 CENT<TAB>CAUGHT IN THE CROSSFIRE<TAB>2010
50 CENT<TAB>THE FROZEN GROUND<TAB>2013
50 CENT<TAB>BEEF III<TAB>2005
50 CENT<TAB>LAST VEGAS<TAB>2013
50 CENT<TAB>GUN<TAB>2010

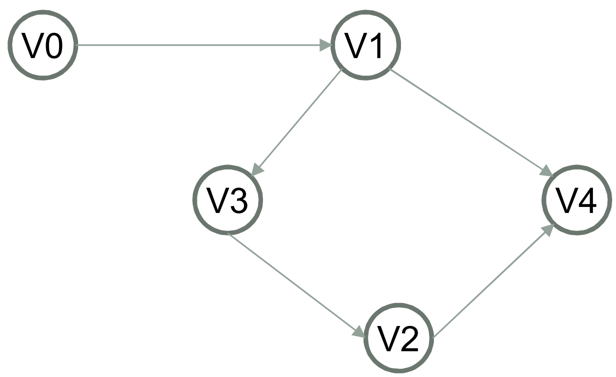
- Take time to come up with at least two ways to store this structure.

Movie graphs: Representation

Actor/Actress<TAB>Movie Year
50 CENT<TAB>BEEF<TAB>2003
50 CENT<TAB>BEFORE I SELF DESTRUCT<TAB>2009
50 CENT<TAB>THE MC: WHY WE DO IT<TAB>2005
50 CENT<TAB>CAUGHT IN THE CROSSFIRE<TAB>2010
50 CENT<TAB>THE FROZEN GROUND<TAB>2013
50 CENT<TAB>BEEF III<TAB>2005
50 CENT<TAB>LAST VEGAS<TAB>2013
50 CENT<TAB>GUN<TAB>2010

Depth First Search for Graph Traversal

- Search as far down a single path as possible before backtracking

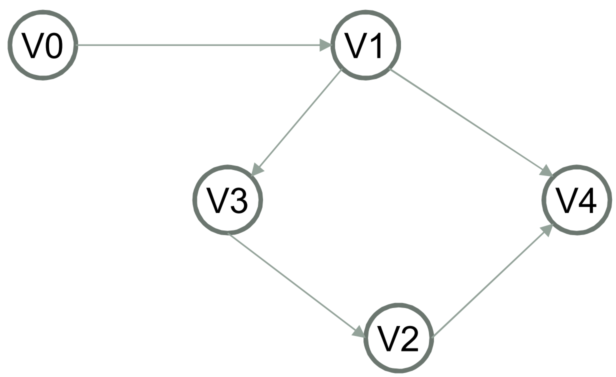


Assuming DFS chooses the lower number node to explore first, in what order does DFS visit the nodes in this graph (start at V0)?

A. V0, V1, V2, V3, V4
B. V0, V1, V3, V4, V2
C. V0, V1, V3, V2, V4
D. Other

Depth First Search for Graph Traversal

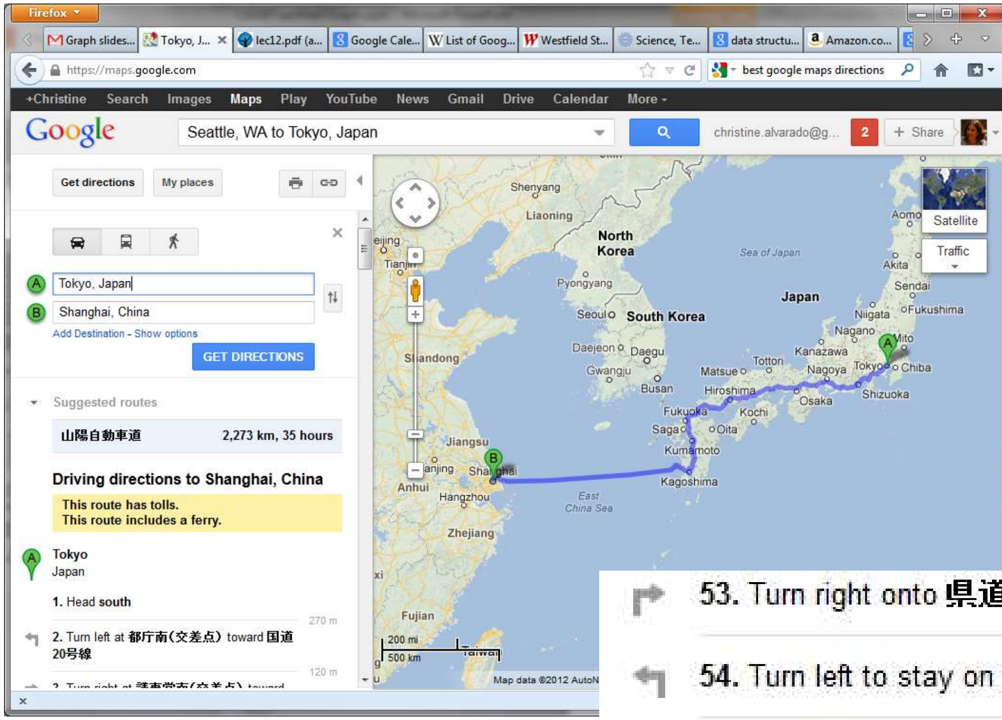
- Search as far down a single path as possible before backtracking



Does DFS always find the shortest path between nodes the first time it encounters a node in its search?

A. Yes
B. No

Shortest Path Algorithms



Finding the shortest route from one city to another is a natural application of graph algorithms!

(Of course there are many other examples)

Shortest Path Algorithms

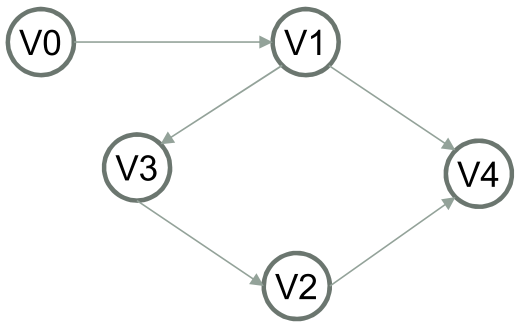
- We’ll look at shortest path algorithms in unweighted and weighted graphs
- These algorithms find the shortest path from a “source” (or start) vertex to every other vertex in the graph (it’s no slower than finding a path to just one destination)
- You will implement some of these algorithms in your PA3

Unweighted Shortest Path

- Input: an unweighted directed graph G = (V, E) and a source vertex *s* in V
- Output: for each vertex *v* in V, a representation of the shortest path in G that starts in *s* and ends at *v*
- This is really just a search problem. We'll look at three algorithms:
  - Depth First Search – inefficient to produce the shortest path
  - Breadth First Search
  - Best-First Search (for weighted graphs)

Breadth First Search

- Explore all the nodes reachable from a given node before moving on to the next node to explore

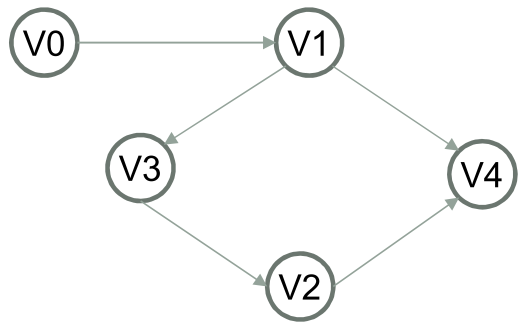


Assuming BFS chooses the lower number node to explore first, in what order does BFS visit the nodes in this graph?

A. V0, V1, V2, V3, V4
B. V0, V1, V3, V4, V2
C. V0, V1, V3, V2, V4
D. Other

BFS Traverse: Idea

- Input: an unweighted directed graph G = (V, E) and a source vertex *s* in V
- Output: for each vertex *v* in V, a representation of the shortest path in G that starts in *s* and ends at *v*

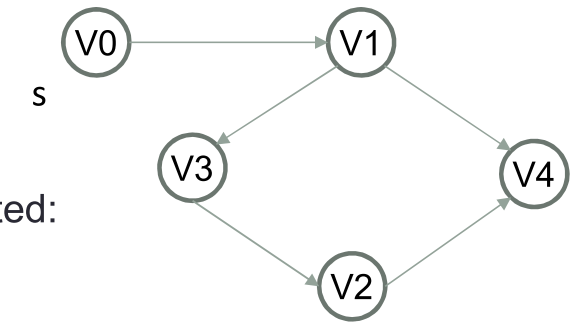


Start at *s*. It has distance 0 from itself.
Consider nodes adjacent to s. They have distance 1. Mark them as visited.
Then consider nodes that have not yet been visited
adjacent to those at distance 1. They have distance 2. Mark them as visited.
Etc. etc. until all nodes are visited.

BFS Traverse: Sketch of Algorithm

The basic idea is a breadth-first search of the graph, starting at source vertex *s*

- Initially, give all vertices in the graph a distance of INFINITY
- Start at *s*; give *s* distance = 0
- Enqueue *s* into a queue
- While the queue is not empty:
  - Dequeue the vertex *v* from the head of the queue
  - For each of *v*'s adjacent nodes that has not yet been visited:
    - Mark its distance as 1 + the distance to *v*
    - Enqueue it in the queue



Queue:

BFS Traverse: Sketch of Algorithm

The basic idea is a breadth-first search of the graph, starting at source vertex *s*

- Initially, give all vertices in the graph a distance of INFINITY
- Start at *s*; give *s* distance = 0
- Enqueue *s* into a queue
- While the queue is not empty:
  - Dequeue the vertex *v* from the head of the queue
  - For each of *v*'s adjacent nodes that has not yet been visited:
    - Mark its distance as 1 + the distance to *v*
    - Enqueue it in the queue

Questions:

- What data do you need to keep track of for each node?
- How can you tell if a vertex has been visited yet?
- This algorithm finds the length of the shortest path from s to all nodes. How can you also find the path itself?

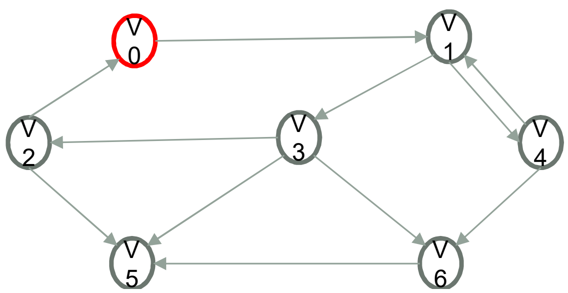
BFS Traverse: Details

source

V0: dist=	**prev=**	**adj: V1**
V1: dist=	**prev=**	**adj: V3, V4**
V2: dist=	**prev=**	**adj: V0, V5**
V3: dist=	**prev=**	**adj: V2, V5, V6**
V4: dist=	**prev=**	**adj: V1, V6**
V5: dist=	**prev=**	**adj:**
V6: dist=	**prev=**	**adj: V5**

The queue (give source vertex dist=0 and prev=-1 and enqueue to start):

HEAD **TAIL**



Representing the graph with `structs`

```
#include <iostream>
#include <limits>
#include <vector>
#include <queue>

using namespace std;

struct Vertex {
  vector<int> adj;  // The adjacency list
  int dist;         // The distance from the source
  int index;  // The index of this vertex
  int prev;   // The index of the vertex previous in the path
};

vector<Vertex*> createGraph() {
…
}
```

Unweighted Shortest Path: C++ code

```
/** Traverse the graph using BFS */
void BFSTraverse( vector<Vertex*> theGraph, int from )
{
  // assume code to initialize each Vertex's dist to INFINITY
  queue<Vertex*> toExplore;
  Vertex* start = theGraph[from];
  // finish the code…

}
```

```
struct Vertex
{
  vector<int> adj;
  int dist;
  int index;
  int prev;
};
```

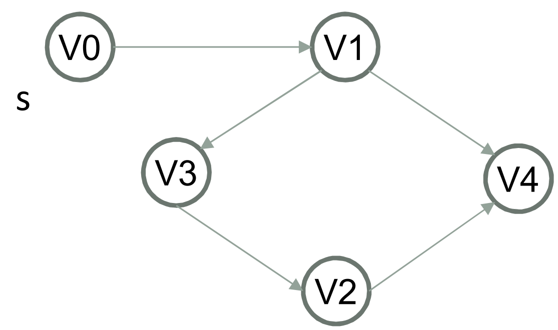
Unweighted Shortest Path: C++ code

```
/** Traverse the graph using BFS */
void BFSTraverse( vector<Vertex*> theGraph, int from )
{
  // assume code to initialize each Vertex's dist to INFINITY
  queue<Vertex*> toExplore;
  Vertex* start = theGraph[from];
  start->dist = 0;
  toExplore.push(start);
  while ( !toExplore.empty() ) {

    Vertex* next = toExplore.front();
    toExplore.pop();
    vector<int>::iterator it = next->adj.begin();
    for ( ; it != next->adj.end(); ++it ) {
      Vertex* neighbor = theGraph[*it];
      if (next->dist+1 < neighbor->dist) {
        neighbor->dist = next->dist + 1;
        neighbor->prev = next->index;
        toExplore.push(neighbor);
      }
    }
  }
}
```

Unweighted Shortest Path: Running Time

The basic idea is a breadth-first search of the graph, starting at source vertex *s*

- Initially, give all vertices in the graph a distance of INFINITY
- Start at *s*; give *s* distance = 0
- Enqueue *s* into a queue
- While the queue is not empty:
  - Dequeue the vertex *v* from the head of the queue
  - For each of *v*'s adjacent nodes that has not yet been visited:
    - Mark its distance as 1 + the distance to *v*
    - Enqueue it in the queue



What is the tightest worst-case time complexity (in terms of |V| and |E|) of this algorithm?

A. O(|V|) B. O(|E|) C. O(|V|+|E|)

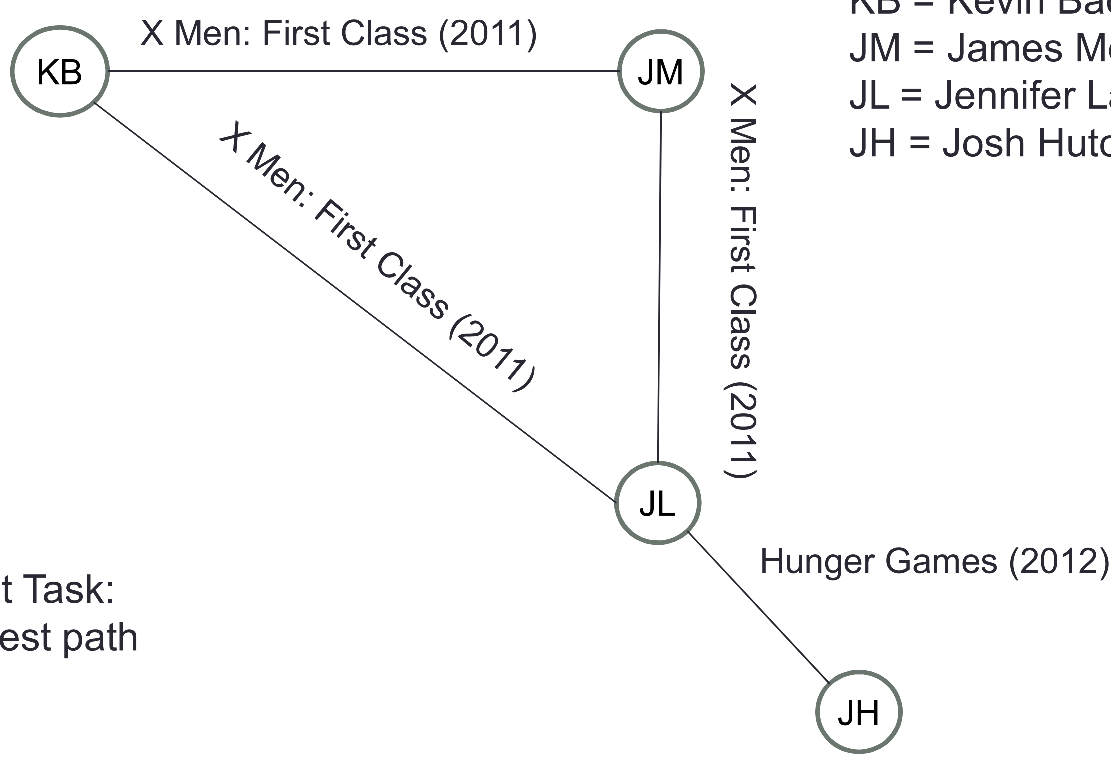
D. O(|V|^2) E. Other

Representing the graph with `structs`

```
#include <iostream>
#include <limits>
#include <vector>
#include <queue>

using namespace std;

struct Vertex
{
  vector<int> adj;  // The adjacency list
  int dist;         // The distance from the source
  int index;  // The index of this vertex
  int prev;   // The index of the vertex previous in the path
};

vector<Vertex*> createGraph()
{ … }
```

Your representation for PA3 will have some similarities and probably some differences.

Movie graphs: Representation hints



KB = Kevin Bacon
JM = James McAvoy
JL = Jennifer Lawrence
JH = Josh Hutcherson

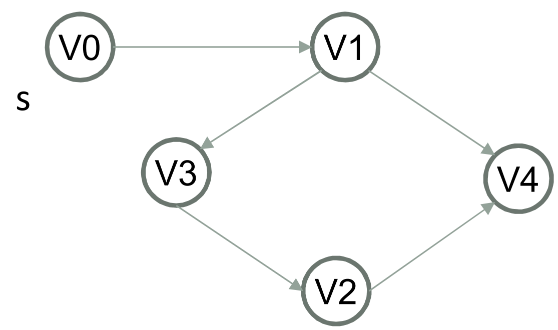
- PA3's First Task:
  - Shortest path

What is this algorithm??

The basic idea is a breadth-first search of the graph, starting at source vertex *s*

- Initially, give all vertices in the graph a distance of INFINITY
- Start at *s*; give *s* distance = 0
- ~~Enqueue~~ Push *s* into a ~~queue~~ stack
- While the ~~queue~~ stack is not empty:
  - ~~Dequeue~~ pop the vertex *v* from the ~~head of the queue~~ top of the stack
  - For each of *v*'s adjacent nodes that has not yet been visited:
    - Mark its distance as 1 + the distance to *v*
    - ~~Enqueue~~ Push it on the ~~queue~~ stack

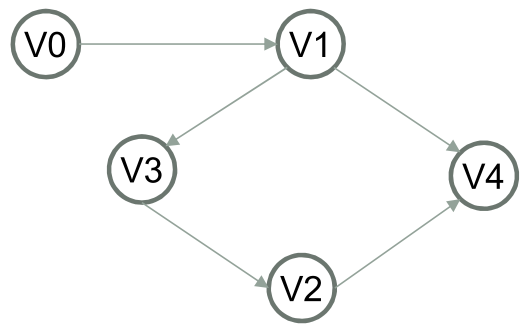
Stack:



A. BFS B. DFS shortest path C. DFS not shortest path D. Dijkstra's algorithm

Breadth First Search

- Explore all the nodes reachable from a given node before moving on to the next node to explore

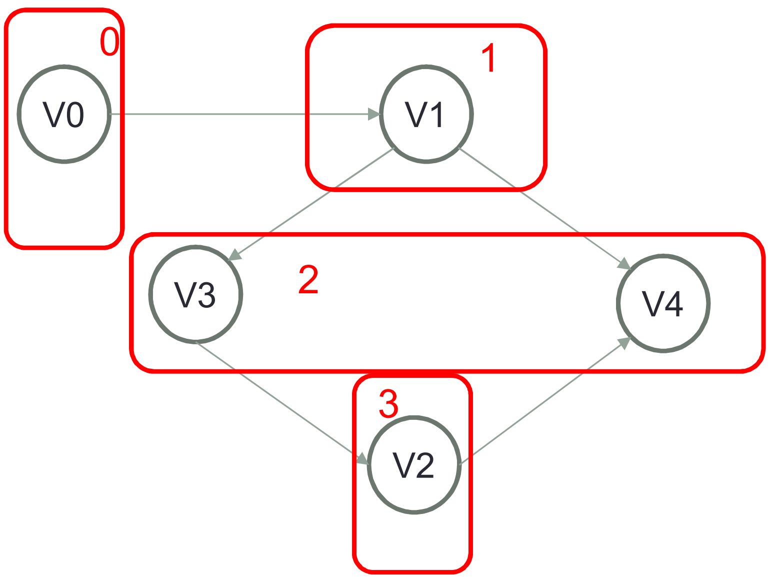


Does BFS always find the shortest path from the source to any node?

A. Yes for unweighted graphs
B. Yes for all graphs
C. No

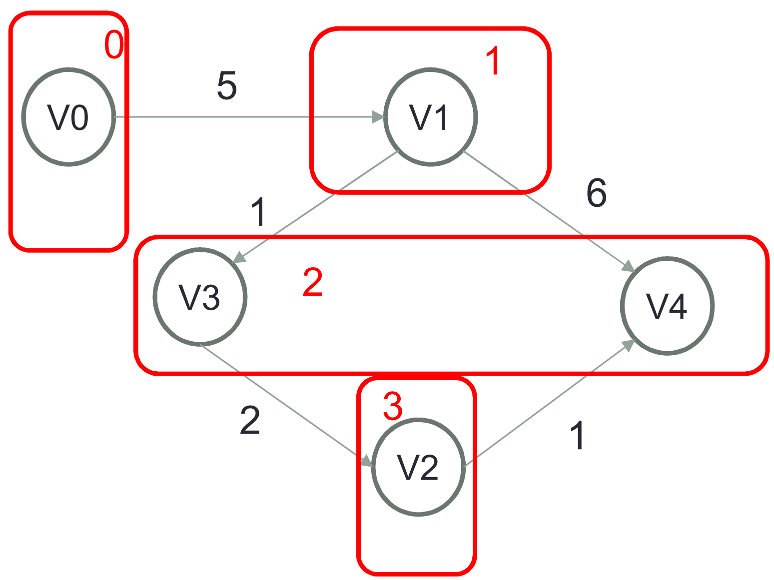
Why Did BFS Work for Shortest Path?

- Vertices are explored in order of their distance away from the source. So we are guaranteed that the first time we see a vertex we have found the shortest path to it.
- The queue (FIFO) assures that we will explore from all nodes in one level before moving on to the next.



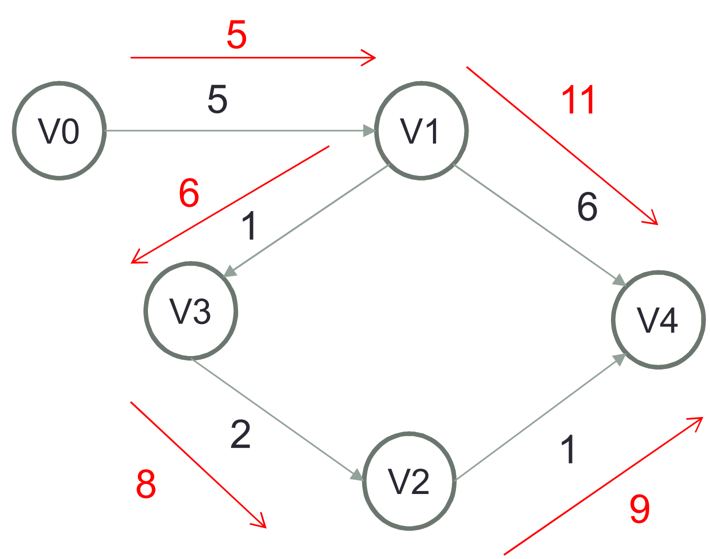
BFS on weighted graphs?

- Run BFS on this weighted graph. What are the weights of the paths that you find to each vertex from v0? Are these the shortest paths?



BFS on weighted graphs?

- In a weighted graph, the number of edges no longer corresponds to the length of the path. We need to decouple path length from edges, and explore paths in increasing *path length* (rather than increasing number of edges).
- In addition, the first time we encounter a vertex may, we may not have found the shortest path to it, so we need to delay committing to that path.



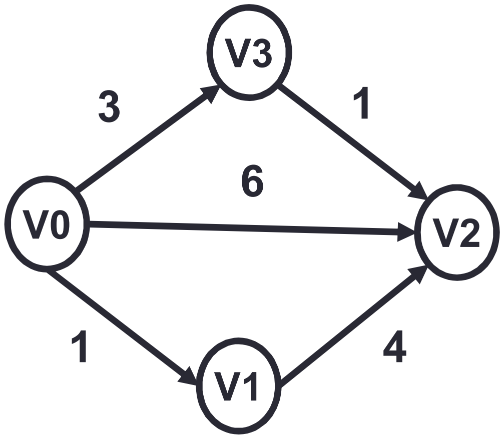
Dijkstra’s Algorithm: Data Structures

- Maintain a sequence (e.g. an array) of vertex objects, indexed by vertex number
  - Vertex objects contain these 3 fields (and others):
    - “dist”: the cost of the best (least-cost) path discovered so far from the start vertex to this vertex
    - “prev”: the vertex number (index) of the previous node on that best path
    - “done”: a boolean indicating whether the “dist” and “prev” fields contain the final best values for this vertex, or not
- Maintain a priority queue
  - The priority queue will contain (*pointer-to-vertex*, *path cost*) pairs
  - *Path cost* is priority, in the sense that low cost means high priority
  - Note: multiple pairs with the same “*pointer-to-vertex*” part can exist in the priority queue at the same time. These will usually differ in the “*path cost*” part

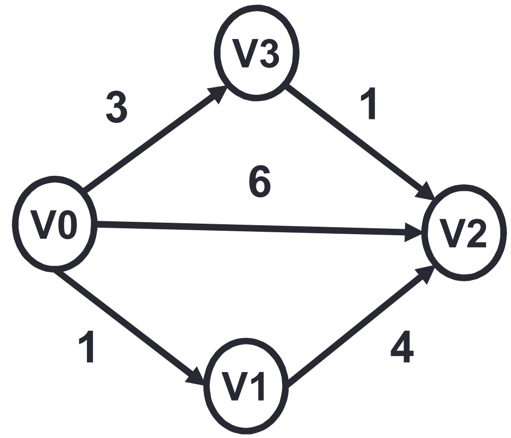
Dijkstra’s Algorithm

Nodes have:
prev
dist
done

```
Dijkstra(S):
  Initialize: Priority queue (PQ), dist fields to infinity,
              prev fields to -1, done fields to false
  Set S's dist to 0
  Enqueue {S, 0} onto the PQ
  while PQ is not empty:
    dequeue node v from front of queue
    if (v is not done)
      set v.done to true
      for each of v's neighbors, w:
        //distance to w through v is:
        c = v.dist + edgeWeight(v, w)
        if c is less than w.dist:
          set w.prev = v and w.dist = c
          enqueue {w, c} into the PQ
```



The array of vertices, which include dist, prev, and done fields (initialize dist to 'INF' and done to 'f'):



V0: dist= **prev=** **done=**

V1: dist= **prev=** **done=**

V2: dist= **prev=** **done=**

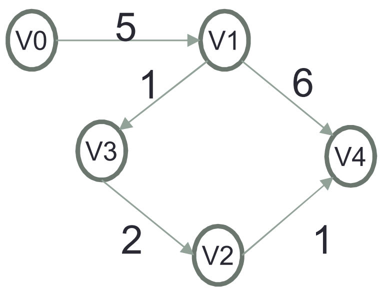
V3: dist= **prev=** **done=**

Initial PQ:	(0, V0)					
PQ after expanding V0:						

Dijkstra's Algorithm: Questions

```
Dijkstra(S):
   Initialize: Priority queue (PQ), dist fields to infinity, prev fields to -1, done fields to false
   Enqueue {S, 0} onto the PQ
   while PQ is not empty:
      dequeue node v from front of queue
      if (v is not done)
         set v.done to true
         for each of v's neighbors, w:
             distance to w through v, c = v.dist + edgeWeight(v, w)
                if c is less than w.dist:
                set w.prev = v and w.dist = c
                enqueue {w, c} into the PQ
```

When a node comes out of the priority queue, how do you know you've found the shortest path to the node?



Dijkstra's Algorithm: Running time

```
Dijkstra(S):
   Initialize: Priority queue (PQ), dist fields to infinity, prev fields to -1, done fields to false
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      dequeue node v from front of queue
      if (v is not done)
         set v.done to true
         for each of v's neighbors, w:
            distance to w through v, c = v.dist + edgeWeight(v, w)
            if c is less than w.dist:
               set w.prev = v and w.dist = c
               enqueue {w, c} into the PQ
```
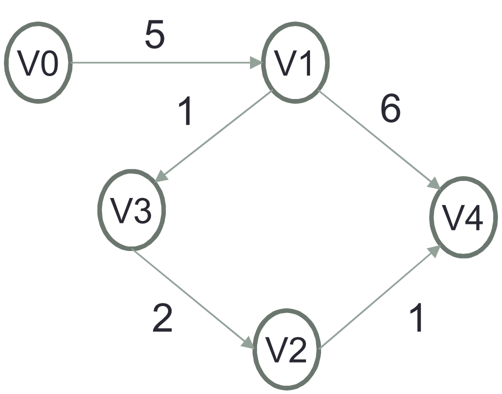

How long does the step in red take?

A. O(1)
B. O(|V|)
C. O(|E|)
D. O(|V|+|E|)
E. Other



Dijkstra's Algorithm: Running time

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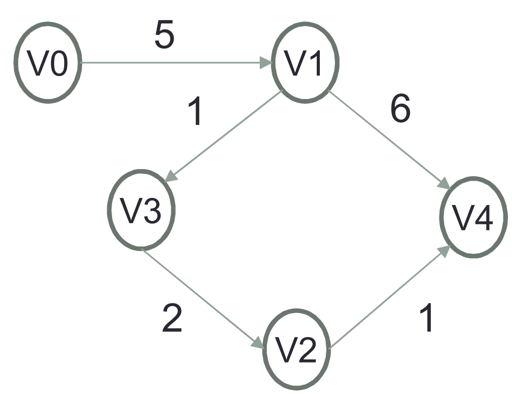
  Initialize: Priority queue (PQ), dist fields to infinity, prev fields to -1, done fields to false
  Enqueue {S, 0} onto the PQ
  while PQ is not empty:
    dequeue node v from front of queue
    if (v is not done)
      set v.done to true
      for each of v's neighbors, w:
        distance to w through v, c = v.dist + edgeWeight(v, w)
        if c is less than w.dist:
          set w.prev = v and w.dist = c
          enqueue {w, c} into the PQ
```

How long does the step in red take?

A. O(1)
B. O(|V|)
C. O(|E|)
D. O(|V|+|E|)
E. Other



Walkthrough

Dijkstra(S, G):

```
Initialize: Priority queue (PQ), dist fields to infinity,
            prev fields to -1, done fields to false
Enqueue {S, 0} onto the PQ
while PQ is not empty:
    dequeue node v from front of queue
    if (v is not done)
        set v.done to true
        for each of v's neighbors, w:
            distance to w through v, c = v.dist + edgeWeight(v, w)
            if c is less than w.dist:
                set w.prev = v and w.dist = c
                enqueue {w, c} into the PQ
```

O(|V|)

O(|1|)

Walkthrough

Dijkstra(S, G):

```
Initialize: Priority queue (PQ), dist fields to infinity,
            prev fields to -1, done fields to false
Enqueue {S, 0} onto the PQ
while PQ is not empty:
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         if c is less than w.dist:
            set w.prev = v and w.dist = c
            enqueue {w, c} into the PQ
```

$O(|V|)$

~~$O(|1|)$~~

$O(|V|\cancel{+1})$

Walkthrough

Dijkstra(S, G):

```
Initialize: Priority queue (PQ), dist fields to infinity,
            prev fields to -1, done fields to false
Enqueue {S, 0} onto the PQ
while PQ is not empty:
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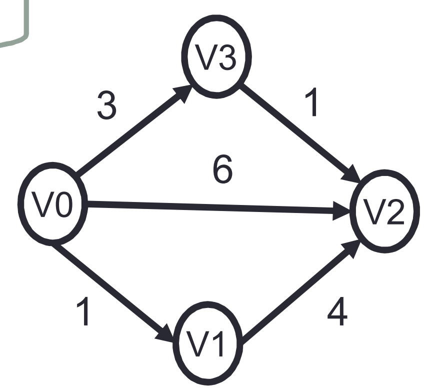
        for each of v's neighbors, w:
            distance to w through v, c = v.dist + edgeWeight(v, w)
            if c is less than w.dist:
                set w.prev = v and w.dist = c
                enqueue {w, c} into the PQ
```

O(|V|)

O(?)



Pairs of (node, cost) go into the priority queue. Can a node go into the priority queue more than once?

A. Yes
B. No

Walkthrough

Dijkstra(S, G):

```
Dijkstra(S, G):
  Initialize: Priority queue (PQ), dist fields to infinity,
              prev fields to -1, done fields to false
  Enqueue {S, 0} onto the PQ
  while PQ is not empty:
    dequeue node v from front of queue
    if (v is not done)
      set v.done to true
      for each of v's neighbors, w:
        distance to w through v, c = v.dist + edgeWeight(v, w)
        if c is less than w.dist:
          set w.prev = v and w.dist = c
          enqueue {w, c} into the PQ
```

O(|V|)

O(?)

The total number of pairs that go into the priority queue is approximately which of the following (in the worst case):

A. |V| (the number of nodes in the graph)
B. |E| (the number of edges in the graph)
C. |V| + |E|
D. |V| * |E|

Walkthrough

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```

O(|V|)

O(?)

So the while loop is making O(|E|) insertions into a priority queue with size at most O(|E|).
What is the total running time for the while loop?

A. O(|E|)
B. O(|E| log |E|)
C. O(|E| * |E|)
D. Other

Walkthrough

```
Dijkstra(S, G):
    Initialize: Priority queue (PQ), dist fields to infinity,
                prev fields to -1, done fields to false
    Enqueue {S, 0} onto the PQ
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                if c is less than w.dist:
                    set w.prev = v and w.dist = c
                    enqueue {w, c} into the PQ
```

O(|V|)

whole loop

O(|E| log |E|)

Walkthrough

Dijkstra(S, G):

```
Initialize: Priority queue (PQ), dist fields to infinity,
            prev fields to -1, done fields to false
Enqueue {S, 0} onto the PQ
while PQ is not empty:
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      if c is less than w.dist:
        set w.prev = v and w.dist = c
        enqueue {w, c} into the PQ
```

$\boxed{O(|V|)}$

whole loop

$\boxed{O(|E| \log |E|)}$

Overall:
O(|E| log |E| + |V|)

Walkthrough

Dijkstra(S, G):

```
Dijkstra(S, G):
    Initialize: Priority queue (PQ), dist fields to infinity,
                prev fields to -1, done fields to false
    Enqueue {S, 0} onto the PQ
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                if c is less than w.dist:
                    set w.prev = v and w.dist = c
                    enqueue {w, c} into the PQ
```

O(|V|)

whole loop

O(|E| log |E|)

Because $|E| <= |V|^2$ and $\log(|V|^2)$ is just $O(\log(|V|))$ we could tighten to:
$O(|E| \log |V| + |V|)$

Overall:
$O(|E| \log |E| + |V|)$

Unweighted Shortest Path: Running Time

```
BFS(S):
  Initialize queue, set dist to INFINITY and prev to null for all nodes
  Add S to queue and set S.dist to 0
  while queue is not empty:
    dequeue node curr from head of queue
    set n.visited = true
    for each of curr's neighbors, n:
      if n.dist > curr.dist+1:
        set n.dist to curr.dist+1
        set n's prev to curr
        enqueue n to the queue
// When we get here then we're done exploring from S
```

What is the time complexity (in terms of |V| and |E|) of this algorithm?

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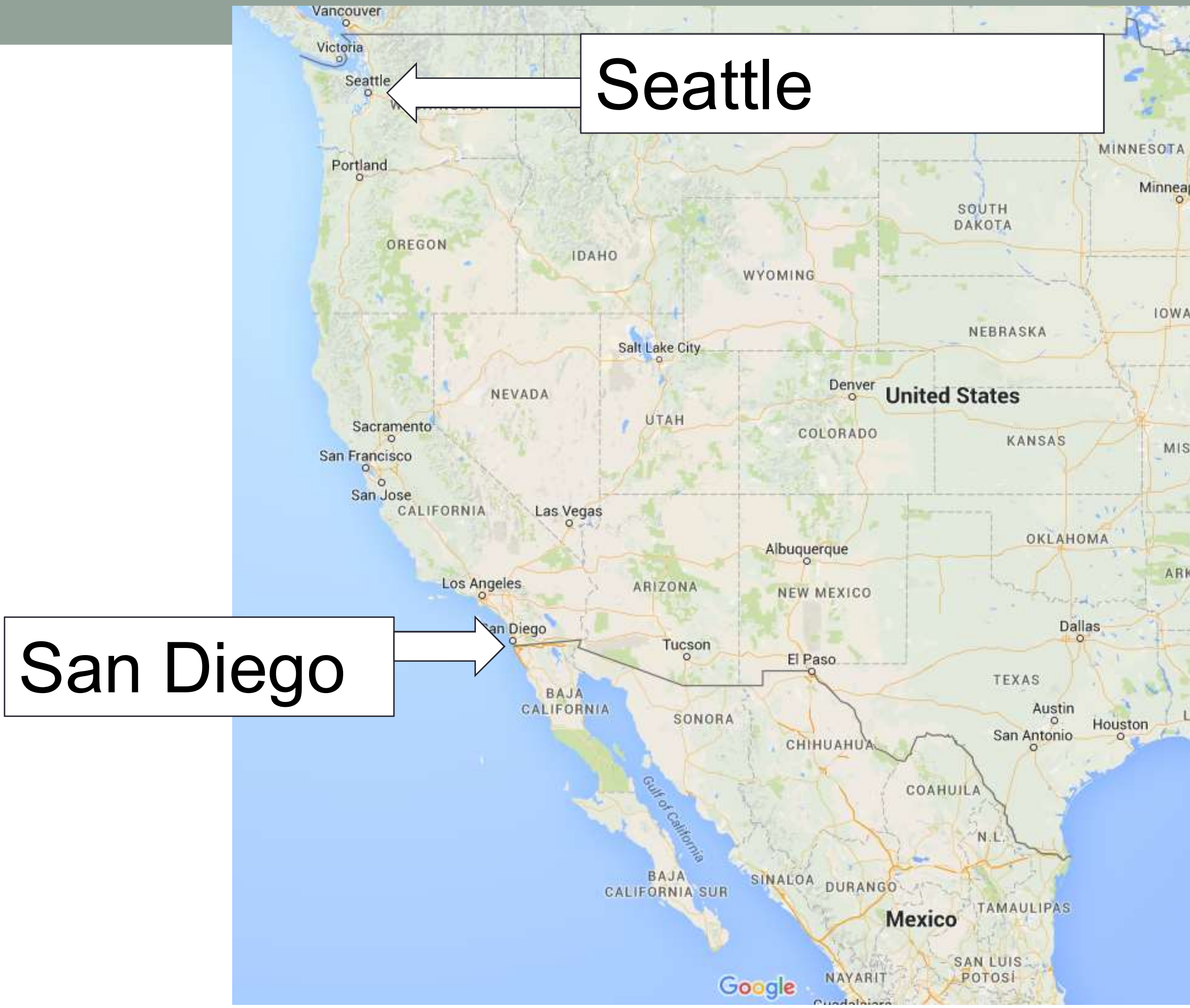
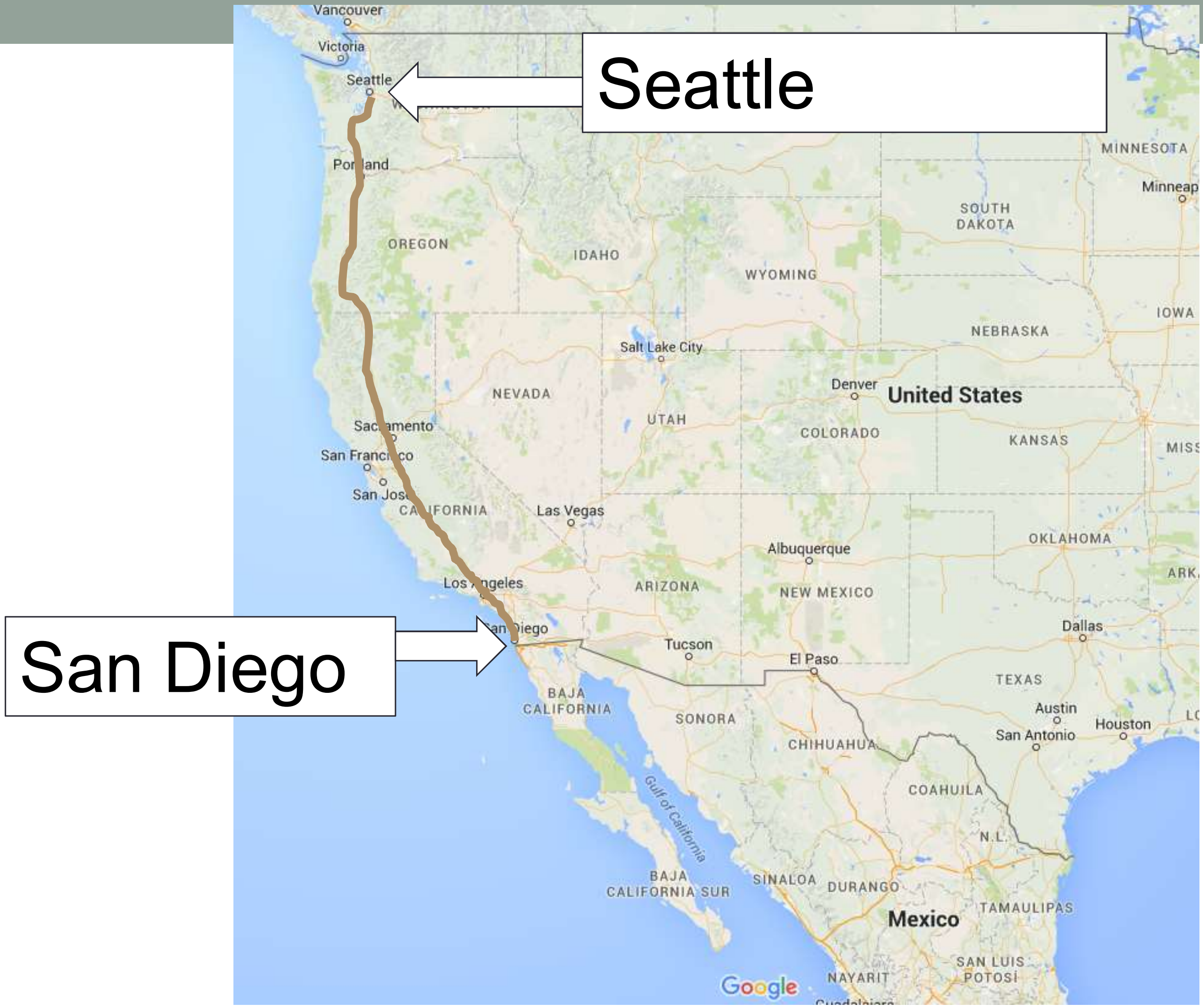


Driving directions from San Diego to Seattle?



maps.google.com

Driving directions from San Diego to Seattle?

maps.google.com

Driving directions from San Diego to Seattle?

1255 miles

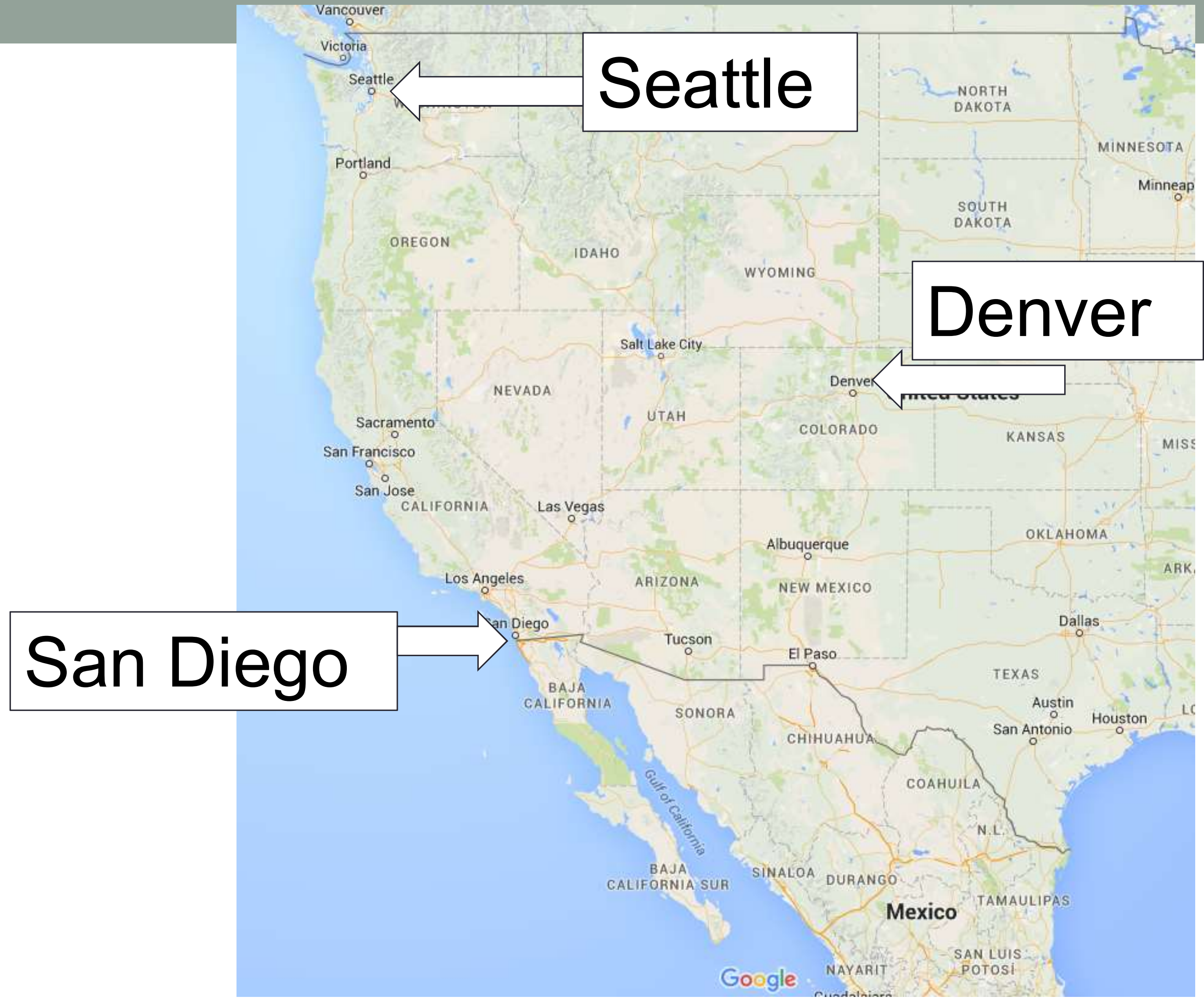
maps.google.com

1255 miles

Dijkstra will find the shortest route. But how?

maps.google.com

Dijkstra will find the shortest route. But how?



Would Dijkstra have you consider Denver in finding the path to Seattle?

A. Yes
B. No
C. Maybe

Why would YOU have never considered Denver?

maps.google.com

Going East is the wrong direction!

maps.google.com

Seattle

We should consider distance from target too!

Dijkstra only considers distance from source

Denver

San Diego

Dijkstra's Algorithm

- Priority Queue ordering is based on:

g(n): the distance (cost) from start vertex to vertex n

A* Algorithm

- Priority Queue ordering is based on:

g(n): the distance (cost) from start vertex to vertex n

AND

h(n): the **heuristic estimated cost** from vertex n to goal vertex

A* Algorithm

- Priority Queue ordering is based on:

g(n): the distance (cost) from start vertex to vertex n

AND

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f(n) = g(n) + h(n)

A* Algorithm

- Priority Queue ordering is based on:

g(n): the distance (cost) from start vertex to vertex n

AND

h(n): the **heuristic estimated cost** from vertex n to goal vertex

f(n) = g(n) + h(n)

Dijkstra can be seen as a special case where h(n)=0

A* Algorithm

- Priority Queue ordering is based on:

g(n): the distance (cost) from start vertex to vertex n

AND

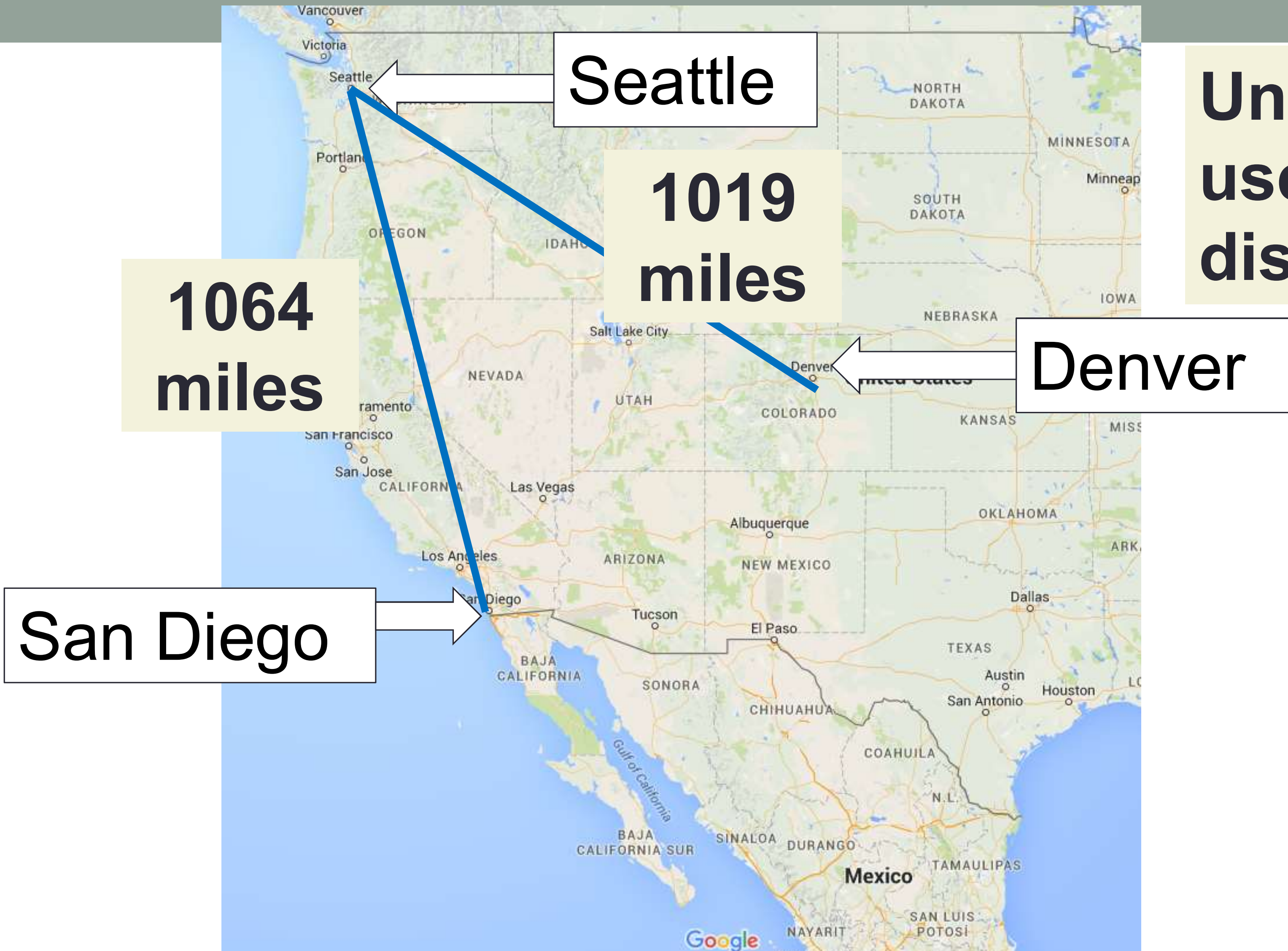
h(n): the **heuristic estimated cost** from vertex n to goal vertex

f(n) = g(n) + h(n)

Guaranteed to find shortest path IF estimate is never an overestimate

maps.google.com

Underestimate: use the exact distance.



http://distancecalculator.globefeed.com/World_Distance_Calculator.asp

A* Algorithm

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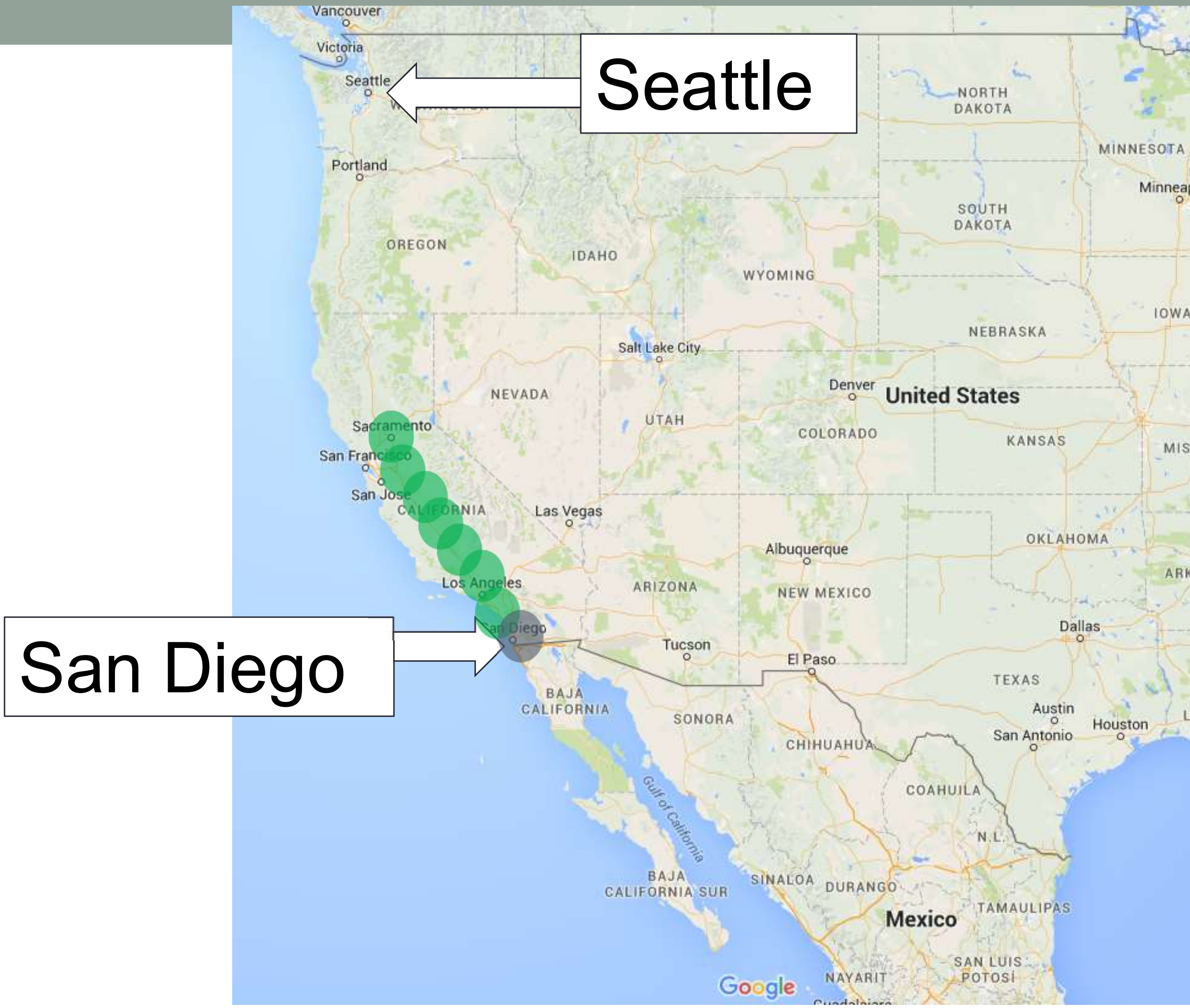
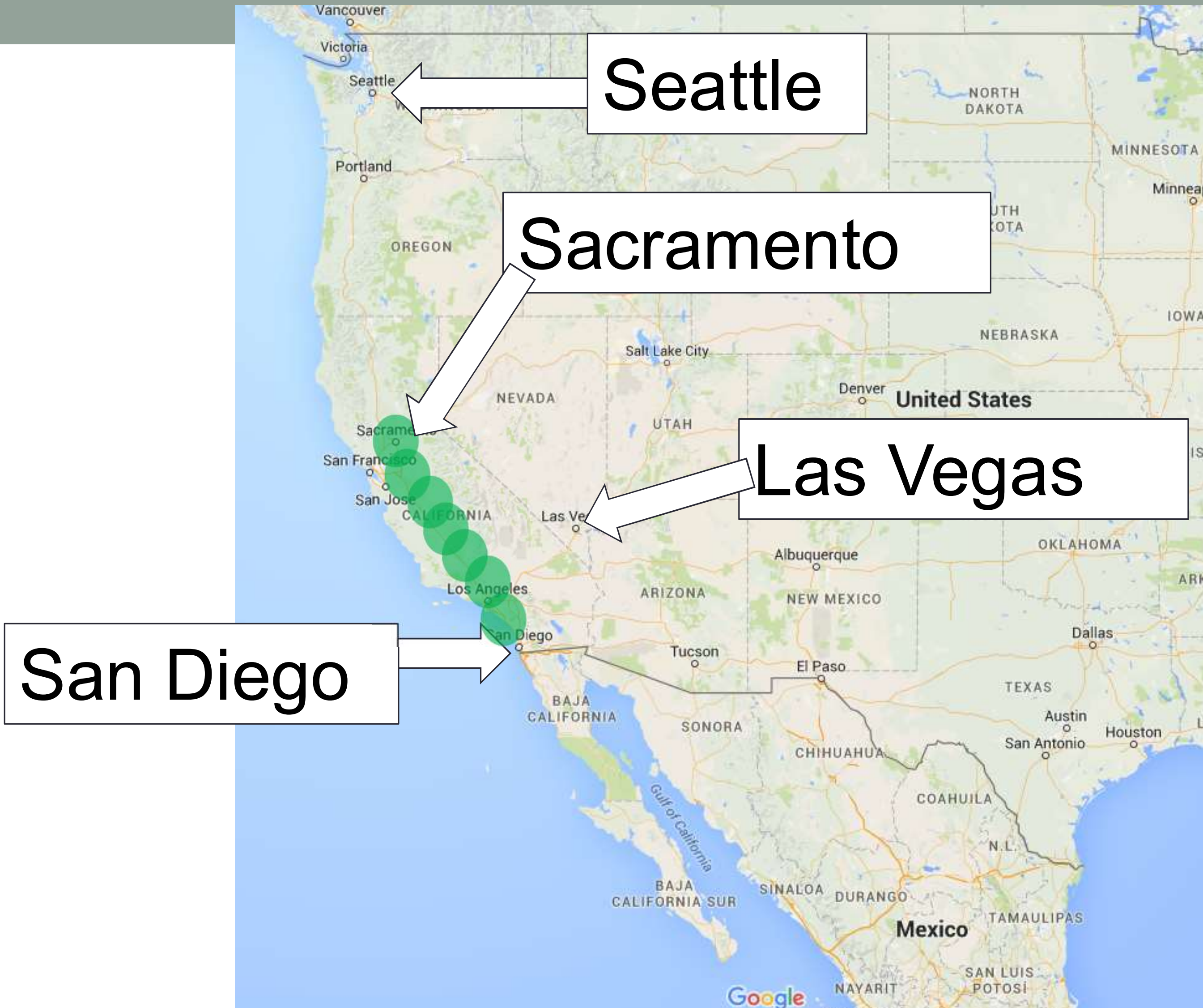
maps.google.com

A*



A*



http://distancecalculator.globefeed.com/World_Distance_Calculator.asp

maps.google.com

A*

Seattle

Sacramento

f(n) = 504 + 625 = 1129

Las Vegas

f(n) = 331 + 871 = 1202

San Diego

http://distancecalculator.globefeed.com/World_Distance_Calculator.asp

A* Algorithm

- Priority Queue ordering is based on:

g(n) the distance (cost) from start vertex to vertex n

AND

h(n) the heuristic estimated cost from vertex n to goal vertex

f(n) = g(n) + h(n)

Just change the priority function!